Modern Control Theory
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Weijun Zhang （张伟军）
Associate Professor, Robotics Institute
School of mechanical Engineering
Office:930 main building of M.E. school

zhangweijun@sjtu.edu.edu.cn
021-34205559

课程概要

- Linear Feedback Control System（5.1）
- Pole-placement approach (5.2)
  - Derivation for state feedback gain matrix
  - Solution of pole-placement problem with Matlab

Linear Feedback Control System

线性系统的分析：包括建模、系统响应、能控性、能观性、稳定性等等

线性系统的综合：包括状态反馈、输出反馈、镇定、解耦状态观测器等等

一、状态反馈

$$u = Kx + v$$

原系统

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

设 v——r×1维参考输入；

K——r×n维状态反馈矩阵；

Linear Feedback Control System

$$\dot{x} = (A + BK)x + Bv$$
$$y = (C + DK)x + Dv$$

D＝0

$$\dot{x} = (A + BK)x + Bv$$
$$y = Cx$$

$$W_k(s) = c[sI - (A + BK)]^{-1}B$$

状态反馈矩阵K，不增加系统的维数，但是可通过K选择系统的特征值。

Linear Feedback Control System

二、输出反馈

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

或

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$u = Hy + v$$

$$\dot{x} = (A + BHC)x + Bv$$
$$y = Cx$$

设 v——r×1维参考输入；

H——r×m维状态反馈矩阵；

$$W_H(s) = c[sI - (A + BHC)]^{-1} B$$

输出反HC相当于状态反馈中的K，但是m<n,故HC的选择的状态变量比K少，只相当于部分状态反馈，仅当C为n×n时，HC的作用才和K的作用相当。

Linear Feedback Control System

三、输出到状态矢量 $\dot{x}$ 的反馈

$$\dot{x} = Ax + Bu \qquad\qquad \dot{x} = Ax + Gy + Bu = (A + GC)x + Bu$$

$$y = Cx \qquad\qquad y = Cx$$

$$W_G(s) = c[sI - (A + GC)]^{-1}B$$

通过选择G能改变闭环系统的特征值，从而影响系统的特性。

四、动态补偿器

1.串连补偿

动态补偿系统阶数发生了变化。

2.反馈补偿

Linear Feedback Control System

五、闭环系统的能控性和能观性

1.状态反馈不改变受控系统$\Sigma_o=(A,B,C)$的能控性，但不能保证系统的能观性不变。

例题1: 对系统

$$\dot{x}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u \qquad K=\begin{bmatrix}-1 & 0\end{bmatrix}$$

$$y=\begin{bmatrix}0 & 1\end{bmatrix}x$$

试分析系统的可控性和可观性

2.输出反馈不改变受控系统$\Sigma_o=(A,B,C)$的能控性和能观性。

课程概要

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  - Derivation for state feedback gain matrix
  - Solution of pole-placement problem with Matlab

Pole Placement

- Fact: The performance of control system are determined by closed-loop poles and zeros.

IF:

