第三讲

短时Fourier变换

Short Time Fourier Transform (STFT)

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Fourier Transform

- 发展简史

Jean Baptiste Joseph Fourier (1768 - 1830)

- 1807 - “All periodic functions can be expressed as a weighted sum of trigonometric function”, Denied publication **by Lagrange, Legendre and Laplace**
- 1822 - The Analytic Theory of Heat published, J.C Maxwell - **Fourier Theory: a great mathematical poem**

…

… Peter Gustav Lejeune Dirichlet, Bernhard Riemann

…

- **1965,** Cooley & Tukey: Fast Fourier Transform – **King of transforms**

Fourier Transform

简介

Fourier级数

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 t}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-jn\omega_0 t}dt$$

$$\omega_0=2\pi/T$$

Fourier变换

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega$$

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

本质思想：线性空间的正交基分解和重构

Fourier Transform

主要性质

线性可加 $$a_1 f_1(t)+a_2 f_2(t)\xleftrightarrow{\mathcal{F}} a_1F_1(j\omega)+a_2F_2(j\omega)$$

时移性质 $$f(t-t_0)\xleftrightarrow{\mathcal{F}} F(j\omega)e^{-j\omega t_0}$$

频移性质 $$f(t)e^{j\omega_0}\xleftrightarrow{\mathcal{F}} F\left[j(\omega-\omega_0)\right]$$

微分性质 $$f(t)\xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \lim_{t\to\pm\infty} f(t)=0$$

$$f'(t)\xleftrightarrow{\mathcal{F}} j\omega F(j\omega)$$

Fourier Transform

主要性质

微分性质

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \lim_{t\to\pm\infty} f(t) = 0$$

$$f^{(n)}(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n F(j\omega)$$

积分性质

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \int_{-\infty}^{\infty} f(t)dt = F(0) = 0$$

$$\mathcal{F}\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{j\omega} F(j\omega)$$

利用Fourier变换，可将微积分方程求解问题变成简单代数运算问题

Fourier Transform

- 示例

$$m\ddot{x}(t)+kx(t)=g(t)$$

$$m\left(\sum_{n=-\infty}^{\infty}\left(-n^2\omega_0^2 c_n e^{jn\omega_0 t}\right)\right)+k\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 t}=\sum_{n=-\infty}^{\infty}d_n e^{jn\omega_0 t}$$

$$\left\langle m\left(\sum_{n=-\infty}^{\infty}\left(-n^2\omega_0^2 c_n e^{jn\omega_0 t}\right)+k\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 t}\right), e^{jr\omega_0 t}\right\rangle=\left\langle\sum_{n=-\infty}^{\infty}d_n e^{jn\omega_0 t}, e^{jr\omega_0 t}\right\rangle$$

$$x(t)=\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 t}$$

$$\ddot{x}(t)=\sum_{n=-\infty}^{\infty}\left(-n^2\omega_0^2 c_n e^{jn\omega_0 t}\right)$$

$$g(t)=\sum_{n=-\infty}^{\infty}d_n e^{jn\omega_0 t}$$

$$-r^2\omega_0^2 c_r m+kc_r=d_r$$

$$c_r=d_r\Big/\left(k-r^2\omega_0^2 m\right)$$

谐波平衡法

Fourier Transform

主要性质

卷积性质

$$f_1(t) * f_2(t) \xleftrightarrow{\mathcal{F}} F_1(j\omega)F_2(j\omega)$$

乘积性质

$$f_1(t) f_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} F_1(j\omega) * F_2(j\omega)$$

卷积定义

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= f_1(t) * f_2(t)$$

Fourier Transform

Matlab 实现

fft(X) is the discrete Fourier transform (DFT) of vector X.
For matrices, the FFT operation is applied to each column.
典型用法： Y = fft(X), % size(Y) = size(X)
Y = fft(X,N) % the N-point FFT

X - 1×xLen ; % 信号的长度为xLen
Fs - Sampling Frequency
Y - 1×xLen ; % 频谱

频率分辨率

$$\Delta F = \frac{Fs}{xLen} = \frac{Fs}{s\Delta t} = \frac{1}{s}$$

s- 采样时间长度 / sec

如果某分量的频率为*f*

$$I = \frac{f}{\Delta F} = fs$$

如果I非整数，则出现能量泄漏

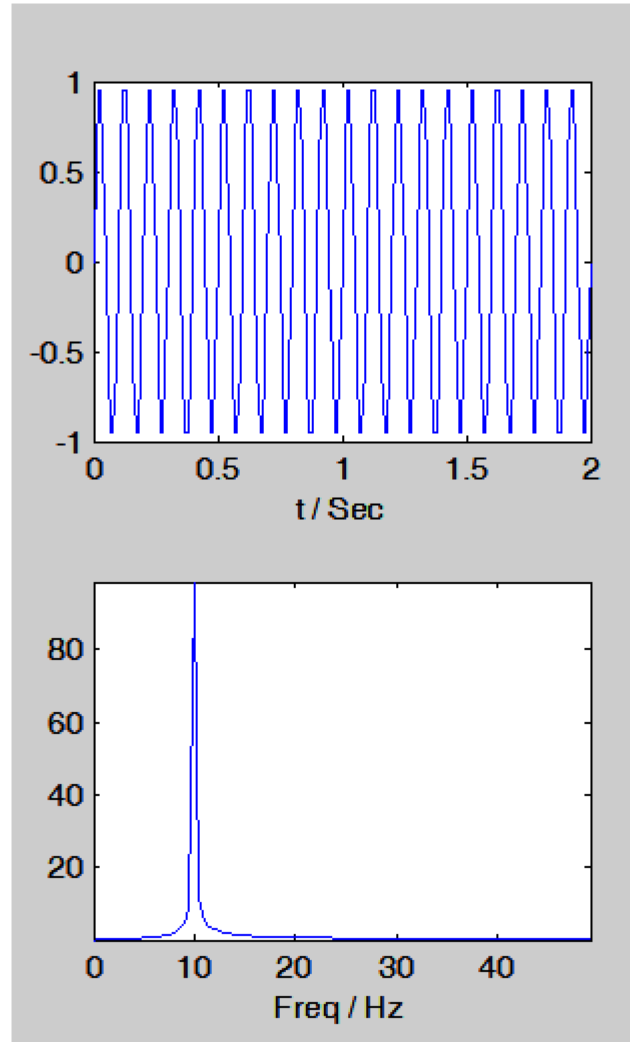
Fourier Transform

Matlab －示例

I非整数

```
function Test_FFT(),

SampFreq = 100;
td = 2;
t = 0:1/SampFreq:td;
%t = t(1:end-1);
Sig = sin(2*pi*10*t);
SigLen = length(Sig);
Spec = fft(Sig, SigLen);
% the lenght of sig is td*SampFreq + 1;
SpecLen = length(Spec);
Freq = 0:SampFreq/SpecLen:SampFreq;
Freq = Freq(1:SpecLen);
subplot(211), plot(t, Sig); xlabel('t / Sec')
subplot(212), plot(Freq(1:end/2), abs(Spec(1:end/2)));
xlabel('Freq / Hz')
axis('tight')
```



Fourier Transform

Matlab －示例

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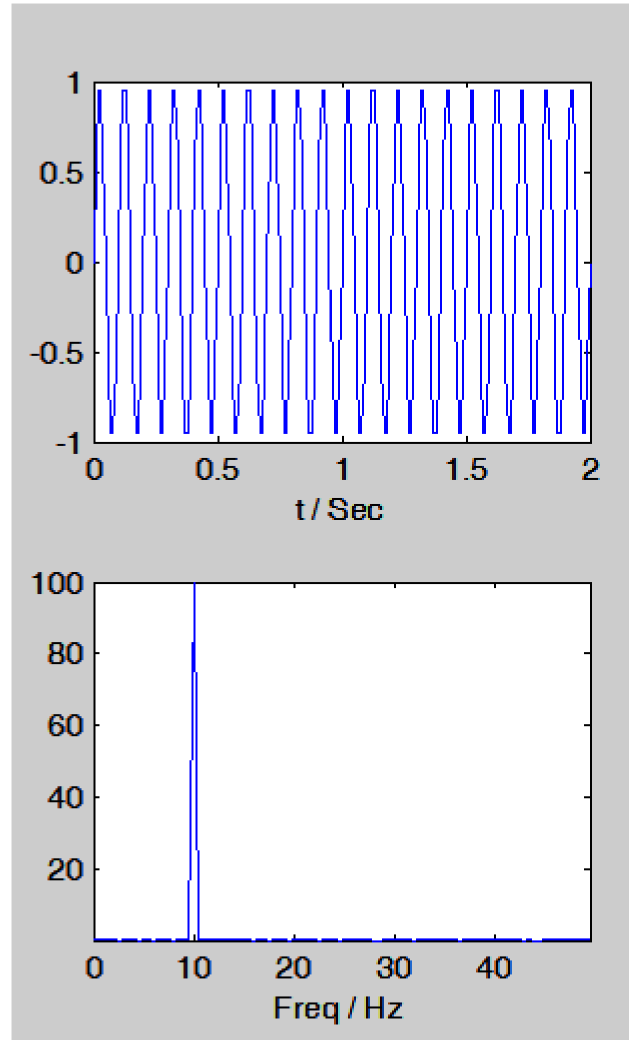

Fourier Transform

Matlab －示例

归一化

```
function Test_FFT(),

SampFreq = 100;
td = 2;
t = 0:1/SampFreq:td;
%t = t(1:end-1);
Sig = sin(2*pi*10*t);
SigLen = length(Sig);
Spec = fft(Sig, SigLen-1);
% the lenght of sig is td*SampFreq + 1;
SpecLen = length(Spec);
Spec = 2*Spec/ SpecLen;
Freq = 0:SampFreq/SpecLen:SampFreq;
Freq = Freq(1:SpecLen);
subplot(211), plot(t, Sig); xlabel('t / Sec')
subplot(212), plot(Freq(1:end/2), abs(Spec(1:end/2)));
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axis('tight')
```
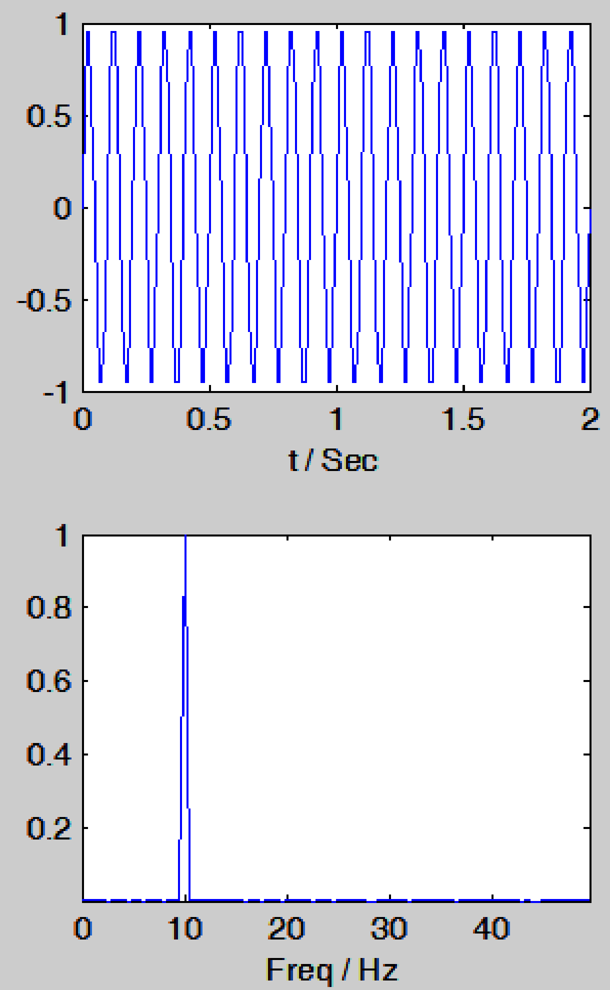



