Modern Control Theory
Spring 2017

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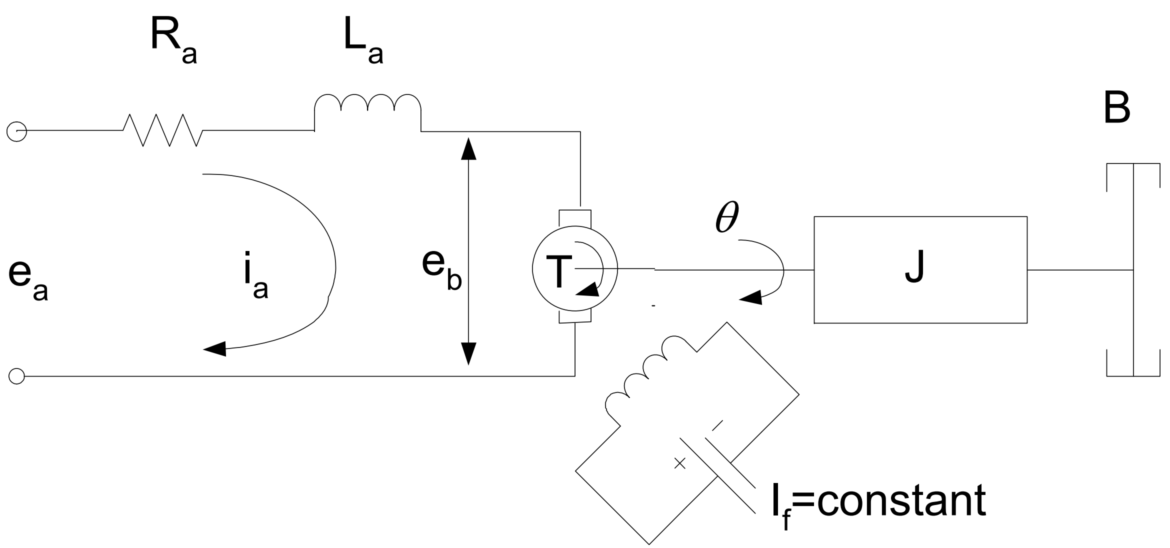
Outline of Today's Lecture

- SS Model of DC Servo Motor
- SS Model of Complex Physical System
- Transformation from TF to SS Model

Building SS Model from Dynamics of Physical System

DC servo motors



R_a=armature resistance, Ω 电枢电阻

La=armature inductance, H, 电枢电感

i_a=armature current, A 电枢电流

i_f=field current, A 激磁电流

Building SS Model from Dynamics of Physical System

e_a=applied armature voltage, V 电枢控制电压

e_b=back emf, V 感应反电动势

θ=angular displacement of the motor shaft, rad 电机轴上的角位移

T=torque developed by the motor. N-m电机力矩。

J=moment of inertia of the motor and load referred to the motor shaft. Kg-m^2 电机转子和负载的惯量。

b=viscous-friction coefficient of the motor and load referred to the motor shaft. N-m/rad/s. 电机转子和负载的粘性摩擦系数。

Building SS Model from Dynamics of Physical System

Electromechanical Systems-DC servo motors

$$\psi = K_f i_f$$

Ψ：激磁绕组形成的磁通链

K_f：常数

$$T = K_f i_f k_1 i_a$$

T：电磁力矩

K_1：常数

$$e_b = K_b \frac{d\theta}{dt}$$

K_b：back emf constant.

反电动势常数

Building SS Model from Dynamics of Physical System

Electromechanical Systems-DC servo motors

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a$$

$$J \frac{d^2\theta}{dt^2} + b \frac{d\theta}{dt} = T = Ki_a$$

Taking the Laplace Transform and assuming zero condition

$$K_b s\theta(s) = E_b(s)$$

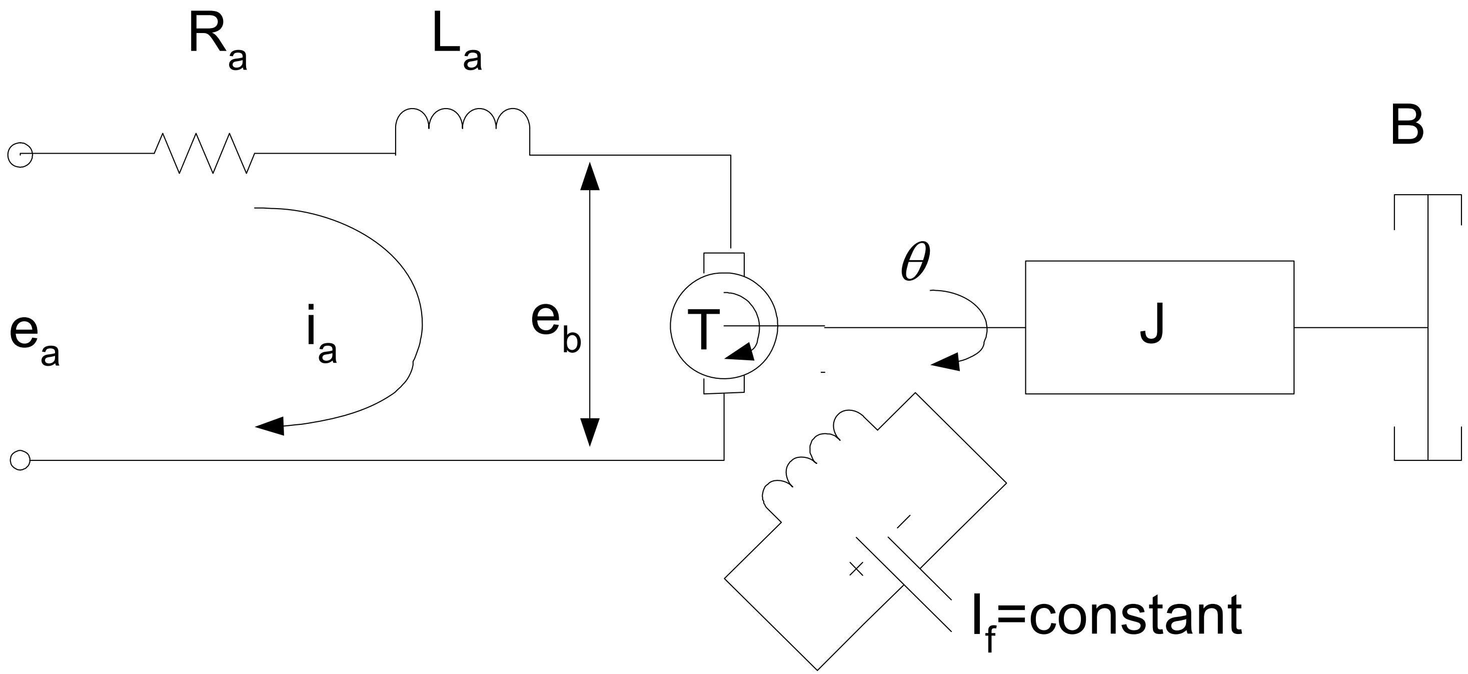
$$(L_a s + R_a) I_a(s) + E_b(s) = E_a(s)$$

$$(Js^2 + bs)\theta(s) = T(s) = KI_a(s)$$

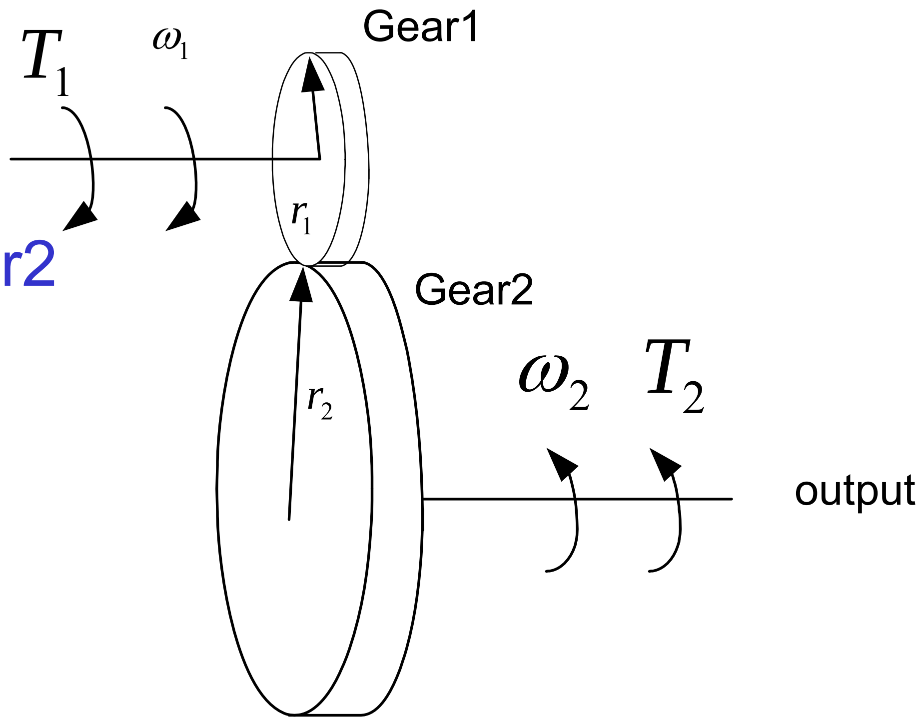
Example: Build the SS Model of DC Servo Motor