1.All state variables are measurable and are available for feedback

2.The system is completely state controllable

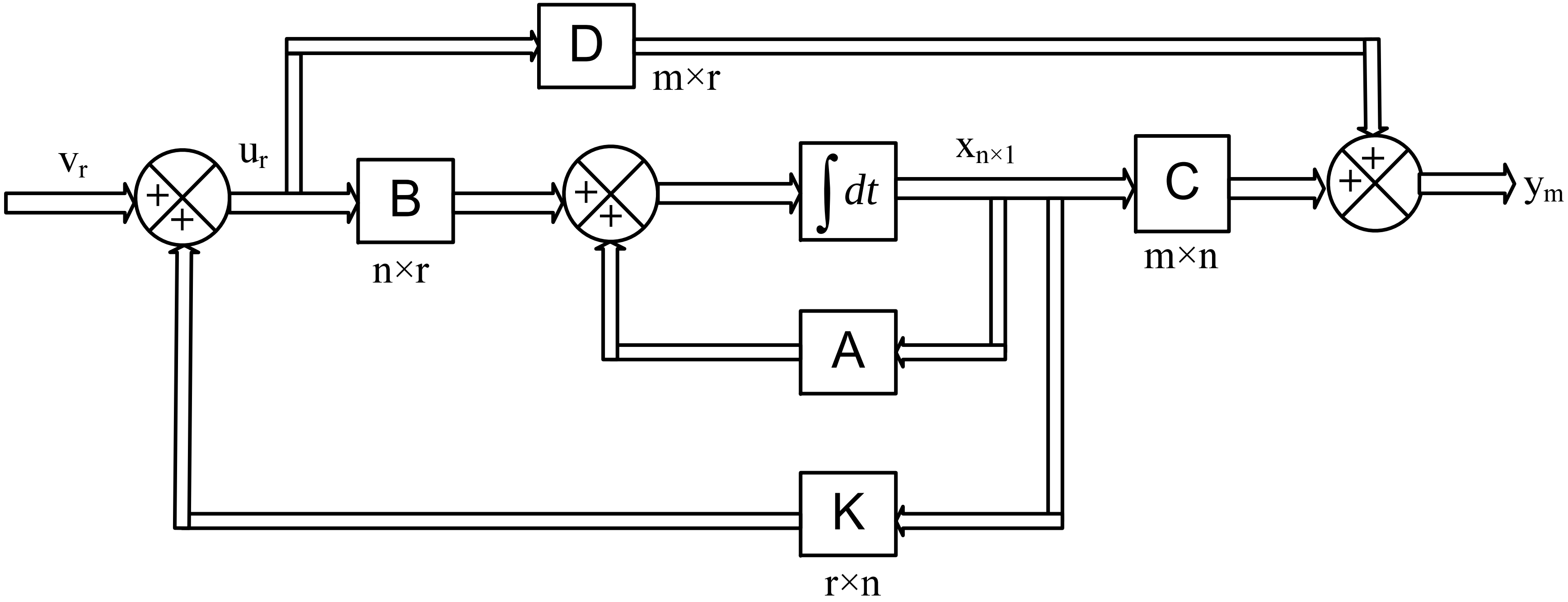
Then

- The poles of closed loop system can be placed at arbitrarily chosen locations

⇓

Pole-placement or Pole-assignment technique

Pole Placement



The original System

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Suppose

$$D = 0$$

$$u = Kx + v$$

K: State Feedback Gain Matrix

Pole Placement

New system

$$\dot{x} = (A + BK)x + Bv \qquad W_k(s) = c[sI - (A + BK)]^{-1} B$$

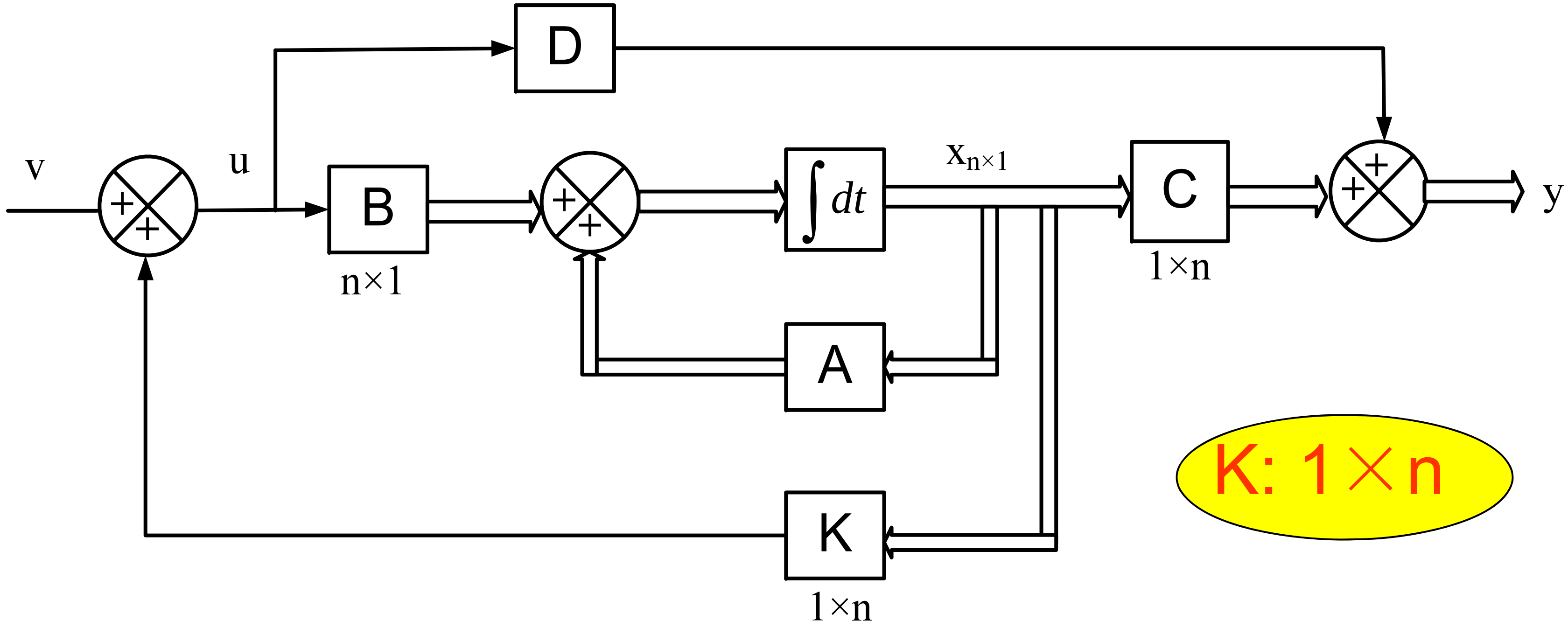
$$y = Cx$$

Results:

1.The eigenvalues of A+BK are called regulator Poles

2.The gain matrix K does not change the dimension of system, but By K the close loop poles can be assigned arbitrarily.

Pole-placement for SISO System



$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

x=State vector (n-vector)

y=output signal (scalar)

u=control signal (scalar)

A: n×n; B: n×1

C: 1×n; D: 1×1

Necessary Condition for Pole Placement

- The necessary and sufficient condition for Pole placement is that the system is completely controllable.

The necessary condition:

completely controllable $\Rightarrow$ Pole placement arbitrarily

Prove:

Not completely controllable $\Rightarrow$ No Pole placement arbitrarily

Necessary Condition for Pole Placement

Not completely controllable $\Rightarrow$ $rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = q < \text{n}$

Let

Define q linearly independent column vectors as $f_1, f_2, \ldots f_q$, Choose n-q vectors $v_{q+1}, v_{q+2}, \ldots V_n$ such that

$$P = [\, f1 \mid f2 \mid \ldots f_q \mid v_{q+1} \mid v_{q+2} \mid \ldots \mid v_n \,]$$

