EE338 Filter Design Assignment

Shubham Kar, 180070058

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1 Student Details

Name	Shubham Kar
Roll Number	180070058
Filter Number	67

2 Part 1

2.1 Band Pass Filter

2.1.1 Un-normalized discrete time filter specifications

Filter Number = 67
$q(m) = [67 \times 0.1] = 6$ where [x] is the greatest integer strictly less than x
$r(m) = (67 - 10q(m)) = 7$
$BL(m) = (25 + 1.7q(m) + 6.1r(m)) = 77.9$
$BH(m) = (BL(m) + 20) = 97.9$
Signals are bandlimited to 160 kHz and the Sampling Frequency is 330 kHz. The Bandpass filter specifications are as follows:

- **PassBand**: 77.9 kHz to 97.9 kHz
- **Transition Band**: 4 kHz on either side of the passband
- **Stopband**: 0-73.9 kHz and 101.9-165 kHz since the sampling frequency is 330 kHz
- **Tolerance**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Monotonic
- **Stopband Nature**: Monotonic

2.1.2 Normalized Digital Filter specifications

We have the sampling frequency as 330 kHz. Since, we want this frequency to represent 2π on the normalized frequency axis, we need to scale all the corresponding filter specifications down to 0 to π:

$$\omega = \left(\frac{2\pi\omega_{unn}}{\omega_{samp}}\right) \quad where \tag{1}$$

$$\begin{aligned} \omega_{unn} &= Unnormalized\ frequency \\ \omega_{samp} &= Sampling\ Frequency \end{aligned}$$

Critical points $\rightarrow$	0	ω_{s1}	ω_{p1}	ω_{p2}	ω_{s2}	π
Un-normalized Specs (kHz)	0	73.9	77.9	97.9	101.9	165
Normalized Specs	0	0.4479π	0.4721π	0.5933π	0.6176π	π

Table 1: Comparison of Unnormalized and Normalized Filter specifications

The passband and the stopband nature remain as monotonic. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications are mentioned in Table 1.

2.1.3 Analog Filter specifications using Bilinear Transformation

We use the Bilinear transformation to map z plane to the s plane where we use:

$$s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{2}$$

$$\Omega = \tan\frac{\omega}{2} \tag{3}$$

The second equation is a corollary of the Bilinear transformation. So, applying the Bilinear transformation to the normalized frequency values, we get:

Critical points for $\downarrow$	0	ω_{s1}	ω_{p1}	ω_{p2}	ω_{s2}	π
Normalized Specs	0	0.4479π	0.4721π	0.5933π	0.6176π	π
Normalized Analog Filter Specs	0	0.8484	0.9159	1.3464	1.4959	∞
Critical points for $\uparrow$	0	Ω_{s1}	Ω_{p1}	Ω_{p2}	Ω_{s2}	∞

Table 2: Comparison of Normalized and Analog Filter specifications

The passband and the stopband nature remain as monotonic. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications for the analog filter are mentioned in Table 2.

2.1.4 Frequency Transformation with relevant parameters

A transformation from the Bandpass filter specifications to that of a low pass filter is needed to effectively design a Butterworth Filter for the same and later transform it back into a Bandpass filter. To do this, we will need the following frequency transformation:

$$\Omega_L = \left(\frac{\Omega^2 - \Omega_o^2}{B\Omega}\right) \tag{4}$$

The parameters in Equation 4 are Ω_o and B. These can be obtained with suitable specifications for the Low Pass Filter we need after transforming it from the Band Pass Filter. For obtaining a symmetric Low Pass Filter, we will need:

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = 1.1105 \tag{5}$$

And, if we need the symmetric passbands of the Low Pass Filter to be at $\Omega_L = 1$, we will need the following condition (essentially modelling bandwidth):

$$B = (\Omega_{p2} - \Omega_{p1}) = 0.4305 \tag{6}$$

Applying this transformation, we get the following Low Pass Filter specifications:

Critical points for ↓	0^+	Ω_{s1}	Ω_{p1}	Ω_o	Ω_{p2}	Ω_{s2}	∞
Normalized BPF Specs	0	0.8484	0.9159	1.1105	1.3464	1.4959	∞
Normalized LPF Specs	0	-1.4057	-1.0001	0	0.9999	1.5598	∞
Critical points for ↑	$-\infty$	Ω_{Ls1}	Ω_{Lp1}	0	Ω_{Lp2}	Ω_{Ls2}	∞

Table 3: Comparison of Normalized and Analog Filter specifications

2.1.5 Analog Low Pass Filter Specifications

We can see that the passband edges for the Low Pass Fiter are symmetric at almost -1 and 1. We select the more stringent stopband specification. This gives us the stopband edge as $\min(|-1.4057|, 1.5598) = 1.4057$. So the final Analog Low Pass Filter specifications are as follows:

- **PassBand Edges** for Ω_L: 1 and -1
- **Stopband Edges** for Ω_L: -1.4057 and 1.4057
- **Tolerances**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Monotonic
- **Stopband Nature**: Monotonic

These specifications follow from the transformation results of Table 3.

2.1.6 Analog Filter LPF transfer function

The passband and the stopband nature required for the Low Pass Filter is monotonic. So, we can design **Butterworth** approximation of the LPF to get the same. The tolerances needed to match are 0.15 in magnitude in both, the passbands and the stopbands, So, we get:

$$\delta_1 \;=\; \delta_2 \;=\; 0.15$$
$$D_1 \;=\; \frac{1}{(1-\delta_1)^2} - 1 \;=\; 0.3841$$
$$D_2 \;=\; \frac{1}{\delta_2^2} - 1 \;=\; 43.4444$$

The minimum order of the Butterworth approximation required is obtained as follows:

$$\Omega_s \;=\; 1.4057, \quad \Omega_p \;=\; 1$$
$$\frac{\Omega_s}{\Omega_p} \;=\; \frac{\Omega_s}{1} \;=\; 1.4057$$
$$N \geq \frac{1}{2} \cdot \frac{\log\left(\frac{D_2}{D_1}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \implies N \geq \frac{1}{2} \cdot \frac{4.7283}{0.3405} \implies N \geq 6.94$$

So, $N = 7$ gives us the minimum order of the Butterworth filter to be constructed for fulfilling the Low Pass Filter Design Specifications. Now, we find the cutoff frequency for our approximation i.e. Ω_c:

$$\left(\frac{\Omega_p^{2N}}{D_1}\right) \leq \Omega_c^{2N} \leq \left(\frac{\Omega_s^{2N}}{D_2}\right)$$
$$\implies 1.0707 \leq \Omega_c \leq 1.0737$$

So, we choose the arithmetic mean of the 2 bounds for Ω_c giving us $\Omega_c \;=\; 1.0722$. we get these bounds because of imposing the condition on N to be an integer. Now, we find the poles for $H_{analog}(s)$ by equating the denominator to 0; specifically:

$$1 + \left(\frac{s}{j\Omega_c}\right)^{2N} \;=\; 1 + \left(\frac{s}{j1.0722}\right)^{14} \;=\; 0 \tag{7}$$

So, the poles can be given in a concise form as shown:

$$s_k \;=\; 1.0722 \exp\left\{j\left(\frac{(2k+1)\pi}{14}\right) + j\frac{\pi}{2}\right\} \quad where \quad k = \{1,2,3,4,5,6,7\}$$

The above solution gives us the following roots which exist on the left half plane of s which are in turn poles of $H_{analog}(s_L)$:

p1	$-0.2386 \ +j\ 1.0453$
p2	$-0.6685 \ +j\ 0.8383$
p3	$-0.966 \ +j\ 0.4652$
p4	-1.0722
p5	$-0.966 \ -j\ 0.4652$
p6	$-0.6685 \ -j\ 0.8383$
p7	$-0.2386 \ -j\ 1.0453$

Table 4: Poles of $H_{analog}(s_L)$

The poles can be shown on the s-plane to lie on a circle centered at 0 and with a radius if $\Omega_c = 1.2879$ which is shown in Figure 1. The expression foe the

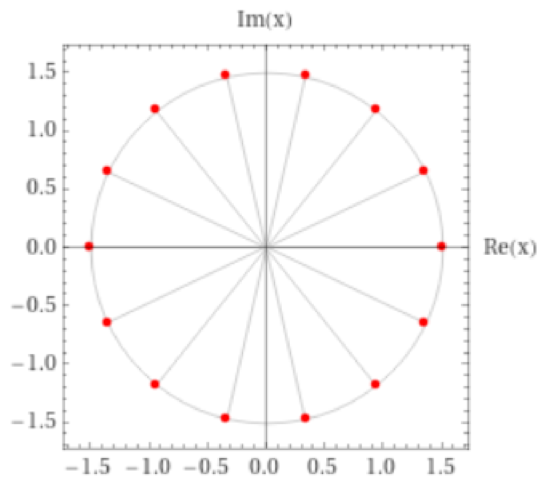

Figure 1: Poles of $H_{analog}(s_L)$ shown on the s plane

transfer function for the Low Pass Filter comes out to be as follows:

$$H_{analog,LPF}(s_L) = \left(\frac{\Omega_c^N}{(s_L-p_1)(s_L-p_2)(s_L-p_3)(s_L-p_4)(s_L-p_5)(s_L-p_6)(s_L-p_7)}\right)$$

$$= \left(\frac{1.6290}{s_L^7+4.8184s_L^6+11.6086s_L^5+17.9861s_L^4+19.2847s_L^3+14.3089s_L^2+6.8278s_L+1.6290}\right)$$

Plotting it on the s_L axis we get the frequency response of the LPF as shown in Figure 2.