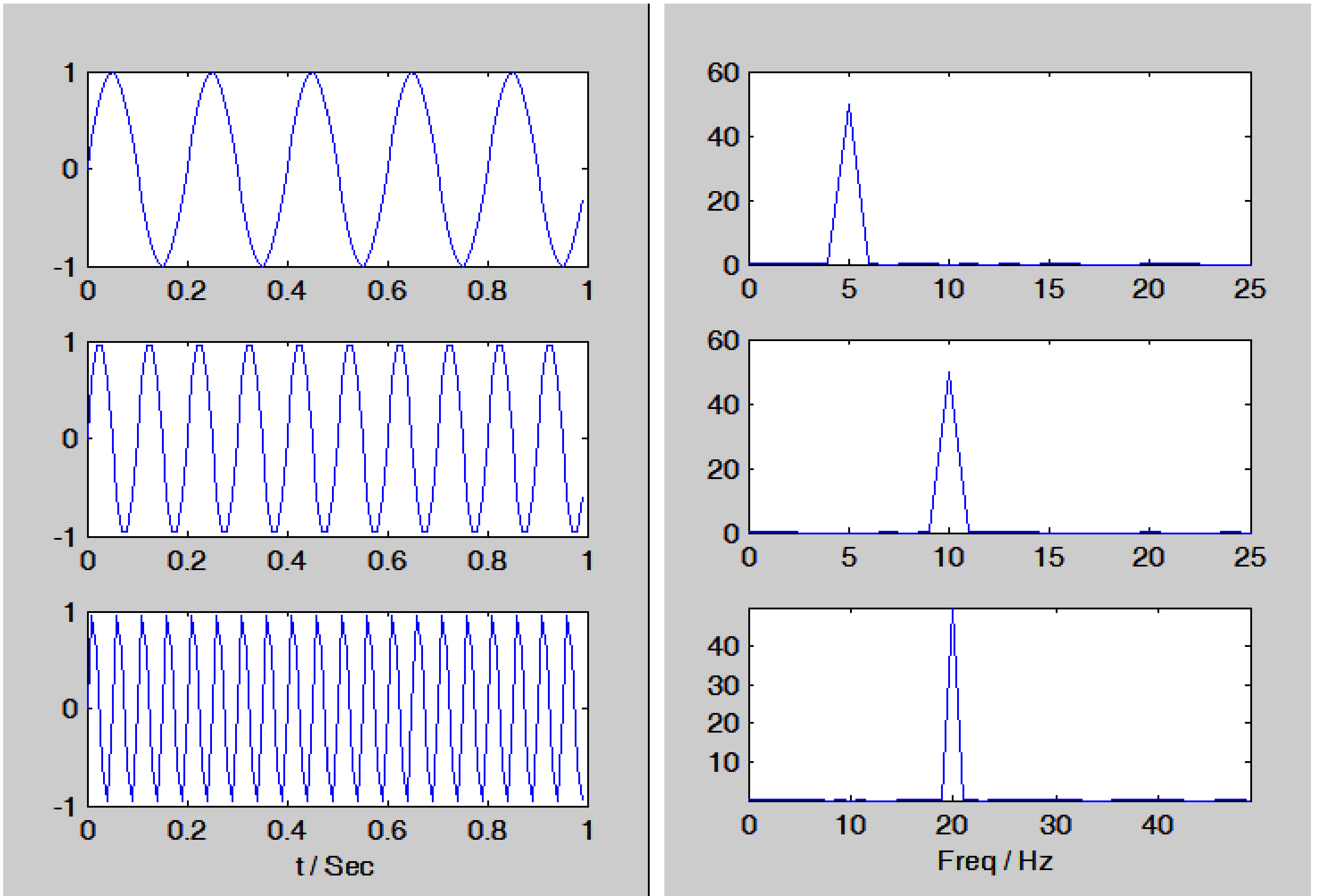
Fourier Transform

- 傅立叶变换缺点

$s_1 = \sin(10\pi t)$

$s_2 = \sin(20\pi t)$

$s_3 = \sin(40\pi t)$

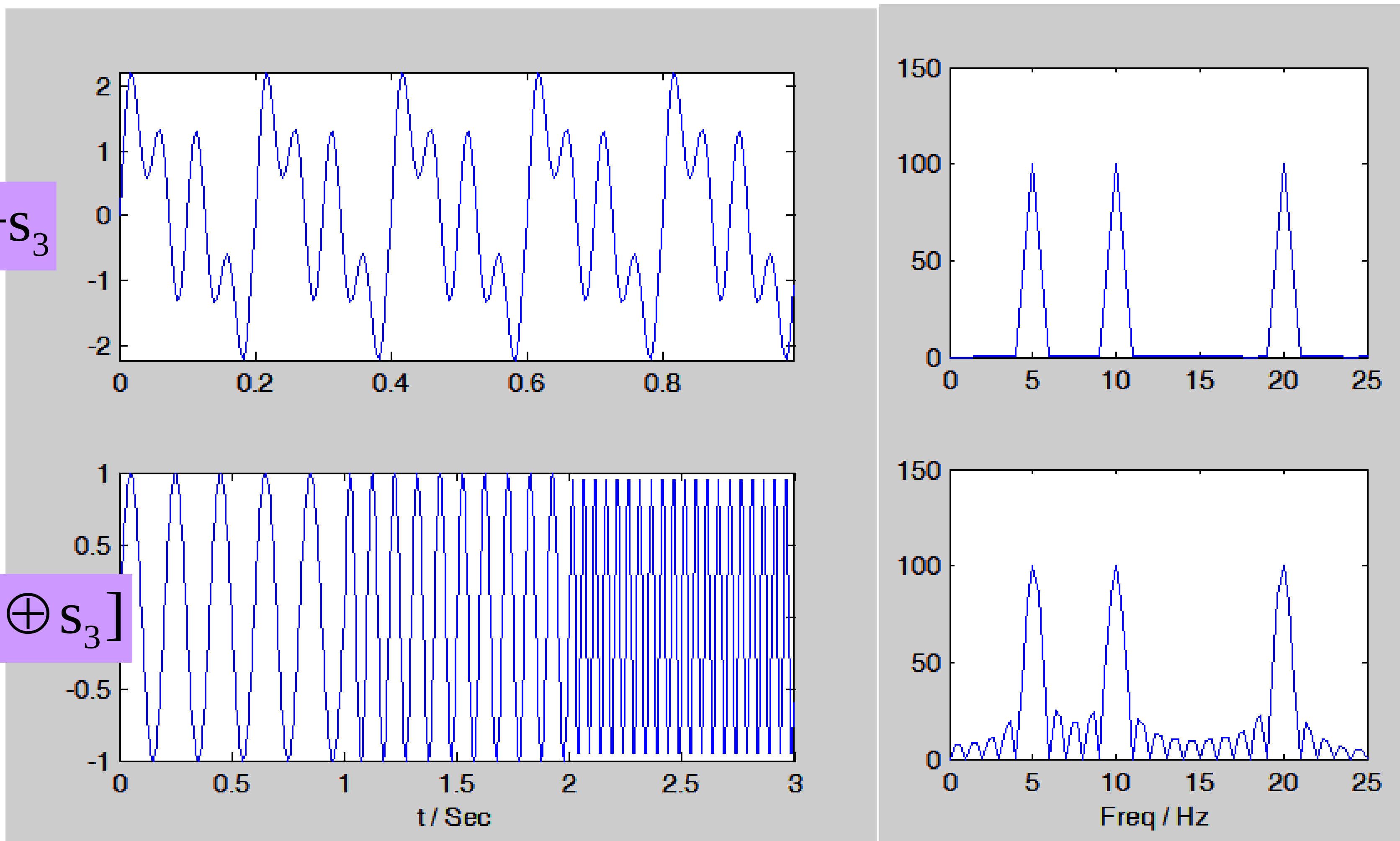


Fourier Transform

- 傅立叶变换缺点

$s_1 + s_2 + s_3$

$[s_1 \oplus s_2 \oplus s_3]$

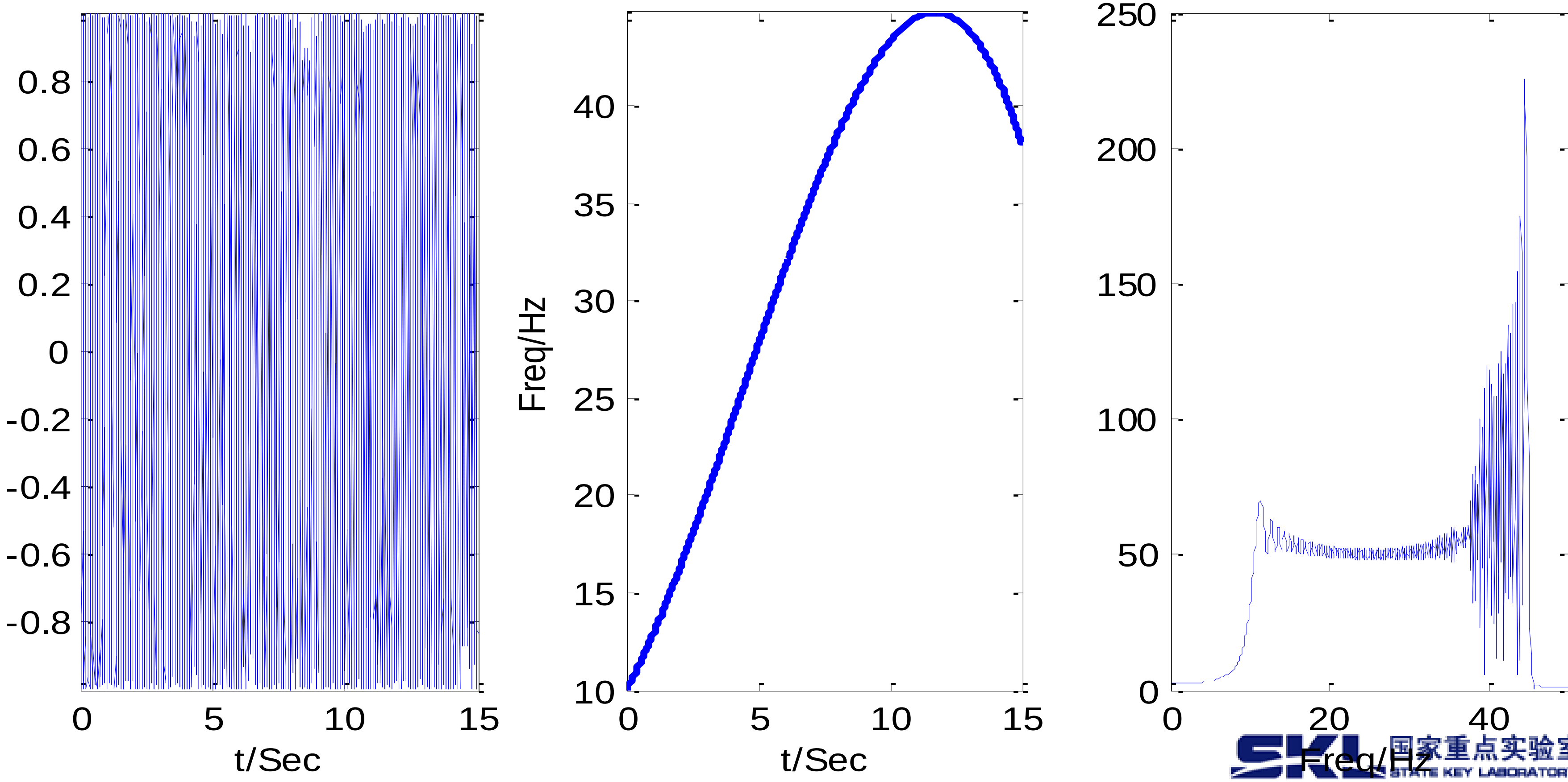


Fourier Transform

傅立叶变换缺点

$$s(t)=\sin\left(2\pi\left(10t+\frac{5}{4}t^2+\frac{1}{9}t^3-\frac{1}{120}t^4\right)\right)$$

$$IF(t)=$$
$$10+2.5t+t^2/3-t^3/30\quad(\mathrm{Hz})$$



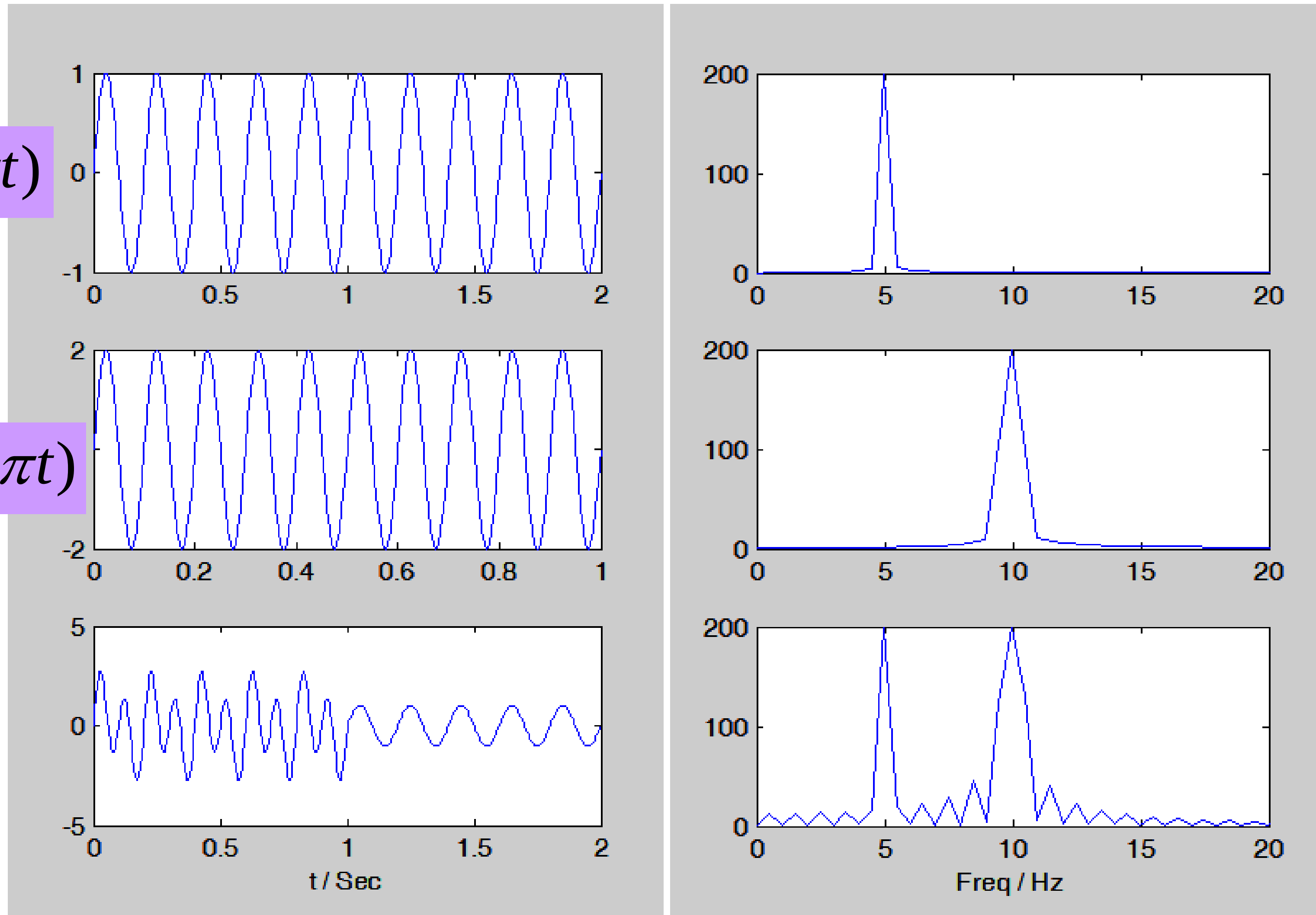
Fourier Transform

- 傅立叶变换缺点

$s_1 = \sin(10\pi t)$

$s_2 = 2\sin(20\pi t)$

s_3



Fourier Transform

傅立叶变换缺点

1) 准确反映信号所含频率分量，不能反映频率分量存在的时间段

2) 能反映调频信号的频率范围，不能反映频率随时间的变化规律

3) 能准确反映信号中某频率分量的总能量，却不能显示频率分量强度随时间的变化规律

Fourier Transform

- 傅立叶变换缺点 全局变换，丢失局部信息

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$\omega_0=2\pi/T$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt=\frac{1}{T}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} f(t)e^{-jn\omega_0 t}dt;$$

