Graphs

Lesson 1.2

REPRESENTING AND COMPARING GRAPHS

Knowledge has Organizing Power

Wholeness of the lecture: Use relationships in a graph to organize adjacency and incidence matrices that give the information contained in the graph in different and useful way. Such information about two different graphs helps us to determine whether the graphs are the same or not. *Science of Consciousness:* Knowledge has organizing power.

Main Points

1. Adjacency and incidence matrices encode all of the information in a given graph into new and useful ways and give different perspectives of the graph. *Science of Consciousness:* Research shows that practice of the Transcendental Meditation technique improves efficiency of visual perception and gives increased freedom from habitual patterns of perception.[2]
2. A graph isomorphism shows that two graphs, even though they may appear to be different, are essentially the same in their structure. *Science of Consciousness:* With regular practice of the Transcendental Meditation technique, we are better able to perceive deeper levels of life where there is greater harmony and connectedness.

Graphs

Ways to Represent a Graph and Isomorphism

i. **Graph Representations:**

- Adjacency lists.
- Adjacency matrices.
- Incidence matrices.

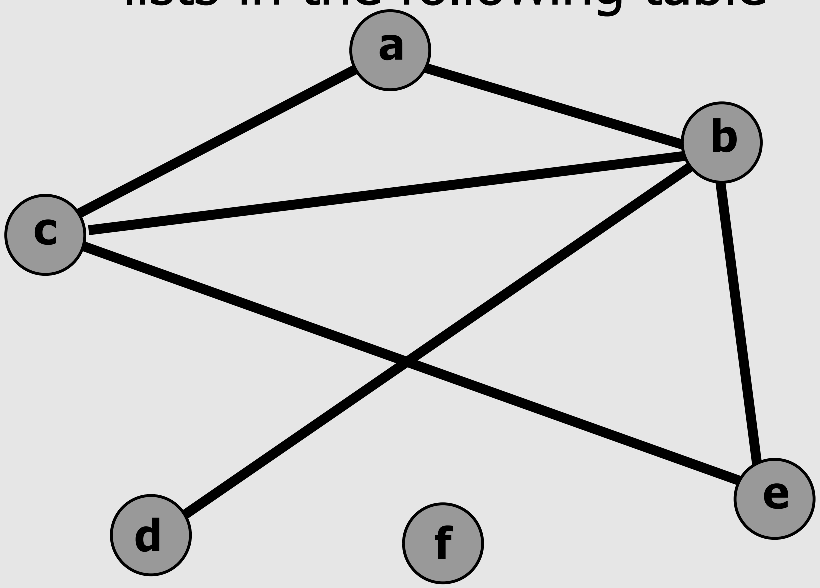
ii. **Graph Isomorphism:**

- Two graphs are said to be isomorphic if and only if they are identical except for their edge names.

Graphs

I. Adjacency Lists:

- A way of representing a graph with no multiple edges is to use adjacency lists.
- The vertices adjacent to each of the vertices of the **undirected graph**, lists in the following table

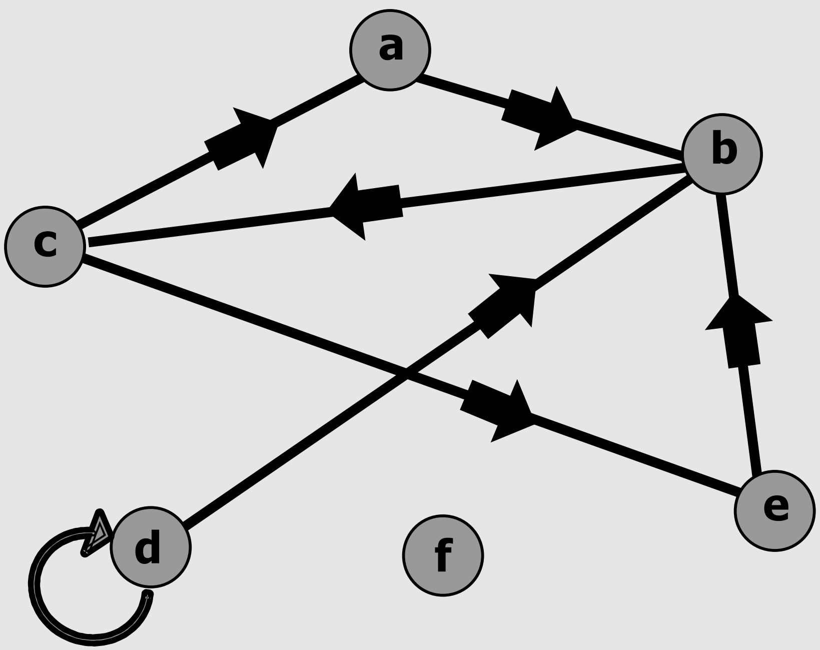


Vertex	Adjacent Vertices
a	b, c
b	a, c, d, e
c	a, b, e
d	b
e	b, c
f	

Graphs

II. Adjacency Lists:

- The vertices adjacent to each of the vertices of the **directed graph** lists in the following table



