Modern Control Theory
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Weijun Zhang （张伟军）
Associate Professor, Robotics Institute
School of Mechanical Engineering
Office:930 Building A of ME school
401 Building B of ME School

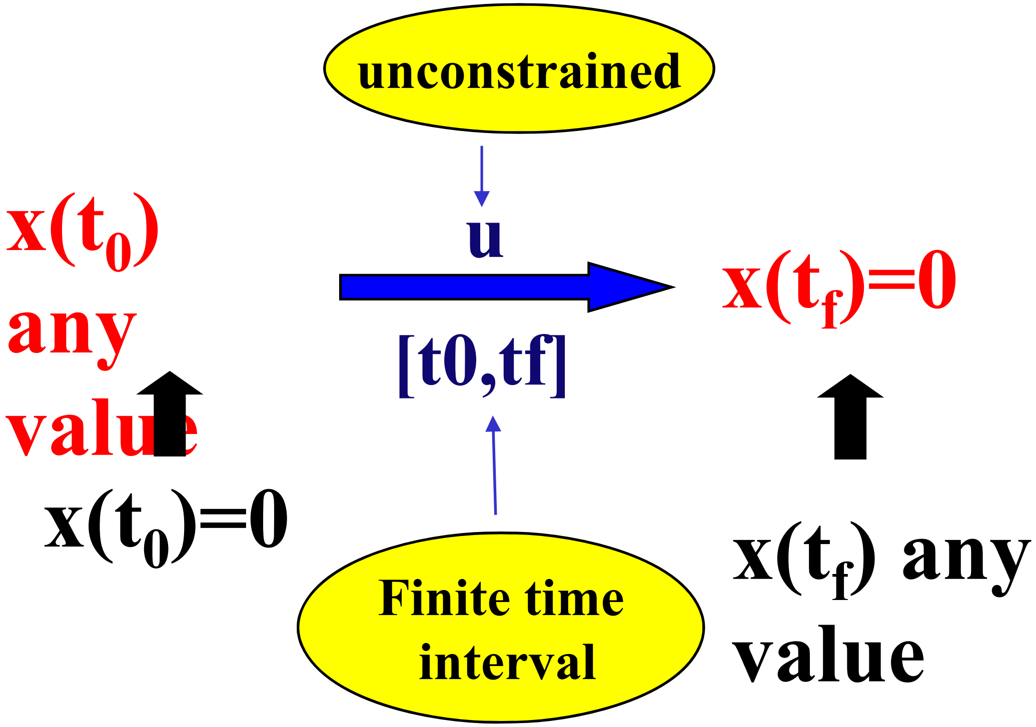
zhangweijun@sjtu.edu.edu.cn
021-34205559

