# Ch1. 矩阵分解

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一. 三角分解. Cholesky 分解

Def. 设 $A\in R^{n\times n}$，若 $\exists$ 单位下三角阵 $L$，

及上三角阵 $U$，使 $\boxed{A=LU}$，

称 $A$ 有三角分解，or $LU$ 分解.

注. 三角分解未必存在.

如. $A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ 不存在三角分解.

设 $\begin{pmatrix}1&0\\a&1\end{pmatrix}\begin{pmatrix}b&d\\0&c\end{pmatrix}=\begin{pmatrix}b&d\\ab&ad+c\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\Rightarrow b=0.$

$\Rightarrow 0=1$ 矛盾

Th. $A$ 可逆，$A$ 存在三角分解

$\Longleftrightarrow$ $A$ 的所有顺序主子式 $\neq 0$.

Th.（LU分解唯一性）

设 $A \in R^{n\times n}$，可逆，且 $A = LU$，则分解唯一.

证. 设 $A = L_1U_1 = L_2U_2$

$$\Rightarrow \underbrace{L_2^{-1}L_1}_{\text{单位下三角}} = \underbrace{U_2U_1^{-1}}_{\text{上三角}} \longrightarrow \text{对角阵}$$

$$\| \quad \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \Rightarrow \begin{matrix} L_1 = L_2 \\ U_1 = U_2 \end{matrix}$$

单位下三角阵之逆仍为单位下三角阵.

Th.（Cholesky 分解）

设 $A \in \mathbb{R}^{n\times n}$，$A \succ 0$.

则 $\exists !$ 对角元为正的下三角阵 $G$，$\ni$ $\boxed{A = GG^T}$

Cholesky 分解

记.

$$
\begin{array}{c}
\begin{pmatrix}
g_{11} & & & & \\
g_{21} & g_{22} & & & \\
\vdots & & \ddots & & \\
g_{i1} & g_{i2} & \cdots & g_{ij} & \\
\vdots & & & & \\
g_{n1} & g_{n2} & \cdots & \cdots & g_{nn}
\end{pmatrix}
\\
\qquad\qquad\quad j \qquad\quad
\end{array}
\begin{pmatrix}
g_{11} & g_{21} & \cdots & g_{j1} & \cdots & g_{n1} \\
 & g_{22} & \cdots & g_{j2} & \cdots & g_{n2} \\
 & & \ddots & & & \\
 & & & g_{jj} & \cdots & g_{nj} \\
 & & & & \ddots & \\
 & & & & & g_{nn}
\end{pmatrix}
$$

$\Rightarrow a_{ij} = \sum_{k=1} g_{ik} g_{jk}$，$i \geq j$.

$\Rightarrow g_{jj} g_{ij} = a_{ij} - \sum_{k=1}^{j-1} g_{ik} g_{jk}$.

## 二. 满秩分解

$A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=r$.

则 $\boxed{A = LR}$, 其中 $L \in \mathbb{R}^{m\times r}$ 列满秩

$R \in \mathbb{R}^{r\times n}$ 行满秩

$$A \xrightarrow{\text{行变换}} H_A = \begin{pmatrix} H_r \\ 0 \end{pmatrix}.$$

设第 $k$ 行第一个非零元 1 所在列为 $j_k$.

则 $j_1 < j_2 < \cdots < j_r$

从而 $A = (A_{j_1}, A_{j_2}, \cdots, A_{j_r})\, H_r$.

# 三、正交三角分解

Gram-Schmidt 正交化方法之直接应用.

Th. 设 $A\in\mathbb{C}^{n\times n}$, A列满秩

则 $\exists!$ 酉阵 $U$, 和对角线元素都大于零

上三角阵 $R$,

使得 $\boxed{A=UR}$

证：$A=(\alpha_1,\cdots,\alpha_n)$ 为满秩，则 $\alpha_1,\cdots,\alpha_n$ 无关.

$$\alpha_1,\cdots,\alpha_n \xrightarrow{\text{正交化方法}} \eta_1,\cdots,\eta_n.$$

$$\begin{cases} \eta_1=\alpha_1. \\ \eta_2=\alpha_2-\dfrac{(\alpha_2,\eta_1)}{(\eta_1,\eta_1)}\eta_1 \\ \quad\vdots \\ \eta_n=\alpha_n-\dfrac{(\alpha_n,\eta_1)}{(\eta_1,\eta_1)}\eta_1-\cdots-\dfrac{(\alpha_n,\eta_{n-1})}{(\eta_{n-1},\eta_{n-1})}\eta_{n-1} \end{cases}$$