Vertex	Adjacent Vertices
a	b
b	c
c	a, e
d	d, b
e	b
f	

Graphs

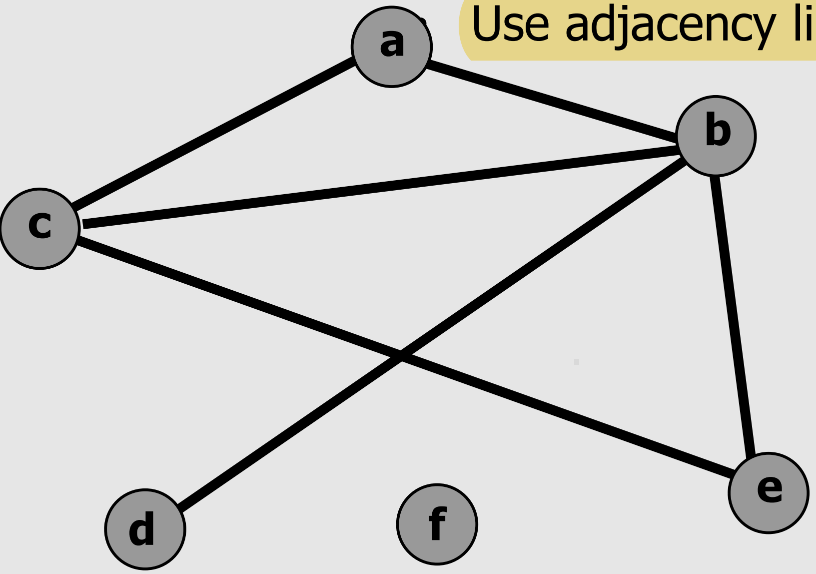
III. Adjacency Matrix:

- Graphs can be represented by matrices to simplify the complex computation.
- Let $G = (V, E)$ is a graph with n vertices listed as $v_1, v_2, v_3, \cdots, v_n$
- The adjacency matrix suppose $A = [a_{ij}]$ of G with respect to the adjacency list of vertices is defined as

$$a_{ij} = \begin{cases} 1, & if\ v_i, v_j\ are\ adjacent\ vertices. \\ 0, & if\ v_i, v_j\ aren't\ adjacent\ vertices. \end{cases}$$

➢ **Example:**

- Use adjacency list to represent graph by adjacency matrix.

