§4. 正交阵与酉阵

一、酉阵 $A^*A = E_n$

正交阵 $AA^{T} = E_n$

$\Big\} \to$ 正规阵

th. 设 A 为酉阵，则 A 之特征值 $\boxed{|\lambda_i| = 1}$.

证. 设 $A = U\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}U^*$，$U$ 为酉阵.

则 $A^*A = U\begin{pmatrix}\lambda_1\bar{\lambda}_1 & & \\ & \ddots & \\ & & \lambda_n\bar{\lambda}_n\end{pmatrix}U^* = E_n$

$\Rightarrow \mathrm{diag}(\lambda_1\bar{\lambda}_1, \cdots, \lambda_n\bar{\lambda}_n) = E_n \Longrightarrow |\lambda_i| = 1$

Th. 设 A 为酉阵(正交阵)

$\Longleftrightarrow$ A 的 n 个列向量或 n 个行向量都是一组标准正交基

证: $A^*A = \begin{pmatrix} A_1^* \\ \vdots \\ A_n^* \end{pmatrix}(A_1, \cdots, A_n) = \begin{pmatrix} A_1^*A_1, & \cdots, & A_1^*A_n \\ \vdots & & \\ A_n^*A_1, & \cdots, & A_n^*A_n \end{pmatrix} = E_n$

$\Rightarrow A_i^*A_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$\Rightarrow A_1, \cdots, A_n$ 两两正交

类似 $A^{(1)}, \cdots, A^{(n)}$ 两两正交

二. 正交阵在正交相似变换下的最简单形式

$A^TA = E_n, \quad |\lambda_i| = 1.$

若 A 的特征值为 ± 1，则 $A \overset{\text{正交}}{\sim} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$

i.e., $\exists$ 正交阵 P，使 $P^TAP = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$

若 A 有复特征值 $\lambda_1 = \cos\theta_1 + i\sin\theta_1$，

则 $\bar{\lambda}_1 = \cos\theta_1 - i\sin\theta_1$ 亦为特征值.

假设A有复特征值2k个，

$\lambda_j=\cos\theta_j+i\sin\theta_j$，$\bar{\lambda}_j=\cos\theta_j-i\sin\theta_j$，$j=1\sim k$

设 $A\alpha_j=\lambda_j\alpha_j$

则 $A\bar{\alpha}_j=\bar{\lambda}_j\bar{\alpha}_j \Longrightarrow \bar{\alpha}_j$ 为对应于 $\bar{\lambda}_j$ 的特征向量.

又A为正交阵，$\lambda_1,\cdots,\lambda_k,\bar{\lambda}_1,\cdots,\bar{\lambda}_k$

设 $\alpha_1,\cdots,\alpha_k,\bar{\alpha}_1,\cdots,\bar{\alpha}_k$ 为两两正交的单位特征向量.

令 $\boxed{\beta_j=\frac{1}{\sqrt{2}}(\alpha_j+\bar{\alpha}_j),\ \gamma_j=\frac{1}{\sqrt{2}i}(\alpha_j-\bar{\alpha}_j)}$ $j=1\sim k$

则 $\beta_1,\gamma_1,\beta_2,\gamma_2,\cdots,\beta_k,\gamma_k$ 为两两正交的单位实向量.

设 $\alpha_{2k+1}, \alpha_{2k+2}, \cdots, \alpha_n$ 为 A 之对应于 1 之相互正交之单位特征向量.

$P = (\beta_1, \gamma_1, \cdots, \beta_k, \gamma_k, \alpha_{2k+1}, \cdots, \alpha_n)$

为正交阵.

$$A\beta_j = \frac{1}{\sqrt{2}} A(\alpha_j + \overline{\alpha_j}) = \frac{1}{\sqrt{2}}(\cos\theta_j + i\sin\theta_j)\alpha_j + \frac{1}{\sqrt{2}}(\cos\theta_j - i\sin\theta_j)\overline{\alpha_j}$$

$$= \cos\theta_j \cdot \frac{\alpha_j + \overline{\alpha_j}}{\sqrt{2}} + \sin\theta_j \cdot \frac{i(\alpha_j - \overline{\alpha_j})}{\sqrt{2}}$$

$$= \underline{\cos\theta_j \beta_j - \sin\theta_j \gamma_j}$$

$$\underline{A\gamma_j} = \frac{1}{\sqrt{2}i} A(\alpha_j - \overline{\alpha_j})$$

$$= \frac{1}{\sqrt{2}i}(\cos\theta_j + i\sin\theta_j)\alpha_j - \frac{1}{\sqrt{2}i}(\cos\theta_j - i\sin\theta_j)\overline{\alpha_j}$$

$$= \sin\theta_j \cdot \frac{\alpha_j + \overline{\alpha_j}}{\sqrt{2}} + \cos\theta_j \cdot \frac{\alpha_j - \overline{\alpha_j}}{\sqrt{2}i}$$

$$= \underline{\sin\theta_j \beta_j + \cos\theta_j \gamma_j}$$

$$AP = A(\beta_1, \gamma_1, \cdots, \beta_k, \gamma_k, \alpha_{2k+1}, \cdots, \alpha_n)$$

$$= (\beta_1, \gamma_1, \cdots, \beta_k, \gamma_k, \alpha_{2k+1}, \cdots, \alpha_n)\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\theta_k & \sin\theta_k & & & \\ & & & -\sin\theta_k & \cos\theta_k & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

i.e., ∃正交阵 P，使

$$\boxed{P^TAP=\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \cos\theta_k & \sin\theta_k & & & & & \\ & & & -\sin\theta_k & \cos\theta_k & & & & & \\ & & & & & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & & \\ & & & & & & & & -1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & -1 \end{pmatrix}}$$

其中 $\lambda_j=\cos\theta_j+i\sin\theta_j$，$\bar{\lambda}_j=\cos\theta_j-i\sin\theta_j$，$j=1\sim k$

为 A 之不同复特征值.

例，$A\in\mathbb{R}^{3\times 3}$，$A^TA=E_3$.

则A可正交相似于下列之一：

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix},\quad \begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix},\quad \begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix},\quad \begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix}$$

$$\begin{pmatrix}\cos\theta&\sin\theta&\\-\sin\theta&\cos\theta&\\&&1\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}\cos\theta&\sin\theta&\\-\sin\theta&\cos\theta&\\&&-1\end{pmatrix}$$

例. $A=\begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$,

$\lambda_1=1$, $\lambda_2=\frac{1+\sqrt{3}i}{2}$, $\lambda_3=\frac{1-\sqrt{3}i}{2}$

$A \sim \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \\ & & 1 \end{pmatrix}$,

对应于 $\lambda_1=1$ 之特征向量 $\alpha=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \to$ 旋转轴

$\theta=\frac{\pi}{3} \to$ 旋转角