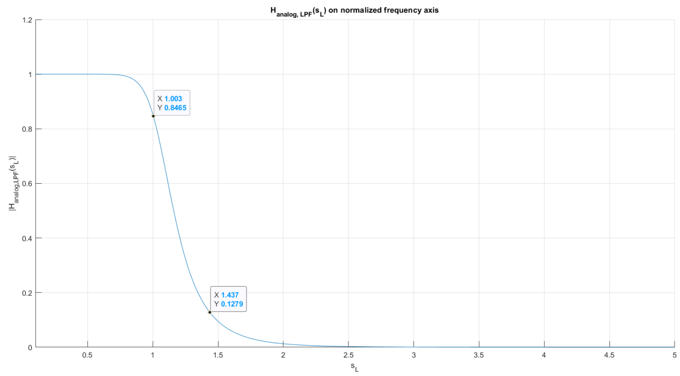

Figure 2: Low Pass Filter Frequency Response

2.1.7 Analog Bandpass Transfer Function

The transformation equation for getting the Bandpass filter transfer function in the Analog domain from the LPF Transfer function thus created is as follows:

$$s_L = \left(\frac{s^2 + \Omega_o^2}{Bs} \right) \tag{8}$$

Substituting s_L from Equation 8 into the $H_{analog,LPF}(s_L)$ created in the previous subsection gives us the Bandpass Transfer function in terms of s:

$$\begin{aligned}
s_L &= \left(\frac{s^2 + 1.2334}{0.4305s} \right) \\
H_{analog,BPF} &= \left(\frac{N(s)}{D(s)} \right), \quad where \\
N(s) &= 0.0045s^7
\end{aligned}$$

$$\begin{aligned}
D(s) = {} & (s^{14} + 2.07s^{13} + 10.78s^{12} + 16.78s^{11} + 45.88s^{10} + 54.62s^9 + 100.9s^8 + \\
& 91.47s^7 + 124.45s^6 + 83.1s^5 + 86.1s^4 + 38.85s^3 + 30.79s^2 + 7.3s + 4.34)
\end{aligned}$$

The frequency response of the Band Pass Filter is shown in Figure 3.

2.1.8 Discrete Time Filter Specifications

We use Bilinear transformation again to obtain the final Discrete time filter from the Analog Bandpass filter thus created. The transformation is:

$$s = \left(\frac{1 - z^-1}{1 + z^-1} \right)$$

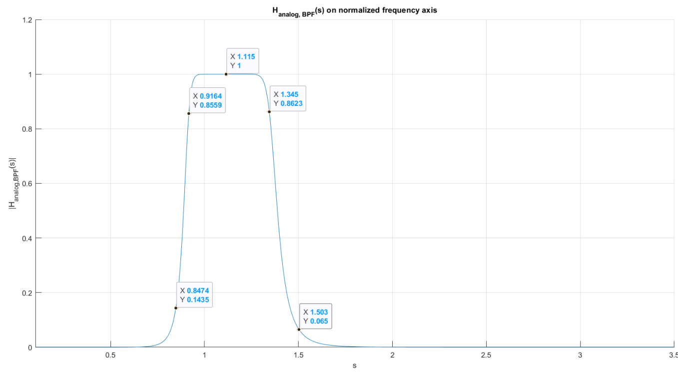

Figure 3: Frequency Response of $H_{analog,BPF}(s)$ on Normalized frequency axis

We finally get the following Discrete Time Filter Transfer function:

$$
\begin{aligned}
H_{discrete,BPF}(z) &= \frac{Num(z)}{Den(z)}, \quad where \\
Num(z) &= (0.0064z^{14} - 0.0447z^{12} + 0.1341z^{10} - 0.2236z^{8} + 0.2236z^{6} - \\
&\qquad 0.1341z^{4} + 0.0447z^{2} - 0.0064) \cdot 10^{-}3, \quad and \\
Den(z) &= (z^{14} + 1.27z^{13} + 5.87z^{12} + 5.89z^{11} + 14.28z^{10} + 11.36z^{9} + 18.81z^{8} + \\
&\qquad 11.66z^{7} + 14.48z^{6} + 6.72z^{5} + 6.51z^{4} + 2.06z^{3} + 1.58z^{2} + 0.26z + 0.16)
\end{aligned}
$$

This discrete time Filter Transfer specification is shown in Figure 4 where the phase response as well as the magnitude response is shown in the log scale in the normalized frequency of the discrete domain.

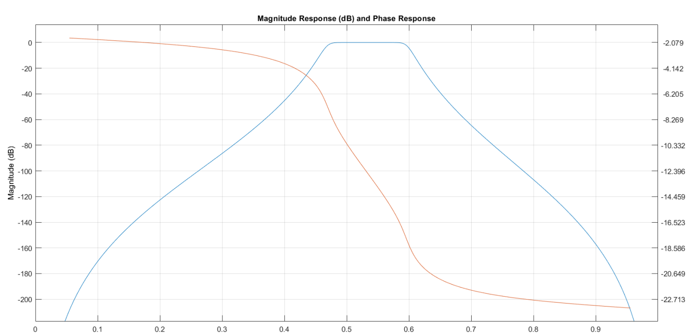

Figure 4: Magnitude and Phase response of the discrete BPF

The transfer function of the discrete filter with the unnormalized frequency

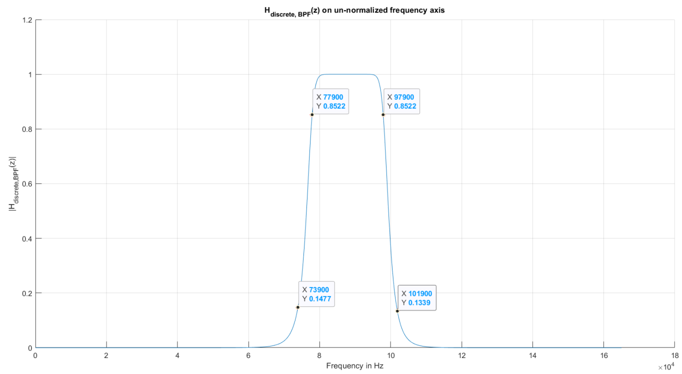

Figure 5: Magnitude response of the discrete BPF with unnormalized frequency

specifications is shown in Figure 5.
We can see from the Pole-Zero Plot of the discrete Band Pass filter in Figure 6 that all the poles are inside the unit circle, thus making the filter stable.

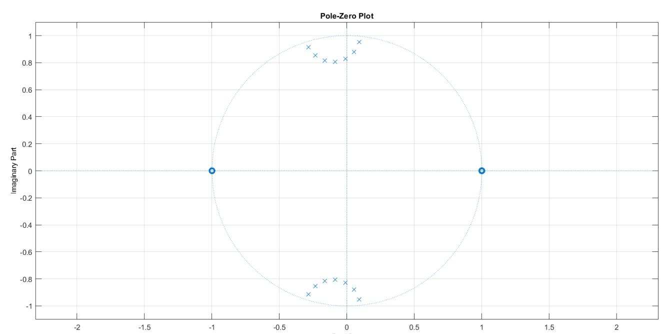

Figure 6: Pole-Zero plot of the discrete BPF

2.1.9 FIR Filter Transfer Function

We have been given that the tolerance of the stopbands is 0.15 in magnitude. We will then obtain an Attenuation constant(A) which is less than 21 dB:

$$A \;=\; -20\log\delta \;=\; 16.478\;dB$$

Since $A < 21$, we obtain the shape parameter α and eventually β as 0.
Now, for determining the order of the FIR filter, we make use of the empirical

inequality:

$$(2N+1) \geq \left(1 + \frac{(A-8)}{2.285\Delta\omega_T}\right), \quad where$$
$$\Delta\omega_T \;=\; (\omega_s - \omega_p) \;=\; 0.0242\pi$$

We get N_{min} to be 25. However, we need N to be 34 in order to fulfill the specifications of our bandpass filter. We get this value by hit and trial. The Kaiser window is itself of length $(2N+1) \;=\; 69$ making the FIR impulse Filter response to be of length 69. Now, the shape parameter α can also be determined using the attenuation constant. As we get $A < 21$, we get $\alpha \;=\; \beta \;=\; 0$. Thus, we get the condition that the Kaiser window is basically a rectangular window which satisfies our tolerance requirements relatively well.
Now, we model the ideal Band Pass Filter as a difference between ideal Low Pass Filters where we take the bandwidth of the ideal LPFs as $(\omega_{p2} + 0.0121\pi)$ and $(\omega_{p1} + 0.0121\pi)$. The Bandpass filter coefficients obtained are shown in Figure 7: These coefficients are obtained by multiplying the ideal BPF impulse

```
bpf_FIR =

  Columns 1 through 15

    0.0175    0.0046   -0.0168   -0.0007    0.0115   -0.0012   -0.0025   -0.0010   -0.0073    0.0074    0.0149   -0.0166   -0.0175    0.0251    0.0147

  Columns 16 through 30

   -0.0290   -0.0081    0.0249    0.0014   -0.0119    0.0003   -0.0081    0.0068    0.0308   -0.0248   -0.0500    0.0524    0.0599   -0.0848   -0.0569

  Columns 31 through 45

    0.1155    0.0408   -0.1375   -0.0148    0.1455   -0.0148   -0.1375    0.0408    0.1155   -0.0569   -0.0848    0.0599    0.0524   -0.0500   -0.0248

  Columns 46 through 60

    0.0308    0.0068   -0.0081    0.0003   -0.0119    0.0014    0.0249   -0.0081   -0.0290    0.0147    0.0251   -0.0175   -0.0166    0.0149    0.0074

  Columns 61 through 69

   -0.0073   -0.0010   -0.0025   -0.0012    0.0115   -0.0007   -0.0168    0.0046    0.0175
```

