第四讲

小波变换

Wavelet Transform

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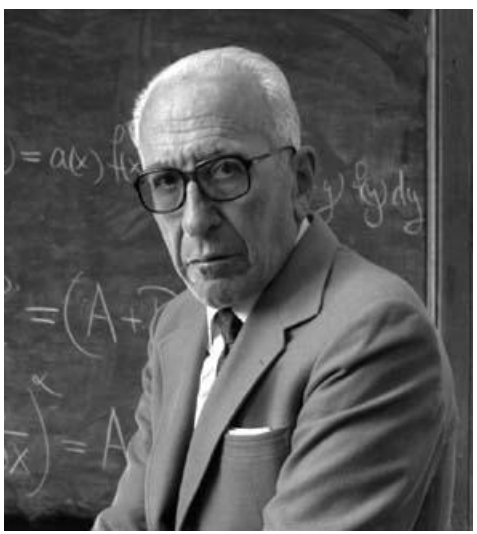
小波变换简史

Jean Morelet
Geophysical Engineer
Elf-Aquitaine Company

- Late 1970s, Morlet problem:
  - Time - frequency analysis of signals with high frequency components for short time spans and low frequency components with long time spans
  - STFT can do one or the other, but not both
    Solution: Use different windowing functions for sections of the signal with different frequency content
  - Windows to be generated from dilation / compression of prototype small, oscillatory signals $\rightarrow$ wavelets
- Criticism for lack of mathematical rigor.

小波变换简史

- Early 1980s, Alex Grossman (theoretical physicist):
  - Recognized the similarity of the Morelet Wavelet to coherent states formalism in quantum mechanics
  - Formalize the transform and devise the inverse transformation → First wavelet transform !



- Rediscovery of Alberto Calderon's 1964 work on harmonic analysis

  Alberto Pedro Calderón (1920 -1998) was one of 20th century leading mathematicians.

小波变换简史

- 1984, Yeves Meyer :
  - Similarity between Morlet and Colderon work, 1984
  - Redundancy in Morlet choice of basis functions
  - 1985, with Lemarie → Orthogonal wavelet basis functions with better time and frequency localization

- Rediscovery of J.O. Stromberg 1980 work the same basis functions (also a harmonic analyst)
- Yet re-rediscovery of Alfred Haar work on orthogonal basis functions, 1909.
  - Simplest known orthonormal wavelets

小波变换简史

Stephane Mallat:

- Multiresolution analysis / Meyer, 1986
  Ph.D. dissertation, 1988
- Discrete wavelet transform
- Cascade algorithm for computing DWT
- Decomposition of a discrete into dyadic frequencies (MRA) , known to EEs under the name of "Quadrature Mirror Filters" , Croisier, Esteban and Galand, 1976
- Matching Pursuit: Using a library of basis functions for decomposition

小波变换简史

Ingrid Daubechies:

- Discretization of time and scale parameters of the wavelet transform
- Wavelet frames, 1986 / Duffin and Schaeffer (1952) – Idea of frame
- Orthonormal bases of compactly supported wavelets (Daubechies wavelets), 1988
- Liberty in the choice of basis functions at the expense of redundancy

G. Battle (1987) and Lamarie (1988)

independently proposed the construction of spline orthogonal wavelet with expoenetial decay

小波变换简史

A. Cohen (1992):

Equivalence of QMF and MRA

Bi-orthogonal wavelet

C.K. Chui and Wang (1991, 1992):

Compactly supported spline wavelets

semi-orthogonal wavelet

Coifman, Meyer and Wickerhauser (1996):

Wavelet packet transform

小波变换简史

D.G Han and D.R Larson (1991):

Super- wavelet

"Frames, bases, and group representations." American Mathematical Soc., 697(2000).

W. Sweldens (1995):

Second generation wavelet

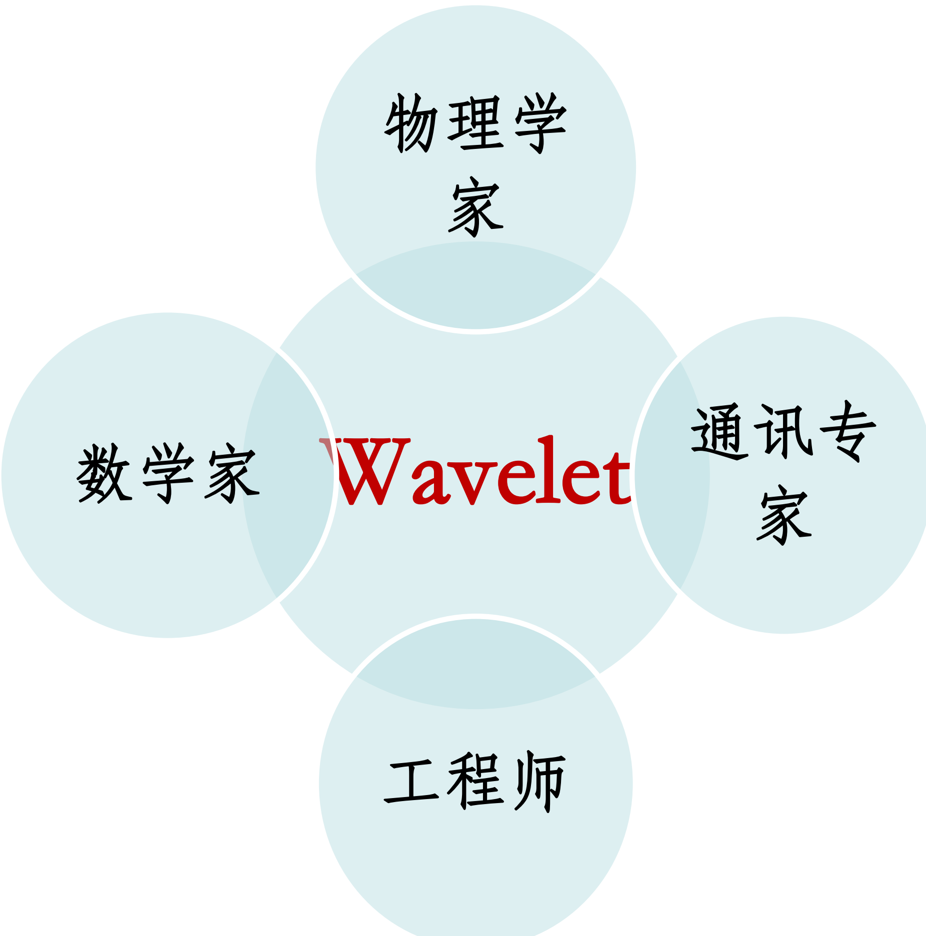
"The lifting scheme: A custom-design construction of biorthogonal wavelets." Applied and computational harmonic analysis 3.2 (1996): 186-200.

小波变换简史

Martin Vetterli & Jelena Kovacevic

- Wavelets and filter banks, 1986
- Perfect reconstruction of signals using FIR filter banks, 1988
- Subband coding
- Multidimensional filter banks, 1992

小波变换简史

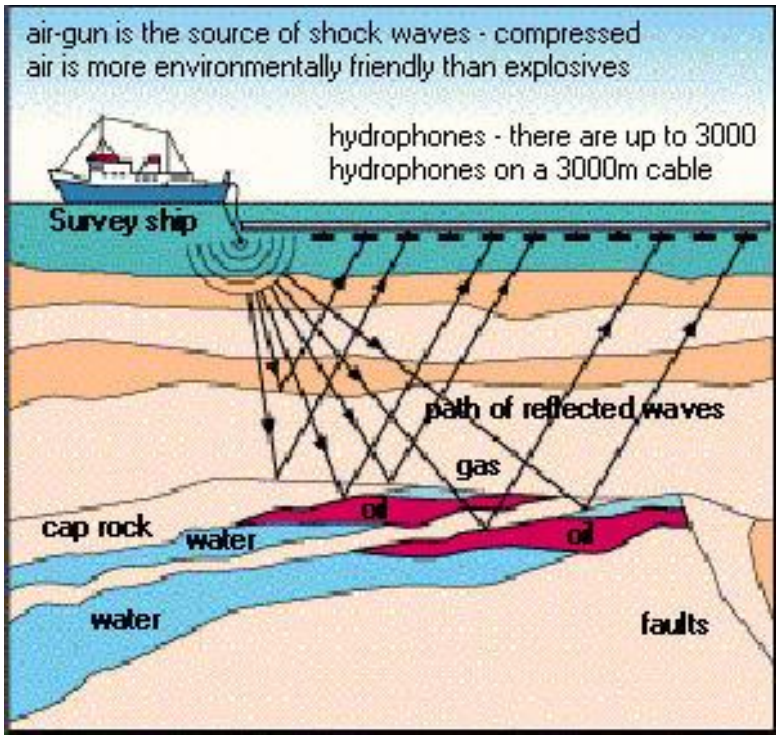


"Jean Morlet launched a scientific program which already offered fruitful alternatives to Fourier analysis and is now moving beyond wavelets."

-Yeves Meyer： A TRIBUTE TO JEAN MORLET

Wavelet Transform

- 石油勘探-地震波



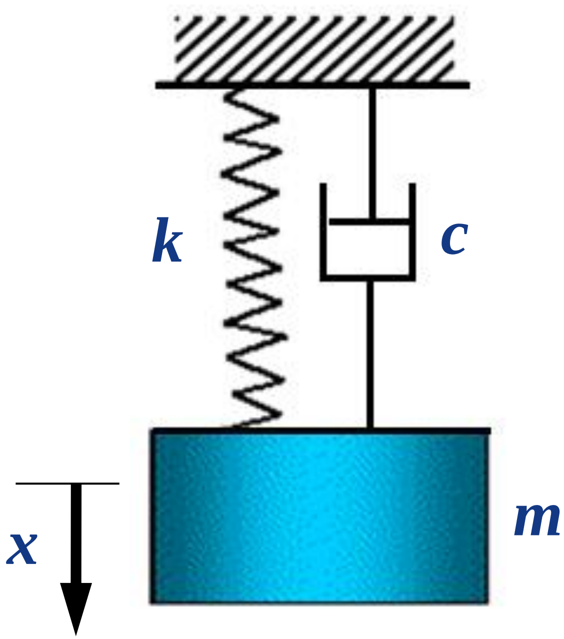
- ➢Compressed-air gun
- ➢Thumper truck
- ➢Explosives

➢高频分量持续时间短
➢低频分量持续时间长

高频分量衰减快

Wavelet Transform

- 自由振动信号



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\omega_n = \sqrt{k/m} \qquad \xi = \frac{c}{2\sqrt{km}}$$

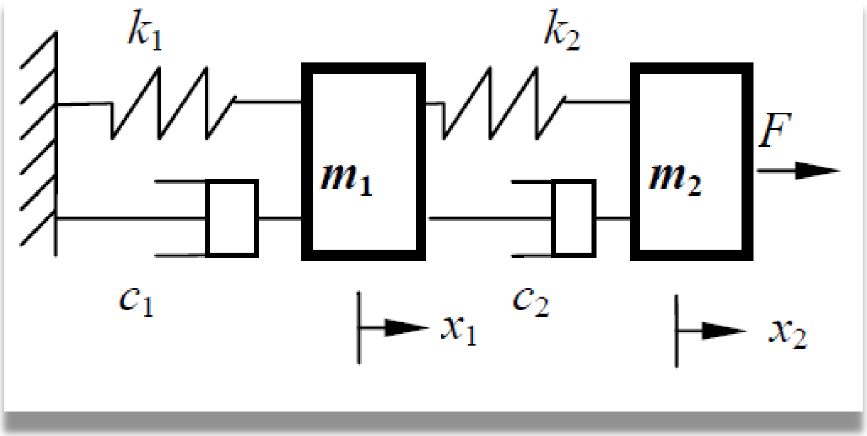
$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

高频分量
衰减快

$$x(t) = \mathrm{e}^{-\xi\omega_n t}\left(B_1\cos\left(\omega_d t\right) + B_2\sin\left(\omega_d t\right)\right)$$

