Modern Control Theory
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Outline of Today's Lecture

- Solution for Nonhomogeneous State Equation
- Solution of State equation of Discrete Control System
- Controllability

Solution for Nonhomogeneous State Equations

$$\dot{x} = Ax + Bu$$

$\Downarrow$ **Laplace Transform**

$$SX(s) - x(0) = AX(s) + BU(s)$$

$$\Downarrow$$

$$(SI - A)X(s) = x(0) + BU(s)$$

$$\Downarrow$$

$$X(s) = (SI - A)^{-1}x(0) + (SI - A)^{-1}BU(s)$$

$$\because (sI - A)^{-1} = L\left[\Phi(t)\right] \Rightarrow x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Solution for Nonhomogeneous State Equations

Example: Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Where u(t) is the unit step function starting at t=0

Solution: The state transition Matrix φ(t) is :

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\Phi(t) = e^{At}x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 d\tau$$

Solution for Nonhomogeneous State Equations

$$\Phi(t) = e^{At}x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 d\tau$$

$$\Downarrow$$

$$\Phi(t) = e^{At}x(0) + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Solution for Nonhomogeneous State Equations

Example A-11-6

Consider the system defined by

$$\dot{x} = Ax + Bu$$

Obtain the response of the system to each of the following inputs:

(a) impulse function u=Kδ(t);

(b) step function u=K×1(t)

(c) Ramp function u=K×t

Solution for Nonhomogeneous State Equations

$$X(s) = (SI - A)^{-1} x(0) + (SI - A)^{-1} BU(s)$$

u=Kδ(t);

$$X(s) = (SI - A)^{-1} x(0) + (SI - A)^{-1} BK$$

$$x(t) = \Phi(t) x(0) + \Phi(t) BK$$

(b) $$x(t) = e^{At} x(0) + A^{-1}(e^{At} - I) BK$$

(c) $$x(t) = e^{At} x(0) + A^{-2}(e^{At} - I - At) BK$$

Matlab Solution

Obtaining response to arbitrary input

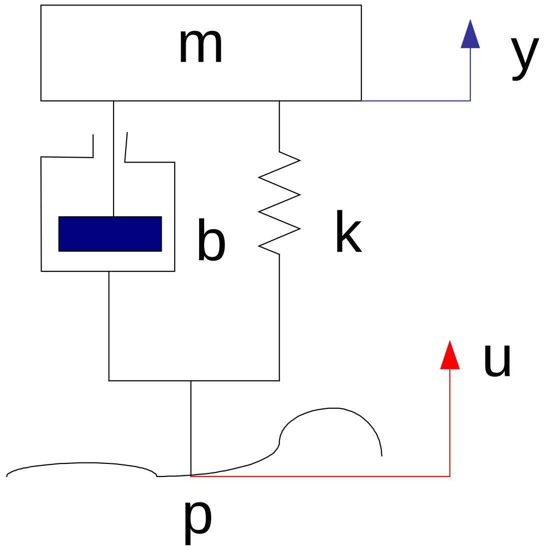
assuming zero initial condition.