$\beta_1$

则
$$\begin{cases} \alpha_1=\eta_1 \\ \alpha_2=\eta_2+\dfrac{(\alpha_2,\eta_1)}{(\eta_1,\eta_1)}\eta_1=\dfrac{(\alpha_2,\eta_1)}{\sqrt{(\eta_1,\eta_1)}}\cdot\dfrac{\eta_1}{\sqrt{(\eta_1,\eta_1)}} \\ \quad\cdots \\ \alpha_n=\eta_n+\dfrac{(\alpha_n,\eta_1)}{(\eta_1,\eta_1)}\eta_1+\cdots+\dfrac{(\alpha_n,\eta_{n-1})}{(\eta_{n-1},\eta_{n-1})}\eta_{n-1} \end{cases}$$

令 $\beta_i = \dfrac{\eta_i}{\sqrt{(\eta_i, \eta_i)}}$, $i = 1 \sim n$

则 $U = (\beta_1, \cdots, \beta_n)$ 为酉阵.

且

$$\alpha_1 = \sqrt{(\eta_1, \eta_1)}\, \beta_1$$

$$\alpha_2 = \sqrt{(\eta_2, \eta_2)}\, \beta_2 + (\alpha_2, \beta_1)\beta_1$$

$$\vdots$$

$$\alpha_n = \sqrt{(\eta_n, \eta_n)}\, \beta_n + (\alpha_n, \beta_1)\beta_1 + \cdots + (\alpha_n, \beta_{n-1})\beta_{n-1}$$

即 $A = (\alpha_1, \cdots, \alpha_n)$

$$= \underbrace{(\beta_1, \cdots, \beta_n)}_{U} \underbrace{\begin{pmatrix} \sqrt{(\eta_1,\eta_1)} & (\alpha_2, \beta_1) & \cdots & (\alpha_n, \beta_1) \\ & \sqrt{(\eta_2,\eta_2)} & \cdots & (\alpha_n, \beta_2) \\ & & \ddots & \vdots \\ & & & \sqrt{(\eta_n,\eta_n)} \end{pmatrix}}_{R}$$

下证唯一性.　　　　$U_1, U_2$ 为酉阵

设 $A = U_1R_1 = U_2R_2$,　　$R_1, R_2$ 对角正, 上三角阵

$$\underbrace{U_2^*U_1}_{\text{酉阵，正规阵}} = \underbrace{R_2R_1^{-1}}_{\text{上三角阵}} \longrightarrow \text{对角阵} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix},\quad |\mu_i| = 1$$

$$R_2 = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} R_1,\qquad \begin{pmatrix} \underset{\sim}{*} & \cdots & * \\ & \underset{\sim}{*} & * \\ & & \ddots \ \underset{\sim}{*} \end{pmatrix} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \begin{pmatrix} \underline{*} & & \\ & \ddots & * \\ & & \end{pmatrix}$$

$$\underbrace{a_{ii}}_{\text{实正}} = \mu_i \underbrace{b_{ii}}_{\text{实正}} \Rightarrow \mu_i = 1$$

$$\Rightarrow R_2 = R_1$$

$$\Rightarrow U_2 = U_1$$

Th. 设 $A\in\mathbb{C}^{n\times r}$, $\operatorname{rank}(A)=r$,

则 $A=UR$, 其中 $U\in\mathbb{C}^{n\times r}$, $U^*U=E_r$

$R\in\mathbb{C}^{r\times r}$ 上三角阵, 对角元为正.

例. 用QR分解求解 $Ax=b$. 其中 $A=\begin{pmatrix}1 & 1 & 3\\ 2 & 1 & 2\\ 2 & -5 & 8\\ -1 & 3 & -7\end{pmatrix}$, $b=\begin{pmatrix}-1\\ 1\\ -1\\ 9\end{pmatrix}$

解. $A=QR$, $Ax=b \Longleftrightarrow QRx=b$

$\Longleftrightarrow Rx=Q^Tb \Longleftrightarrow x=R^{-1}Q^Tb$

$(R,\ Q^Tb) \xrightarrow{\text{行变换}} (E,\ R^{-1}Q^Tb)$

# 四. 奇异值分解

Th.（奇异值分解定理）

设 $A\in\mathbb{C}^{m\times n}$，$\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_r>0$，

则存在 m阶 和 n阶 酉阵 $U, V$

使得 $\boxed{A=U\Sigma V^*}$，

其中 $\Sigma=\mathrm{diag}(\sigma_1,\cdots,\sigma_r,0,\cdots,0)_{m\times n}$.

这里 $\sigma_1,\cdots,\sigma_r$ 称为 A 的奇异值.