Wavelet Transform

- 自由振动信号



$$m_1 = m_2 = 1$$

$$k_1 = k_2 = 3.9748 \times 10^3$$

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

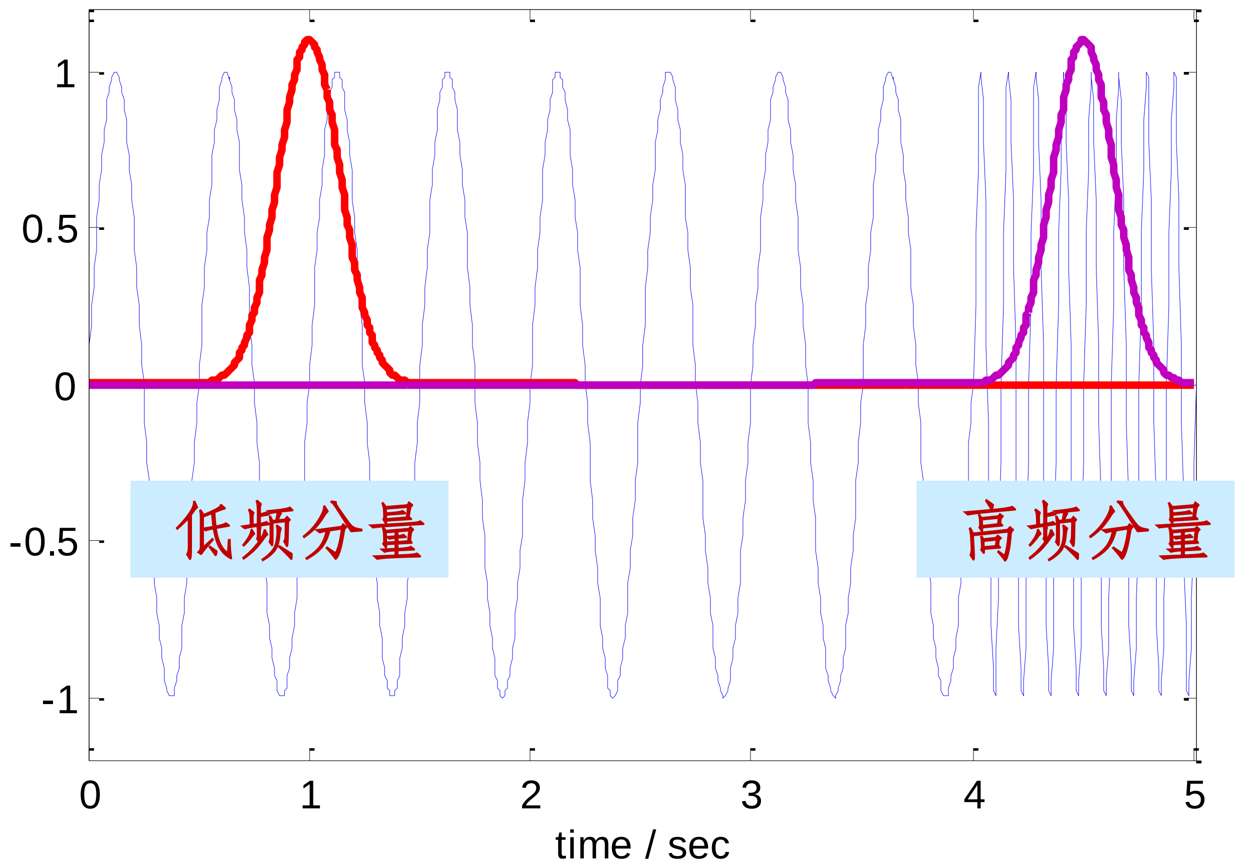
$$C = 10^{-4} K$$

Mode Number	Modal Frequency (rad/s)	Modal Damping ($\times 10^{-2}$)
Mode 1	38.8322	0.1942
Mode 2	101.6641	0.5083

高频分量衰减快

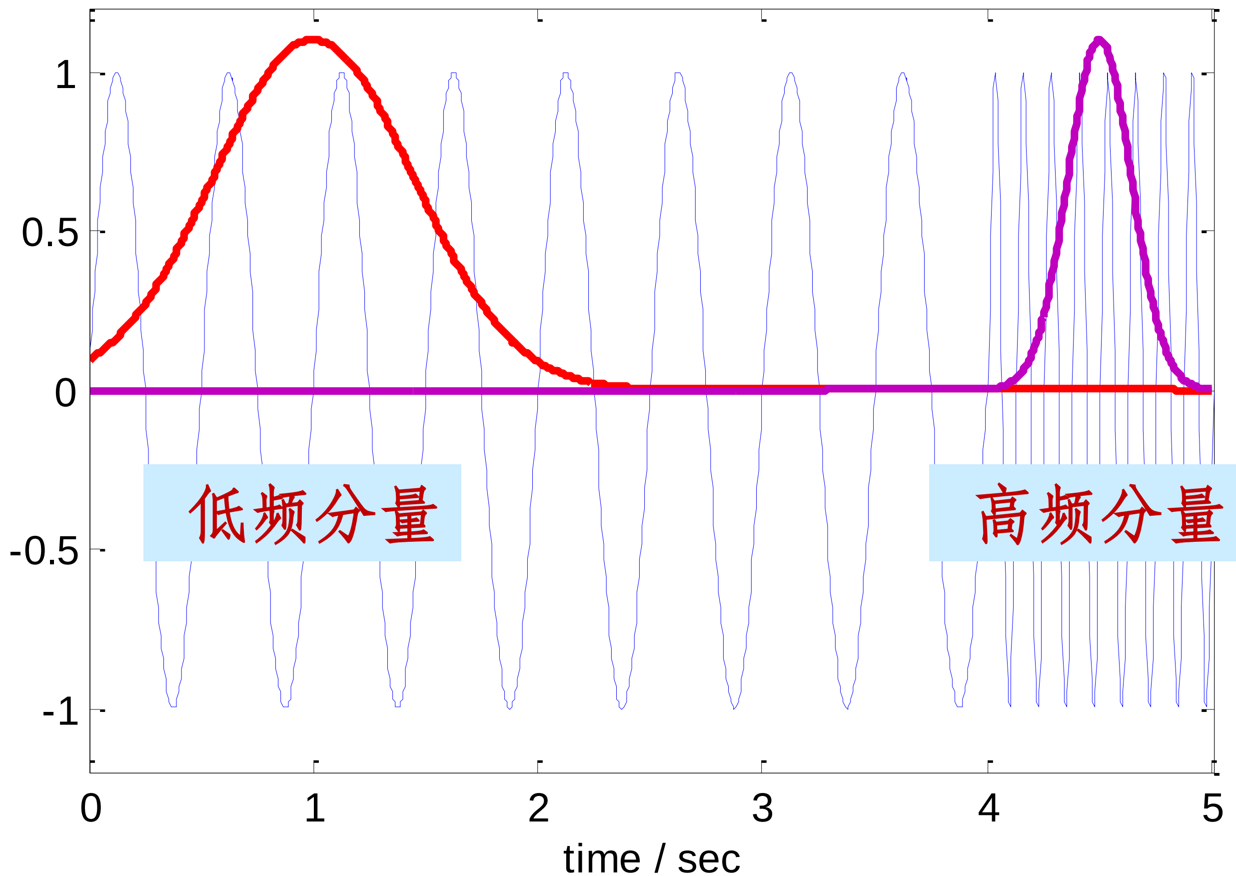
Wavelet Transform

- 由STFT到小波变换



Wavelet Transform

- 由STFT到小波变换



Wavelet Transform

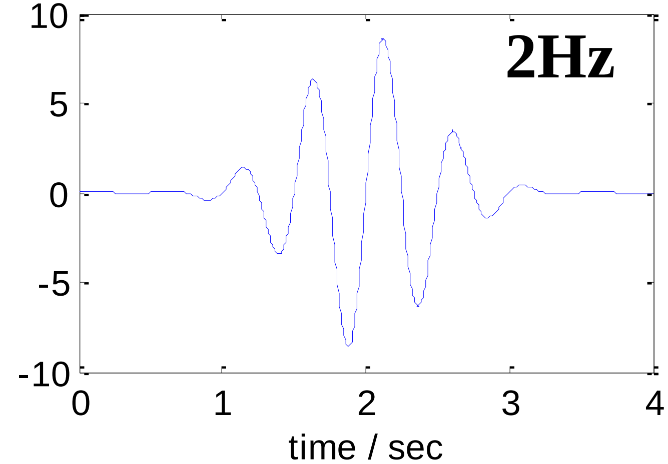
- 由STFT到小波变换

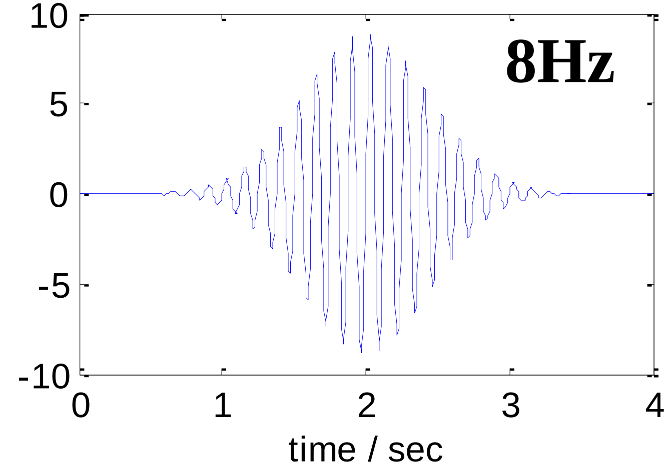
$$\mathrm{STFT}_x(\tau,\omega) = \int\left[x(t)\frac{1}{2\sqrt{\pi a}}e^{-(t-\tau)^2/4a}\right]e^{-j\omega t}dt$$

$$\psi(\tau,\omega) = \frac{1}{2\sqrt{\pi a}}e^{-(t-\tau)^2/4a}e^{-j\omega t}$$

$$\mathrm{STFT}_x(\tau,\omega) = \left\langle x(t),\psi(\tau,\omega)\right\rangle$$

***a*= 0.1**





Wavelet Transform

- 由STFT到小波变换

$$\psi\left(\tau,\frac{\omega}{a}\right)=\frac{1}{2\sqrt{\pi a}}e^{-(t-\tau)^2/4a}e^{-j\omega t/a}$$

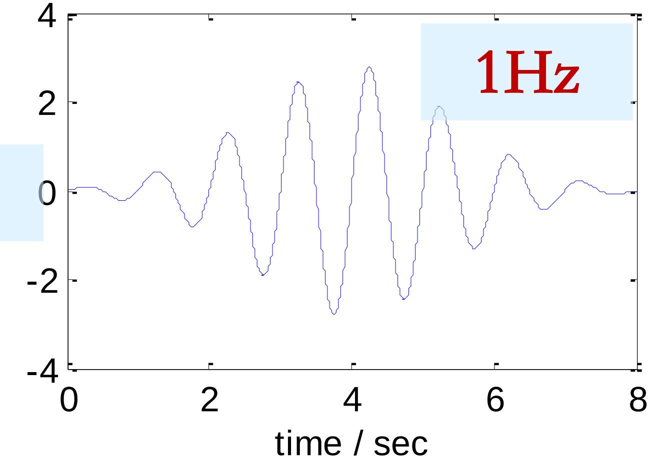
小波变换

$$\mathrm{WT}_x\left(\tau,\frac{\omega}{a}\right)=\left\langle x(t),\psi\left(\tau,\frac{\omega}{a}\right)\right\rangle$$

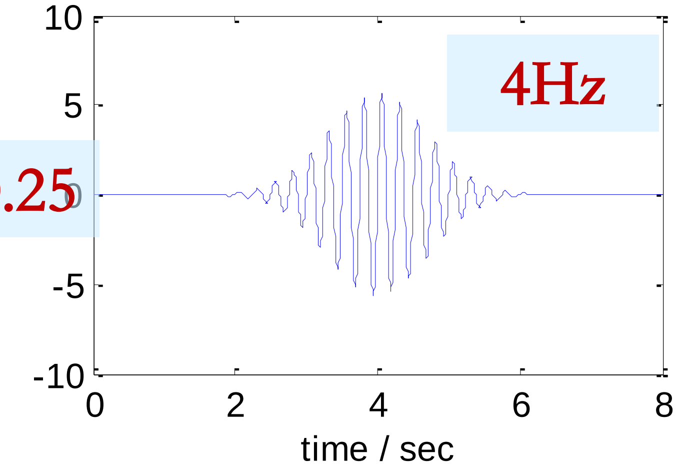
$$\psi(t)=\pi^{-1/4}e^{-t^2/2}e^{-j\omega_0 t}$$

