Modern Control Theory
Spring 2017

Weijun Zhang （张伟军）
Associate Professor, Robotics Institute
School of Mechanical Engineering
Office:922 Building A of ME school
319 Building B of ME School

zhangweijun@sjtu.edu.edu.cn
021-34205559

Outline of Today's Lecture

- Transformation from TF to SS Model
- Linear Transformation between different SS Models
- Diagonal Canonical Form and Jordan Canonical Form
- The matlab example of control system

Similarity Transformation

- Given one state space representation, any nonsingular linear transformation of that state, such as **x= Tz** (T is nonsingular), is called a similarity transformed pair of the original representation.

$$\dot{x} = Ax + Bu\,;\;\; x(0) = x_0$$
$$y = Cx + Du$$

$x = Tz \quad z = T^{-1}x$ Nonsingular Matrix T

$$\dot{z} = T^{-1}ATz + T^{-1}Bu\,;\;\; z(0) = T^{-1}x(0) = T^{-1}x_0$$
$$y = CTz + Du$$

Transformation of System Models using Matlab

Example: Find the eigenvalues and eigenvector of Matrix A

The SS model of system is

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} x$$

The initial state is: $x_1(0)=1, x_2(0)=1$

The transformation Matrix is

$$P = \begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix}$$

Find the new system in SS model.

Transformation of System Models using Matlab

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \qquad P = \begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix} \qquad x_1(0)=1, x_2(0)=1$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} x$$

```
>> P=[6 2; 2 0];

>> P1=inv(P)

P1 =

        0    0.5000

   0.5000   -1.5000
```

```
>> A=[0 2;1 -3]; A1=P1*A*P

A1 =

     0     1

     2    -3
```

```
>> B=[2,0]'；   >> B1=P1*B

B1 =

     0

     1
```

Transformation of System Models using Matlab

```
>> C=[0 3]

C =

     0     3

>> C1=C*P

C1 =

     6     0
```

```
>> x0=[1,1]';

>> z0=P1*x0

z0 =

    0.5000

   -1.0000
```

Invariance of Eigenvalues

- Eigenvalues of n×n Matrix A

The eigenvalues of an n×n matrix A are the roots of the characteristic equation

$$\left|\lambda I - A\right| = 0$$

Example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} =$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda+1)(\lambda+2)(\lambda+3)$$

Eigenvalues are: -1,-2,-3

Invariance of Eigenvalues

- The eigenvalues of matrix A and $T^{-1}AT$ (T is nonsigular linear matrix) are same. (Invariance of the eigenvalues under a linear transformation)

$$
\begin{aligned}
\left|\lambda I - T^{-1}AT\right| &= \left|\lambda T^{-1}T - T^{-1}AT\right| \\
&= \left|T^{-1}(\lambda I - A)T\right| = \left|T^{-1}\right|\left|\lambda I - A\right|\left|T\right| \\
&= \left|\lambda I - A\right|
\end{aligned}
$$

- Nonuiqueness of a Set of State Variables

$$x = Tz \quad z = T^{-1}x$$

Characteristic Polynomial

系统特征方程:

$$\left|\lambda I - T^{-1}AT\right| = 0$$

特征多项式:

$$\left|\lambda I - A\right| = \lambda^{n} + a_{n-1}\lambda^{n-1} + \ldots + a_{1}\lambda + a_{0}$$

系统特征多项式的系数 $a_{n-1},\ldots,a_1,a_0$ 为系统的不变量

特征向量:

满足 $Ap_i = \lambda p_i$ 的矢量p_i为A的对应于λ_i的特征矢量。

Transformation of System Models using Matlab

- [V,D]=eig(A)   v: Eigenvector D: eigenvalues

Example: Find the eigenvalues and eigenvector of Matrix A

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

```
>> A=[0,1,-1;-6,-11,6;-6,-11,5];
>> [V,D]=eig(A)
```

```
V =

   0.7071   -0.2182   -0.0921
   0.0000   -0.4364   -0.5523
   0.7071   -0.8729   -0.8285
```

```
D =

  -1.0000         0         0
        0   -2.0000         0
        0         0   -3.0000
```

Outline of Today's Lecture

- Transformation from TF to SS Model
- Linear Transformation between different SS Model
- Diagonal Form and Jordan Form
- The Matlab Example of Control System

Jordan Canonical Form

1.Problem Description

$$\dot{x} = Ax + Bu\,;\;\; x(0) = x_0 \qquad \Rightarrow \qquad \dot{z} = Jz + T^{-1}Bu\,;\;\; z(0) = T^{-1}x_0$$

$$y = Cx \qquad\qquad y = CTz$$

$$J = \Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

$$J = \Lambda = \left[\begin{array}{cccc|ccc} \lambda_1 & 1 & & 0 & & & \\ & \lambda_1 & \ddots & & & 0 & \\ & & \ddots & 1 & & & \\ 0 & & & \lambda_1 & & & \\ \hline & & & & \lambda_{q+1} & & 0 \\ & 0 & & & & \ddots & \\ & & & & 0 & & \lambda_n \end{array}\right]$$

T ?

Diagonal Canonical Form

The state space realization in *diagonal canonical form*

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$=b_n+\frac{c_1}{s+p_1}+\frac{c_2}{s+p_2}+\ldots+\frac{c_n}{s+p_n}$$

$$c_i=(s+p_i)G(s)\Big|_{s=-p_i}$$

Diagonal Canonical Form (Cont.)

$$Y(s) = b_n U(s) + \frac{c_1}{s+p_1} U(s) + \frac{c_2}{s+p_2} U(s) + \ldots + \frac{c_n}{s+p_n} U(s)$$