Figure 7: Coefficients of the FIR Band Pass Filter

repsonse with the kaiser window impulse response. The magnitude and the phase response of the FIR filter is shown in Figure 8. The magnitude response

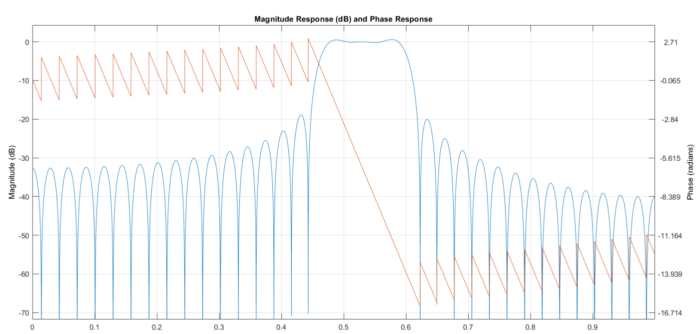

Figure 8: Magnitude and Phase response of the FIR Band Pass Filter

of the filter on un-normalized frequency is shown in 9 where we can see that the discrete time filter specifications are indeed being met.

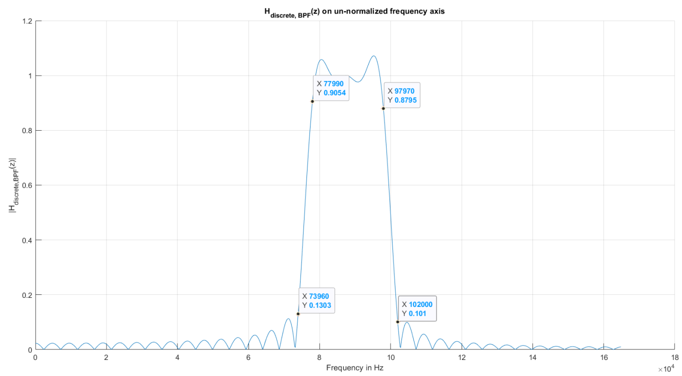

Figure 9: Magnitude response of the BPF filter on un-normalized frequencies

2.1.10 Comparison between FIR and IIR filters

We note the following points between the 2 types of filters:

- The phase response of the FIR filter is piece-wise linear whereas that of the IIR filter is far from linear. Even the Butterworth filter which is supposed to give a better phase generates a non-linear phase response for the Band Pass Filter.
- The number of delay lines required to realize the IIR Band Pass Filter is $(7 + 14) = 21$ whereas we require 58 delay lines for realizing the FIR filter. This implies that FIR filter requires more hardware than the IIR filters at the cost of providing linear phase response to the filters.

2.2 Band Stop Filter

2.2.1 Un-normalized discrete time filter specifications

Filter Number = 67
$q(m) = [67 \times 0.1] = 6$ where [x] is the greatest integer strictly less than x
$r(m) = (67 - 10q(m)) = 7$
$BL(m) = (25 + 1.9q(m) + 4.1r(m)) = 65.1$
$BH(m) = (BL(m) + 20) = 85.1$
Signals are bandlimited to 120 kHz and the Sampling Frequency is 260 kHz. The Bandpass filter specifications are as follows:

- **Stopband**: 65.1 kHz to 85.1 kHz
- **Transition Band**: 4 kHz on either side of the passband

- **Passband**: 0-61.1 kHz and 85.1-130 kHz since the sampling frequency is 260 kHz
- **Tolerance**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Equiripple
- **Stopband Nature**: Monotonic

2.2.2 Normalized Digital Filter specifications

We have the sampling frequency as 260 kHz. Since, we want this frequency to represent 2π on the normalized frequency axis, we need to scale all the corresponding filter specifications down to 0 to π:

$$\omega = \left(\frac{2\pi\omega_{unn}}{\omega_{samp}}\right) \quad where \tag{9}$$

$$\omega_{unn} = Unnormalized\ frequency$$
$$\omega_{samp} = Sampling\ Frequency$$

Critical points $\rightarrow$	0	ω_{p1}	ω_{s1}	ω_{s2}	ω_{p2}	π
Un-normalized Specs (kHz)	0	61.1	65.1	85.1	89.1	165
Normalized Specs	0	0.47π	0.5008π	0.6546π	0.6854π	π

Table 5: Comparison of Unnormalized and Normalized Filter specifications

The passband and the stopband nature remain as equiripple and montonic respectively. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications are mentioned in Table 5.

2.2.3 Analog Filter specifications using Bilinear Transformation

We use the Bilinear transformation to map z plane to the s plane where we use:

$$s = \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{10}$$

$$\Omega = \tan\frac{\omega}{2} \tag{11}$$

The second equation is a corollary of the Bilinear transformation. So, applying the Bilinear transformation to the normalized frequency values, we get:
The passbands and the stopband retain their nature as equiripple and monotonic respectively . Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications for the analog filter are mentioned in Table 6.

Critical points for ↓	0	ω_{p1}	ω_{s1}	ω_{s2}	ω_{p2}	π
Normalized Specs	0	0.47π	0.5008π	0.6546π	0.6854π	π
Normalized Analog Filter Specs	0	0.9099	1.0025	1.6586	1.8561	∞
Critical points for ↑	0	Ω_{p1}	Ω_{s1}	Ω_{s2}	Ω_{p2}	∞

Table 6: Comparison of Normalized and Analog Filter specifications

2.2.4 Frequency Transformation with relevant parameters

A transformation from the Band Stop Filter specifications to that of a low pass filter is needed to effectively design a Chebyshev Filter for the same and later transform it back into a Band Stop Filter. To do this, we will need the following frequency transformation:

$$\Omega_L = \left(\frac{B\Omega}{\Omega_o^2 - \Omega^2}\right) \tag{12}$$

The parameters in Equation 12 are Ω_o and B. These can be obtained with suitable specifications for the Low Pass Filter we need after transforming it from the Band Pass Filter. For obtaining a symmetric Low Pass Filter, we will need:

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = 1.2996 \tag{13}$$

And, if we need the symmetric passbands of the Low Pass Filter to be at $\Omega_L = 1$, we will need the following condition (essentially modelling bandwidth):

$$B = (\Omega_{p2} - \Omega_{p1}) = 0.9462 \tag{14}$$

Applying this transformation, we get the following Low Pass Filter specifications:

Critical points for ↓	0^+	Ω_{s1}	Ω_{p1}	Ω_o^-	Ω_o^+	Ω_{p2}	Ω_{s2}	∞
Normalized BPF Specs	0	0.9099	1.0025	1.2966	1.2966	1.6586	1.8561	∞
Normalized LPF Specs	0^+	1.009	1.4029	∞	$-\infty$	-1.4670	-0.9956	0^-
Critical points for ↑	0^+	Ω_{Lp1}	Ω_{Ls1}	∞	∞	Ω_{Ls2}	Ω_{Lp2}	0^-

Table 7: Comparison of Normalized and Analog Filter specifications

2.2.5 Analog Low Pass Filter Specifications

We can see that the passband edges for the Low Pass Fiter are symmetric at almost -1 and 1. We select the more stringent stopband specification. This gives us the stopband edge as $\min(|-1.4670|, 1.4029) = 1.4029$. So the final Analog Low Pass Filter specifications are as follows:

- **PassBand Edges** for Ω_L: 1 and -1
- **Stopband Edges** for Ω_L: -1.4029 and 1.4029
- **Tolerances**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Equiripple
- **Stopband Nature**: Monotonic

These specifications follow from the transformation results of Table 7.

2.2.6 Analog Filter LPF transfer function

The passband nature required for the Low Pass Filter is equiripple whereas that required for the stopband is monotonic. So, we can design **Chebyshev** approximation of the LPF to get the same natures. The tolerances needed to match are 0.15 in magnitude in both, the passbands and the stopbands, So, we get:

$$\delta_1 = \delta_2 = 0.15$$
$$D_1 = \frac{1}{(1-\delta_1)^2} - 1 = 0.3841$$
$$D_2 = \frac{1}{\delta_2^2} - 1 = 43.4444$$

We select the value of ϵ to be $\sqrt{D_1}$ in order to minimize N. So,

$$\epsilon = \sqrt{D_1} = 6.5912$$

The minimum order of the Chebyshev approximation required is obtained as follows:

$$\Omega_s = 1.4029, \quad \Omega_p = 1 \tag{15}$$
$$\frac{\Omega_s}{\Omega_p} = \frac{\Omega_s}{1} = 1.4029 \tag{16}$$
$$N \geq \cdot \frac{\cosh^{-1}\left(\sqrt{\frac{D_2}{D_1}}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \implies N \geq 3.5117 \tag{17}$$

So, $N = 4$ gives us the minimum order of the Chebyshev filter to be constructed for fulfilling the Low Pass Filter Design Specifications. We can see right here that the order is much less for almost the same parameters for Butterworth filter. This may however, not be a good comparison as these are two different filters but the values used for calculating the minimum order of the filter are almost the same and thus the observation.