Morlet 小波函数

a= 1



a= 0.25



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Wavelet Transform

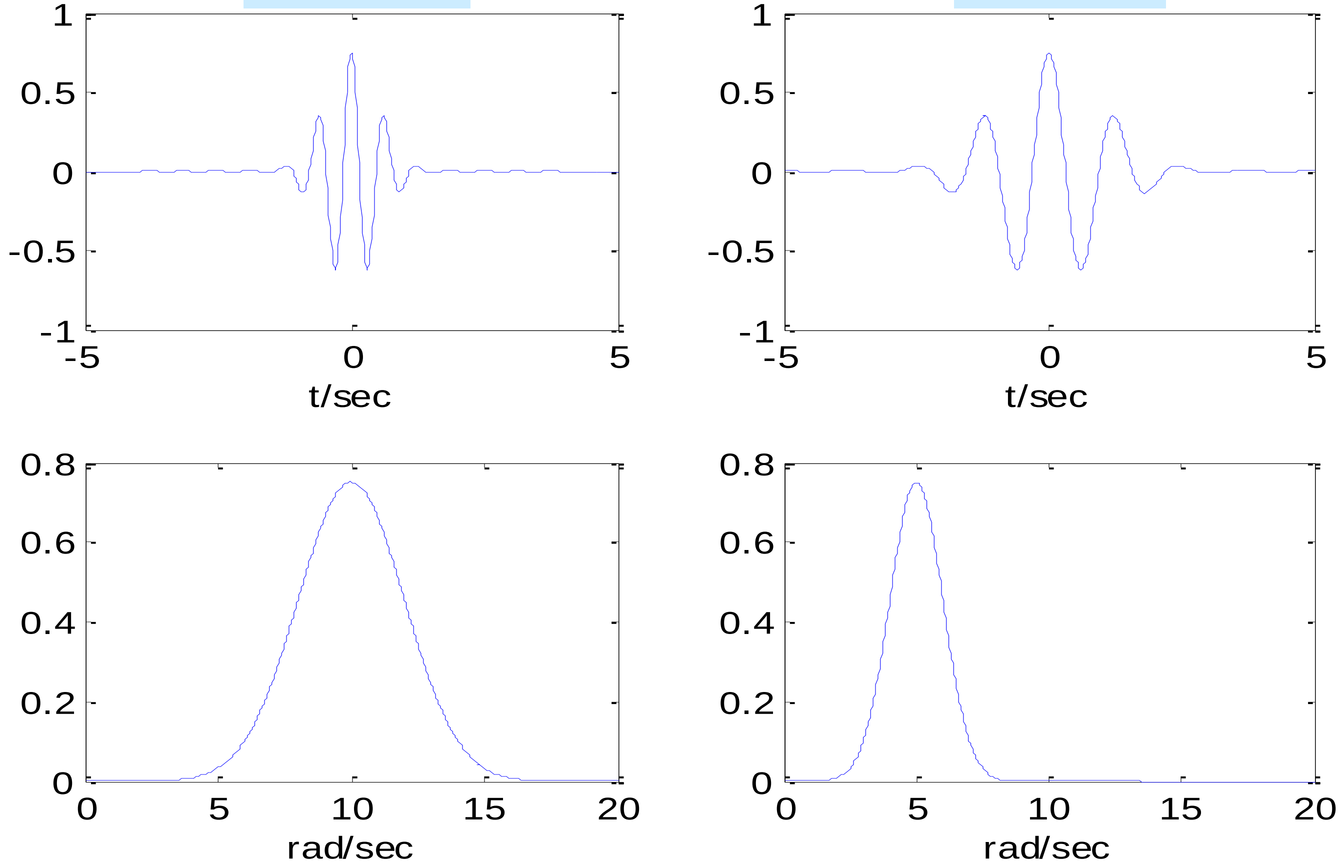
- 小波变换

$$\psi(t) \rightarrow \hat{\psi}(\omega) \quad \Longrightarrow \quad \psi(t/a) \rightarrow \frac{1}{|a|}\hat{\psi}(a\omega)$$

$$\psi(t) = \pi^{-1/4} e^{-t^2/2} e^{-j\omega_0 t}$$

$$\hat{\psi}(\omega) = \pi^{-1/4} e^{-(\omega-\omega_0)^2/2}$$

$a = 0.5$ $\quad\omega_0 = 5\quad$ $a = 1$



Wavelet Transform

小波变换定义

$$W_x(a,b) = \left\langle x(t),\ \psi_{a,b}(t) \right\rangle = \int_{-\infty}^{\infty} x(t)\psi_{a,b}(t)dt; \quad a > 0$$

$$\psi_{a,b}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right)$$

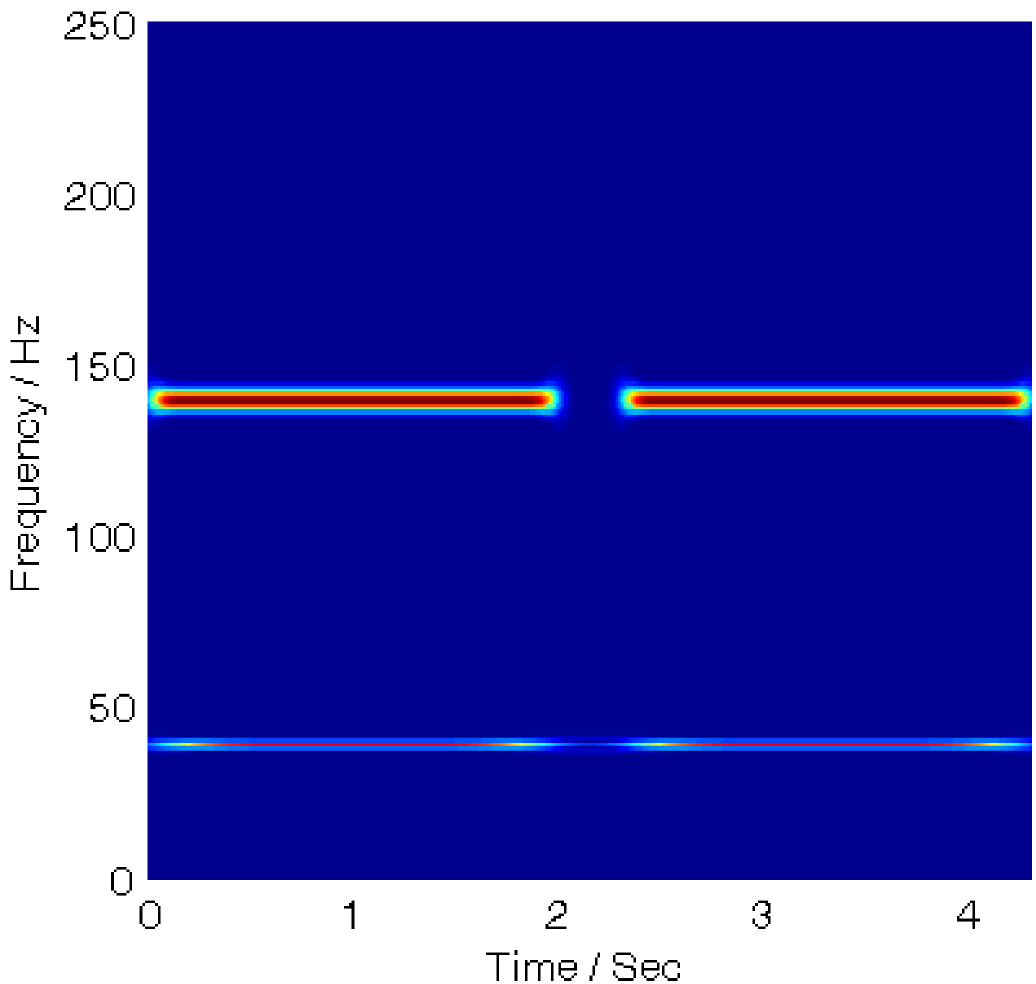
$$\| \psi_{a,b}(t) \|^2 = \int_{-\infty}^{\infty} | \psi_{a,b}(t) |^2 \ dt = \int_{-\infty}^{\infty} | \psi(t) |^2 \ dt$$

- 紧支撑性
- 小波容许条件

$$\leftarrow \begin{cases} \int_{-\infty}^{\infty} \psi(t)dt = 0 \\ \int_{-\infty}^{\infty} t^k \psi(t)dt = 0; \quad k = \ 0,1,\ldots N-1 \\ C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \end{cases}$$

Wavelet Transform

小波变换示例



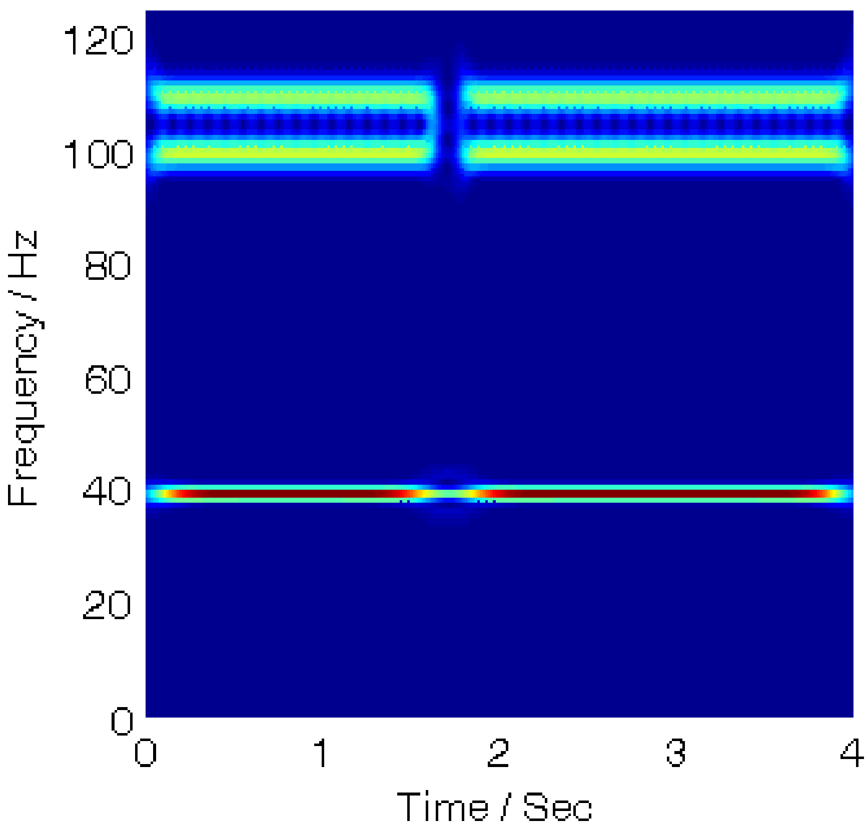
尺度和中心频率的转换关系

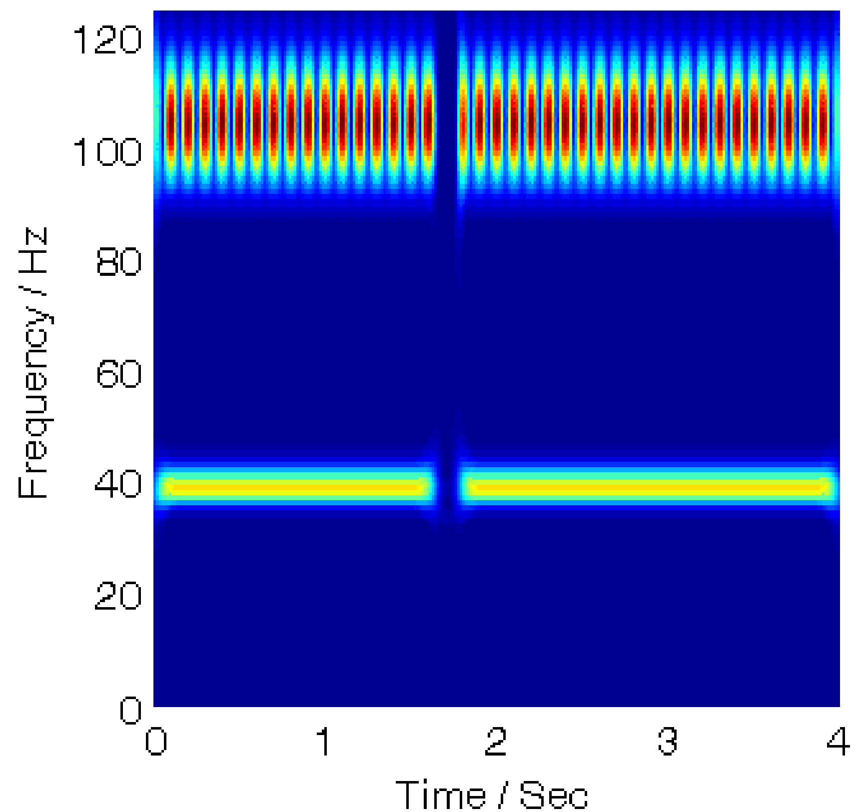
$$\psi(t)=\pi^{-1/4}e^{-t^2/2}e^{-j\omega_0 t}$$

$$\omega=\frac{\omega_0}{a}$$

Wavelet Transform

- 小波变换示例





Wavelet Transform

- 反小波变换

$$x(t)=\frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a^{-2}W_x(a,b)\psi_{a,b}(t)dadb$$

$$\langle x(t),x(t)\rangle=\int_{-\infty}^{\infty}|x(t)|^2dt$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{a^2C_\psi}|W_x(a,b)|^2\,dbda$$

$$|W_x(a,b)|^2$$

尺度谱

$$\langle x(t),y(t)\rangle=\frac{1}{C_\psi}\int_{-\infty}^{\infty}a^{-2}da\int_{-\infty}^{\infty}W_x(a,b)\overline{W_y(a,b)}db$$

$W_x(a,b)\overline{W_y(a,b)}\rightarrow$ 互尺度谱

Wavelet Transform

小波变换谱

$$\langle x(t), x(t) \rangle = \frac{1}{C_\psi} \int_{-\infty}^{\infty} a^{-2} da \int_{-\infty}^{\infty} |W_x(a,b)|^2 \, db$$

$$= \frac{1}{C_\psi} \int_{-\infty}^{\infty} a^{-2} E_x(a) da$$

$$E_x(a) = \int_{-\infty}^{\infty} |W_x(a,b)|^2 \, db$$

小波能谱

$$E_{xy}(a) = \int_{-\infty}^{\infty} W_x(a,b) \overline{W_y(a,b)} db$$

小波互谱

和功率谱、互功率谱的关系?

