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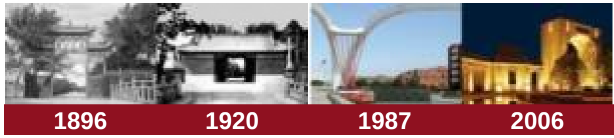

ME6011 弹性塑性力学

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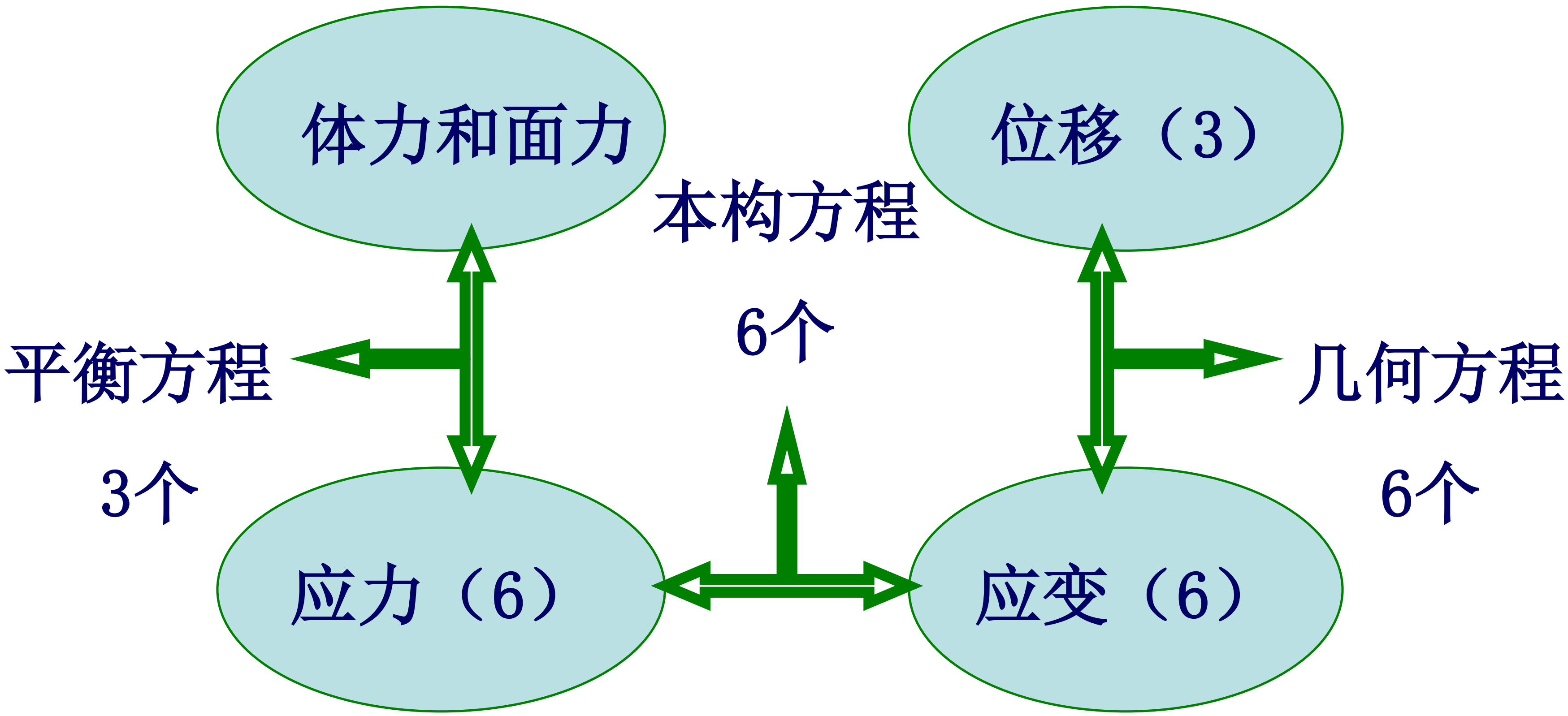
第五章 弹性与塑形力学的解题方法

- 按位移求解弹性力学问题
- 按应力求解弹性力学问题
- 平面问题和应力函数
- 逆解法和半逆解法
- 边界上φ及其导数的力学意义
- 平面问题的极坐标解法
- 塑性力学的解题方法

弹性力学基本方程



弹性力学基本方程

弹性力学的一般问题中，共包含15个未知函数，将用15方程来求解。

对于各向同性的弹性体：

- 3个平衡微分方程
- 6个几何方程（微分方程）
- 6个物理方程（广义胡克定律）
- 边界条件（与上述方程组成封闭的定解问题）

弹塑性力学的基本方程

柯西应力公式

$$p_{vx}=l\sigma_x+m\tau_{yx}+n\tau_{zx}$$
$$p_{vy}=l\tau_{xy}+m\sigma_y+n\tau_{zy}$$
$$p_{vz}=l\tau_{xz}+m\tau_{yz}+n\sigma_z$$

变形协调方程

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad \frac{\partial}{\partial x}\left(\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{yz}}{\partial x}\right)=2\frac{\partial^2\varepsilon_x}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}\right)=2\frac{\partial^2\varepsilon_y}{\partial x\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial y\partial z}\qquad \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xz}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)=2\frac{\partial^2\varepsilon_z}{\partial x\partial y}$$

Navier平衡微分方程

$$\boxed{\frac{\partial\sigma_{ij}}{\partial x_j}+f_i=0}$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0=\rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+f_y=0=\rho\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+f_z=0=\rho\frac{\partial^2 w}{\partial t^2}$$

柯西几何方程

$$\boxed{\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$$

$$\varepsilon_x=\frac{\partial u}{\partial x}\qquad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\varepsilon_y=\frac{\partial v}{\partial y}\qquad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}$$

$$\varepsilon_z=\frac{\partial w}{\partial z}\qquad \gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$

弹性力学本构方程

广义胡克定律

$$\varepsilon_x = \frac{1}{E}[(1+\nu)\sigma_x - \nu\Theta] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1}{E}[(1+\nu)\sigma_y - \nu\Theta] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_z = \frac{1}{E}[(1+\nu)\sigma_z - \nu\Theta] \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \theta = 3\varepsilon_0$$

$$\sigma_x + \sigma_y + \sigma_z = \Theta = 3\sigma_0$$

$$G = E/2(1+\nu)$$

$$\left.\begin{aligned} \sigma_x &= \lambda\theta + 2G\varepsilon_x & \tau_{xy} &= G\gamma_{xy} \\ \sigma_y &= \lambda\theta + 2G\varepsilon_y & \tau_{yz} &= G\gamma_{yz} \\ \sigma_z &= \lambda\theta + 2G\varepsilon_z & \tau_{zx} &= G\gamma_{zx} \end{aligned}\right\}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

体积胡克定律

$$\Theta = 3K\theta$$

$$K = \frac{E}{3(1-2\nu)}$$

弹性力学基本方程

弹性力学问题的分类：

- 力的边值问题：在物体的全部表面上给定表面力
- 位移边值问题：在物体的全部表面上给定位移
- 混合边值问题：在物体的一部分表面上给定表面力，而另一部分表面上给定位移

弹性力学问题解法的分类：

- 取位移作为基本未知量 ——位移法
- 取应力作为基本未知量 ——应力法

按位移求解弹性力学问题

要点：按位移求解弹性力学问题时，取u，v，w作为基本未知量，将各个方程中的应力、应变一概用位移表示。先求位移，再求应力和应变。

按位移求解弹性力学问题

$$\left.\begin{aligned}\sigma_x &= \lambda\theta + 2G\varepsilon_x & \tau_{xy} &= G\gamma_{xy} \\ \sigma_y &= \lambda\theta + 2G\varepsilon_y & \tau_{yz} &= G\gamma_{yz} \\ \sigma_z &= \lambda\theta + 2G\varepsilon_z & \tau_{zx} &= G\gamma_{zx}\end{aligned}\right\}$$

代入

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} & \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\end{aligned}$$

用位移表示的应力分量

$$\begin{aligned}\sigma_x &= \lambda\theta + 2G\frac{\partial u}{\partial x} & \tau_{xy} &= G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \sigma_y &= \lambda\theta + 2G\frac{\partial v}{\partial y} & \tau_{yz} &= G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \sigma_z &= \lambda\theta + 2G\frac{\partial w}{\partial z} & \tau_{zx} &= G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\end{aligned}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$2G = \frac{E}{1+\nu} \qquad \theta = 3\varepsilon_0$$

按位移求解弹性力学问题

考虑平衡微分方程 $$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f_x=0$$

$$\lambda\frac{\partial \theta}{\partial x}+2G\frac{\partial^2 u}{\partial x^2}+G(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y})+G(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 \omega}{\partial x\partial z})+f_x=0$$

$$\lambda\frac{\partial \theta}{\partial x}+G(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2})+G(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 \omega}{\partial x\partial z})+f_x=0$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad \theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial z}$$

$$\lambda\frac{\partial \theta}{\partial x}+G\nabla^2 u+G\frac{\partial \theta}{\partial x}+f_x=0$$

按位移求解弹性力学问题

Lamé位移方程

$$(\lambda+G)\frac{\partial\theta}{\partial x}+G\nabla^2 u+f_x=0$$

$$(\lambda+G)\frac{\partial\theta}{\partial y}+G\nabla^2 v+f_y=0$$

$$(\lambda+G)\frac{\partial\theta}{\partial z}+G\nabla^2 w+f_z=0$$

用位移表示的平衡微分方程

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial x}+G\nabla^2 u+f_x=0$$

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial y}+G\nabla^2 v+f_y=0$$

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial z}+G\nabla^2 w+f_z=0$$

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial z}$$

$$\lambda+G=\frac{E\nu}{(1+\nu)(1-2\nu)}+G=\frac{G}{1-2\nu}$$

按位移求解弹性力学问题

物体表面位移已知，边界条件容易提出！

物体表面力已知，则需要求得边界条件：

$$l\sigma_x + m\tau_{yx} + n\tau_{zx} = F_x$$
$$l\tau_{yx} + m\sigma_y + n\tau_{yz} = F_y$$
$$l\tau_{zx} + m\tau_{zy} + n\sigma_z = F_z$$

$$\sigma_x = \lambda\theta + 2G\frac{\partial u}{\partial x} \qquad \tau_{xy} = G(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})$$
$$\sigma_y = \lambda\theta + 2G\frac{\partial v}{\partial y} \qquad \tau_{yz} = G(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})$$
$$\sigma_z = \lambda\theta + 2G\frac{\partial w}{\partial z} \qquad \tau_{zx} = G(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$$

边界条件

$$l(\lambda\theta + 2G\frac{\partial u}{\partial x}) + mG(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + nG(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) = F_x$$
$$lG(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + m(\lambda\theta + 2G\frac{\partial v}{\partial y}) + nG(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) = F_y$$
$$lG(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + mG(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) + n(\lambda\theta + 2G\frac{\partial w}{\partial z}) = F_z$$

按位移求解弹性力学问题

按位移求解弹性力学问题的步骤：

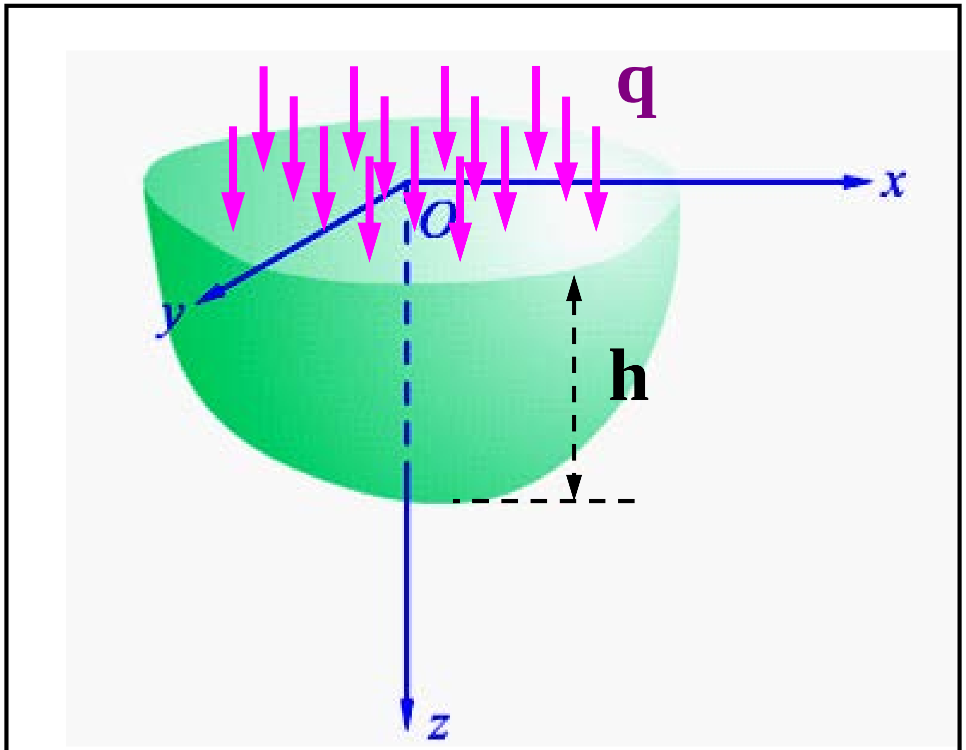
1. 位移函数u, v, w在物体内部满足拉梅位移方程
2. 在边界上满足求得或直接给出的位移边界
3. 将求得的u, v, w代入几何方程可求得应变
4. 由胡克定律求得应力；

- 优点：未知函数的个数比较少，即仅有三个未知量u, v, w；
- 缺点：必须求解三个联立的二阶偏微分方程；

- 按位移求解问题是普遍适用的方法，特别是在数值解中得到了广泛的应用，例如在有限元法，差分法等数值计算方法中，得到了很好的应用。

例题1

设有半空间体，单位体积的质量为ρ，在水平边界上受均布压力q的作用，试用位移法求各位移分量和应力分量，假设在z=h处z方向的位移w=0。



解：

由于载荷和弹性体对z轴对称，并且为半空间体，可以假设

$$u=0, v=0, w=w(z)$$

拉梅位移方程：

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial x}+G\nabla^2 u+f_x=0$$

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial y}+G\nabla^2 v+f_y=0$$

$$\frac{G}{1-2\nu}\frac{\partial\theta}{\partial z}+G\nabla^2 w+f_z=0$$

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial\omega}{\partial z}=\frac{\partial\omega}{\partial z}$$

$$\frac{\partial\theta}{\partial x}=\frac{\partial^2\omega}{\partial z\partial x}=0 \qquad \frac{\partial\theta}{\partial y}=\frac{\partial^2\omega}{\partial z\partial y}=0$$

$$\frac{\partial\theta}{\partial z}=\frac{\partial^2\omega}{\partial z^2} \qquad \nabla^2 u=\nabla^2 v=0$$

$$\nabla^2\omega=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\omega=\frac{\partial^2\omega}{\partial z^2}$$

前两式恒等，第三式为：

$$(\lambda+2G)\frac{d^2w}{dz^2}+\rho g=0$$

即：

$$\frac{d^2w}{dz^2}=\frac{\rho g}{\lambda+2G}$$

积分： $$w=-\frac{1-2\mu}{4(1-\mu)G}\rho g z^2+Az+B$$

力的边界条件（上表面）

$$l(\lambda\theta+2G\frac{\partial u}{\partial x})+mG(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})+nG(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})=F_x$$

$$lG(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})+m(\lambda\theta+2G\frac{\partial v}{\partial y})+nG(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z})=F_y$$

$$lG(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x})+mG(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})+n(\lambda\theta+2G\frac{\partial w}{\partial z})=F_z$$

$$l=m=0,\ \ n=-1$$

$$F_x=F_y=0,\ \ F_z=q$$

$$\Rightarrow\ -(\lambda\frac{dw}{dz}+2G\frac{dw}{dz})_{z=0}=q\ \Rightarrow\ (\frac{dw}{dz})_{z=0}=-\frac{1-2\mu}{2(1-\mu)G}q$$

$$\Rightarrow\ [\frac{1-2\mu}{2(1-\mu)G}\rho g z+A]_{z=0}=-\frac{1-2\mu}{2(1-\mu)G}q\ \Rightarrow\ A=-\frac{1-2\mu}{2(1-\mu)G}q$$

位移边界条件 $(w)_{z=h}=0$

可得： $$B=\frac{1-2\mu}{2(1-\mu)G}qh+\frac{1-2\mu}{4(1-\mu)G}\rho gh^2$$

位移分量

$$u=0,\quad v=0$$

$$w=\frac{1-2\mu}{4(1-\mu)G}[\rho g(h^2-z^2)+2q(h-z)]$$

应力分量

$$\sigma_x=\sigma_y=-\frac{\mu}{1-2\mu}(q+\rho gz)$$

$$\sigma_z=-(q+\rho gz)$$

$$\tau_{xy}=\tau_{yz}=\tau_{zx}=0$$

按应力求解弹性力学问题

要点：对于物体边界上给定了表面力的问题，以6个应力分量为基本未知量，将各个方程中的位移、应变一概用应力表示。以应力方程和变形协调方程为求解对象.

