ME2 Computing- Coursework summary

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A) What physics are you trying to model and analyse? (Describe clearly, in words, what physical phenomenon you wish to analyse)

We aim to model the wave phenomenon in a vibrating string, including wave propagation, reflection, interference and eventually demonstrate the formation of stationary wave at certain fundamental frequency. Specifically, we simulate the motion of a string that is fixed at one end while a harmonic wave is introduced from the other end via an oscillator.

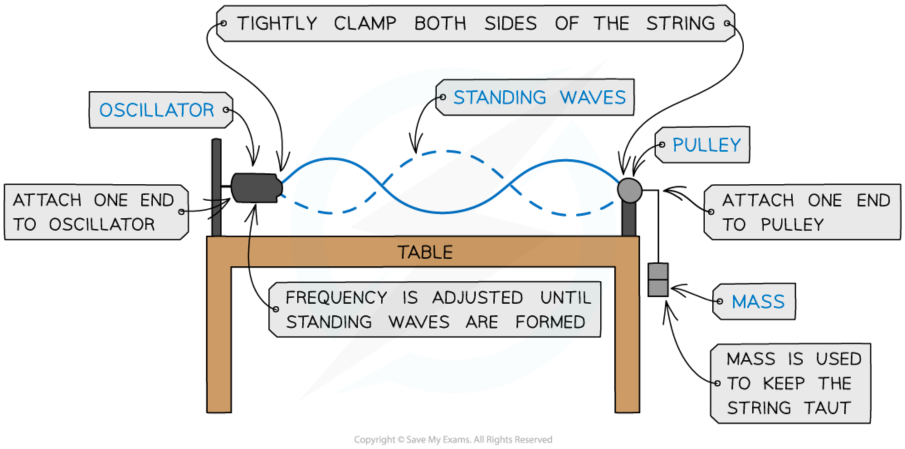

Figure 1: Stationary wave on a stretched string (Katie, 2024).

B) What PDE are you trying to solve, associated with the Physics described in A? (write the PDE)

The motion of vibrating string can be described using the one-dimensional wave equation, which is a hyperbolic PDE:

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$

Where x is the horizontal distance along the string, t is the time elapsed, u is the vertical displacement of the string from equilibrium position, and c is the speed of wave travelling.

C) Boundary value and/or initial values for my specific problem: (be CONSISTENT with what you wrote in A)

The wave equation is second order in both space and time derivatives; hence two sets of initial and boundary conditions will be needed. For travelling pulses send to the string from one end,
the boundary conditions are:

$$u(t, x = 0) = A_0 \cdot \sin(\omega t) \quad \{t < T\}$$
$$u(t, x = L) = 0$$

where A_0 is the amplitude of the excitation wave and ω is the excitation frequency (rads^{-1}); note that in the code, we express ω = 2πf, where f is the excitation frequency in Hertz (Hz).

And the initial conditions:

$$u(t = 0, x) = 0$$
$$\left.\frac{du}{dt}\right|_{t=0,x} = 0$$

Because the wave is initially stationary.

D) What numerical method are you going to deploy and why? (Describe, in words, which method you intend to apply and why you have chosen it as opposed to other alternatives)

For the explicit method, the central finite difference scheme is used. This method is second-order accurate in both space and time. Other alternatives, such as applying the Method of Lines to solve the PDE as a system of ODEs, were considered. However, while this approach allows for the use of higher-accuracy ODE methods (e.g., RK4), it increases algorithmic complexity and is deemed unnecessary for this simulation.

For the implicit method, we have chosen Crank Nicholson method due to its unconditional stability. It is second order accurate in both space and time. To avoid complexity in algebra while constructing the matrix, we have implemented logic statements to enforce boundary conditions on the edge nodes and only enforce initial condition on the initial time step. Since the quality of the wave simulation can be refined with finer spatial discretisation, the coefficient matrix will have a greater size, thus we have employed a more efficient method to solve for banded linear systems [Ax = B] by using LU decomposition rather than using Gaussian elimination. We have used a customised function from the SciPy library to solve for large and sparse matrix.