Wavelet Transform

- 小波变换谱

$$E_x(a)=\int_{-\infty}^{\infty}|W_x(a,b)|^2\,db$$

和功率谱的关系

$$E_x(a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)S_{\psi_a}(\omega)d\omega$$

和功率谱的关系

$$E_{xy}(a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xy}(\omega)S_{\psi_a}(\omega)d\omega$$

Wavelet Transform

小波变换分辨率

小波函数中心

$(t^0_{\psi_{a,b}}, \omega^0_{\hat{\psi}_{a,b}})$

$$\begin{cases} t^0_{\psi_{a,b}} = \dfrac{\int_{-\infty}^{\infty} t\,|\psi_{a,b}(t)|^2\,dt}{\int_{-\infty}^{\infty} |\psi_{a,b}(t)|^2\,dt} \\ \omega^0_{\psi_{a,b}} = \dfrac{\int_0^{\infty} \omega\,|\hat{\psi}_{a,b}(\omega)|^2\,d\omega}{\int_0^{\infty} |\hat{\psi}_{a,b}(\omega)|^2\,d\omega} \end{cases}$$

$$t^0_{\psi_{a,b}} = a t^0_{\psi_{1,0}} + b$$

$$\omega^0_{\psi_{a,b}} = \frac{1}{a}\omega^0_{\psi_{1,0}}$$

小波窗函数时宽

小波窗函数频宽

$$\begin{cases} \Delta\psi_{a,b} = \\ \left(\int_{-\infty}^{\infty} (t - t^0_{\psi_{a,b}})^2\,|\psi_{a,b}(t)|^2\,dt\right)^{1/2} \\ \Delta\hat{\psi}_{a,b} = \\ \left(\int_{-\infty}^{\infty} (\omega - \omega^0_{\psi_{a,b}})^2\,|\hat{\psi}_{a,b}(\omega)|^2\,d\omega\right)^{1/2} \end{cases}$$

$$\Delta\psi_{a,b} = a\Delta\psi_{1,0}$$

$$\Delta\hat{\psi}_{a,b} = \frac{1}{a}\Delta\hat{\psi}_{1,0}$$

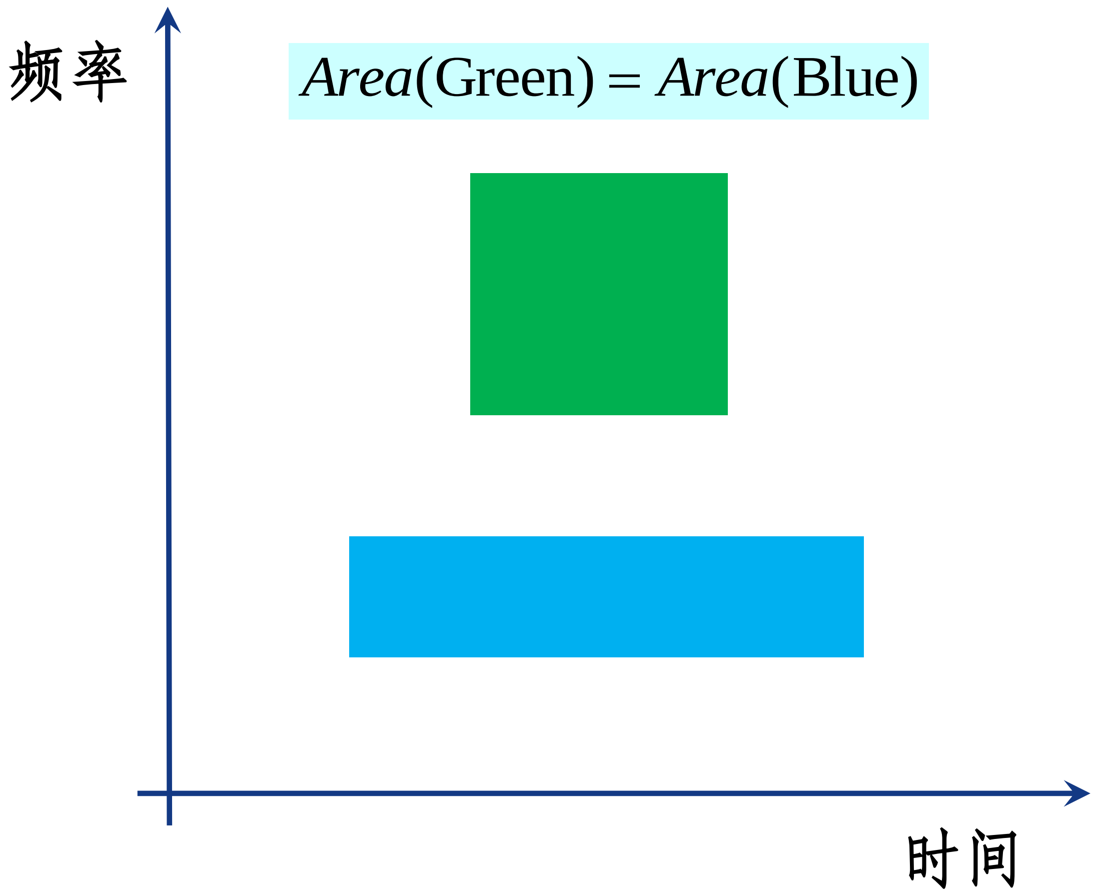
$$\Delta\psi_{a,b}\Delta\hat{\psi}_{a,b} = \Delta\psi_{1,0}\Delta\hat{\psi}_{1,0}$$

Wavelet Transform

小波变换分辨率

$$\Delta\psi_{a,b}\Delta\hat{\psi}_{a,b} = \Delta\psi_{1,0}\Delta\hat{\psi}_{1,0}$$

$$Area(\text{Green}) = Area(\text{Blue})$$



$$t^0_{\psi_{a,b}} = a t^0_{\psi_{1,0}} + b$$

$$\omega^0_{\psi_{a,b}} = \frac{1}{a}\omega^0_{\psi_{1,0}}$$

$$\Delta\psi_{a,b} = a\Delta\psi_{1,0}$$

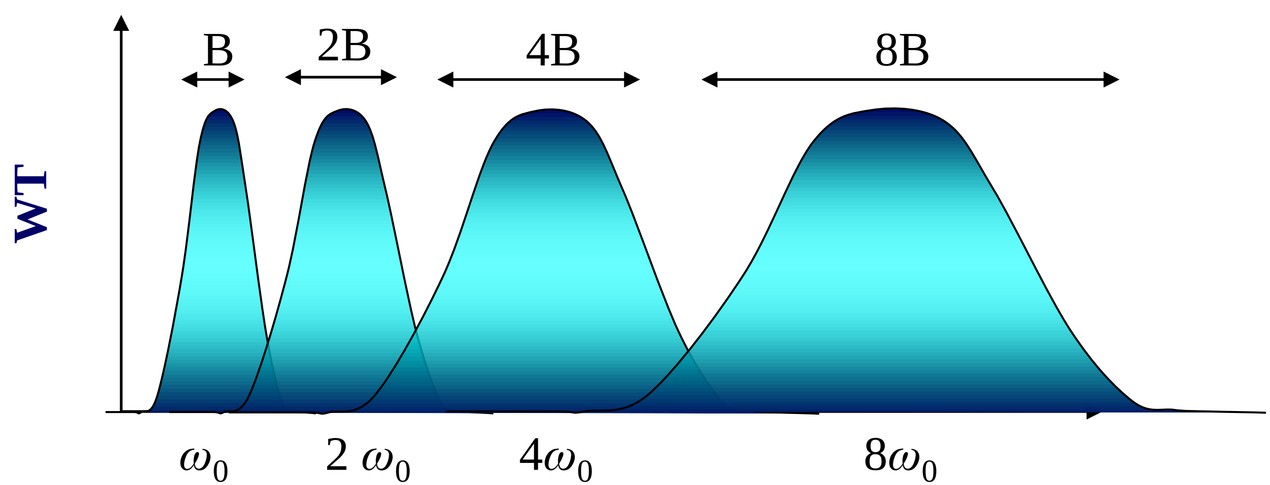
$$\Delta\hat{\psi}_{a,b} = \frac{1}{a}\Delta\hat{\psi}_{1,0}$$

Wavelet Transform

- 小波变换分辨率

$$\omega^0_{\psi_{a,b}} = \frac{1}{a}\omega^0_{\psi_{1,0}} \qquad \Delta\hat{\psi}_{a,b} = \frac{1}{a}\Delta\hat{\psi}_{1,0}$$

$$Q = \frac{\text{中心频率}}{\text{带宽}} = \frac{\omega^0_{\psi_{a,b}}}{2\Delta\hat{\psi}_{a,b}} = \frac{\omega^0_{\psi_{1,0}}}{2\Delta\hat{\psi}_{1,0}} = \text{常数}$$



Wavelet Transform

小波变换性质

- 线性：小波变换定义为内积运算
- 平移和伸缩共变性：

$$x(t) \to W_x(a,b) \Rightarrow x(t-b_0) \to W_x(a,b-b_0)$$

$$x(t) \to W_x(a,b) \Rightarrow x(a_0 t) \to \frac{1}{\sqrt{a_0}} W_x(a_0 a, a_0 b)$$

- 微分运算可交换性

$$\frac{\partial}{\partial b} W_x(a,b) = W_{\frac{\partial x}{\partial t}}(a,b)$$

Wavelet Transform

小波变换性质

- 微分运算

$$W_{\frac{\partial^m x(t)}{\partial t^m}}(a,b) = (-1)^m \int_{-\infty}^{+\infty} x(t) \frac{\partial^m}{\partial t^m} \overline{\left[\psi_{a,b}(t)\right]} dt$$