$$x_1 = \theta, x_2 = \dot{\theta}, x_3 = \ddot{\theta}$$

Input=Ea, Output=θ



Gear Trains



n1: the number of teeth on Gear1;
n2:the number of teeth on Gear2

r1: the radius of Gear1
r2:the radius of Gear2

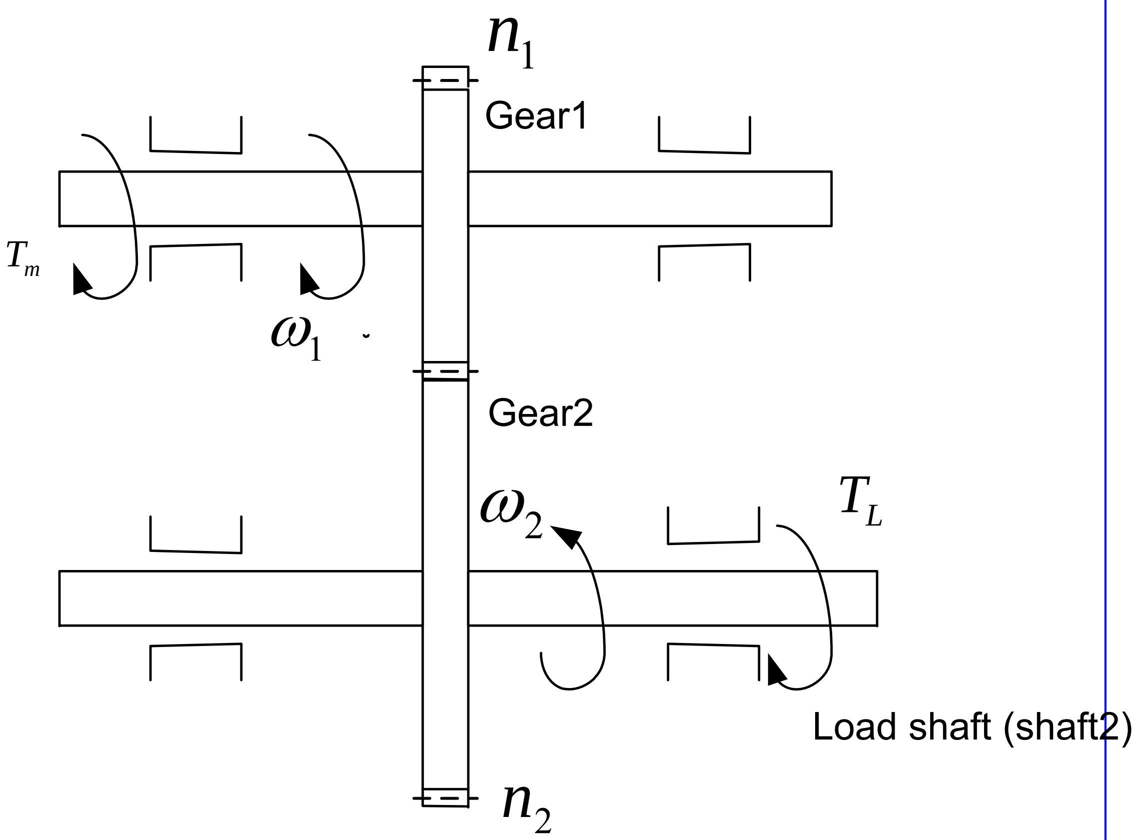
$$\frac{r_1}{r_2} = \frac{n_1}{n_2}$$

$$r_1\omega_1 = r_2\omega_2 \Rightarrow \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

$$T_1\omega_1 = T_2\omega_2$$

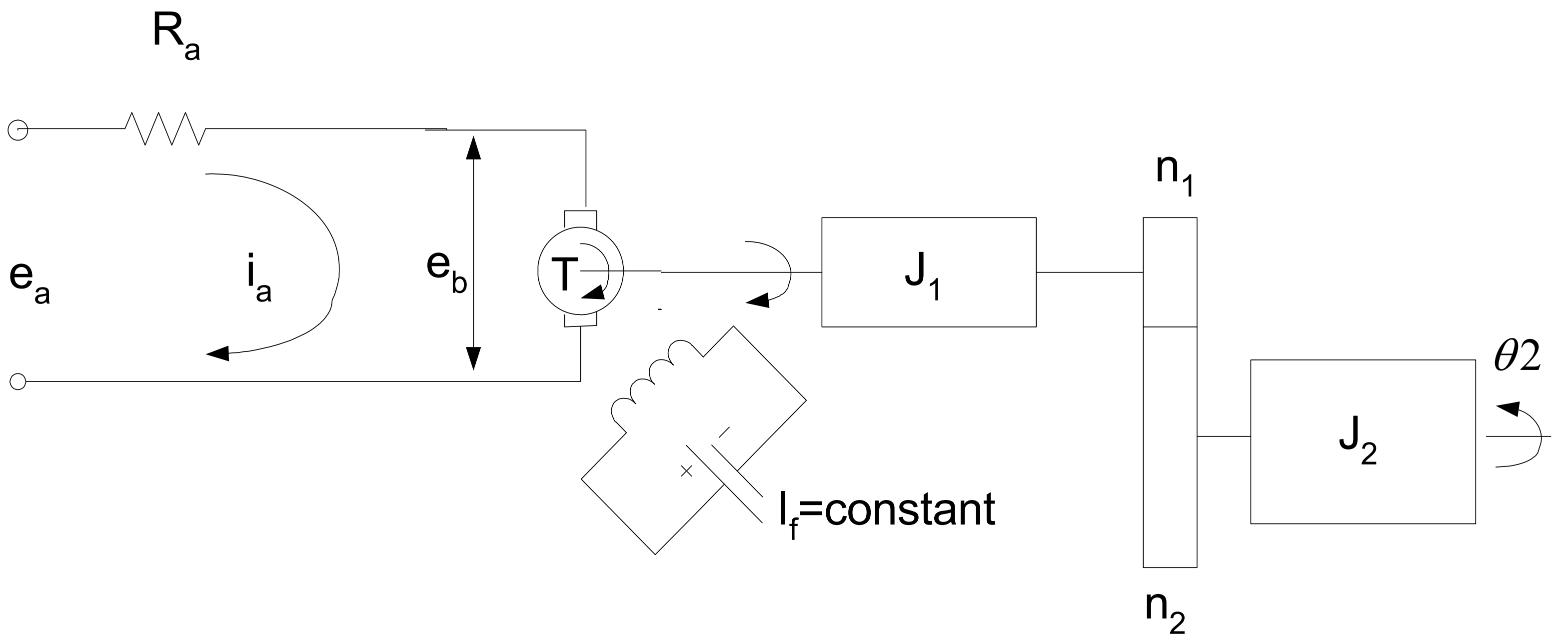
Example

Load is driven by a motor through the gear train.
Assume the stiffness of the gear train is infinite, no backlash and deformation.
Find the equivalent inertia and equivalent friction referred to the motor shaft and those referred to the load shaft.



For motor shaft: J1,b1(including gear1)
For load shaft: J2,b2(including gear2)

Example



Obtain the SS Model for the DC Servo System, output θ_2 and the input e_a.

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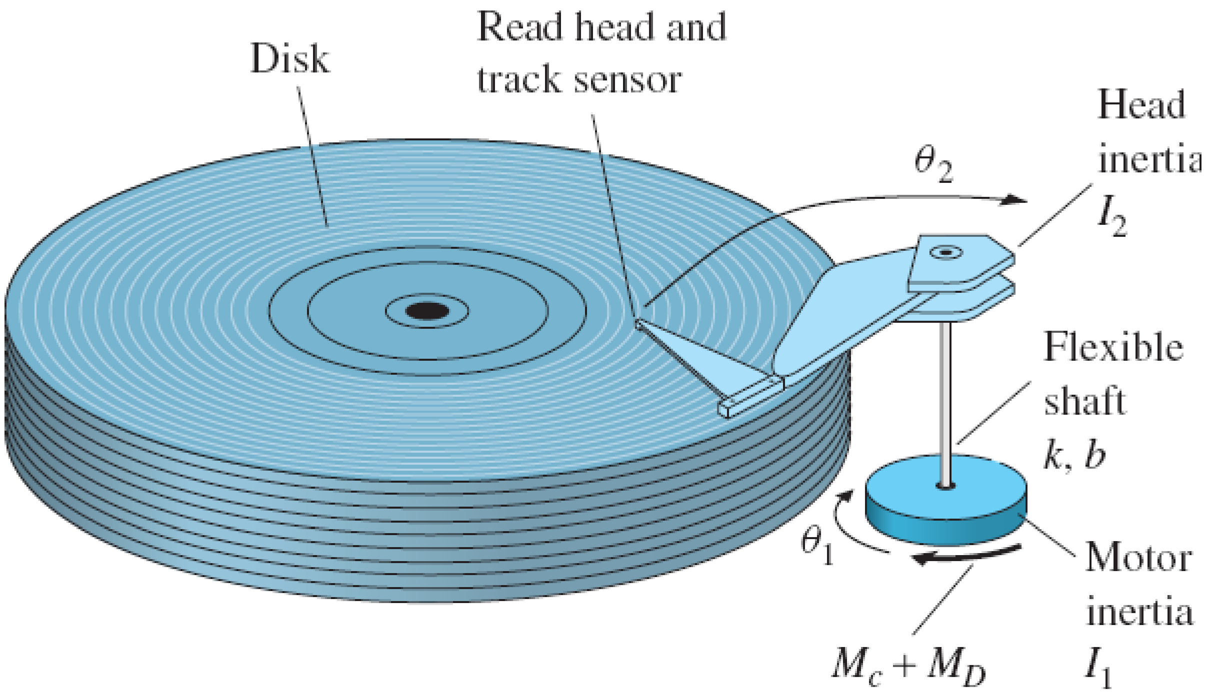
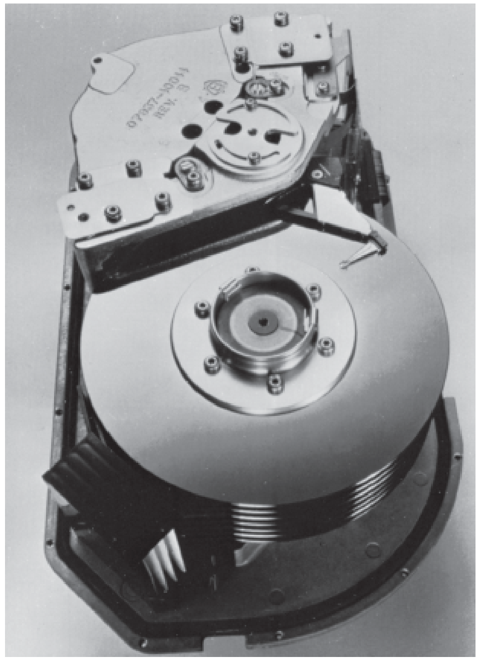
Mechanical System Example 1