Now, we find the poles for $H_{analog}(s)$ by equating the denominator to 0; specifically:

$$1 + \epsilon^2 \cos^2\left(N \cos^{-1}\left(\frac{s}{j}\right)\right) \;=\; 1 + D_1 \cos^2\left(N_{min} \cos^{-1}\left(\frac{s}{j}\right)\right) \;=\; 0 \qquad (18)$$

So, the poles can be given in a concise form as shown:

$$\begin{aligned} s_k &= (\Sigma_k + j\Omega_k), \quad \textit{where} \\ \Sigma_k &= -\Omega_p \sin A_k \sinh B \;=\; -\sin A_k \sinh B, \quad \textit{and} \\ \Omega_k &= \Omega_p \cos A_k \cosh B \;=\; \cos A_k \cosh B, \quad \textit{where} \\ B &= \frac{1}{4} \cdot \sinh^{-1}\left(\frac{1}{\epsilon}\right), \quad \textit{and} \\ A_k &= \left(\frac{(2k+1)\pi}{8}\right) \end{aligned}$$

where we select only those A_k for which $\sin A_k$ is positive as we require only those poles where the real part of s_k is negative. Since, $\sinh B$ is always positive, we require $k = 0, 1, 2, 3$ so that we make a complete revolution around the ellipse on which the poles lie as shown in Figure 10. The above solution gives us the following roots which exist on the left half plane of s which are in turn poles of $H_{analog,BSF}(s_L)$:

p1	$-0.1222 \;+\; j0.9698$
p2	$-0.2949 \;+\; j0.4017$
p3	$-0.2949 \;-\; j0.4017$
p4	$-0.1222 \;-\; j0.9698$

Table 8: Poles of $H_{analog,BSF}(s_L)$

The poles can be shown on the s-plane to lie on an ellipse centered at 0 which is shown in Figure 10. The expression for the transfer function for the Low Pass Filter comes out to be as follows (Keeping in mind that it is an even order Chebyshev approximation):

$$\begin{aligned} H_{analog,LPF}(s_L) \;&=\; \left(\frac{(-1)^4 p_1 p_2 p_3 p_4}{\sqrt{1+D_1}(s_L - p_1)(s_L - p_2)(s_L - p_3)(s_L - p_4)}\right) \\ &= \left(\frac{0.2017}{s_L^4 + 0.8342 s_L^3 + 1.3479 s_L^2 + 0.6243 s_L^2 + 0.2373}\right) \end{aligned}$$

Plotting it on the s_L axis we get the frequency response of the LPF as shown in Figure 11.

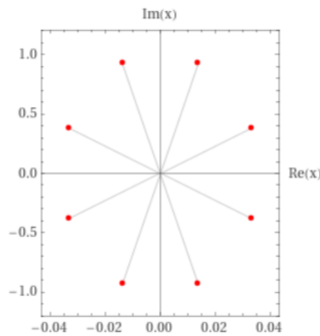

Figure 10: Poles of $H_{analog}(s_L)$ shown on the s plane

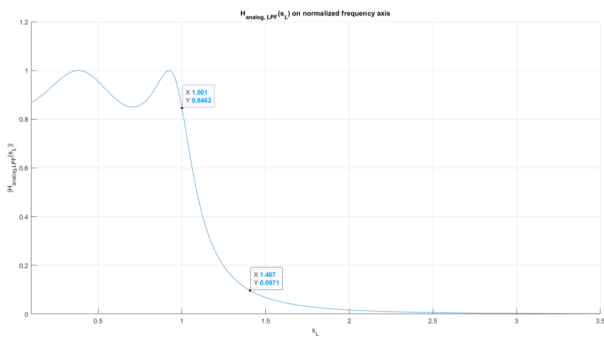

Figure 11: Low Pass Filter Frequency Response

2.2.7 Analog Band Stop Transfer Function

The transformation equation for getting the Band Stop Filter transfer function in the Analog domain from the LPF Transfer function thus created is as follows:

$$s_L = \left(\frac{Bs}{s^2 + \Omega_o^2} \right) \tag{19}$$

Substituting s_L from Equation 19 into the $H_{analog,LPF}(s_L)$ created in the previous subsection gives us the Band Stop Transfer function in terms of s:

$$\begin{aligned} s_L &= \left(\frac{s^2 + 1.2996}{0.9462s} \right) \\ H_{analog,BSF} &= \left(\frac{N(s)}{D(s)} \right), \quad where \\ N(s) &= (0.85s^8 + 5.742s^6 + 14.5461s^4 + 16.3773s^2 + 6.9146) \\ D(s) &= (s^8 + 2.49s^7 + 11.84s^6 + 15.59s^5 + 37.66s^4 + \\ &\qquad 26.32s^3 + 33.77s^2 + 11.99s + 8.13) \end{aligned}$$

The frequency response of the Band Stop Filter is shown in Figure 12.

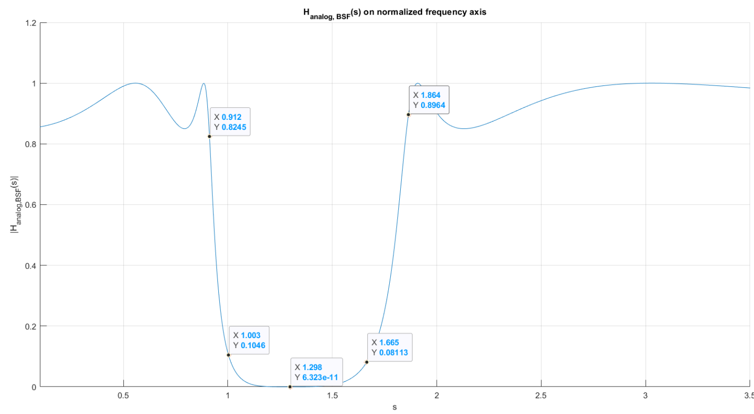

Figure 12: Frequency Response of $H_{analog,BSF}(s)$ on Normalized frequency axis

2.2.8 Discrete Time Filter Specifications

We use Bilinear transformation again to obtain the final Discrete time filter from the Analog Band Stop filter thus created. The transformation is:

$$s = \left(\frac{1 - z^{-}1}{1 + z^{-}1} \right)$$

We finally get the following Discrete Time Filter Transfer function:

$$\begin{aligned} H_{discrete,BSF}(z) &= \frac{Num(z)}{Den(z)}, \quad where \\ Num(z) &= (0.2986z^8 - 0.612z^7 + 1.6647z^6 + 1.9966z^5 + 2.7528z^4 - \\ &\qquad 1.9966z^3 + 1.6647z^2 + 0.612z + 0.2986), \quad and \\ Den(z) &= (z^8 + 1.5005z^7 + 2.7315z^6 + 2.5566z^5 + 2.751z^4 + \\ &\qquad 1.6349z^3 + 1.1337z^2 + 0.4457z + 0.2421) \end{aligned}$$

This discrete time Filter Transfer specification is shown in Figure 13 where the phase response as well as the magnitude response is shown in the log scale in the normalized frequency of the discrete domain.
The transfer function of the discrete filter with the un-normalized frequency specifications is shown in Figure 14.
We can see from the Pole-Zero Plot of the discrete Band Pass filter in Figure 15 that all the poles are inside the unit circle, thus making the filter stable.

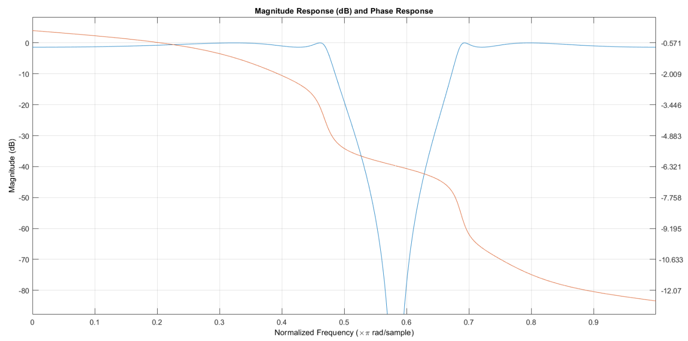

Figure 13: Magnitude and Phase response of the discrete BSF

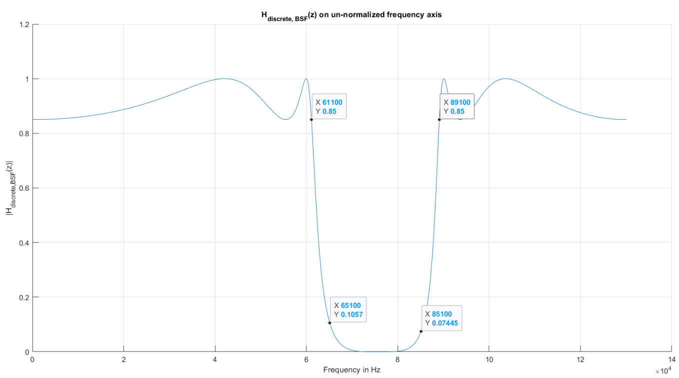

Figure 14: Magnitude response of the discrete BSF with un-normalized frequency

2.2.9 FIR Filter Transfer Function

We have been given that the tolerance of the stopbands is 0.15 in magnitude. We will then obtain an Attenuation constant(A) which is less than 21 dB:

$$A \;=\; -20\log\delta \;=\; 16.478\,dB$$

Since $A < 21$, we obtain the shape parameter α and eventually β as 0.
Now, for determining the order of the FIR filter, we make use of the empirical inequality:

$$(2N+1) \geq \left(1 + \frac{(A-8)}{2.285\Delta\omega_T}\right), \quad where$$

$$\Delta\omega_T \;=\; (\omega_s - \omega_p) \;=\; 0.0308\pi$$

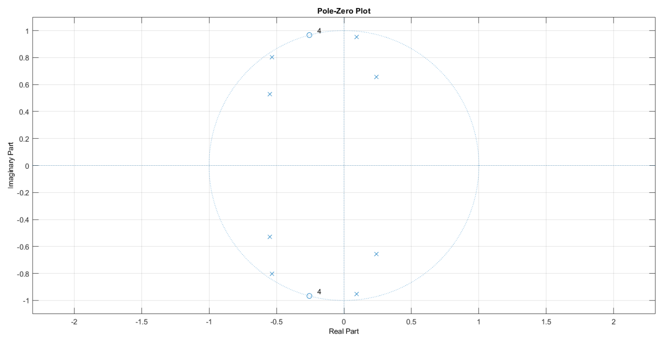

Figure 15: Pole-Zero plot of the discrete BSF

We get N_{min} to be 20. However, we need N to be 26 in order to fulfill the specifications of our Band Stop Filter. We get this value by hit and trial. The Kaiser window is itself of length $(2N + 1) = 53$ making the FIR impulse Filter response to be of length 53. Now, the shape parameter α can also be determined using the attenuation constant. As we get $A < 21$, we get $\alpha = \beta = 0$. Thus, we get the condition that the Kaiser window is basically a rectangular window which satisfies our tolerance requirements relatively well.
Now, we model the ideal Band Stop Filter as a difference between ideal Low Pass Filters and add this difference to the lpf signal with π and $-\pi$ as its bounds in the frequency domain. We take the bandwidth of the ideal LPFs as $(\omega_{p2} - 0.0154\pi)$ and $(\omega_{p1} + 0.0154\pi)$. The Band Stop Filter coefficients obtained are shown in Figure 16: These coefficients are obtained by multiplying the ideal

```
bsf_FIR =

  Columns 1 through 11

    0.0230   -0.0044   -0.0143    0.0062    0.0011    0.0062    0.0022   -0.0239    0.0106    0.0306   -0.0292

  Columns 12 through 22

   -0.0188    0.0344    0.0004   -0.0163    0.0009   -0.0122    0.0299    0.0213   -0.0812    0.0120    0.1183

  Columns 23 through 33

   -0.0822   -0.1075    0.1536    0.0435    0.8163    0.0435    0.1536   -0.1075   -0.0822    0.1183    0.0120

  Columns 34 through 44

   -0.0812    0.0213    0.0299   -0.0122    0.0009   -0.0163    0.0004    0.0344   -0.0188   -0.0292    0.0306

  Columns 45 through 53

    0.0106   -0.0239    0.0022    0.0062    0.0011    0.0062   -0.0143   -0.0044    0.0230
```

Figure 16: Coefficients of the FIR Band Stop Filter

BSF impulse response with the kaiser window impulse response. The magnitude and the phase response of the FIR filter is shown in Figure 17. The magnitude response of the filter on un-normalized frequency is shown in 18 where we can see that the discrete time filter specifications are indeed being met.

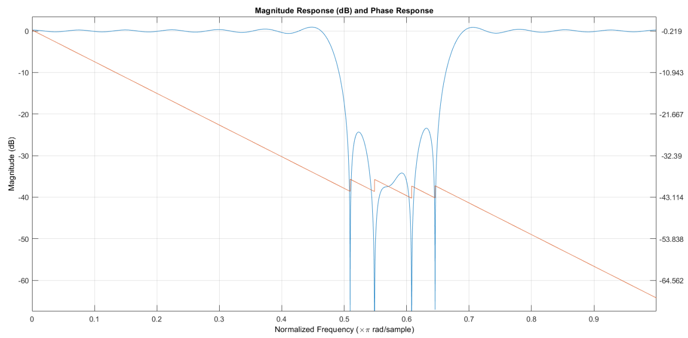

Figure 17: Magnitude and Phase response of the FIR Band Stop Filter

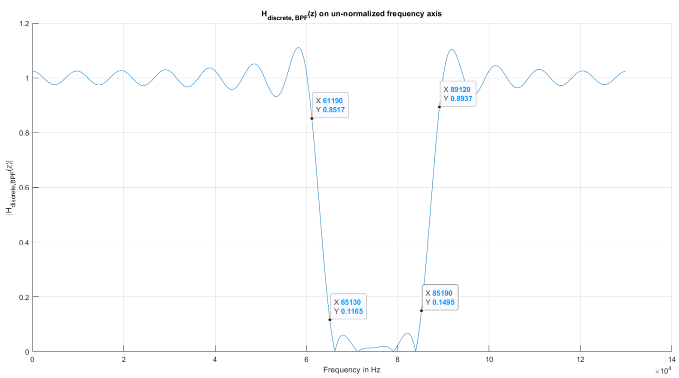

Figure 18: Magnitude response of the BPF filter on un-normalized frequencies

2.2.10 Comparison between FIR and IIR filters

We note the following points between the 2 types of filters:

- The phase response of the FIR filter is piece-wise linear whereas that of the IIR filter is far from linear.
- The number of delay lines required to realize the IIR Band Stop Filter is $(8+8) = 16$ whereas we require 52 delay lines for realizing the FIR filter. This implies that FIR filter requires more hardware than the IIR filters at the cost of providing linear phase response to the filters.

3 Part 2

3.1 Elliptic Band Pass Filter

3.1.1 Un-normalized discrete time filter specifications

Filter Number = 67
$q(m) = [67 \times 0.1] = 6$ where [x] is the greatest integer strictly less than x
$r(m) = (67 - 10q(m)) = 7$
$BL(m) = (25 + 1.7q(m) + 6.1r(m)) = 77.9$
$BH(m) = (BL(m) + 20) = 97.9$
Signals are bandlimited to 160 kHz and the Sampling Frequency is 330 kHz. The Bandpass filter specifications are as follows:

- **PassBand**: 77.9 kHz to 97.9 kHz
- **Transition Band**: 4 kHz on either side of the passband
- **Stopband**: 0-73.9 kHz and 101.9-165 kHz since the sampling frequency is 330 kHz
- **Tolerance**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Equiripple
- **Stopband Nature**: Equiripple

3.1.2 Normalized Digital Filter specifications

We have the sampling frequency as 330 kHz. Since, we want this frequency to represent 2π on the normalized frequency axis, we need to scale all the corresponding filter specifications down to 0 to π:

$$\omega = \left(\frac{2\pi\omega_{unn}}{\omega_{samp}}\right) \quad where \tag{20}$$

$$\omega_{unn} = Unnormalized\ frequency$$
$$\omega_{samp} = Sampling\ Frequency$$

Critical points →	0	ω_{s1}	ω_{p1}	ω_{p2}	ω_{s2}	π
Un-normalized Specs (kHz)	0	73.9	77.9	97.9	101.9	165
Normalized Specs	0	0.4479π	0.4721π	0.5933π	0.6176π	π

Table 9: Comparison of Unnormalized and Normalized Filter specifications

The passband and the stopband nature remain as monotonic. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications are mentioned in Table 9.

3.1.3 Analog Filter specifications using Bilinear Transformation

We use the Bilinear transformation to map z plane to the s plane where we use:

$$s = \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{21}$$

$$\Omega = \tan\frac{\omega}{2} \tag{22}$$

The second equation is a corollary of the Bilinear transformation. So, applying the Bilinear transformation to the normalized frequency values, we get:

Critical points for ↓	0	ω_{s1}	ω_{p1}	ω_{p2}	ω_{s2}	π
Normalized Specs	0	0.4479π	0.4721π	0.5933π	0.6176π	π
Normalized Analog Filter Specs	0	0.8484	0.9159	1.3464	1.4959	∞
Critical points for ↑	0	Ω_{s1}	Ω_{p1}	Ω_{p2}	Ω_{s2}	∞

Table 10: Comparison of Normalized and Analog Filter specifications

The passband and the stopband nature remain as monotonic. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications for the analog filter are mentioned in Table 10.

3.1.4 Frequency Transformation with relevant parameters

A transformation from the Bandpass filter specifications to that of a low pass filter is needed to effectively design a Butterworth Filter for the same and later transform it back into a Bandpass filter. To do this, we will need the following frequency transformation:

$$\Omega_L = \left(\frac{\Omega^2 - \Omega_o^2}{B\Omega}\right) \tag{23}$$

The parameters in Equation 23 are Ω_o and B. These can be obtained with suitable specifications for the Low Pass Filter we need after transforming it from the Band Pass Filter. For obtaining a symmetric Low Pass Filter, we will need:

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = 1.1105 \tag{24}$$

And, if we need the symmetric passbands of the Low Pass Filter to be at $\Omega_L = 1$, we will need the following condition (essentially modelling bandwidth):

$$B = (\Omega_{p2} - \Omega_{p1}) = 0.4305 \tag{25}$$

Applying this transformation, we get the following Low Pass Filter specifications:

Critical points for $\downarrow$	0^+	Ω_{s1}	Ω_{p1}	Ω_o	Ω_{p2}	Ω_{s2}	∞
Normalized BPF Specs	0	0.8484	0.9159	1.1105	1.3464	1.4959	∞
Normalized LPF Specs	0	-1.4057	-1.0001	0	0.9999	1.5598	∞
Critical points for $\uparrow$	$-\infty$	Ω_{Ls1}	Ω_{Lp1}	0	Ω_{Lp2}	Ω_{Ls2}	∞

Table 11: Comparison of Normalized and Analog Filter specifications

3.1.5 Analog Low Pass Filter Specifications

We can see that the passband edges for the Low Pass Fiter are symmetric at almost -1 and 1. We select the more stringent stopband specification. This gives us the stopband edge as $\min(|-1.4057|, 1.5598) = 1.4057$. So the final Analog Low Pass Filter specifications are as follows:

- **PassBand Edges** for Ω_L: 1 and -1
- **Stopband Edges** for Ω_L: -1.4057 and 1.4057
- **Tolerances**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Equiripple
- **Stopband Nature**: Equiripple

These specifications follow from the transformation results of Table 11.