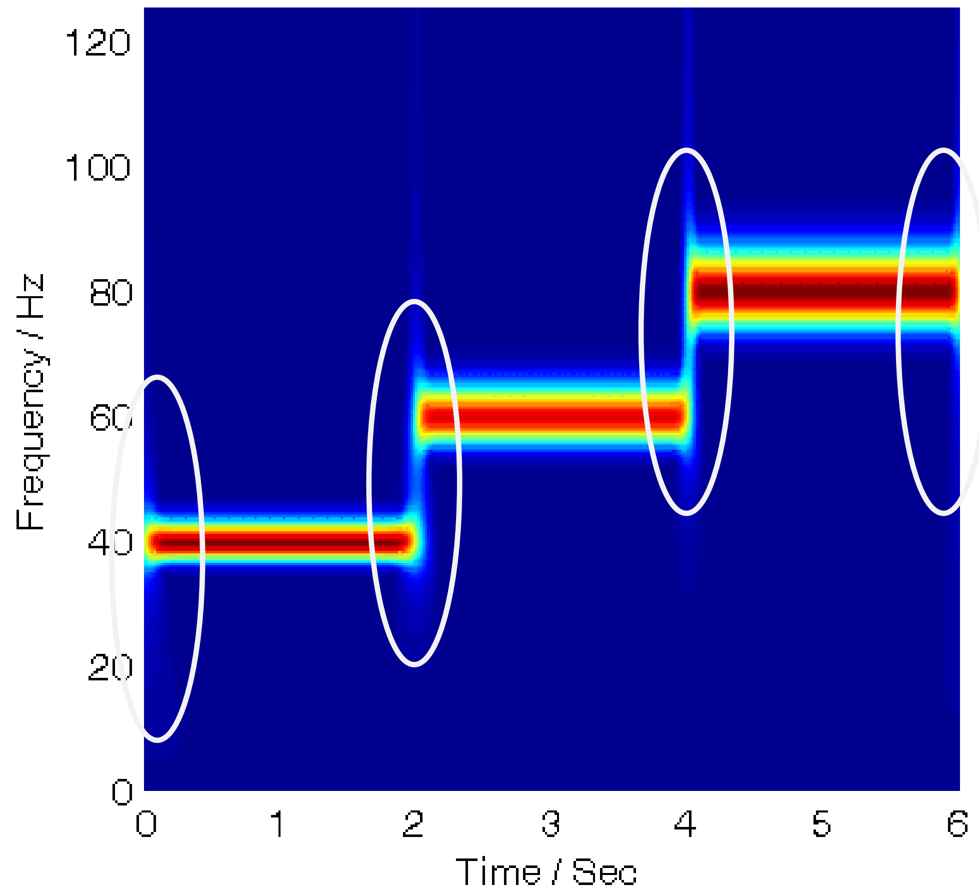
- 局部正则性

如果 $x(t) \in C^m(t_0)$ 则

$$W_x(a,t_0) \le a^{m+1} a^{1/2}$$

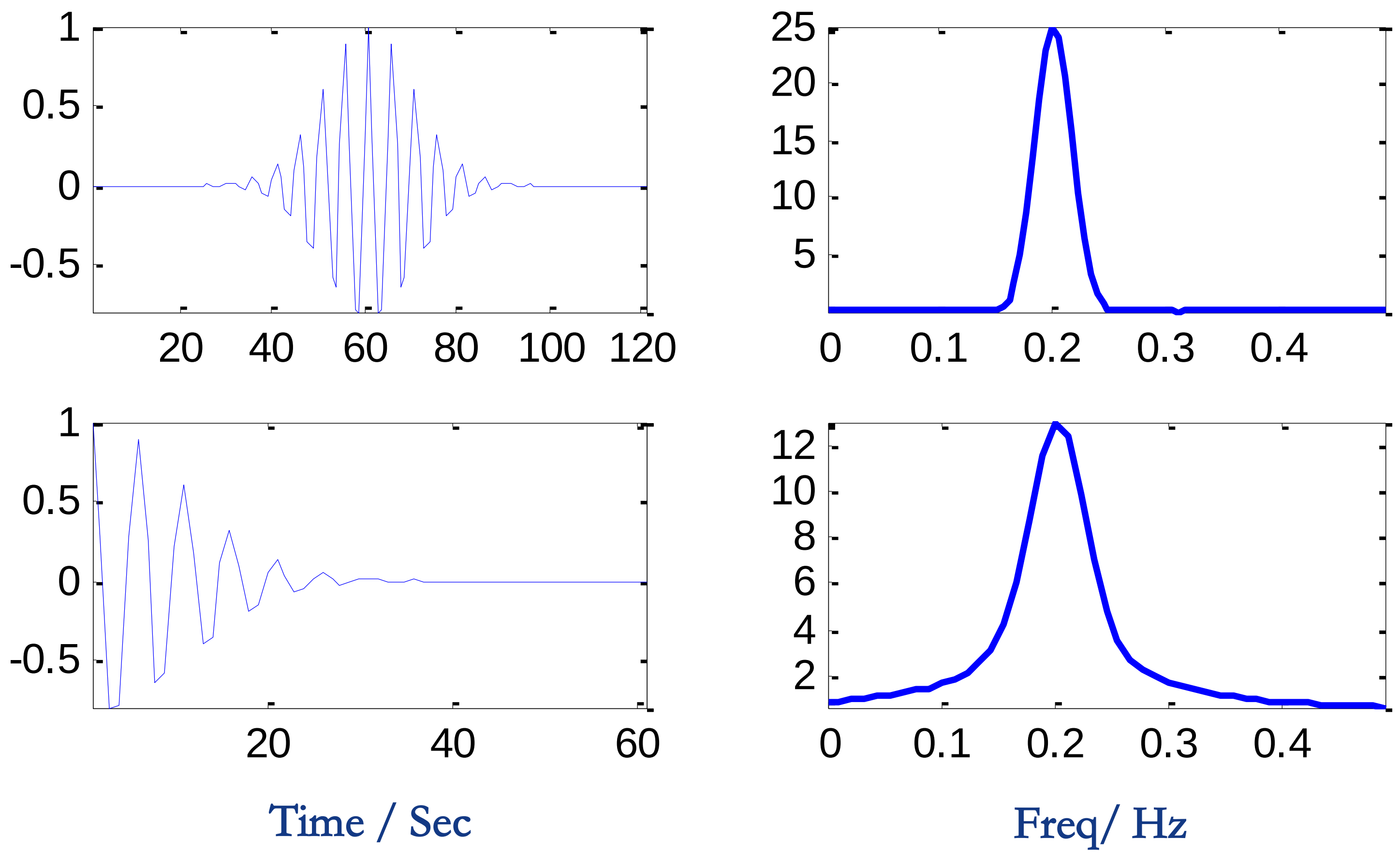
Wavelet Transform

- 小波变换边界扭曲现象



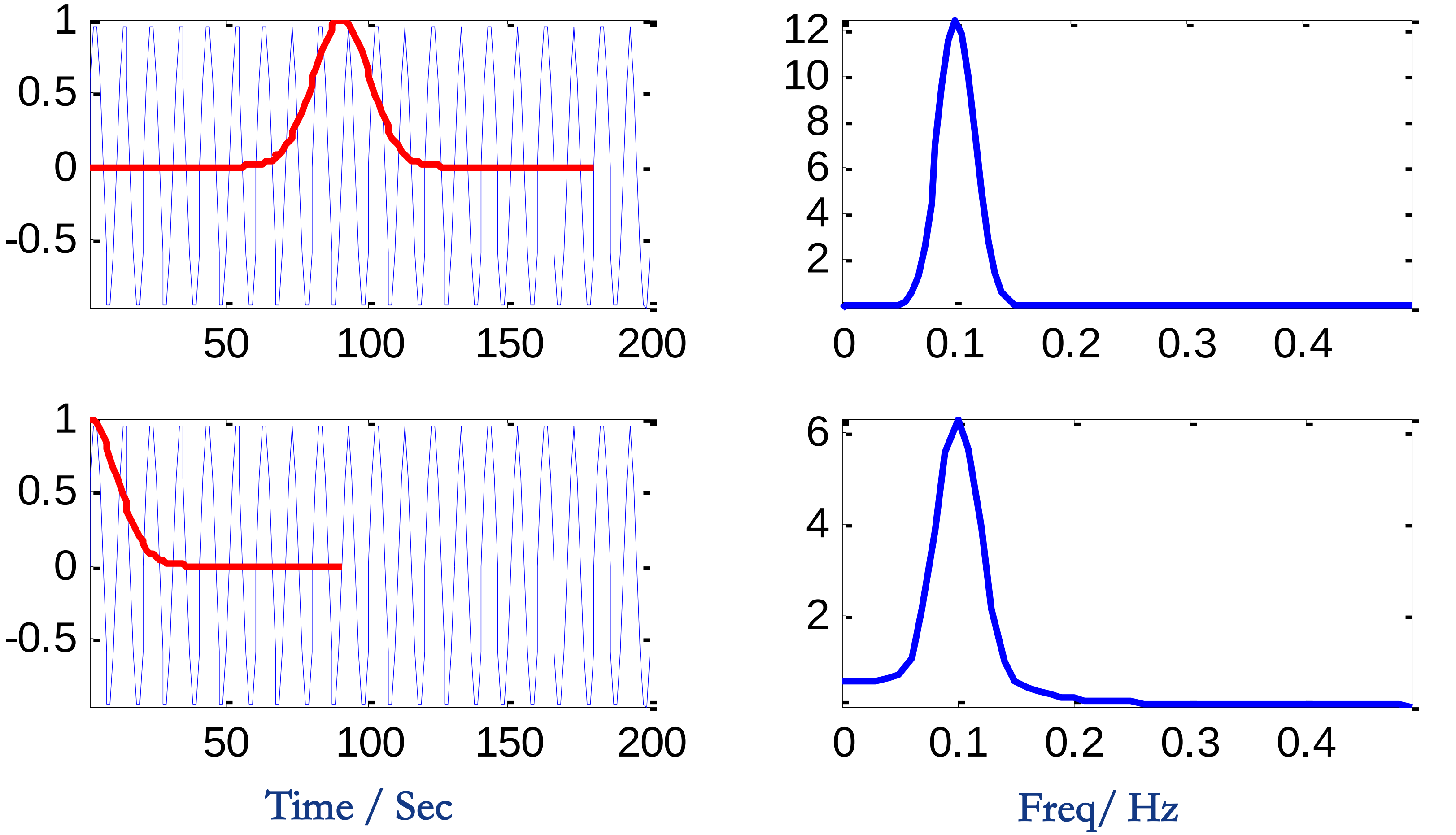
Wavelet Transform

- 小波变换边界扭曲现象



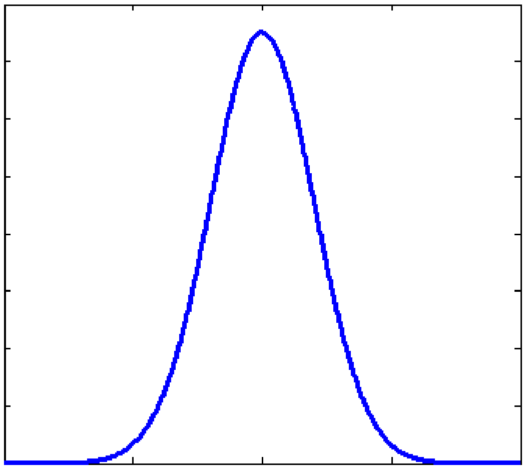
Wavelet Transform

- 小波变换边界扭曲现象



Wavelet Transform

小波变换计算



$$\psi(t)=\pi^{-1/4}e^{-t^2/2}e^{-j\omega_0 t}$$

$$\omega_0=2\pi f_0$$

$$T_s=\frac{1}{F_s}$$

$$\omega_H=\omega_0/a+\frac{\ln 2}{a}$$

$$\omega_L=\omega_0/a-\frac{\ln 2}{a}$$

$$2\pi F_s=\frac{2\pi}{T_s}\geq 2\omega_H=2\left(\frac{\omega_0+\ln 2}{a}\right)$$

$$a_{\min}=\frac{(\omega_0+\ln 2)T_s}{\pi}$$

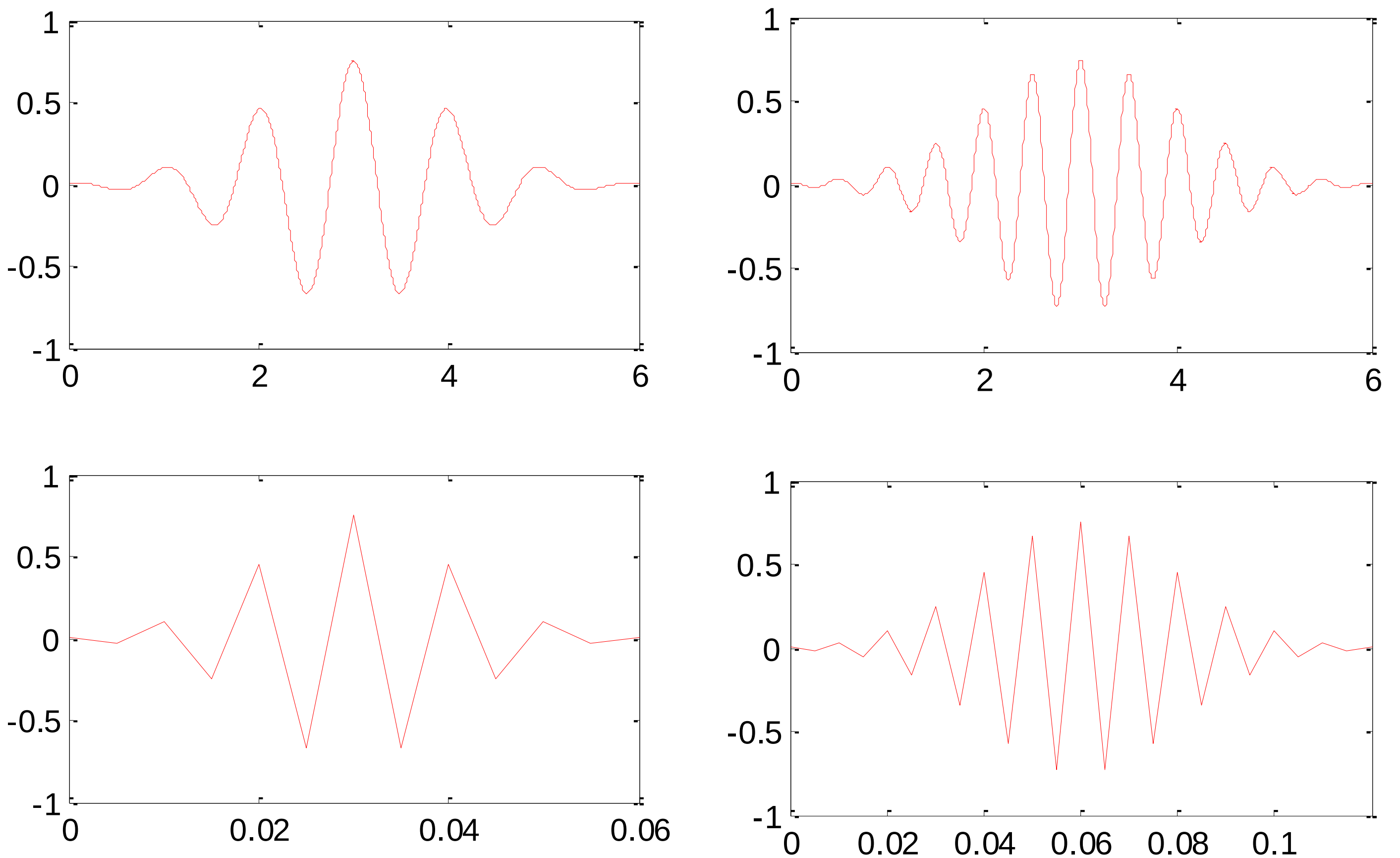
如何离散a?