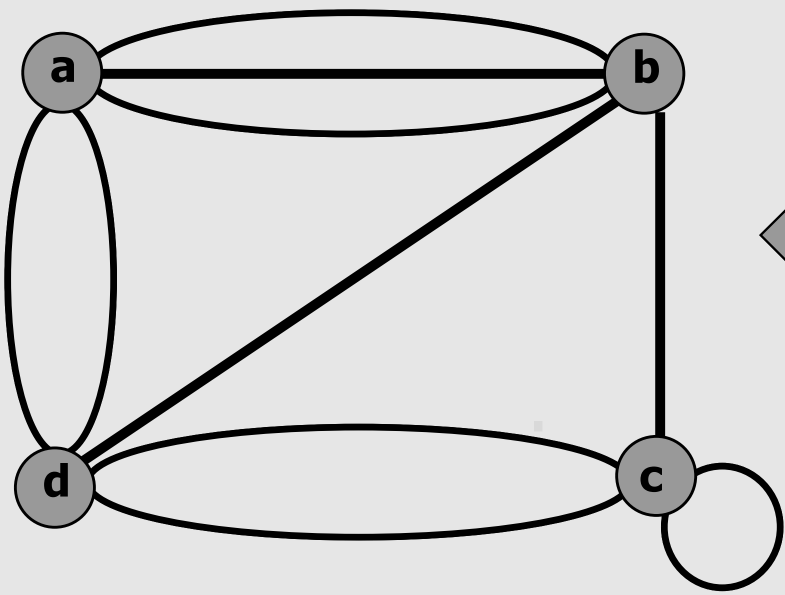


$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Graphs

Example

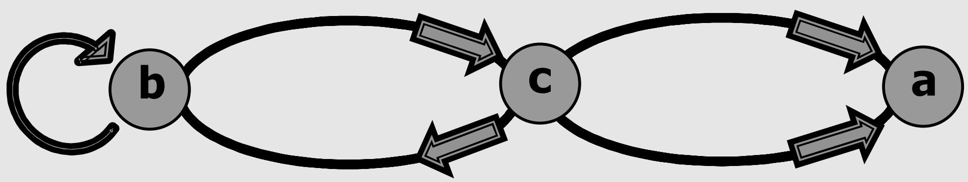
- Use adjacency matrix to represent the graph.



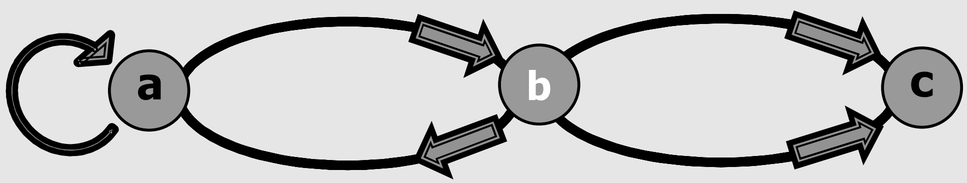
$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}
\end{array}
$$

Adjacency Matrix of a Graph

- Write into adjacency matrix.



$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

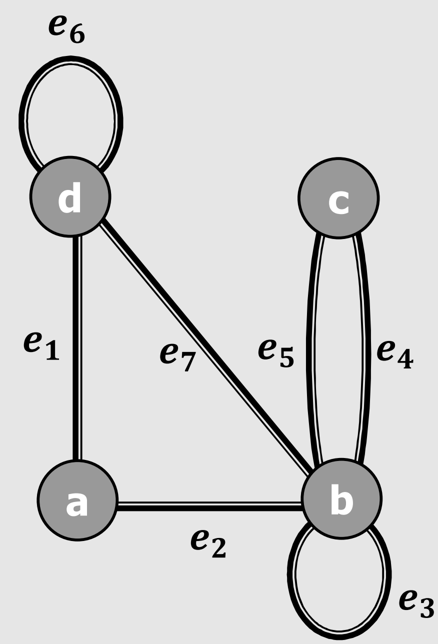


$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix

- Draw a graph with the adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$



Graphs

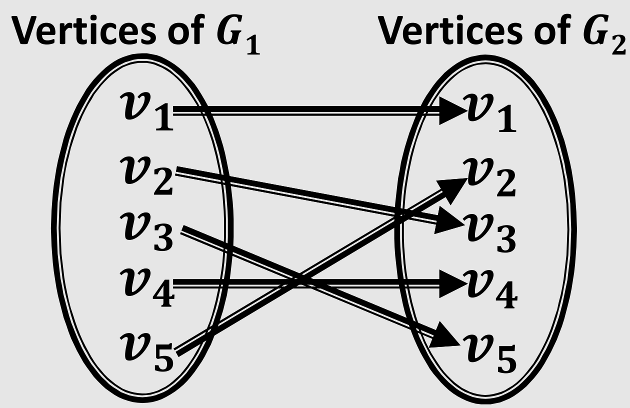
Adjacency Matrix

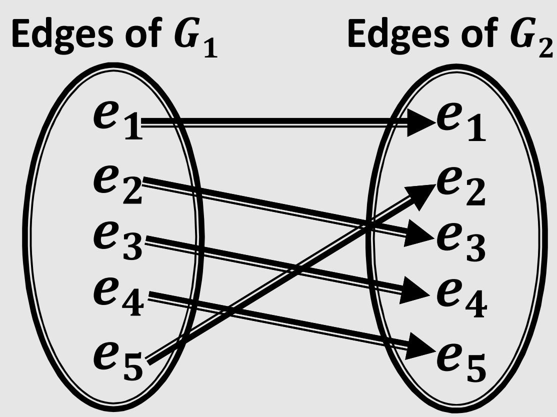
- Draw a graph with the adjacency matrix.