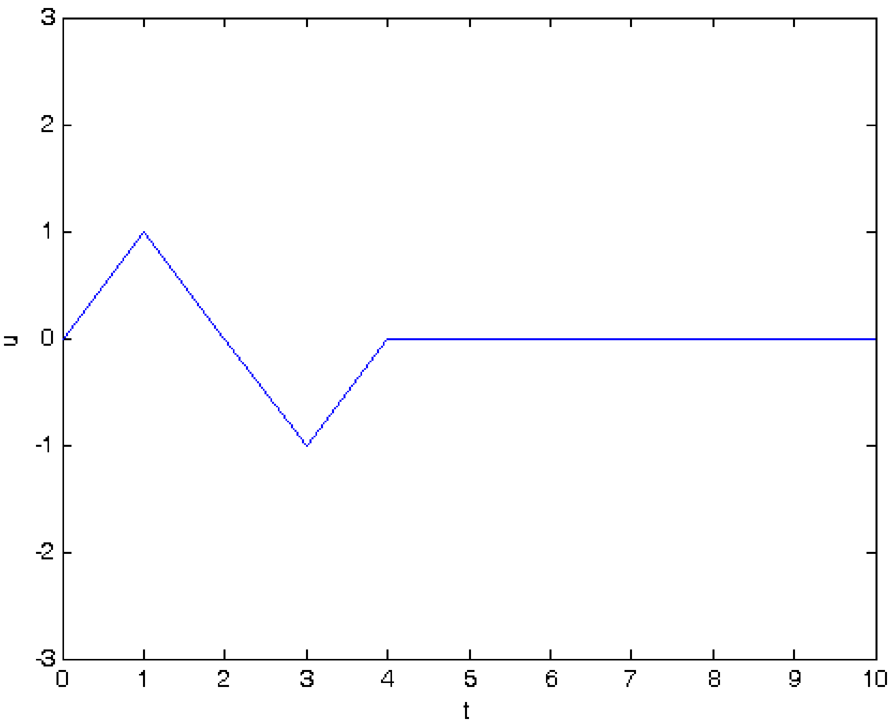
1. **lsim(sys,u,t) or lsim(num,den,u,t).**
2. **y=lsim(sys,u,t) or y=lsim(num,den,u,t)**
3. **[y,t]=lsim(sys,u,t) Or [y,t]=lsim(num,den,u,t)**
4. **lsim(sys1,sys2,…,u,t)**

assuming nonzero initial condition.

1. **Lsim(sys,u,t,x0)**
2. **[y,t]=lsim(sys,u,t,x0)**

Example A-4-18 Ogata P159





Assuming the motion u is a small bump,as shown in figure. Obtain the response y(t) of the system.
M=100kg,b=200N-s/m, k=1000N/m

Matlab Solution

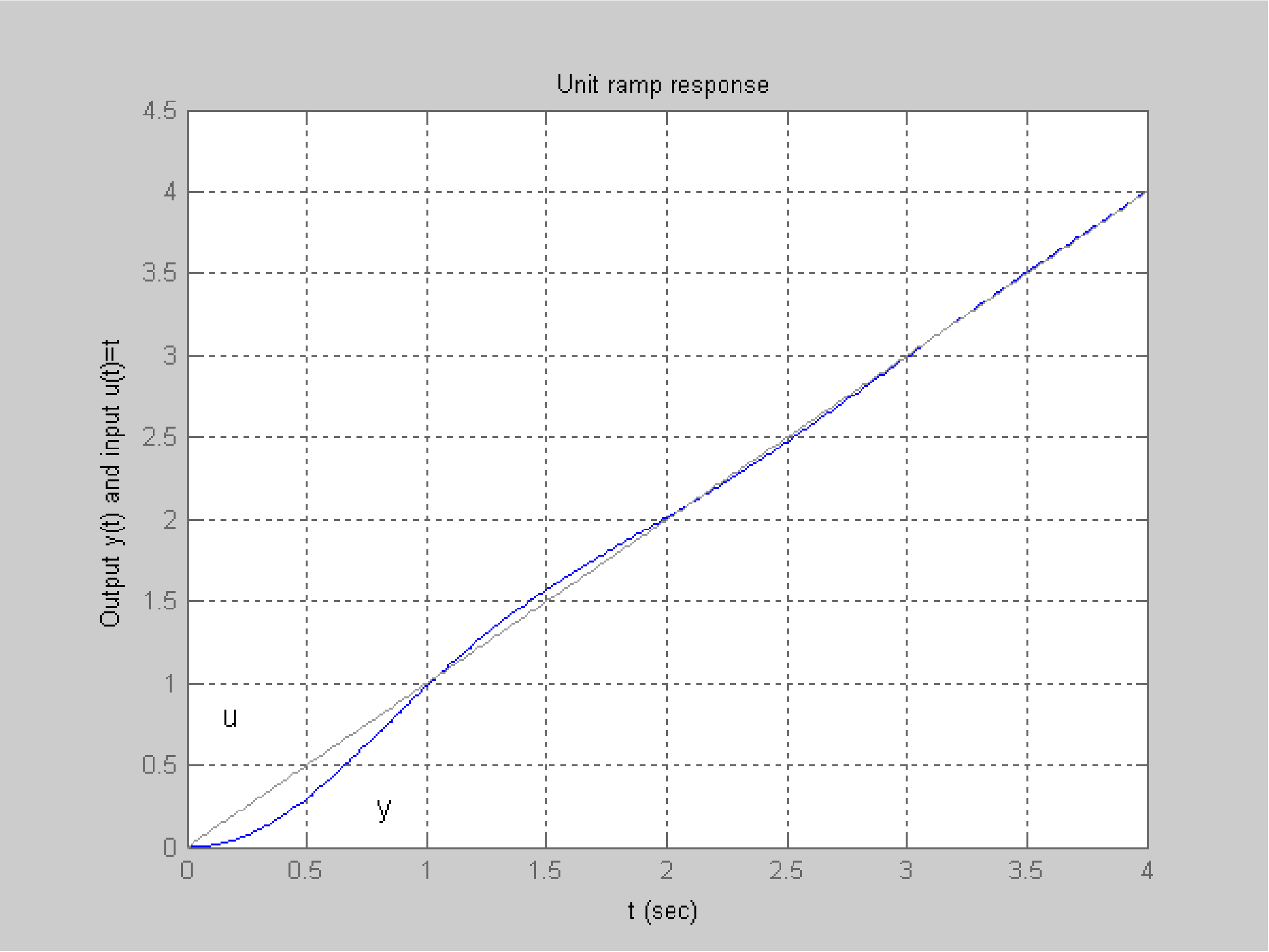
Example: **plots the unit-ramp response curve y(t) and input ramp function u(t)**