Outline of Today's Lecture

- Controllability
- Observability
- Principle of Duality

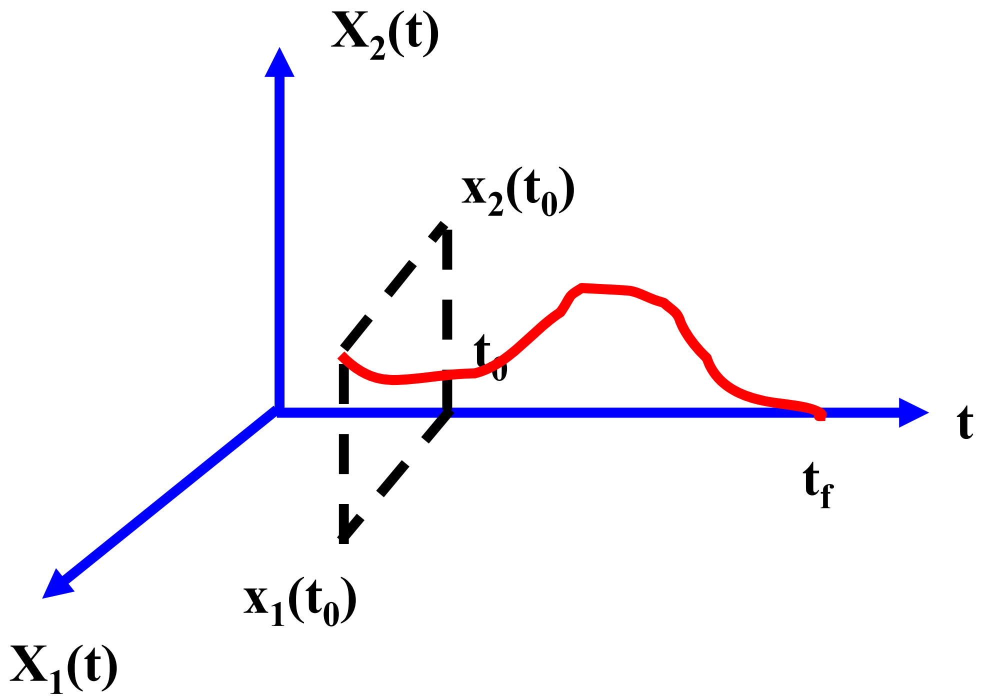
Controllability

- A System is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.



Controllability

- The controllability answers "whether the state vector can be controlled to any value? " not for "how to control".



Controllability

- Controllability analysis for Diagonal Form;
- Suppose the SS model has a diagonal form:

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix}$$

Condition for Controllability Diagonal Form

For any SS model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Suppose n×n matrix A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, corresponding eigenvectors are $P_1, P_2 \ldots P_n$, then Matrix A can be diagonalized by Matrix $T = [P_1, P_2 \ldots P_n]$, New SS model are:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu;\ \ z(0) = T^{-1}x(0) = T^{-1}x_0$$

$$y = CTz + Du$$

Condition for Controllability (I) Diagonal Form

The new System are:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \cdot \\ \cdot \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_{n-1} \\ z_n \end{bmatrix} + T^{-1}Bu$$

Let

$$T^{-1}B = F = (f_{ij})$$

Condition for Controllability
Diagonal Form

Suppose we have r inputs

$$\dot{z}_1 = \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \ldots f_{1r}u_r$$

$$\dot{z}_2 = \lambda_2 z_2 + f_{21}u_1 + f_{22}u_{22} + \ldots + f_{2r}u_r$$

$$\dot{z}_3 = \lambda_3 z_3 + f_{31}u_1 + f_{32}u_{22} + \ldots + f_{3r}u_r$$

$$\ldots$$

$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_{22} + \ldots + f_{nr}u_r$$

Condition for Controllability
Diagonal Form

Results: The elements of any row of $T^{-1}B$ that corresponds to distinct eigenvalues are not all zero.

Condition for Controllability
Jordan Form

For any SS model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Suppose n×n matrix A has one 3-order repeated eigenvalues λ_1 and n-3 distinct eigenvalues $\lambda_4\,\lambda_5 \ldots \lambda_n$, the eigenvectors are $P_1, P_2 \ldots P_n$, then Matrix A can be diagonalized by Matrix $T=[P_1, P_2 \ldots P_n]$, New SS model are:

Condition for Controllability
Jordan Form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ . \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & . & 0 \\ 0 & \lambda_1 & 1 & 0 & . & \\ 0 & 0 & \lambda_1 & 0 & . & \\ 0 & 0 & 0 & \lambda_4 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ . \\ z_n \end{bmatrix} + T^{-1}Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = Jz + T^{-1}Bu$$

Let

$$T^{-1}B = F = (f_{ij})$$

Condition for Controllability
Jordan Form

Suppose we have r inputs

$$\dot{z}_1 = \lambda_1 z_1 + z_2 + f_{11}u_1 + f_{12}u_2 + \ldots f_{1r}u_r$$

$$\dot{z}_2 = \lambda_1 z_2 + z_3 + f_{21}u_1 + f_{22}u_{22} + \ldots + f_{2r}u_r$$

$$\dot{z}_3 = \lambda_1 z_3 + f_{31}u_1 + f_{32}u_{22} + \ldots + f_{3r}u_r$$

$$\ldots$$

$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_{22} + \ldots + f_{nr}u_r$$

Condition for Controllability
Jordan Form

Results1:The elements of any row of $T^{-1}B$ that correspond to the last row of each Jordan Block are not all zero. (means f31 or f32 or f33 not equal to zero).

