Modern Control Theory
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Weijun Zhang （张伟军）

Associate Professor, Robotics Institute

School of Mechanical Engineering

Office:922 Building A of ME school

319 Building B of ME School

zhangweijun@sjtu.edu.edu.cn

021-34205559

Outline of Today's Lecture

- Linearization
- SS model of discrete control system
- Solution of Homogeneous State Equations

Linearization of a Nonlinear Function

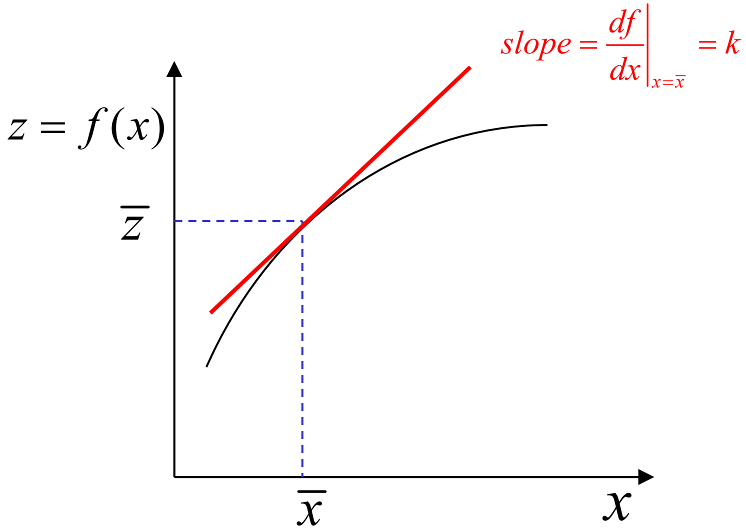
- Given a nonlinear function $z(t) = f(x(t))$, find a linear approximation.
- Basic idea is Taylor series expansion

$$z(t) = f(x(t)) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) + \left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} \frac{(x-\bar{x})^2}{2!} + \cdots\cdots$$

$$\approx f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) \equiv \bar{z} + k(x-\bar{x})$$

$$\Rightarrow \quad \boxed{z - \bar{z} = k(x-\bar{x})}$$

$$\boxed{\Delta z = k\Delta x}$$



Multiple Independent Variables

- Given a nonlinear function $z(t) = f(x(t), y(t))$, find a linear approximation.

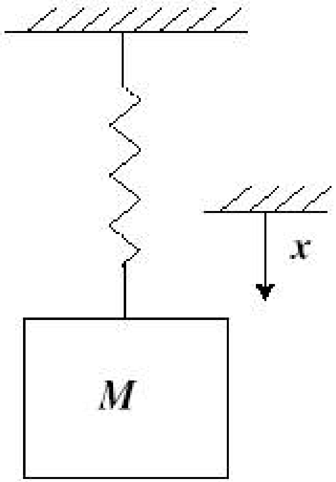
$$z(t) = f(\bar{x}, \bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} (x - \bar{x}) + \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} (y - \bar{y}) + \cdots\cdots$$

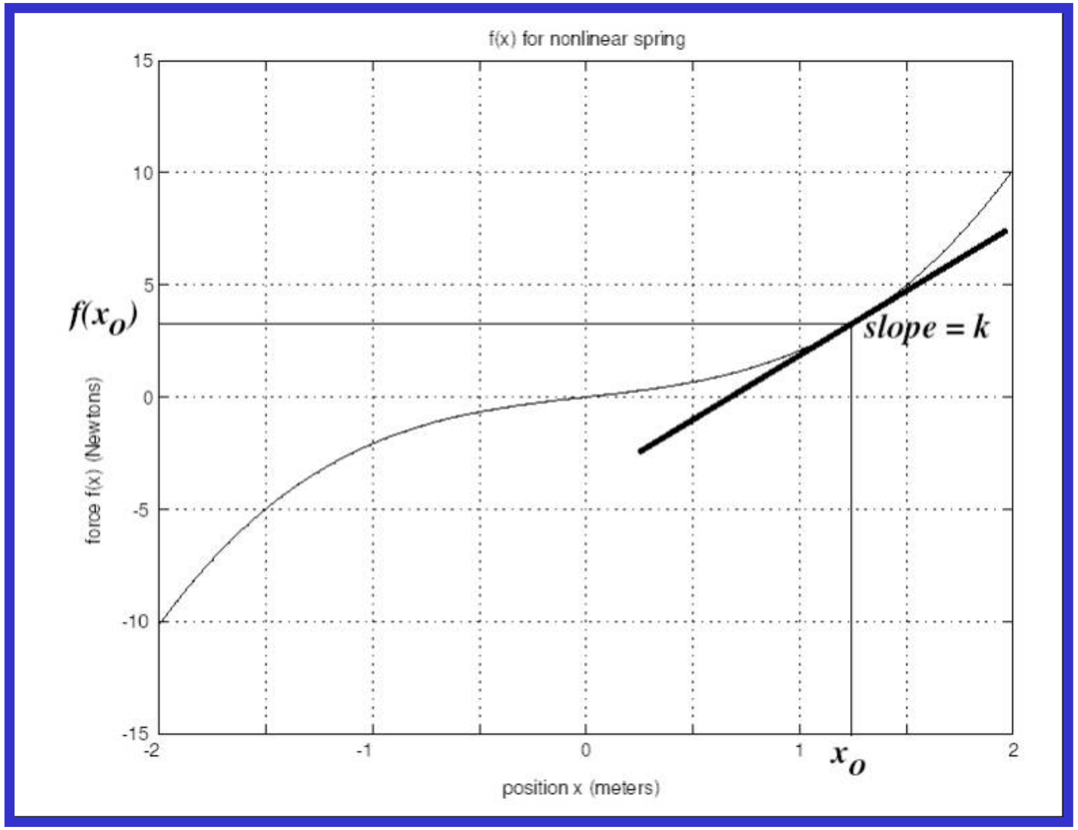
$$\approx f(\bar{x}, \bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} (x - \bar{x}) + \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} (y - \bar{y}) \equiv \bar{z} + a(x - \bar{x}) + b(y - \bar{y})$$

$$\boxed{\Delta z = a\Delta x + b\Delta y}$$

Ex.1 Nonlinear spring

1.1 Nonlinear spring





$$f(x) \approx f(x_o) + k\delta x$$

where k is the slope of the force/displacement curve at the operating point x_o.