$$\frac{Y(s)}{U(s)} = \frac{2s+10}{s^2+2s+10}$$

```
>> num=[2 10]
>> den=[1 2 10];
>> sys=tf(num,den);
>> t=0:0.01:4;
>> u=t;
```

```
>> lsim(sys,u,t)
>> grid
>> title('Unit ramp response')
>> xlabel('t')
>> ylabel('Output y(t) and input u(t)=t')
>> text(0.8,0.25,'y')
>> text(0.15,0.8,'u')
```

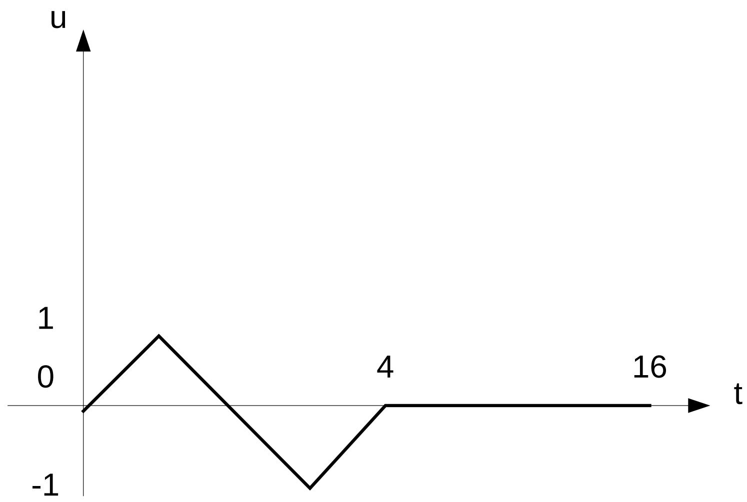
Matlab Solution



Matlab Solution

Example: **Find the response of function listed below.**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -10 & -15 & -12 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 5\\ -15\\ 40\end{bmatrix}u \qquad y=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+0u$$