按应力求解弹性力学问题

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z}+\frac{\partial \gamma_{zx}}{\partial y}-\frac{\partial \gamma_{yz}}{\partial x}\right)=2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial y^2}=\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z}+\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{zx}}{\partial y}\right)=2\frac{\partial^2 \varepsilon_y}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial z^2}=\frac{\partial^2 \gamma_{zx}}{\partial y \partial z} \qquad \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right)=2\frac{\partial^2 \varepsilon_z}{\partial x \partial y}$$

+

$$\varepsilon_x=\frac{1}{E}[(1+\nu)\sigma_x-\nu\Theta] \qquad \gamma_{xy}=\frac{\tau_{xy}}{G}$$

$$\varepsilon_y=\frac{1}{E}[(1+\nu)\sigma_y-\nu\Theta] \qquad \gamma_{yz}=\frac{\tau_{yz}}{G}$$

$$\varepsilon_z=\frac{1}{E}[(1+\nu)\sigma_z-\nu\Theta] \qquad \gamma_{zx}=\frac{\tau_{zx}}{G}$$

用应力表示的应变协调方程

$$\frac{\partial^2 \sigma_x}{\partial y^2}+\frac{\partial^2 \sigma_y}{\partial x^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 \Theta}{\partial x^2}+\frac{\partial^2 \Theta}{\partial y^2}\right)=2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \sigma_y}{\partial z^2}+\frac{\partial^2 \sigma_z}{\partial y^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 \Theta}{\partial z^2}+\frac{\partial^2 \Theta}{\partial y^2}\right)=2\frac{\partial^2 \tau_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \sigma_z}{\partial x^2}+\frac{\partial^2 \sigma_x}{\partial z^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 \Theta}{\partial x^2}+\frac{\partial^2 \Theta}{\partial z^2}\right)=2\frac{\partial^2 \tau_{xz}}{\partial x \partial z}$$

$$\frac{\partial^2 \sigma_x}{\partial y \partial z}-\frac{\nu}{1+\nu}\frac{\partial^2 \Theta}{\partial y \partial z}=\frac{\partial}{\partial x}\left(\frac{\partial \tau_{zx}}{\partial y}+\frac{\partial \tau_{xy}}{\partial z}-\frac{\partial \tau_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 \sigma_y}{\partial z \partial x}-\frac{\nu}{1+\nu}\frac{\partial^2 \Theta}{\partial z \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial z}+\frac{\partial \tau_{yz}}{\partial x}-\frac{\partial \tau_{zx}}{\partial y}\right)$$

$$\frac{\partial^2 \sigma_z}{\partial x \partial y}-\frac{\nu}{1+\nu}\frac{\partial^2 \Theta}{\partial x \partial y}=\frac{\partial}{\partial z}\left(\frac{\partial \tau_{xz}}{\partial y}+\frac{\partial \tau_{yz}}{\partial x}-\frac{\partial \tau_{xy}}{\partial z}\right)$$

按应力求解弹性力学问题

用应力表示的应变协调方程（相容方程）

推导过程
P135-136

$$\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=-2\frac{\partial f_x}{\partial x}-\frac{\nu}{1-\nu}(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z})$$

$$\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=-2\frac{\partial f_y}{\partial y}-\frac{\nu}{1-\nu}(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z})$$

$$\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=-2\frac{\partial f_z}{\partial z}-\frac{\nu}{1-\nu}(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z})$$

$$\nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\frac{\partial f_x}{\partial y}-\frac{\partial f_y}{\partial x}$$

$$\nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=-\frac{\partial f_y}{\partial z}-\frac{\partial f_z}{\partial y}$$

$$\nabla^2\tau_{zx}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z}=-\frac{\partial f_x}{\partial z}-\frac{\partial f_z}{\partial x}$$

$$\Theta=\sigma_x+\sigma_y+\sigma_z$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

按应力求解弹性力学问题

体积力f_i为零或为常量

$$(1+\nu)\nabla^2\sigma_x+\frac{\partial^2\Theta}{\partial x^2}=0$$

$$(1+\nu)\nabla^2\sigma_y+\frac{\partial^2\Theta}{\partial y^2}=0$$

$$(1+\nu)\nabla^2\sigma_z+\frac{\partial^2\Theta}{\partial z^2}=0$$

$$(1+\nu)\nabla^2\tau_{xy}+\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$(1+\nu)\nabla^2\tau_{yz}+\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$(1+\nu)\nabla^2\tau_{zx}+\frac{\partial^2\Theta}{\partial x\partial z}=0$$

由 $\frac{2(1-\nu)}{1+\nu}\nabla^2\Theta=-2(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z})$　P135 公式(4-7)

可知：应力第一不变量Θ是调和函数

$$\nabla^2\Theta=0$$

对左式两边分别作Laplace运算可得：

$$\nabla^2\nabla^2\sigma_x=0\qquad \nabla^2\nabla^2\tau_{xy}=0$$

$$\nabla^2\nabla^2\sigma_y=0\qquad \nabla^2\nabla^2\tau_{yz}=0$$

$$\nabla^2\nabla^2\sigma_z=0\qquad \nabla^2\nabla^2\tau_{zx}=0$$

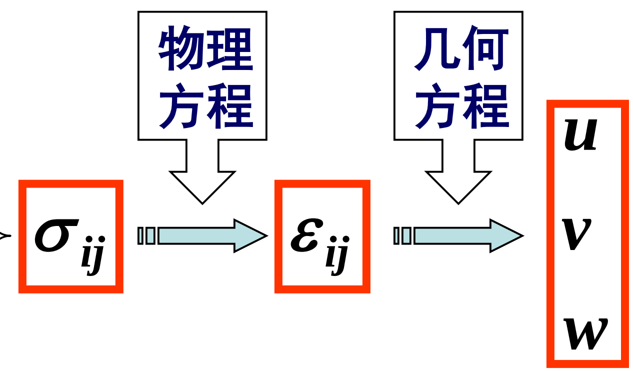
应力分量是双调和函数，并且满足应力平衡方程！

按应力求解弹性力学问题

按应力求解弹性力学问题的步骤：

1. 所求的应力分量应满足平衡微分方程和协调方程；
2. 应力分量在边界上应满足力的边界条件；
3. 求得应力分量后，根据应力应变关系求解出应变分量；
4. 根据几何方程求解获得位移；

$$\left.\begin{aligned}&\frac{\partial \sigma_{ij}}{\partial x_j}+f_i=0\\&(1+\mu)\nabla^2\sigma_{ij}+\frac{\partial^2\Theta}{\partial x_i\partial x_j}=0\\&\sigma_{ij}n_j=F_i\end{aligned}\right\}$$



按应力求解弹性力学问题

注意位移单值性的问题：

物体内任一点的位移必须是单值（唯一）的，因为由应变求位移时，需要进行积分运算，这就会涉及到积分的连续条件。

- **单连体（内部无洞）**：只具有一个连续边界的物体。满足平衡方程和相容方程，也满足应力边界条件，则应力分量完全确定，即解是唯一确定的。
- **多连体（内部有洞）**：具有多个连续边界的物体。除了满足方程和边界条件，还要考虑位移的单值性条件，这样才能完全确定应力分量。

按应力求解弹性力学问题

优点：边界条件比较简单，并且得到的应力表达式在大多数具体问题中比位移表达式简单。

缺点：未知函数较多，所要求解的二阶偏微分方程比较复杂。

按应力求解比按位移求解一般来说容易些。

但就解决弹性体问题的普遍性而言，按位移求解更具有普遍性。

对于实际问题，适当的选择求解方法。

平面问题

平面问题即二维问题，是弹性力学中比较简单的一类问题，它可以分为两类：一类是平面应力问题，另一类是平面应变问题。

任何一个弹性体都是空间物体，一般的外力都是空间力系，因此，严格地说，任何一个实际的弹性力学问题都是空间问题。

如果所考察的弹性体具有某种特殊的形状，并且承受的是某种特殊的外力，就可以把空间问题简化为近似的平面问题。这样处理，分析和计算的工作量将大大地减少，而所得的成果都仍然能满足工程上对精度的要求。

平面问题的基本方程

平衡方程

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + f_y = 0 \end{cases}$$

边界条件

$$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$$

几何方程

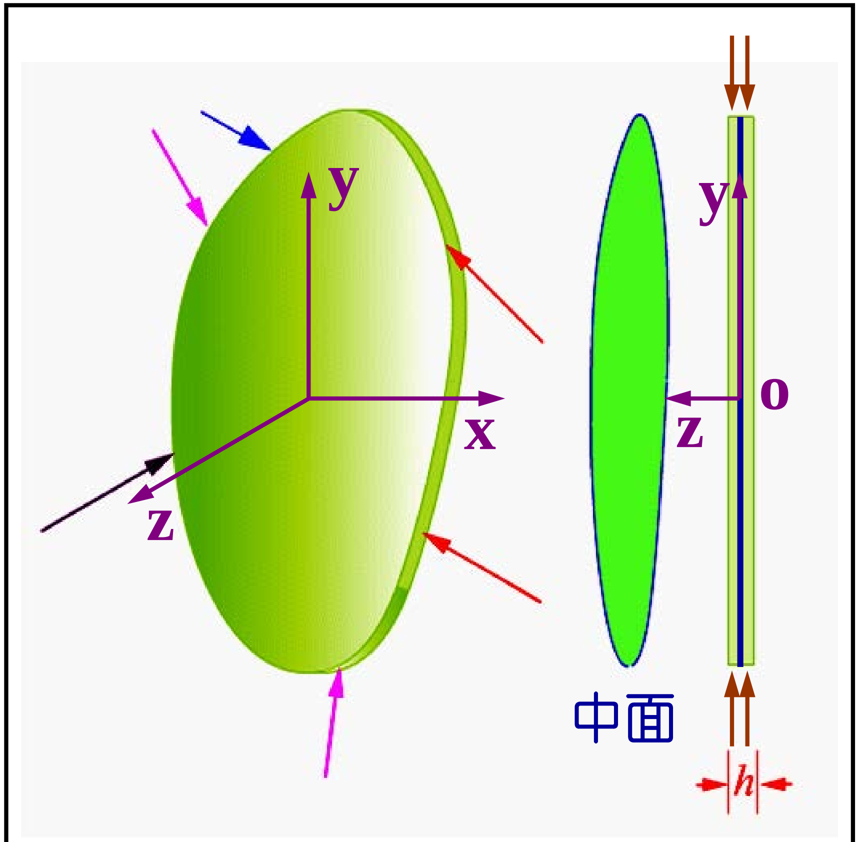
$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{cases}$$

物理方程

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \mu\sigma_y\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \mu\sigma_x\right] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases}$$

平面问题

1、平面应力问题



构件几何形状特征：薄板外力平行于中面，沿厚度均匀分布，表面不受外力作用。

表面面力边界条件：

$$\sigma_z\Big|_{z=\pm\frac{h}{2}}=0$$

$$\tau_{xz}\Big|_{z=\pm\frac{h}{2}}=0,\quad \tau_{yz}\Big|_{z=\pm\frac{h}{2}}=0$$

由于薄板厚度很小，应力分量均匀分布，薄板各点都有

$$\sigma_z=0,\quad \tau_{yz}=0,\quad \tau_{xz}=0$$

平面问题

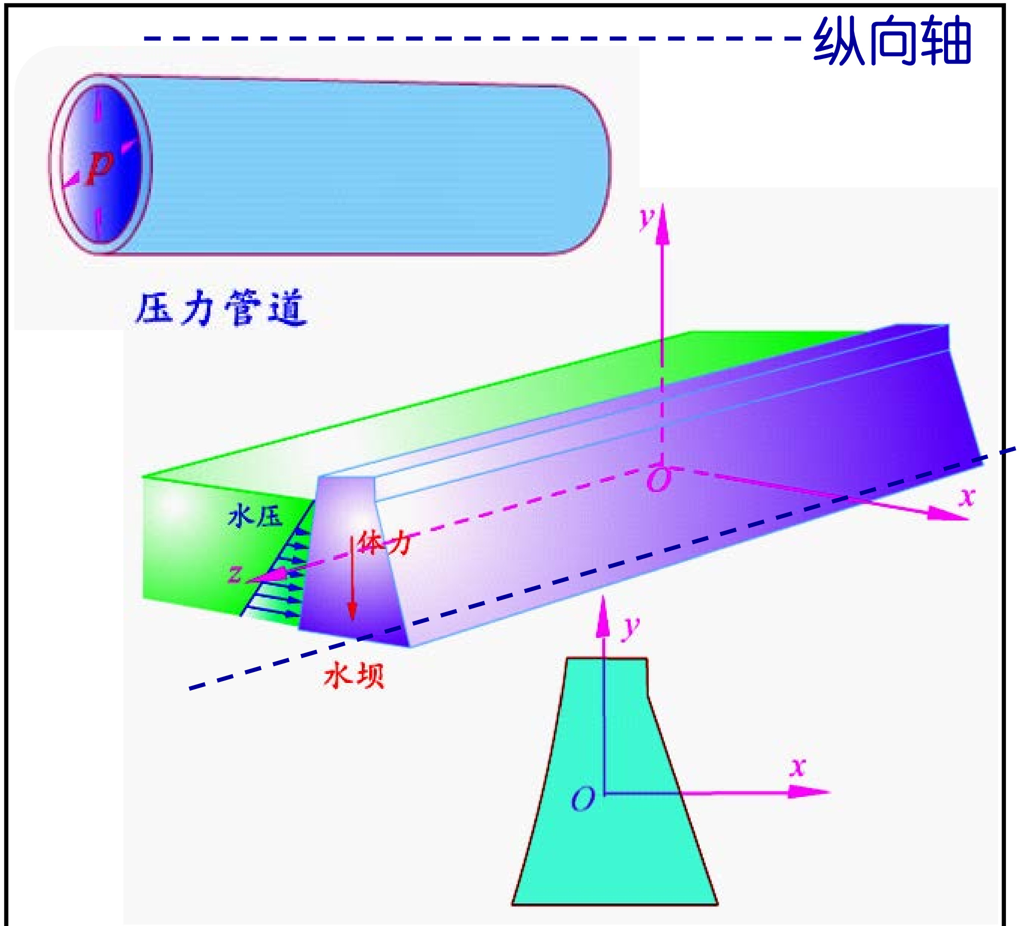
同时，也因为板很薄，以及分析问题时必须要考虑的形变分量和位移分量，都可以是不沿厚度变化的，即：它们只是x和y的函数，不随z 而变化。