Ex.1 Nonlinear spring (Cont)

The equilibrium is where the spring force balances gravity.

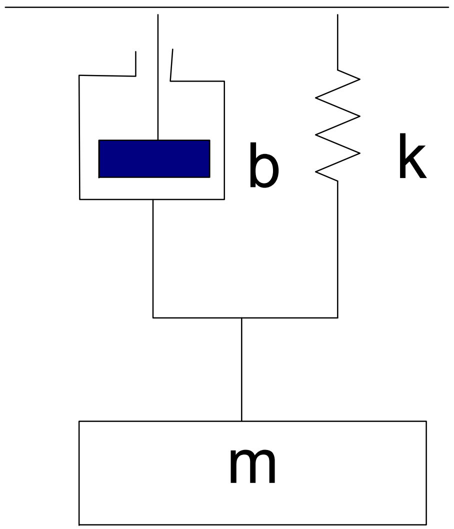
$$\dot{x}_o = 0$$

$$\ddot{x}_o = 0$$

$$Mg - f(x_o) = 0$$

$$f(x_o) = Mg$$

Ex.1 Nonlinear spring (Cont)



$$Mg - b\dot{x} - f(x) = M\ddot{x}$$

At the operating point x_o, the extension of the spring where $f(x_o)$ = Mg. we do the Taylor expansion:

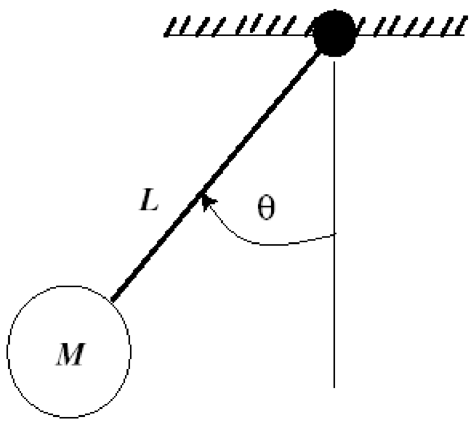
$$Mg - b(x_o \dot{+} \delta x) - \left(f(x_o) + \left.\frac{\partial f}{\partial x}\right|_{x=x_o} \cdot \delta x\right) = M(x_o \ddot{+} \delta x)$$

$$Mg - b\dot{\delta x} - f(x_o) - k\delta x = M\ddot{\delta x}$$

$$M\ddot{\delta x} + b\dot{\delta x} + k\delta x = 0$$

Ex.2 Pendulum

The torque is a nonlinear function of the angle .
We can use the operating point $\theta_0 = 0$, where the pendulum is hanging straight down.



$$f(\theta) = \tau = MgL\sin\theta$$

$$\begin{aligned}
\tau &\approx MgL\sin\theta_o + \frac{\partial}{\partial\theta}(MgL\sin\theta)\Big|_{\theta=\theta_o} \cdot \delta\theta \\
&\approx 0 + MgL\cos\theta_o\delta\theta \\
&\approx MgL\delta\theta
\end{aligned}$$

Ex.2 Pendulum (Cont.)

The equilibrium point:

$$\Sigma\text{torques} = J\ddot{\theta} = \tau - MLg\sin\theta$$

$$\dot{\theta}_o = 0$$

$$\ddot{\theta}_o = 0$$

$$\tau_o - MgL\sin\theta_o = 0$$

If there is no nominal applied torque, $\tau_0 = 0$, then we have

$$\theta_o = 0 \quad \textbf{OR} \quad \theta_o = \pi$$

Ex.2 Pendulum (Cont.)

$$\tau - MgL\sin\theta = ML^2\ddot{\theta}$$

$$\theta \rightarrow \theta_o + \delta\theta, \tau \rightarrow \tau_o + \delta\tau$$

$$(\tau_o + \delta\tau) - (MgL\sin\theta_o + MgL\cos\theta_o \cdot \delta\theta) = ML^2\ddot{\theta_o + \delta\theta}$$

$$\delta\tau - MgL\cos\theta_o \cdot \delta\theta = ML^2\ddot{\delta\theta}$$

(1)Hanging pendulum: $\theta_o = 0$

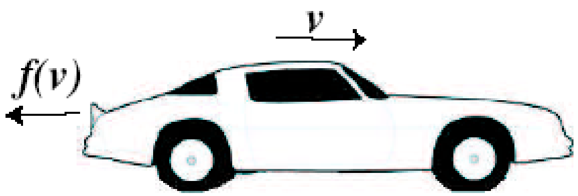
$$ML^2\ddot{\delta\theta} + MgL\delta\theta = \delta\tau$$

(2)Inverted pendulum: $\theta_o = \Pi$

$$ML^2\ddot{\delta\theta} - MgL\delta\theta = \delta\tau$$

Ex.3 Aerodynamic drag

The force f is a nonlinear function of the velocity v; typically, f is proportional to v^2. We use the operating point v_o.



$$f(v) = av^2$$

$$f(v) \approx f(v_o) + \left.\frac{\partial f}{\partial v}\right|_{v=v_o} \cdot \delta v$$

$$= av_o^2 + 2av\big|_{v=v_o} \cdot \delta v$$

$$= av_o^2 + (2av_o)\delta v$$

Ex.3 Aerodynamic drag (Cont.)

The equilibrium point:

$$\Sigma \text{forces} = M\dot{v} = F - f(v)$$

For v_o = constant to be a solution, we have

$$\dot{v}_o = 0$$

$$F_o - f(v_o) = 0$$

$$F_o - av_o^2 = 0$$

$$v_o = \sqrt{\frac{F_o}{a}}$$

Ex.3 Aerodynamic drag (Cont.)

$$F - av^2 = M\dot{v}$$

$$v \longrightarrow v_o + \delta v, F \longrightarrow F_o + \delta F$$

$$(F_o + \delta F) - \left(av_o^2 + (2av_o)\delta v\right) = M(v_o \dot{+} \delta v)$$

$$F_o + \delta F - av_o^2 - (2av_o)\delta v = M\dot{\delta v}$$

Now, noting that $F_o = av_o^2$, we get

$$M\dot{\delta v} + (2av_o)\delta v = \delta F$$

Note that $\delta v = 0, \delta F = 0$ is a solution of the linearized equation.