最大尺度参数确定

$$3\Delta\psi_{a,b}\leq \text{信号长度}/2$$

Wavelet Transform

- 小波变换计算



Discrete Wavelet Transform

- 尺度参数离散化：如何离散尺度参数和平移参数，使得小波变换没有信息损失，并且冗余度尽量小

$$a = a_0^m,\ (m \in z; a_0 > 1);\quad b = nb_0 a_0^m,\quad (b_0 > 0)$$

$$\psi_{m,n}(t) = \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{t - nb_0 a_0^m}{a_0^m}\right) = a_0^{-m/2} \psi\left(a_0^{-m} t - nb_0\right)$$

$$DWT_x(m,n) = \left\langle x(t),\ \psi_{m,n}(t) \right\rangle = a_0^{-m/2} \int_{-\infty}^{+\infty} x(t) \psi\left(a_0^{-m} t - nb_0\right) dt$$

Discrete Wavelet Transform

- 尺度参数

$$a_0 = 2;\ b_0 = 1$$

$$\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m/2}t - n\right)$$

二进制小波

$$DWT_x(m,n) = 2^{-m/2}\int_{-\infty}^{+\infty} x(t)\psi\left(2^{-m/2}t - n\right)dt$$

二进制小波变换

二进制正交小波基

$$\int_{-\infty}^{\infty}\psi_{m,n}\left(t\right)\psi_{m',n'}\left(t\right)dt = \delta\left(m - m'\right)\delta\left(n - n'\right)$$

伸缩平移均正交

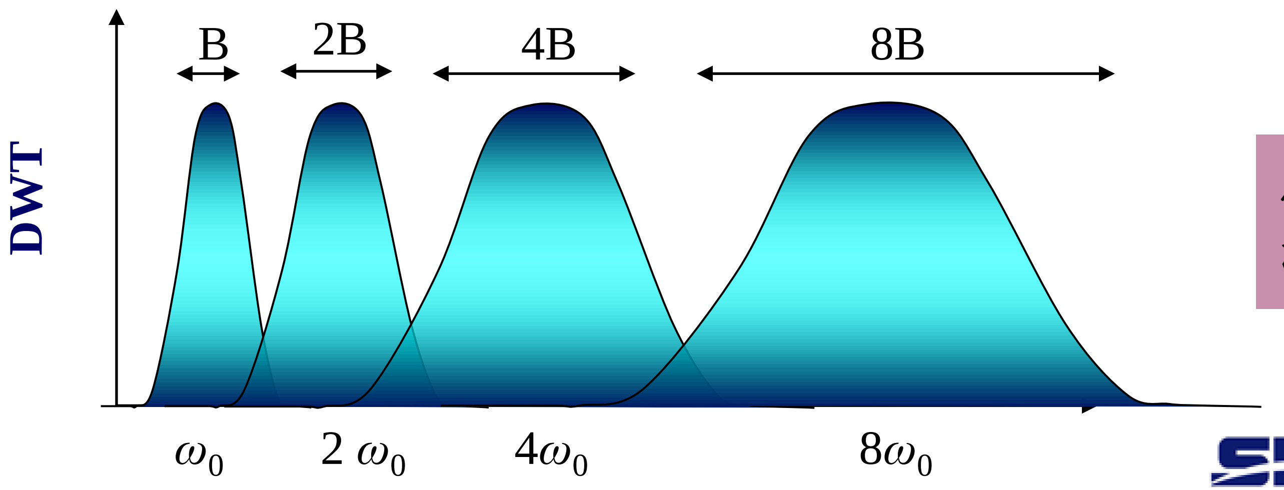
Discrete Wavelet Transform

- 正交分解与重构

$$DWT_x(m,n) = \left\langle x(t),\ \psi_{m,n}(t) \right\rangle$$

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} DWT_x(m,n)\psi_{m,n}(t)$$

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle x(t),\ \psi_{m,n}(t) \right\rangle \psi_{m,n}(t)$$



小波函数
带通性质

Discrete Wavelet Transform

- 正交分解与重构

$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle \psi_{m,n}(t)$$

$$\Downarrow$$

$$x(t)=\sum_{m=m_0+1}^{\infty}\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle \psi_{m,n}(t)$$

低频部分：模糊像

$$+\sum_{m=-\infty}^{m_0}\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle \psi_{m,n}(t)$$

高频部分：细节补充

Discrete Wavelet Transform

- 正交分解与重构

$$\sum_{m=m_0+1}^{\infty}\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle\psi_{m,n}(t)\to\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \varphi_{m_0,n}(t)\right\rangle\varphi_{m_0,n}(t)$$

$$x(t)=\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \varphi_{m_0,n}(t)\right\rangle\varphi_{m_0,n}(t)$$

低频部分：模糊像

$$+\sum_{m=-\infty}^{m_0}\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle\psi_{m,n}(t)$$

高频部分：细节补充

$\varphi_{m,n}(t)$ 尺度函数：1) 低通滤波器; 2) 平移正交

Discrete Wavelet Transform

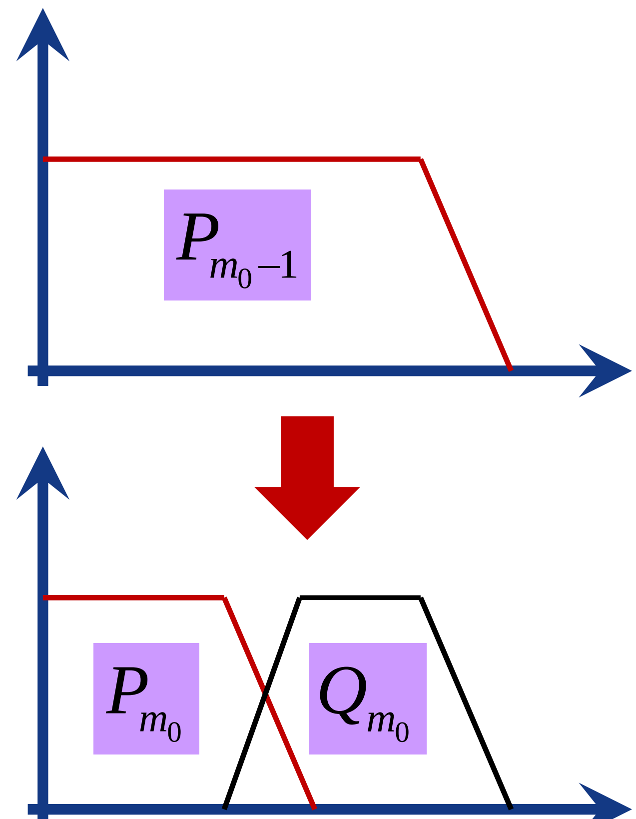
镜像滤波

$$P_{m_0}x(t)=\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \varphi_{m_0,n}(t)\right\rangle \varphi_{m_0,n}(t)$$

$$Q_{m}x(t)=\sum_{n=-\infty}^{\infty}\left\langle x(t),\ \psi_{m,n}(t)\right\rangle \psi_{m,n}(t)$$

$$P_{m_0-1}x(t)=P_{m_0}x(t)+Q_{m_0}x(t)$$

$$x(t)=P_{m_0}x(t)+\sum_{m=-\infty}^{m_0}Q_{m}x(t)$$



Discrete Wavelet Transform

- 多分辨率分析(MRA)

$$\mathbf{V}_m = \overline{\mathrm{Span}\left\{\varphi_{m,n}(t), n \in Z\right\}}$$

$$\mathbf{W}_m = \overline{\mathrm{Span}\left\{\psi_{m,n}(t), n \in Z\right\}}$$

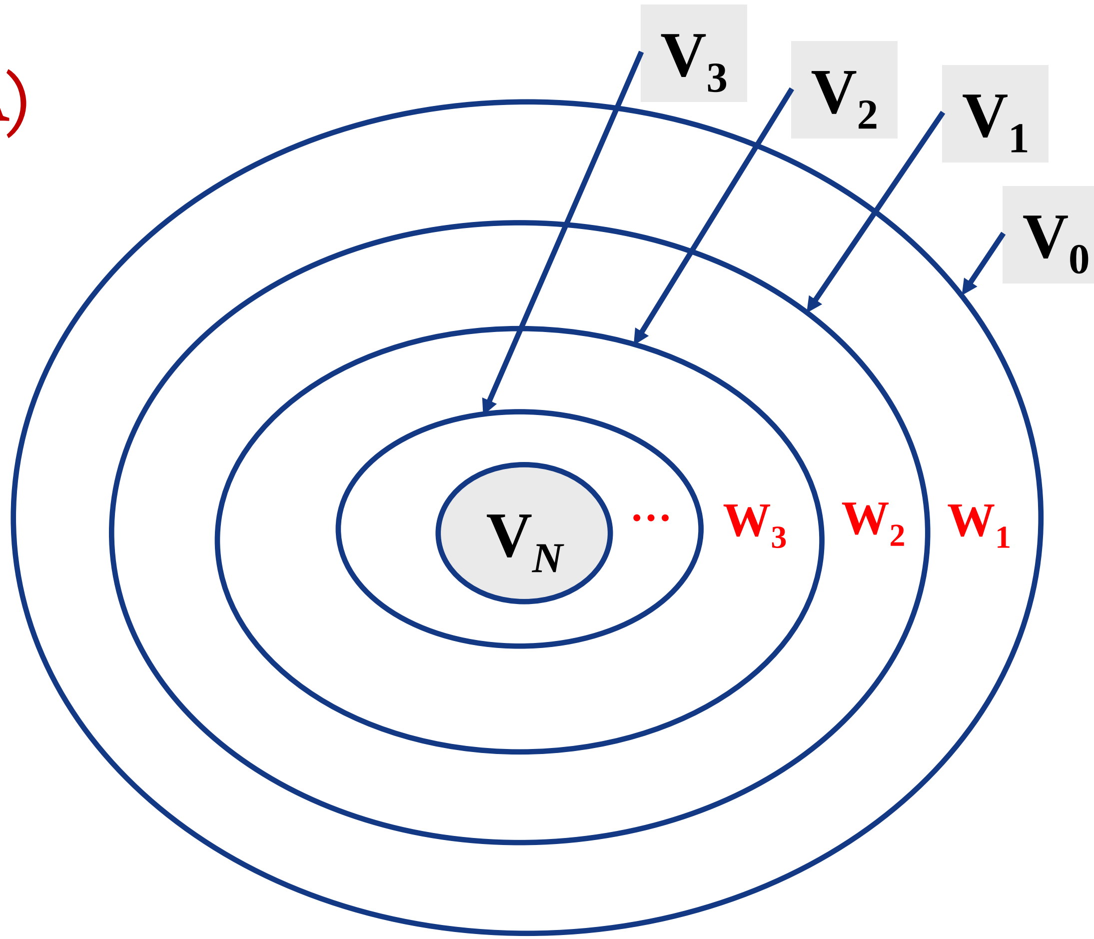
$$\mathbf{V}_{m-1} = \mathbf{V}_m \oplus \mathbf{W}_m$$

$$\mathbf{V}_m \perp \mathbf{W}_m$$

$$\mathbf{W}_{m-1} \perp \mathbf{W}_m$$

$$\mathbf{V}_0 = \mathbf{V}_N \bigcup_{k=1}^{N} \oplus \mathbf{W}_m$$

$$\lim_{m \to \infty} \mathbf{V}_m = 0$$



$$x(t) = x_N^A(t) + \sum_{m=-\infty}^{N} x_m^D(t)$$

Discrete Wavelet Transform

- 问题：如何构造尺度函数?

1. 尺度函数具有低通滤波特性；小波函数具有带通滤波特性

$$\int_{-\infty}^{+\infty} \varphi(t) = 1; \qquad \|\varphi(t)\| = 1$$

$$\int_{-\infty}^{+\infty} \psi(t) = 0; \qquad \|\psi(t)\| = 1$$

加权平均

2. 尺度函数对所有小波函数正交

$$\int_{-\infty}^{\infty} \varphi_{m,n}(t) \psi_{m',n'}(t) dt = 0; \quad m > m'$$

3. 尺度函数间平移正交

$$\int_{-\infty}^{\infty} \varphi_{m,n}(t) \varphi_{m,n'}(t) dt = 0$$

Discrete Wavelet Transform

- 问题：如何构造尺度函数?

$$\mathbf{V}_{m-1} = \mathbf{V}_m \oplus \mathbf{W}_m$$

4. 某一尺度上的尺度函数可由自身在下一个尺度上的线性组合得到: 双尺度差分方程

$$\varphi_{m,n}(t) = \sum_{n=-\infty}^{\infty} \langle \varphi_{m,n}(t),\ \varphi_{m-1,n}(t) \rangle \varphi_{m-1,n}(t)$$

$$\varphi(t) = \sqrt{2} \sum_{n \in Z} h_n \varphi(2t - n)$$

$\boldsymbol{h_n}$ - 尺度系数

5. 某一尺度上的小波函数可由自身在下一个尺度上的线性组合得到

$$\psi(t) = \sqrt{2} \sum_{n \in Z} g_n \varphi(2t - n)$$

Discrete Wavelet Transform

- 问题：如何构造尺度函数?