所以有：

$$\begin{cases} \sigma_x = F_1(x, y) & \tau_{xy} = F_3(x, y) \\ \sigma_y = F_2(x, y) & \tau_{yz} = 0 \\ \sigma_z = 0 & \tau_{zx} = 0 \end{cases}$$

平面问题

2、平面应变问题



构件几何形状特征：具有很长纵向轴的柱体

在柱上受有平行于横截面而且不沿长度变化的面力，同时，体力也平行于横截面而且不沿长度变化。

假想该柱形体为无限长，以任一横截面为xy面，任一纵线为z轴，则所有一切应力分量，形变分量和位移分量都不沿z轴方向变化，而只是x和y的函数。

平面问题

在这一情况下，z方向的位移w=0。因为所有各点的位移分量都平行于xy面对称（任一横截面都看作是对称面），所以各点都只会沿x和y方向移动而不会，所以这种问题称为“平面位移问题”，但在习惯上常称为“平面应变问题”。

对于平面应变问题： $u=\varphi_1(x,y), v=\varphi_2(x,y), w=0$

根据几何方程： $\varepsilon_z=0,\quad \gamma_{xz}=\gamma_{yz}=0$

根据物理方程： $\varepsilon_z=\dfrac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]=0$

$$\sigma_z=\nu(\sigma_x+\sigma_y)$$

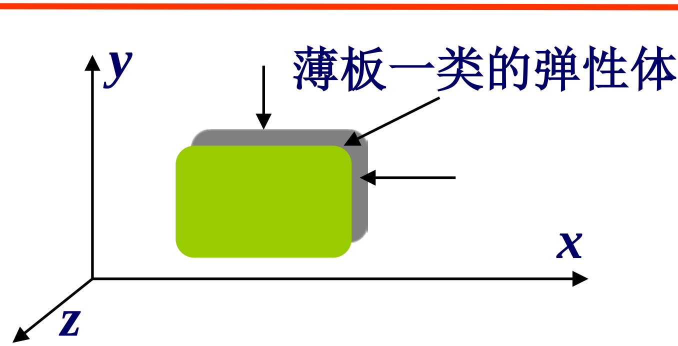
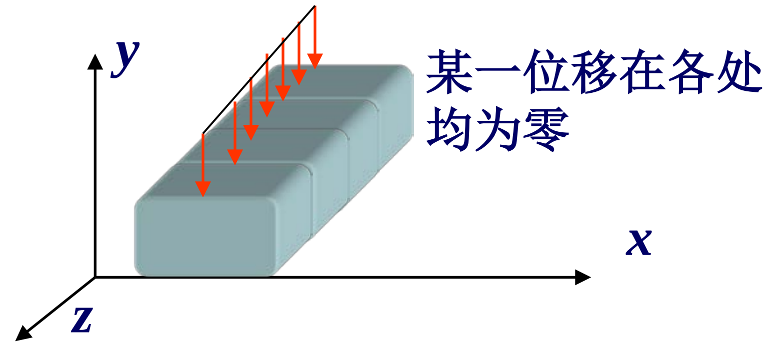
平面问题

则对于平面应变问题有：

$$\begin{cases}\sigma_x = F_1(x,y) & \tau_{xy} = F_3(x,y) \\ \sigma_y = F_2(x,y) & \tau_{yz} = 0 \\ \sigma_z = \nu(\sigma_x + \sigma_y) & \tau_{zx} = 0\end{cases}$$

应变是平面的，应力是空间的。

平面问题

	平面应力问题：	平面应变问题：
构件特征：		
受力特点：	平行于板面，板面上无载荷	载荷与 z 轴垂直沿 z 轴不变
应力分量：	$\sigma_z = \tau_{xz} = \tau_{zy} = 0$ $\sigma_x, \sigma_y, \tau_{xy}(x,y)$	$\sigma_x, \sigma_y, \tau_{xy}(x,y)$ $\tau_{xz} = \tau_{zy} = 0, \sigma_z = \mu(\sigma_x + \sigma_y)$
应变分量：	$\gamma_{yz} = \gamma_{xz} = 0$ $\varepsilon_x, \varepsilon_y, \gamma_{xy}(x,y); \varepsilon_z$	$\varepsilon_z = \gamma_{yx} = \gamma_{zx} = 0$ $\varepsilon_x, \varepsilon_y, \gamma_{xy}(x,y)$
位移分量：	$u(x,y), v(x,y); w$	$u(x,y), v(x,y); w=0$

按位移求解平面问题

对于平面应力问题有平衡微分方程为:

$$
\begin{cases}
\dfrac{E}{1-\nu^2}\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{1-\nu}{2}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{1+\nu}{2}\dfrac{\partial^2 u}{\partial x\partial y}\right)+f_x=0 \\
\dfrac{E}{1-\nu^2}\left(\dfrac{\partial^2 v}{\partial y^2}+\dfrac{1-\nu}{2}\dfrac{\partial^2 v}{\partial x^2}+\dfrac{1+\nu}{2}\dfrac{\partial^2 u}{\partial x\partial y}\right)+f_y=0
\end{cases}
$$

用位移表示的应力边界条件为:

$$
\begin{cases}
\dfrac{E}{1-\nu^2}\left[l\left(\dfrac{\partial u}{\partial x}+\nu\dfrac{\partial v}{\partial y}\right)+m\dfrac{1-\nu}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)\right]=F_x \\
\dfrac{E}{1-\nu^2}\left[m\left(\dfrac{\partial v}{\partial y}+\nu\dfrac{\partial u}{\partial x}\right)+l\dfrac{1-\nu}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right]=F_y
\end{cases}
$$

按应力求解平面问题

对于平面应力问题所有平衡微分方程为：

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + f_x = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + f_y = 0 \end{cases}$$

相容方程为：

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

当体力为常量时有，相容方程为： $\nabla^2(\sigma_x + \sigma_y) = 0$

按应力求解平面问题

常体力平衡微分方程式是一个非齐次微分方程组，它的解包含两部分，即任意一个特解及对应的齐次微分方程的通解。

特解可取为： $\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = -yf_x - xf_y$

通解对应的齐次方程为：
$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 \end{cases}$$

应力函数

在平面问题中，引进应力函数的概念，往往使求解问题变得简单。

无体力存在时：

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0 \qquad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

假定：

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2},\ \sigma_y=\frac{\partial^2\varphi}{\partial x^2},\ \tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}$$

平衡方程将自然满足

应力分量能够用一个函数表示： $\varphi(x,y)$ — Airy应力函数

只需求解以应力函数表示的协调方程

应力函数

平面应力问题的广义胡克定律

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{2G(1+\nu)}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{2G(1+\nu)}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

两类问题具有相同的应力-应变关系，只是对于平面应变问题，需要用v`代替v，因此，两类问题在数学处理上的方法是一样的。

平面应变问题的广义胡克定律

$$\varepsilon_x = \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y]$$

$$\varepsilon_y = \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\xrightarrow[E = 2G(1+\nu)]{\nu` = \frac{\nu}{1-\nu}}$$

$$\varepsilon_x = \frac{1}{2G(1+\nu`)}(\sigma_x - \nu`\sigma_y)$$

$$\varepsilon_y = \frac{1}{2G(1+\nu`)}(\sigma_y - \nu`\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

应力函数

以平面应力问题为例进行推导：

用应力函数表示物理方程

$$\varepsilon_x=\frac{1}{2G(1+\nu)}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)$$

$$\varepsilon_y=\varepsilon_x=\frac{1}{2G(1+\nu)}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)$$

$$\gamma_{xy}=-\frac{1}{G}\cdot\frac{\partial^2\varphi}{\partial x\partial y}$$

满足协调方程

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

$$\Rightarrow\quad \frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0$$

即

$$\boxed{\nabla^2\nabla^2\varphi=0}$$

边界条件

$$l\frac{\partial^2\varphi}{\partial y^2}-m\frac{\partial^2\varphi}{\partial x\partial y}=F_x$$

$$-l\frac{\partial^2\varphi}{\partial x\partial y}+m\frac{\partial^2\varphi}{\partial x^2}=F_y$$

平面问题归结为求解满足双调和方程和给定边界条件的应力函数 $\varphi(x,y)$

应力函数

对于平面应变问题，将 $\nu` = \frac{\nu}{1-\nu}$ 代入E和G之间的表达式，可得：

$$E' = 2G(1+\nu') = 2G(1+\frac{\nu}{1+\nu}) = \frac{2G(1+\nu)}{(1-\nu)(1+\nu)} = \frac{E}{1-\nu^2}$$

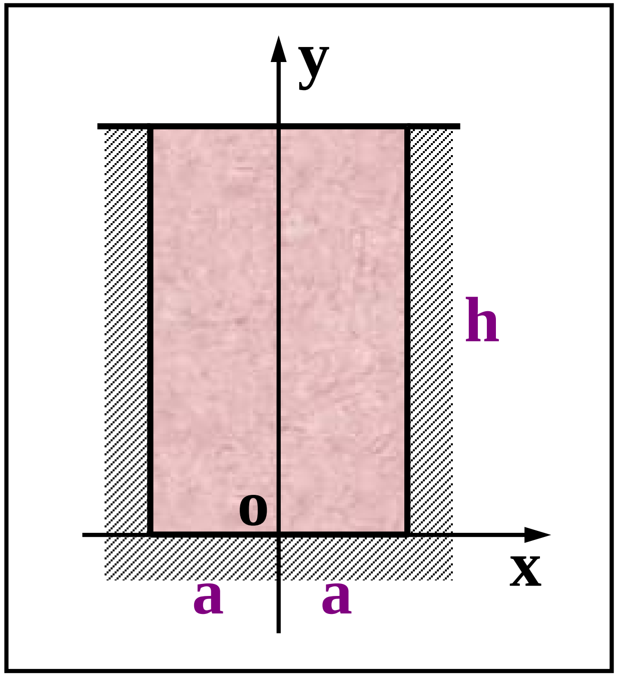
因此，只需要将平面应力问题的有关公式中的E和v用E`和v`替代即可求得平面应变问题的相关公式。

✍ 例题2

图示很长的矩形柱体，材料的比重为ρ，将其放入形状相同的刚性槽内若不考虑摩擦力，设应力函数的形式为

$$\varphi = Ax^2y + By^3 + Cy^2 + Dx^2$$

试求各应力分量、应变分量以及位移分量。



$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2},\quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y}$$

满足

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} + f_x = 0 \\ \dfrac{\partial\tau_{yx}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + f_y = 0 \end{cases}$$

解： 根据Airy应力函数可得通解

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \varphi}{\partial y^2} = 6By + 2C \\ \sigma_y = \dfrac{\partial^2 \varphi}{\partial x^2} = 2Ay + 2D \\ \tau_{xy} = -\dfrac{\partial^2 \varphi}{\partial x \partial y} = -2Ax \end{cases}$$

构造一个特解：

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + f_x = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + f_y = 0 \end{cases}$$

已知：

$$f_x = 0$$
$$f_y = -\rho$$

可得：

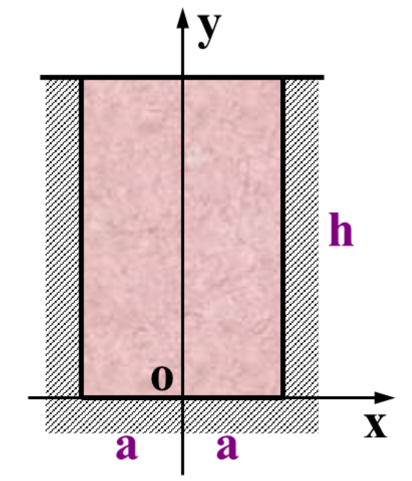
$$\sigma_x = 0, \quad \sigma_y = 0$$
$$\tau_{xy} = -yf_x - xf_y = \rho x$$

因此可得全解：

$$\sigma_x = 6By + 2C, \quad \sigma_y = 2Ay + 2D, \quad \tau_{xy} = -2Ax + \rho x$$

应力边界条件

$y=h$ 处， $\tau_{xy}=0$, $\sigma_y=0$ $\Rightarrow$ $A=\rho/2$, $D=-\rho h/2$



$$\sigma_x = 6By+2C$$
$$\sigma_y = 2Ay+2D$$
$$\tau_{xy} = -2Ax+\rho x$$

刚性槽的条件 $\int_{-a}^{a}\varepsilon_x dx = 0$ $\xrightarrow{\varepsilon_x\ □□□\ x\ □□}$ $\varepsilon_x = 0$

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\mu)\sigma_x - \nu\sigma_y] = 0 \quad \Rightarrow \quad \sigma_x = \frac{\nu}{1-\nu}\sigma_y$$

$$\sigma_y = \rho(y-h) \quad \sigma_x = \frac{\nu}{1-\nu}\rho(y-h) \quad \Rightarrow \quad B = \frac{\nu}{1-\nu}\cdot\frac{\rho}{6} \quad C = -\frac{\nu}{1-\nu}\cdot\frac{\rho h}{2}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x] = \frac{1}{E}\frac{(1+\nu)(1-2\nu)}{1-\nu}\rho(y-h)$$

$$\begin{cases} u = \int \varepsilon_x dx = 0 \\ v = \int \varepsilon_y dy = \dfrac{1}{E}\dfrac{(1+\nu)(1-2\nu)}{1-\nu}\rho(\dfrac{y^2}{2} - hy) + K \end{cases}$$

$y=0$ □ □$v=0$

$\Rightarrow K = 0$

逆解法和半逆解法

求解弹性力学问题在数学上是比较复杂的，因此不得不采用逆解法和半逆解法。

优点：在数学上比较容易；

缺点：带有一定的偶然性和相当大的局限性，有时需要进行多次反复试算。

逆解法

- 先假设物体内部的应力分布规律；
- 然后分析它所对应的边界条件，以确定这样的应力分布规律是什么问题的解答。

逆解法解题思路：

$\boxed{设定\varphi(x,y)} \Rightarrow \boxed{满足\nabla^4\varphi = 0} \Rightarrow \boxed{\begin{cases}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{cases}} \Rightarrow \boxed{边界条件}$