Equilibrium Point

- In the application to dynamic systems, linearizations were usually performed at an "equilibrium point" for a dynamic system, i.e., for a nonlinear system, $\dot{X} = F(X,U)$ at a point (X_0,U_0) such that $F(X_0,U_0) = 0$

$$\Downarrow$$

$$\dot{X} = 0$$

Linearization in state space model

-The time-varying system and Nonlinear system

$$\dot{x} = A(t)x + B(t)u \qquad \dot{x} = f(x,u,t)$$

$$y = c(t)x + D(t)u \qquad y = g(x,u,t)$$

$$\dot{x}_0 = f(x_0, u_0)$$ Taylor series expansion at equilibrium point

$$f(x,u) = f(x_0,u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \delta u + \alpha(\delta x, \delta u)$$

$$g(x,u) = g(x_0,u_0) + \left.\frac{\partial g}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial g}{\partial u}\right|_{x_0,u_0} \delta u + \beta(\delta x, \delta u)$$

Linearization in state space model

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \qquad \frac{\partial g}{\partial u} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_r} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_m}{\partial u_1} & \frac{\partial g_m}{\partial u_2} & \cdots & \frac{\partial g_m}{\partial u_r} \end{bmatrix}$$

Linearization in state space model

$$\delta\dot{x} = \dot{x} - \dot{x}_0 = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \delta u$$

$$\delta y = y - y_0 = \left.\frac{\partial g}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial g}{\partial u}\right|_{x_0,u_0} \delta u$$

$$\hat{x} = \delta x \quad \hat{u} = \delta u \quad \hat{y} = \delta y$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} = A \quad \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} = B$$

$$\left.\frac{\partial g}{\partial x}\right|_{x_0,u_0} = C \quad \left.\frac{\partial g}{\partial u}\right|_{x_0,u_0} = D$$

$$\dot{\hat{x}} = A\hat{x} + B\hat{u}$$

$$\hat{y} = C\hat{x} + D\hat{u}$$

Example

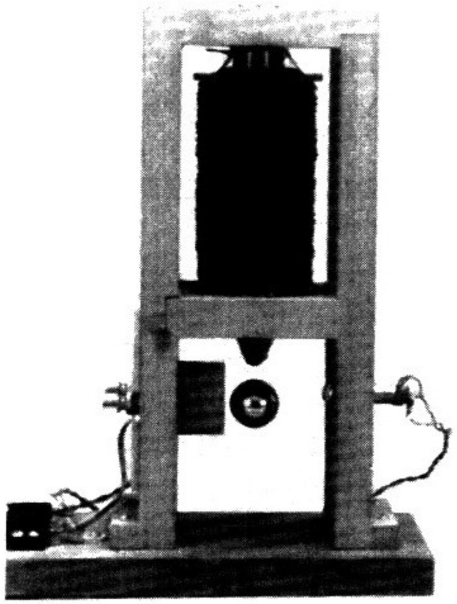
- Linearization at x0=0

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1 + x_2 + x_2^3 + 2u$$

$$y = x_1 + x_2^2$$

Ex. 2-22 (Franklin. pp. 70)

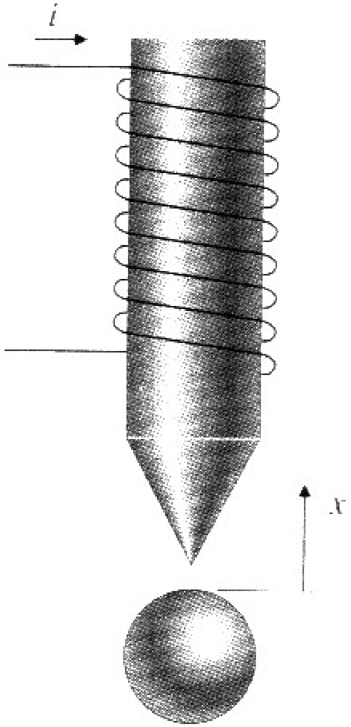


The simplified verson of a megnetic bearing that can be built in a laboratory is shown in figure,where one electromagnet is used to levitate a metal ball. The equation of motion of the ball is:

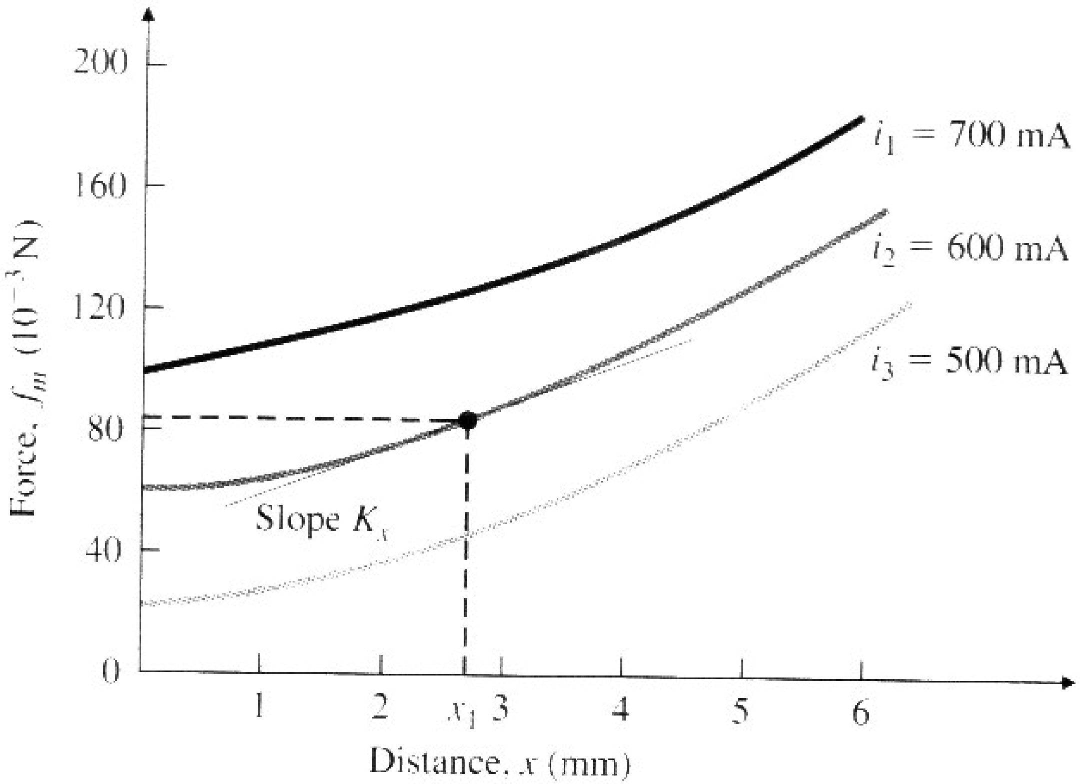
$$m\ddot{x} = f_m(x,i) - mg$$

Where the force fm is caused by the field of the electromagnet falls off with an inverse square relationship to the distance from the magnet. A experimental curves for a ball is shown in figure.

Ex. 2-22 (Cont.)



i
x



200
160
120
80
40
0
Force, f_m (10^{-3} N)
i_1 = 700 mA
i_2 = 600 mA
i_3 = 500 mA
Slope K_x
1
2
x_1 3
4
5
6
Distance, x (mm)

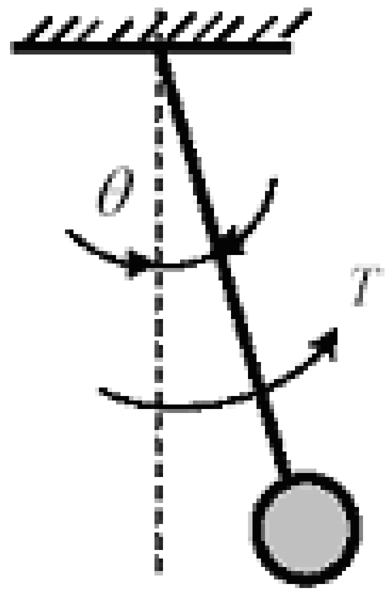
Ex. 2-22 (Cont.)

At the value for the current of i_2 =600mA and magnetic force f_m just cancels the gravity force $mg=82\times10^{-3}N$. Therefore the point (x_1,i_2) represents an equilibrium. Using the data, find the linearized equations of motion about the equilibrium point.

Keynotes-1: Linearization

Observe the pendulum shown in figure, moving in a gaseous fluid in which the aerodynamic drag is proportional to the square of the velocity of the body that moves through it, with b being the constant of proportionality i.e. aerodynamic drag is equal to

$$F_d = b(l\dot{\theta})^2 = bl^2\dot{\theta}^2$$



where θ is the angular pendulum distance away from its lowest point and l is the length of the pendulum). The pendulum is assumed to be moving only under the influence of gravity, with g being the gravity constant (around 9.8 N/Kg).

Keynotes-1: Linearization

1. **Write out equation of motion in terms of the angle θ, denoting angular pendulum distance away from its lowest point. (8 points)**

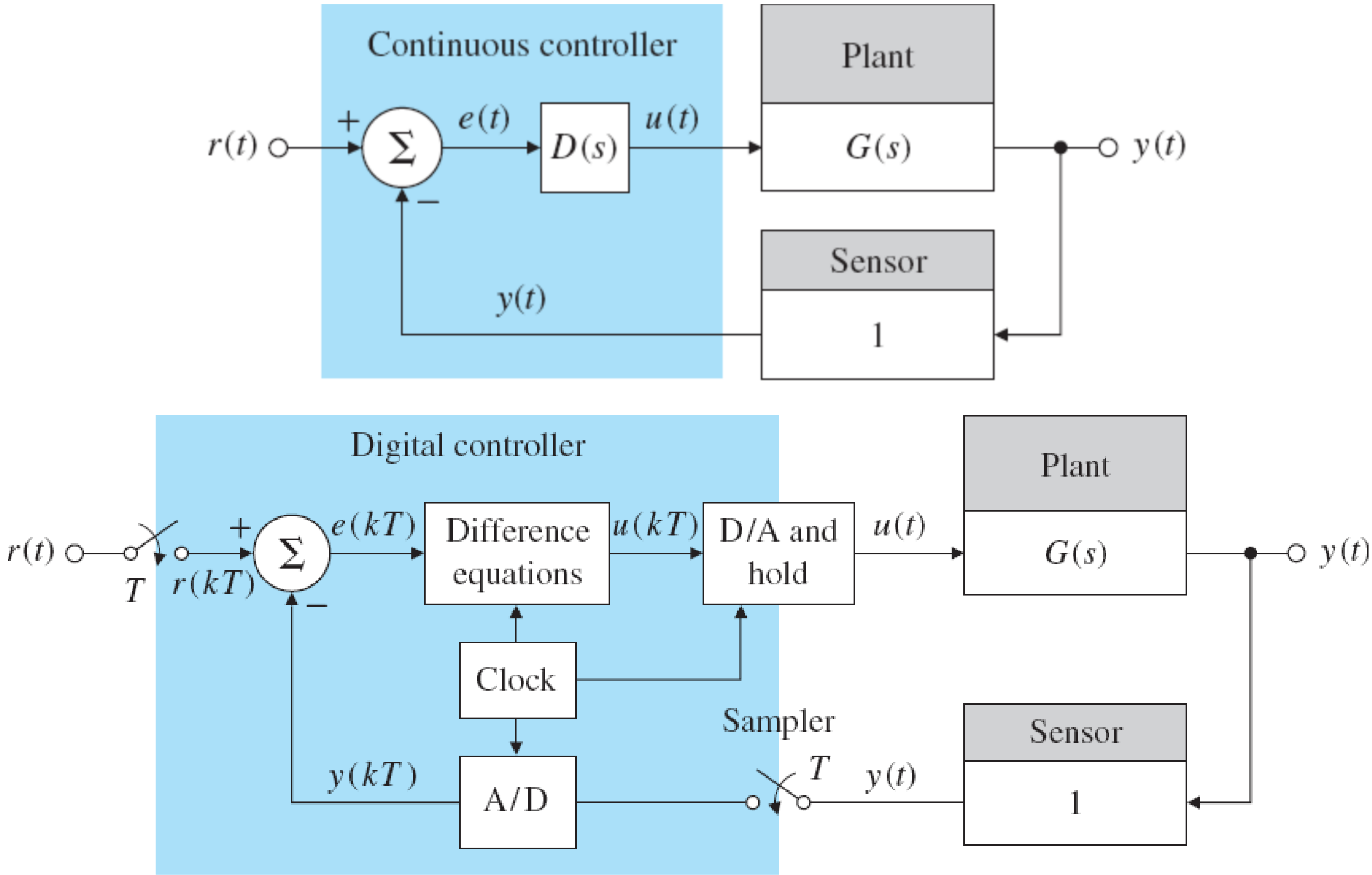
2. **Identify the equilibrium point. (6 points)**