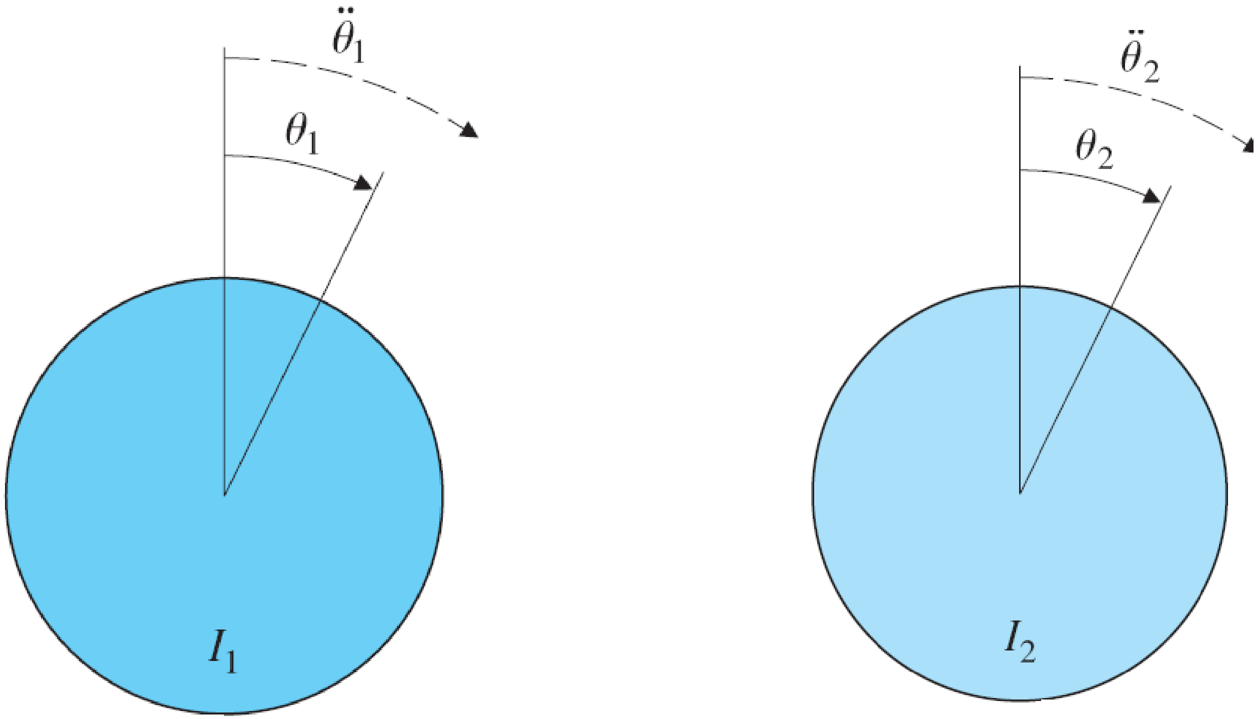
Ex 2.4 (Franklin,5th Edition) Assume that there is some flexibility between the read head and drive motor in figure below, Find the equation of motion relating the motion of the read head to a torque applied to the base.





M_C: applied torque M_D: disturbance torque

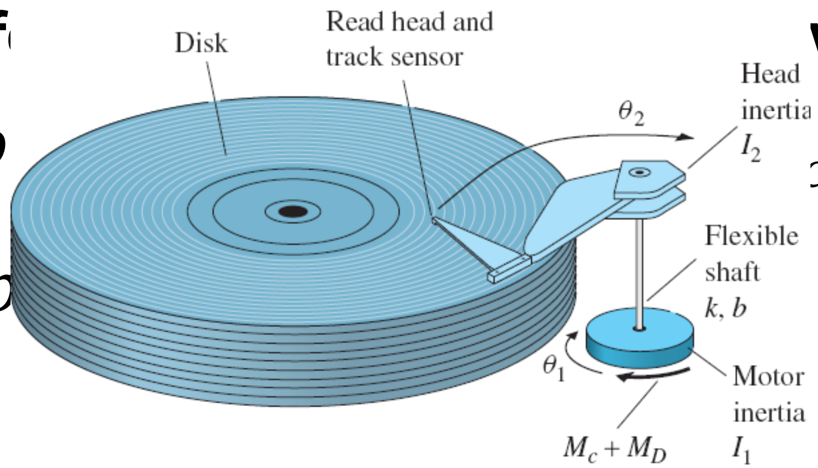
Mechanical System Example 1



Solution: The diff **wo free bodies are:**

$$I_1\ddot{\theta}_1 + b \quad \ldots \quad + M_D$$

$$I_2\ddot{\theta}_2 + b \quad \ldots$$



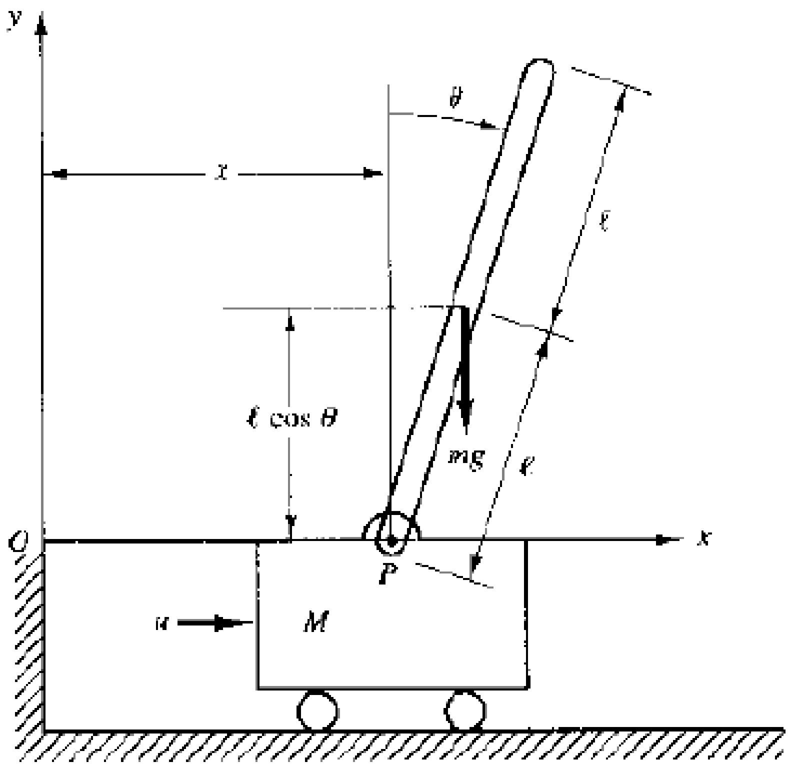
Mechanical System Example 1

$$\frac{\Theta_2(s)}{M_c(s)} = \frac{k}{I_1 I_2 s^2 (s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$$

$$\frac{\Theta_1(s)}{M_c(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2 (s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$$

Mechanical System Example 2

Ex 3.8 (Ogata,4th Edition) An inverted pendulum on a motor-driven cart is shown in figure below. The control force u is applied to the cart, Obtain a mathematical model for the system, The input is U(s) and the output is $\Theta(s)$



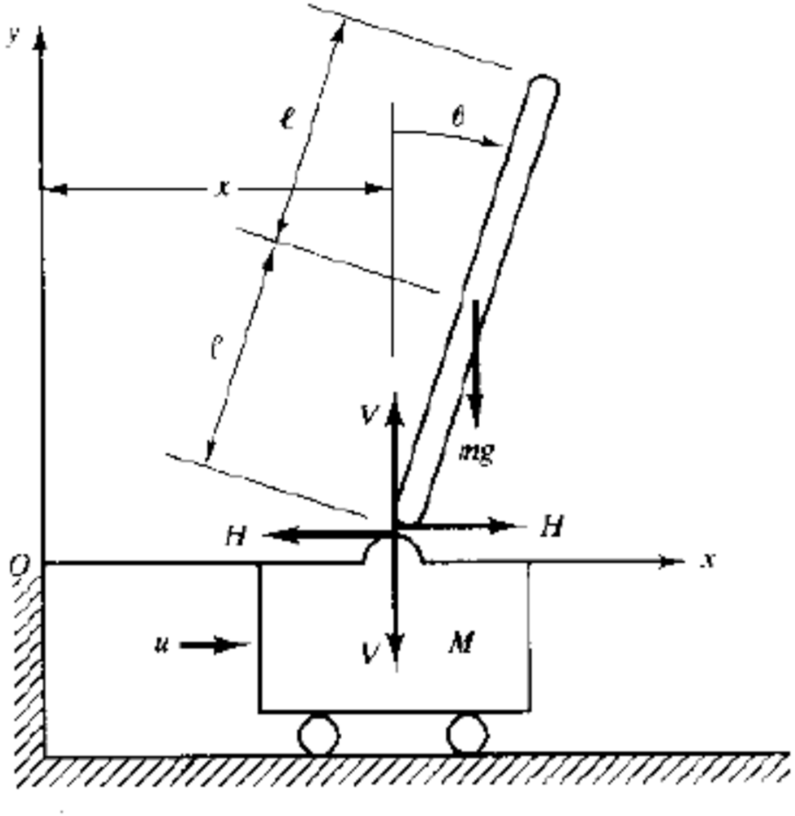
Mechanical System Example 2

Solution:

(1) The coordinates of the center of gravity of the pendulum rod as (x_G, y_G) is:

$$x_G = x + l\sin\theta$$

$$y_G = l\cos\theta$$



(2) The rotation motion of the pendulum rod about its center of gravity is:

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta$$

(3) The horizontal motion of the pendulum rod is :

$$m\frac{d}{dt^2}(x + l\sin\theta) = H \quad \Rightarrow \quad m(\ddot{x} + l\ddot{\theta}) = H$$

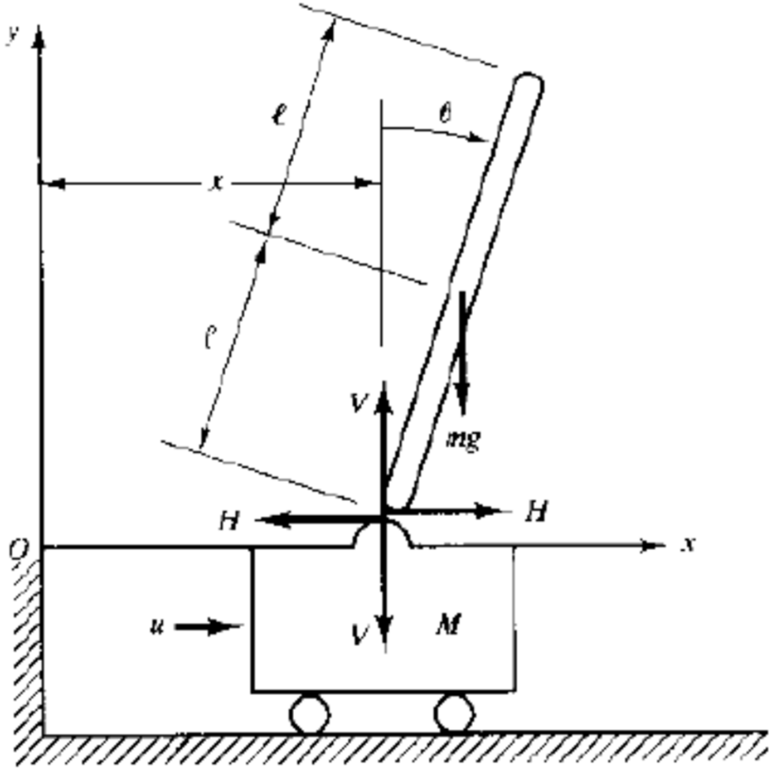
Mechanical System Example 2

(4) The Vertical motion of the center of gravity of pendulum rod is :

$$m\frac{d}{dt^2}(l\cos\theta) = V - mg$$

$$0 = V - mg$$

(5) The horizontal of the cart is:



$$M\frac{d^2x}{dt^2} = u - H$$

$$(3) \quad m(\ddot{x} + l\ddot{\theta}) = H$$

$$(M + m)\ddot{x} + ml\ddot{\theta} = u$$

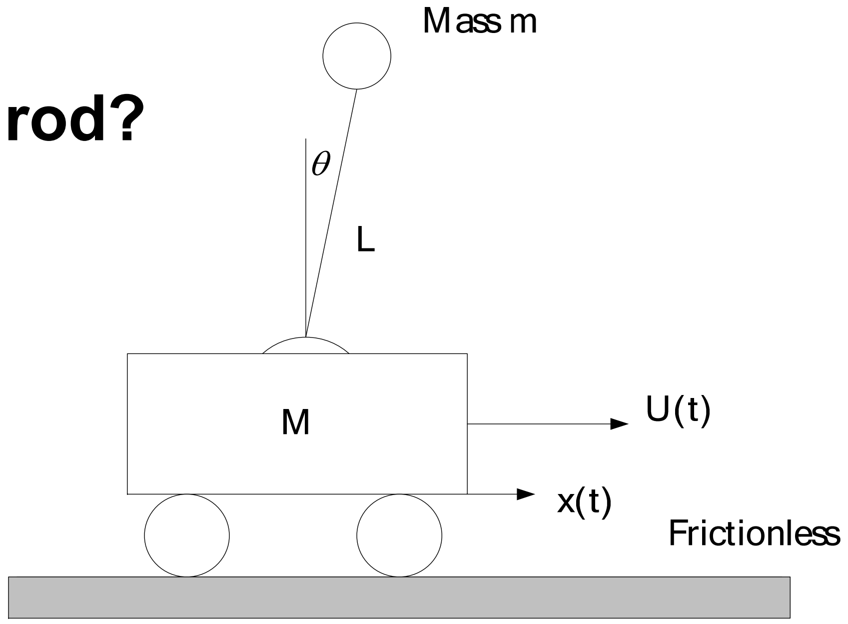
$$(1) \quad I\ddot{\theta} = Vl\theta - Hl$$

$$(4) \quad 0 = V - mg$$

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta$$

Mechanical System Example 3

Problem: If the Mass is concentrated at the top of the rod?



$$(I+ml^2)\ddot{\theta}+ml\ddot{x}=mgl\theta$$

$$ml^2\ddot{\theta}+ml\ddot{x}=mgl\theta$$

$$(M+m)\ddot{x}+ml\ddot{\theta}=u$$

$$M\ddot{x}=u-mg\theta$$

$$ml^2\ddot{\theta}=(M+m)g\theta-u$$

State Space Representation

$$x_1=\theta$$

$$x_2=\dot{\theta}$$

$$x_3=x$$

$$x_4=\dot{x}$$

$$\dot{x}_1=x_2$$

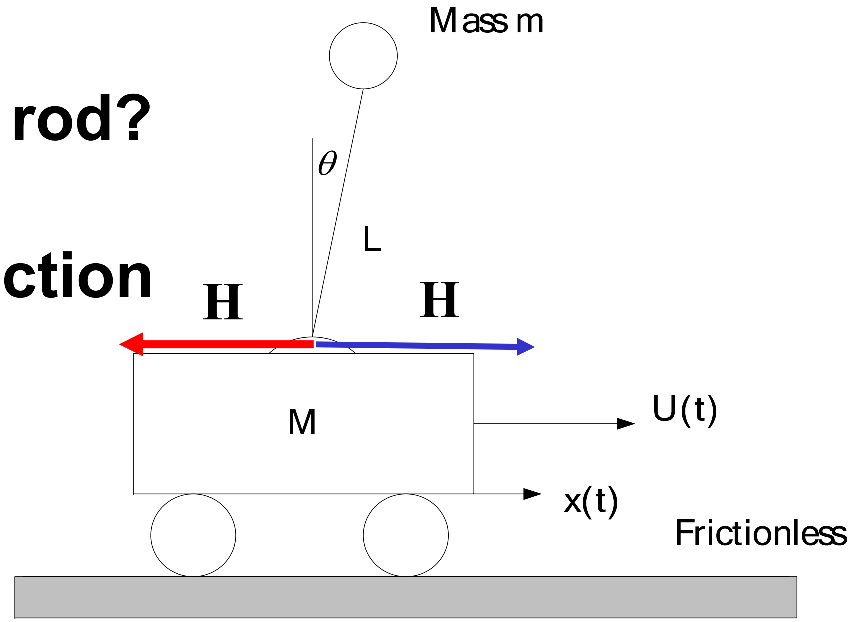
$$\dot{x}_2=\frac{M+m}{Ml}gx_1-\frac{1}{Ml}u$$

$$\dot{x}_3=x_4$$

$$\dot{x}_4=-\frac{m}{M}gx_1+\frac{1}{M}u$$

Mechanical System Example 3

Problem: If the Mass is concentrated at the top of the rod?



For the cart: In horizontal Direction

$$M\ddot{x} = u - H$$

For the Mass m

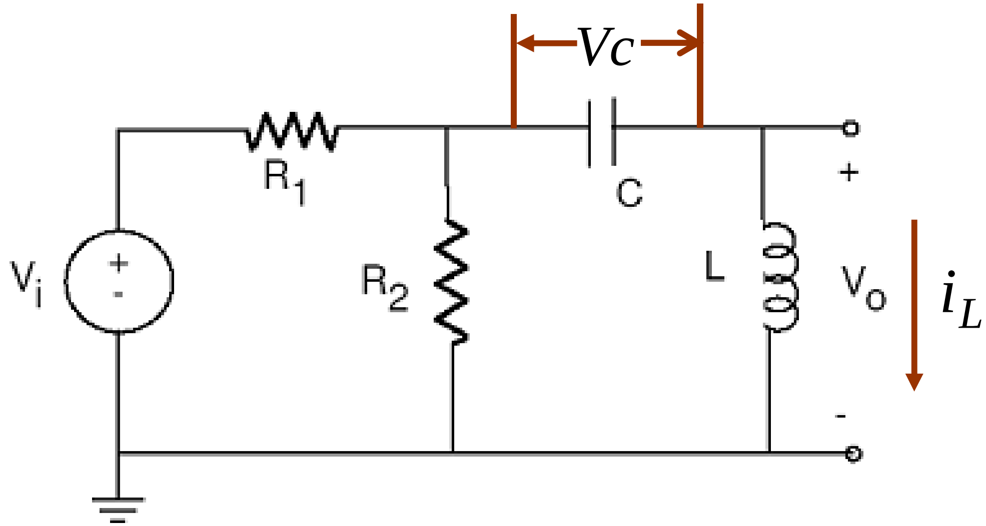
$$m\frac{d}{dt^2}(x + L\sin\theta) = H$$