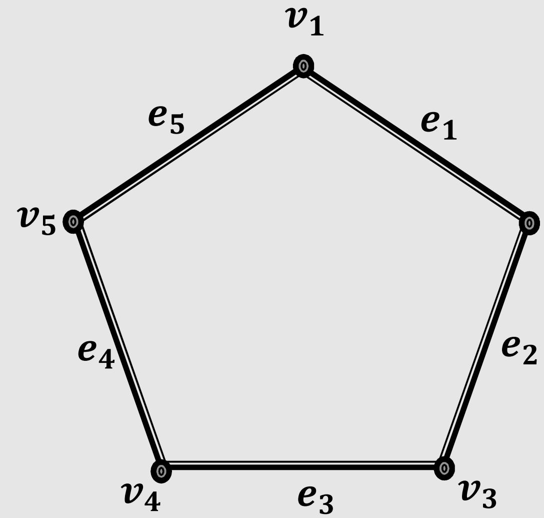
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Graphs

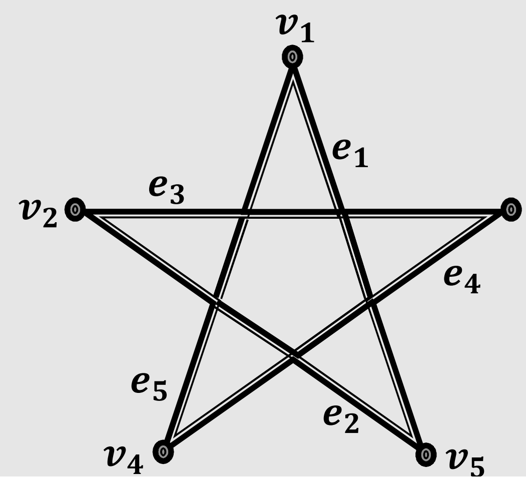
2. Isomorphisms of Graphs:







G_1



G_2

Graphs

2. Graph Isomorphism:

- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are ***isomorphic*** $(G_1 \cong G_2) \Leftrightarrow$ a mapping f from $V_1 \rightarrow V_2$ is bijective with the property that a and b are adjacent in $G_1 \Leftrightarrow f(a)$ and $f(b)$ are adjacent in G_2. Such a function f is called an isomorphism.

or

- Two simple graphs G_1 and G_2 are isomorphic $(G_1 \cong G_2)$ if
  - Both graphs have same no. of vertices.
  - 1-1 mapping f exists between the vertices and edges of G_1 to vertices and edges of G_2.
- "If two simple graphs are not ***isomorphic***, then they are said ***nonisomorphic.***