3. **Linearize equations of motion obtained in part (1) around the angle θ_0=0, increment of the torque T as the system input, and increment of the angle θ as the system output. (10 points)**

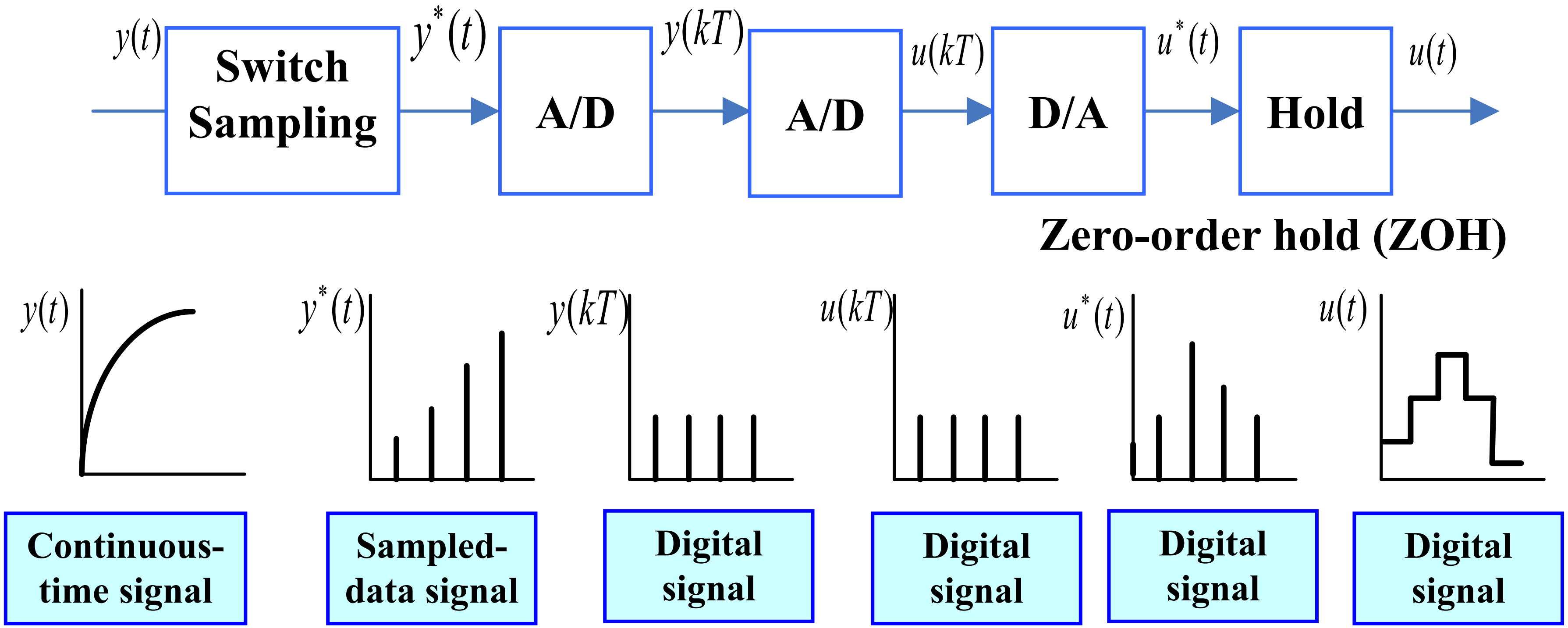
Outline of Today's Lecture

- Linearization
- SS model of discrete control system
- Solution of Homogeneous State Equations