Suppose

$$\begin{cases} \dfrac{1}{s+p_1} = x_1 \longrightarrow \dot{x}_1 = 1 - p_1 x_1 \\ \dfrac{1}{s+p_2} = x_2 \longrightarrow \dot{x}_2 = 1 - p_2 x_2 \\ \ldots\ldots \\ \dfrac{1}{s+p_n} = x_2 \longrightarrow \dot{x}_n = 1 - p_n x_n \end{cases} \Rightarrow$$

$$y = b_n u + c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

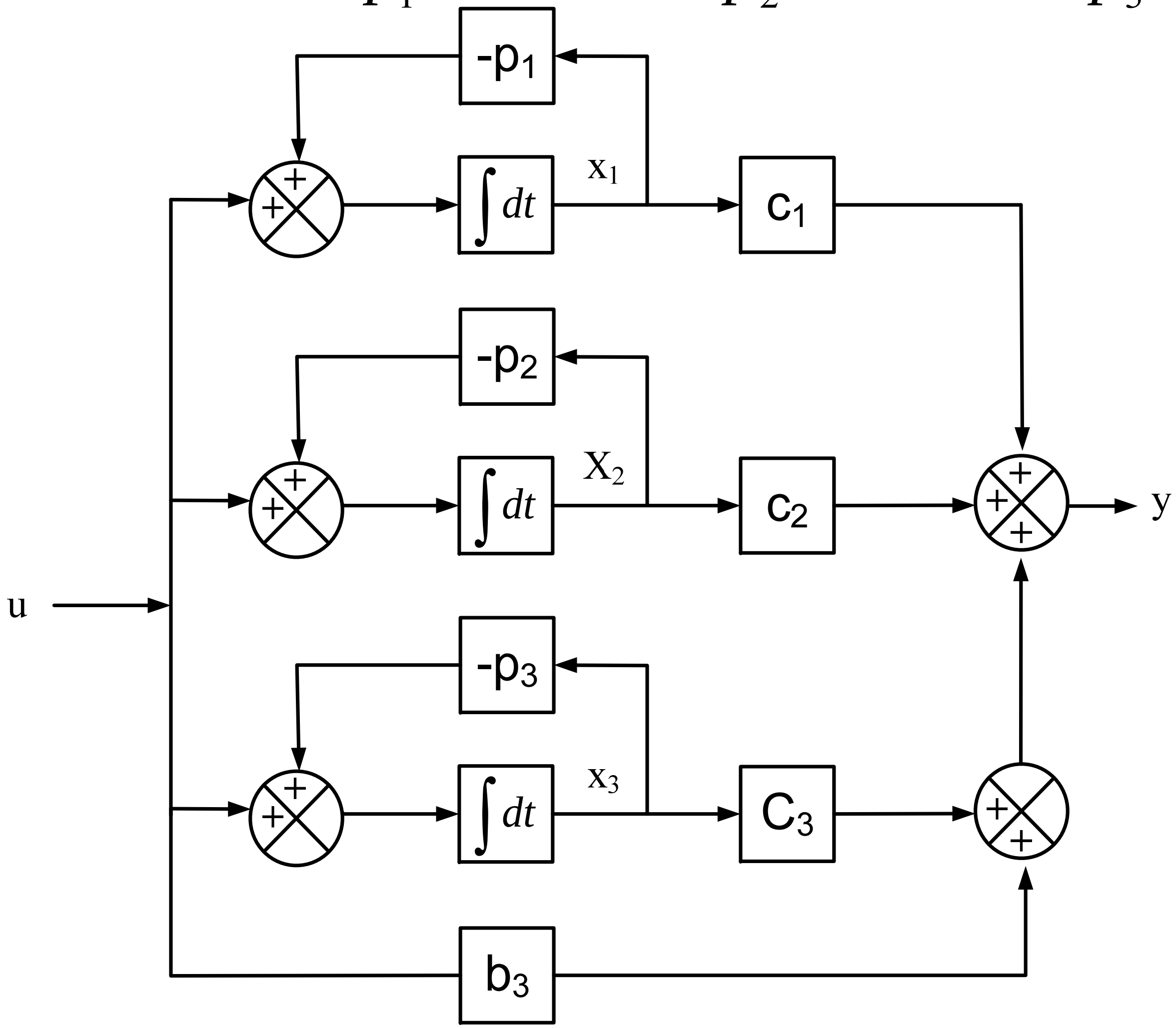
Diagonal Canonical Form (Cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & \dots & 0 & 0 \\ 0 & -p_2 & \dots & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \dots & -p_{n-1} & 0 \\ 0 & 0 & \dots & 0 & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & . & . & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_n u$$

The Block Diagram of Diagonal Canonical Form

$$Y(s) = b_3 U(s) + \frac{c_1}{s+p_1} U(s) + \frac{c_2}{s+p_2} U(s) + \frac{c_3}{s+p_3} U(s)$$



Jordan Canonical Form

（1）特征根无重根时

结论：设 λ_i 是A的n个互异根（i＝1，2，…,n）

则变换阵由特征矢量$p_1, p_2, \ldots, p_n$构成。即

$$T = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}$$

例题

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

化为对角线标准型

Jordan Canonical Form

```
>> A=[0 1 0;0 0 1;2 -1 2];
>> [V,D]=eig(A)
V =
  -0.0000 + 0.5774i  -0.0000 - 0.5774i  -0.2182
  -0.5774             -0.5774             -0.4364
   0.0000 - 0.5774i   0.0000 + 0.5774i  -0.8729
D =
   0.0000 + 1.0000i        0                 0
        0             0.0000 - 1.0000i       0
        0                  0            2.0000
```

Jordan Canonical Form

（1）特征根有重根时

结论：设A的特征根有q个是 λ_i 的重根，余下为单根。则重根对应的特征向量计算如下：

$$\lambda_1 P_1 - AP_1 = 0$$

$$\lambda_1 P_2 - AP_2 = -P_1$$

$$\cdots$$

$$\lambda_1 P_q - AP_q = -P_{q-1}$$

例题

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Jordan Canonical Form

```
>> A=[0 1 0;0 0 1; 2 3 0];
>> [V,J]=jordan(A)
```

V =

0.1111	0.6667	0.8889
0.2222	-0.6667	-0.2222
0.4444	0.6667	-0.4444

J =

2	0	0
0	-1	1
0	0	-1

Jordan Canonical Form

The state space realization in *Jordan canonical form*

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_n s^n+b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{(s+p_1)^3(s+p_4)(s+p_5)\ldots(s+p_n)}$$

$$=b_n+\frac{c_1}{(s+p_1)^3}+\frac{c_2}{(s+p_1)^2}+\frac{c_3}{s+p_1}+\frac{c_4}{s+p_4}+\ldots+\frac{c_n}{s+p_n}$$

$$c_1=(s+p_1)^3G(s)\Big|_{s=-p_1} \qquad c_2=\frac{d}{ds}(s+p_1)^3G(s)\Big|_{s=-p_1}$$

$$c_3=\frac{1}{2!}\frac{d^2}{ds^2}(s+p_1)^3G(s)\Big|_{s=-p_1}$$

Diagonal Canonical Form (Cont.)