$\Downarrow$

$\boxed{位移分量} \Leftarrow \boxed{应变分量} \Leftarrow \boxed{确定待定常数}$

需了解满足 $\nabla^2\nabla^2\varphi = 0$ 的各种形式的应力函数。

逆解法

（不计体力）

1. $\varphi = ax + by + c$

满足 $\nabla^2\nabla^2\varphi = 0$

$$\begin{cases} \sigma_x = \dfrac{\partial^2\varphi}{\partial y^2} \\ \sigma_y = \dfrac{\partial^2\varphi}{\partial x^2} \\ \tau_{xy} = -\dfrac{\partial^2\varphi}{\partial x\partial y} \end{cases}$$

$$\begin{cases} \sigma_x = 0 \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

$$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$$

$$\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$$

- 不论弹性体何种形状，不论坐标轴如何选择，线性应力函数对应于无面力、无应力的状态。
- 在应力函数中加上或减去一个线性函数并不影响应力。

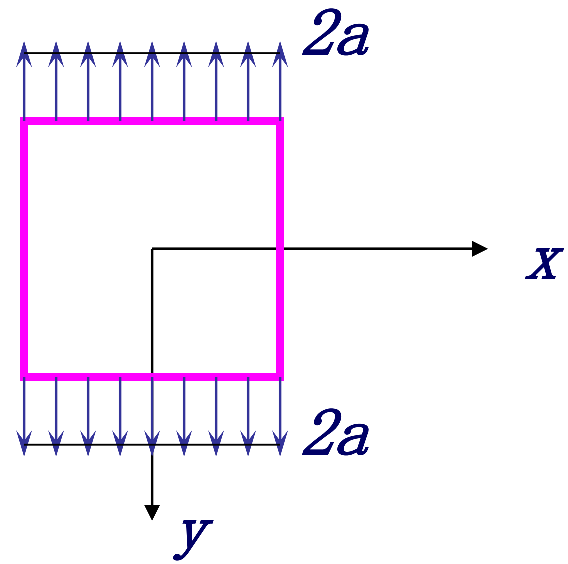
逆解法

2. $\varphi = ax^2$

满足 $\nabla^2 \nabla^2 \varphi = 0$

$$\begin{cases} \sigma_x = 0 \\ \sigma_y = 2a \\ \tau_{xy} = 0 \end{cases}$$

边界条件：$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$



左右边界：$\begin{cases} l = \pm 1 \\ m = 0 \end{cases}$ $\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$

上下边界：$\begin{cases} l = 0 \\ m = \pm 1 \end{cases}$ $\begin{cases} F_x = 0 \\ F_y = \pm 2a \end{cases}$

矩形板在 y 方向受均匀拉伸（压缩）。

逆解法

3. $\varphi = cy^2$

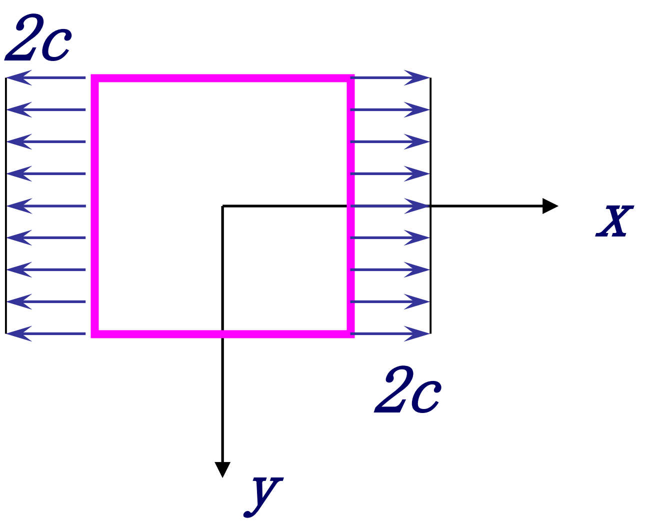
满足 $\nabla^2\nabla^2\varphi = 0$

$$\begin{cases} \sigma_x = 2c \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

边界条件: $\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$

左右边界: $\begin{cases} l = \pm 1 \\ m = 0 \end{cases}$ $\begin{cases} F_x = \pm 2c \\ F_y = 0 \end{cases}$

上下边界: $\begin{cases} l = 0 \\ m = \pm 1 \end{cases}$ $\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$



矩形板在 x 方向受均匀拉伸（压缩）。

逆解法

4. $\varphi = bxy$

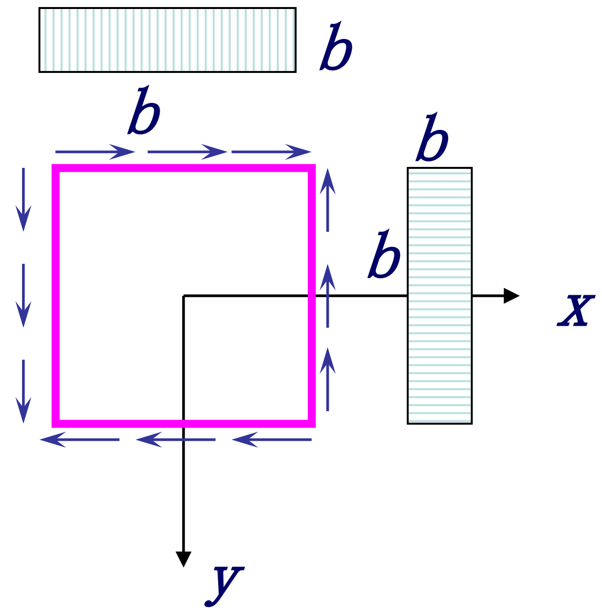
满足$\nabla^2\nabla^2\varphi = 0$

$$\begin{cases} \sigma_x = 0 \\ \sigma_y = 0 \\ \tau_{xy} = -b \end{cases}$$

边界条件：$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$

左右边界：$\begin{cases} F_x = 0 \\ F_y = \pm b \end{cases}$

上下边界：$\begin{cases} F_x = \pm b \\ F_y = 0 \end{cases}$



矩形板在 四周受均布剪应力作用。

逆解法

5. $\varphi = Ay^3$

满足$\nabla^2\nabla^2\varphi = 0$

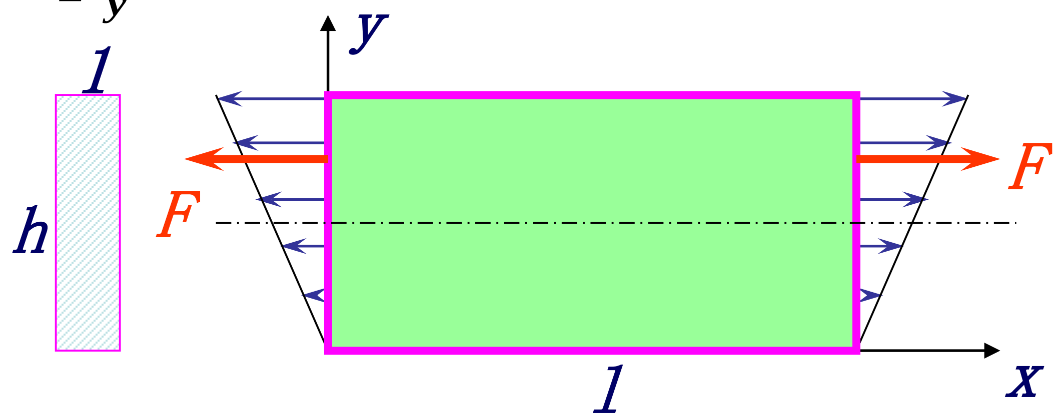
$$\begin{cases} \sigma_x = 6Ay \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

边界条件:$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$

上下边界: $\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$

左边界: $\begin{cases} F_x = -6Ay \\ F_y = 0 \end{cases}$

右边界: $\begin{cases} F_x = 6Ay \\ F_y = 0 \end{cases}$



$$F = \int_0^h F_x dy = 3Ah^2$$

矩形板受偏心拉力作用。

逆解法

5. $\varphi = Ay^3$

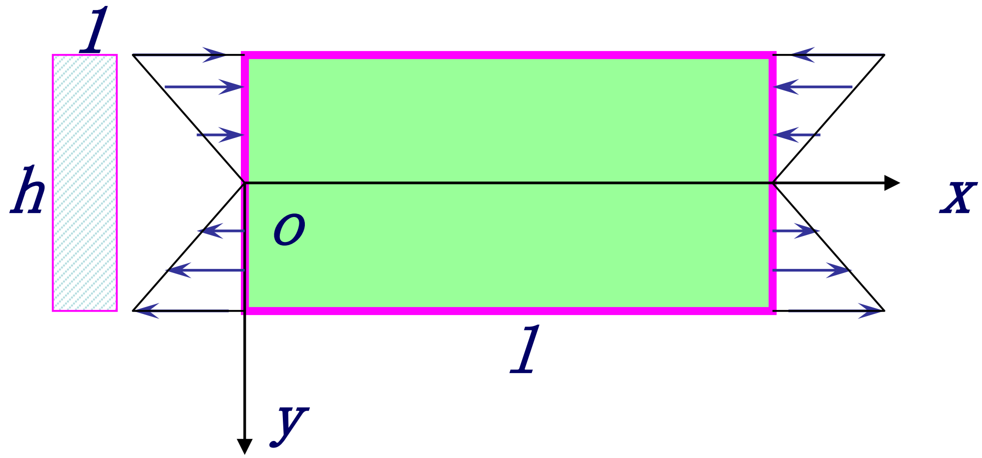
满足 $\nabla^2\nabla^2\varphi = 0$

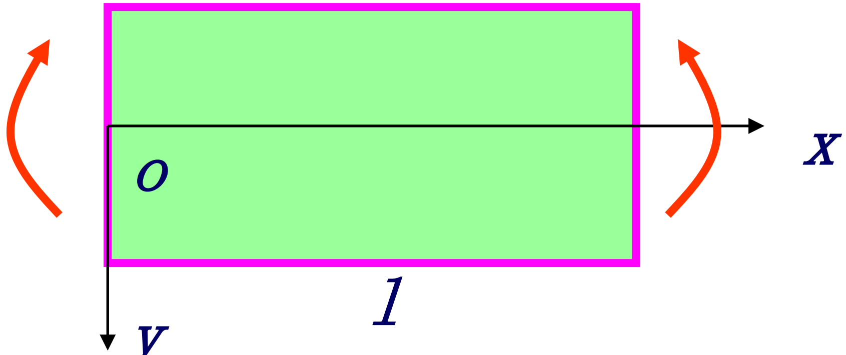
$$\begin{cases} \sigma_x = 6Ay \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