Brief Introduction of Discrete Control System



Introduction on Digital control



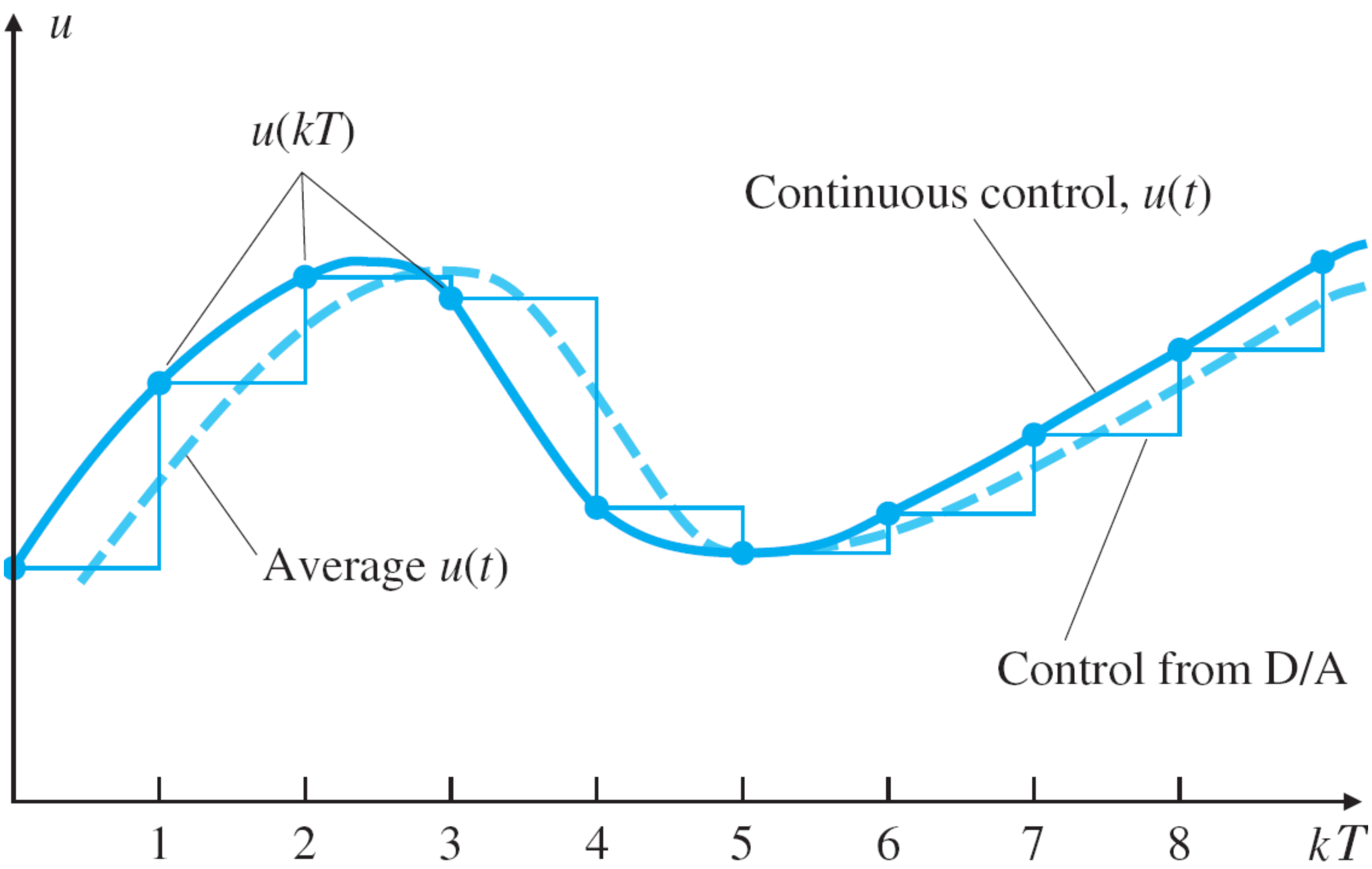
- **Sample period**
- **Sample rate**

Brief Introduction of Discrete Control System

- Digital Control Systems operate on samples of the sensed plant output rather than on the continuous signal.
- Digital Control Systems are represented by algebraic recursive equations called difference equations.

$$
\begin{aligned}
& y(k+n)+a_{n-1}y(k+n-1)+\ldots+a_1 y(k+1)+a_0 y(k) \\
& = b_n u(k+n)+b_{n-1}u(k+n-1)+\ldots+b_1 u(k+1)+b_0 u(k)
\end{aligned}
$$

Brief Introduction of Discrete Control System



Brief Introduction of Discrete Control System

- Continuous-time signal
- Discrete-time signal
- Sampled-data signal
- Digital signal
- Sample period
- Sample rate
- Zero-order hold (ZOH)
- Discrete control system

State Space Representation for Discrete Control System

传递函数为：

$$W(z)=\frac{Y(s)}{U(s)}=\frac{b_n z^n+b_{n-1}z^{n-1}+\ldots+b_1 z+b_0}{z^n+a_{n-1}z^{n-1}+\ldots+a_1 z+a_0}$$

等效的状态空间表达式:

$$x(k+1)=Gx(k)+hu(k)$$
$$y(k)=C^T x(k)+du(k)$$

State Space Representation for Discrete Control System

根据连续系统的推导，从模拟结构图可推导一种实现。

$$x(k+1)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x(k)+\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k)=[\mathrm{b}_0-a_0 b_n \quad b_1-a_1 b_n \quad \cdots \quad b_{n-1}-a_{n-1} b_n]x(k)$$

Outline of Today's Lecture

- Linearization
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- Solution of Homogeneous State Equations

Solution of Homogeneous State Equations

- **The evolution of state variable under the initial state by assuming no input.**

$$x(t_0) = x_0 \longrightarrow \boxed{\dot{x} = Ax} \longrightarrow x(t) = e^{A(t-t_0)}x_0; t \geq t_0$$

Normally t=0 and x(0)=x_0

$$\boxed{x(t) = e^{At}x_0; t \geq 0}$$

Review: the solution of scalar differential equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_0 u$$

Solution of Homogeneous State Equations

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_0 u$$

homogeneous solution

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = 0$$

particular solution

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_0 u$$

Solution of scalar differential equation
=homogeneous solution + one particular solution

Solution of Homogeneous State Equations

$$x(0) = x_0 \longrightarrow \boxed{\dot{x} = Ax} \longrightarrow x(t) = e^{A(t-t_0)}x_0; t \geq t_0$$

Firstly we consider the scalar differential equation :

$$\dot{x} = \mathrm{a}x$$

Suppose the solution has a form of a polynomial of t

$$x(t) = b_0 + b_1 t + \cdots + b_k t^k + \cdots$$

$$\dot{x}(t) = b_1 + 2b_2 t \cdots + kb_k t^{k-1} + \cdots$$

Solution of Homogeneous State Equations

Compare the coefficients of equal power of t :

$$\begin{cases} b_1 = ab_0 \\ b_2 = \dfrac{1}{2}ab_1 = \dfrac{1}{2}a^2 b_0 \\ b_3 = \dfrac{1}{3}ab_2 = \dfrac{1}{3\times 2}a^3 b_0 \\ \cdots \\ b_k = \dfrac{1}{k}ab_{k-1} = \dfrac{1}{k!}a^k b_0 \\ \cdots \end{cases}$$

$$x(t) = b_0 + b_1 t + \cdots + b_k t^k + \cdots$$

t=0,x(t)=x(0)

b_0=x(0)

$$x(t) = (1 + at + \frac{1}{2!}a^2 t^2 + \cdots + \frac{1}{k!}a^k t^k + \cdots)x(0)$$