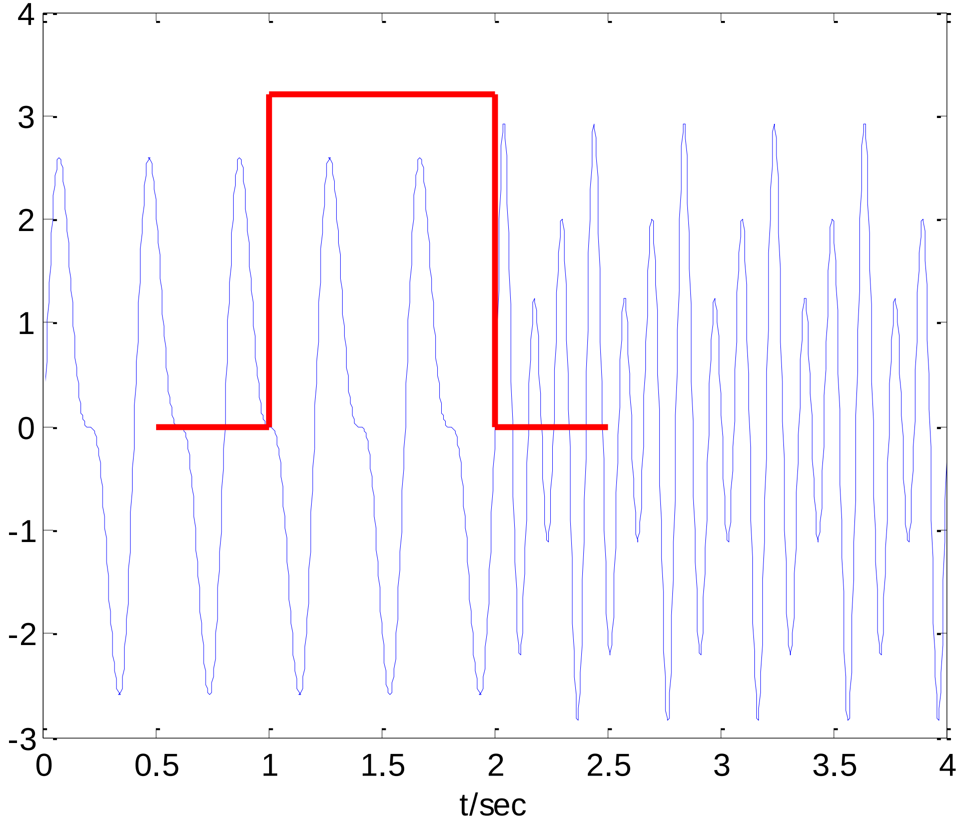
$$t_0=-T/2; t_n=T/2$$

可看出，只要信号$f(t)$任一段$[t_k, t_{k+1}]$内包含有频率为$n\omega_0$的分量，则c_n非零。

Short Time Fourier Transform

- 基本思想：加窗Fourier变换

通过加窗处理，取出信号片段进行Fourier变换，以观察该片段内信号的频率分量



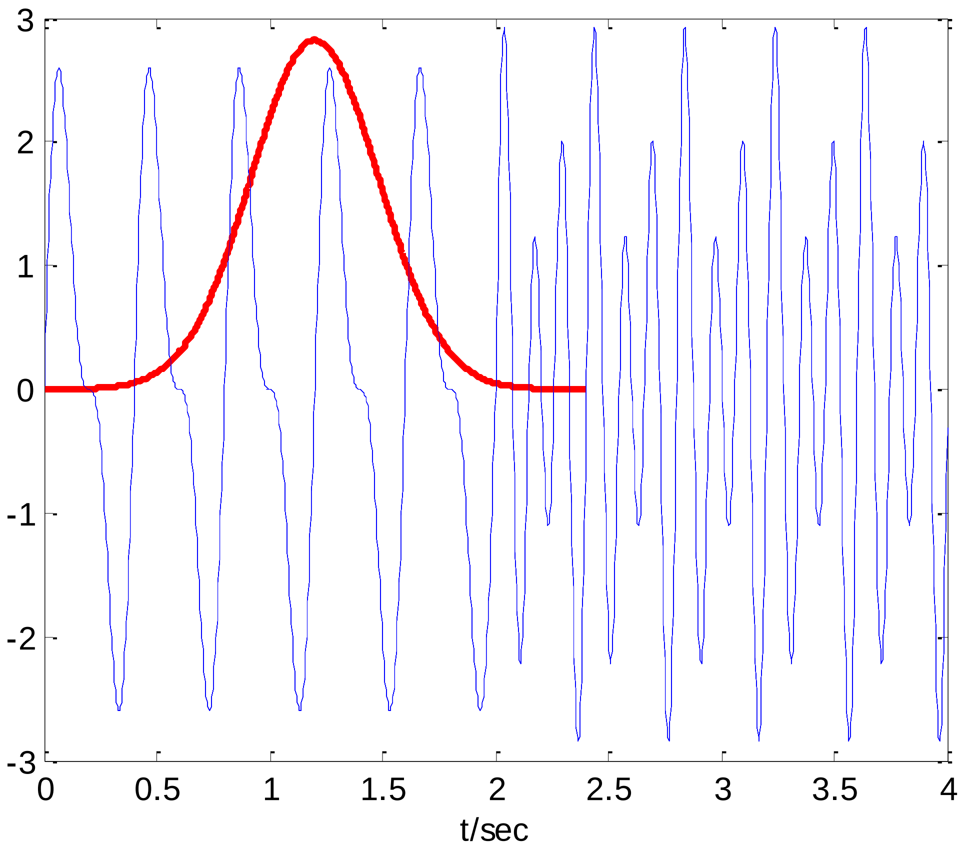
矩形窗

窗口中的每一点在计算中的贡献是等同的

Short Time Fourier Transform

- 基本思想：加窗Fourier变换

通过加窗处理，取出信号片段进行Fourier变换，以观察该片段内信号的频率分量

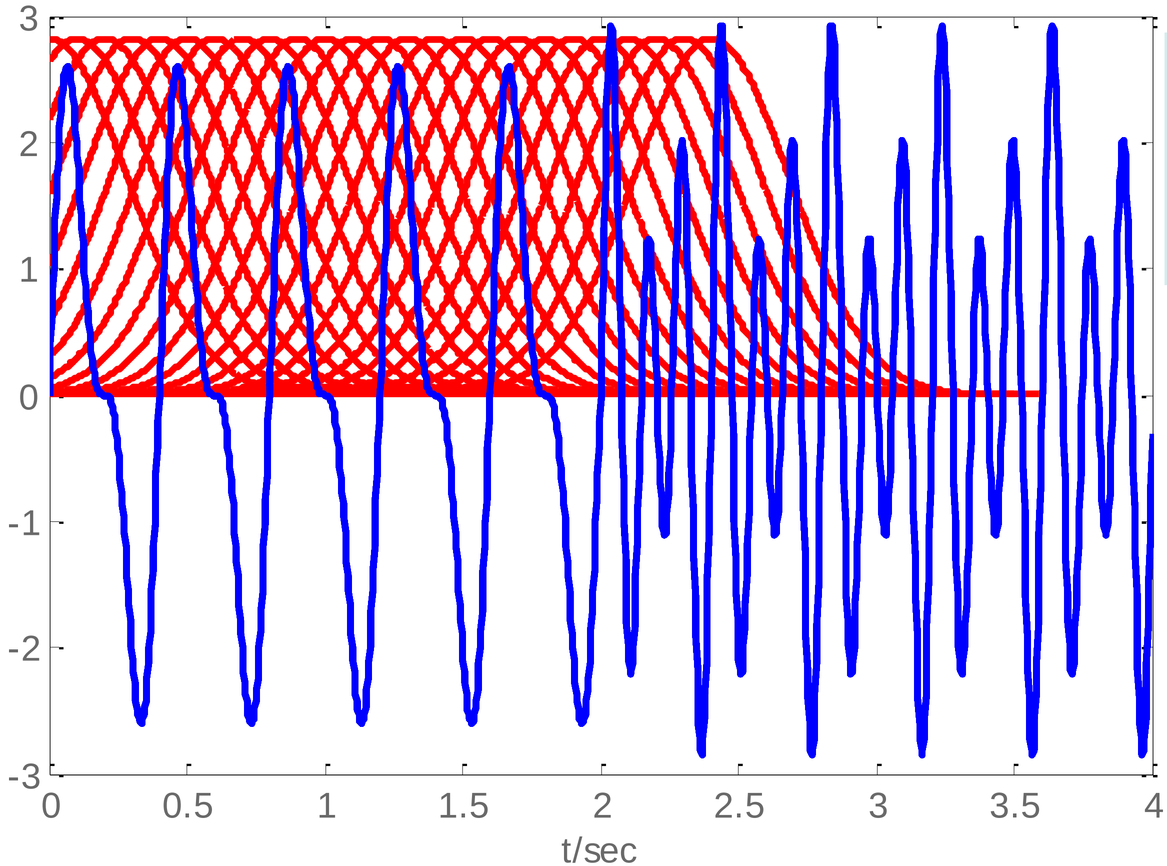


钟形窗
突出窗口中的中间点在计算中的贡献

Short Time Fourier Transform

基本思想：加窗Fourier变换

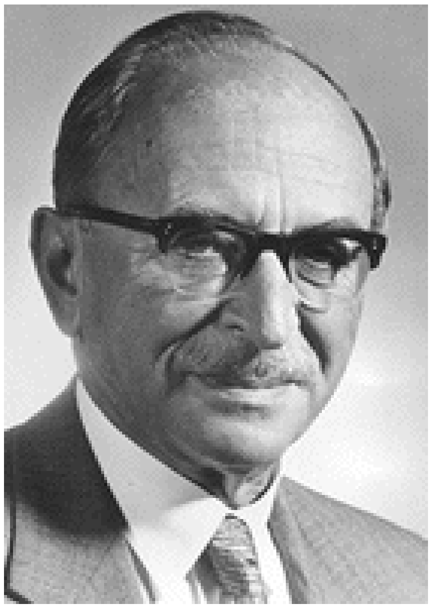
通过加窗处理，取出信号片段进行Fourier变换，以观察该片段内信号的频率分量



逐点移动钟形窗
计算各个点附近片段的Fourier变换

Short Time Fourier Transform

- STFT定义



Gabor (1946)

全息照相技术1971获诺贝尔奖

Time parameter

Frequency parameter

Signal to be analyzed

FT Kernel (basis function)

$$\mathrm{STFT}_x(\tau,\omega)=\int_t\left[x(t)W(t-\tau)\right]e^{-j\omega t}dt$$

STFT of signal x(t): Computed for each window centered at $t=\tau$

Windowing function

Windowing function centered at $t=\tau$

Short Time Fourier Transform

STFT性质

线性可加 $$a_1 f_1(t) + a_2 f_2(t) \xleftrightarrow{STFT} a_1 \mathrm{STFT}_{x1} + a_2 \mathrm{STFT}_{x2}$$

时移性质 $$f(t-t_0) \xleftrightarrow{STFT} \mathrm{STFT}_x(\tau - t_0, \omega)$$

频移性质 $$f(t)e^{j\omega_0} \xleftrightarrow{STFT} \mathrm{STFT}_x(\tau, \omega - \omega_0)$$

众多Fourier变换满足的性质

Short Time Fourier Transform

STFT性质

特别是，当W为高斯窗时 $W(t)=\frac{1}{2\sqrt{\pi a}}e^{-t^2/4a}$

$$\int \mathrm{STFT}_x(\tau,\omega)d\tau = \int\int_t [x(t)W(t-\tau)]e^{-j\omega t}dtd\tau = \hat{x}(\omega)$$

$$\int\int \left|\mathrm{STFT}_x(\tau,\omega)\right|^2 d\tau d\omega = \int \left|x(t)\right|^2 dt$$

$$x(t) = \frac{1}{2\pi}\int\int \mathrm{STFT}_x(\tau,\omega)W(t-\tau)e^{j\omega t}d\omega d\tau$$

Short Time Fourier Transform

- 离散STFT

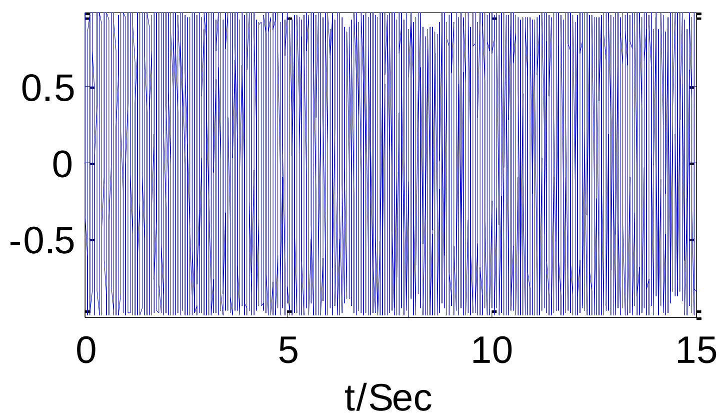
$T \rightarrow$ 采样间隔 $f = 1/s$ s- 信号长度 / sec

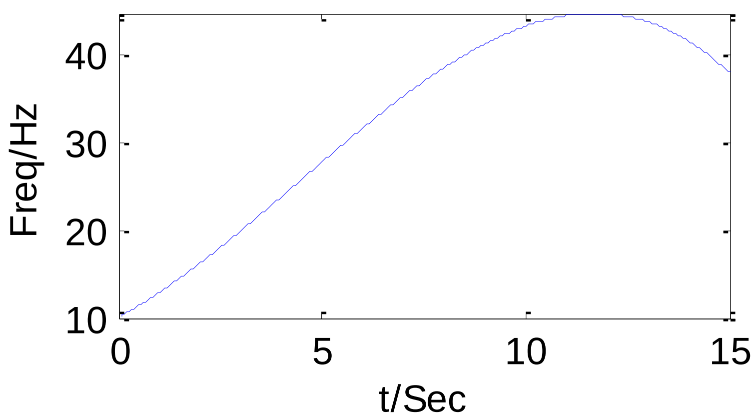
$$STFT_x(n,k) = \int x(t)W(t-nT)e^{-j2\pi kft}dt$$