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1$$

$$H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega-\pi)} = 0$$

$$g_n = (-1)^{1-n}\overline{h}_{1-n}, \quad n \in Z$$

如何构造尺度函数是小波变换理论研究中的重要方向，其实质是双尺度差分方程求解，常用涉及迭代过程。

Discrete Wavelet Transform

- Mallat算法（快速DWT）

  - 不同的尺度函数对应的尺度系数$\boldsymbol{h_n}$不同

$$\varphi(t)=\sqrt{2}\sum_{n\in Z}h_n\varphi(2t-n)$$

$$\psi(t)=\sqrt{2}\sum_{n\in Z}g_n\varphi(2t-n)$$

  - 如何在不知道尺度函数和小波函数具体结构的情况下，只根据已知的尺度系数计算DWT

Discrete Wavelet Transform

- Mallat算法 － 分解

$$\left\langle x,\varphi_{m,n}\right\rangle=\left\langle x,\sum_k \bar{h}_{k-2n}\varphi_{m-1,k}\right\rangle=\sum_k \bar{h}_{k-2n}\left\langle x,\varphi_{m-1,k}\right\rangle$$

$$\left\langle x,\psi_{m,n}\right\rangle=\left\langle x,\sum_k \bar{g}_{k-2n}\varphi_{m-1,k}\right\rangle=\sum_k \bar{g}_{k-2n}\left\langle x,\varphi_{m-1,k}\right\rangle$$

$$a_n^1=\sum_k \bar{h}_{k-2n}\left\langle x,\varphi_{0,k}\right\rangle=\sum_k \bar{h}_{k-2n}a_k^0$$

$$d_n^1=\sum_k \bar{g}_{k-2n}\left\langle x,\varphi_{0,k}\right\rangle=\sum_k \bar{g}_{k-2n}a_k^0$$

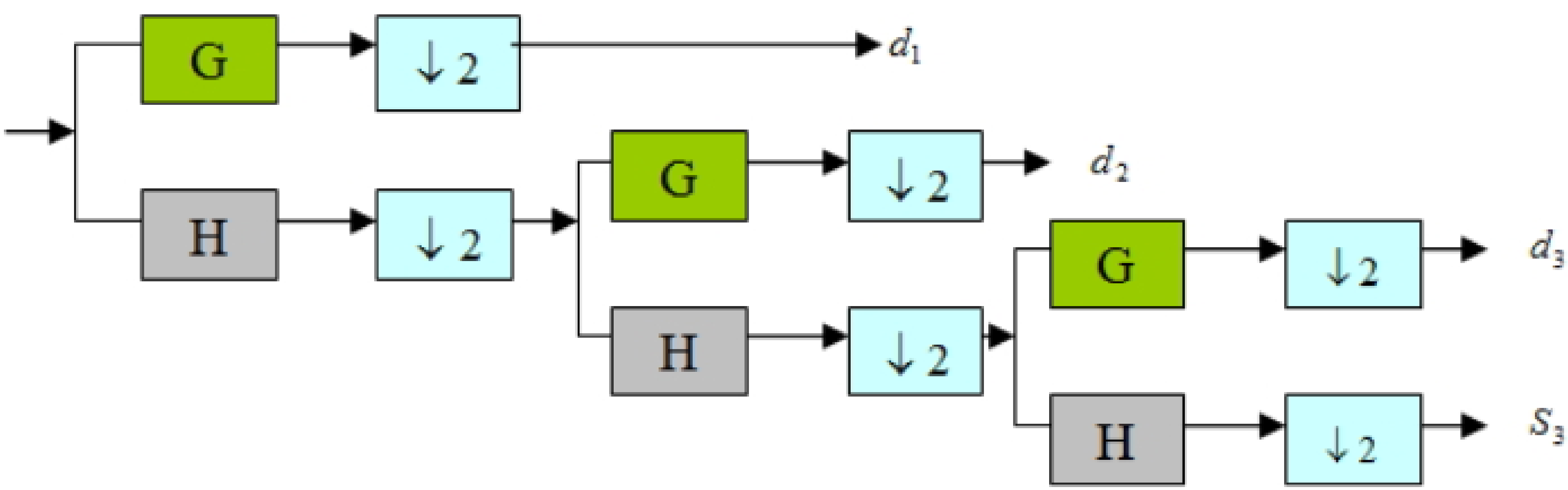
Discrete Wavelet Transform

- Mallat算法 – 分解

$$a_n^1 = \sum_k \overline{h}_{k-2n} \left\langle x, \varphi_{0,k} \right\rangle = \sum_k \overline{h}_{k-2n} a_k^0$$

$$d_n^1 = \sum_k \overline{g}_{k-2n} \left\langle x, \varphi_{0,k} \right\rangle = \sum_k \overline{g}_{k-2n} a_k^0$$

$$a_n^m = \sum_k \overline{h}_{k-2n} a_k^{m-1} \qquad d_n^m = \sum_k \overline{g}_{k-2n} a_k^{m-1}$$



Discrete Wavelet Transform

- Mallat算法- 重构

$$x(t) = P_0 x(t) = P_1 x(t) + Q_1 x(t)$$

$$x(t) = \sum_n a_n^0 \varphi(t-n) = 2^{-1/2}\left[\sum_n a_n^1 \varphi(2^{-1}t-n) + \sum_n d_n^1 \psi(2^{-1}t-n)\right]$$

$$= \sum_n a_n^1 \sum_k h_k \varphi(t-2n-k) + \sum_n d_n^1 \sum_k g_k \varphi(t-2n-k)$$

$$= \sum_n a_n^1 \sum_j h_{j-2n} \varphi(t-j) + \sum_n d_n^1 \sum_j g_{j-2n} \varphi(t-j)$$

$$= \sum_n \left(\sum_k a_k^1 h_{n-2k} + \sum_k d_k^1 g_{n-2k} \right) \varphi(t-n)$$

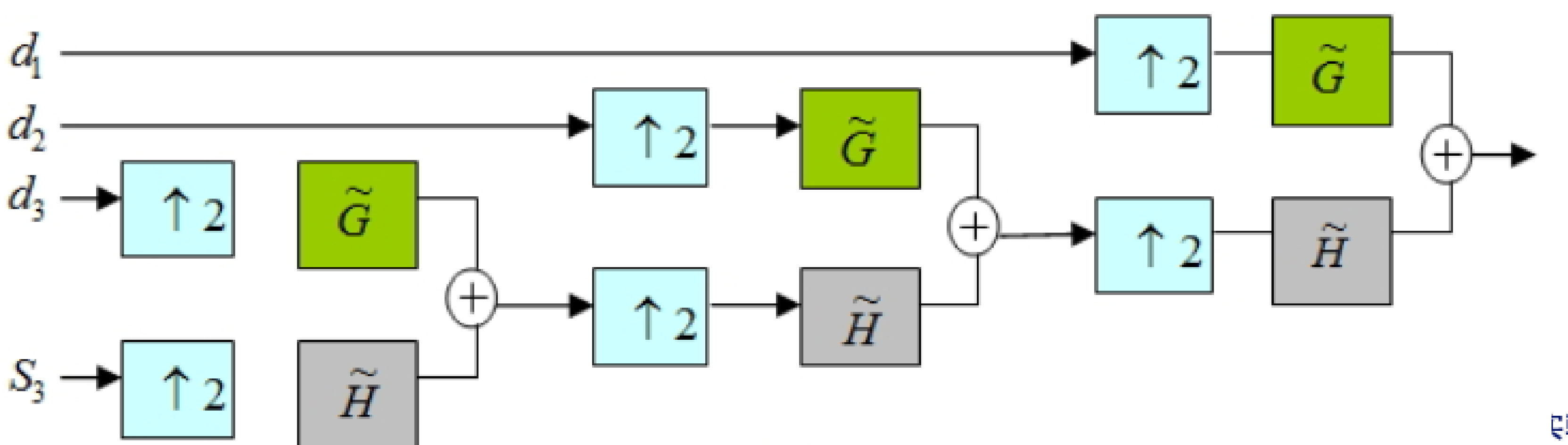
Discrete Wavelet Transform

- **Mallat算法**- 重构

$$\sum_n a_n^0 \varphi(t-n) = \sum_n \left(\sum_k a_k^1 h_{n-2k} + \sum_k d_k^1 g_{n-2k} \right) \varphi(t-n)$$

$$a_n^0 = \sum_k a_k^1 h_{n-2k} + \sum_k d_k^1 g_{n-2k}$$

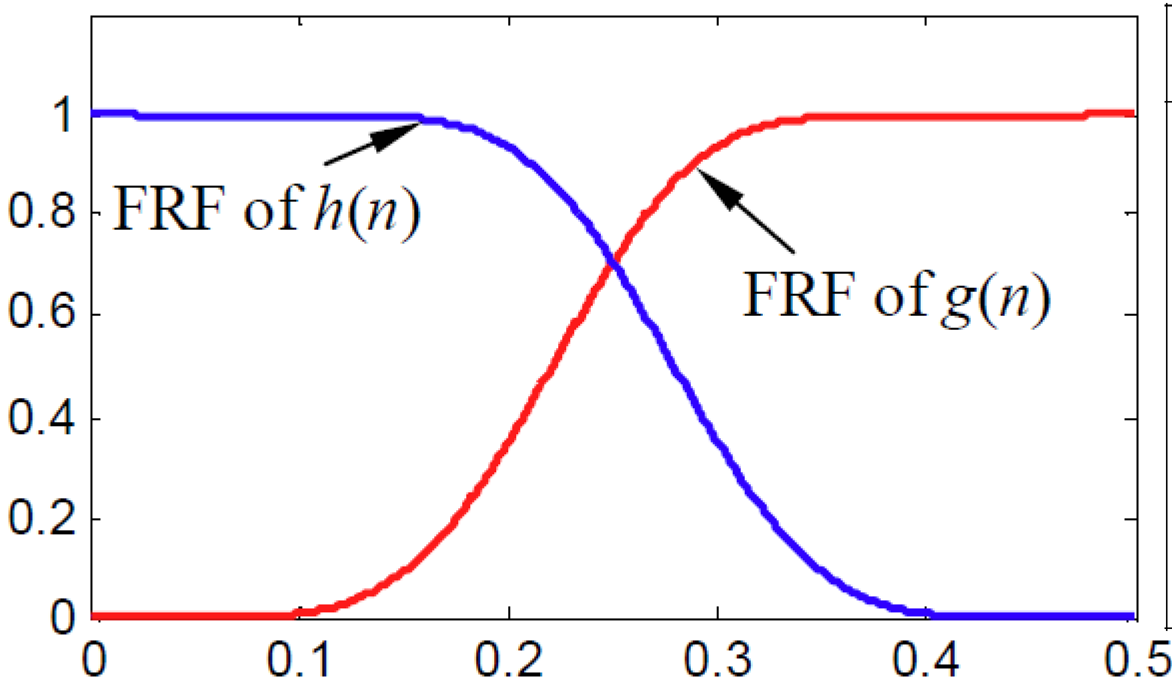
$$a_n^{m-1} = \sum_k a_k^m h_{n-2k} + \sum_k d_k^m g_{n-2k}$$



Discrete Wavelet Transform

- 镜像滤波器组

$$H(\omega)H'(\omega)+G(\omega)G'(\omega)=I$$

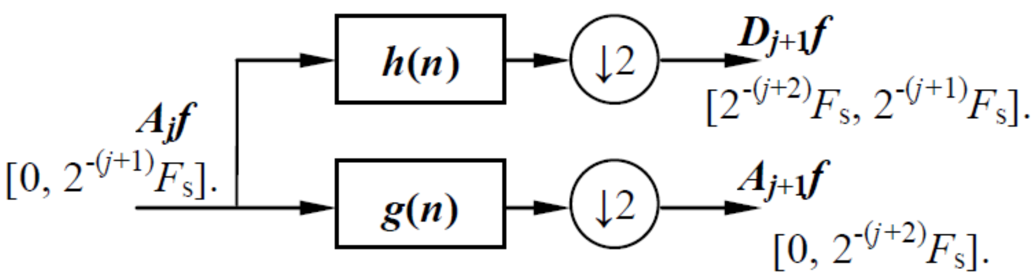


Matlab Codes

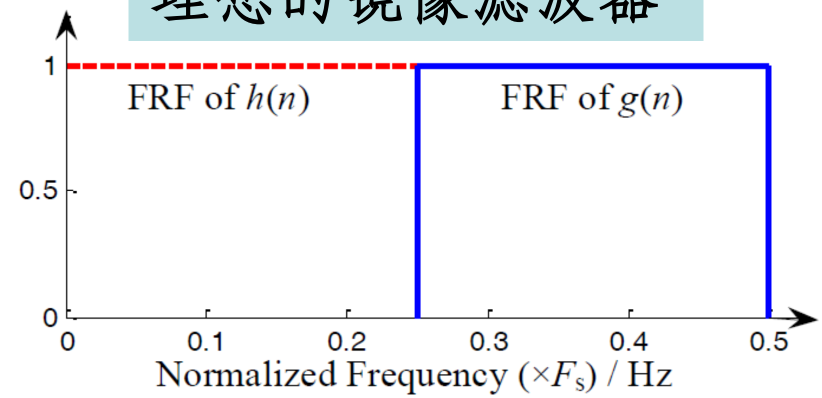
```
[Lo_D, Hi_D] = wfilters('db7');
[Hi, Fq1] = freqz(Hi_D);
[Lo, Fq2] = freqz(Lo_D);
plot(Fq1/2/pi, abs(Hi)/sqrt(2));
hold on
plot(Fq2/2/pi, abs(Lo)/sqrt(2));
set(gca, 'XLim', [0, 0.5])
```