上下边界： $\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$

左边界： $\begin{cases} F_x = -6Ay \\ F_y = 0 \end{cases}$

右边界： $\begin{cases} F_x = 6Ay \\ F_y = 0 \end{cases}$





矩形板受纯弯曲作用。

同一应力函数在不同的坐标系中解决的问题也不同。

逆解法

$6.\ \varphi = ax^2 + bxy + cy^2$

$$\sigma_x = 2c, \sigma_y = 2a, \tau_{xy} = -b$$

均匀应力状态

$b = 0, a > 0, c > 0$ 双向受拉

$b \neq 0, a = 0, c = 0$ 纯剪切

$7.\ \varphi = ax^3 + bx^2y + cxy^2 + dy^3$

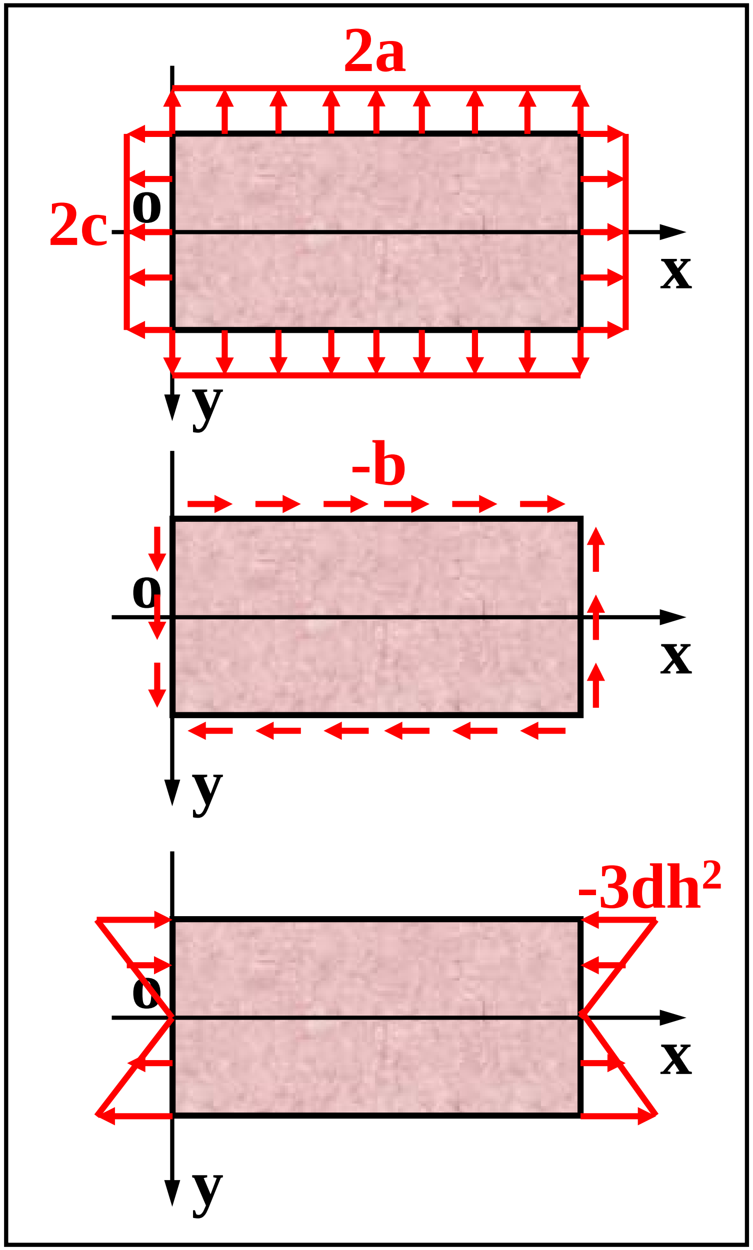
$$\sigma_x = 2cx + 6dy$$

$$\sigma_y = 6ax + 2by$$

$$\tau_{xy} = -2(bx + cy)$$

复杂应力状态
应用叠加原理
可分解为简单
应力状态

$a = b = c = 0, d \neq 0$ 纯弯曲



逆解法

$8.\ \varphi = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$

满足 $\nabla^2\nabla^2\varphi = 0$ ⇨ $\boxed{e = -(a + \frac{c}{3})}$

$$\sigma_x = 2cx^2 + 6dxy - 12(a + \frac{c}{3})y^2$$

$$\sigma_y = 12ax^2 + 2bxy + 2cy^2$$

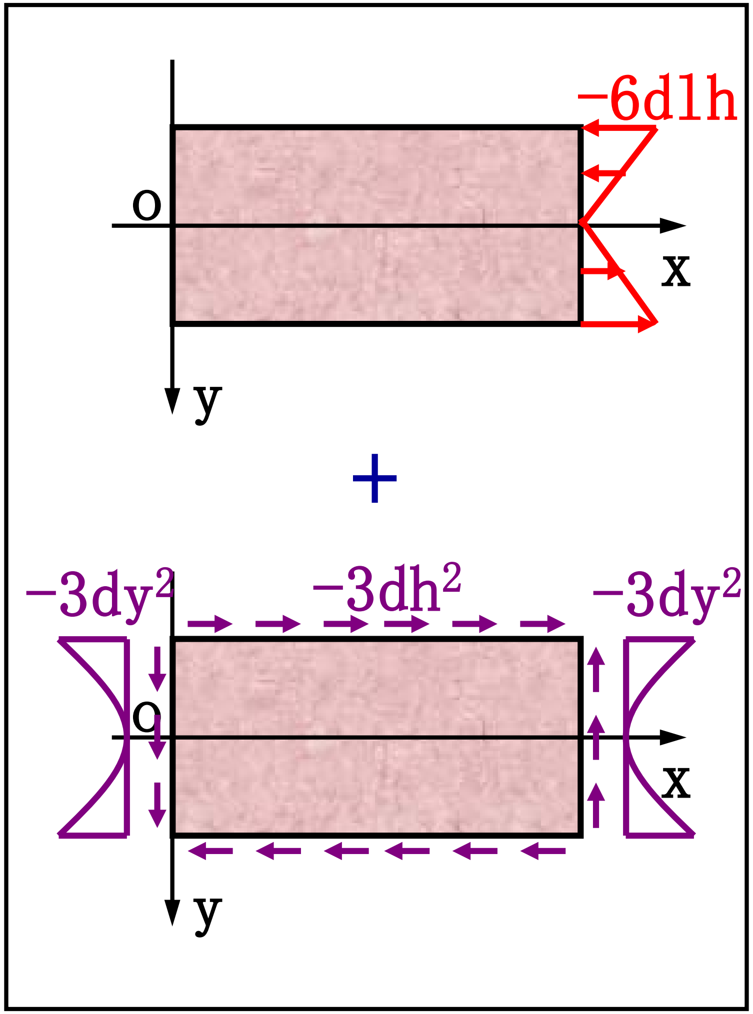
$$\tau_{xy} = -(3bx^2 + 4cxy + 3dy^2)$$

$$a = b = c = 0,\ d \neq 0$$

$$\sigma_x = 6dxy,\ \sigma_y = 0,\ \tau_{xy} = -3dy^2$$

$9.\ \varphi = Axy^3$

$$\sigma_x = 6Axy,\ \sigma_y = 0,\ \tau_{xy} = -3Ay^2$$

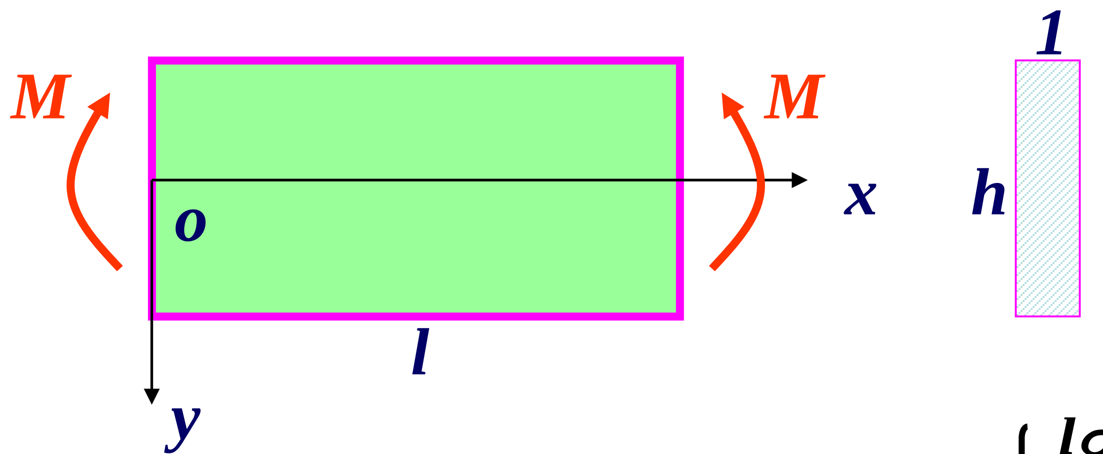


逆解法

- 如果以上个弹性体边界上确实作用着如图所示的面力，那么以上结果即为该问题的解；
- 在实际问题中，只有较简单的问题才容易找到应力函数的形式；
- 对一个应力函数而言，对于不同形状的弹性体，或者选用不同的坐标系，均对应着不同的面力分布；
- 掌握了许多简单应力函数所对应的应力特点，可以用叠加原理去解决实际上比较复杂的问题。

例题

单位厚度的矩形截面梁，受到单位厚度的力偶矩M作用，试求应力分量和位移分量。



$$\varphi = Ay^3$$

满足 $\nabla^2 \nabla^2 \varphi = 0$

应力分量：
$$\begin{cases} \sigma_x = 6Ay \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

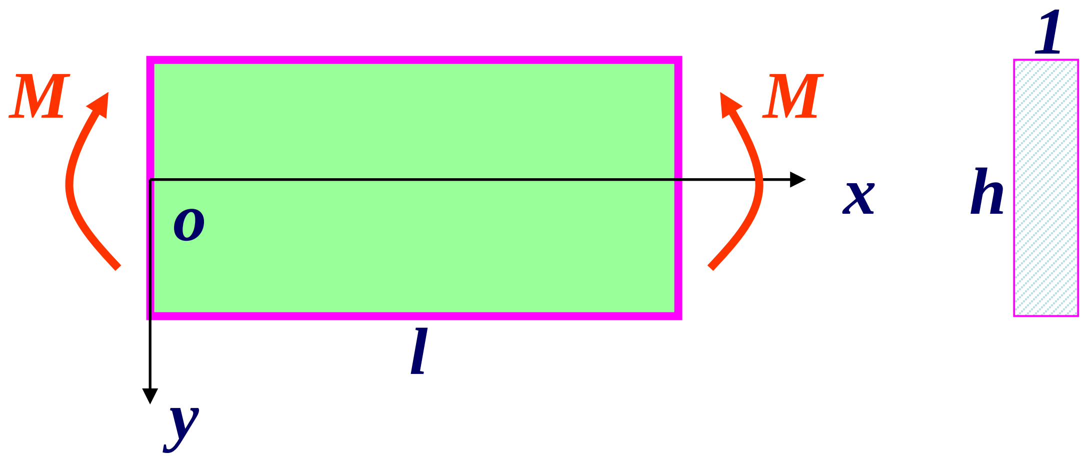
边界条件：
$$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$$

上下边界：

$$y = \pm\frac{h}{2}; l = 0, m = \pm 1;$$

$$F_x = 0$$
$$F_y = 0$$

自然满足



$$\begin{cases}\sigma_x = 6Ay \\ \sigma_y = 0 \\ \tau_{xy} = 0\end{cases}$$

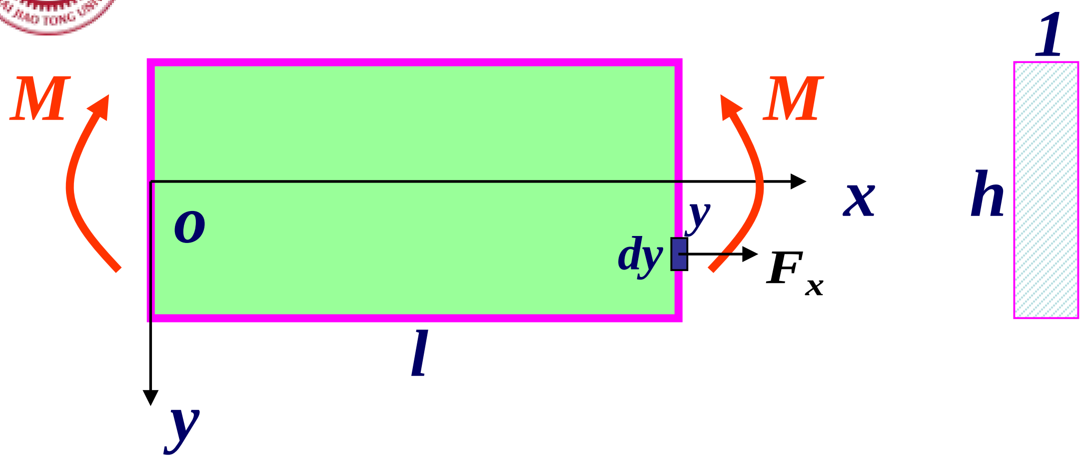
$$\begin{cases} l\sigma_x + m\tau_{yx} = F_x \\ l\tau_{xy} + m\sigma_y = F_y \end{cases}$$

左右边界：$x = l; l = 1, m = 0$　　$F_x = 6Ay \neq 0$　$F_y = 0$　不满足！

静力等效边界条件（*Saint-Venant principle*）：

把物体的一小部分边界上的面力，改为具体分布不同，但静力等效的面力，只影响近处应力分布，对远处影响很小。

静力等效：主矢量相等、主矩相等。



次边界：$x = l; l = 1, m = 0$

$$F_x = 6Ay \neq 0$$

$$F_y = 0$$

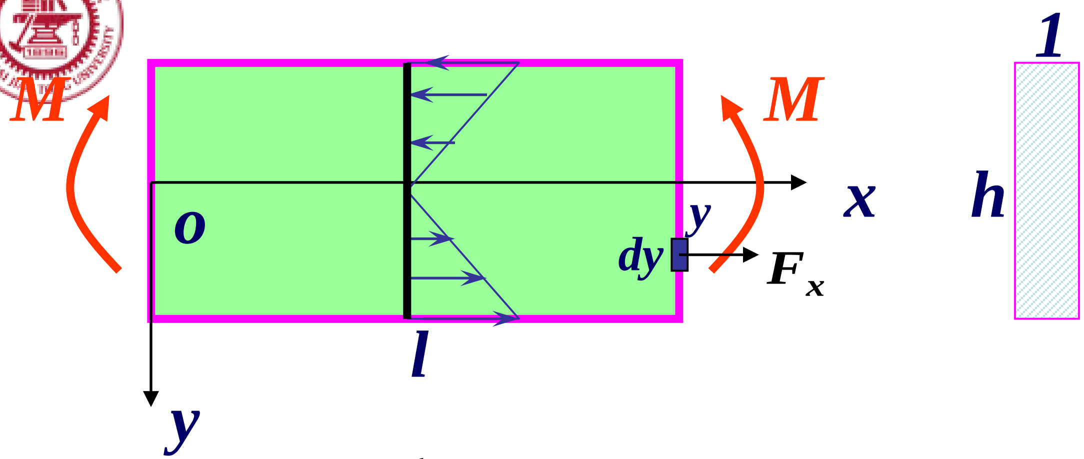
不满足！

主矢量相等： $\int_{-h/2}^{h/2} F_x dy = 0$ $\int_{-h/2}^{h/2} 6Aydy = 0$

$$\int_{-h/2}^{h/2} F_y dy = 0$$

主矩相等： $\int_{-h/2}^{h/2} F_x ydy = M$ $\int_{-h/2}^{h/2} 6Ay^2 dy = M$

$$\frac{Ah^3}{2} = M \qquad A = \frac{2M}{h^3}$$



$$\begin{cases} \sigma_x = 6Ay \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases} \qquad A = \frac{2M}{h^3}$$

应力分量：

$$\begin{cases} \sigma_x = \dfrac{12M}{h^3} y = \dfrac{My}{I} \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

弯曲截面系数

$$I = \frac{1 \cdot h^3}{12}$$

应变分量：

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\sigma_y\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\sigma_x\right] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases} \Rightarrow \begin{cases} \varepsilon_x = \dfrac{My}{EI} \\ \varepsilon_y = -\dfrac{\mu My}{EI} \\ \gamma_{xy} = 0 \end{cases}$$

位移分量:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{cases} \qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{My}{EI} \\ \dfrac{\partial v}{\partial y} = -\dfrac{\mu My}{EI} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \end{cases}$$

$$u = \frac{Mxy}{EI} + f_1(y)$$

$$v = -\frac{\mu My^2}{2EI} + f_2(x)$$

$$\frac{Mx}{EI} + \frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} = 0$$

$$\frac{df_2(x)}{dx} + \frac{Mx}{EI} = -\frac{df_1(y)}{dy} = \omega \quad \Rightarrow \quad f_2(x) = -\frac{Mx^2}{2EI} + \omega x + v_0$$

$$-\frac{df_1(y)}{dy} = \omega \qquad f_1(y) = -\omega y + u_0$$

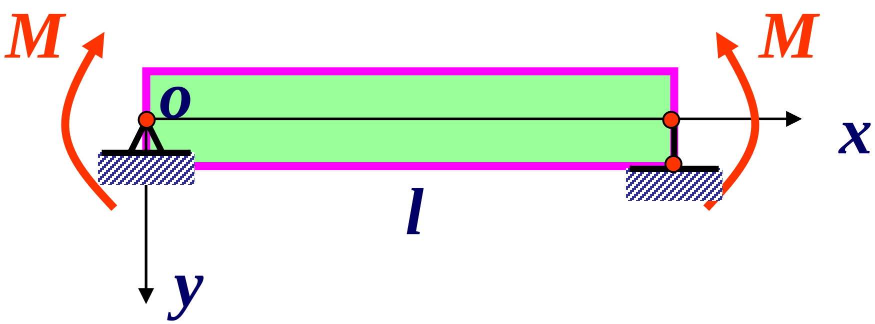
$$u = \frac{Mxy}{EI} - \omega y + u_0 \qquad v = -\frac{\mu My^2}{2EI} - \frac{Mx^2}{2EI} + \omega x + v_0$$

位移分量：

$$u = \frac{Mxy}{EI} - \omega y + u_0$$

$$v = -\frac{\mu My^2}{2EI} - \frac{Mx^2}{2EI} + \omega x + v_0$$

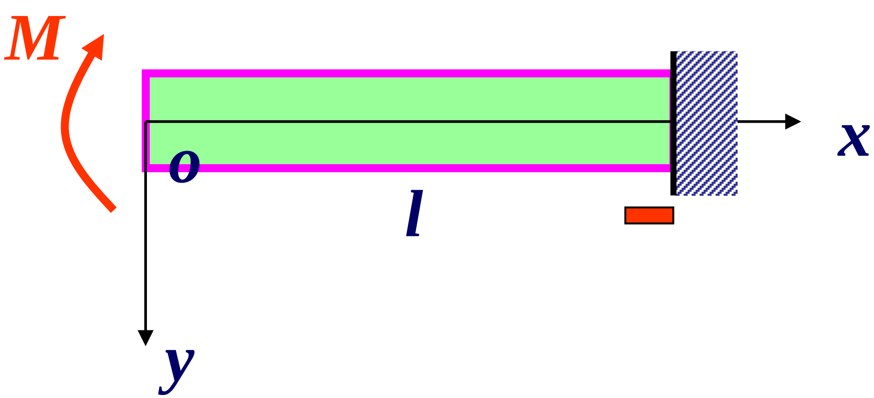
位移边界条件：



$$u\big|_{x=0,y=0} = 0; u_0 = 0$$

$$v\big|_{x=0,y=0} = 0; v_0 = 0$$

$$v\big|_{x=l,y=0} = 0; \omega = \frac{Ml}{2EI}$$



$$u\big|_{x=l,y=0} = 0; u_0 = 0$$

$$v\big|_{x=l,y=0} = 0; -\frac{Ml^2}{2EI} + \omega l + v_0 = 0$$

$$\frac{\partial v}{\partial x}\bigg|_{x=l,y=0} = 0; -\frac{Ml}{EI} + \omega = 0$$