E) I am going to discretise my PDE as the following (show the steps from continuous to discrete equation and boundary/initial conditions:

To discretise the wave equation, the practical central finite difference for second order derivative is used to replace the second order partial derivatives.

The partial derivative of u with respect to x is discretised as:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\Delta x^2}$$

Similarly for partial derivative with respect to t,

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{\Delta t^2}$$

For notations, k is the index for time step, i is the index for spatial steps in x-direction.

Hence the discretised wave equation is:

$$\frac{1}{\Delta t^2}\left(u_i^{k+1} - 2u_i^k + u_i^{k-1}\right) = \frac{c^2}{\Delta x^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right)$$

Which can be expressed explicitly as:

$$u_{i,j}^{k+1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) + 2u_i^k - u_i^{k-1}$$

Note that for the explicit method, CFL condition must be satisfied for the solution to converge:

$$\frac{c\Delta t}{\Delta x} \leq C_{max} = 1$$

Implicit Discretisation:

$$\frac{1}{\Delta t^2}\left(u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1}\right) = \frac{c^2}{2\Delta x^2}\left(u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1} + u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k\right)$$

$$\left(u_i^{k+1} - 2u_i^k + u_i^{k-1}\right) = \frac{\Delta t^2 c^2}{2\Delta x^2}\left(u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1} + u_{i+1}^k - 2u_i^k + u_{i-1}^k\right)$$

$$r = \frac{\Delta t^2 c^2}{2\Delta x^2}$$

$$\left(u_i^{k+1} - 2u_i^k + u_i^{k-1} +\right) = r\left(u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1} + u_{i+1}^k - 2u_i^k + u_{i-1}^k\right)$$

$$(1+2r)\, u_i^{k+1} - r\left(u_{i+1}^{k+1} + u_{i-1}^{k+1}\right) = (2-2r)u_i^k + r(u_{i+1}^k + u_{i-1}^k) - u_i^{k-1}$$

E) I am going to discretise my PDE as the following (cont…)

First time step (k=0):

$$(1 + 2r)\, u_i^1 \ - r(u_{i+1}^1 + \ u_{i-1}^1) = (2 - 2r)u_i^0 \ + \ r(u_{i+1}^0 + u_{i-1}^0) - u_i^{-1}$$

Ghost node is u_i^{-1}, it can be eliminated by discretising the initial time derivative (ie. initial speed) with central finite difference, where the derivative can be a function of x or simply a constant.

$$\left.\frac{du_i}{dt}\right|_{t=0} = \frac{1}{2\Delta t}\,(\,u_i^1 \ - \ u_i^{-1}) = g(x) = constant, \text{G}$$

For our application, the initial speed is zero throughout the length of the string, but we will use constant C to represent a general discretisation and substituting back into the equation to remove ghost node.

$$u_i^{-1} \ = \ u_i^1 \ - \ G * 2\Delta t$$

$$(1 + 2r)\, u_i^1 \ - r(u_{i+1}^1 + \ u_{i-1}^1) = (2 - 2r)u_i^0 \ + \ r(u_{i+1}^0 + u_{i-1}^0) - \ u_i^1 \ + \ G * 2\Delta t$$

$$(2 + 2r)\, u_i^1 \ - r(u_{i+1}^1 + \ u_{i-1}^1) = (2 - 2r)u_i^0 \ + \ r(u_{i+1}^0 + u_{i-1}^0) + \ G * 2\Delta t$$

To construct matrices for solving unknowns in each step in time, the coefficient matrix and the known vector are reconstructed using the knowns at the boundaries. Hence, the nearest unknown nodes (i = 1 and i = N-1, where N is the number of nodes in the space domain) to both boundaries have a different set of coefficient matrix and known vector. We apply Dirichlet boundary conditions.