Important Properties of State Transition Matrix

For equation of Matix A $\dot{x} = Ax$

$$x(t) = (I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots)x(0)$$

N×N Matrix

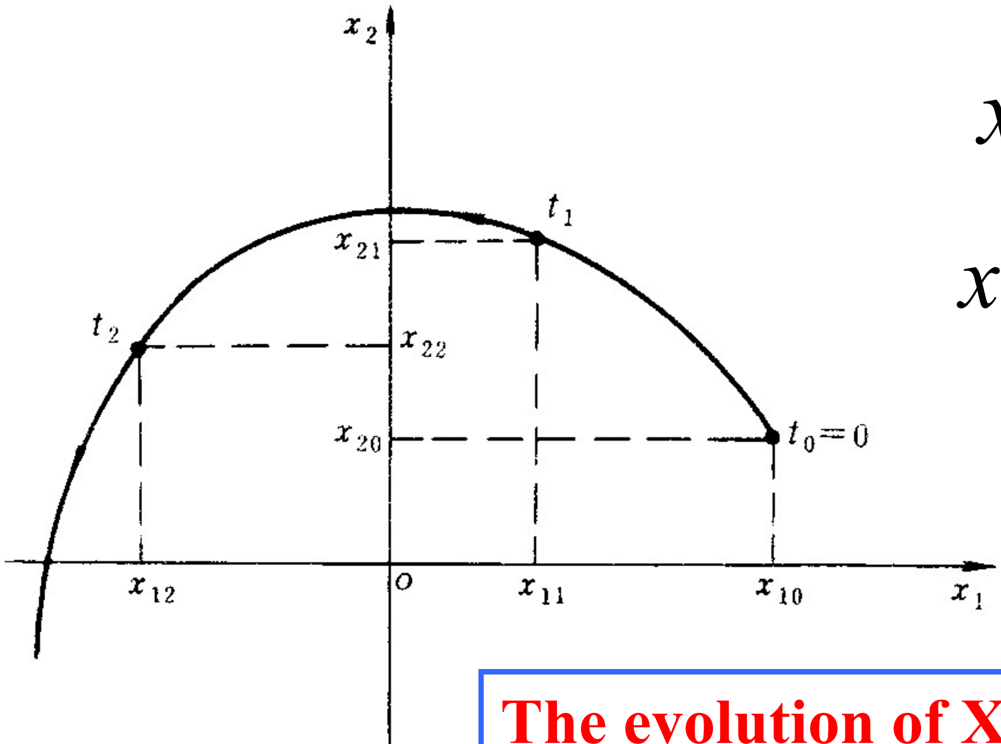
Define Matrix Exponential

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

State Transition Matrix

Important Properties of State Transition Matrix

$$\dot{x} = Ax \Rightarrow e^{At} \xrightarrow{deifne} \Phi(t)$$

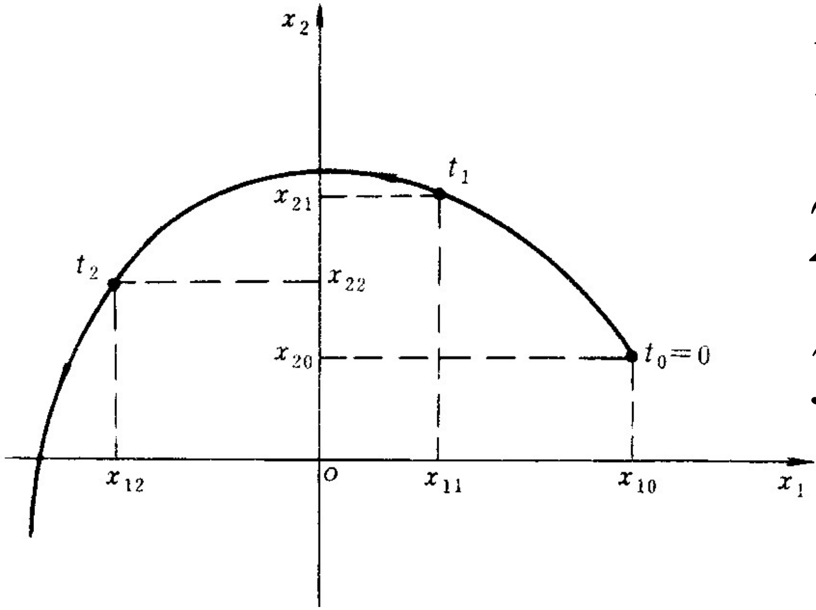


$$x(t) = \Phi(t)x_0$$

$$x(t_2) = \Phi(t_2 - t_1)x(t_1)$$

The evolution of X(t) can be solved by the state transition matrix and any given state.

Important Properties of State Transition Matrices



$$1.\Phi(t+\tau)=\Phi(t)\Phi(\tau)$$

$$2.\Phi(0)=I$$

$$3.[\Phi(t)]^{-1}=\Phi(-t)$$

$$4.[\Phi(t)]^{n}=\Phi(nt)$$

$$5.\Phi(t_2-t_1)\Phi(t_1-t_0)=\Phi(t_2-t_0)$$

$$=\Phi(t_1-t_0)\Phi(t_2-t_1)$$

Laplace transform Approach to the solution of Homogeneous Equations

$$\dot{x} = Ax$$

$$\Downarrow$$

$$sX(s) - x(0) = AX(s)$$

$$\Downarrow$$

$$(sI - A)X(s) = x(0)$$

$$\Downarrow$$

$$X(s) = (sI - A)^{-1}x(0)$$

$$\Downarrow$$

$$X(s) = L^{-1}[(sI - A)^{-1}]x(0)$$

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

Laplace transform Approach to the solution of Homogeneous Equations

Example: Obtain the state transition matrix φ(t) of the following system:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Solution: The state transition Matrix φ(t) is given by:

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

Laplace transform Approach to the solution of Homogeneous Equations

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{(s+1)} - \dfrac{1}{(s+2)} & \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)} \\ \dfrac{-2}{(s+1)} + \dfrac{2}{(s+2)} & \dfrac{-1}{(s+1)} + \dfrac{2}{(s+2)} \end{bmatrix}$$