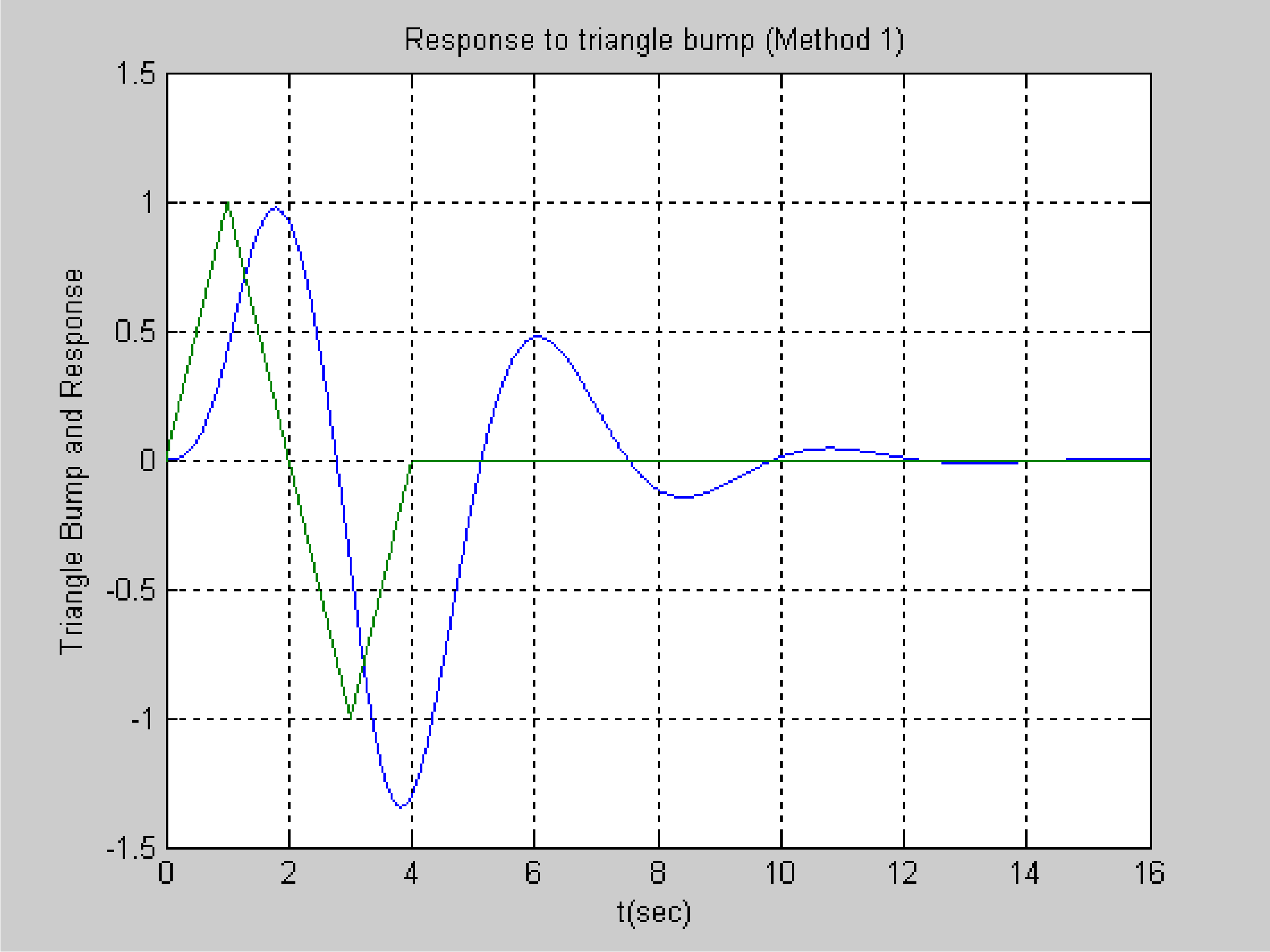

```
>> t=0:0.01:16;
>> A=[0 1 0 0;0 0 1 0;0 0 0 1;-10 -15 -12 -6];
>> B=[0;5;-15;40];
>> C=[1 0 0 0];
>> D=0;
```

Matlab Solution

```
>> sys=ss(A,B,C,D);
>> u1=[0:0.01:1];
>> u2=[0.99:-0.01:-1];
>> u3=[-0.99:0.01:0];
>> u4=0*[4.01:0.01:16];
>> u=[u1 u2 u3 u4];
>> y=lsim(sys,u,t);
```

```
>> plot(t,y,t,u)
>> v=[0 16 -1.5 1.5];
>> axis(v)
>> grid
>> title('Response to triangle bump (Method 1)')
>> xlabel('t(sec)')
>> ylabel('Triangle Bump and Response')
```