$$Y(s) = b_n U(s) + \frac{c_1}{(s+p_1)^3}U(s) + \frac{c_2}{(s+p_1)^2}U(s) + \frac{c_3}{(s+p_1)}U(s)$$

$$+\frac{c_4}{s+p_4}U(s)\ldots+\frac{c_n}{s+p_n}U(s)$$

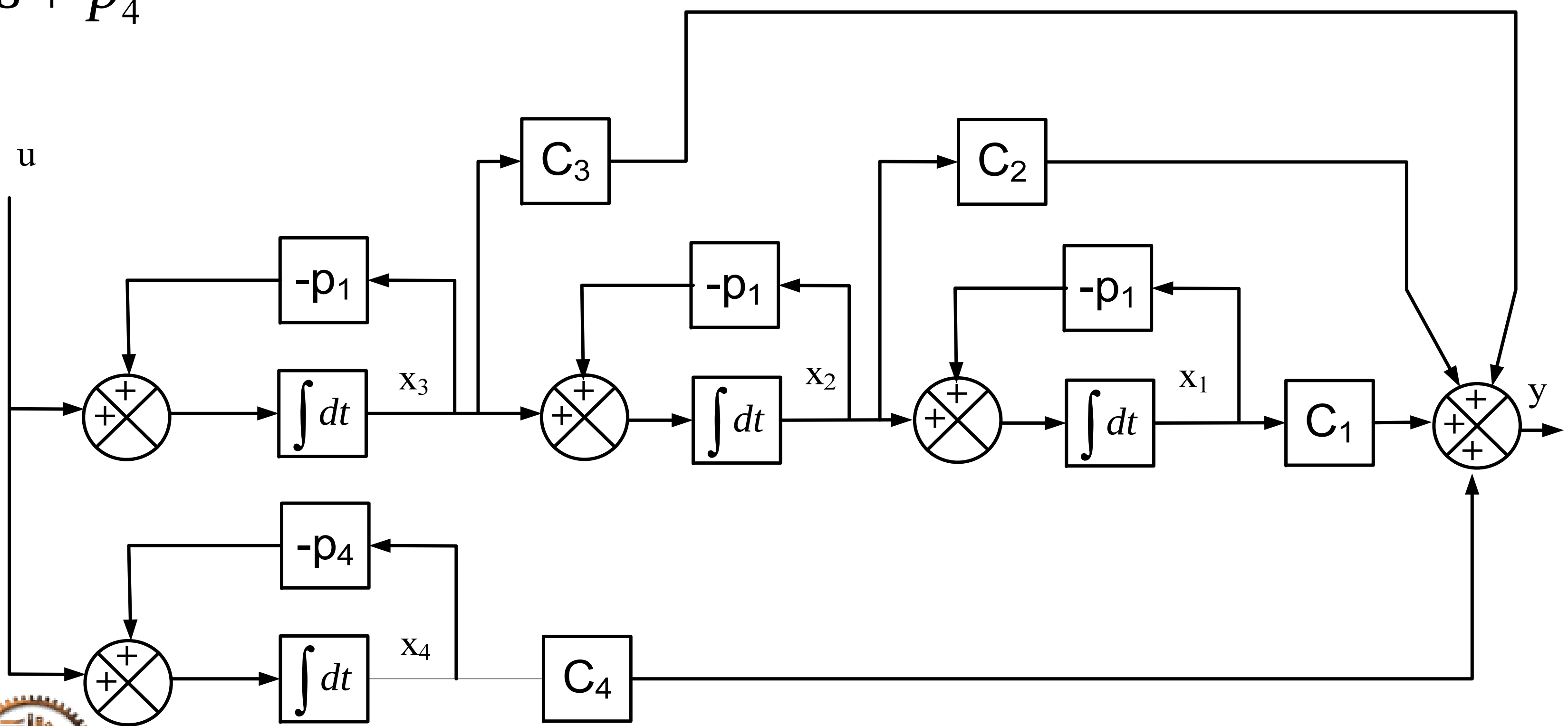
Suppose

$$\begin{cases} \dfrac{U(s)}{s+p_1} = x_3 \longrightarrow \dot{x}_3 = u - p_1 x_3 \\ \dfrac{U(s)}{(s+p_1)^2} = \dfrac{1}{(s+p_1)}x_3 = x_2 \longrightarrow \dot{x}_2 = x_3 - p_1 x_2 \\ \dfrac{U(s)}{(s+p_1)^3} = \dfrac{1}{(s+p_1)}x_2 = x_1 \longrightarrow \dot{x}_1 = x_2 - p_1 x_1 \\ \cdots\cdots \\ \dfrac{U(s)}{s+p_n} = x_2 \longrightarrow \dot{x}_n = u - p_n x_n \end{cases} \Rightarrow$$

The Block Diagram of Diagonal Canonical Form

$$Y(s) = b_3 U(s) + \frac{c_1}{(s+p_1)^3} U(s) + \frac{c_2}{(s+p_1)^2} U(s) + \frac{c_3}{(s+p_1)} U(s)$$

$$+ \frac{c_4}{s+p_4} U(s)$$



Jordan Canonical Form

（3）范德蒙德（Vandermonde）矩阵

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

无重根时

$$T=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

设有 λ_1 三重根

$$T=\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \cdots & \lambda_4 & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 1 & \cdots & \lambda_4^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \dfrac{d}{d\lambda_1}(\lambda_1^{n-1}) & \dfrac{1}{2}\dfrac{d^2}{d\lambda_1}(\lambda_1^{n-1}) & \cdots & \lambda_4^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Jordan Canonical Form

例题4.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} x$$

```
>> A=[0 1 0; 0 0 1; 2 -5 4];
>> [V,D]=jordan(A)
V =

     1    -2     0
     2    -2    -2
     4    -2    -4
```

```
D =

     2     0     0
     0     1     1
     0     0     1
```

$$T = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$$

Jordan Canonical Form

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

```
>> A=[0 1 0;0 0 1;2 -5 4];

>> P=[1 0 1;1 1 2;1 2 4];

>> P1=inv(P);

>> J=P1*A*P

J =

     1     1     0

     0     1     0

     0     0     2
```

