Uniformization of Multi-Valued Functions

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1 Multi-Valued Functions

Definition 1.1. Let X and Y be two sets, $P(Y)$ is the set of all of subsets contained in Y. The Function f is called a multi-valued function, if $f : X \to P(Y)$. Such f can be denoted by $f : X \rightarrowtail Y$. The domain of definition of f is denoted by $D(f)$.

Example 1.2. The function $\sqrt{-} : \mathbb{C} \to \mathbb{C}$ is a multi-valued function.

Definition 1.3. Let X and Y be two topology spaces, $f : X \rightarrowtail Y$ is a multi-valued function. For $x \in D(f)$, x is called a regular point, if there exists neighborhood U of x, a disjoint collection $\{V_i : i \in I\}$ of open sets in Y and a collection of continous maps $\{f_i : U \to V_i : i \in I\}$, such that for any $y \in U$, we have $f(y) = \{f_i(y) : i \in I\}$.

All of regular points in $D(f)$ forms a open set $R(f)$, and points in $D(f) - R(f)$ are called branching points.

Example 1.4. Consider $P_n = \{$all the monic polynomials over $\mathbb{C}$ with degree $d \leq n$ $\}$, we view it as $\mathbb{C}^n$ with Euclidean topology. Then we define $\Psi : P_n \rightarrowtail \mathbb{C}$, $\Psi(f) = \{$the roots of $f\}$. We find that $R(P_n) = \{$all the separable polynomails$\}$, denoted by P_n^s.

2 Regular Path Lifting Theorem

Definition 2.1. Consider multi-valued function $f : X \rightarrowtail Y$. A path $\gamma : I \to R(f)$ in $R(f)$ is a regular path. The path $\tilde{\gamma} : I \to Y$ is a lift of γ, if $\tilde{\gamma}(t) \in f(\gamma(t))$ for any $t \in I$.

Theorem 2.2. There is a multi-valued function between topology spaces $f : X \rightarrowtail Y$. Suppose $\gamma : I \to R(f)$ is a regular path, $x_0 = \gamma(0)$. Choose $y_0 \in f(x_0)$, then there exists a unique lift $\tilde{\gamma}$ of γ, such that $\tilde{\gamma}(t) \in f(\gamma(t))$ for any $t \in I$ and $\tilde{\gamma}(0) = y_0$.

Proof. $\forall x \in \gamma(I), \exists$ a neighborhood U_x of x, a disjoint collection $\{V_i : i \in I\}$ of open sets in Y and a collection of continous maps $\{f_i^x : U_x \to V_i : i \in f(x)\}$, such that for any $y \in U_x$, we have $f(y) = \{f_i^x(y) : i \in f(x)\}$. The collection $\{U_x : x \in \gamma(I)\}$ forms a open covering of subspace $\gamma(I)$. By Lebesgue's Lemma, $\exists n \in \mathbb{N}$, such that $\gamma([\frac{i-1}{n}, \frac{i}{n}]) \subset U_{x_i}$ for some $x_i \in \gamma(I)$, where $1 \leq i \leq n$.

Then for $\frac{i-1}{n} \leq t \leq \frac{i}{n}$, we can define $\tilde{\gamma}(t) = f^{x_i}_{\tilde{\gamma}(\frac{i-1}{n})}(\gamma(t))$ inductively, the basic case holds because we fix $y_0 = \tilde{\gamma}(0)$. By Pasting Lemma we can check $\tilde{\gamma}$ is a path in Y. Also we can check $\tilde{\gamma}(t) \in f(\gamma(t))$ for any $t \in I$. Thus $\tilde{\gamma}$ is the lift we need and it is unique.

$\square$

Using the same techique, we can also proof the following theorem.

Theorem 2.3. If two paths η and γ are homotopic rel endpoints, then their lifts $\tilde{\eta}$ and $\tilde{\gamma}$ are also homotopic rel endpoints.

3 Uniformization Groups

Definition 3.1. For a multi-valued function $f : X \rightarrowtail Y$ and a path $\gamma : I \to X$ from x_0 to x_1, we define $M(f,\gamma) : f(x_0) \to f(x_1)$, $y_0 \in f(x_0) \mapsto \tilde{\gamma}_{y_0}(1)$.