i = 1:

$$(1 + 2r)\, u_1^{k+1} \ - r(u_2^{k+1} \ + \ u_0^{k+1}) = (2 - 2r)u_1^k \ + \ r(u_2^k \ + u_0^k) - u_1^{k-1}$$

$$u_0^{\forall k} \ = A_0 \cdot \sin(\omega t_k) = A_k$$

$$(1 + 2r)\, u_1^{k+1} \ - r(u_2^{k+1} \ + A_{k+1}) = (2 - 2r)u_1^k \ + \ r(u_2^k \ + A_k) - u_1^{k-1}$$

$$(1 + 2r)\, u_1^{k+1} \ - r\, u_2^{k+1} \ = (2 - 2r)u_1^k \ + \ r(u_2^k \ + A_k) - u_1^{k-1} + \ rA_{k+1}$$

$$(1 + 2r)\, u_1^{k+1} \ - r\, u_2^{k+1} \ = (2 - 2r)u_1^k \ + \ r(u_2^k \ + A_k) - u_1^{k-1} + \ rA_{k+1}$$

i = N-1:

$$(1 + 2r)\, u_{N-1}^{k+1} - r(u_N^{k+1} \ + \ u_{N-1}^{k+1}) = (2 - 2r)u_{N-1}^k + \ r(u_N^k \ + u_{N-2}^k) - u_{N-1}^{k-1}$$

$$u_N^{\forall k} \ = B_k$$

$$(1 + 2r)\, u_{N-1}^{k+1} - r\, u_{N-1}^{k+1} \ = (2 - 2r)u_{N-1}^k + \ r(B_k \ + u_{N-2}^k) - u_{N-1}^{k-1} + rB_{k+1}$$

F) Plot the numerical results comprehensively and discuss them (discuss how the results describe the physics and comment on any discrepancies or unexpected behaviours). Use multiple types of visual graphs.

<u>Numerical results obtained with explicit method:</u>

The following plots are plotted with Δt = 0.05 and Δx= 0.05/0.005/0.0049 respectively, demonstrating the influence of the size of discretisation steps on numerical results.

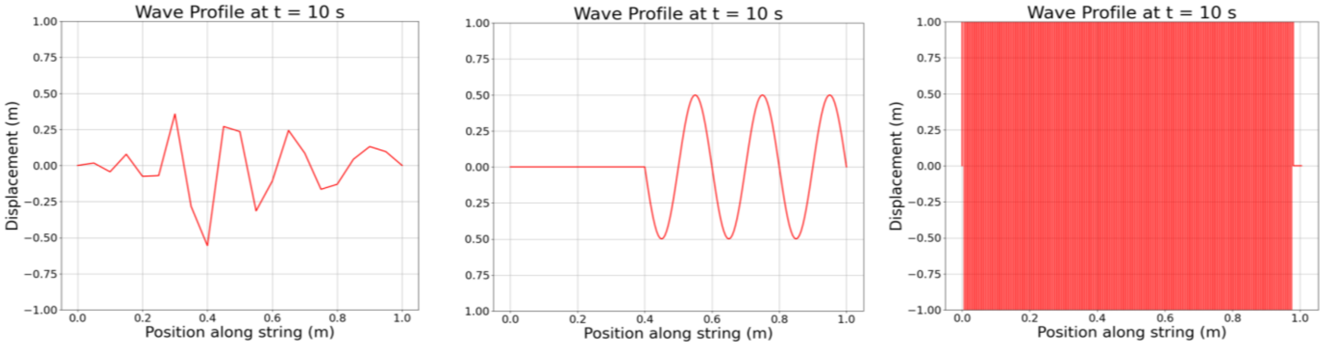

Figure 2: Displacement against time plot for numerical results obtained using explicit method

<u>Numerical results obtained with implicit method:</u>

The following plots are plotted with Δt = 0.05 and Δx= 0.05/0.005/0.0049 respectively,