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- The matlab example of control system

Step Response In State Space Model

Build the state space model

$$sys = ss(A, B, C, D)$$

Unit Step Response

$$step(sys)$$

```
>> A=[-1 -1;6.5 0];
>> B=[1 1;1 0];
>> C=[1 0; 0 1];
>> D=[0 0;0 0];
```

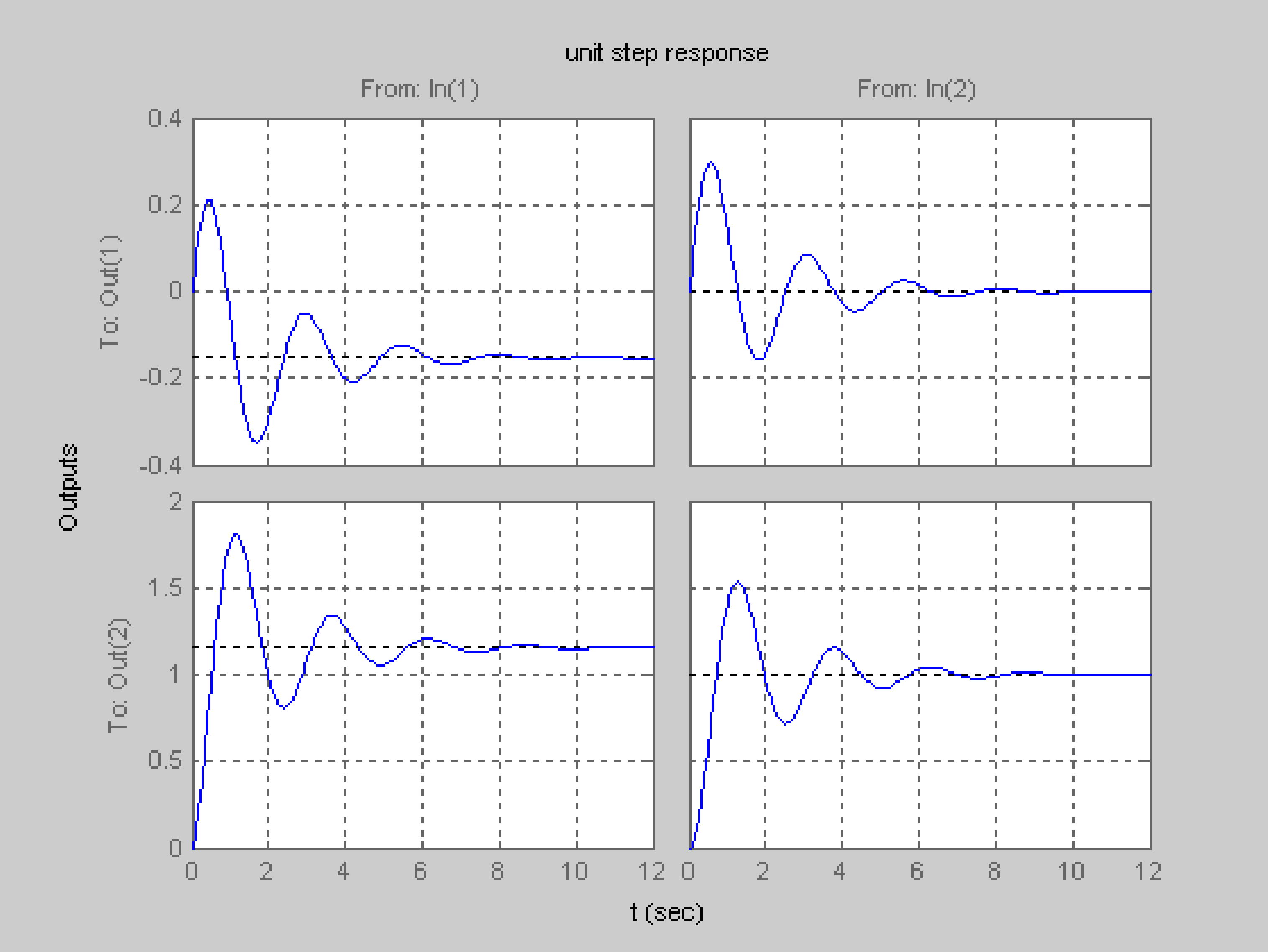
```
sys=ss(A,B,C,D)

a =
        x1    x2
  x1    -1    -1
  x2   6.5     0

b =
       u1  u2
  x1    1   1
  x2    1   0
```

Step Response In State Space Model

```
c =
      x1  x2
  y1   1   0
  y2   0   1
d =
      u1  u2
  y1   0   0
  y2   0   0
```