By theorem 2.2 we can check this map is well-defined. Theorem 2.3 shows that $M(f,\gamma)$ depends on the homotopy class of γ, and $M(f,\gamma)$ is bijective obviously.

Theorem 3.2. Consider multi-valued function $f : X \rightarrowtail Y$. Fix $x_0 \in R(f)$, then we have group homomorphism:

$$\rho : \pi_1(R(f), x_0) \to \mathsf{S}_{f(x_0)}$$
$$[\gamma] \mapsto M(f,\gamma)$$

Such ρ is called the uniformization homomorphism of f, and the image of ρ is called the uniformization group of f, denoted by $M(f,x_0)$

Proof. By theorem 2.3, we know that if $[\gamma_1] = [\gamma_2]$, then $M(f,\gamma_1) = M(f,\gamma_2)$. Since $M(f,\gamma)$ is bijective, $M(f,\gamma) \in \mathsf{S}_{f(x_0)}$.

Since $\rho([\gamma])\rho(\eta) = M(f,\gamma)M(f,\eta) = M(f,\gamma\cdot\eta) = \rho([\gamma\cdot\eta]) = \rho([\gamma])\rho([\eta])$, ρ is a group homomorphism. □

We have the following proposition.

Proposition 3.3. If x_1 and x_2 are in the same path connected component of $R(f)$, then $M(f,x_1) \cong M(f,x_2)$.

Definition 3.4. Let G be a group. The commutant of G, $G^{'}$, is the subgroup $\{aba^{-1}b^{-1} : \forall a,b \in G\}$.Then we define $G^{(1)} = G/G^{'}$ and $G^{(k)} = G^{(k-1)}/(G^{(k-1)})^{'}$ for $2 \le k$. The length of G, $I(G)$, is the smallest integer n such that $G^{(n)} = (e)$.

Definition 3.5. For a multi-valued function $f : X \rightarrowtail Y$, the complexity of f is defined as follow

$$c(f) = max\{I(M(f,x_0)) : \forall x_0 \in X\}$$

Example 3.6. Consider $\sqrt[n]{-} : \mathbb{C} \rightarrowtail \mathbb{C}$. $R(f) = \mathbb{C} - \{0\}$. Choose base point $x_0 = 1$, $\pi_1(R(\sqrt[n]{-}), x_0)$ is generated by $[\gamma]$, where $\gamma(t) = e^{i2\pi t}$. And $\sqrt[n]{x_0} = \{1,\xi,...,\xi^{n-1}\}$, where $\xi = e^{\frac{2\pi i}{n}}$. Then γ has n lifts $\tilde{\gamma}_{\xi^k}(t) = \xi^k e^{\frac{2\pi i}{n}}$, thus $M(\sqrt[n]{-},\gamma) = (1\xi\xi^2...\xi^{n-1})$. Hence, $M(\sqrt[n]{-},x_0) \cong \mathbb{Z}_n$ and $c(\sqrt[n]{-}) = 1$

Example 3.7. $M(\Psi_n,x_0) \cong \mathsf{S}_n$, where Ψ_n maps a polynomial to its roots and x_0 is $p(z) = z^n - 1$.

Proof. The polynomial x_0 has roots $\{1,\xi,...,\xi^{n-1}\}$. For ξ^j and ξ^k, choose a path η from ξ^j to ξ^k and a path λ from ξ^k to ξ^j, such that $\eta(0,1) \cap \Psi_n = \varnothing$ and $\lambda(0,1) \cap \Psi_n = \varnothing$. Consider the polynomial

$$p_t(z) = (z - \eta(t))(z - \lambda(t)) \prod_{i \neq j,k} (z - \xi^i)$$

We can define a closed path $\gamma : I \to P_n^s$, $t \mapsto p_t(z)$, which start from x_0 and $(\xi^j\xi^k) = M(\Psi_n,\gamma) \in M(\Psi_n)$. Hence $M(\Psi_n,x_0) \cong \mathsf{S}_n$. □

4 The Solvability of Multi-Valued Functions

For the discussion, we will directly give the following definitions and theorems.