Is of rank N

Necessary and Sufficient Condition for Pole Placement

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \quad \xrightarrow{\boxed{x = P\hat{x}}} \quad \begin{aligned}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u\\ y &= \hat{C}\hat{x}\end{aligned}$$

$$\hat{A} = P^{-1}AP = \begin{array}{c} \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline 0 & \hat{A}_{22} \end{array}\right] \end{array} \begin{array}{l} q \\ n-q \end{array} \qquad \hat{B} = P^{-1}B = \left[\begin{array}{c} \hat{B}_1 \\ \hline 0 \end{array}\right] \begin{array}{l} q \\ n-q \end{array}$$

$$\overbrace{\mathrm{q}} \quad \overbrace{n-q}$$

$$\hat{C} = CP = \left[\begin{array}{c|c} \underbrace{\hat{C}_1}_{q} & \underbrace{\hat{C}_2}_{n-q} \end{array}\right] \qquad \hat{x} = \left[\begin{array}{c} \hat{x}_1 \\ \hline \hat{x}_2 \end{array}\right]^{q}_{(n-q)}$$

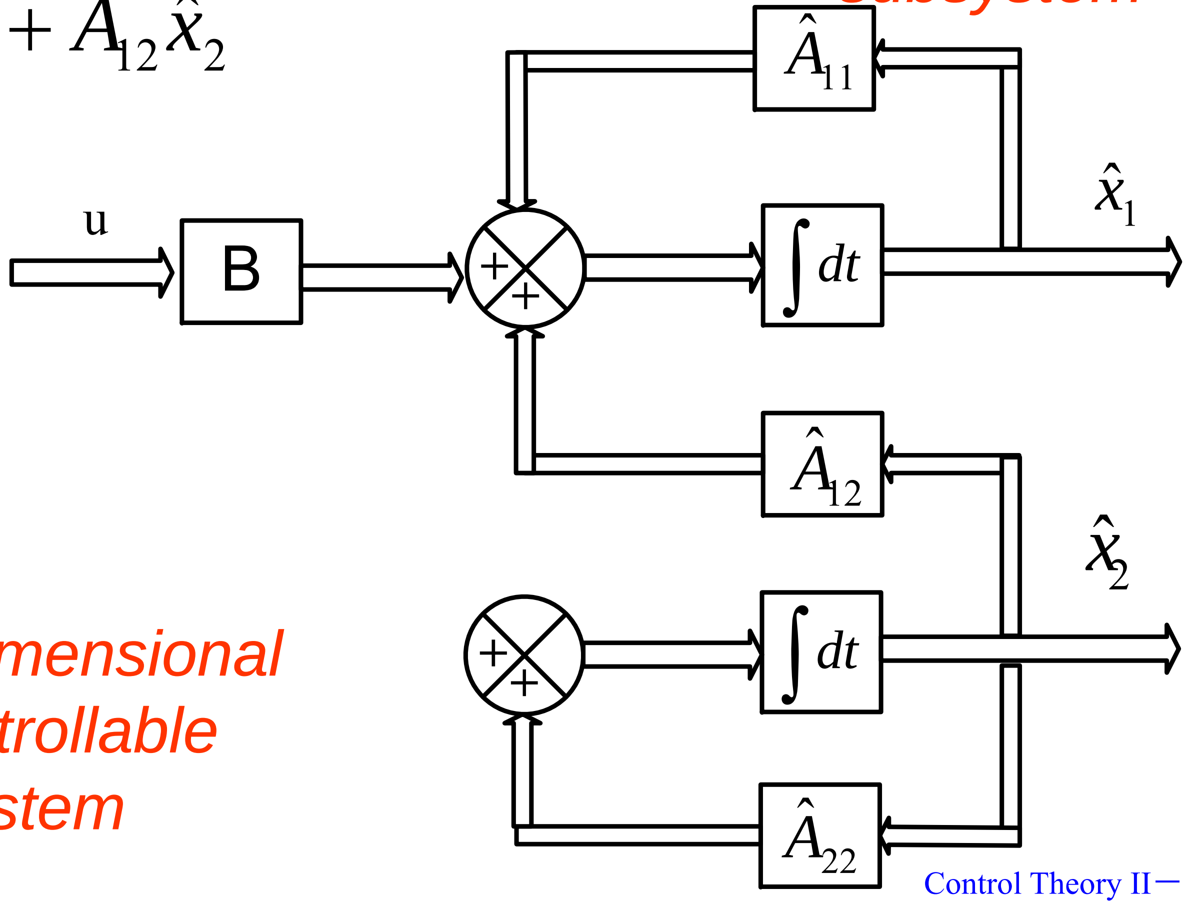
Necessary and Sufficient Condition for Pole Placement

The state space is divided into two parts

q dimensional Controllable subsystem

$$\dot{\hat{x}}_1 = \hat{A}_{11}\hat{x}_1 + \hat{B}_1 u + \hat{A}_{12}\hat{x}_2$$

$$\dot{\hat{x}}_2 = \hat{A}_{22}\hat{x}_2$$



N-q dimensional uncontrollable subsystem

Necessary and Sufficient Condition for Pole Placement

Define $\hat{K} = KP = [k_1 \; k_2]$

$$\left|sI - (A + BK)\right| = \left|P^{-1}(sI - A - BK)P\right|$$

$$= \left|sI - P^{-1}AP - P^{-1}BKP\right|$$

$$= \left|sI - \hat{A} - \hat{B}\hat{K}\right|$$

$$= \left| sI - \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline 0 & \hat{A}_{22} \end{array}\right] - \left[\begin{array}{c} \hat{B}_1 \\ \hline 0 \end{array}\right] [\hat{k}_1 \quad \hat{k}_2] \right|$$

$$= \begin{vmatrix} sI_q - \hat{A}_{11} - \hat{B}_{11}\hat{k}_1 & -\hat{A}_{12} - \hat{B}_{11}\hat{k}_2 \\ 0 & sI_{n-q} - A_{22} \end{vmatrix}$$