The torque for the Joint: (Note the direction of the angle)

$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta$$

Electrical System Example 1

Ex. Consider the circuit shown in the figure below:



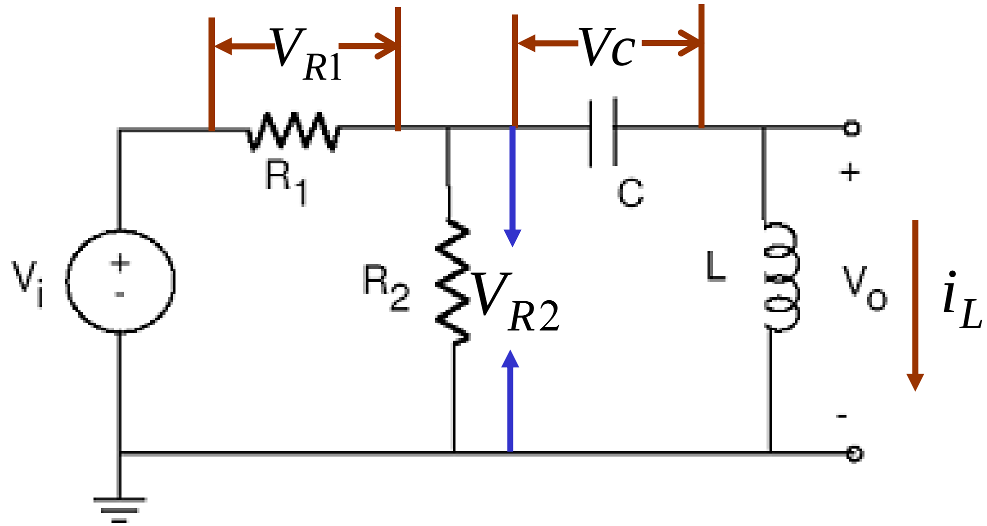
(1) Choose an appropriate set of the state variables, and label them on the circuit.(6%)

$$x_1 = V_c$$
$$x_2 = i_L$$

$$\Rightarrow x_2 = i_L = C\frac{d}{dt}V_c = C\frac{d}{dt}x_1 \Rightarrow \dot{x}_1 = \frac{1}{C}x_2$$

Electrical System Example 1

(2) Write the dynamic equations for the system using the state variables you found in (1). The input is Vi and the output is Vo. (6%)



$$V_{R2} = Vc + V_o = x_1 + L\frac{d}{dt}x_2$$

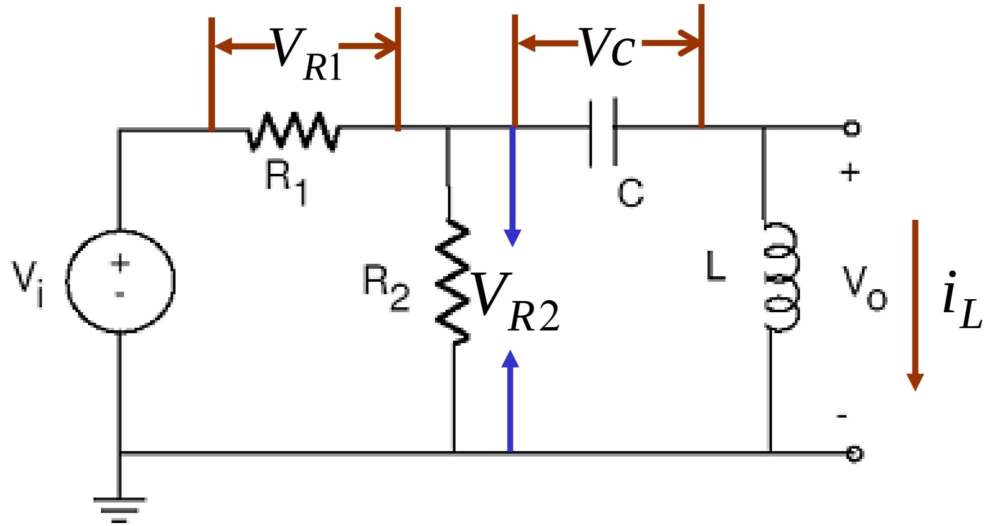
$$V_{R1} = R_1(\frac{V_{R2}}{R_2} + x_2) = R_1(\frac{x_1 + L\dot{x}_2}{R_2} + x_2)$$

$$V_i = R_1(\frac{x_1 + L\dot{x}_2}{R_2} + x_2) + x_1 + L\dot{x}_2$$

$$\dot{x}_2 = -\frac{1}{L}x_1 - \frac{R_1 R_2}{L(R_1 + R_2)}x_2 + \frac{R_2}{L(R_1 + R_2)}V_i$$

Electrical System Example 1

(3) Put the equation from part (2) into state-space form, The input is Vi and the output is Vo.(8%)



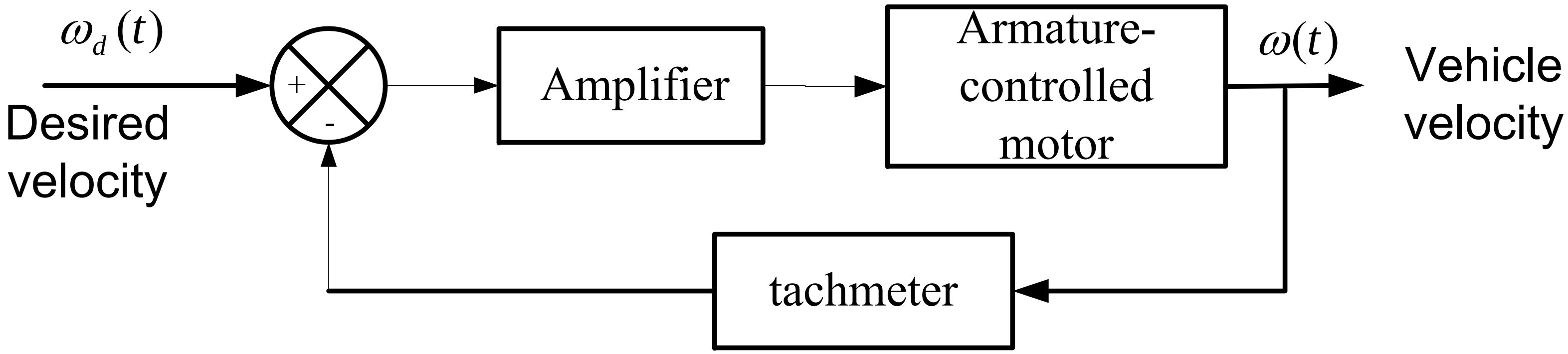
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & \dfrac{1}{C}\\ -\dfrac{1}{L} & -\dfrac{R_1R_2}{L(R_1+R_2)}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{R_2}{L(R_1+R_2)}\end{bmatrix}V_i$$

$$y=V_o=L\dot{x}_2= \quad -x_1-\frac{R_1R_2}{(R_1+R_2)}x_2+\frac{R_2}{(R_1+R_2)}V_i$$

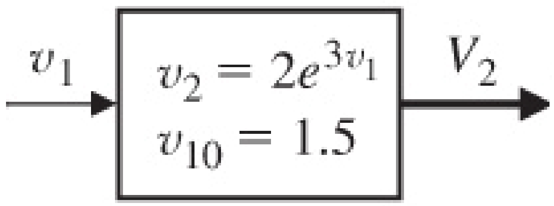
$$=\begin{bmatrix}-1 & \dfrac{R_1R_2}{(R_1+R_2)}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\frac{R_2}{(R_1+R_2)}V_i$$

Electrical System Example 2

Ex. The electrical motor drive for a railway vehicle is shown in in block diagram form in figure below, incorporating the necessary control of the velocity of the vehicle.



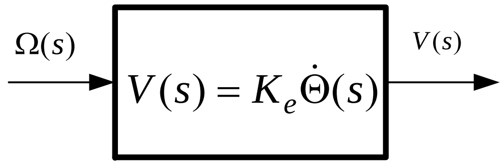
Problem1:Suppose the amplify is nonlinear and the operating point is V_{10}=1.5V, the block diagram of the amplify is shown in below:



Find the linear representation of δV_2 and δV_1

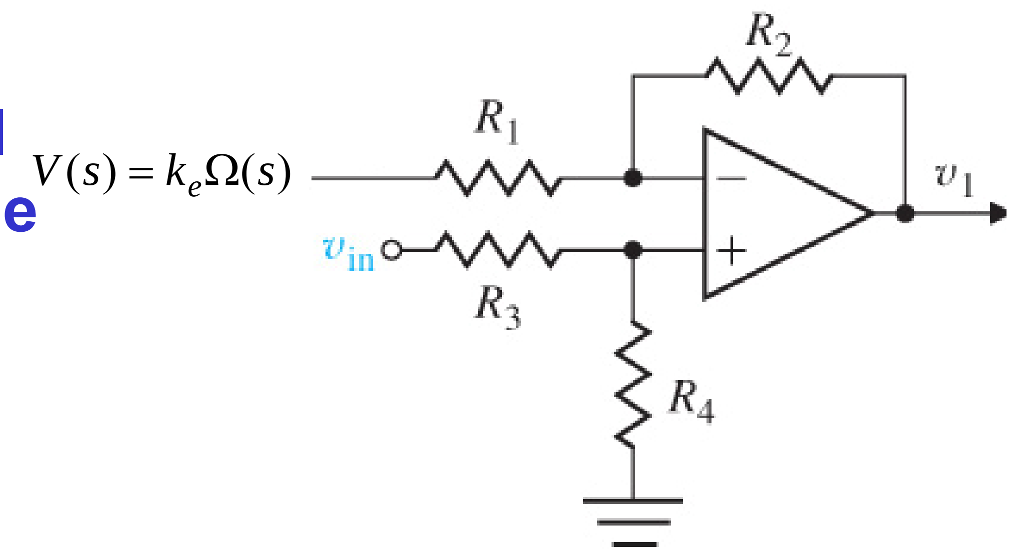
Electrical System Example 2

Problem2: Suppose we use a tachmeter to measure the velocity of the vehicle, and the block diagram for the tachmeter is:



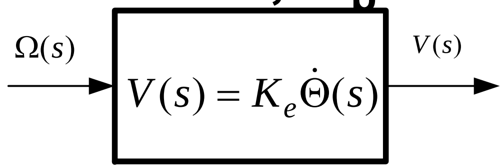
Where K_e=0.1,suppose we use the Op-Amps shown in below to realize the the function of comparator, select appropriate resistors R_1,R_2,R_3,R_4.(give one numerical design)

(Hints:We wish to obtain an input control sets $\omega_d(t)=V_{in}$ where the units of $\omega_d(t)$ are rad/s and the units of V_{in} are volts.)