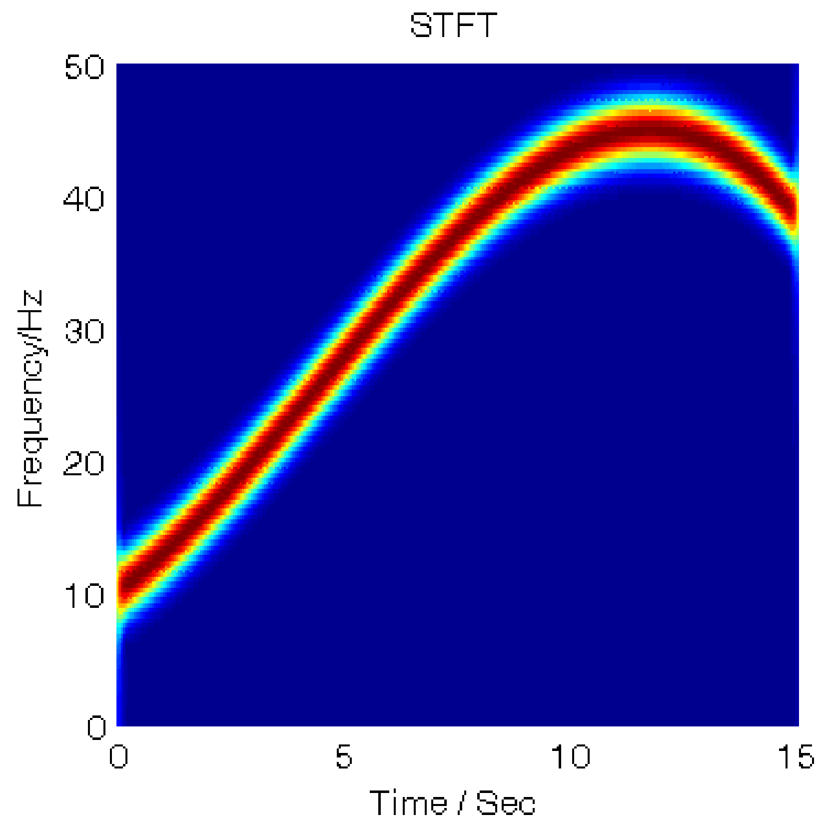
$$x(t) = \sum_n \sum_k STFT_x(n,k)W(t-nT)e^{j2\pi kft}$$

Short Time Fourier Transform

- 示例 $$s(t) = \sin\left(2\pi\left(10t + \frac{5}{4}t^2 + \frac{1}{9}t^3 - \frac{1}{120}t^4\right)\right)$$