引理1. $\boxed{\mathrm{rank}(A)=\mathrm{rank}(A^*A)=\mathrm{rank}(AA^*)}$

证: $Ax=0 \Rightarrow A^*Ax=0$

$A^*Ax=0 \Rightarrow x^*A^*Ax=0 \Rightarrow Ax=0$

即: $Ax=0$与$A^*Ax=0$之解空间一致

从而 $\mathrm{rank}(A)=\mathrm{rank}(A^*A)$.

引理2. $A^*A$与$AA^*$之特征值都大于或等于零.

证: $A^*A \geq 0, \Rightarrow \boxed{\lambda_i(A^*A)\geq 0.}$

引理3. 设 $A\in\mathbb{C}^{m\times n}$，$\operatorname{rank}(A)=r$

则，$A^*A$ 与 $AA^*$ 有完全相同之非零特征值 $\lambda_1,\cdots,\lambda_r$.

证：$A^*A$ 与 $AA^*$ 正规阵 ～ 酉对角阵，

对角元素为特征值，$\boxed{\text{个数均为 } r.}$

设 $\lambda$ 为 $A^*A$ 之特征值，$\boxed{A^*A\alpha_i=\lambda\alpha_i}$，$i=1\sim k$.

（代数重数 = 几何重数 = $k$）

$\Rightarrow$ $\boxed{AA^*A\alpha_i=\lambda A\alpha_i}$，$i=1\sim k$.（下证 $A\alpha_1,\cdots,A\alpha_k$ 无关）

设 $l_1A\alpha_1+\cdots+l_kA\alpha_k=0$

$\Rightarrow l_1\lambda\alpha_1+\cdots+l_k\lambda\alpha_k=0 \Rightarrow l_1\alpha_1+\cdots+l_k\alpha_k=0\Rightarrow l_i=0$

故 $\lambda$ 作为 $AA^*$ 之特征值之重数 $\geq$ 作为 $A^*A$ 之特征值之重数

反之亦然

Def. $A\in\mathbb{C}^{m\times n}$, $\mathrm{rank}(A)=r$, 设 $A^*A$ 之正特征值 $\lambda_1,\cdots,\lambda_r$.

称 $\sqrt{\lambda_1},\cdots,\sqrt{\lambda_r}$ 为 $A$ 之 $\boxed{\text{奇异值}}$.

引理4. 设 $\lambda$ 为 $A^*A$ 之正特征值,

$\alpha_1,\cdots,\alpha_k$ 为 $\lambda$ 之两两正交之单位特征向量.

则 $A\alpha_1,\cdots,A\alpha_k$ 为 $AA^*$ 之对应于 $\lambda$ 之相互正交之特征.

且 $\boxed{\sqrt{(A\alpha_i,A\alpha_i)}=\sqrt{\lambda}}$, $i=1\sim k$.

证. $A^*A\alpha_i=\lambda\alpha_i \Longrightarrow AA^*A\alpha_i=\lambda A\alpha_i$, $i=1\sim k$

$$(A\alpha_i,A\alpha_j)=\alpha_j^*A^*A\alpha_i=\lambda\alpha_j^*\alpha_i=\begin{cases}\lambda, & i=j\\ 0, & i\neq j\end{cases}$$

$\Longrightarrow A\alpha_1,\cdots,A\alpha_k$ 两两正交, $\sqrt{(A\alpha_i,A\alpha_i)}=\sqrt{\lambda}$.

Th. $A \in \mathbb{C}^{m\times n}$，$\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_r}$ 为 $A$ 之奇异值，

则 $\exists$ 酉阵 $U, V$，$\ni$ $A = U\Sigma V^* = U\begin{pmatrix} \sqrt{\lambda_1} & & & & \\ & \ddots & & & \\ & & \sqrt{\lambda_r} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}_{m\times n} V^*$.

证：设 $\alpha_1, \cdots, \alpha_n$ 为 $A^*A$ 之两两正交之单位特征向量.

$\boxed{A^*A\alpha_i = \lambda_i\alpha_i,\ i=1\sim r,\quad A^*A\alpha_i = 0,\ i=r+1,\cdots,n.}$

设 $\beta_1, \cdots, \beta_n$ 为 $AA^*$ 之两两正交之单位特征向量.