Electrical System Example 2

Problem3: Suppose the parameters of a large DC motor is : K_t=10 Nm/A, Ra=1,La=1H,J=2 Nm/rad/s^2 ,b=0.5 Nm/rad/s, K_b=0.1 V/rad/s

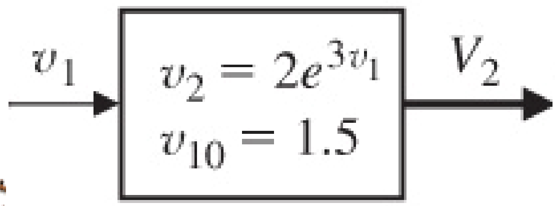


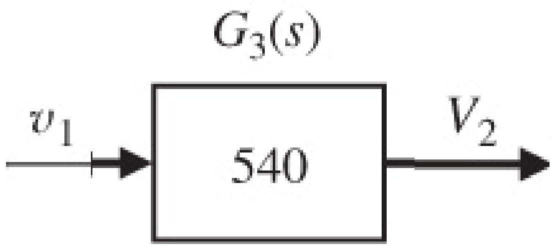
Write the complete block diagram of the system and derive the TF of ω (s)/V_{in}(s)

Solution1: for amplify function:

$$\delta v_2 = \frac{d}{dv_1}(2e^{3v_1})\Big|_{v_1=1.5}\delta v_1$$

$$= 2\times 3\times e^{3v_1}\Big|_{v_1=1.5}\delta v_1 = 540.10\delta v_1 \approx 540\delta v_1$$

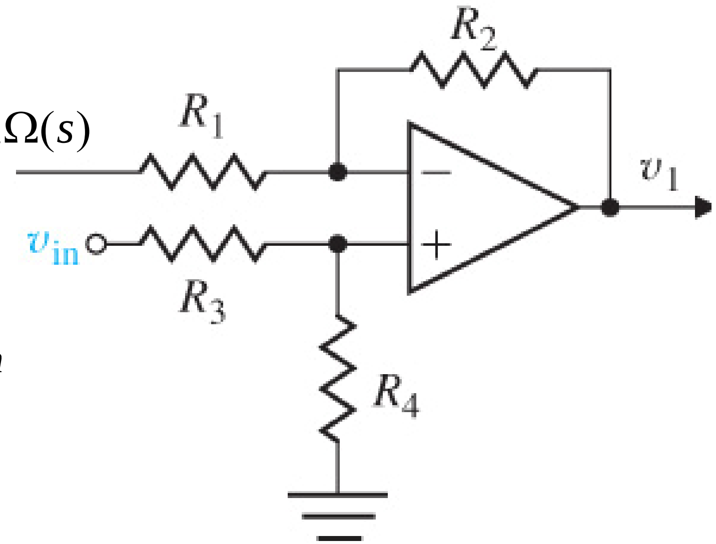




Electrical System Example 2

Solution 2: for the Op-Amps, we have:

$$V(s) = k_e \Omega(s)$$



$$v_1 = \frac{1+\dfrac{R_2}{R_1}}{1+\dfrac{R_3}{R_4}} v_{in} - \frac{R_2}{R_1} K_e \omega = \frac{1+\dfrac{R_2}{R_1}}{1+\dfrac{R_3}{R_4}} v_{in} - \frac{R_2}{R_1} K_e v_{in}$$

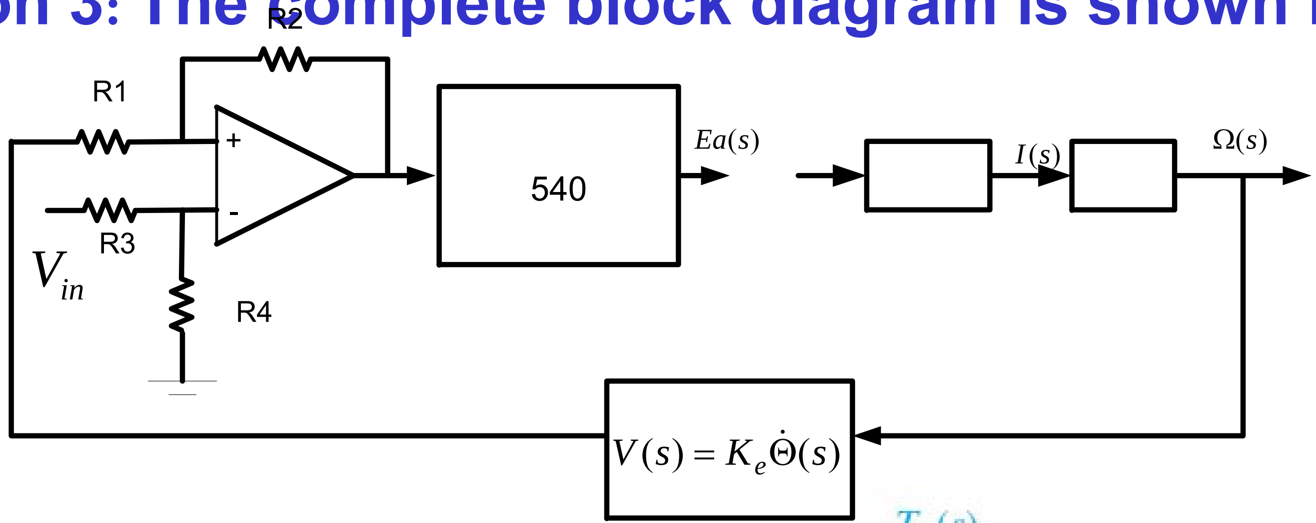
When the system in balance, v_1=0, k_e=0.1

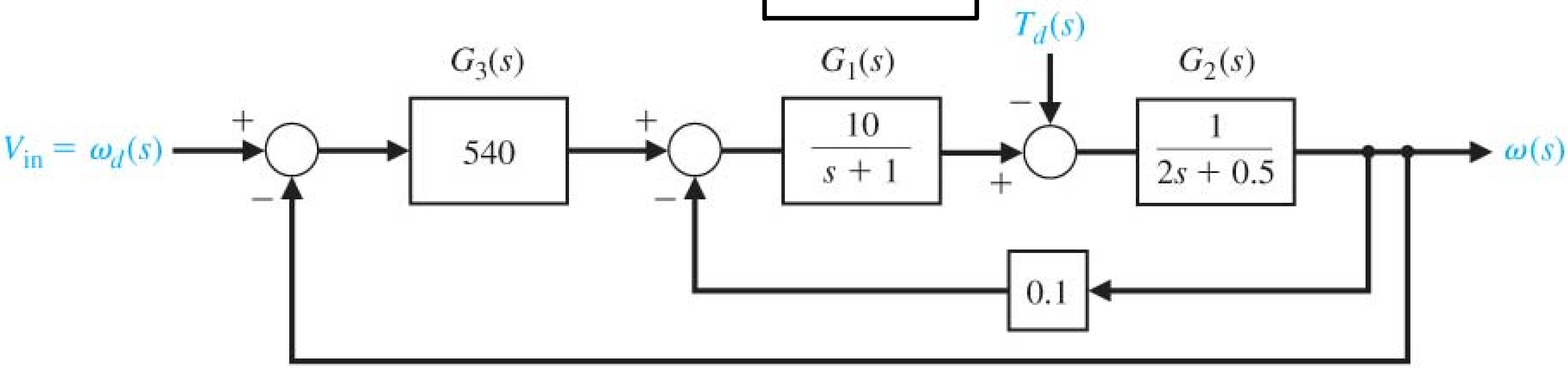
$$\frac{1+\dfrac{R_2}{R_1}}{1+\dfrac{R_3}{R_4}} = \frac{R_2}{R_1} 0.1$$

if we R2/R1=10,R3/R4=10, it's one solution.

Electrical System Example 2

Solution 3: The complete block diagram is shown in below:





$$\frac{\omega(s)}{\omega_d(s)} = \frac{2700}{s^2 + 1.25s + 2700.75}$$

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- SS Model of DC Servo Motor
- SS Model of Complex Physical System
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From TF to SS– Canonical Forms

- Since the SS representation of a system is not unique, given the TF of a discrete-time system, there can be infinite number of state space representations. In the following discussion, we will introduce one canonical representation that is often used.