Matlab Solution



Outline of Today's Lecture

- Solution for Nonhomogeneous State Equation
- Solution of State equation of Discrete Control System

Solution of State equation of Discrete Control System

1.递推法：适用于数值解法

方程

$$\begin{aligned} x(k+1) &= Gx(k)+Hu(k) \\ x(k)\big|_{k=0} &= x(0) \end{aligned}$$

的解为：

$$x(k)=G^{k}x(0)+\sum_{j=0}^{k-1}G^{k-j-1}Hu(j)$$

或

$$x(k)=G^{k}x(0)+\sum_{j=0}^{k-1}G^{j}Hu(k-j-1)$$

Solution of State equation of Discrete Control System

用矩阵方法表示:

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(k) \end{bmatrix} = \begin{bmatrix} G \\ G^2 \\ G^3 \\ \vdots \\ G^k \end{bmatrix} x(0) + \begin{bmatrix} H & 0 & 0 & \cdots & 0 \\ GH & H & 0 & \cdots & 0 \\ G^2H & GH & H & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ G^{k-1}H & G^{k-2}H & G^{k-3}H & \cdots & H \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(k-1) \end{bmatrix}$$

若从k=h开始计算，则:

$$x(k) = G^{k-h}x(h) + \sum_{j=h}^{k-1} G^j Hu(k-j-1)$$

或

$$x(k) = G^{k-h}x(h) + \sum_{j=h}^{k-1} G^{k-j-1} Hu(j)$$

Solution of State equation of Discrete Control System

例题：离散时间系统的状态方程

$$x(k+1) = Gx(k) + Hu(k)$$

$$x(k)\big|_{k=0} = x(0)$$

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad u(k) = 1$$

求系统 $\Phi(k)$ $x(k)$

Solution of State equation of Discrete Control System

连续时间状态空间表达式的离散化

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

U(t)=U(KT) when

KT<=t<KT+1

$$x(k+1) = G(T)x(k) + H(T)u(k)$$
$$y(k) = Cx(k) + Du(k)$$

其中： $G(T) = e^{AT}$ $\qquad H(T) = \int_0^T e^{At}dtB$

近似离散化

当采样时间为系统最小时间常数的1/10时，离散化的状态方程可近似表示为：

Solution of State equation of Discrete Control System

$$G(T) \approx TA + 1, H(T) \approx TB$$

Determination of Sampling Period

1. Open Loop Control in the Sampling Period

2. Sampling Frequency should satisfy:

$$\omega = 2\pi \frac{1}{T} > 2\omega_c$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Find the Solution of Discrete Control System with the Sampling Period T

Appendix1 Programming and Data Types

if, else, and elseif

```
if logical_expression
    statements
end
```

```
if rem(a,2) = = 0
    disp('a is even')
    b = a/2;
end
```

If the logical expression is true (1), MATLAB executes all the statements between the if and end lines. It resumes execution at the line following the end statement. If the condition is false (0), MATLAB skips all the statements between the if and end lines, and resumes execution at the line following the end statement.

```
if X
    statements
end
```

Appendix1 Programming and Data Types

While

The while loop executes a statement or group of statements repeatedly as long as the controlling expression is true (1). Its syntax is

```
while expression
    statements
end
```

If the expression evaluates to a matrix, all its elements must be 1 for execution to continue. To reduce a matrix to a scalar value, use the all and any functions.

```
n = 1;
while prod(1:n) < 1e100
    n = n + 1;
end
```

Appendix1 Programming and Data Types

For

For example, this loop executes five times.

```
for i = 2:6
    x(i) = 2*x(i-1);
end
```

You can nest multiple for loops.

```
for i = 1:m
    for j = 1:n
        A(i,j) = 1/(i + j - 1);
    end
end
```

The for loop executes a statement or group of statements a predetermined number of times. Its syntax is:

```
for index = start:increment:end
    statements
end
```

The default increment is 1. You can specify any increment, including a negative one. For positive indices, execution terminates when the value of the index exceeds the end value; for negative increments, it terminates when the index is less than the end value.