其中 $AA^*\beta_i = \lambda_i\beta_i$，$i=1\sim r$，$\Rightarrow$ $\boxed{\beta_i = \dfrac{A\alpha_i}{\sqrt{\lambda_i}}}$

令 $U = (\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_n)$，$V = (\alpha_1, \cdots \alpha_r, \cdots \alpha_n)$

则 $U^*AV = (\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_n)^* A(\alpha_1, \cdots \alpha_r, \alpha_{r+1}, \cdots \alpha_n)$

$= (\beta_1, \cdots, \beta_n)^*(\sqrt{\lambda_1}\beta_1, \cdots, \sqrt{\lambda_r}\beta_r, 0, \cdots, 0)$

$$= (\beta_1, \cdots, \beta_n)^* (\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_n) \begin{pmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

即 $A = U \Sigma V^*$.

例. 求 $A=\begin{pmatrix}1&0&1\\0&1&-1\end{pmatrix}$ 的奇异值分解.

解. $AA^*=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$, $|\lambda E-AA^*|=(\lambda-1)(\lambda-3)$.

$\lambda_1=1$, $\beta_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$, $\lambda_2=3$, $\beta_2=\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

则 $A^*A$ 特征值: $\lambda_1=1$, $\alpha_1=\frac{1}{\sqrt{\lambda_1}}A^*\beta_1=A^*\beta_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}$

$\lambda_2=3$, $\alpha_2=\frac{1}{\sqrt{\lambda_2}}A^*\beta_2=\frac{1}{\sqrt6}\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\lambda_3=0$, $\alpha_3=\frac{1}{\sqrt3}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $(A^*Ax=0)$

令 $U=(\beta_1,\beta_2)$, $V=(\alpha_1,\alpha_2,\alpha_3)$

则 $A=U\begin{pmatrix}1&0&0\\0&\sqrt3&0\end{pmatrix}V^*$

# 五. 谱分解

## (一) 正规阵的谱分解

$A$ 为正规阵 $\Rightarrow \exists$ 酉阵 $U$, 使 $A = U\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{pmatrix}U^*$

$$\underline{\underline{A}} = (u_1, \cdots, u_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix} = \sum_{i=1}^{n} \lambda_i u_i u_i^*$$

$$= (u_{1,1}, \cdots, u_{1,k_1}, \cdots, u_{s,1}, \cdots, u_{s,k_s})\begin{pmatrix} \gamma_1 & & & & & \\ & \ddots & & & & \\ & & \gamma_1 & & & \\ & & & \ddots & & \\ & & & & \gamma_s & \\ & & & & & \ddots \\ & & & & & & \gamma_s \end{pmatrix}\begin{pmatrix} u_{1,1}^* \\ \vdots \\ u_{1,k_1}^* \\ \vdots \\ u_{s,1}^* \\ \vdots \\ u_{s,k_s}^* \end{pmatrix}$$

$$= \sum_{i=1}^{s} \gamma_i \underbrace{\sum_{j=1}^{k_i} u_{i,j} u_{i,j}^*}_{P_i} = \underline{\underline{\sum_{i=1}^{s} \gamma_i P_i}}.$$

其中 $\gamma_1, \cdots, \gamma_s$ 为 $A$ 的两两不同的特征值.

注意到：

$$\left.\begin{array}{ll}(u_i u_i^*)^* = u_i u_i^*, & i=1\sim n \\ (u_i u_i^*)^2 = u_i u_i^*, & i=1\sim n \\ (u_i u_i^*)(u_j u_j^*) = 0, & i\neq j\end{array}\right\} \Longrightarrow \begin{array}{ll} P_i^* = P_i, & i=1\sim s \\ P_i^2 = P_i, & i=1\sim s \\ P_i P_j = 0, & i\neq j\end{array}$$

称 $P$ 为正交投影阵.

Th. 如果 $A$ 为正规阵，则 $A=\sum_{i=1}^{s}\lambda_i P_i$，

其中 $\lambda_i$ 为不同特征值，$P_i^*=P_i$，$P_i^2=P_i$，$P_iP_j=0$.

注. 若 $A$ 为正规阵且可逆，则 $A^{-1}=\sum_{i=1}^{n}\frac{1}{\lambda_i}u_i u_i^*$.

例. 已知 $A=\begin{pmatrix}8 & 4 & -1\\ 4 & -7 & 4\\ -1 & 4 & 8\end{pmatrix}$, 试求谱分解, 及 $A^{2016}$

解. $|\lambda E-A| = (\lambda-9)^2(\lambda+9)$. $A^T=A$.

$\lambda_1=9$ (2重), $\alpha_1=\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$,

$\lambda_2=-9$ (1重), $\alpha_3=\frac{1}{6}\begin{pmatrix}\sqrt{2}\\-4\sqrt{2}\\\sqrt{2}\end{pmatrix}$.