- The transfer function considered is assumed to be

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

Controllable Canonical Form

Let

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_n}\cdot\boxed{\frac{X_1(s)}{X_1(s)}} \qquad \textbf{(*)}$$

$$Y(s)=\left(b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n\right)X_1(s)$$

$$U(s)=\left(s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_n\right)X_1(s)$$

Choose

$$\boxed{\begin{array}{c} sX_1(s)=X_2(s)\rightarrow \dot{x}_1(t)=x_2(t) \\ sX_2(s)=X_3(s)\rightarrow \dot{x}_2(t)=x_3(t) \\ sX_3(s)=X_4(s)\rightarrow \dot{x}_3(t)=x_4(t) \\ \vdots \\ sX_{n-1}(s)=X_n(s)\rightarrow \dot{x}_{n-1}(t)=x_n(t) \end{array}}$$

$$s^nX_1=\left(-a_1s^{n-1}-a_2s^{n-2}-\cdots-a_n\right)X_1+U(s)$$

$$\Rightarrow\ \boxed{\dot{x}_n(t)=-a_1x_n(t)-a_2x_{n-1}(t)-\cdots-a_nx_1(t)+u(t)}$$

Controllable Canonical Form (cont.)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_n}\cdot\boxed{\frac{X_1(s)}{X_1(s)}}$$

$$Y(s)=\left(b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n\right)X_1(s)$$

$$=b_0\left\{\left(-a_1s^{n-1}-a_2s^{n-2}-\cdots-a_n\right)X_1(s)+U(s)\right\}+\left(b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n\right)X_1(s)$$

$$=\left(b_n-a_nb_0\right)X_1(s)+\left(b_{n-1}-a_{n-1}b_0\right)X_2(s)+\cdots+\left(b_1-a_1b_0\right)X_n(s)+b_0U(s)$$

$$\Rightarrow \quad \boxed{y(t)=\left(b_n-a_nb_0\right)x_1(t)+\left(b_{n-1}-a_{n-1}b_0\right)x_2(t)+\cdots+\left(b_1-a_1b_0\right)x_n(t)+b_0u(t)}$$

Controllable Canonical Form (cont.)

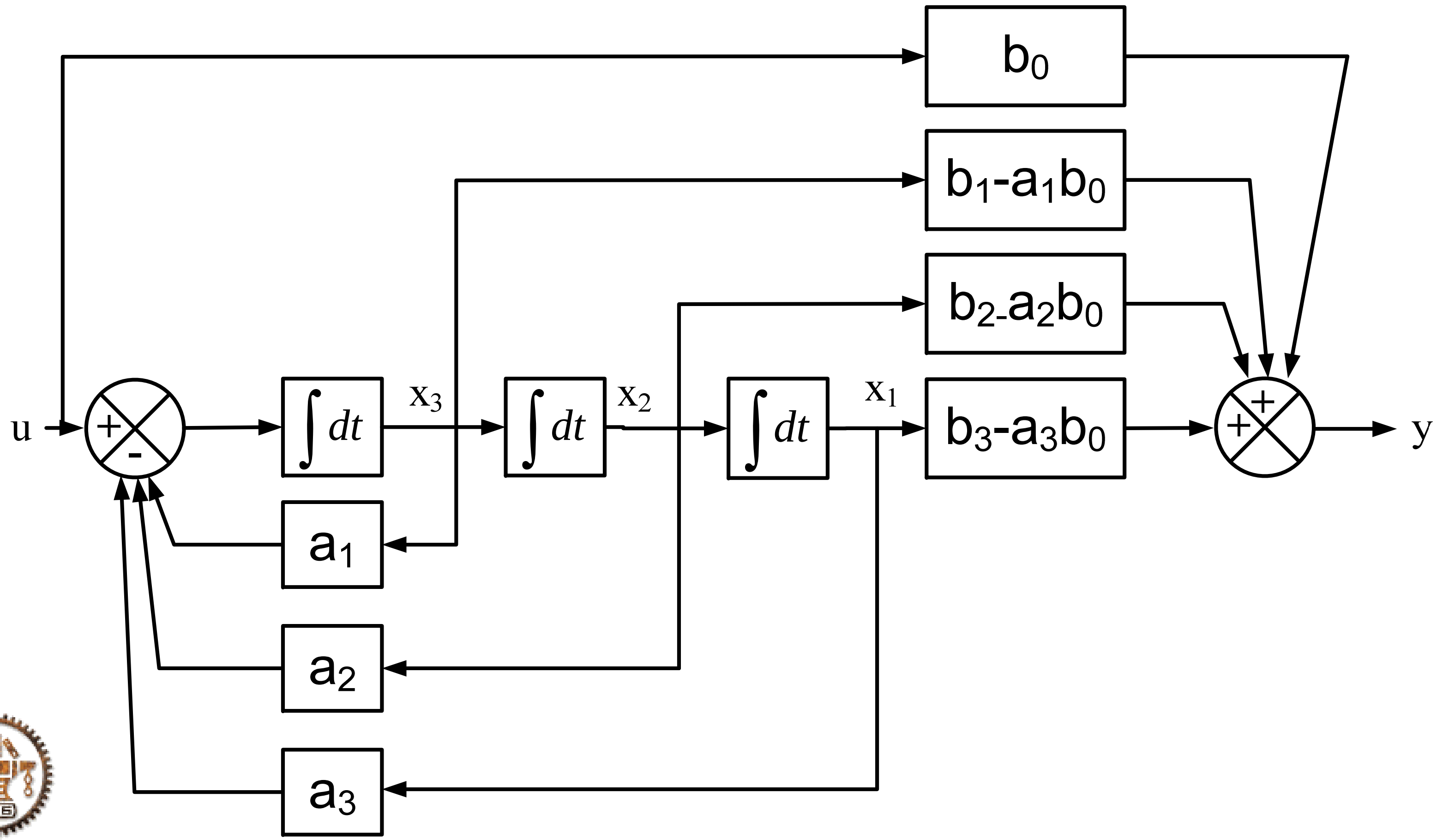
The state space realization in *controllable canonical form*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \cdots & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + b_0 u(t)$$

The Block Diagram of Controllable Canonical Form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0 s^3+b_1 s^2+b_2 s+b_3}{s^3+a_1 s^2+a_2 s+a_3}$$



Observable Canonical Form

The state space realization in *Observable canonical form*

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_n}$$

$$Y(s)=b_0U(s)+\frac{1}{s}[b_1U(s)-a_1Y(s)]+\cdots$$

$$+\frac{1}{s^{n-1}}[b_{n-1}U(s)-a_{n-1}Y(s)]+\frac{1}{s^n}[b_nU(s)-a_nY(s)]$$

Suppose

$$\begin{cases} X_1(s)=\frac{1}{s}[b_nU(s)-a_nY(s)] \\ X_2(s)=\frac{1}{s}[b_{n-1}U(s)-a_{n-1}Y(s)+X_1(s)] \\ \cdots\cdots \\ X_{n-1}(s)=\frac{1}{s}[b_2U(s)-a_2Y(s)+X_{n-2}(s)] \\ X_n(s)=\frac{1}{s}[b_1U(s)-a_1Y(s)+X_{n-1}(s)] \end{cases} \Rightarrow Y(s)=b_0U(s)+X_n(s)$$

Observable Canonical Form (Cont.)

$$
\begin{cases}
X_1(s) = \dfrac{1}{s}[b_n U(s) - a_n Y(s)] \\
X_2(s) = \dfrac{1}{s}[b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)] \\
\cdots\cdots \\
X_{n-1}(s) = \dfrac{1}{s}[b_2 U(s) - a_2 Y(s) + X_{n-2}(s)] \\
X_n(s) = \dfrac{1}{s}[b_1 U(s) - a_1 Y(s) + X_{n-1}(s)] \\
Y(s) = b_0 U(s) + X_n(s)
\end{cases}
$$

$$sX_1(s) = [b_n U(s) - a_n b_0 U(s) - a_n X_n(s)] \Rightarrow \dot{x}_1 = -a_n x_n + (b_n - a_n b_0)u$$

$$sX_2(s) = [b_{n-1} U(s) - a_{n-1} b_0 U(s) - a_{n-1} X_n(s) + X_1(s)] \Rightarrow \dot{x}_2 = x_1 - a_{n-1} x_n + (b_{n-1} - a_{n-1} b_0)u$$

$$sX_n(s) = [b_1 U(s) - a_1 b_0 U(s) - a_1 X_n(s) + X_{n-1}(s)] \Rightarrow \dot{x}_n = x_{n-1} - a_1 x_n + (b_1 - a_1 b_0)u$$