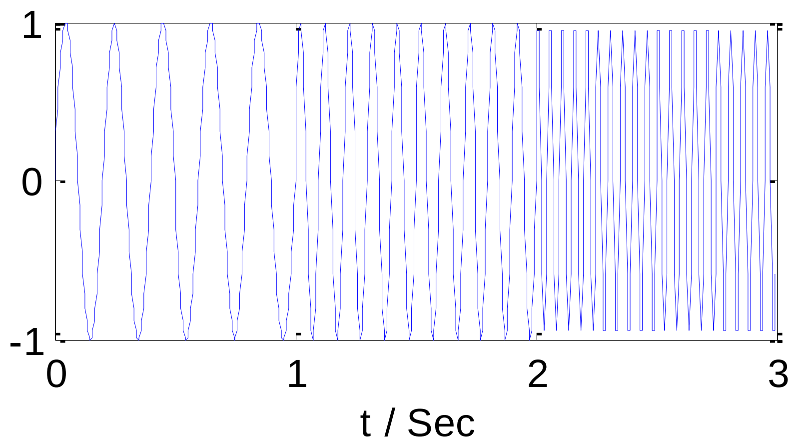


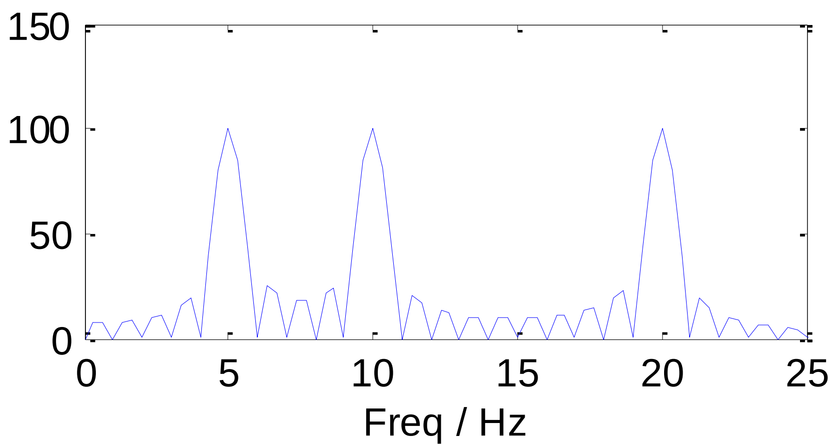


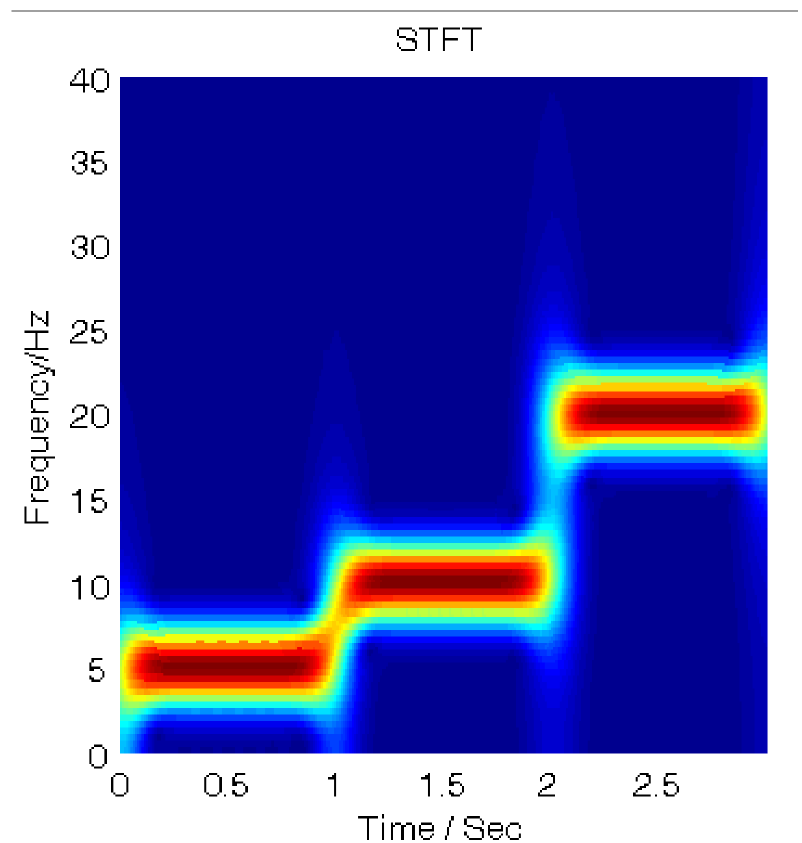


Short Time Fourier Transform

- 示例 $[s_1 \oplus s_2 \oplus s_3]$

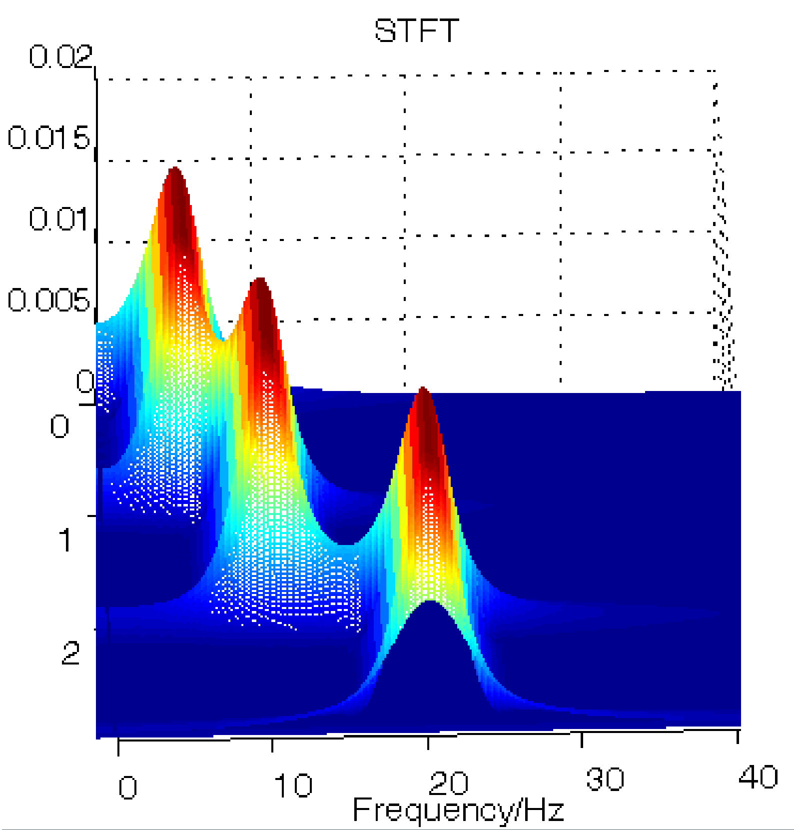


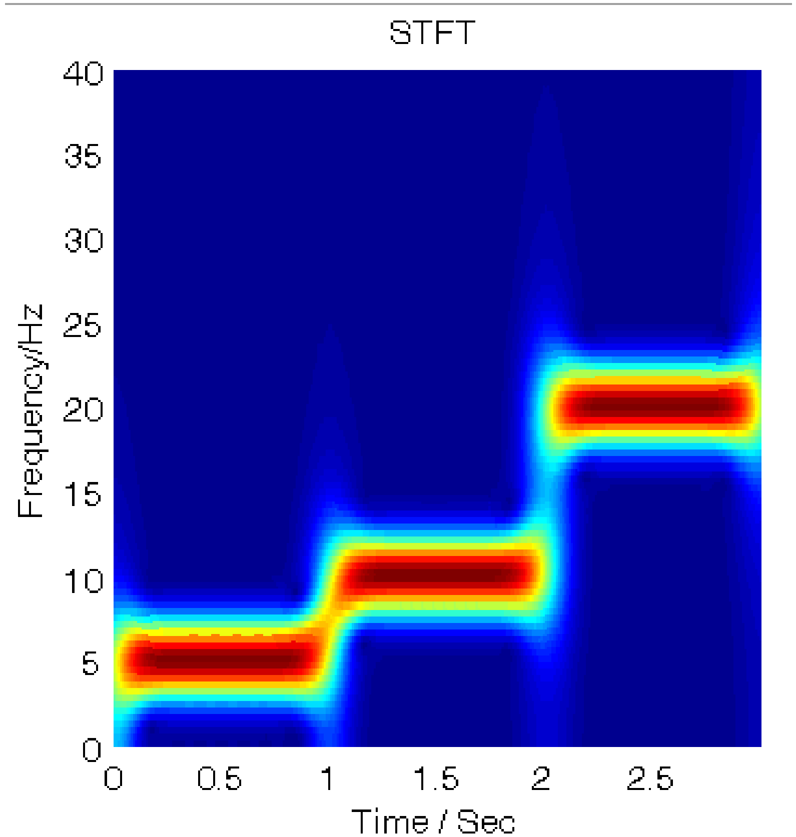




Short Time Fourier Transform

- 示例 $[s_1 \oplus s_2 \oplus s_3]$

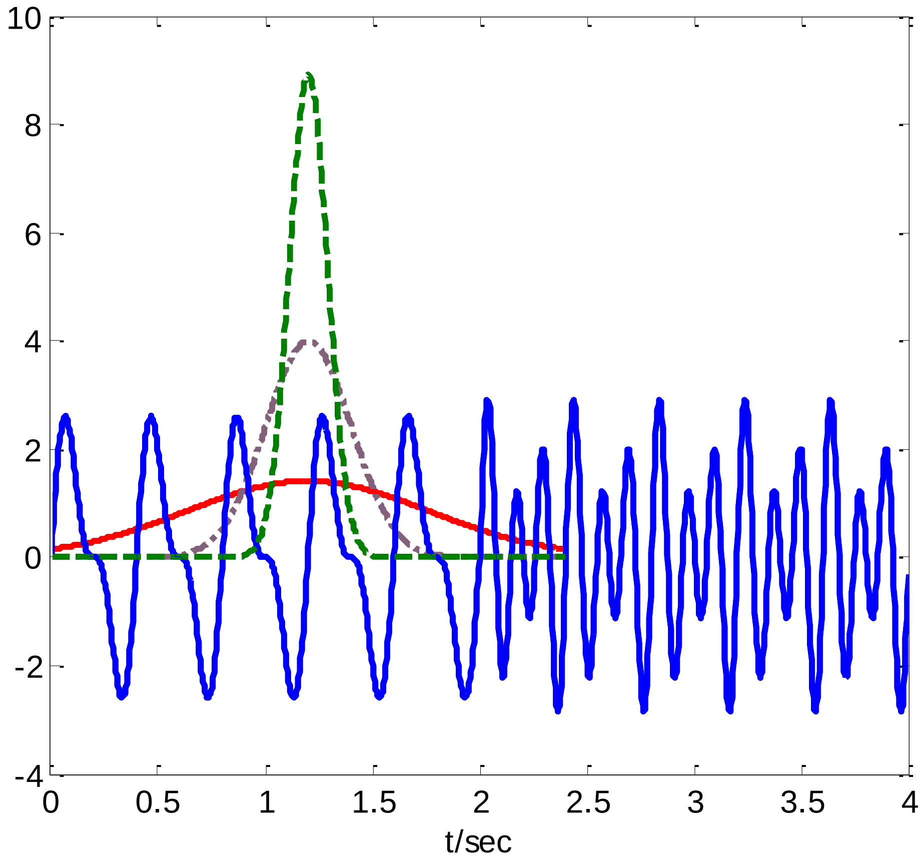




Short Time Fourier Transform

- 问题：窗口宽度的选择

是否可以将窗口取得非常小，以确定信号在每个瞬间所包含的频率分量?



极端情况

1. 窗宽为信号长度
2. 窗宽为单点($\Delta w = 1/Fs$)

Short Time Fourier Transform

- 问题：窗口宽度的选择

是否可以将窗口取得非常小，以确定信号在每个瞬间所包含的频率分量?

$$\mathrm{STFT}_x(\tau,\omega)=\int_t \left[x(t)W(t-\tau)\right]e^{-j\omega t}dt$$

$f(t)=x(t)W(t-\tau)$ 记窗口宽度$\Delta\mathbf{w}$ $\omega_0=2\pi/\Delta\mathbf{w}$

$$\mathrm{STFT}_x(\tau,n\omega_0)=\frac{1}{T}\int_{\tau-\Delta\mathbf{w}/2}^{\tau+\Delta\mathbf{w}/2} f(t)e^{-jn\omega_0 t}dt$$

频域离散： $\{n\omega_0 << \mathrm{Fs}/2;\quad n=0,\pm1,\pm2,\ldots\}$

频域分辨率： $\Delta\omega=\omega_0=2\pi/\Delta\mathbf{w}$

Short Time Fourier Transform

- 问题：窗口宽度的选择

时域分辨率： $\Delta t = \Delta \mathbf{w}$

频域分辨率： $\Delta\omega = \omega_0 = 2\pi / \Delta \mathbf{w}$

$$\boxed{\Delta\omega\Delta t = 2\pi}$$

- 提高时域分辨率，必然使频域分辨率下降
- 提高频域分辨率，必然使时域分辨率下降

极端情况

1. 窗宽为信号长度 $\Delta\omega = \dfrac{2\pi}{\mathrm{s}}$ s- 信号长度 / sec

2. 窗宽为单点($\Delta \mathbf{w} = 1/\mathrm{Fs}$) $\Delta\omega = 2\pi \mathrm{Fs}$