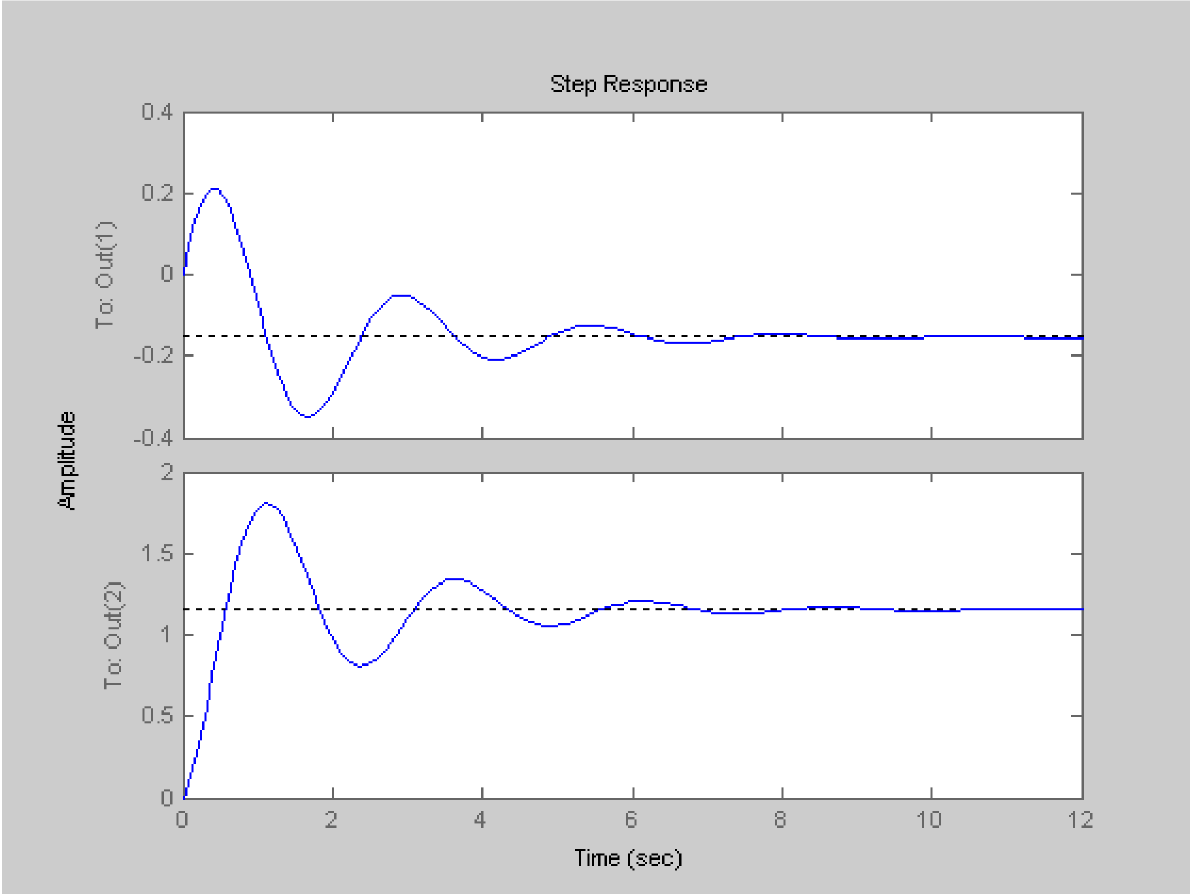
```
>> step(sys)
>> grid
>> title('unit step response')
>> xlabel('t')
>> ylabel('Outputs')
```

Step Response In State Space Model



Step Response In State Space Model

>> step(A,B,C,D,1)

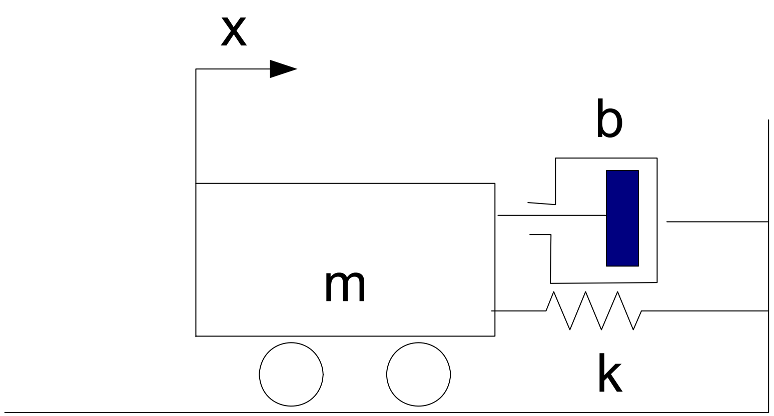


Unit step response when u1 is input and u2=0

Response to initial condition (No input)

Method1:

- Deriving transfer function with initial condition.
- Apply step response function or impulse response function



M=1kg,b=3N-s/m k=2N/m

$$x(0) = 0.1m$$

$$\dot{x}(0) = 0.05m/s$$

Response to initial condition (No input)

Solution:

$$m\ddot{x} + b\dot{x} + kx = 0$$

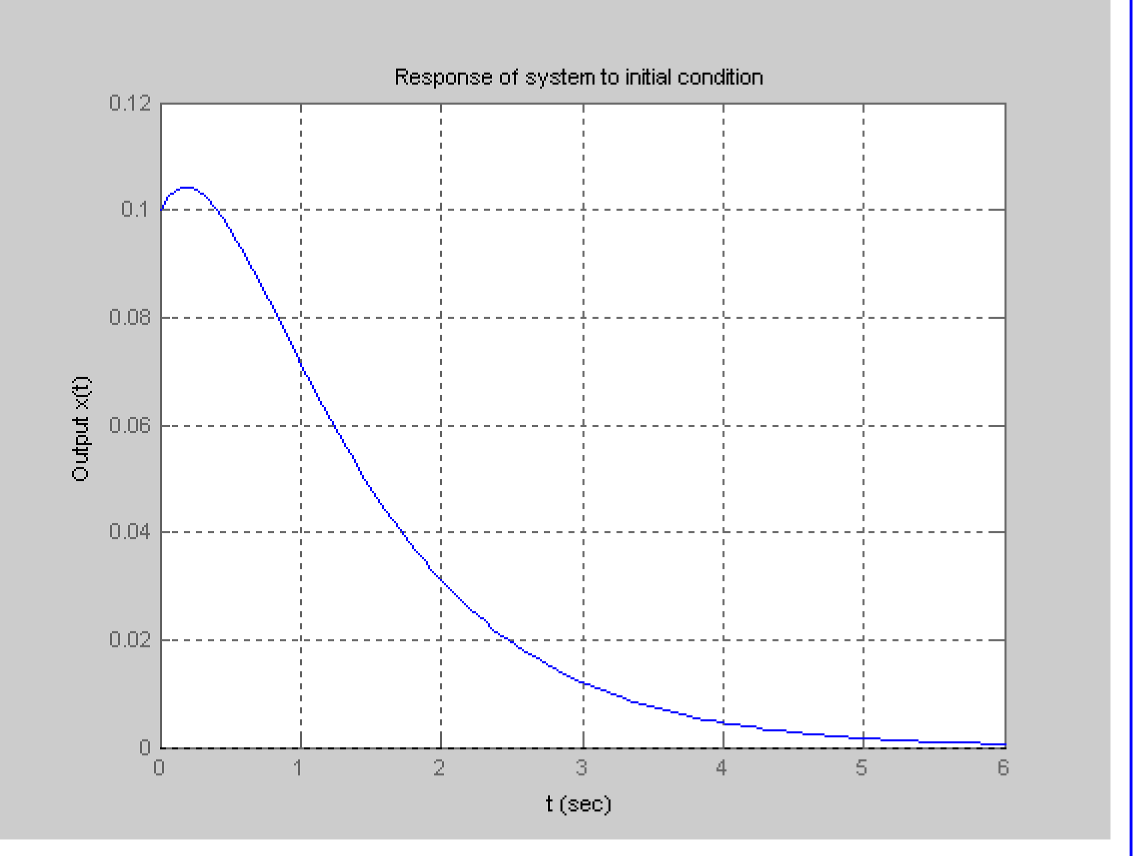
$$m[s^2 X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + kX(s) = 0$$

$$X(s) = \frac{mx(0)s + m\dot{x}(0) + bx(0)}{ms^2 + bs + k}$$

$$X(s) = \frac{0.1s + 0.35}{s^2 + 3s + 2} \qquad X(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}\frac{1}{s}$$