$$\omega = \frac{Ml}{EI} \qquad v_0 = -\frac{Ml^2}{2EI}$$

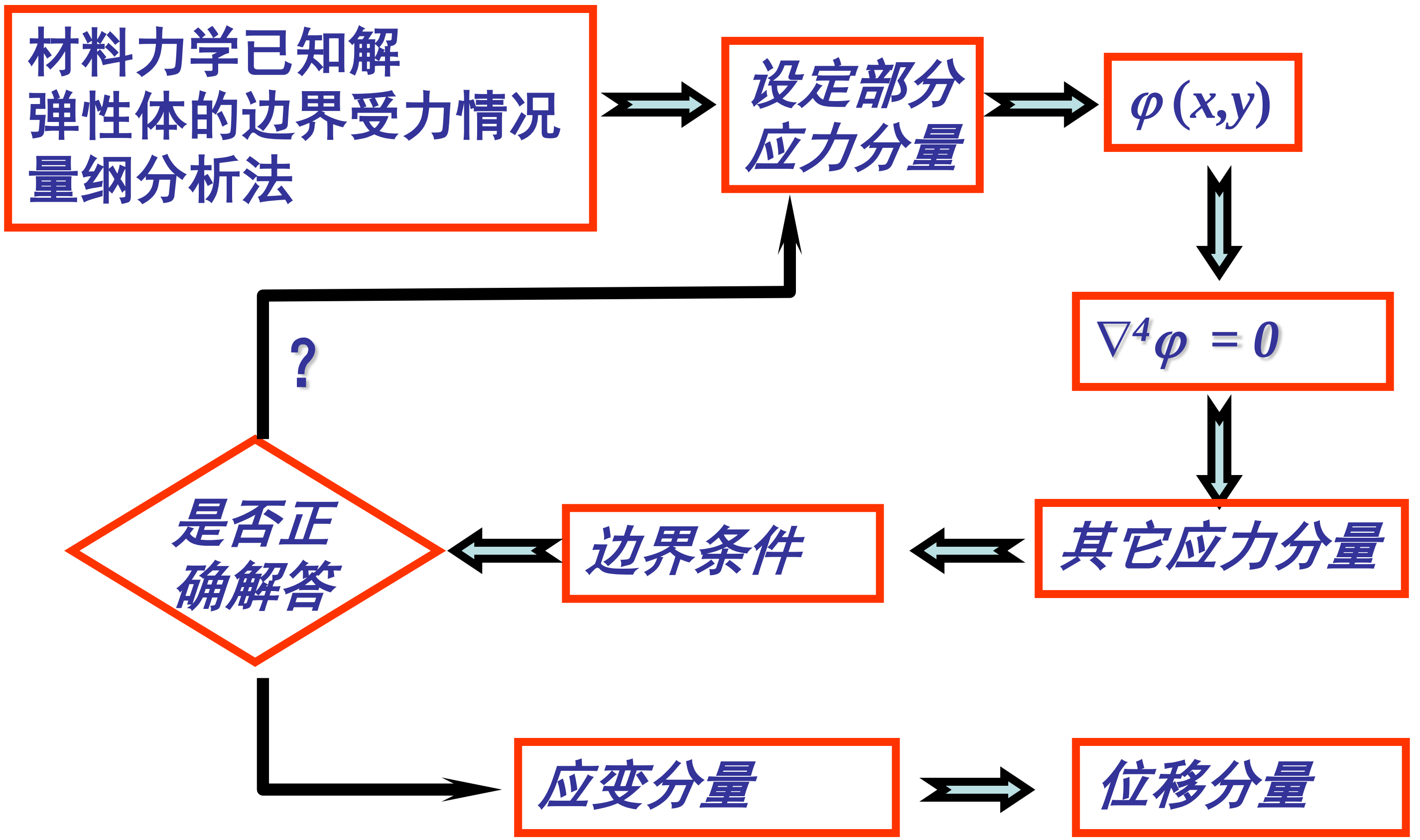
半逆解法

- 针对求解的问题，根据材料力学已知解或弹性体的边界形状和受力情况，假设部分应力为某种形式的函数，从而推断出应力函数；
- 然后用方程和边界条件确定尚未求出的应力分量，或完全确定原来假设的尚未全部定下来的应力。
- 如能满足弹性力学的全部条件，则这个解就是正确的解答，如不能满足全部条件，则需另外假定，重新求解。

由于根据已有解或经验作了一定假设，使得问题的求解过程得以大大简化。

半逆解法

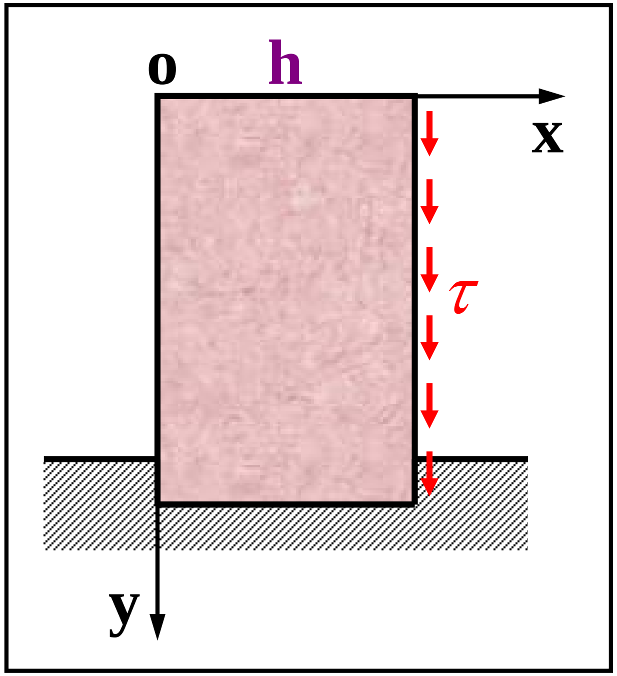
解题思路：



材料力学已知解
弹性体的边界受力情况
量纲分析法
设定部分应力分量
$\varphi(x,y)$
$\nabla^4\varphi = 0$
其它应力分量
边界条件
是否正确解答
?
应变分量
位移分量

例题

图示立柱（厚度为单位厚度），在其侧面上，作用有均布剪力 τ，试用半逆解法求其应力分布规律。



解：假定纵向纤维互不挤压

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$\Longrightarrow \varphi(x, y) = f_1(x) y + f_2(x)$$

代入 $\nabla^2 \nabla^2 \varphi = 0$

$$\Longrightarrow y \frac{d^4 f_1(x)}{dx^4} + \frac{d^4 f_2(x)}{dx^4} = 0$$

上式对于y取任何值均应成立

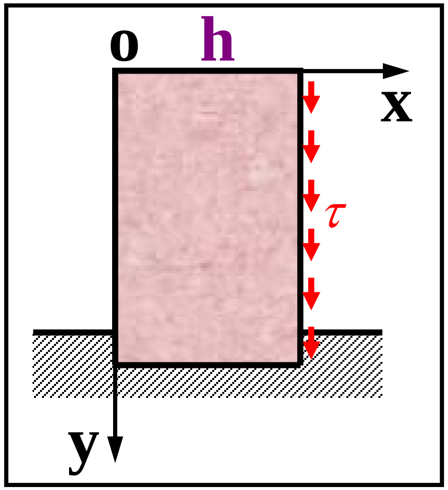
$$\frac{d^4 f_1(x)}{dx^4} = 0, \quad \frac{d^4 f_2(x)}{dx^4} = 0$$

$$f_1(x) = Ax^3 + Bx^2 + Cx \boxed{+ I}$$

$$f_2(x) = Dx^3 + Ex^2 \boxed{+ Jx + K}$$

$$\varphi = y(Ax^3 + Bx^2 + Cx) + Dx^3 + Ex^2$$

对应力分量无影响



$$\sigma_x = 0$$

$$\sigma_y = y(6Ax + 2B) + 6Dx + 2E$$

$$\tau_{xy} = -3Ax^2 - 2Bx - C$$

边界条件：

在x=0处， $\sigma_x = 0,\ \tau_{xy} = 0 \Longrightarrow C = 0$

在x=h处， $\sigma_x = 0,\ \tau_{xy} = \tau \Longrightarrow -3Ah^2 - 2Bh = \tau$

（主要边界条件，需精确满足）

在y=0处， $\int_0^h \tau_{xy} dx = 0 \Longrightarrow [Ax^3 + Bx^2]\Big|_0^h = 0$

$$\left.\begin{array}{l} -3Ah^2 - 2Bh = \tau \\ [Ax^3 + Bx^2]\Big|_0^h = 0 \end{array}\right\} \quad A = -\frac{\tau}{h^2}, \quad B = \frac{\tau}{h}$$

在y=0处，

$$\int_0^h \tau_{xy} dx = 0 \Longrightarrow [Ax^3 + Bx^2]\Big|_0^h = 0$$

$$\left.\begin{aligned} \int_0^h \sigma_y dx = 0 &\Longrightarrow 3Dh^2 + 2Eh = 0 \\ \int_0^h \sigma_y x dx = 0 &\Longrightarrow 2Dh^3 + Eh^2 = 0 \end{aligned}\right\} D = E = 0$$

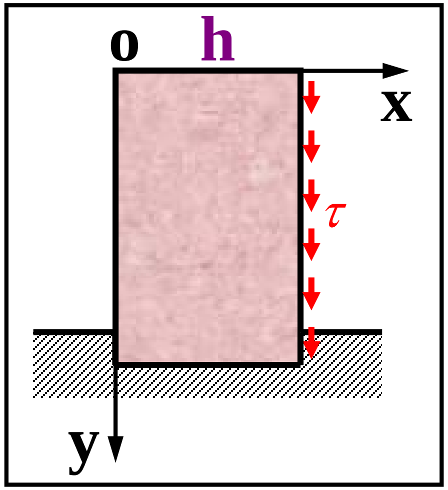
（次要边界条件，使用圣维南原理建立）

应力分量：

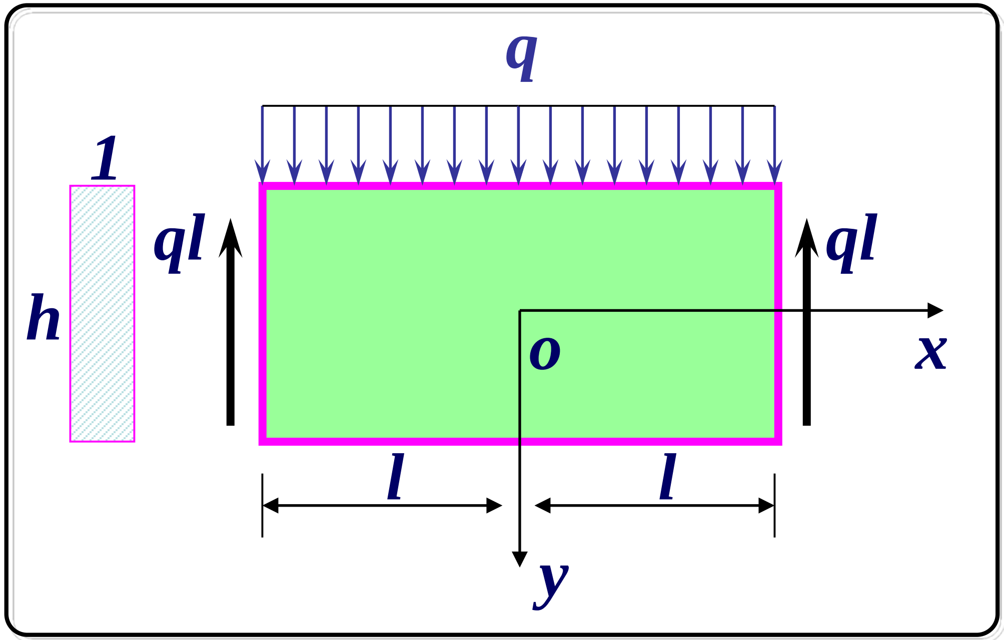
$$\sigma_x = 0$$

$$\sigma_y = \frac{2\tau}{h}(1 - \frac{3x}{h})y$$

$$\tau_{xy} = \frac{\tau}{h}(\frac{3x}{h} - 2)x$$



例：单位厚度的矩形截面梁，受到均布力作用，试求应力分量。（不计体力）



受力分析：面力在 y 方向有变化，

$$F_y\Big|_{y=-\frac{h}{2}} = -\sigma_y\Big|_{y=-\frac{h}{2}} = q$$

$$F_y\Big|_{y=\frac{h}{2}} = \sigma_y\Big|_{y=\frac{h}{2}} = 0$$

解：（一）确定应力函数：

$$\sigma_y = f(y) \qquad \sigma_y = \frac{\partial^2\varphi}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2\varphi}{\partial x^2} = f(y)$$

$$\varphi = \frac{x^2}{2} f(y) + x f_1(y) + f_2(y)$$

$$\nabla^4\varphi = 0$$

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0$$

$$\frac{\partial^4\varphi}{\partial x^4} = 0 \qquad \frac{\partial^4\varphi}{\partial x^2\partial y^2} = \frac{df^2(y)}{dy^2}$$

$$\frac{\partial^4\varphi}{\partial y^4} = \frac{x^2}{2}\frac{d^4 f(y)}{dy^4} + x\frac{d^4 f_1(y)}{dy^4} + \frac{d^4 f_2(y)}{dy^4}$$

$$\varphi = \frac{x^2}{2} f(y) + x f_1(y) + f_2(y)$$

$$\frac{1}{2}\frac{d^4 f(y)}{dy^4} x^2 + \frac{d^4 f_1(y)}{dy^4} x + \frac{d^4 f_2(y)}{dy^4} + 2\frac{d^2 f(y)}{dy^2} = 0$$

$$\frac{1}{2}\frac{d^4 f(y)}{dy^4} = 0 \qquad f(y) = Ay^3 + By^2 + Cy + D$$

$$\frac{d^4 f_1(y)}{dy^4} = 0 \qquad f_1(y) = Ey^3 + Fy^2 + Gy \boxed{+H}$$

$$\frac{d^4 f_2(y)}{dy^4} = -2\frac{d^2 f(y)}{dy^2} = -12Ay - 4B \qquad f_2(y) = -\frac{A}{10}y^5 - \frac{B}{6}y^4 + Ky^3 + Ly^2 \boxed{+My + N}$$

$$\varphi = \frac{x^2}{2}\left(Ay^3 + By^2 + Cy + D\right) + x\left(Ey^3 + Fy^2 + Gy\right) - \frac{A}{10}y^5 - \frac{B}{6}y^4 + Ky^3 + Ly^2$$

（二）应力分量：

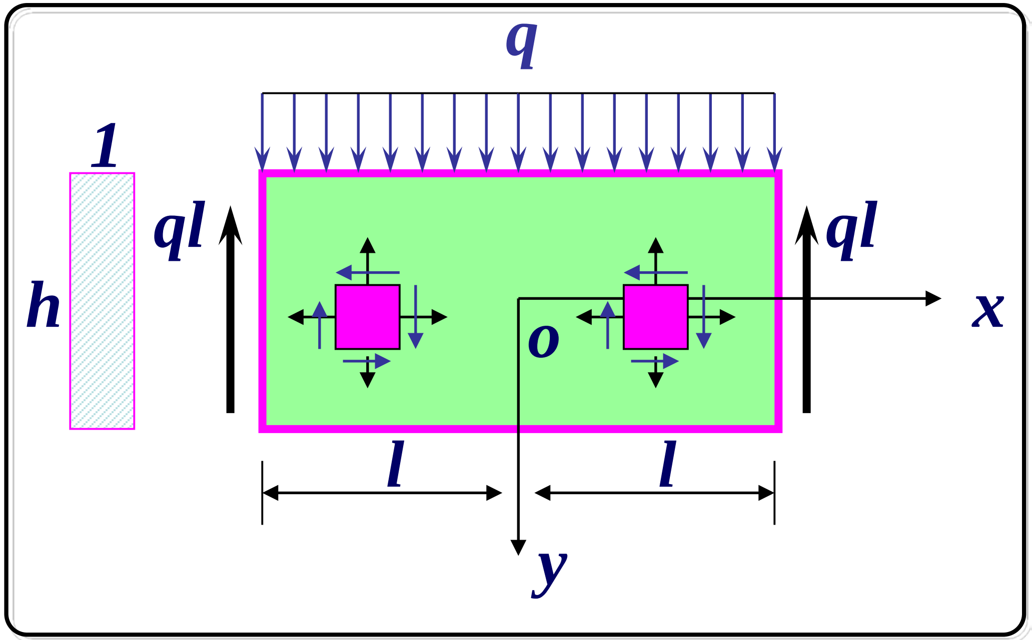
$$\begin{cases} \sigma_x = \dfrac{\partial^2 \varphi}{\partial y^2} \\ \sigma_y = \dfrac{\partial^2 \varphi}{\partial x^2} \\ \tau_{xy} = -\dfrac{\partial^2 \varphi}{\partial x \partial y} \end{cases}$$

$$\varphi = \frac{x^2}{2}\left(Ay^3 + By^2 + Cy + D\right) + x\left(Ey^3 + Fy^2 + Gy\right) - \frac{A}{10}y^5 - \frac{B}{6}y^4 + Ky^3 + Ly^2$$

$$\begin{cases} \sigma_x = \dfrac{x^2}{2}(6Ay + 2B) + x(6Ey + 2F) - 2Ay^3 - 2By^2 + 6Ky + 2L \\ \sigma_y = Ay^3 + By^2 + Cy + D \\ \tau_{xy} = -x\left(3Ay^2 + 2By + C\right) - \left(3Ey^2 + 2Fy + G\right) \end{cases}$$