Discrete Wavelet Transform

- 能量泄漏



理想的镜像滤波器



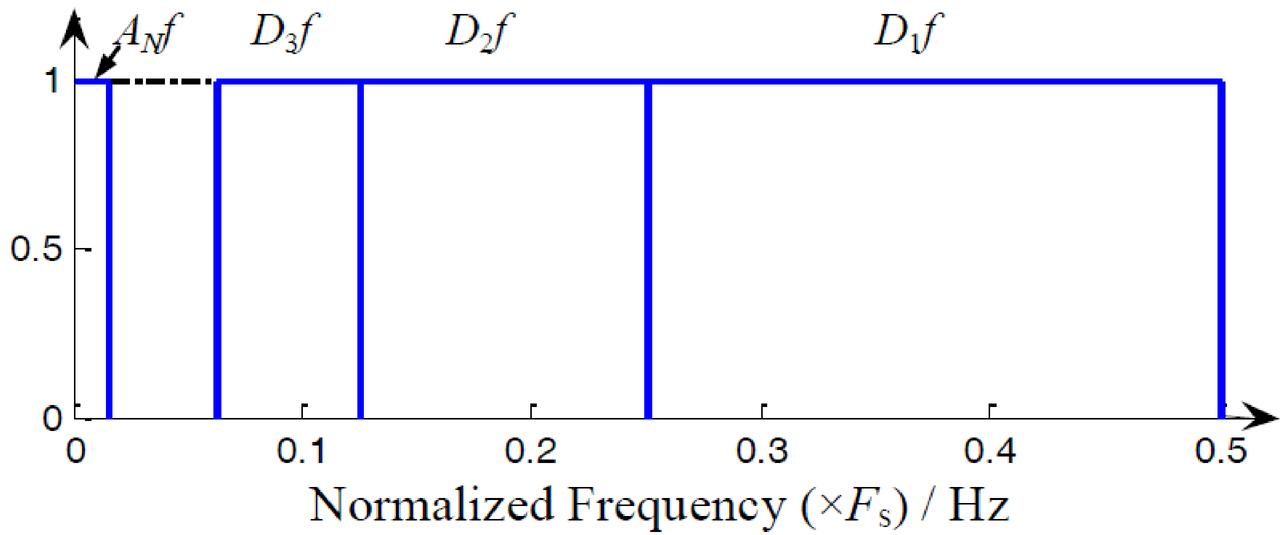


Fig 2, the ideal frequency domain division of DWT

Discrete Wavelet Transform

能量泄漏

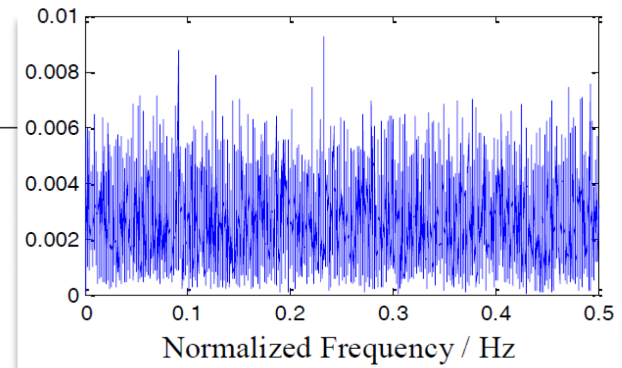


Fig 5, the FFT spectrum of the random signal

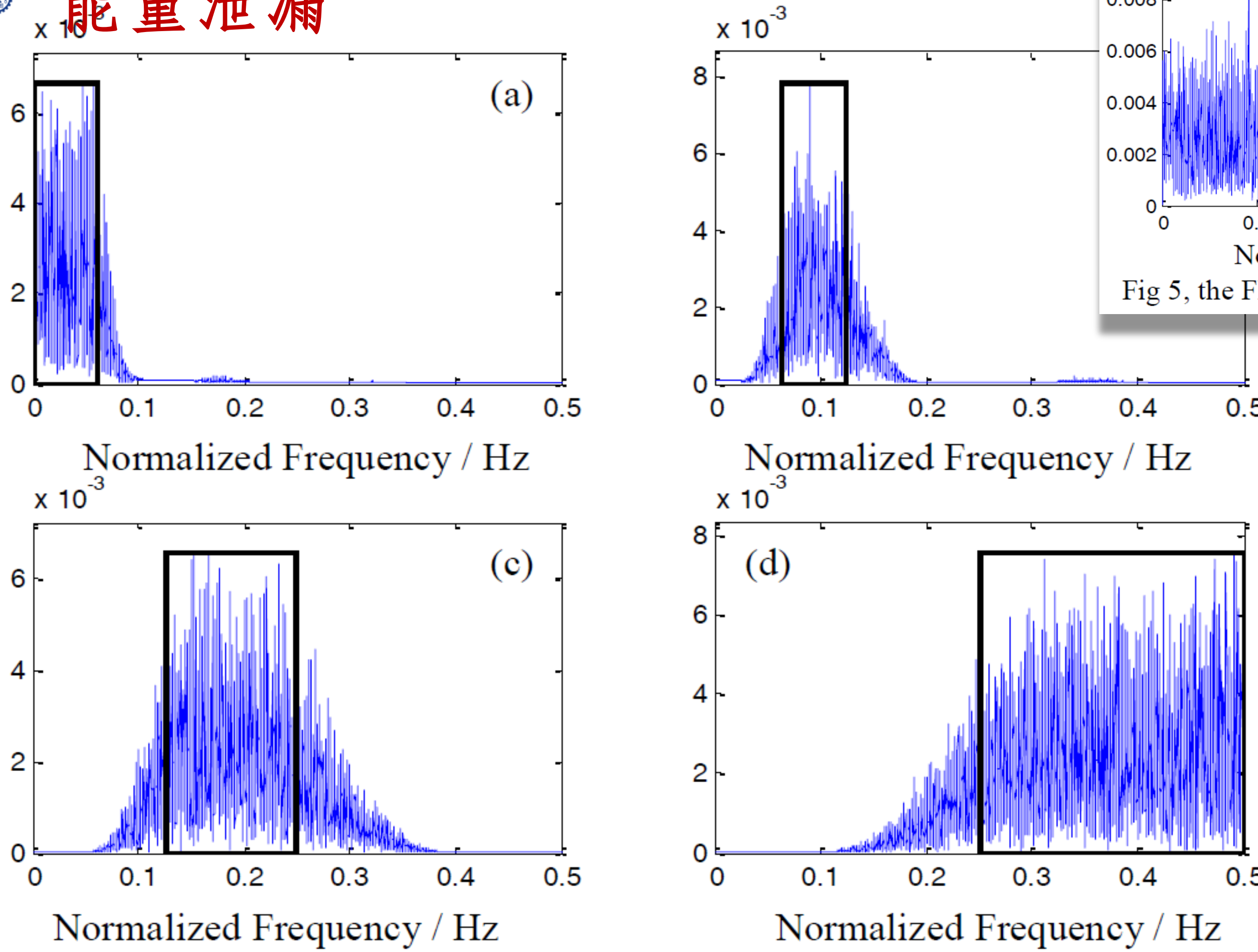
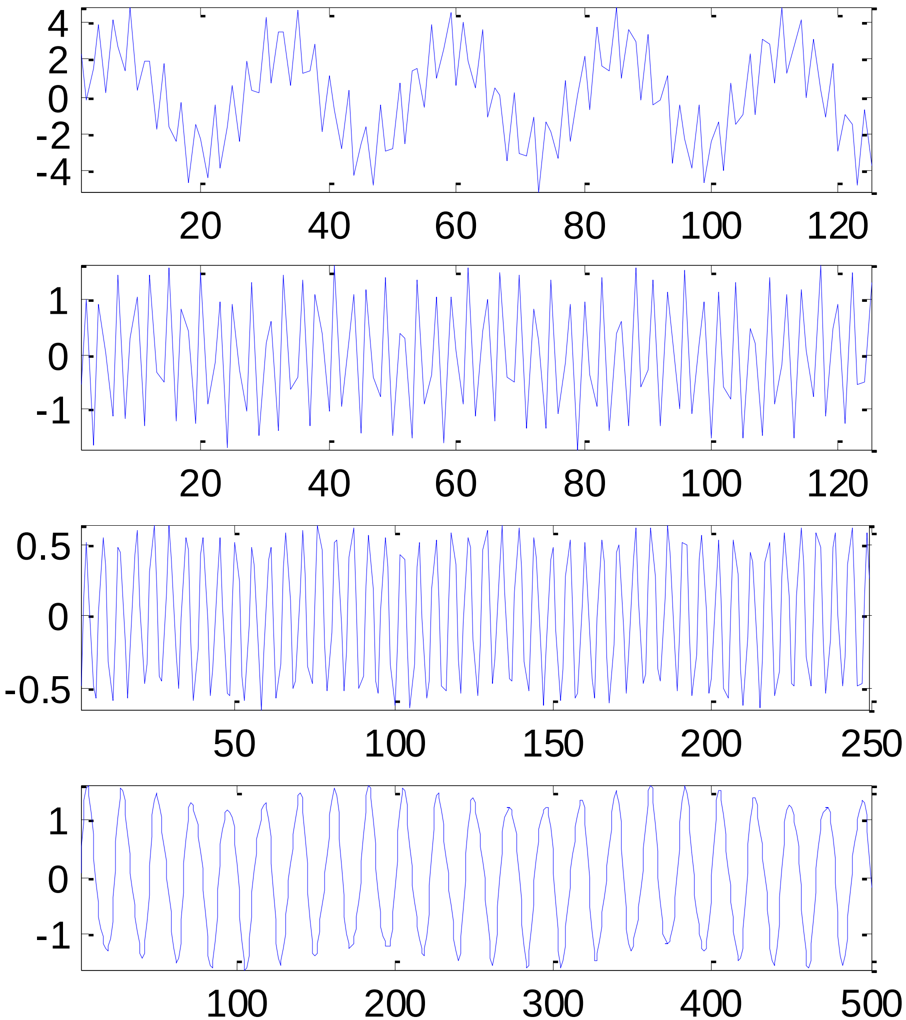


Fig 6, the FFT spectra of the approximation and three details

[(a) A_3f ; (b) D_3f; (c) D_2f; (d) D_1f]

Discrete Wavelet Transform

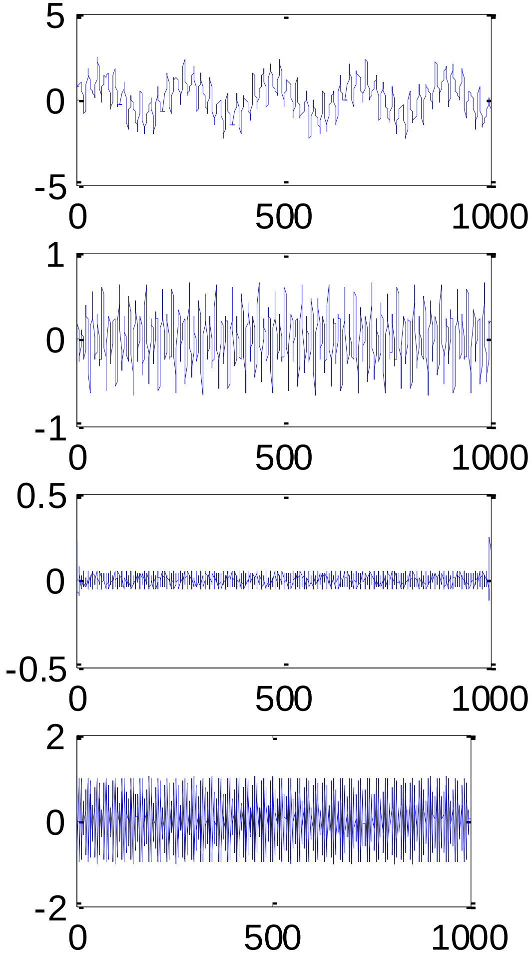
- Matlab示例 - wavedec

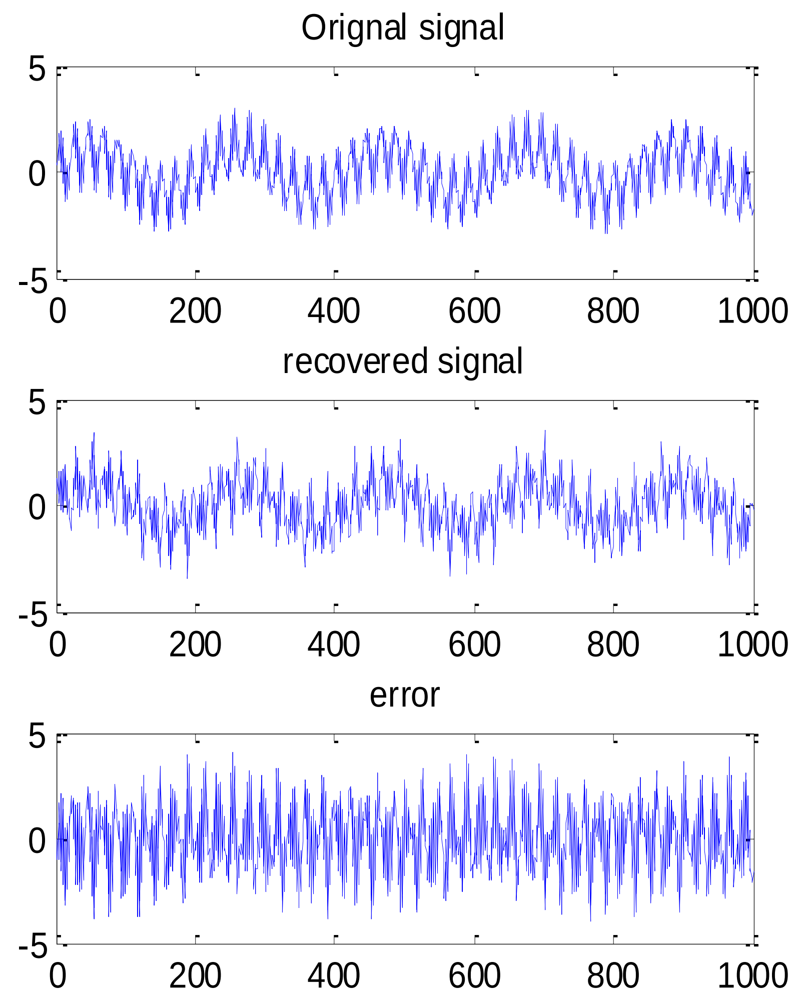



```
load sumsin; s = sumsin;
[c,l] = wavedec(s,3,'db1');
[b,nLevel ]= size(l);
iStart = 1;
figure(1)
for k=1:nLevel-1,
    iEnd = iStart + l(k)-1;
    subplot(4,1,k)
    sig = c(iStart:iEnd);
    plot(sig);
    axis tight
    iStart = iEnd  + 1;
end
```

Discrete Wavelet Transform

- Matlab示例 - waverec





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