Laplace transform Approach to the solution of Homogeneous Equations

$$\Phi(s) = \begin{bmatrix} \dfrac{2}{(s+1)} - \dfrac{1}{(s+2)} & \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)} \\ \dfrac{-2}{(s+1)} + \dfrac{2}{(s+2)} & \dfrac{-1}{(s+1)} + \dfrac{2}{(s+2)} \end{bmatrix}$$

$$\Downarrow$$

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Laplace transform Approach to the solution of Homogeneous Equations

Example: Obtain the Homogeneous solution of the following system:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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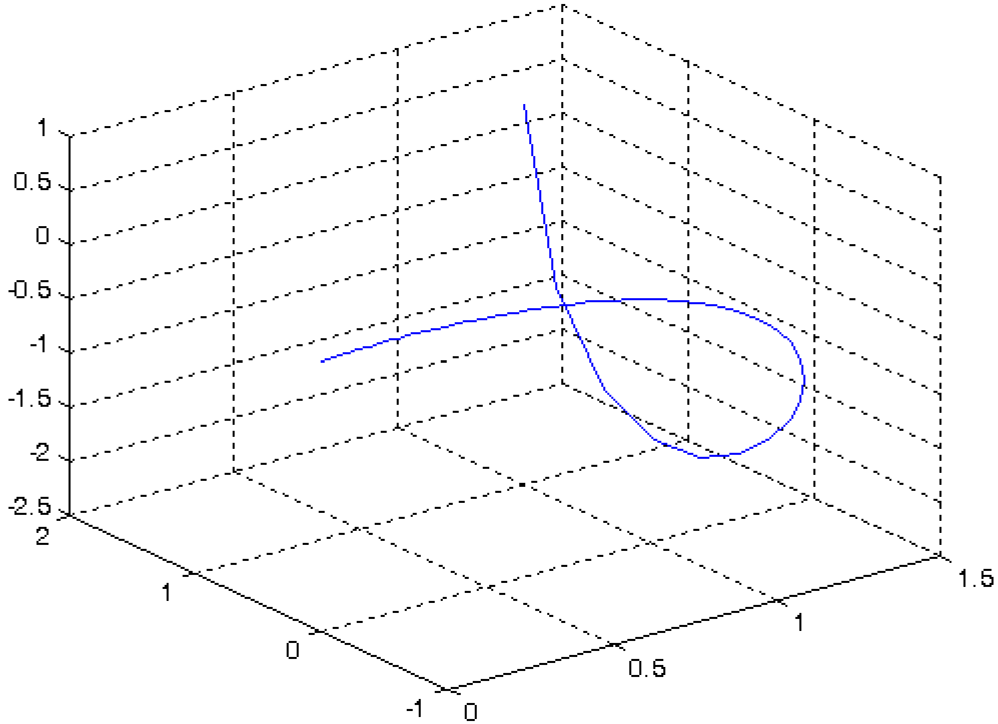
```
a=[0 1 0;0 0 1;-6 -11 -6];
 x0=[1;1;1];
 t=0:0.1:10;
 for i=1:length(t)
x(:,i)=expm(a*t(i))*x0;
end
 plot3(x(1,:),x(2,:),x(3,:));
grid on
```



Matrix Transformation Method for solution of e^{At}

几个特殊矩阵的矩阵指数函数

1. 若A为对角矩阵

$$A = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad e^{At} = \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Matrix Transformation Method for solution of e^{At}

2. 若A能通过非奇异变换对角化 $T^{-1}AT = \Lambda$

$$e^{At} = \Phi(t) = T\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

Matrix Transformation Method for solution of e^{At}

3. 若A为Jordan阵

$$A = J = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \quad e^{Jt} = \Phi(t) = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ 0 & 1 & t & \cdots & \frac{1}{(n-2)!}t^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix Transformation Method for solution of e^{At}

1. 根据定义计算—数值解法

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

例题1　已知

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$ 求 e^{At}

2. 变换A为Jordan标准型

(1). 特征根互异

$$T^{-1}AT = \Lambda \quad e^{At} = \Phi(t) = T\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

例题2　采用线性变换法求 e^{At}

Matrix Transformation Method for solution of e^{At}

3.变换A为Jordan标准型

(2). 特征根重根 $T^{-1}AT = J \qquad e^{At} = Te^{jt}T^{-1}$

例题2 采用线性变换法求 e^{At} $\qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

4.采用拉氏反变换法

可以证明 $e^{At} = \Phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$

例题3 A阵同例1，采用拉氏反变换法求 e^{At}

Matrix Transformation Method for solution of e^{At}

5.应用凯莱一哈密顿定理

$$e^{At} = a_{n-1}(t)A^{n-1} + a_{n-2}(t)A^{n-2} + \dots + a_1(t)A + a_0 I$$

Case1 特征根互异

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \cdots \\ a_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ & & \cdots & \cdots & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \cdots \\ e^{\lambda_n t} \end{bmatrix}$$

Case2 特征根有重根

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \cdots \\ \cdots \\ a_{n-2}(t) \\ a_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & (n-1)\lambda_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots & \frac{(n-1)(n-2)}{2!}\lambda_1^{n-3} \\ 0 & 1 & 2\lambda_1 & \cdots & \cdots & (n-1)\lambda_1^{n-2} \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-2} & \lambda_1^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{(n-1)!}t^{n-1}e^{\lambda_1 t} \\ \frac{1}{(n-2)!}t^{n-2}e^{\lambda_1 t} \\ \cdots \\ \cdots \\ te^{\lambda_1 t} \\ e^{\lambda_1 t} \end{bmatrix}$$

Matrix Transformation Method for solution of e^{At}

例题4　A阵同例1，采用凯莱—哈密顿法求 e^{At}

例题5　A阵同例2，采用凯莱—哈密顿法求 e^{At}

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Laplace transform Approach to the solution of Homogeneous Equations

Example: A given system

$$\dot{x} = Ax + Bu$$

To satisfy

$$u = 0, x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$u = 0, x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad x(t) = \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix}$$

Find the system transition matrix $\Phi(t)$ **and system matrix A**