Short Time Fourier Transform

- 问题：窗口宽度的选择

极端情况

1. 窗宽为信号长度

$$\Delta\omega = \frac{2\pi}{s}$$

$$W(\tau) = 1 \rightarrow \mathrm{STFT}_x(\tau,\omega) = \hat{x}(\omega)$$

2. 窗宽为单点($\Delta w = 1/Fs$)

$$\Delta\omega = 2\pi \mathrm{Fs}$$

$$W(\tau) = \delta(\tau) \rightarrow \mathrm{STFT}_x(\tau,\omega) = x(\tau)e^{-j\omega\tau}$$

Short Time Fourier Transform

- **Heisenberg测不准原理**

如果 $W(t)\in L^2(R)$ and $tW(t)\in L^2(R)$

则 $W(t)$为窗口函数，窗口中心点定义为(t_0,ω_0)

$$\begin{cases} t_0=\dfrac{1}{\|W(t)\|^2}\displaystyle\int_{-\infty}^{\infty} t\,|W(t)|^2\,dt \\ \omega_0=\dfrac{1}{\|\widehat{W}(\omega)\|^2}\displaystyle\int_{-\infty}^{\infty} \omega\,|\widehat{W}(\omega)|^2\,d\omega \end{cases}$$

$$\widehat{W}(\omega)=F\left[W(t)\right]$$

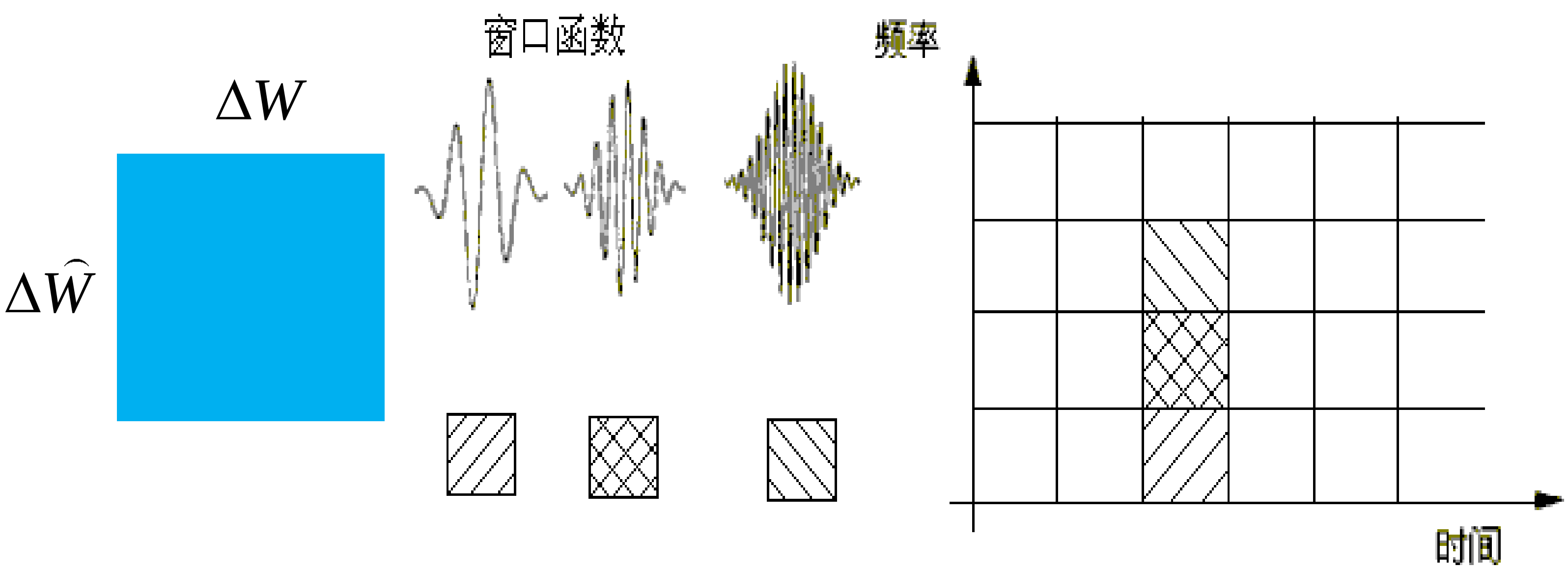
窗口大小

$$\begin{cases} \Delta W=\left(\dfrac{1}{\|W(t)\|^2}\displaystyle\int_{-\infty}^{\infty}(t-t_0)^2|W(t)|^2\,dt\right)^{1/2} \\ \Delta\widehat{W}=\left(\dfrac{1}{\|\widehat{W}(\omega)\|^2}\displaystyle\int_{-\infty}^{\infty}(\omega-\omega_0)^2\,|\widehat{W}(\omega)|^2\,d\omega\right)^{1/2} \end{cases}$$

$$\Delta W\Delta\widehat{W}\geq\frac{1}{2}$$

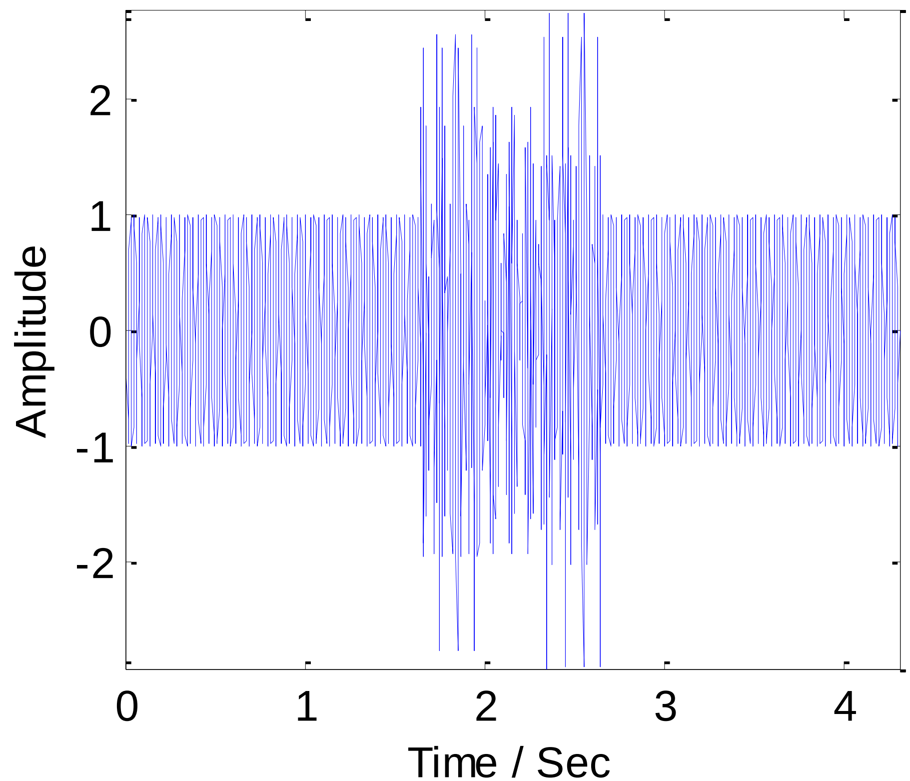
Short Time Fourier Transform

- Heisenberg测不准原理
  - 窗口面积必须大于一个常数 $\Delta W \Delta \widehat{W} \geq \frac{1}{2}$
  - 窗口大小不受中心点位置影响 $(\Delta W, \Delta \widehat{W})$ (t_0, ω_0)