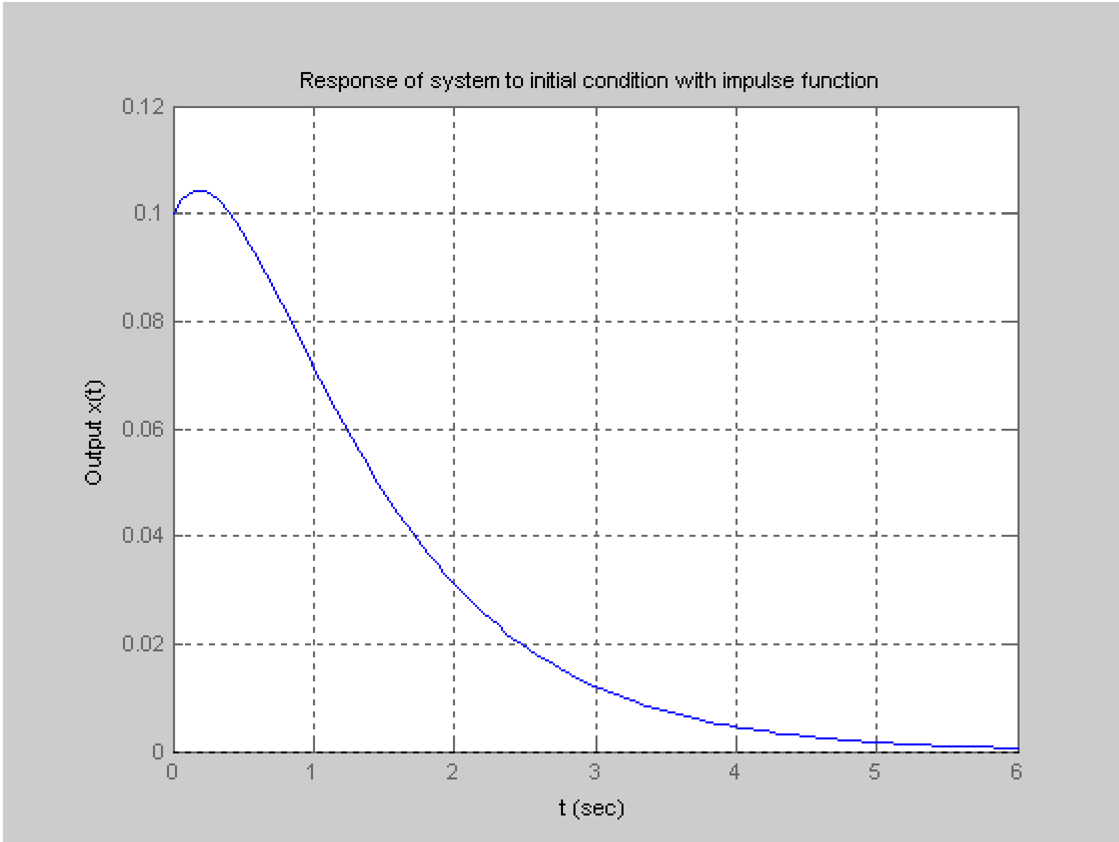
Response to initial condition (No input)

```
>> num=[0.1 0.35 0];
>> den=[1 3 2];
>> sys=tf(num,den)
>> step(sys)
>> title('Response of system to initial condition')
>> xlabel('t')
>> ylabel('Output x(t)')
```



Response to initial condition (No input)

```
num=[0 0.1 0.35]

den=[1 3 2]

sys=tf(num,den)

impulse(sys)

title('Response of system to initial condition with impulse function ')

xlabel('t')

ylabel('Output x(t)')
```

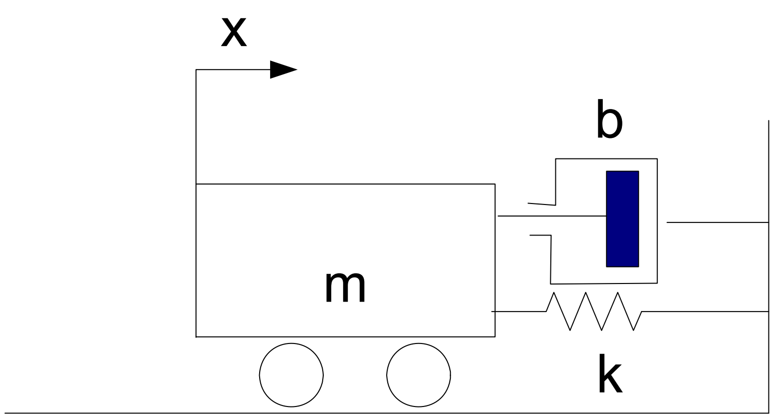


Response to initial condition (No input)

Method2:

- Deriving state space model;

- Using initial function.



$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

Response to initial condition (No input)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