Appendix1 Programming and Data Types

mesh(X,Y,Z)

mesh(Z)

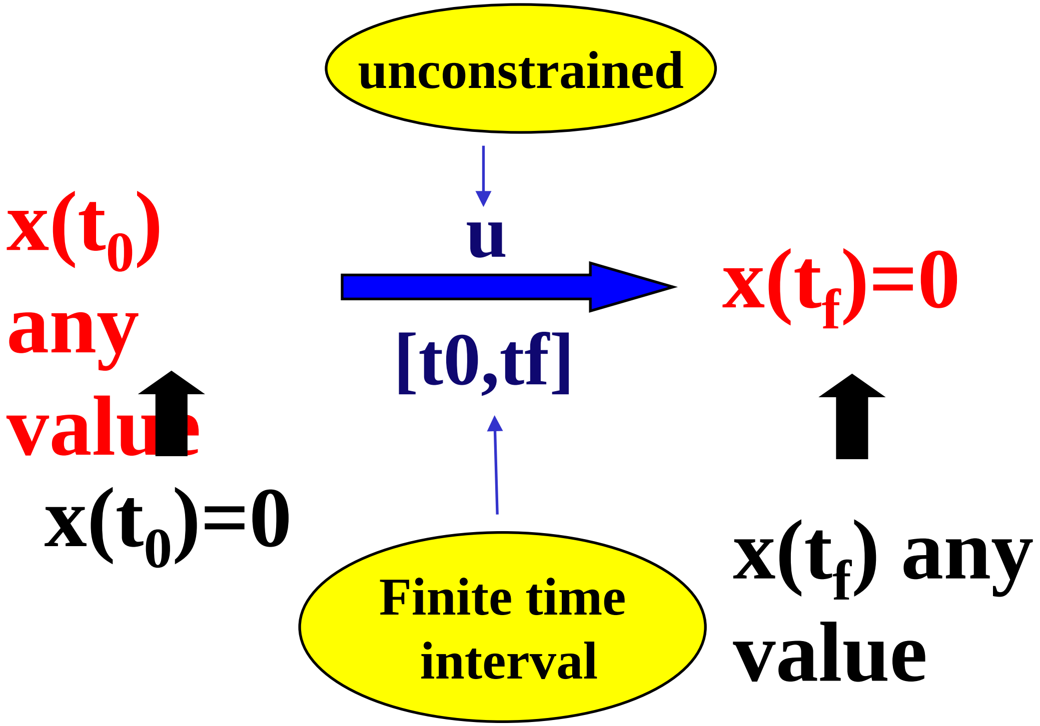
mesh(X,Y,Z) draws a wireframe mesh with color determined by Z, so color is proportional to surface height.
If X and Y are vectors, length(X) = n and length(Y) = m, where [m,n] = size(Z). In this case, are the intersections of the wireframe grid lines; X and Y correspond to the columns and rows of Z, respectively.
If X and Y are matrices, are the intersections of the wireframe grid lines. mesh(Z) draws a wireframe mesh using X = 1:n and Y = 1:m, where [m,n] = size(Z). The height, Z, is a single-valued function defined over a rectangular grid. Color is proportional to surface height.

Outline of Today's Lecture

- Solution for Nonhomogeneous State Equation
- Solution of State equation of Discrete Control System
- Controllability