（三）确定待定系数：

$$\begin{cases} \sigma_x = \dfrac{x^2}{2}(6Ay + 2B) \quad - 2Ay^3 - 2By^2 + 6Ky + 2L \\ \sigma_y = Ay^3 + By^2 + Cy + D \\ \tau_{xy} = -x\left(3Ay^2 + 2By + C\right) \end{cases}$$



对称性：

$$\sigma_x(-x, y) = \sigma_x(x, y)$$

$$6Ey + 2F = 0 \qquad \boxed{E = F = 0}$$

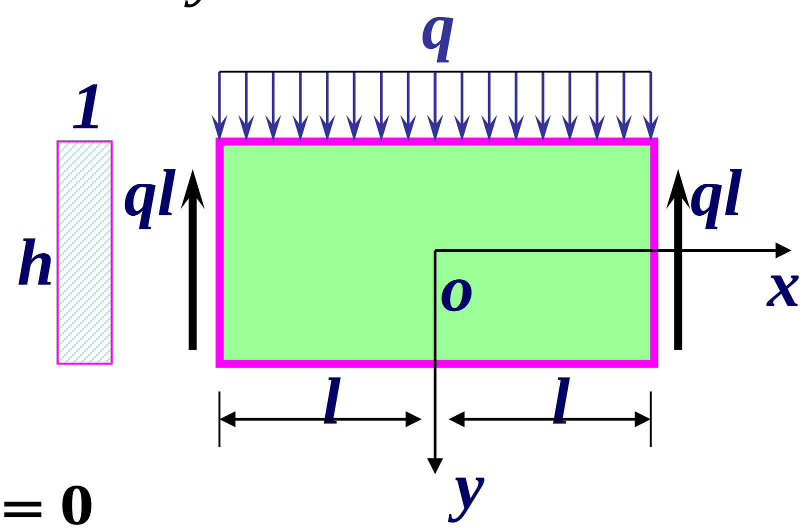
$$\sigma_y(-x, y) = \sigma_y(x, y)$$

$$\tau_{xy}(-x, y) = -\tau_{xy}(x, y)$$

$$3Ey^2 + 2Fy + G = 0 \qquad \boxed{G = 0}$$

边界条件：A、B、C、D、K、L

$$\begin{cases} \sigma_x = \dfrac{x^2}{2}(6Ay+2B) - 2Ay^3 - 2By^2 + 6Ky + 2L \\ \sigma_y = Ay^3 + By^2 + Cy + D \\ \tau_{xy} = -x(3Ay^2 + 2By + C) \end{cases}$$



$$y = \frac{h}{2}: \ l = 0, m = 1$$

$$\tau_{xy}\big|_{y=h/2} = 0 \qquad \frac{3}{4}Ah^2 + Bh + C = 0$$

$$\sigma_y\big|_{y=h/2} = 0 \qquad \frac{1}{8}Ah^3 + \frac{1}{4}Bh^2 + \frac{1}{2}Ch + D = 0$$

$$y = -\frac{h}{2}: \quad l = 0, m = -1$$

$$\tau_{xy} = 0 \qquad \frac{3}{4}Ah^2 - Bh + C = 0$$

$$\sigma_y = -q \qquad -\frac{1}{8}Ah^3 + \frac{1}{4}Bh^2 - \frac{1}{2}Ch + D = -q$$

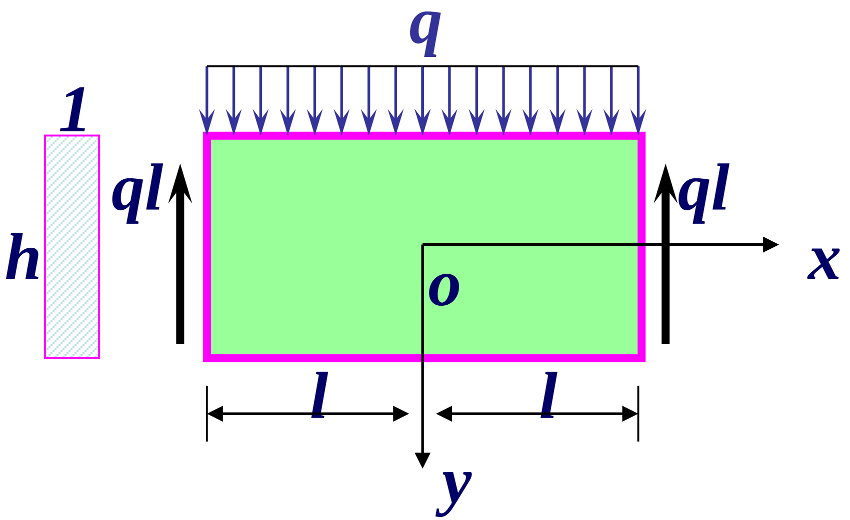
$$A = -\frac{2q}{h^3}$$

$$B = 0$$

$$C = \frac{3q}{2h}$$

$$D = -\frac{q}{2}$$

$$\begin{cases} \sigma_x = -\dfrac{6qx^2y}{h^3} + \dfrac{4qy^3}{h^3} + \dfrac{ql^2}{h^3} - \dfrac{q}{10h} \\ \sigma_y = -\dfrac{2qy^3}{h^3} + \dfrac{3qy}{2h} - \dfrac{q}{2} \\ \tau_{xy} = \dfrac{6qxy^2}{h^3} - \dfrac{3qx}{2h} \end{cases}$$



$x = l:$

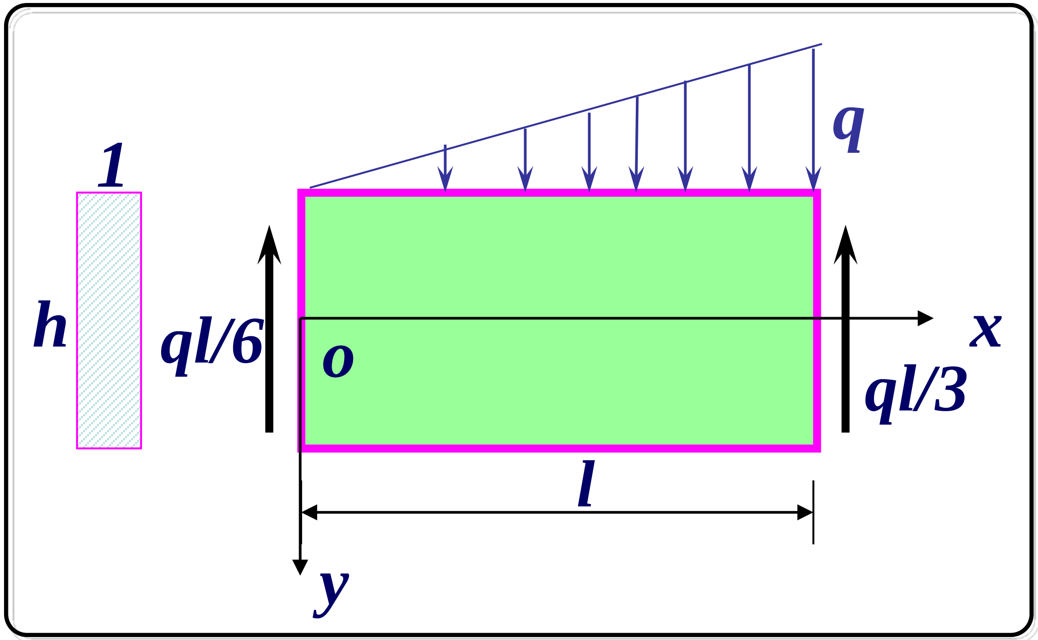
$$\sigma_x\big|_{x=l} = F_x \qquad \tau_{xy}\big|_{x=l} = F_y$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\big|_{x=l} dy = 0 \qquad \boxed{L = 0}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\big|_{x=l} dy = -ql$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\big|_{x=l}\, y dy = 0 \qquad \boxed{K = \frac{ql^2}{h^3} - \frac{q}{10h}}$$

例：单位厚度的矩形截面梁，受到 线性分布力作用，试求应力分量。(不计体力)



解：（一）确定应力函数：

$$\sigma_y = xf(y) \qquad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = xf(y)$$

$$\varphi = \frac{x^3}{6} f(y) + xf_1(y) + f_2(y)$$

$$\nabla^4 \varphi = 0$$

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0$$

$$\frac{\partial^4 \varphi}{\partial x^4} = 0 \qquad \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} = x\frac{df^2(y)}{dy^2}$$

$$\frac{\partial^4 \varphi}{\partial y^4} = \frac{x^3}{6}\frac{d^4 f(y)}{dy^4} + x\frac{d^4 f_1(y)}{dy^4} + \frac{d^4 f_2(y)}{dy^4}$$

受力分析：面力在 y 方向有变化

$$F_y\Big|_{y=-\frac{h}{2}} = -\sigma_y\Big|_{y=-\frac{h}{2}} = \frac{qx}{l}$$

$$F_y\Big|_{y=\frac{h}{2}} = \sigma_y\Big|_{y=\frac{h}{2}} = 0$$

$$\frac{1}{6}\frac{d^4 f(y)}{dy^4}x^3+\left(\frac{d^4 f_1(y)}{dy^4}+2\frac{d^2 f(y)}{dy^2}\right)x+\frac{d^4 f_2(y)}{dy^4}=0$$

$$\frac{1}{6}\frac{d^4 f(y)}{dy^4}=0 \qquad f(y)=Ay^3+By^2+Cy+D$$

$$\frac{d^4 f_1(y)}{dy^4}=-12Ay-4B \qquad f_1(y)=-\frac{A}{10}y^5-\frac{B}{6}y^4+Ey^3+Ey^2+Gy$$

$$\frac{d^4 f_2(y)}{dy^4}=0 \qquad f_2(y)=Hy^3+Ky^2$$

$$\varphi=\frac{x^3}{6}\left(Ay^3+By^2+Cy+D\right)+x\left(-\frac{A}{10}y^5-\frac{B}{6}y^4+Ey^3+Fy^2+Gy\right)$$
$$+Hy^3+Ky^2$$

（二）应力分量：

$$\begin{cases} \sigma_x = \dfrac{x^3}{6}(6Ay+2B) + x\left(-2Ay^3 - 2By^2 + 6Ey + 2F\right) + 6Hy + 2K \\ \sigma_y = x(Ay^3 + By^2 + Cy + D) \\ \tau_{xy} = -\dfrac{x^2}{2}\left(3Ay^2 + 2By + C\right) + \dfrac{A}{2}y^4 + \dfrac{2B}{3}y^3 - \left(3Ey^2 + 2Fy + G\right) \end{cases}$$

（三）边界条件：

$y = \dfrac{h}{2}$：

$$\tau_{xy}\big|_{y=h/2} = 0$$

$$\sigma_y\big|_{y=h/2} = 0$$

$y = -\dfrac{h}{2}$：

$$\tau_{xy} = 0$$

$$\sigma_y = -q\frac{x}{l}$$

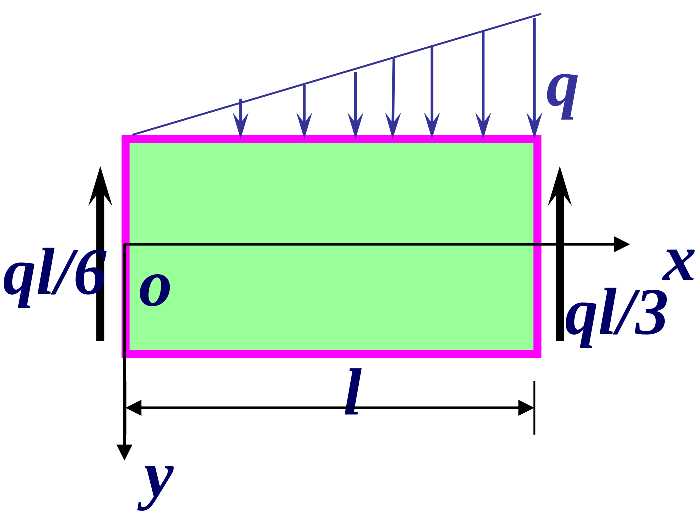
$$A = -\frac{2q}{lh^3}$$

$$B = 0$$

$$C = \frac{3q}{2lh}$$

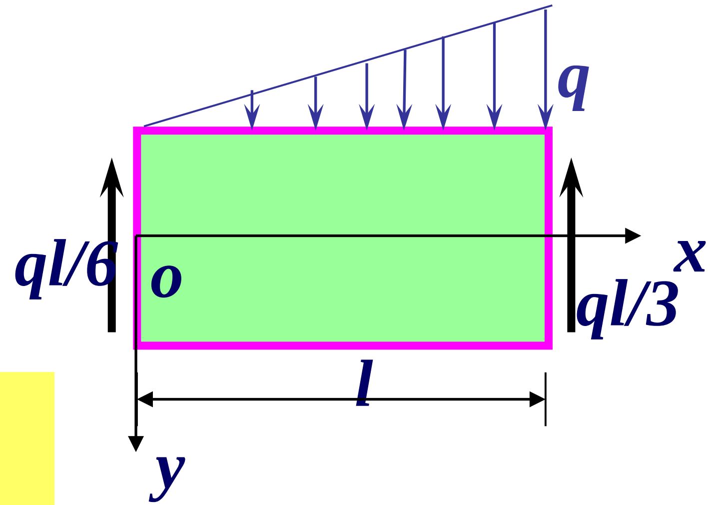
$$D = -\frac{q}{2l}$$

$$F = 0$$



$$x = l: \quad \sigma_x\big|_{x=l} = F_x$$

$$\tau_{xy}\big|_{x=l} = F_y$$



$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\big|_{x=l} dy = 0$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\big|_{x=l} dy = -\frac{ql}{3}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\big|_{x=l}\, y dy = 0$$

$$K = 0$$

$$H = 0$$

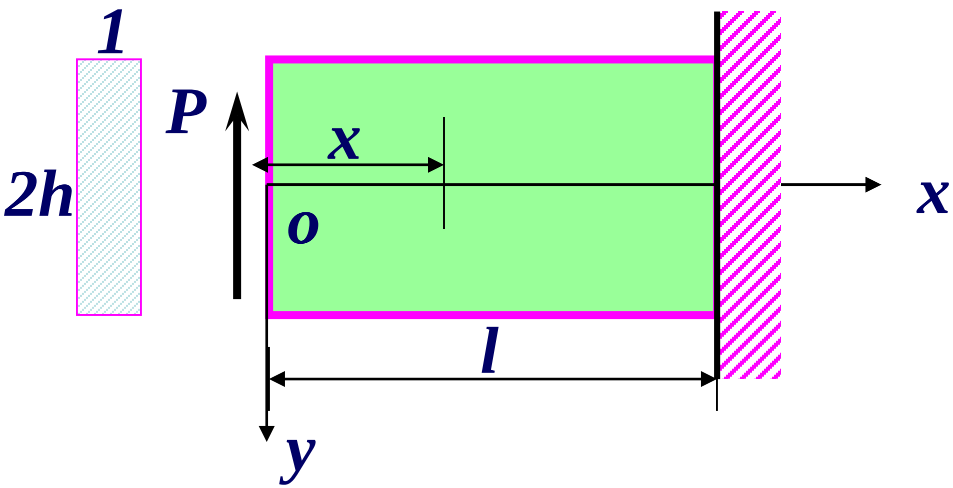
$$E = \frac{ql}{3h^3} - \frac{q}{10lh}$$

$$G = \frac{qh}{80l} - \frac{ql}{4h}$$

应力分量:

$$\begin{cases} \sigma_x = \dfrac{2qxy}{h^3 l}\left(2y^2 - x^2 - l - \dfrac{3h^3}{10}\right) \\ \sigma_y = \dfrac{qx}{2lh^3}\left(3yh^2 - 4y^3 - h^3\right) \\ \tau_{xy} = \dfrac{6q(h^2 - 4y^2)}{4lh^3}\left(-3x^2 - y^2 + l^2 - \dfrac{h^2}{20}\right) \end{cases}$$