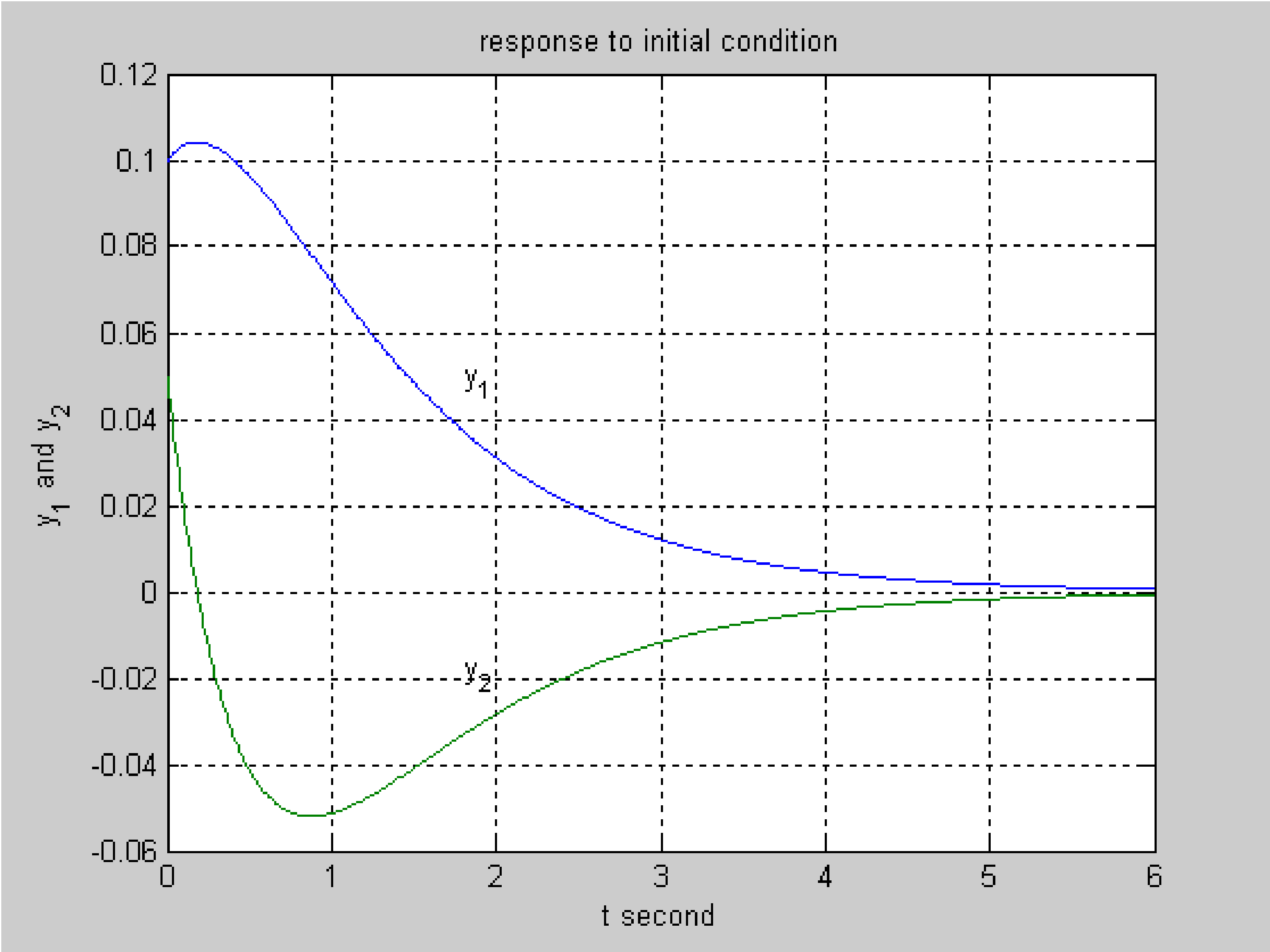
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}$$

Response to initial condition (No input)

```
t=0:0.01:6;
A=[0 1;-2 -3]
B=[0;0]
C=[1 0;0 1]
D=[0;0]
[y,x,t]=initial(A,B,C,D,[0.1;0.05],t)
y1=y(:,1)
```

```
y2=y(:,2)
plot(t,y1,t,y2)
grid
title('response to initial condition')
xlabel('t second')
ylabel('y_1 and y_2')
gtext('y_1')
gtext('y_2')
```

Response to initial condition (No input)



Obtaining response to arbitrary input & nonzero initial condition

Method:

- Lsim(sys,u,t,x0)

- [y,t] = lsim(sys,u,t,x0)

Transformation from state space to transfer function

Usage:

[num,den]=ss2tf(A,B,C,D,iu)

iu=1:input2=input3=…=0

iu=2:input1=input3=…=0

…

Transformation from transfer function to state space

[A,B,C,D]=tf2ss(num,den)

Ex. Transformation from TF to SS

$$G(s) = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
>> num=[0 0 1 0];
>> den=[1 14 56 160];
[A,B,C,D]=tf2ss(num,den)
A =
  -14  -56  -160
    1    0     0
    0    1     0
B =    [1   0   0]'
C =    0    1    0
D =    0
```

Ex1. The transient of 2nd order system

Considering unit step function of $G(s) = \dfrac{s}{s^2 + 2\zeta s + 1}$

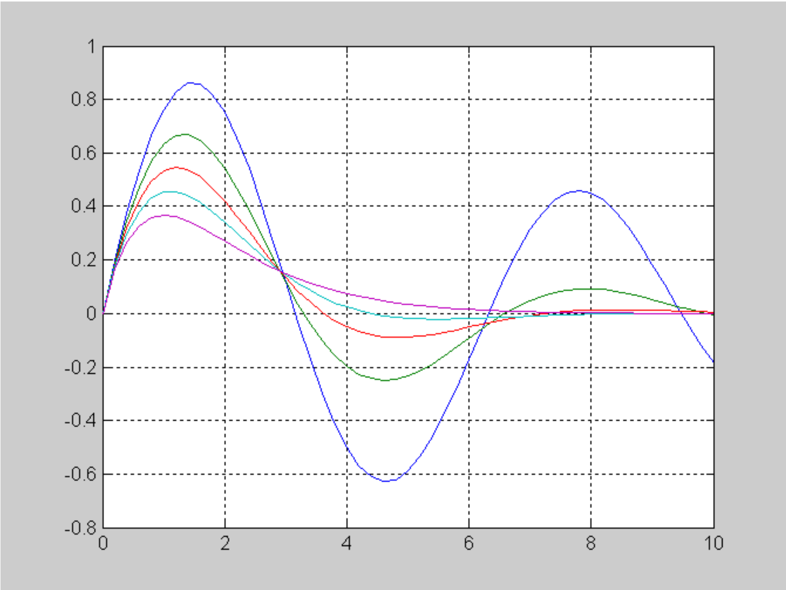
With $\zeta = 0.1, 0.3, 0.5, 0.7 \text{ and } 1.0$ in one figure

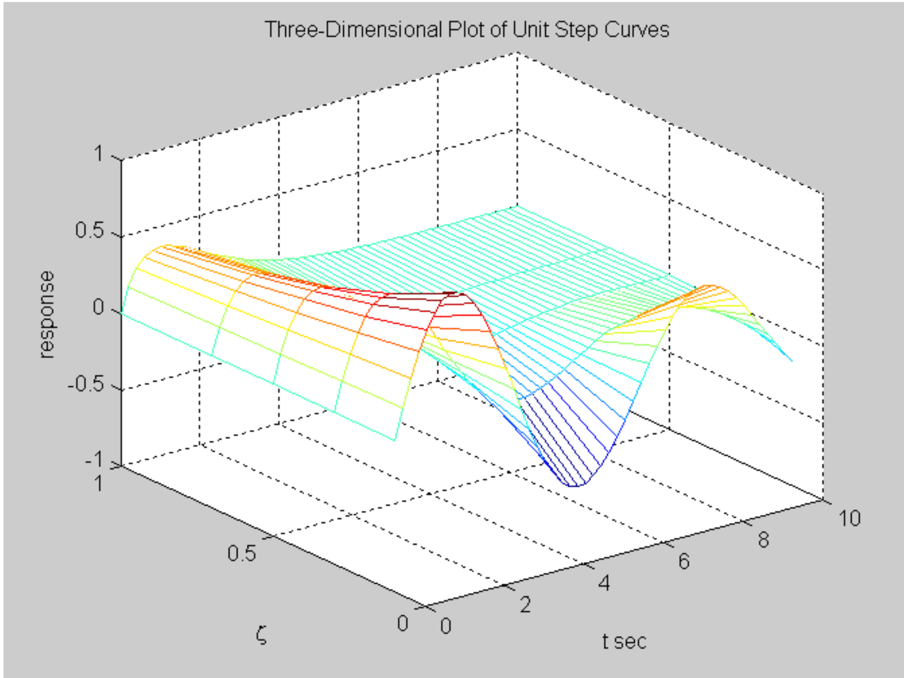
Ex1. The transient Response of 2nd order system