Definition 4.1. Given a multi-valued function $f : X \rightarrowtail Y$ and a regular closed path $\gamma : I \to R(f)$. The path γ, is called 1-st commutant path if there exists regular closed paths γ_1 and γ_2, such that $\gamma = \gamma_1\gamma_2\bar{\gamma}_1\bar{\gamma}_2$.
The path γ is called k-th commutant path if there exists (k-1)-th commutant paths γ_1 and γ_2, such that $\gamma = \gamma_1\gamma_2\bar{\gamma}_1\bar{\gamma}_2$.

Definition 4.2. A regular closed path γ is liftable if $\rho(\gamma) : f(x_0) \to f(x_0)$ is identity.

Obviously, if the complexity of $f : X \rightarrowtail Y$ is 1, then there exists a regular closed path which is unliftable, and each 1-st commutant path is liftable, since $M(f, x_0)$ is an abelian group. Then by induction, we can proof the following theorem.

Theorem 4.3. $c(f) = k \Leftrightarrow$ All k-th commutant paths are liftable, and there exists a (k-1)-th unliftable commutant path.
$c(f) = \infty \Leftrightarrow \forall k > 0$, there exists a k-th unliftable commutant path.

Definition 4.4. Given two multi-valued functions $f : X \rightarrowtail Y$ and $g : Y \rightarrowtail Z$. The composition of f and g is $g \circ f : X \rightarrowtail Z, x \mapsto \bigcup_{y \in f(x)} g(y)$.

Theorem 4.5. If $R(g) = Y$, then $c(g \circ f) \leq c(g) + c(f)$.

Proof. Without the loss of generality, we assume $c(g) = 1$, and suppose $c(f) = 1$. Choose a (k+1)-th commutant path γ in $R(g \circ f)$ base at x_0, i.e. there exists k-th commutant path γ_1 and γ_2 such that $\gamma = \gamma_1\gamma_2\bar{\gamma}_1\bar{\gamma}_2$. Since $c(f) = k$, then γ_1 and γ_2 can be lifted to $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ by f, and endpoint is $y_0 \in f(x_0)$. Thus γ has a lift $\tilde{\gamma}$ in Y, satisfying $\tilde{\gamma} = \tilde{\gamma}_1\tilde{\gamma}_2\bar{\tilde{\gamma}}_1\bar{\tilde{\gamma}}_2$. Since $c(g) = 1$, $\tilde{\gamma}$ has a lift $\hat{\gamma}$ in Z by g with endpoint $z_0 \in g(y_0) \subset g \circ f(x_0)$. Hence, by theorem 4.3, $c(g \circ f) \leq k + 1$. □

Definition 4.6. Given $f, g : X \rightarrowtail Y$, we say that $f \subset g$ if $f(x) \subset g(x) \forall x \in X$.

Theorem I. f $f \subset g$ and $R(f) \subset R(g)$, then $c(f) \leq c(g)$.

Proof. Using theorem 4.3 and definition 4.4, the proof is trivial. □

In summary, we have the following useful theorem.

Theorem 4.7. Suppose $f : X \rightarrowtail Y$ is contained the composition of multi-valued functions $X \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = Y$, and each component function $f_i : X_{i-1} \rightarrowtail X_i$ is regular. Then $c(f) \leq \sum_{i=1}^{n} c(f_i)$.

Definition 4.8. A multi-valued function f is solvable if $c(f) < \infty$, otherwise it is unsolvable.

Immediately, an unsolvable function is not a composition of finite many solvable function.

5 A Topology Proof of Abel Theorem

Before proofing Abel Theorem, we need to introduce some facts about Zariski topology on $\mathbb{C}^n$.

Definition 5.1. Subset $A \subset \mathbb{C}^n$ is a Zariski closed set, if there exists finite many polynomials $f_1, ..., f_k$, such that every point in A is the root of one if the above polynomials.

Proposition 5.2. We have some facts about Zariski topology.

1. Arbitrary intersection of Zarisiki closed set is closed.
2. Finte union of Zarisiki closed set is closed.
3. The inverse image of a Zariski closed(open) set under polynomial map is closed(open).
4. Finite intersection of nonempty Zariski open sets is nonempty.
5. Nonempty Zariski open sets are path connected.