Necessary and Sufficient Condition for Pole Placement

$$= \begin{vmatrix} sI_q - \hat{A}_{11} - \hat{B}_{11}\hat{k}_1 & -\hat{A}_{12} - \hat{B}_{11}\hat{k}_2 \\ 0 & sI_{n-q} - \hat{A}_{22} \end{vmatrix}$$

$$= \left| sI_q - \hat{A}_{11} - \hat{B}_{11}\hat{k}_1 \right| \left| sI_{n-q} - \hat{A}_{22} \right|$$

I_q:q-dimensional identity matrix

Results: K can not control the eigenvalues of A_{22}

-> the close loop poles can not be placed arbitrarily

Derivation for state feedback gain matrix

Step1：问题&目标

$\Sigma_o = (A, B, C)$ 的特征多项式为

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

$\Sigma_k = (A + Bk, B, C)$ 的特征多项式为

$$f^*(\lambda) = \lambda^n + a_{n-1}{}^*\lambda^{n-1} + \cdots + a_1{}^*\lambda + a_0{}^*$$

$$= \prod_{i=1}^{n}(\lambda - \lambda_i^*)$$ 代表着期望极点的分布

Derivation for state feedback gain matrix

Step2：规范化成能控I型

$$
\begin{aligned}
e_1 &= A^{n-1}b + a_{n-1}A^{n-2}b + a_{n-2}A^{n-3}b + \ldots + a_1 b \\
e_2 &= A^{n-2}b + a_{n-1}A^{n-3}b + \ldots + a_2 b \\
&\vdots \\
e_{n-1} &= Ab + a_{n-1}b \\
e_n &= b
\end{aligned}
$$

$$
= \begin{bmatrix} A^{n-1}b & A^{n-2}b & \cdots & b \end{bmatrix}
\begin{bmatrix}
1 & & & & \\
a_{n-1} & 1 & & 0 & \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_2 & a_3 & \cdots & 1 & \\
a_1 & a_2 & a_3 & \cdots & 1
\end{bmatrix}
$$

Derivation for state feedback gain matrix

$$\overline{A} = T_{c1}^{-1} A T_{c1}$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad \overline{b} = T_{c1}^{-1} b \qquad \overline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\overline{C} = CT_{c1} = C\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

$$= \mathrm{C}\begin{bmatrix} A^{n-1}b & A^{n-2}b & \cdots & b \end{bmatrix} \begin{bmatrix} 1 & & & & \\ a_{n-1} & 1 & & \mathbf{0} & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & 1 & \\ a_1 & a_2 & a_3 & \cdots & 1 \end{bmatrix}$$

Derivation for state feedback gain matrix

加入状态反馈后 $\bar{K}=\begin{bmatrix}\bar{k}_0 & \bar{k}_1 & \cdots & \bar{k}_{n-1}\end{bmatrix}$

$$\dot{x}=(\bar{A}+\bar{b}\bar{K})\bar{x}+\bar{b}u$$

$$y=\bar{C}\bar{x}$$

$$\bar{A}+\bar{b}\bar{K}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_0-\bar{k}_0) & -(a_1-\bar{k}_1) & \cdots & \cdots & -(a_{n-1}-\bar{k}_{n-1}) \end{bmatrix}$$

Derivation for state feedback gain matrix

$$f(\lambda) = \left|\lambda I - (\overline{A} + \overline{b}\,\overline{K})\right|$$

$$= \lambda^n + (a_{n-1} - \overline{k}_{n-1})\lambda^{n-1} + \cdots + (a_1 - \overline{k}_1)\lambda + (a_0 - \overline{k}_0)$$

闭环传递函数的变化

能控I型的传递函数

$$\Sigma_o = (\overline{A}, \overline{B}, \overline{C})$$

$$W_0(s) = \overline{C}(sI - \overline{A})^{-1}\overline{b}$$

$$= \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Derivation for state feedback gain matrix

$$W_k(s) = \overline{C}(sI - (\overline{A} + \overline{B}\overline{K})^{-1}\overline{b}$$

$$\Sigma_o = (\overline{A} + \overline{B}\overline{K}, \overline{B}, \overline{C}) \quad = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_1 s + \beta_0}{s^n + (a_{n-1} - \overline{k}_{n-1})s^{n-1} + \cdots + (a_1 - k_1)s + (a_0 - k_0)}$$

结论：变换前后分子不变，分母发生变化

因此可观性发生变化　　Step3 求解新K阵

使闭环系统与给定的期望极点相符合，必须满足：

$$a_i - \overline{k}_i = a_i^*$$

Derivation for state feedback gain matrix

$$\bar{K} = \begin{bmatrix} a_0 - a_0^* & a_1 - a_1^* & \cdots & a_{n-1} - a_{n-1}^* \end{bmatrix}$$