```
>> t=0:0.2:10;
>> zeta=[0.1,0.3,0.5,0.7,1];
>> for n=1:5
num=[0 1 0];
den=[1 2*zeta(n) 1];
[y(1:51,n),t]=step(num,den,t);
end
>> plot(t,y)
>> grid
```

```
>> mesh(t,zeta,y')
>> title('Three-Dimensional Plot of Unit Step Curves ')
>> ylabel('t sec')
>> xlabel('t sec')
>> ylabel(‘\zeta')
>> zlabel('response')
```

Ex1. The transient Response of 2nd order system

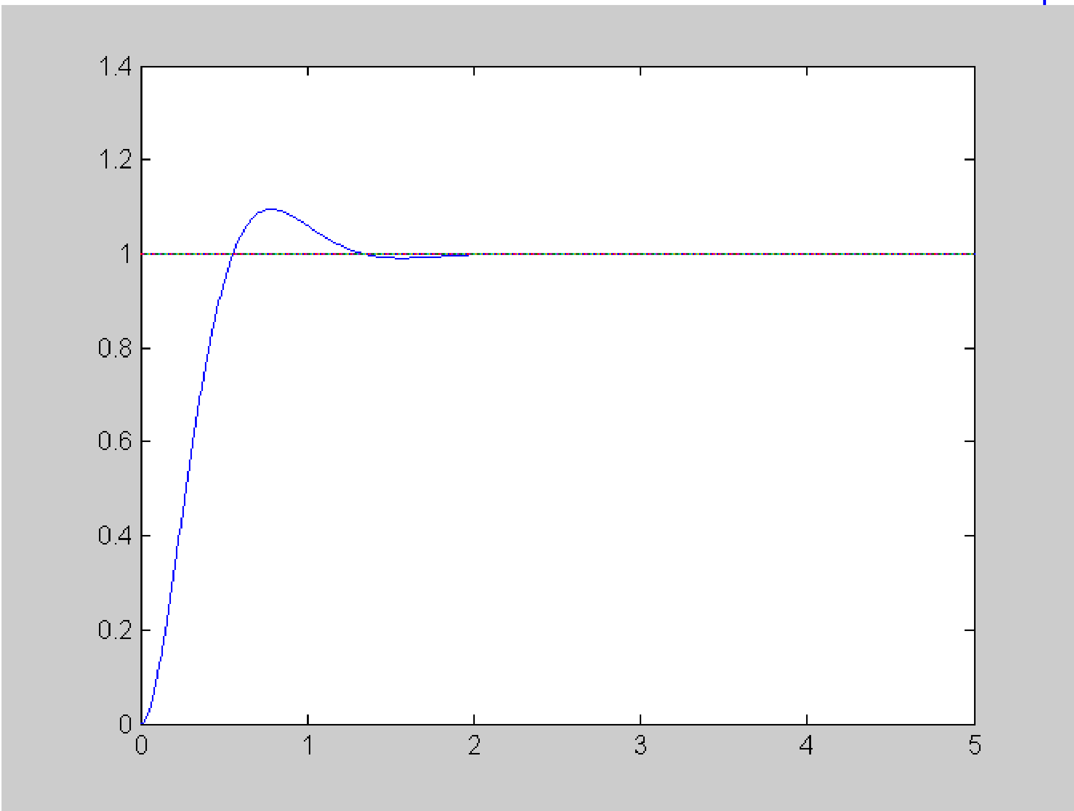




Ex2. The transient Response of 2nd order system

Find the rise-time,peak-time, overshoot and setting time for unit step response of

$$G(s) = \frac{25}{s^2 + 6s + 25}$$




```
num=[0 0 25];

den=[1 6 25];

t=0:0.005:5;

[y x t]=step(num,den,t)

        x=1

        plot(t,y,t,x)
```

Ex2. The transient Response of 2nd order system

```
>> r=1;
while(y(r)<1.0001);r=r+1;end;

>> r

r =

   112

>> rise_time=(r-1)*0.005

rise_time =

    0.5550
```

```
>> [ymax,tp]=max(y);

>> peak_time=tp*0.005

peak_time =

    0.7900

>> max_overshoot=ymax-1

max_overshoot =

    0.0948
```

Ex2. The transient Response of 2nd order system

```
>> s=1001;while(y(s)>0.98 & y(s)<1.02);s=s-1;end;

>> s

s =

   238

>> setting_time=(s-1)*0.005

setting_time =

    1.1850
```