令 $P_1=\alpha_1\alpha_1^*+\alpha_2\alpha_2^*$, $P_2=\alpha_3\alpha_3^*$

则 $A=9P_1-9P_2$

$A^{2016}=9^{2016}(P_1-P_2)^{2016}=9^{2016}(P_1+P_2)$.

## (二) 一般可对角化阵的谱分解

设 $A$ 可对角化，$s$ 个不同特征值 $\lambda_1, \cdots, \lambda_s$，重数为 $k_1, \cdots, k_s$.

$$\boxed{A\alpha_{i,j} = \lambda_i \alpha_{i,j}, \quad \begin{matrix} i=1\sim s \\ j=1,\cdots,k_i \end{matrix}}$$

令 $P = (\alpha_{1,1}, \cdots, \alpha_{1,k_1}, \cdots, \alpha_{s,1}, \cdots, \alpha_{s,k_s})$

则 $P^{-1}AP = \mathrm{diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{k_1\text{个}}, \cdots, \underbrace{\lambda_s, \cdots, \lambda_s}_{k_s\text{个}})$

即 $A = P\,\mathrm{diag}(\lambda_1, \cdots, \lambda_1, \cdots, \lambda_s, \cdots, \lambda_s)P^{-1}$

令 $P^{-1} = (\beta_{1,1}, \cdots, \beta_{1,k_1}, \cdots, \beta_{s,1}, \cdots, \beta_{s,k_s})^T$

则 $\underline{\underline{A}} = \sum_{i=1}^{s} \lambda_i (\underbrace{\alpha_{i,1}\beta_{i,1}^T + \cdots + \alpha_{i,k_i}\beta_{i,k_i}^T}_{P_i})$

$= \underline{\underline{\sum_{i=1}^{s} \lambda_i P_i}}$. 称为 $A$ 之谱分解.

注到 $PP^{-1} = E_n \Longrightarrow \sum_{i=1}^{s} (\alpha_{i,1}\beta_{i,1}^T + \cdots + \alpha_{i,k_i}\beta_{i,k_i}^T) = E_n$

即 $\boxed{\sum_{i=1}^{s} P_i = E_n}$

又 $P^{-1}P = E_n \Longrightarrow \beta_{i_1,j_1}^T \alpha_{i_2,j_2} = \delta_{i_1,i_2}\delta_{j_1,j_2}$

$\Longrightarrow \boxed{P_i P_j = 0,\ i \neq j}$

且 $\boxed{P_i^2 = (\alpha_{i,1}\beta_{i,1}^T + \cdots + \alpha_{i,k_i}\beta_{i,k_i}^T)(\alpha_{i,1}\beta_{i,1}^T + \cdots + \alpha_{i,k_i}\beta_{i,k_i}^T) = P_i}$

又 $\sum_{i=1}^{s} P_i = E_n \Rightarrow n = \text{rank}(E_n) \le \sum_{i=1}^{s} \text{rank}(P_i)$

$\le k_1 + \cdots + k_s = n$

$\Rightarrow \text{rank}(P_i) = k_i$.

Th.(谱分解定理).

设 $A$ 可对角化，$A$ 有 $s$ 个不同特征值 $\lambda_1, \cdots, \lambda_s$

重数分别为 $k_1, \cdots, k_s$. 则 $\exists$ $s$ 个 $n$ 阶方阵 $P_i$, $i=1 \sim s$,

$\exists$: $\boxed{A = \sum_{i=1}^{s} \lambda_i P_i, \quad P_i^2 = P_i, \quad P_i P_j = 0 \ (i \ne j) \quad \sum_{i=1}^{s} P_i = E_n, \quad \text{rank}(P_i) = k_i.}$

$$A = P\Sigma P^{-1} \Rightarrow A^T = P^{-T}\Sigma P^T \Rightarrow A^T P^{-T} = P^{-T}\Sigma.$$

故 $P^{-T} = (\beta_{1,1}, \cdots, \beta_{1,k_1}, \cdots, \beta_{s,1}, \cdots, \beta_{s,k_s})$ 的列

为 $A^T$ 对应于 $\lambda_1, \cdots, \lambda_1, \cdots, \lambda_s, \cdots, \lambda_s$ 之特征向量.

i.e. $\boxed{A^T \beta_{i,j} = \lambda_i \beta_{i,j}, \quad \begin{matrix} i = 1 \sim s \\ j = 1, \cdots, k_i \end{matrix}}$

或 $\beta_{i,j}^T A = \lambda_i \beta_{i,j}^T$

称 $\beta_{i,j}^T$ 为 $A$ 之左特征向量.

$\alpha_{i,j}$ …… 右 ……