Step4 求解原K阵 $K = \bar{K} T_{c1}^{-1}$

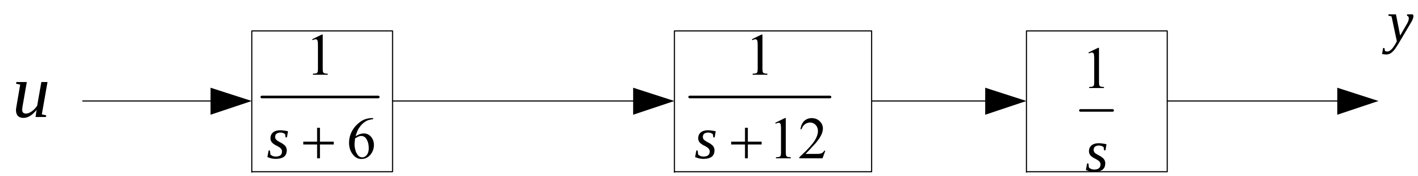
例题2：设系统的传递函数为：

$$W(s) = \frac{10}{s(s+1)(s+2)}$$

试设计状态反馈阵，使闭环系统极点为－2，－1±j

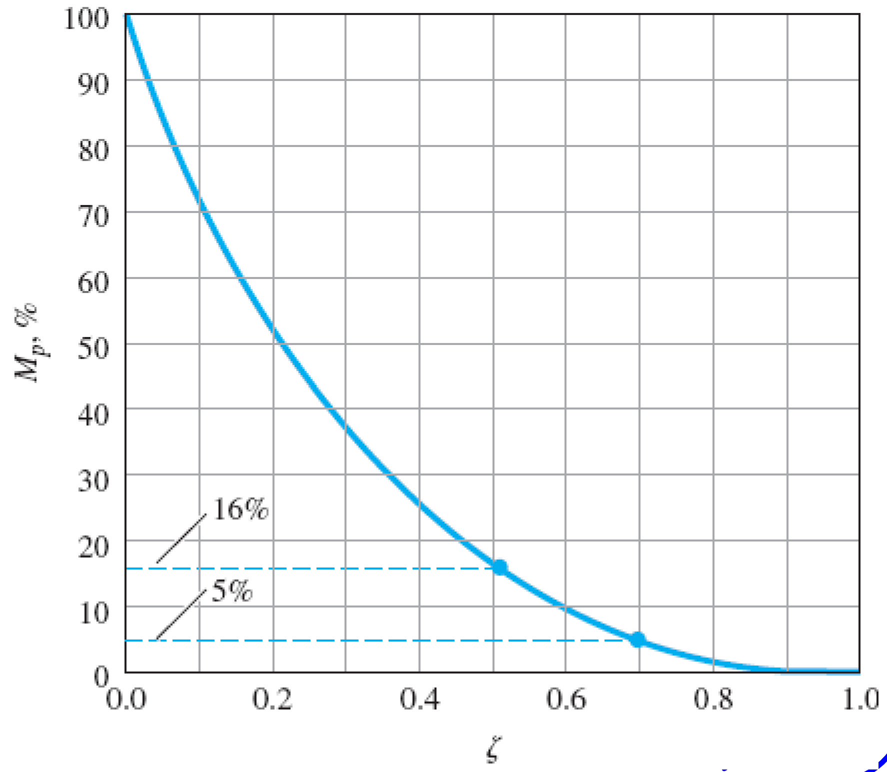
Derivation for state feedback gain matrix

例题3：设系统如下所示



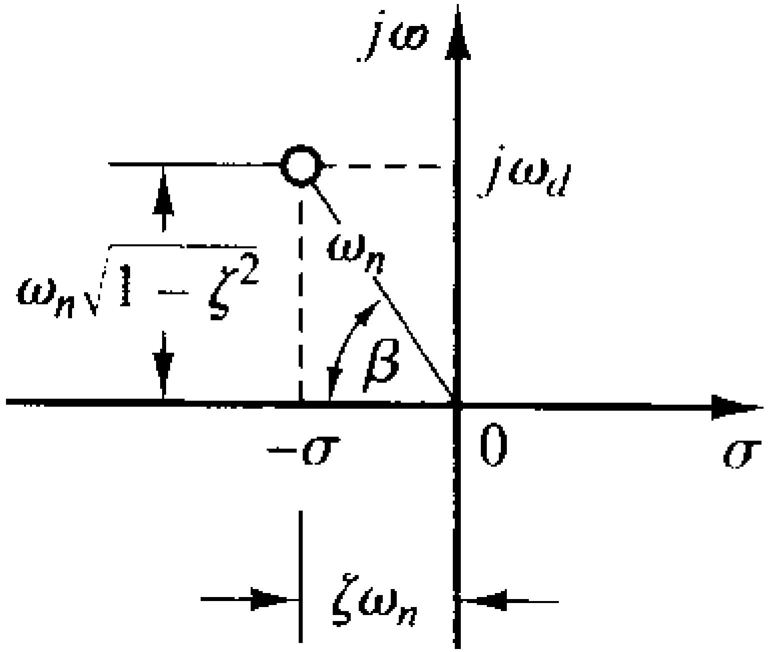
1.输出超调量 $Mp \le 5\%$

2.调整时间 $t_s \le 0.5s$



Derivation for state feedback gain matrix

Graphical Interpretation



Two facts:

- Magnitude of complex conjugate poles = ω_n

- $\theta = \sin^{-1}\zeta$

Significance of ω_d: the apparent oscillation frequency

Derivation for state feedback gain matrix

- Given desired time domain spec.
  - Maximum overshoot < 10%
  - 2% settling time < 2 seconds
  - Rise time < 0.9sec

Find allowable pole locations

Maximum overshoot < 10% ⇨ $\zeta > 0.6$

2% settling time < 2 seconds ⇨ $\sigma > 2$

Rise time < 0.9 seconds ⇨ $\omega_d > \dfrac{2.21}{0.9}$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