3.1.6 Elliptic Functions Information

The passband and the stopband nature required for the Low Pass Filter is equiripple. So, we can design **Elliptic** approximation of the LPF to get the same. The tolerances needed to match are 0.15 in magnitude in both, the passbands and the stopbands, So, we get:

$$\delta_1 = \delta_2 = 0.15$$
$$D_1 = \frac{1}{(1-\delta_1)^2} - 1 = 0.3841$$
$$D_2 = \frac{1}{\delta_2^2} - 1 = 43.4444$$

The Elliptic Analog LPF function can be given in the follwoing form:

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 F_N^2(w)} \quad where \tag{26}$$

$$F_N(\omega) = cd(NuK1, k1) \; and \; w = cd(uK, k) = \frac{\Omega}{\Omega_{Lp}} = \Omega \tag{27}$$

cd used in Equation 27 is a Jacobian Elliptic function with elliptic modulus k and real quarter period K. cd and K are defined as follows:

$$K(k) \;=\; \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad and \quad z \;=\; \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}$$

Here, we can see that K is just a special case of z when $\phi = \frac{\pi}{2}$. We also define K' for $k = \sqrt{1-k^2}$. This is also known as a quarter period and k' is referred to as the complementary elliptic modulus.

Although there exist 12 Jacobian elliptic functions, we make use of only 4 in elliptic filter design as far as this assignment is concerned. Those are defined as follows:

$$sn(z,k) \;=\; \sin(\phi(z,k)) \tag{28}$$

$$cn(z,k) \;=\; \cos(\phi(z,k)) \tag{29}$$

$$dn(z,k) \;=\; \sqrt{1-k^2sn^2(z,k)} \tag{30}$$

$$cd(z,k) \;=\; \frac{cn(z,k)}{dn(z,k)} \tag{31}$$

Some other properties worth mentioning are as follows:

$$sn^2(z,k)+cn^2(z,k) \;=\; 1 \tag{32}$$

$$sn^2 \;=\; \frac{1-cd^2(z,k)}{1\;-\;k^2cd^2(z,k)} \tag{33}$$

$$K(k) \;=\; \frac{\pi}{2}\; if\; k=0 \; and \; \infty \; if \; k=1 \tag{34}$$

$$\tag{35}$$

sn and cd are doubly periodic functions in the z-plane with a period of $4K$ and a complex period of $2jK'$. cd and sn are also interchangeable analogous to sin and cos as follows:

$$cd(z,k) \;=\; sn(z+K,k) \;=\; sn(K-z,k) \tag{36}$$

$$cd(z+(2i-1)K,k) \;=\; (-1)^i sn(z,k)\; for\; any\; integer\; i \tag{37}$$

$$cd(z+2iK,k) \;=\; (-1)^i cd(z,k)\; for\; any\; integer\; i \tag{38}$$

$$cd(z+jK',k) \;=\; \frac{1}{kcd(z,k)} \implies cd(jK',k) \;=\; \frac{1}{k} \tag{39}$$

$$cd(jz,k) \;=\; \frac{1}{dn(z,k')} \tag{40}$$

$cd(z,k)$ has zeroes at $z \;=\; ((2m+1)K \;=\; 2njK')$ for any integers m and n. It has it poles at $z \;=\; ((2m+1)K \;+\; (2n+1)jK')$ again with any integer m and n.

3.1.7 Analog Filter LPF transfer function

The minimum degree of the filter is decided by the constraint put on the filter to have a magnitude spectrum more than $(1-\delta_1)$ at $\Omega = 1$ and less than $(1-\delta_2)$ at $\Omega_L s$. The second constraint gives us the following constraint:

$$N \geq \frac{K_1' K}{K_1 K'} \tag{41}$$

where K_1, K, K_1' *and* K' are the respective complete elliptic integrals as defined before where $k = \frac{\Omega_p}{\Omega_s}$ and $k_1 = \sqrt{\frac{D_1}{D_2}}$. We get N equal to 3 from this equation as N needs to be a positive integer.
Now, we need to find the zeroes and poles of $F_N(w)$ now as these give the zeroes and further information for $|H(j\Omega)|^2$. The zeroes of $F_N(w)$ correspond to the zeroes of the Jacobian elliptic function $cd(3uK, k)$ where $u = \frac{cd^-1(\Omega,k)}{K}$. This is given as follows:

$$3u_iK_1 \;=\; (2i-1)K_1 \tag{42}$$

$$u_i \;=\; \frac{2i-1}{3}\; for\; i = 1 \tag{43}$$

$$\Omega \;=\; cd(u_iK, k) \;=\; \zeta_i,\; i = 1 \tag{44}$$

We have the following condition on $F_N(w)$:

$$F_N(\frac{1}{kw}) \;=\; \frac{1}{k_1F_N(w)}$$

using 39.This gives us the simple solution that $\Omega = \frac{1}{k\zeta_i}$ are the poles of $F_N(w)$. The poles of $F_N(w)$ correspond to the zeroes of $|H(j\Omega)|^2$ and the poles of $|H(j\Omega))|^2$ can be found out by placing the condition that $(1 + D_1F_N^2(w)) \;=\; 0$. This gives us the poles as follows:

$$p_{ai} \;=\; jcd((u_i - jv_0)K, k),\; i = 1 \quad where$$

$$sn(jv_0NK_1, k_1) \;=\; \frac{j}{\sqrt{D_1}}$$

These poles correspond to the poles in the second quadrant of the s-plane. The conjugate of the above poles are also the poles of the same. Since N is odd, we have another pole as follows:

$$p_{a0} \;=\; jcd((1 - jv_0)K, k) \;=\; jsn(jv_0K, k)$$

The zeroes and poles obtained are shown below in Table 16: The $H_{analog,LPF}(s)$ is shown as follows:

$$H_{analog,LPF}(s) \;=\; \frac{(0.3925s^2 + 0.6235)}{(s^3 + 0.8538s^2 + 1.1443s + 0.6235)}$$

We see that the degree of the filter is less than those of the Chebyshev and the Butterworth filters for the same specifications. This lessening of order comes from the increasing non-linearity of phase of the Elliptic filters as visible in 21. The LPF magnitude response is shown in figure 19.

$Zero_1$	- j1.2604
$Zero_2$	j1.2604
$Pole_1$	-0.1153 - j0.9936
$Pole_2$	-0.1153 + j0.9936
$Pole_3$	-0.6232

Table 12: Zeroes and Poles of the Elliptic Low Pass Filter

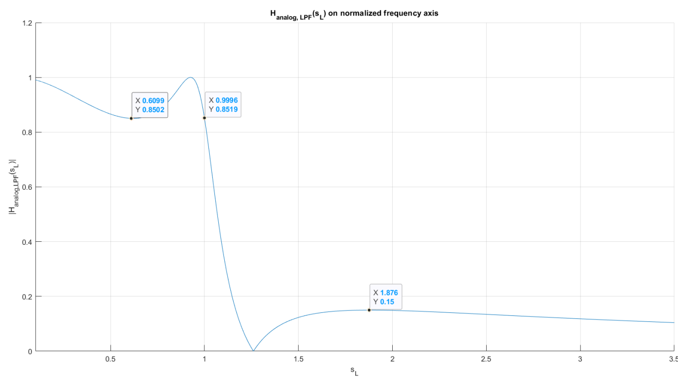

Figure 19: Low Pass Filter with the Elliptic approximation

3.1.8 Analog Bandpass Transfer Function

The transformation equation for getting the Bandpass filter transfer function in the Analog domain from the LPF Transfer function thus created is as follows:

$$s_L \ = \ \left(\frac{s^2 + \Omega_o^2}{Bs}\right) \tag{45}$$

Substituting s_L from Equation 45 into the $H_{analog,LPF}(s_L)$ created in the previous subsection gives us the Bandpass Transfer function in terms of s:

$$\begin{aligned}
s_L \ &= \ \left(\frac{s^2 + 1.2334}{0.4305s}\right) \\
H_{analog,BPF} \ &= \ \left(\frac{N(s)}{D(s)}\right), \quad where \\
N(s) \ &= \ 0.169s^4 + 0.4665s^2 + 0.257 \\
D(s) \ &= \ (s^6 + 0.3675s^5 + 3.9124s^4 + 0.9564s^3 + 4.8257s^2 + 0.5592s + 1.8766)
\end{aligned}$$

The frequency response of the Band Pass Filter is shown in Figure 20.