Results2:The elements of any row of $T^{-1}B$ that correspond to distinct eigenvalues are not all zero.

Condition for Controllability
Diagonal Form

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} u$$

Condition for Controllability
Diagonal Form

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u$$

Controllability

Complete State Controllability of continuous-time System:

$$\dot{x} = Ax + Bu$$

Suppose:

x=state vector (n×1) u=control signal (scalar)

A=n×n matrix B=n×1 matrix

$$x(0)=\text{any value} \xrightarrow[{[0,t_1]}]{u} x(t_1)=0$$

Controllability

The solution of the x(t):

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

x(0)=any value $\xrightarrow[\;[0,t_1]\;]{u}$ **x(t_1)=0**

$$0 = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

$$\Downarrow$$

$$x(0) = -\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

Controllability

Applying Gayley-Hamilton Theorem:

$$\left|\lambda I - A\right| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

The matrix A satisfies its own characteristic equation

$$A^n + a_1 A^{n-1} + \ldots + a_{n-1}A + a_n = 0$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^k t^k + \cdots$$

$$\Downarrow$$

$$e^{At} = \alpha_{n-1}(t)A^{n-1} + \alpha_{n-2}(t)A^{n-2} + \ldots + \alpha_1(t)A + \alpha_0 I$$

Controllability

$$x(0)=\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

$$\Downarrow$$

$$x(0)=-\int_0^{t_1}\sum_{k=0}^{n-1}\alpha_k(\tau)A^kBu(\tau)d\tau$$

$$\Downarrow$$

$$x(0)=-\sum_{k=0}^{n-1}A^kB\int_0^{t_1}\alpha_k(\tau)u(\tau)d\tau$$

Let

$$\int_0^{t_1}\alpha_k(\tau)u(\tau)d\tau=\beta_k$$

Controllability

$$x(0) = -\sum_{k=0}^{n-1} A^k B \beta_k$$

$$\Downarrow$$

$$x(0) = -\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdot \\ \beta_{n-1} \end{bmatrix} (\#1)$$

Results1: The system is completely controllable, then, given any initial state x(0), The equation #1 should be satisfied. This requires that the rank of the n×n matrix

Controllability

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} (\#2)$$

be n. (or the vector of B AB … A^{n-1}B are linearly independent)

Results2: if u is an r-vector, then the condition for complete state controllability is that the n×nr matrix

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

be of rank n. (or contain n linearly independent vector)

Controllability

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Is called *controllability matrix*

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

Controllability

Example:

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

```
>> A=[1 1 0;0 1 0;0 1 1];
>> B=[0 1; 1 0; 0 1];n=3;
>> M=ctrb(A,B);
>> rankM=rank(M);
>> if rankM ==n
disp('system is controllable')
else
disp('system is uncontrolable')
End
system is uncontrolable
```

Output Controllability

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Where

x=state vector (n×1) u=control vector (r-vector)

y=output A=n×n matrix; B=n×r matrix;

C=m×n matrix; D=m×r matrix

A System is said to be output controllable if it is possible by means of an unconstrained control vector to transfer the system from any initial output ~~$y(t_0)$~~ to any final output ~~$y(t_1)$~~ in a finite inter~~val of~~ time $t_0<t<t_1$

Output Controllability

- It can be proved that the system is said to be output controllable if and only of m$\times$(n+1)r matrix:

$$[CB \;\; CAB \;\; \ldots \; A^{n\text{-}1}B \;\; D]$$

Is of rank m

Condition for Complete State Controllability in the s plane

- It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation in the transfer function or transfer matrix.