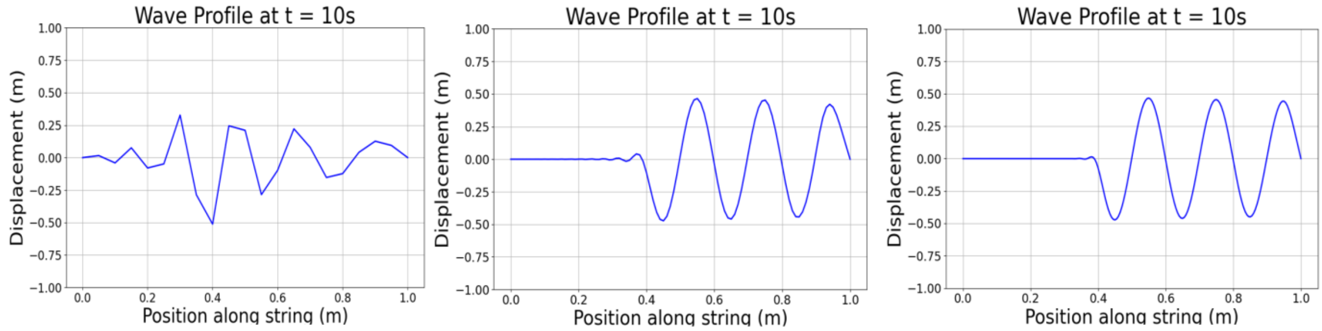

Figure 3: Displacement against time plot for numerical results obtained using implicit method

With large Δx, the CFL stability condition is easily satisfied, but the simulation quality is poor. A smaller Δx better captures the wave behaviour but increases the CFL number, imposing stricter stability constraint on Δt. When CFL condition is violated, the explicit method becomes unstable. To satisfy CFL condition, Δt needs to be reduced as we refine Δx. Although reducing Δt ensures stability, it also increase the number of evaluations required, leading to higher computational costs. In contrast, the implicit method remains unconditionally stable for small steps. As shown in Figure 2 and Figure 3, when $\Delta x = 0.0049$, the CFL number reaches 1.02, the explicit solution diverges while implicit method remains stable and solution converges. This highlights the key advantage of implicit method over explicit despite higher memory requirement.

F) Plot the numerical results comprehensively and discuss them (discuss how the results describe the physics and comment on any discrepancies or unexpected behaviours). Use multiple types of visual graphs.

To demonstrate wave behaviour, a sinusoidal wave is sent to the string with 0.5Hz frequency speed 0.1 ms^{-1}.

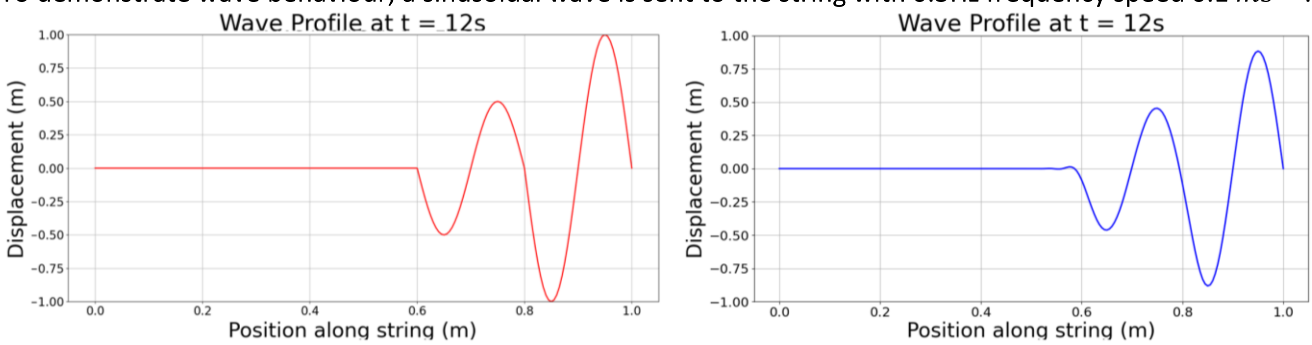

Figure 4: Displacement against position graph showing the snapshot of wave profile at t = 12s

Figure 4 illustrates the wave superposition phenomenon. In the explicit result, constructive interference is evident. Initially, the incident wave amplitude was 0.5m. When the first pulse reached the fixed end, it reflects and travels in the opposite direction (see [animation]). As it superposed with the incident wave, constructive and destructive interference occurred.