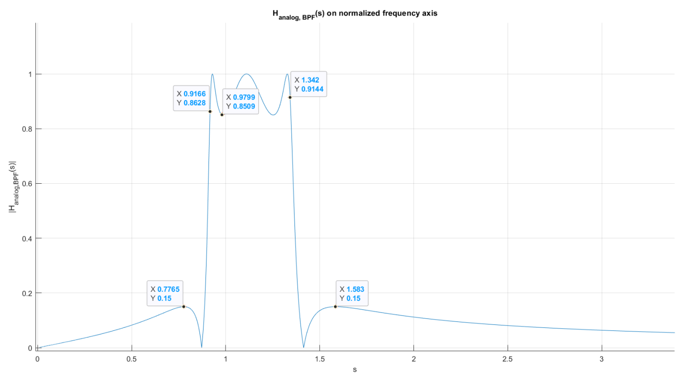

Figure 20: Frequency Response of $H_{analog,BPF}(s)$ on Normalized frequency axis

3.1.9 Discrete Time Filter Specifications

We use Bilinear transformation again to obtain the final Discrete time filter from the Analog Bandpass filter thus created. The transformation is:

$$s = \left(\frac{1 - z^{-}1}{1 + z^{-}1}\right)$$

We finally get the following Discrete Time Filter Transfer function:

$$H_{discrete,BPF}(z) = \frac{Num(z)}{Den(z)}, \quad where$$

$$\begin{aligned} Num(z) &= (0.0661z^6 + 0.0261z^5 + 0.0541z^4 - 0.0541z^2 - 0.0261z - 0.0661), \quad and \\ Den(z) &= (z^6 + 0.5818z^5 + 2.68z^4 + 1.0282z^3 + 2.4186z^2 + 0.4682z + 0.721) \end{aligned}$$

This discrete time Filter Transfer specification is shown in Figure 21 where the phase response as well as the magnitude response is shown in the log scale in the normalized frequency of the discrete domain.
The transfer function of the discrete filter with the unnormalized frequency specifications is shown in Figure 22.
We can see from the Pole-Zero Plot of the discrete Band Pass filter in Figure 23 that all the poles are inside the unit circle, thus making the filter stable.

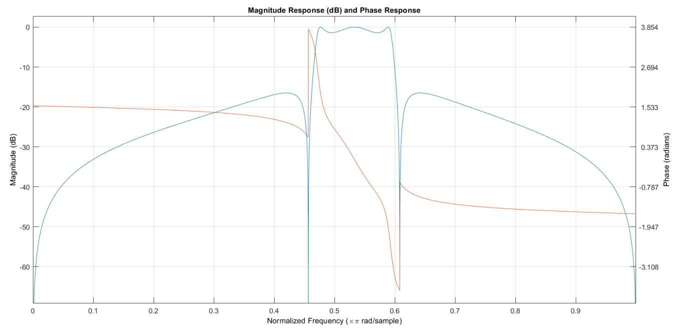

Figure 21: Magnitude and Phase response of the discrete BPF

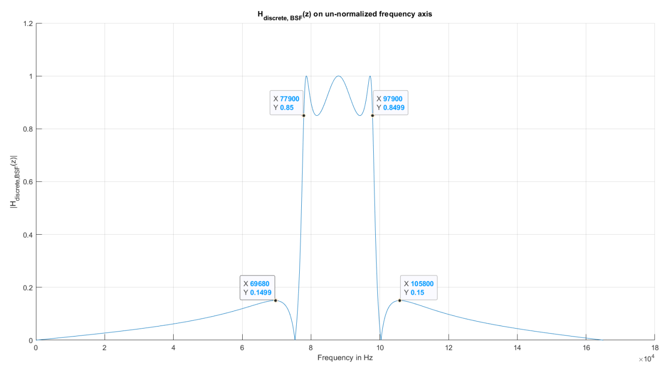

Figure 22: Magnitude response of the discrete BPF with unnormalized frequency

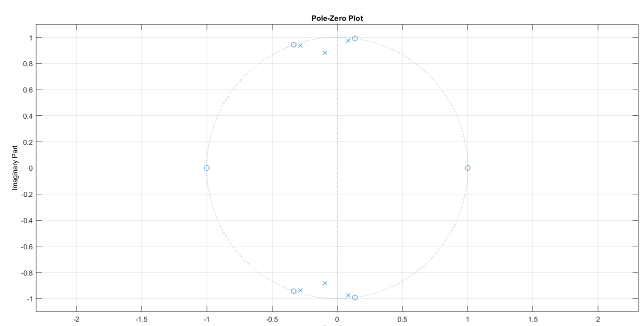

Figure 23: Pole-Zero plot of the discrete BPF

3.2 Band Stop Filter

3.2.1 Un-normalized discrete time filter specifications

Filter Number = 67
$q(m) = [67 \times 0.1] = 6$ where [x] is the greatest integer strictly less than x
$r(m) = (67 - 10q(m)) = 7$
$BL(m) = (25 + 1.9q(m) + 4.1r(m)) = 65.1$
$BH(m) = (BL(m) + 20) = 85.1$
Signals are bandlimited to 120 kHz and the Sampling Frequency is 260 kHz. The Bandpass filter specifications are as follows:

- **Stopband**: 65.1 kHz to 85.1 kHz
- **Transition Band**: 4 kHz on either side of the passband
- **Passband**: 0-61.1 kHz and 85.1-130 kHz since the sampling frequency is 260 kHz
- **Tolerance**: Both, the passband and the stopband tolerances are 0.15
- **Passband Nature**: Equiripple
- **Stopband Nature**: Equiripple

3.2.2 Normalized Digital Filter specifications

We have the sampling frequency as 260 kHz. Since, we want this frequency to represent 2π on the normalized frequency axis, we need to scale all the corresponding filter specifications down to 0 to π:

$$\omega = \left(\frac{2\pi\omega_{unn}}{\omega_{samp}} \right) \quad where \tag{46}$$

$$\omega_{unn} = Unnormalized\ frequency$$
$$\omega_{samp} = Sampling\ Frequency$$

Critical points →	0	ω_{p1}	ω_{s1}	ω_{s2}	ω_{p2}	π
Un-normalized Specs (kHz)	0	61.1	65.1	85.1	89.1	165
Normalized Specs	0	0.47π	0.5008π	0.6546π	0.6854π	π

Table 13: Comparison of Unnormalized and Normalized Filter specifications

The passband and the stopband nature remain as equiripple and montonic respectively. Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications are mentioned in Table 13.

3.2.3 Analog Filter specifications using Bilinear Transformation

We use the Bilinear transformation to map z plane to the s plane where we use:

$$s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{47}$$

$$\Omega = \tan\frac{\omega}{2} \tag{48}$$

The second equation is a corollary of the Bilinear transformation. So, applying the Bilinear transformation to the normalized frequency values, we get:

Critical points for ↓	0	ω_{p1}	ω_{s1}	ω_{s2}	ω_{p2}	π
Normalized Specs	0	0.47π	0.5008π	0.6546π	0.6854π	π
Normalized Analog Filter Specs	0	0.9099	1.0025	1.6586	1.8561	∞
Critical points for ↑	0	Ω_{p1}	Ω_{s1}	Ω_{s2}	Ω_{p2}	∞

Table 14: Comparison of Normalized and Analog Filter specifications

The passbands and the stopband retain their nature as equiripple and monotonic respectively . Tolerances for passband and stopband are both 0.15 in magnitude. The other normalized specifications for the analog filter are mentioned in Table 14.

3.2.4 Frequency Transformation with relevant parameters

A transformation from the Band Stop Filter specifications to that of a low pass filter is needed to effectively design a Chebyshev Filter for the same and later transform it back into a band Stop Filter. To do this, we will need the following frequency transformation:

$$\Omega_L = \left(\frac{B\Omega}{\Omega_o^2 - \Omega^2}\right) \tag{49}$$

The parameters in Equation 49 are Ω_o and B. These can be obtained with suitable specifications for the Low Pass Filter we need after transforming it from the Band Pass Filter. For obtaining a symmetric Low Pass Filter, we will need:

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = 1.2996 \tag{50}$$

And, if we need the symmetric passbands of the Low Pass Filter to be at $\Omega_L = 1$, we will need the following condition (essentially modelling bandwidth):

$$B = (\Omega_{p2} - \Omega_{p1}) = 0.9462 \tag{51}$$

Applying this transformation, we get the following Low Pass Filter specifications:

Critical points for ↓	0^+	Ω_{s1}	Ω_{p1}	Ω_o^-	Ω_o^+	Ω_{p2}	Ω_{s2}	∞
Normalized BPF Specs	0	0.9099	1.0025	1.2966	1.2966	1.6586	1.8561	∞
Normalized LPF Specs	0^+	1.009	1.4029	∞	$-\infty$	-1.4670	-0.9956	0^-
Critical points for ↑	0^+	Ω_{Lp1}	Ω_{Ls1}	∞	∞	Ω_{Ls2}	Ω_{Lp2}	0^-