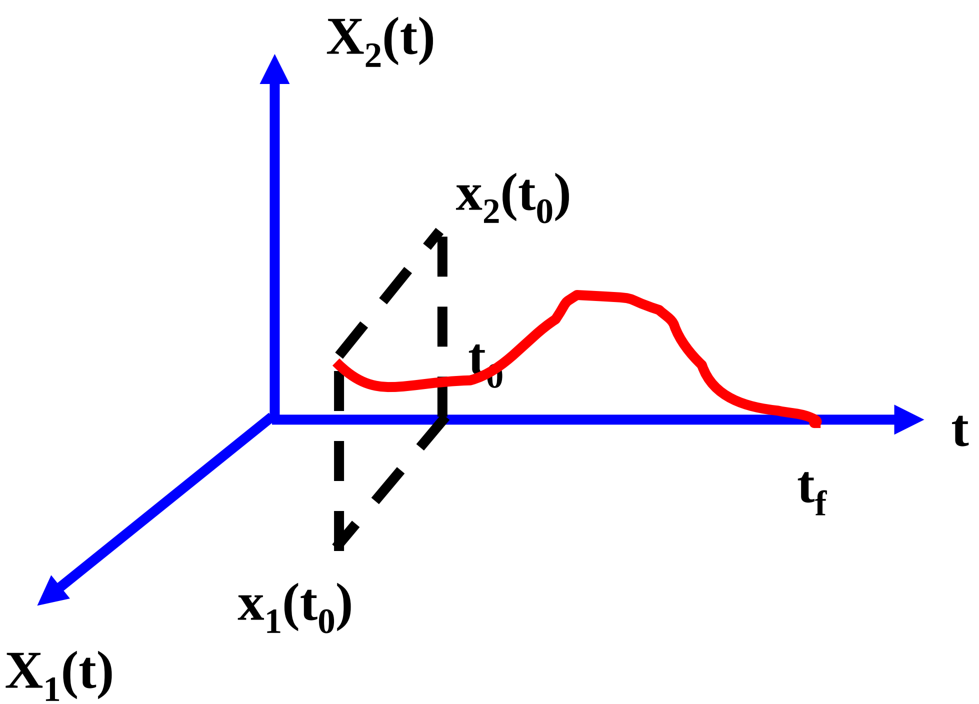
Controllability

- A System is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.



Controllability

- The controllability answers "whether the state vector can be controlled to any value? " not for "how to control".



Controllability

- Controllability analysis for Diagonal Form;
- Suppose the SS model has a diagonal form:

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \ldots & \lambda_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix}$$

Condition for Controllability
Diagonal Form

For any SS model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Suppose n×n matrix A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, corresponding eigenvectors are $P_1, P_2 \ldots P_n$, then Matrix A can be diagonalized by Matrix $T=[P_1, P_2 \ldots P_n]$, New SS model are:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu; \;\; z(0) = T^{-1}x(0) = T^{-1}x_0$$

$$y = CTz + Du$$

Condition for Controllability (I)
Diagonal Form

The new System are:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \cdot \\ \cdot \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_{n-1} \\ z_n \end{bmatrix} + T^{-1}Bu$$

Let

$$T^{-1}B = F = (f_{ij})$$

Condition for Controllability
Diagonal Form

Suppose we have r inputs

$$\dot{z}_1 = \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \ldots f_{1r}u_r$$

$$\dot{z}_2 = \lambda_2 z_2 + f_{21}u_1 + f_{22}u_{22} + \ldots + f_{2r}u_r$$

$$\dot{z}_3 = \lambda_3 z_3 + f_{31}u_1 + f_{32}u_{22} + \ldots + f_{3r}u_r$$

$$\ldots$$

$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_{22} + \ldots + f_{nr}u_r$$

Condition for Controllability
Diagonal Form

Results: The elements of any row of $T^{-1}B$ that corresponds to distinct eigenvalues are not all zero.

Condition for Controllability
Jordan Form

For any SS model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Suppose n$\times$n matrix A has one 3-order repeated eigenvalues λ_1 and n-3 distinct eigenvalues λ_4 λ_5…λ_n, the eigenvectors are P_1,P_2… P_n, then Matrix A can be diagonalized by Matrix T=[P_1,P_2… P_n], New SS model are:

Condition for Controllability
Jordan Form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ . \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & . & 0 \\ 0 & \lambda_1 & 1 & 0 & . & \\ 0 & 0 & \lambda_1 & 0 & . & \\ 0 & 0 & 0 & \lambda_4 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ . \\ z_n \end{bmatrix} + T^{-1}Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = Jz + T^{-1}Bu$$

Let

$$T^{-1}B = F = (f_{ij})$$

Condition for Controllability
Jordan Form

Suppose we have r inputs

$$\dot{z}_1 = \lambda_1 z_1 + z_2 + f_{11}u_1 + f_{12}u_2 + \ldots f_{1r}u_r$$

$$\dot{z}_2 = \lambda_1 z_2 + z_3 + f_{21}u_1 + f_{22}u_{22} + \ldots + f_{2r}u_r$$

$$\dot{z}_3 = \lambda_1 z_3 + f_{31}u_1 + f_{32}u_{22} + \ldots + f_{3r}u_r$$

$$\ldots$$

$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_{22} + \ldots + f_{nr}u_r$$

Condition for Controllability
Jordan Form

Results1:The elements of any row of $T^{-1}B$ that correspond to the last row of each Jordan Block are not all zero. (means f31 or f32 or f33 not equal to zero).