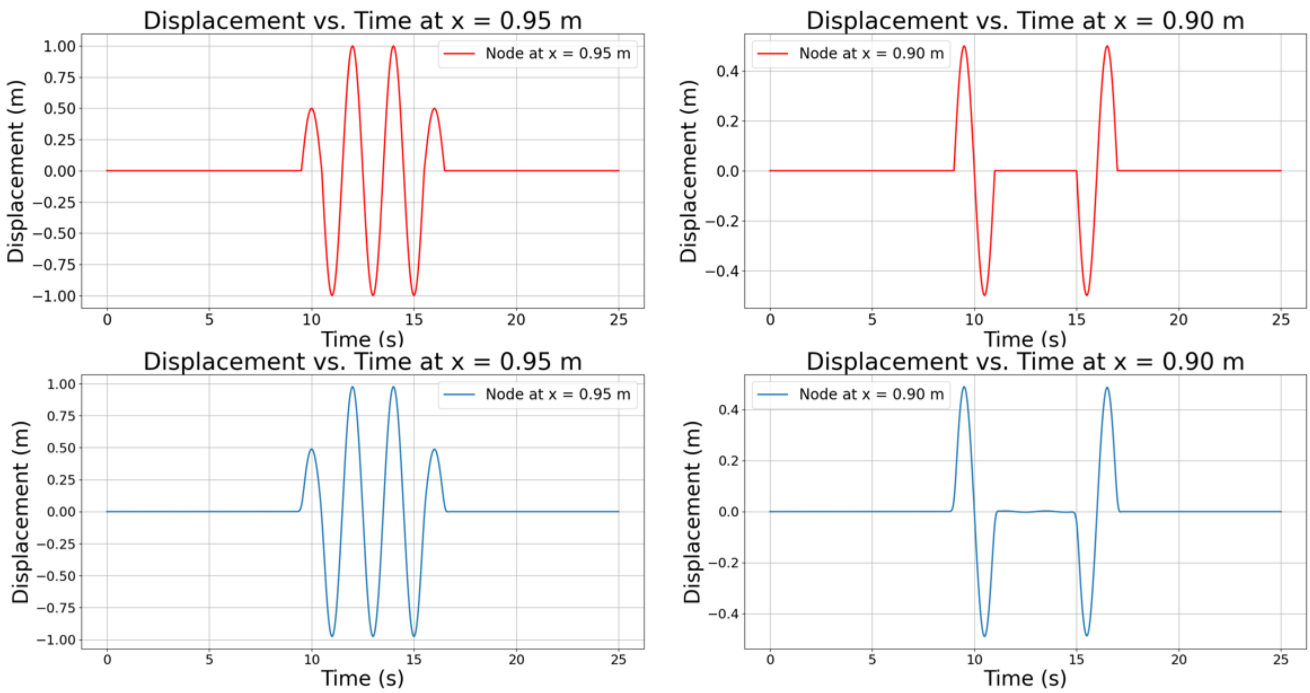

Figure 5: Displacement against time graph showing the change of amplitude with respect to time at specific nodes (x=0.90 and x=0.95), Explicit - Red; Implicit - Blue

Figure 5 illustrates wave amplitude at a specific node to demonstrate part of the “standing wave” simulated by our solutions. Standing wave is formed due to superposition of two identical waves traveling into each other (see Figure 6). The reflection of wave at the fixed end creates this effect, causing a synchronised constructive and destructive interference effect of waves.

F) Plot the numerical results comprehensively and discuss them (discuss how the results describe the physics and comment on any discrepancies or unexpected behaviours). Use multiple types of visual graphs.

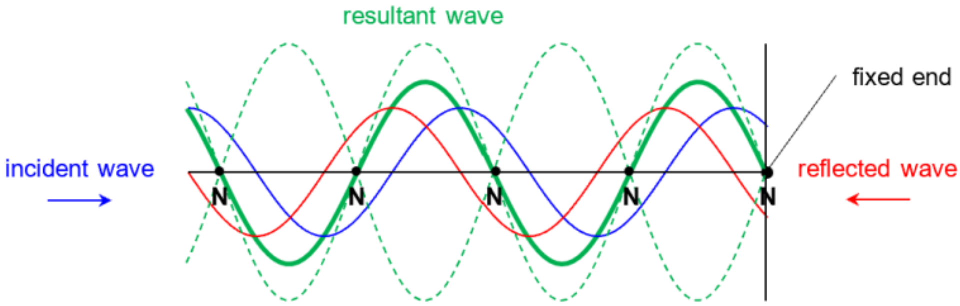

Figure 6: Standing wave phenomenon on a string with fixed end (XM Physics, 2023).