$$W_{ux}(s) = (sI - A)^{-1} b$$

Example:

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} u}$$

Stabilizability

- For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Controllability

离散系统的可控性判别

例题：

设离散系统状态方程为：

$$x(k+1)=\begin{bmatrix}-1 & 1 & 1\\ 1 & 0 & -2\\ 1 & 0 & 1\end{bmatrix}x(k)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(k)$$

试分析能否找到控制作用**u(0)**、**u(1)**，**…**，将初始状态

$$x_0=\begin{bmatrix}2\\1\\1\end{bmatrix}$$ 转移到零状态

例题：

试分析所示系统能否找到控制序列，使初始状态

$$x_0=\begin{bmatrix}1\\1\\1\end{bmatrix}$$ 转移到零状态

Controllability

5 离散系统的可控性判别

$$x(k+1) = Gx(k) + Hu(k)$$

U(k) –标量控制作用

H—n维列矢量

G－系统矩阵

X－状态向量（n×1）

当矩阵

$$M = [h \quad Gh \quad \ldots \quad G^{n-2}h \quad G^{n-1}h]$$

秩为**n**时，系统可控

$$x(k+1) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}$$

Outline of Today's Lecture

- Controllability
- Observability
- Principle of Duality

Observability

能观性的概念

能观性表示的是输出y(t)反映状态变量x(t) 的能力，因此，研究此问题时不考虑输入u(t).

线性连续定常系统 $\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned}$,如果对任意给定的输入**u(t)**,

在有限观测时间区间$[t_0,t_f]$内，从系统的输出**y(t)**能唯一确定系统的初始状态$x(t_0)$,则称$x(t_0)$这一时刻状态是可观的，如果系统所有状态都是可观的，则称此系统是完全能观的。

对问题的基本思考：

1.假设**m=n,u=0,**结论如何？

2.假设$m<n$**,u=0,**结论如何？

3. 观测**x0**和观测任意一点是相同的

Condition for Observability Diagonal From

- Suppose A has a diagonal form (or has been transformed to diagonal form)

$$A = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ & & \cdots & \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{bmatrix}$$

Condition for Observability Diagonal From

$$x = \begin{bmatrix} e^{\lambda_1 t} x_1(0) \\ e^{\lambda_2 t} x_2(0) \\ . \\ e^{\lambda_n t} x_n(0) \end{bmatrix}$$

$$y_1 = c_{11} e^{\lambda_1 t} x_1(0) + c_{12} e^{\lambda_2 t} x_2(0) + \ldots + c_{1n} e^{\lambda_n t} x_n(0)$$

$$y_2 = c_{21} e^{\lambda_1 t} x_1(0) + c_{22} e^{\lambda_2 t} x_2(0) + \ldots + c_{2n} e^{\lambda_n t} x_n(0)$$

$$\ldots$$

$$y_m = c_{m1} e^{\lambda_1 t} x_1(0) + c_{m2} e^{\lambda_2 t} x_2(0) + \ldots + c_{mn} e^{\lambda_n t} x_n(0)$$

Condition for Observability Diagonal From

Results:

The system is completely observable if none of the column of the $m \times n$ matrix C consists of all zero elements.

Condition for Observability
Jordan From

- Suppose A has a Jordan form (or has been transformed to Jordan form)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ . \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & . & 0 \\ 0 & \lambda_1 & 1 & 0 & . & \\ 0 & 0 & \lambda_1 & 0 & . & \\ 0 & 0 & 0 & \lambda_4 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ . \\ z_n \end{bmatrix}$$

Condition for Observability Jordan From

$$
e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}
$$

$$
x = \begin{bmatrix} e^{\lambda_1 t}x_1(0) + te^{\lambda_1 t}x_2(0) + \frac{1}{2}t^2 e^{\lambda_1 t}x_3(0) \\ e^{\lambda_1 t}x_2(0) + te^{\lambda_1 t}x_3(0) \\ e^{\lambda_1 t}x_3(0). \\ e^{\lambda_4 t}x_4(0). \\ \cdots \\ e^{\lambda_n t}x_n(0) \end{bmatrix}
$$

Condition for Observability
Jordan From

Results:

1) The system is completely observable if no two Jordan blocks in J are associated with the same eigenvalues

2) no columns of C that corresponding to the first row of each Jordan block consist of zero elements

3) no columns of C that corresponding to distinct eigenvalues consist of zero elements.