例：单位厚度的悬臂矩形截面梁，受集中力作用，试求应力分量和位移分量。（不计体力）



解：（一）确定应力函数：

$$\sigma_x = Axy \qquad \sigma_x = \frac{\partial^2 \varphi}{\partial y^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = Axy$$

$$\varphi = \frac{Axy^3}{6} + yf_1(x) + f_2(x)$$

$$\nabla^4 \varphi = 0$$

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0$$

$$\frac{\partial^4 \varphi}{\partial y^4} = 0 \qquad \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} = 0$$

$$\frac{\partial^4 \varphi}{\partial x^4} = \frac{d^4 f_1(x)}{dx^4} y + \frac{d^4 f_2(x)}{dx^4}$$

材料力学：

$$\sigma_x = \frac{My}{I}$$

$$M = Px$$

$$\sigma_x = \frac{Pxy}{I}$$

$$\varphi = \frac{Axy^3}{6} + yf_1(x) + f_2(x)$$

$$\frac{d^4 f_1(x)}{dx^4} y + \frac{d^4 f_2(x)}{dx^4} = 0$$

$$\frac{d^4 f_1(x)}{dx^4} = 0 \qquad f_1(x) = Bx^3 + Cx^2 + Dx$$

$$\frac{d^4 f_2(x)}{dx^4} = 0 \qquad f_2(x) = Ex^3 + Fx^2$$

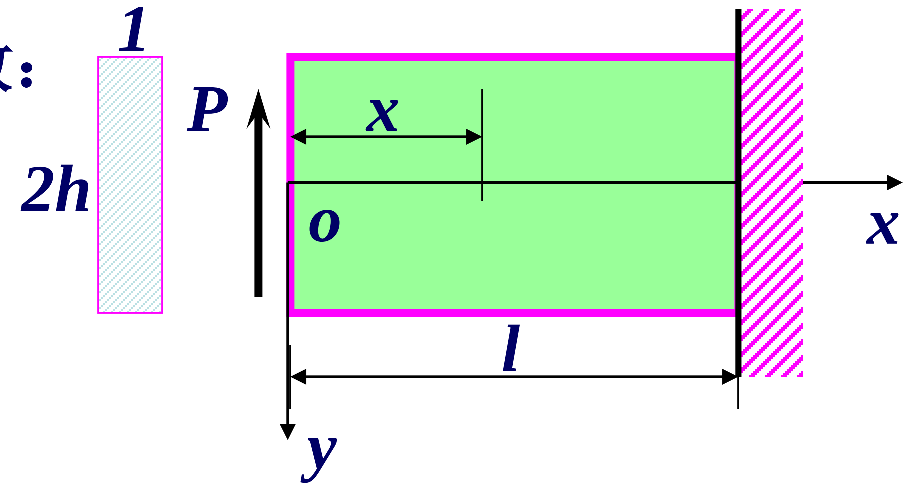
$$\varphi = \frac{A}{6} xy^3 + (Bx^3 + Cx^2 + Dx)y + (Ex^3 + Fx^2)$$

（二）应力分量:

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \varphi}{\partial y^2} \\ \sigma_y = \dfrac{\partial^2 \varphi}{\partial x^2} \\ \tau_{xy} = -\dfrac{\partial^2 \varphi}{\partial x \partial y} \end{cases} \Rightarrow \begin{cases} \sigma_x = Axy \\ \sigma_y = 6Bx + 2Cy + 6Ex + 2F \\ \tau_{xy} = -\left(\dfrac{A}{2} y^2 + 3Bx^2 + 2Cx + D \right) \end{cases}$$

（三）利用边界条件确定待定系数：

$$\begin{cases} \sigma_x = Axy \\ \sigma_y = 6Bxy + 2Cy + 6Ex + 2F \\ \tau_{xy} = -\left(\dfrac{A}{2}y^2 + 3Bx^2 + 2Cx + D\right) \end{cases}$$



$$y = -h : \tau_{xy} = 0; -\left(\frac{A}{2}h^2 + 3Bx^2 + 2Cx + D\right) = 0$$

$$y = h : \tau_{xy} = 0; \ -\left(\frac{A}{2}h^2 + 3Bx^2 + 2Cx + D\right) = 0$$

$$y = -h : \sigma_y = 0; \ \ -6Bxh - 2Ch + 6Ex + 2F = 0$$

$$y = h : \sigma_y = 0; \quad 6Bxh + 2Ch + 6Ex + 2F = 0$$

$$B = 0$$

$$C = 0$$

$$D + \frac{A}{2}h^2 = 0$$

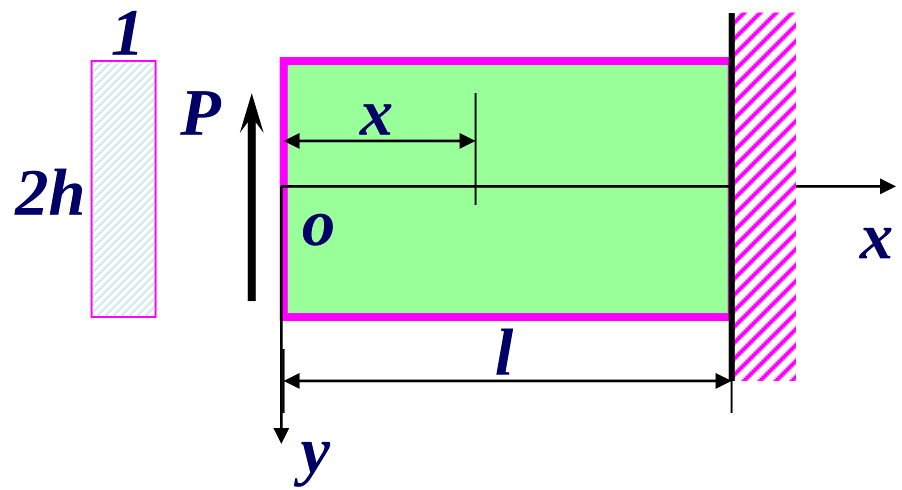
$$E = 0$$

$$F = 0$$

$$\sigma_x = Axy$$

$$\sigma_y = 6Bxy + 2Cy + 6Ex + 2F$$

$$\tau_{xy} = -\left(\frac{A}{2}y^2 + 3Bx^2 + 2Cx + D\right)$$



$$x = 0, l = -1, m = 0$$

$$F_x = -\sigma_x = 0$$

$$F_y = -\tau_{xy} = \frac{A}{2}y^2 + D$$

$$\int_{-h}^{h} F_y dy = -P \qquad \int_{-h}^{h} \tau_{xy} dy = P$$

$$\frac{A}{3}h^3 + 2Dh = -P$$

$$\sigma_x = \frac{3P}{2h^3}xy = \frac{pxy}{I}$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\left(\frac{3P}{4h^3}y^2 - \frac{3P}{4h}\right) = -\frac{P}{2I}\left(y^2 - h^2\right)$$

$$A = \frac{3P}{2h^3}$$

$$D = -\frac{3P}{4h}$$

（四）应变分量：

$$\varepsilon_x = \frac{pxy}{EI} \qquad \varepsilon_y = -\frac{\mu pxy}{EI} \qquad \gamma_{xy} = -\frac{(1+\mu)P}{EI}\left(y^2 - h^2\right)$$

（五）位移分量：

$$\frac{\partial u}{\partial x} = \frac{pxy}{EI} \qquad u = \frac{px^2 y}{2EI} + f_1(y)$$

$$\frac{\partial v}{\partial y} = -\frac{\mu pxy}{EI} \qquad v = -\frac{\mu pxy^2}{2EI} + f_2(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{(1+\mu)P}{EI}\left(y^2 - h^2\right)$$

$$\frac{Px^2}{2EI} + \frac{df_1(y)}{dy} - \frac{\mu Py^2}{2EI} + \frac{df_2(x)}{dx} = -\frac{(1+\mu)P}{EI}\left(y^2 - h^2\right)$$

$$\frac{Px^2}{2EI}+\frac{df_2(x)}{dx}=-\frac{df_1(y)}{dy}+\frac{\mu Py^2}{2EI}-\frac{(1+\mu)P}{EI}\left(y^2-h^2\right)=\omega$$

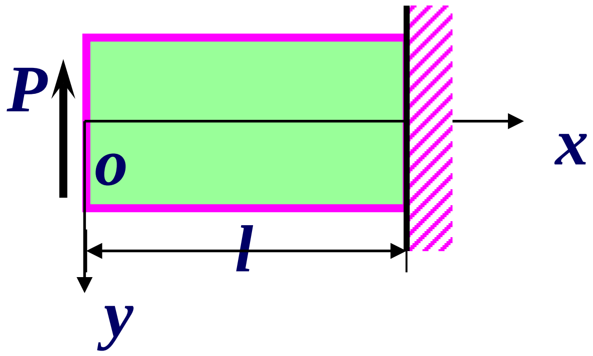
$$\frac{Px^2}{2EI}+\frac{df_2(x)}{dx}=\omega \qquad f_2(x)=-\frac{Px^3}{6EI}+\omega x+v_0$$

$$-\frac{df_1(y)}{dy}+\frac{\mu Py^2}{2EI}-\frac{(1+\mu)P}{EI}\left(y^2-h^2\right)=\omega$$

$$f_1(y)=-\frac{(2+\mu)Py^3}{6EI}+\frac{(1+\mu)Ph^2y}{EI}-\omega y+u_0$$

$$u=\frac{px^2y}{2EI}-\frac{(2+\mu)Py^3}{6EI}+\frac{(1+\mu)Ph^2y}{EI}-\omega y+u_0$$

$$v=-\frac{\mu pxy^2}{2EI}-\frac{Px^3}{6EI}+\omega x+v_0$$



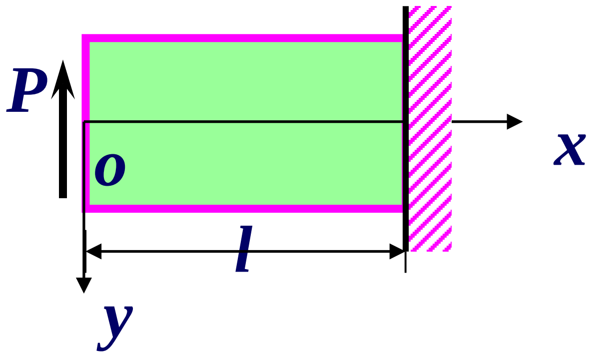
$$u=\frac{px^2y}{2EI}-\frac{(2+\mu)Py^3}{6EI}+\frac{(1+\mu)Ph^2y}{EI}-\frac{Pl^2y}{2EI}$$

$$v=-\frac{\mu pxy^2}{2EI}-\frac{Px^3}{6EI}+\frac{Pl^2x}{2EI}-\frac{Pl^3}{3EI}$$

$$u\Big|_{\substack{x=l\\y=0}}=0 \qquad u_0=0$$

$$v\Big|_{\substack{x=l\\y=0}}=0 \qquad v_0=\frac{Pl^3}{6EI}-\omega l$$

$$\frac{\partial v}{\partial x}\Big|_{\substack{x=l\\y=0}}=0 \qquad \omega=\frac{Pl^2}{2EI}$$



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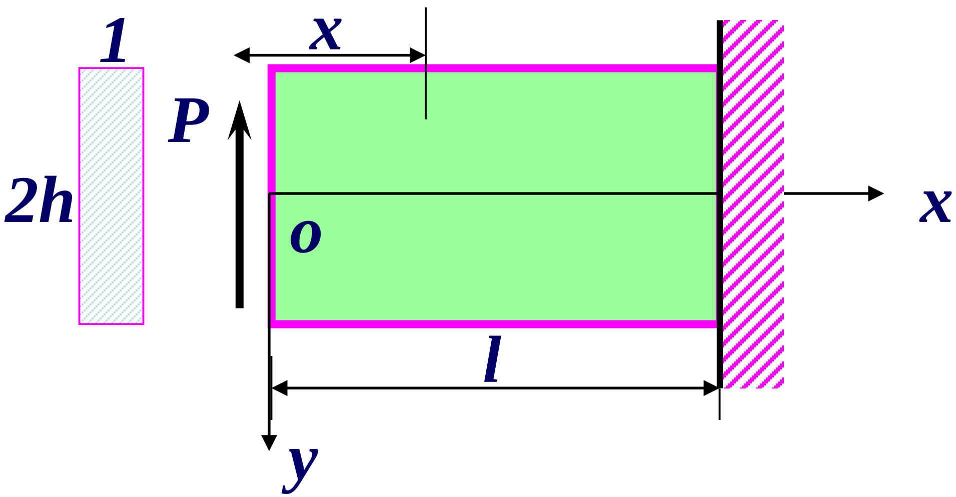

解法 （二）

$$\sigma_y = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 0$$

$$\varphi = x f_1(y) + f_2(y)$$

$$\nabla^4 \varphi = 0$$

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0$$

$$\frac{d^4 f_1(y)}{dy^4} x + \frac{d^4 f_2(y)}{dy^4} = 0$$

受力分析：

$$\sigma_y \big|_{y=\pm h} = 0$$

$$f_1(y) = By^3 + Cy^2 + Dy$$

$$f_2(y) = Ey^3 + Fy^2$$

$$\varphi = (By^3 + Cy^2 + Dy)x + (Ey^3 + Fy^2)$$

$$\sigma_x = 6Bxy + 2Cx + 6Ey + 2F$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\left(3By^2 + 2Cy + D\right)$$

$$B = \frac{P}{4h^3}$$

$$C = 0$$

$$D = -\frac{3P}{4h}$$

$$E = 0$$

$$F = 0$$

$$\sigma_x = \frac{3P}{2h^3}xy = \frac{pxy}{I}$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\left(\frac{3P}{4h^3}y^2 - \frac{3P}{4h}\right) = -\frac{P}{2I}\left(y^2 - h^2\right)$$

应力函数不同，但应力分量的表达式相同。

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谢 谢 各 位！

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