With the boundary conditions employed in our code, one end of the boundary is always a node: amplitude always remain zero. Referring to 6, we know that the next node(N) should occur at half a wavelength from the previous node. Since the wavelength is 0.2 m, x=0.9 m is expected to be a node. Moreover, an antinode – where the amplitude is maximum – appears exactly between two nodes; hence, x=0.95 m would be an antinode. These two behaviours can be observed in Figure 5, where the amplitude at x=0.95 m is 1 m (maximum amplitude) and amplitude at x=0.9 m remains zero when the waves interfere from t = 11 to t =16s. This evidence proves that the wave undergoes a 180° phase shift when reflecting from a fixed end, which aligns with real-world physics (see animation).

G) Other remarks (limits of the model, convergence problems, possible alternative approaches, anything you find relevant and important to mention):

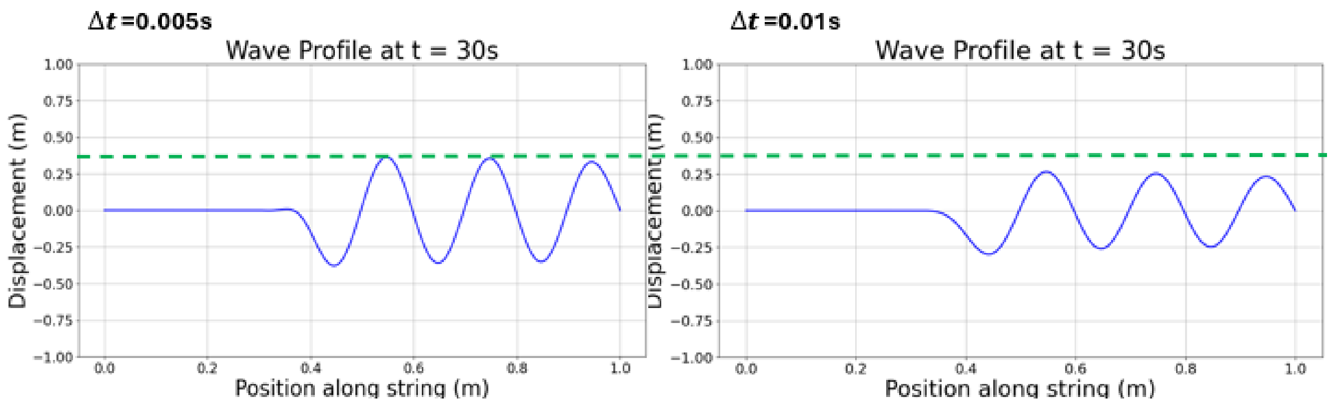

Figure 7: Numerical damping shown in Crank Nicolson method.

Limitation:
Numerical damping is shown in Crank Nicolson method but the damping effect will reduce with finer time steps. The problems that arise from numerical damping is obvious damping as the wave propagates through in time and there will be phase shifts observed which will disrupts the periodic motion of the wave. Since this method is taking the average of the current (implicit) and previous (explicit) time steps, the amount of numerical damping depends on the amplification factor of each method. In this case the overall effect is dampened.

Watch animation for obvious damping when $\Delta t = 0.05s$.

Reference:
Katie M. (2024) *Stationary waves*, Save My Exams. Available at: https://www.savemyexams.com/a-level/physics/ocr/17/revision-notes/4-electrons-waves-and-photons/4-9-superposition-and-stationary-waves/4-9-7-stationary-waves/ (Accessed: 21 March 2025).
XM Physics (2023) *Wave reflections*, XM Physics. Available at: https://xmphysics.com/2023/01/02/10-7-1-wave-reflections/ (Accessed: 21 March 2025).