Condition for Controllability
Diagonal Form

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

Condition for Controllability
Diagonal Form

Example:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}
=
\left[\begin{array}{ccc:cc}
-2 & 1 & 0 & & 0 \\
0 & -2 & 1 & & \\
0 & 0 & -2 & & \\
\hdashline
 & & & -5 & 1 \\
0 & & & 0 & -5
\end{array}\right]
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
=
\begin{bmatrix} 1 & 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 \end{bmatrix}
$$

Condition for Observability

Results: The system is completely Observable, then, given the output y(t) over a finite time interval, x(0) is uniquely determined. It can be shown that this requires the rank of nm×n matrix

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

be n.

Condition for Observability

or we use n×nm matrix

$$\left[c^T \quad A^T C^T \quad . \quad . \quad (A^T)^{n-1} C^T\right]$$

is of rank n

Observability matrix

Example

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u$$

Observable?

Controllable?

$$y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x$$

Condition of Observability for Discrete Control System

离散系统的能观性判别

对离散系统方程：

$$x(k+1) = Gx(k)$$

$$y(k) = Cx(k)$$

y:m维列矢量，**C**：**m**×**n**矩阵

若通过有限周期输出**y(t)**，能唯一确定初始状态矢量**x(0),**则系统是能观的。

其能观性判别矩阵为：$N = \begin{bmatrix} G & CG & \cdots & CG^{n-1} \end{bmatrix}^T$

Condition of Observability for Discrete Control System

5. 采样周期对离散系统可控性和可观性的影响

例题：设连续系统的状态方程和输出方程为

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

试判别此系统和将其离散化的离散系统的状态能控性。

Condition of Observability for Discrete Control System

例题7：

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x$$

判别可控性和可观性。

```
>> A=[-3 1;1 -3];B=[1 1;1 1]; C=[1 1; 1 -1];D=[0];

>> M=ctrb(A,B)

M =

     1     1    -2    -2

     1     1    -2    -2
```

Condition of Observability for Discrete Control System

```
>> rankM=rank(M)
rankM =     1

>> N=obsv(A,C)

N =

     1     1

     1    -1

    -2    -2

    -4     4
```

```
>> rankN=rank(N)

rankN =

     2
```

Condition for Complete State Observability in the s plane

- It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation in the transfer function or transfer matrix.

$$W_{yx}(s) = C(sI - A)^{-1}$$

Outline of Today's Lecture

- Controllability
- Observability
- Principle of Duality

Principle of Duality

The relation between controllability and observability:

System 1
$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1 x_1$$

System 2
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2 x_2$$

if satisfy
$$A_2 = A_1^T$$
$$B_2 = C_1^T$$
$$C_2 = B_1^T$$
$\Rightarrow$ **System 1 is the dual system of system2**

Principle of Duality

The principle of duality states that the system 1 is completely state controllable (Observable) if and only if system 2 is completely observable (controllable)

（1）对偶系统传递函数阵是互为转置的。

（2）互为对偶的系统其特征方程是相同的。

系统 $\Sigma 1$ $\quad \dot{x}_1 = A_1 x_1 + B_1 u_1$, $\quad y_1 = C_1 x_1$

系统 $\Sigma 2$ $\quad \dot{x}_2 = A_2 x_2 + B_2 u_2$, $\quad y_2 = C_2 x_2$

互为对偶，则 $\Sigma 1$ 的可控性等价于 $\Sigma 2$ 的可观性

$\Sigma 1$ 的可观性等价于 $\Sigma 2$ 的可控性