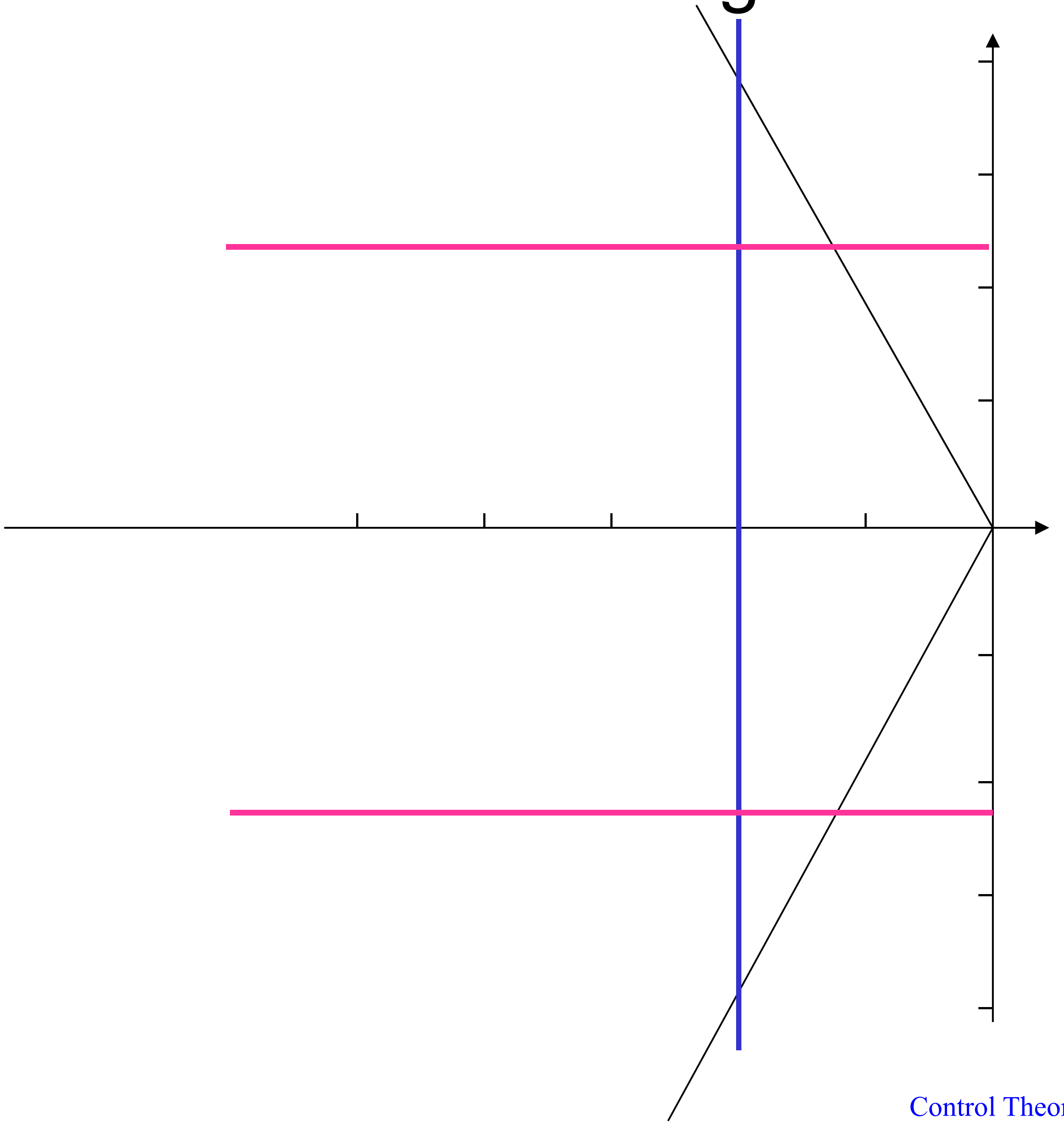
$$t_r = \frac{\frac{\pi}{2} + \sin^{-1}\zeta}{\omega_d}$$

Derivation for state feedback gain matrix

$$\zeta > 0.6$$

$$\sigma > 2$$

$$\omega_d > \frac{2.21}{0.9}$$



Derivation for state feedback gain matrix

Example:

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The desired close-loop poles:

$$s_1 = -2 + j4 \qquad s_2 = -2 - j4 \qquad s_3 = -10$$

State feedback control u=v-Kx, find K

Derivation for state feedback gain matrix

Solution:

1.Check the controllability matrix of the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

Derivation for state feedback gain matrix

Solution:

2. The desired characteristic equation

$$(s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

$$(s+2-j4\mu_1)(s+2+j4)(s+10) = s^3 + 14s^2 + 60s + 200$$

3. Determine K

$$\begin{aligned}\hat{K} &= \begin{bmatrix}\alpha_3 - a_3 & \alpha_2 - a_2 & \alpha_1 - a_3 \end{bmatrix}\\ &= \begin{bmatrix}200-1 & 60-5 & 14-6\end{bmatrix}\\ &= \begin{bmatrix}199 & 55 & 8\end{bmatrix}\end{aligned}$$

$$K = \hat{K}T^{-1} = \hat{K}$$

Derivation for state feedback gain matrix

- A simple method for n<=3

For a 3 order system, suppose K=[k_1 k_2 k_3]

- Substituting of matrix K into the desired characteristic polynomial:

$$\left| sI - A + BK \right| = (s-\mu_1)(s-\mu_2)(s-\mu_3)$$

- By equating the coefficients of the like powers of s on both sides to determine K

Derivation for state feedback gain matrix

Example: $$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The desired close-loop poles:

$$s_1 = -2 + j4 \qquad s_2 = -2 - j4 \qquad s_3 = -10$$

State feedback control u=v-Kx, find K

Derivation of K Using Ackerman's Formula

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\dot{x} = (A - BK)x + Bv$$
$$y = Cx$$

The desired characteristic equation is :

$$(s-\mu_1)(s-\mu_2)\ldots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n = 0$$