Results2:The elements of any row of $T^{-1}B$ that correspond to distinct eigenvalues are not all zero.

Condition for Controllability
Diagonal Form

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} u$$

Condition for Controllability
Diagonal Form

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u$$

Controllability

Complete State Controllability of continuous-time System:

$$\dot{x} = Ax + Bu$$

Suppose:

x=state vector (n×1) u=control signal (scalar)

A=n×n matrix B=n×1 matrix

$$\mathbf{x(0)=any\ value} \xrightarrow[\mathbf{[0,t_1]}]{\mathbf{u}} \mathbf{x(t_1)=0}$$

Controllability

The solution of the x(t):

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

x(0)=any value $\xrightarrow[{[0,t_1]}]{\mathbf{u}}$ **x(t₁)=0**

$$0 = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

$$\Downarrow$$

$$x(0) = -\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

Controllability

Applying Gayley-Hamilton Theorem:

$$\left|\lambda I - A\right| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

The matrix A satisfies its own characteristic equation

$$A^n + a_1 A^{n-1} + \ldots + a_{n-1}A + a_n = 0$$

$$e^{At} = I + At + \frac{1}{2!}A^2 t^2 + \cdots + \frac{1}{k!}A^k t^k + \cdots$$

$$\Downarrow$$

$$e^{At} = \alpha_{n-1}(t)A^{n-1} + \alpha_{n-2}(t)A^{n-2} + \ldots + \alpha_1(t)A + \alpha_0 I$$

Controllability

$$x(0) = \int_0^{t_1} e^{-A\tau} Bu(\tau)d\tau$$

$$\Downarrow$$

$$x(0) = -\int_0^{t_1} \sum_{k=0}^{n-1} \alpha_k(\tau) A^k Bu(\tau)d\tau$$

$$\Downarrow$$

$$x(0) = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k(\tau)u(\tau)d\tau$$

Let

$$\int_0^{t_1} \alpha_k(\tau)u(\tau)d\tau = \beta_k$$

Controllability

$$x(0) = -\sum_{k=0}^{n-1} A^k B \beta_k$$

$$x(0) = -\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdot \\ \beta_{n-1} \end{bmatrix} (\#1)$$

Results1: The system is completely controllable, then, given any initial state x(0), The equation #1 should be satisfied. This requires that the rank of the n×n matrix

Controllability

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} (\#2)$$

be n. (or the vector of B AB … A^{n-1}B are linearly independent)

Results2: if u is an r-vector, then the condition for complete state controllability is that the n×nr matrix

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

be of rank n. (or contain n linearly independent vector)

Controllability

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Is called *controllability matrix*

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

Controllability

```
>> A=[1 1 0;0 1 0;0 1 1];
>> B=[0 1; 1 0; 0 1];n=3;
>> M=ctrb(A,B);
>> rankM=rank(M);
>> if rankM ==n
disp('system is controllable')
else
disp('system is uncontrolable')
End
system is uncontrolable
```

Example:

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

Output Controllability

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Where

x=state vector (n×1) u=control vector (r-vector)

y=output   A=n×n matrix; B=n×r matrix;

C=m×n matrix; D=m×r matrix

A System is said to be output controllable if it is possible by means of an unconstrained control vector to transfer the system from any initial output $y(t_0)$ to any final output $y(t_1)$ in a finite interval of time $t_0<t<t_1$

Output Controllability

- It can be proved that the system is said to be output controllable if and only of m×(n+1)r matrix:

$$[CB \;\; CAB \;\; \ldots \; A^{n\text{-}1}B \;\; D]$$

Is of rank m

Condition for Complete State Controllability in the s plane

- It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation in the transfer function or transfer matrix.

$$W_{ux}(s) = (sI - A)^{-1} b$$

Example:

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} u}$$

Stabilizability

- For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable.

Example:

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u}$$