To translate the algebraic language of Abel Theorm into the language of multi-valued function, we need to define the rational function and the radical function.

Definition 5.3. Consider multivariate polynomial functions $f_1, ..., f_m, g_1, ..., g_m : \mathbb{C}^n \to \mathbb{C}$, we call the function

$$(\frac{f_1}{g_1}, ..., \frac{f_m}{g_m}) : U \to \mathbb{C}^m$$

$$(z_1, ..., z_n) \mapsto (\frac{f_1(z_1, ..., z_n)}{g_1(z_1, ..., z_n)}, ..., \frac{f_m(z_1, ..., z_n)}{g_m(z_1, ..., z_n)})$$

rational function, where $U = \{(z_1, ..., z_n) \in \mathbb{C}^n : g_i(z_1, ..., z_n) \neq 0 \forall 1 \leq i \leq m\}$ is a Zariski open set.

Definition 5.4. Let $k_1, ..., k_n$ be integers. We call the function

$$f : \mathbb{C}^n \rightarrowtail \mathbb{C}^n$$

$$(z_1, ..., z_n) \mapsto \{(y_1, ..., y_n) \in \mathbb{C}^n : y_i \in \sqrt[k_i]{z_i} \forall 1 \leq i \leq n\}$$

multivariate radical function. And $R(f) = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_i \neq 0\}$.

Definition 5,5. (Roots-Finding Formula for Equation of Degree n) Let U be a nonempty Zariski open set in $P_n \cong \mathbb{C}^n$. Consider composition multi-valued function

$$f : U \rightarrowtail \mathbb{C}^{n_1} \rightarrowtail ... \rightarrowtail \mathbb{C}$$

where each $f_i : \mathbb{C}^{n_{i-1}} \rightarrowtail \mathbb{C}^{n_i}$ is either a rational function, or a multivariate radical function. If $\Psi_n|_U \subset f$, then we call f the roots-finding formula for equation of degree n.

Theorem 5.6. (Abel-Ruffini Theorem) If $n \geq 5$, there doesn't exist the roots-finding formula.

Proof. First we will proof $M(\Psi_n|_U) \cong \mathsf{S}_n$.

Consider function $\sigma : \mathbb{C}^n \to P_n$, $(z_1, ..., z_n) \mapsto (z - z_1)...(z - z_n)$ and $g \in \mathsf{S}_n$. Choose $p_0 \in U \cap R(\Psi_n)$, p_0 has roots $r_1, ..., r_n$. Then we have $\sigma(r_1, r_2, ..., r_n) = \sigma(r_{g(1)}, r_{g(2)}, ..., r_{g(n)}) = p_0$. Since σ is continous, then $(r_1, r_2, ..., r_n)$, $(r_{g(1)}, r_{g(2)}, ..., r_{g(n)}) \in \sigma^{-1}(U \cap R(\Psi_n))$ can be connected by a path γ. Thus we obtain a regular path $\sigma \circ \gamma$ base at p_0 and $M(\Psi_n, \sigma \circ \gamma) = g$. Hence $M(\Psi_n|_U) = \mathsf{S}_n$.

Let $n \geq 5$, suppose exists roots-finding formula, i.e. $\Psi_n|_U \subset f : U \rightarrowtail \mathbb{C}^{n_1} \rightarrowtail ... \rightarrowtail \mathbb{C}$. Define $V_i = R(f_i)$, then each V_i is nonempty Zariski open set. Then construct $U_i = V_i \cap f_i^{-1}(U_{i+1})$ by induction, we have $\Psi_n|_{U_1} \subset f|_{U_1} : U_1 \overset{f_1|_{U_1}}{\rightarrowtail} U_2 \overset{f_2|_{U_2}}{\rightarrowtail} ... \overset{f_k|_{U_k}}{\rightarrowtail} \mathbb{C}$, where each $f_i|_{U_i}$ is regular and solvable, but $\Psi_n|_{U_1}$ is unsolvable, then by theorem 4.7 this implies contradiction. $\square$