Gaylay-Hamilton Theorem

$$\Phi(\hat{A}) = \hat{A}^n + \alpha_1 \hat{A}^{n-1} + \ldots + \alpha_{n-1}\hat{A} + \alpha_n I = 0$$

$$\hat{A} = A - BK$$

Using A-BK, suppose n=3

$$= (A-BK)^3 + \alpha_1 (A-BK)^2 + \alpha_2 (A-BK) + \alpha_3 I = 0$$

Derivation of K Using Ackerman's Formula

$$I = I$$

$$\hat{A} = A - BK$$

$$\hat{A}^2 = (A - BK)^2 = A^2 - ABK - BK(A - BK)$$

$$= A^2 - ABK - BK\hat{A}$$

$$\hat{A}^3 = (A - BK)^3 = A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2$$

$$\phi(\hat{A}) = \alpha_3 I + \alpha_2 \hat{A} + \alpha_1 \hat{A}^2 + \alpha_0 \hat{A}^3$$

Derivation of K Using Ackerman's Formula

$$= (A - BK)^3 + \alpha_1 (A - BK)^2 + \alpha_2 (A - BK) + \alpha_3 I = 0$$

$$= \alpha_3 I + \alpha_2 (A - BK) + \alpha_1 (A^2 - ABK - BK\hat{A})$$

$$+ (A^3 - A^2 BK - ABK\hat{A} - BK\hat{A}^2) = 0$$

$$\phi(A) = B(\alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2) + AB(\alpha_1 K + K\hat{A}) + A^2 BK$$

$$= \begin{bmatrix} B & AB & A^2 B \end{bmatrix} \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}$$

Derivation of K Using Ackerman's Formula

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix}^{-1} \phi(A) = \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}$$

$$\Downarrow$$

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & A^2B \end{bmatrix}^{-1} \phi(A)$$

$\Downarrow$ For system with order n

$$K = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}^{-1} \phi(A)$$

Derivation of K Using Ackerman's Formula

Example: $$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The desired close-loop poles:

$$s_1 = -2 + j4 \qquad s_2 = -2 - j4 \qquad s_3 = -10$$

State feedback control u=v-Kx, find K

Derivation of K Using Ackerman's Formula

Solution:

1. The desired characteristic equation

$$(s-\mu_1)(s-\mu_2)(s-\mu_3)=s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3$$

$$(s+2-j4\mu_1)(s+2+j4)(s+10)=s^3+14s^2+60s+200$$

$$\phi(A)=A^3+14A^2+60A+200I$$

$$=\begin{bmatrix}199 & 55 & 8\\ -8 & 159 & 7\\ -7 & -43 & 117\end{bmatrix}$$

Derivation of K Using Ackerman's Formula

Solution:

2. The controllability matrix is

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

3. Find k

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} M^{-1} \phi(A)$$

$$K = \begin{bmatrix} 199 & 58 & 8 \end{bmatrix}$$

Output Feedback

采用输出反馈

定理：对完全能控的单输入－单输出系统$\Sigma_o = (A, B, C)$，不能采用输出线性反馈来实现闭环系统极点的任意配置

证明：对单输入－单输出反馈系统， $\Sigma_h = [(A + BhC), b, c]$
闭环传递函数为

$$W_h(s) = \frac{W_0(s)}{1 + hW_0(s)}$$

闭环特征方程：$1 + hW_0(s) = 0$

由于该方程的限制，方程的特征根受h的控制，但仅限制在根轨迹上，而不是任意的。

Output Feedback to Derivative of x

从输出到 $\dot{x}$ 的反馈

定理：对系统 $\Sigma_o=(A,B,C)$，采用输出到 $\dot{x}$ 的 线性反馈实现闭环系统极点的任意配置的充要条件是 $\Sigma_o=(A,B,C)$ 能观。

证明：根据对偶原理，若 $\Sigma_o=(A,B,C)$ 能观，则 $\tilde{\Sigma}_o=(A^T,C^T,B^T)$ 能控 因而可以任意配置 $(A^T+C^TG^T)$ 的特征值。由于 $(A^T+C^TG^T)$ 和 $(A^T+C^TG^T)^T$ 特征值相同，

$$(A^T+C^TG^T)^T=A+GC$$

Output Feedback to Derivative of x

即可以对 $A+GC$ 任意配置极点。

Step1. 化为能观标准II型

$$T_{o2}=\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & 1 & \cdots & a_3 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} CA^{n-1} \\ CA^{n-2} \\ \vdots \\ CA \\ C \end{bmatrix}$$

$$\overline{A}=T_{o2}^{-1}AT_{o1}=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \qquad \overline{C}=CT_{o2}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\overline{b}=T_{o2}^{-1}b=\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

Output Feedback to Derivative of x

Step2. 引入反馈阵

$$\overline{G}=\begin{bmatrix} \overline{g}_0 \\ \overline{g}_1 \\ \vdots \\ \overline{g}_{n-1} \end{bmatrix} \quad \overline{A}+\overline{G}\,\overline{C}=\begin{bmatrix} 0 & \cdots & \cdots & 0 & -(a_0-\overline{g}_0) \\ 1 & 0 & \cdots & 0 & -(a_1-\overline{g}_1) \\ 0 & 1 & \cdots & 0 & -(a_2-\overline{g}_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -(a_{n-1}-\overline{g}_{n-1}) \end{bmatrix}$$

$$f(\lambda)=\left|\lambda I-(\overline{A}+\overline{G}\,\overline{C})\right|$$

$$=\lambda^n+(a_{n-1}-\overline{g}_{n-1})\lambda^{n-1}+\cdots+(a_1-\overline{g}_1)\lambda+(a_0-\overline{g}_0)$$