Short Time Fourier Transform

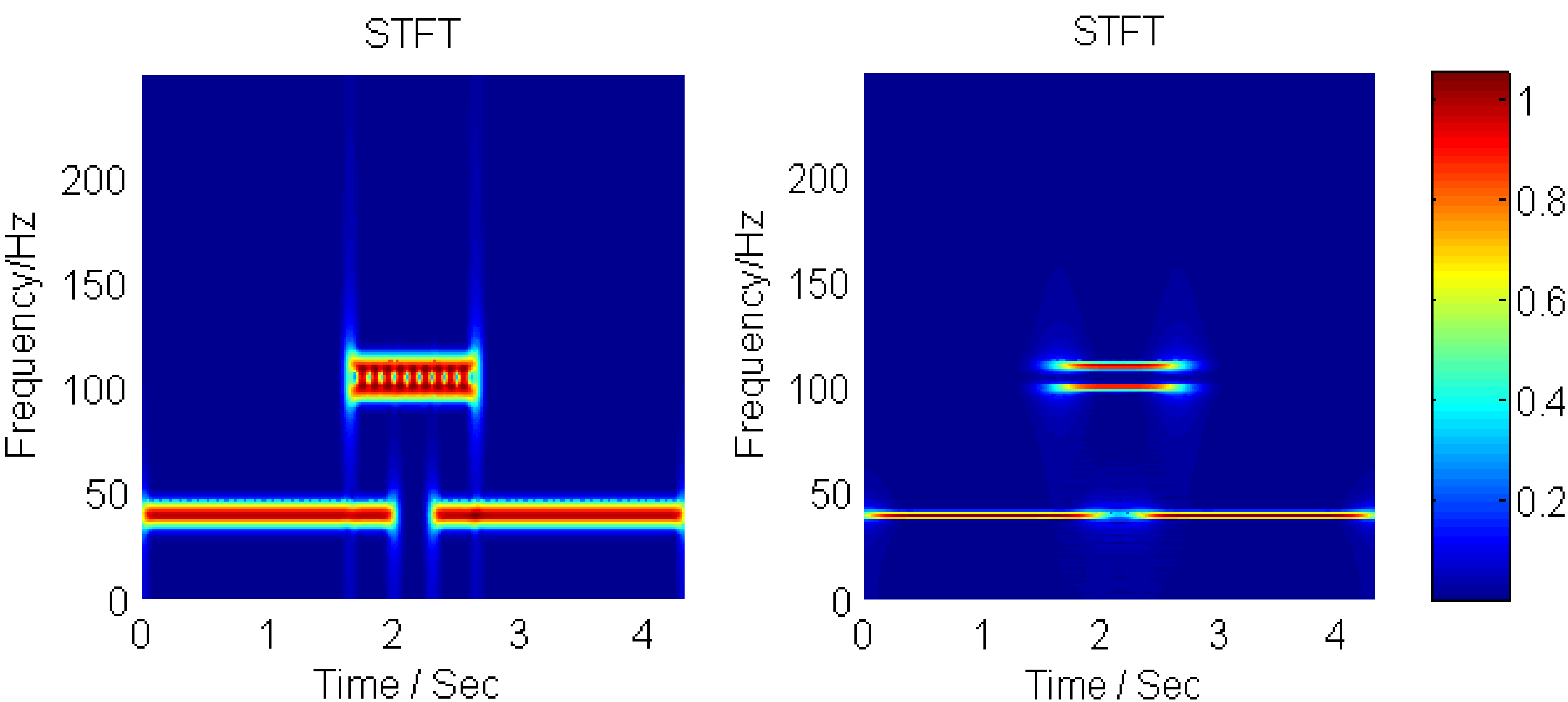
- 示例



The three components are: 1) 40Hz over the whole time span apart from a narrow disconnection of 0.3 second in the middle, 2) 100Hz occupying the middle part with a 1 second duration, and 3) 110Hz occupying the middle part with a 1 second duration. The sampling frequency is 500Hz.

Short Time Fourier Transform

- 示例



窗长128点 窗长512点

Short Time Fourier Transform

- STFT的滤波器观点解释

$$\mathrm{STFT}_x(\tau,\omega)=\int_t\left[x(t)W(t-\tau)\right]e^{-j\omega t}dt$$

$$f(t,\tau)=x(t)W(t-\tau)$$

$$\mathrm{STFT}_x(\tau,\omega)=\int_t f(t,\tau)e^{-j\omega t}dt$$

$$\mathrm{STFT}_x(\tau,\omega)=\hat{x}(\omega)\otimes\left[\hat{W}(\omega)e^{-j\omega\tau}\right]$$

$$\mathrm{STFT}_x(\tau,\omega)=e^{-j\omega\tau}\int\hat{x}(\bar{\omega})\hat{W}(\omega-\bar{\omega})e^{j\bar{\omega}\tau}d\bar{\omega}$$

Short Time Fourier Transform

- STFT的滤波器观点解释

$$\mathrm{STFT}_x(\tau,\omega)=e^{-j\omega\tau}\int\hat{x}(\bar{\omega})\widehat{W}(\omega-\bar{\omega})e^{j\bar{\omega}\tau}d\bar{\omega}$$

$$x(t)\xrightarrow{\mathcal{F}}\hat{x}(\bar{\omega})\xrightarrow[\text{中心频率}\omega]{\text{带通滤波}\widehat{W}}\hat{x}(\bar{\omega})\widehat{W}(\bar{\omega}\text{-}\omega)$$

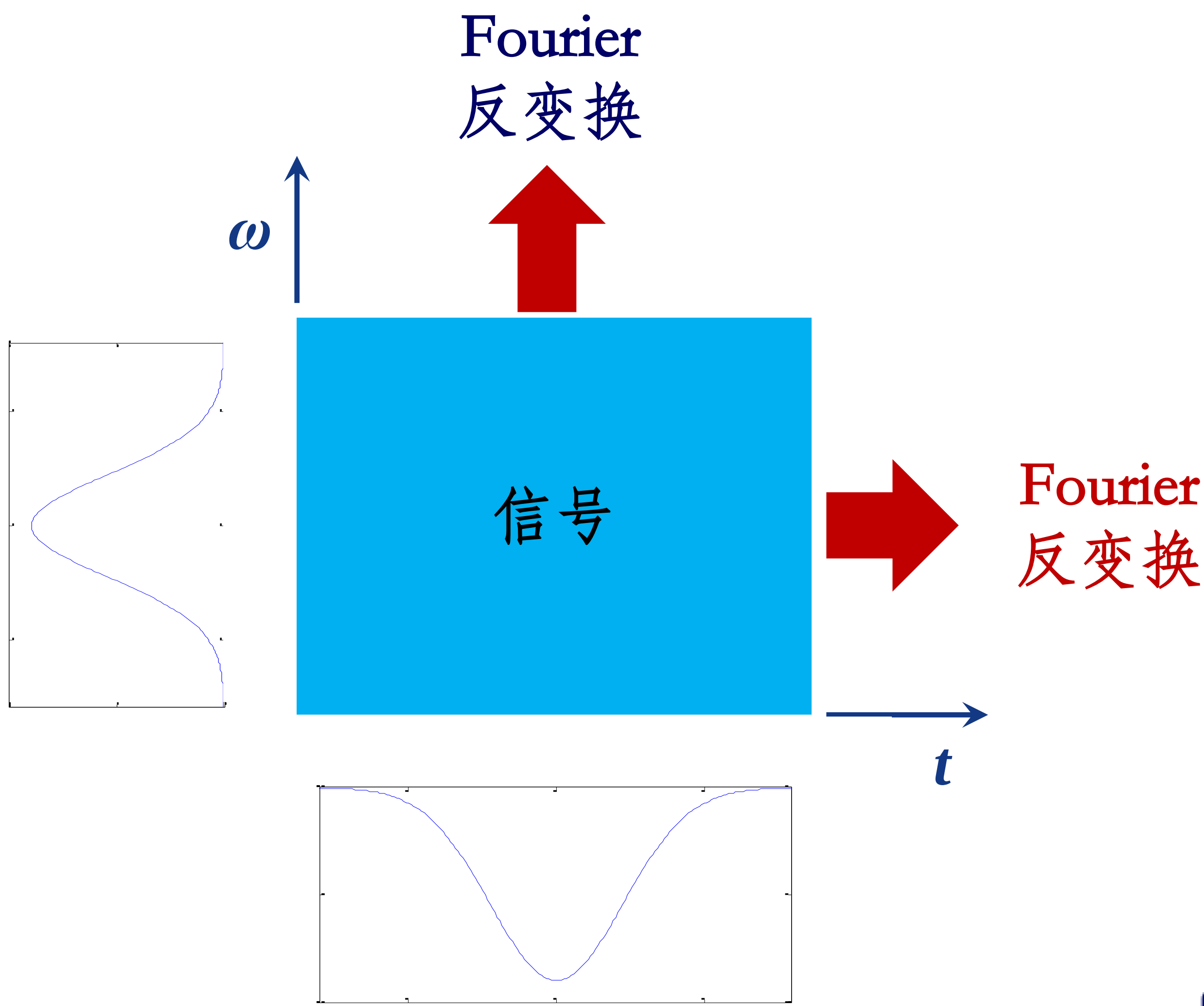
Fourier 反变换

$$\mathrm{STFT}_x(\tau,\omega)\leftarrow\otimes\hat{f}(\tau,\omega)$$

$$\uparrow e^{-j\omega\tau}$$

Short Time Fourier Transform

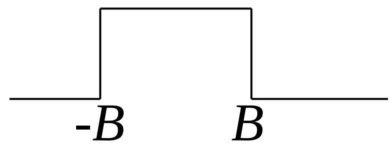
- STFT的滤波器观点解释



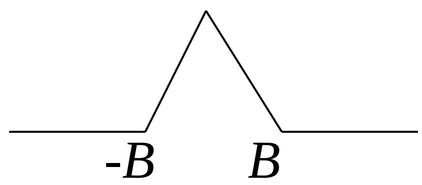
Short Time Fourier Transform

窗类型的影响

(1) Rectangle



(2) Triangle



(3) Hanning

$$W(t)=\begin{cases} 0.5+0.5\cos(\pi t/B) & \text{when } |t|\le B \\ 0 & \text{otherwise} \end{cases}$$