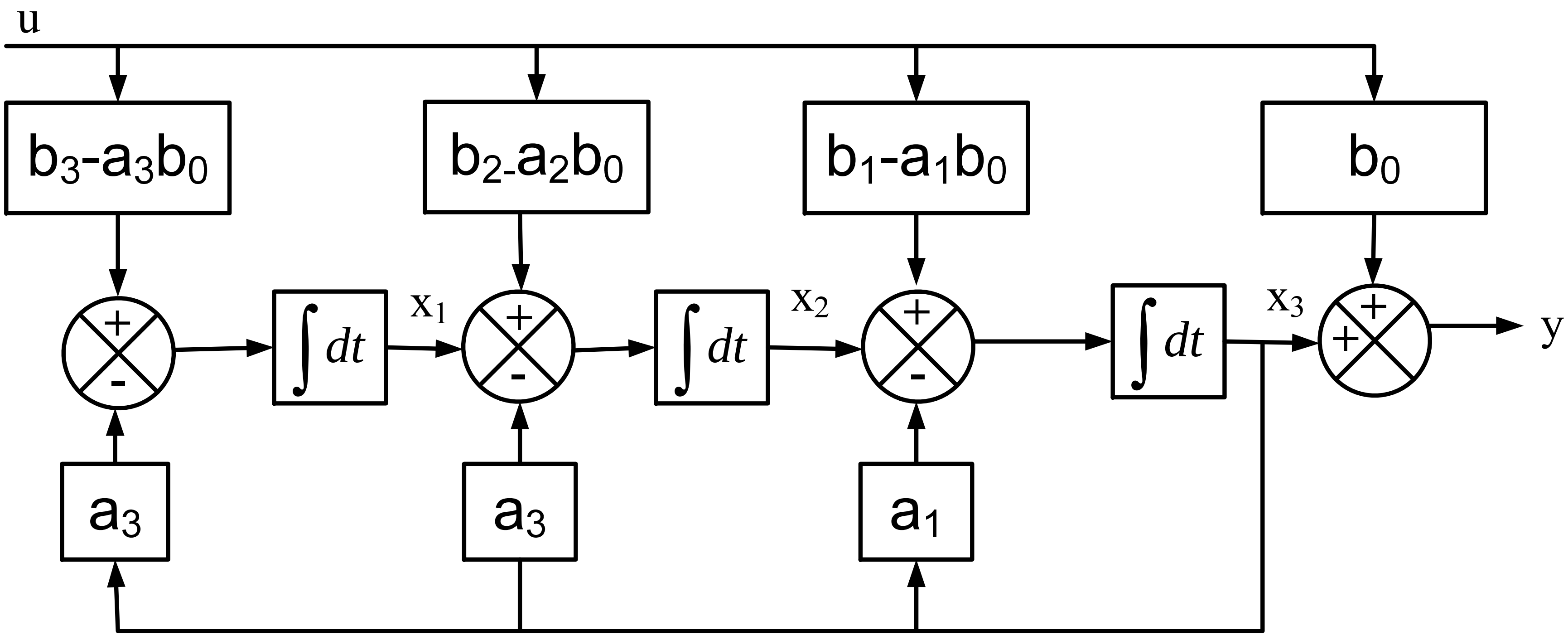
Observable Canonical Form (Cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & . & . & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

The Block Diagram of Observable Canonical Form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^3+b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$



Diagonal Canonical Form

The state space realization in *diagonal canonical form*

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{(s + p_1)(s + p_2)\ldots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \ldots + \frac{c_n}{s + p_n}$$

$$c_i = (s + p_i) G(s) \Big|_{s = -p_i}$$

Diagonal Canonical Form (Cont.)

$$Y(s) = b_0 + \frac{c_1}{s+p_1}U(s) + \frac{c_2}{s+p_2}U(s) + \ldots + \frac{c_n}{s+p_n}U(s)$$

Suppose

$$\begin{cases} \dfrac{1}{s+p_1} = x_1 \longrightarrow \dot{x}_1 = 1 - p_1 x_1 \\ \dfrac{1}{s+p_2} = x_2 \longrightarrow \dot{x}_2 = 1 - p_2 x_2 \\ \cdots\cdots \\ \dfrac{1}{s+p_n} = x_n \longrightarrow \dot{x}_n = 1 - p_n x_n \end{cases} \Rightarrow$$

$$y = b_0 + c_1 x_1 + c_2 x_2 + \ldots + c_n x_n(s)$$

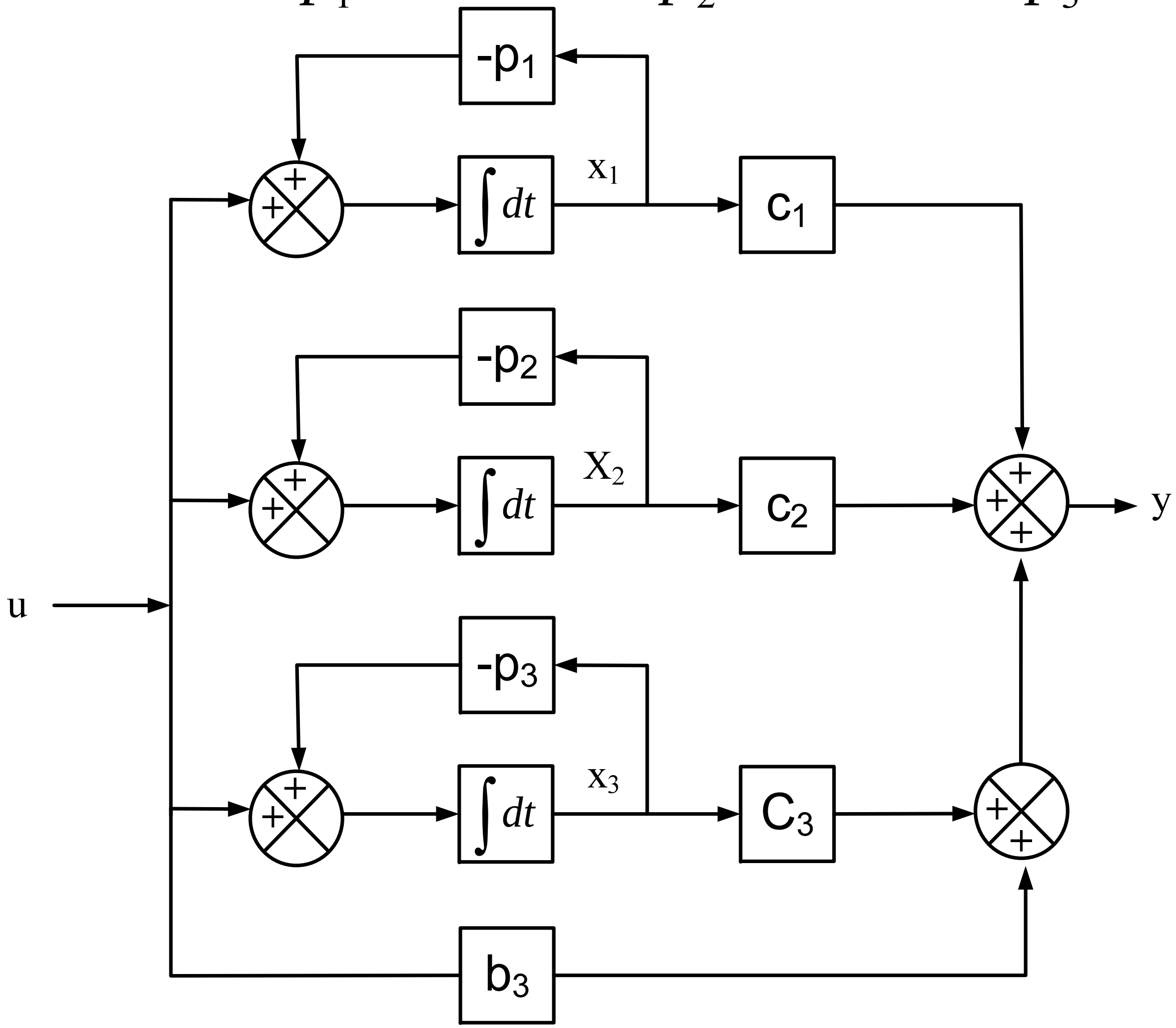
Diagonal Canonical Form (Cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & \dots & 0 & 0 \\ 1 & -p_2 & \dots & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \dots & -p_{n-1} & 0 \\ 0 & 0 & \dots & 0 & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & . & . & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

The Block Diagram of Diagonal Canonical Form

$$Y(s) = b_0 + \frac{c_1}{s+p_1}U(s) + \frac{c_2}{s+p_2}U(s) + \frac{c_3}{s+p_3}U(s)$$



Jordan Canonical Form

The state space realization in *Jordan canonical form*

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{(s+p_1)^3 (s+p_4)(s+p_5)\ldots(s+p_n)}$$

$$= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \ldots + \frac{c_n}{s+p_n}$$

$$c_1 = (s+p_1)^3 G(s)\Big|_{s=-p_1} \qquad c_2 = \frac{d}{ds}(s+p_1)^3 G(s)\Big|_{s=-p_1}$$

$$c_3 = \frac{1}{2!}\frac{d^2}{ds^2}(s+p_1)^3 G(s)\Big|_{s=-p_1}$$

Diagonal Canonical Form (Cont.)

$$Y(s)=b_0+\frac{c_1}{(s+p_1)^3}U(s)+\frac{c_2}{(s+p_1)^2}U(s)+\frac{c_3}{(s+p_1)}U(s)$$

$$+\frac{c_4}{s+p_4}U(s)\ldots+\frac{c_n}{s+p_n}U(s)$$

Suppose

$$\begin{cases} \dfrac{U(s)}{s+p_1}=x_3 \longrightarrow \dot{x}_3=u-p_1x_3 \\ \dfrac{U(s)}{(s+p_1)^2}=\dfrac{1}{(s+p_1)}x_3=x_2 \longrightarrow \dot{x}_2=x_3-p_1x_2 \\ \dfrac{U(s)}{(s+p_1)^3}=\dfrac{1}{(s+p_1)}x_2=x_1 \longrightarrow \dot{x}_1=x_2-p_1x_1 \\ \cdots\cdots \\ \dfrac{1}{s+p_n}=x_2 \longrightarrow \dot{x}_n=1-p_nx_n \end{cases}$$

Jordan Canonical Form (Cont.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & . & 0 \\ 0 & -p_1 & 1 & 0 & . & \\ 0 & 0 & -p_1 & 0 & . & \\ 0 & 0 & 0 & -p_4 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & . & . & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

The Block Diagram of Diagonal Canonical Form

$$Y(s) = b_0 + \frac{c_1}{(s+p_1)^3}U(s) + \frac{c_2}{(s+p_1)^2}U(s) + \frac{c_3}{(s+p_1)}U(s)$$

$$+\frac{c_4}{s+p_4}U(s)$$