Output Feedback to Derivative of x

Step3. 由期望极点得到期望特征多项式

$$f^*(\lambda) = \lambda^n + a_{n-1}{}^* \lambda^{n-1} + \cdots + a_1{}^* \lambda + a_0{}^*$$

$$= \prod_{i=1}^{n} (\lambda - \lambda_i{}^*)$$

Step4. 比较特征多项式系数

$$\overline{G} = \begin{bmatrix} a_0 - a_0^* & a_1 - a_1^* & \cdots & a_{n-1} - a_{n-1}^* \end{bmatrix}$$

Output Feedback to Derivative of x

Step5. 变换回原系统下

$$G = T_{o2}\overline{G}$$

例题：设系统

$$\dot{x} = \begin{bmatrix} 0 & \omega_s^2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

试选择状态反馈矩阵G，使极点配置为－5，－8

课程概要

- Linear Feedback Control System（5.1）
- Pole-placement approach (5.2)
  - Derivation for state feedback gain matrix
  - Solution of pole-placement problem with Matlab

Derivation of K Using Matlab

- Two matlab command:

K=acker(A,B,J) for SISO system;

K=place(A,B,J) for both SISO and MIMO System.

A: matrix   B:matrix $J = [\mu_1 \quad \mu_2 \quad \dots \quad \mu_n]$

$\mu_1 \quad \mu_2 \quad \dots \quad \mu_n$ are desired colse loop poles

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{aligned} s_1 &= -2 + j4 \\ s_2 &= -2 - j4 \\ s_3 &= -10 \end{aligned}$$

Derivation of K Using Matlab

```
A=[0 1 0;0 0 1;-1 -5 -6];
B=[0;0;1];
J=[-2+j*4,-2-j*4,-10];
K=acker(A,B,J)

K =

199    55    8
```

```
K=place(A,B,J)

K =

199.0000   55.0000   8.0000
```

Stabilization of Control System

1. 对受控系统 $\Sigma_o = (A, B, C)$,通过反馈使其极点具有负实部，以保证系统是渐进稳定的。这种方法称为系统镇定。

2. 分为状态反馈镇定和输出反馈镇定两种情况。

3. 镇定是极点配置的一种特殊情况，只需把极点配置到根平面的左半平面即可。

定理：对系统$\Sigma_o = (A, B, C)$，采用状态反馈使系统镇定的充要条件是其不能控子系统为渐进稳定。

Stabilization of Control System

Step1.按可控性对系统进行分解

$$x = R_c \hat{x} \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} < \mathrm{n}$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \hat{B}u \\ y &= \hat{C}\hat{x} \end{aligned} \qquad \hat{x} = \begin{array}{c} \begin{bmatrix} \hat{x}_1 \\ \hline \hat{x}_2 \end{bmatrix} \end{array}\begin{array}{l} n_1 \\ (n-n_1) \end{array} \qquad A = R_c^{-1} A R_c = \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline 0 & \hat{A}_{22} \end{array}\right] \begin{array}{l} n_1 \\ n-n_1 \end{array}$$

$$\hat{B} = R_c^{-1} B = \begin{bmatrix} B_1 \\ \hline 0 \end{bmatrix} \begin{array}{l} n_1 \\ n-n_1 \end{array} \qquad \hat{C} = CR_c^{-1} = \left[\begin{array}{c|c} \underbrace{\hat{C}_1}_{n_1} & \underbrace{\hat{C}_2}_{n-n_1} \end{array}\right]$$

$$\overbrace{\mathrm{n}_1} \qquad \overbrace{n-n_1}$$

Stabilization of Control System

$$\widehat{\Sigma}_c = (\widehat{A}_{11}, \hat{B}_1, \hat{C}_1)$$ 能控子系统

$$\hat{\Sigma}_{\bar{c}} = (\hat{A}_{22}, 0, \hat{C}_2)$$ 不能控子系统

Step2. $\widehat{\Sigma} = (\hat{A}, \widehat{B}, \hat{C})$ 与 $\Sigma_o = (A, B, C)$ 在稳定性和可控性上等价

$$\widehat{K} = \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}$$

$$\widehat{A} + \widehat{B}\widehat{K} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix} + \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{A}_{11} + \widehat{B}_1\widehat{K}_1 & \widehat{A}_{12} + \widehat{B}_1\widehat{K}_2 \\ 0 & \widehat{A}_{22} \end{bmatrix}$$

Stabilization of Control System

特征多项式

$$\det[sI-(\hat{A}+\hat{B}\hat{K})]=\det\left[sI_1-(\hat{A}_{11}+\hat{B}_1\hat{K}_1)\right]\bullet\det\left[sI_2-\hat{A}_{22}\right]$$

从特征根分布看，可控的部分可以通过K阵调整根的分布，不可控部分则需要自身稳定，系统才可保证镇定。

输出反馈镇定

定理：对系统$\Sigma_o=(A,B,C)$，采用状态反馈使系统镇定的充要条件是能控且能观子系统是输出反馈能镇定的；其余子系统是渐进稳定的。

Stabilization of Control System

定理：对一个能控且能观系统，不能保证一定能够通过输出反馈镇定。

例题： 设系统
$$\dot{x}=\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}x+\begin{bmatrix}0\\1\\0\end{bmatrix}u$$
$$y=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}x$$

试证明不能通过输出反馈使之镇定。

Stabilization of Control System

定理：对系统 $\Sigma_o=(A,B,C)$，采用输出到 $\dot{x}$ 的线性反馈实现镇定的充要条件是 $\Sigma_o=(A,B,C)$ 不能观子系统为渐进稳定。