(4) Hamming

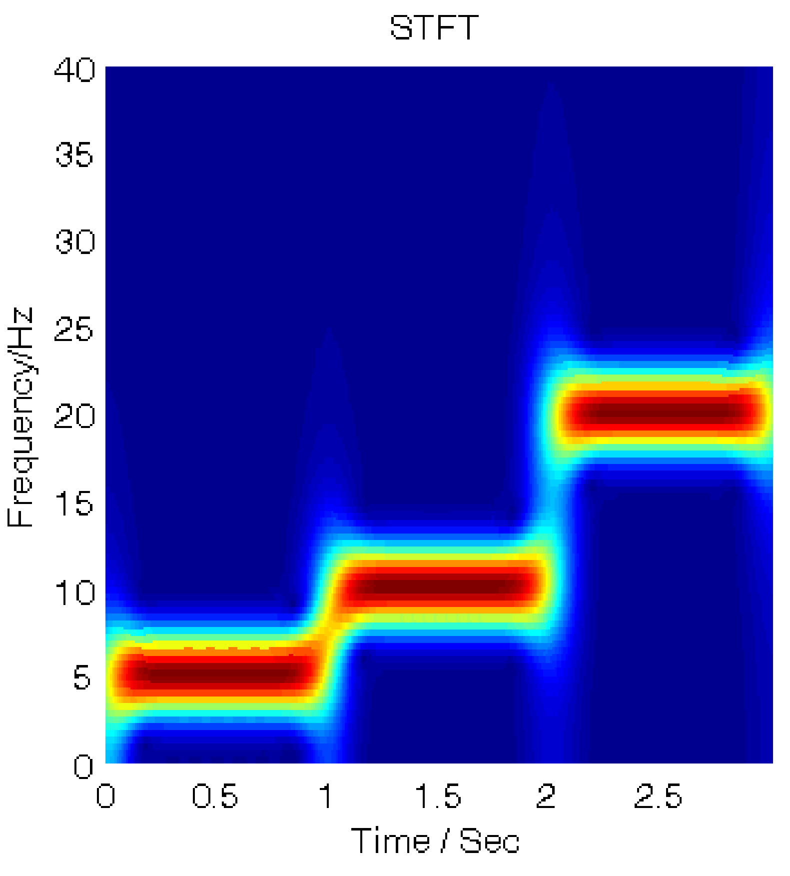
$$W(t)=\begin{cases} 0.54+0.46\cos(\pi t/B) & \text{when } |t|\le B \\ 0 & \text{otherwise} \end{cases}$$

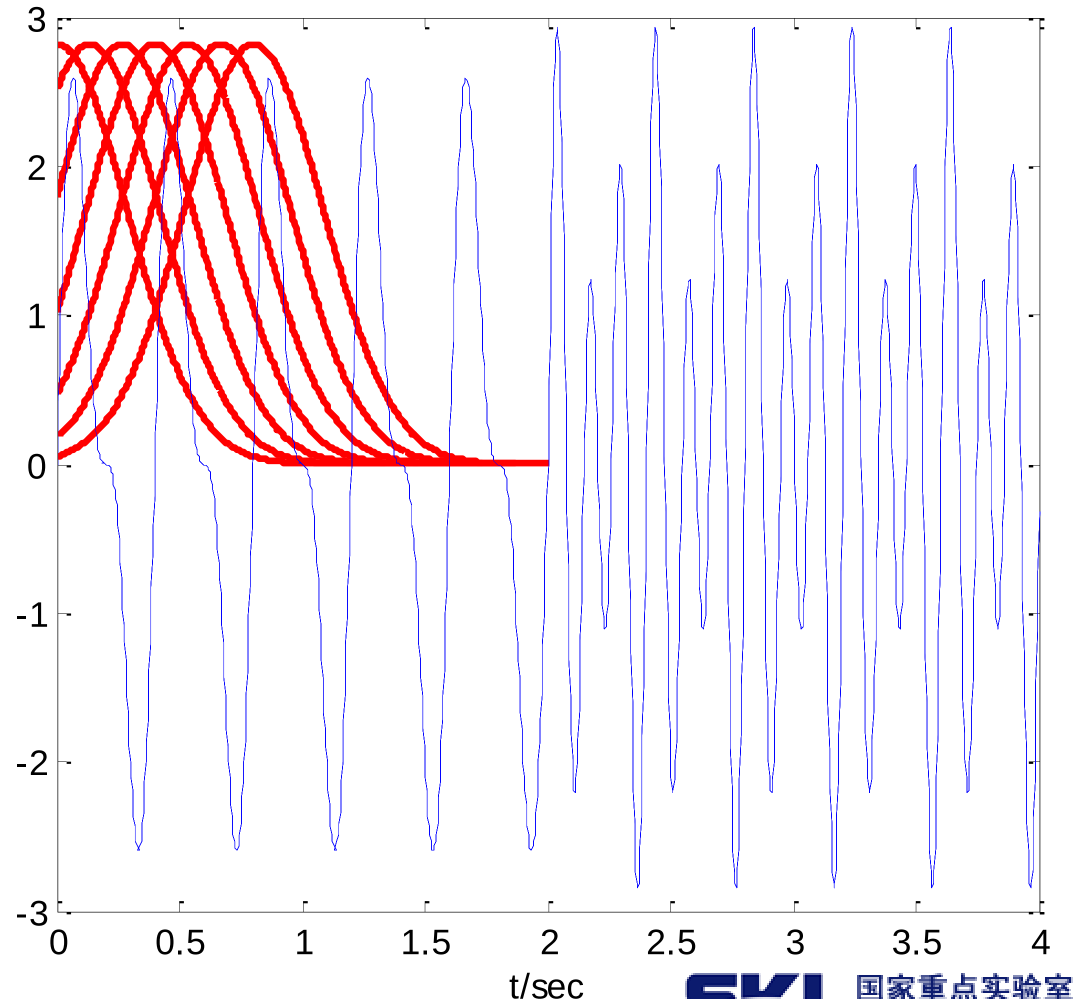
(5) Gaussian

$$W(t)=\frac{1}{2\sqrt{\pi a}}e^{-t^2/4a}$$

Short Time Fourier Transform

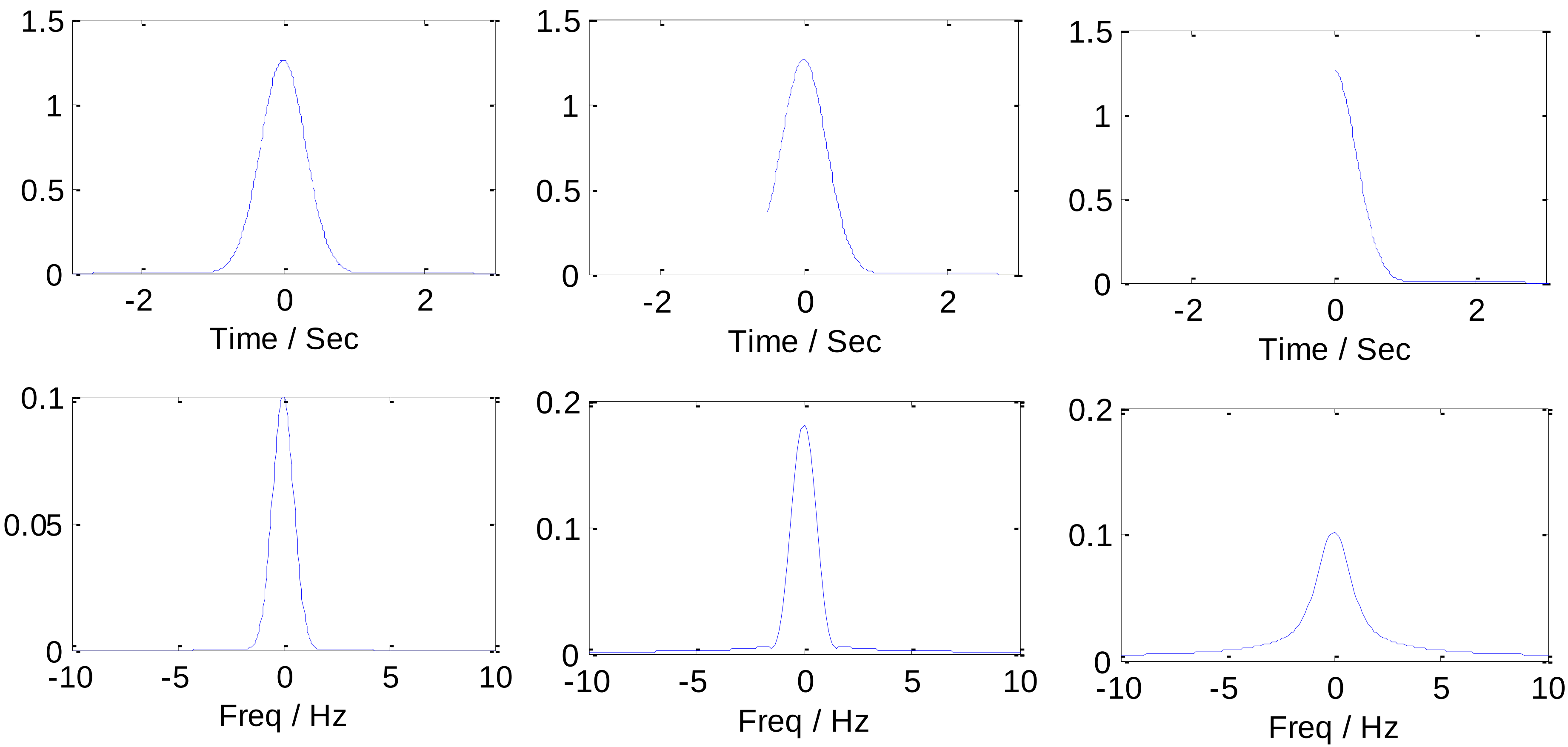
- STFT的边界扭曲





Short Time Fourier Transform

- STFT的边界扭曲



不完整的窗不再严格满足带通滤波器性质

Short Time Fourier Transform

- STFT的边界扭曲

改进方法

> 边界延拓
>
> 镜像拓展
> 补零
> 边界预测
> …

有效?

Short Time Fourier Transform

- 讨论

> Fourier变换的适用性
>
> STFT的适用性

> 平稳信号
>
> 非平稳信号

谢谢聆听
欢迎交流