Table 15: Comparison of Normalized and Analog Filter specifications

3.2.5 Analog Filter LPF transfer function

The minimum degree of the filter is decided by the constraint put on the filter to have a magnitude spectrum more than $(1-\delta_1)$ at $\Omega = 1$ and less than $(1-\delta_2)$ at $\Omega_L s$. The second constraint gives us the following constraint:

$$N \geq \frac{K_1' K}{K_1 K'} \tag{52}$$

where K_1, K, K_1' *and* K' are the respective complete elliptic integrals as defined before where $k = \frac{\Omega_p}{\Omega_s}$ and $k_1 = \sqrt{\frac{D_1}{D_2}}$. We get N equal to 3 from this equation as N needs to be a positive integer.
Now, we need to find the zeroes and poles of $F_N(w)$ now as these give the zeroes and further information for $|H(j\Omega)|^2$. The zeroes of $F_N(w)$ correspond to the zeroes of the Jacobian elliptic function $cd(3uK, k)$ where $u = \frac{cd^-1(\Omega,k)}{K}$. This is given as follows:

$$3u_i K_1 = (2i-1)K_1 \tag{53}$$

$$u_i = \frac{2i-1}{3}\ for\ i = 1 \tag{54}$$

$$\Omega = cd(u_i K, k) = \zeta_i,\ i = 1 \tag{55}$$

We have the following condition on $F_N(w)$:

$$F_N(\frac{1}{kw}) = \frac{1}{k_1 F_N(w)}$$

using 39.This gives us the simple solution that $\Omega = \frac{1}{k\zeta_i}$ are the poles of $F_N(w)$. The poles of $F_N(w)$ correspond to the zeroes of $|H(j\Omega)|^2$ and the poles of $|H(j\Omega))|^2$ can be found out by placing the condition that $(1 + D_1 F_N^2(w)) = 0$. This gives us the poles as follows:

$$p_a i = jcd((u_i - jv_0)K, k),\ i = 1 \quad where$$
$$sn(jv_0 N K_1, k_1) = \frac{j}{\sqrt{D_1}}$$

These poles correspond to the poles in the second quadrant of the s-plane. The conjugate of the above poles are also the poles of the same. Since N is odd, we

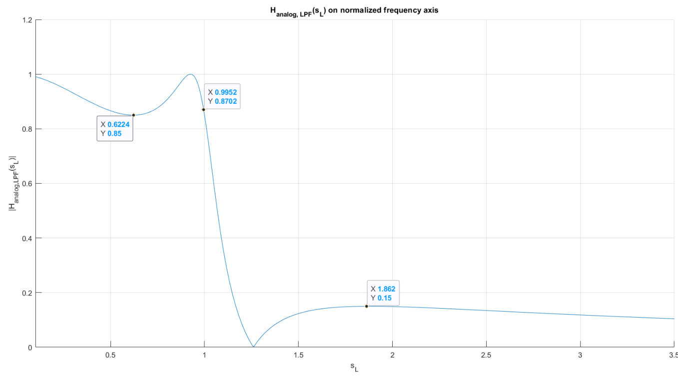

Figure 24: Low Pass Filter with the Elliptic approximation

have another pole as follows:

$$p_a0 \ = \ jcd((1 - jv_0)K, k) \ = \ jsn(jv_0K, k)$$

The zeroes and poles obtained are shown below in Table 16: The $H_{analog,LPF}(s)$

$Zero_1$	- j1.2604
$Zero_2$	j1.2604
$Pole_1$	-0.1153 - j0.9936
$Pole_2$	-0.1153 + j0.9936
$Pole_3$	-0.6232

Table 16: Zeroes and Poles of the Elliptic Low Pass Filter

is shown as follows:

$$H_{analog,LPF}(s) \ = \ \frac{(0.3925s^2 + 0.6235)}{(s^3 + 0.8538s^2 + 1.1443s + 0.6235)}$$

We see that the degree of the filter is less than those of the Chebyshev and the Butterworth filters for the same specifications. This lessening of order comes from the increasing non-linearity of phase of the Elliptic filters as visible in 21. The LPF magnitude response is shown in figure 24.

3.2.6 Analog Bandpass Transfer Function

The transformation equation for getting the Bandpass filter transfer function in the Analog domain from the LPF Transfer function thus created is as follows:

$$s_L \ = \ \left(\frac{s^2 + \Omega_o^2}{Bs}\right) \tag{56}$$

Substituting s_L from Equation 56 into the $H_{analog,LPF}(s_L)$ created in the previous subsection gives us the Bandpass Transfer function in terms of s:

$$
\begin{aligned}
s_L &= \left(\frac{s^2 + 1.2334}{0.4305s}\right) \\
H_{analog,BPF}(s) &= \left(\frac{N(s)}{D(s)}\right), \quad where \\
N(s) &= (s^6 + 5.6299s^4 + 9.5081s^2 + 4.8168) \\
D(s) &= (s^6 + 1.7363s^5 + 6.2922s^4 + 7.2228s^3 + 10.6265s^2 + 4.9523s + 4.8168)
\end{aligned}
$$

The frequency response of the Band Stop Filter is shown in Figure 25.

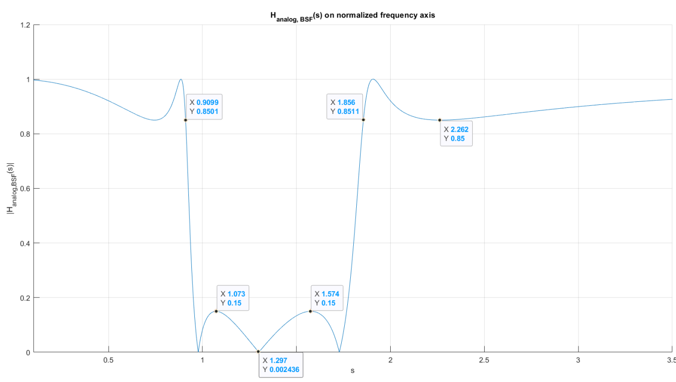

Figure 25: Frequency Response of $H_{analog,BSF}(s)$ on Normalized frequency axis

3.2.7 Discrete Time Filter Specifications

We use Bilinear transformation again to obtain the final Discrete time filter from the Analog Band Stop filter thus created. The transformation is:

$$
s = \left(\frac{1 - z^-1}{1 + z^-1}\right)
$$

We finally get the following Discrete Time Filter Transfer function:

$$
\begin{aligned}
H_{discrete,BSF}(z) &= \frac{Num(z)}{Den(z)}, \quad where \\
Num(z) &= (0.5718z^6 + 0.8366z^5 + 1.9678z^4 - 1.6597z^3 + 1.9678z^2 + 0.8366z + 0.5718), \quad and \\
Den(z) &= (z^6 + 1.2125z^5 + 2.2405z^4 + 1.61z^3 + 1.5979z^2 + 0.5104z + 0.2408)
\end{aligned}
$$

This discrete time Filter Transfer specification is shown in Figure 26 where the phase response as well as the magnitude response is shown in the log scale in

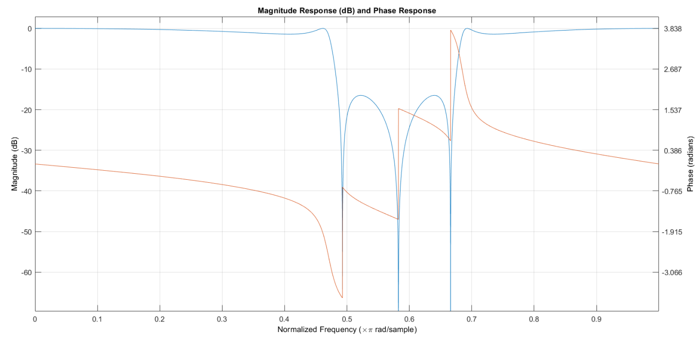

Figure 26: Magnitude and Phase response of the discrete BSF

the normalized frequency of the discrete domain.
The transfer function of the discrete filter with the unnormalized frequency specifications is shown in Figure 27.

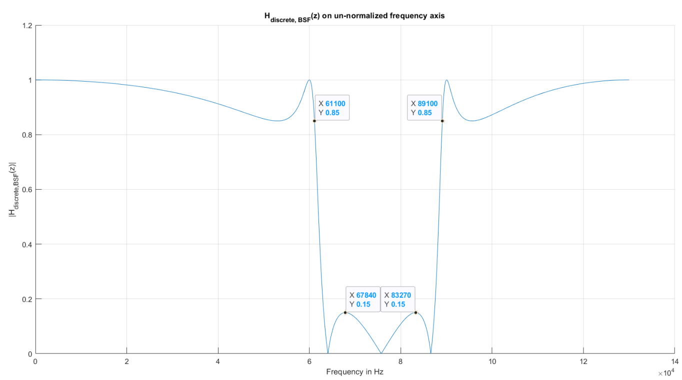

Figure 27: Magnitude response of the discrete BSF with unnormalized frequency

We can see from the Pole-Zero Plot of the discrete Band Stop filter in Figure 28 that all the poles are inside the unit circle, thus making the filter stable.

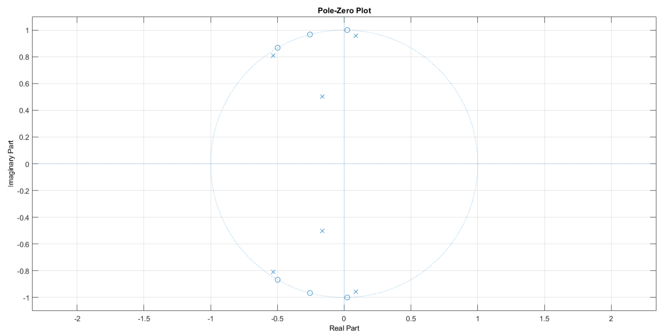

Figure 28: Pole-Zero plot of the discrete BSF

4 Conclusion

We conclude that the Elliptic Filters require a lower order than their counterparts for the same filter specifications using Chebyshev or Butterworth approximations which is evident from the designs depicted earlier. Also, we see that the passband and stopband of the Elliptic filters are equiripple rather than either of them being monotonic like it is observed in Chebyshev or Butterworth filters. We also observe that the phase response of the Elliptic filter is the furthest from linearity whereas Butterworth filters have a mcuh more linear phase response than the Chebyshev Filters and the Chebyshev filters much more than the Equiripple filters.