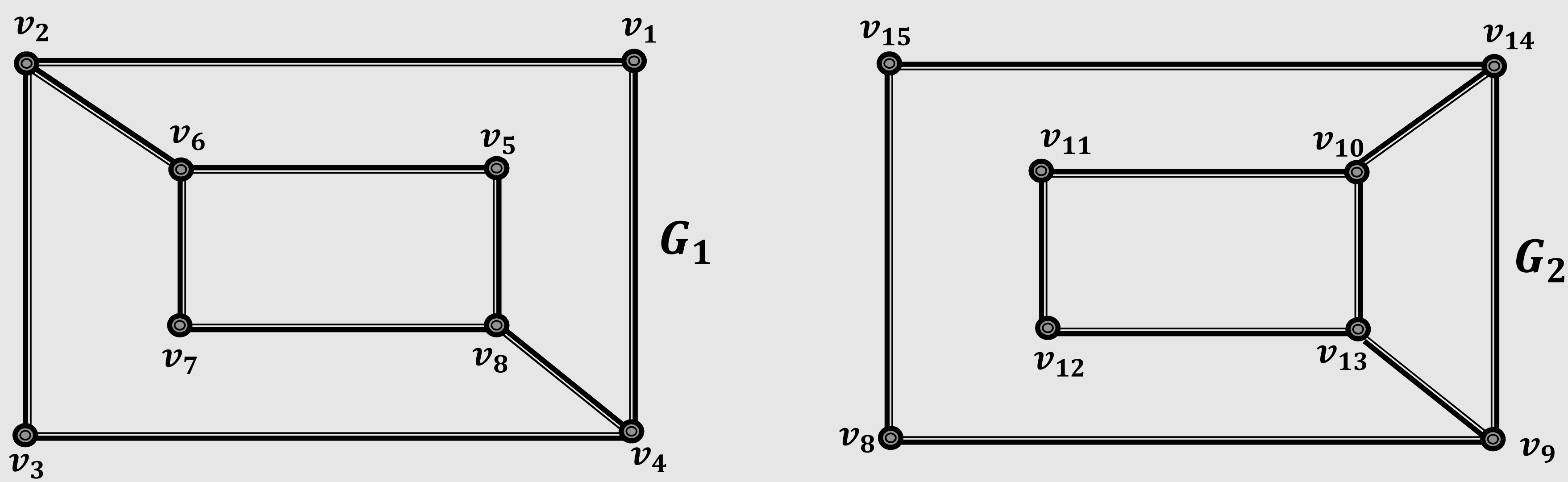
Graphs

What we notice? When two graphs are isomorphic

- *Necessary* but not *sufficient* conditions for $G_1 = (V_1, E_1)$ isomorphic to $G_2 = (V_2, E_2)$.
  - No. of vertices and edges must be same in both graphs i.e., $|V_1| = |V_2|$ and $|E_1| = |E_2|$.
  - Corresponding vertices have the same degree i.e., $\deg(V_1) = \deg(V_2)$.
  - The no. of vertices with degree *n* is same in both graphs.
  - Same no. of triangles and quadrilaterals in both graphs.
  - If two vertices are joined by a path of length n in one graph, there must be a path of length n joining the two corresponding vertices in the other graph.

Graphs

Determine whether the graphs shown below are isomorphic or not?

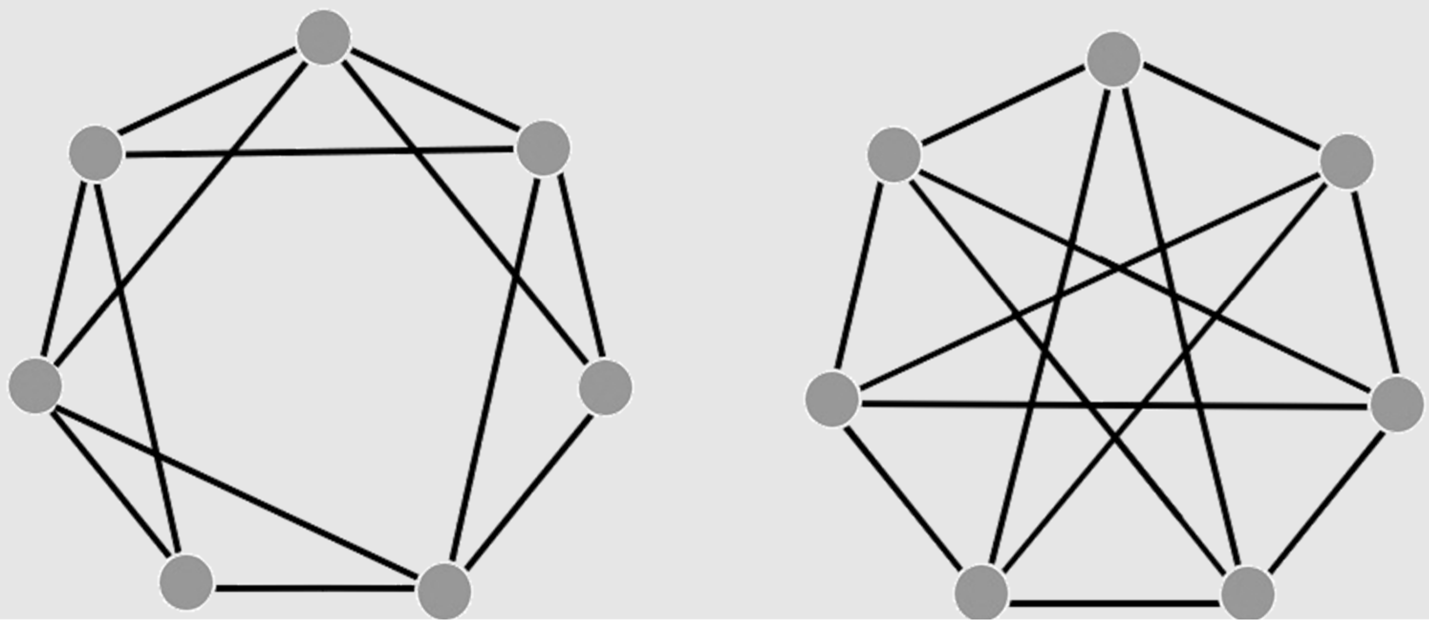


Since both graphs have 8-vertices and 10-edges. Both have 4-vertices of degree 2 and 4 of degree 3. However, G_1 and G_2 are not isomorphic. Because deg(v_1) = 2 in G_1, then v_1 must correspond to either v_{15}, v_8, v_{11} or v_{12} in G_2, the reason is these all vertices of degree 2 in G_2. Each of these vertices are adjacent to another vertex of degree 2 in G_2, which is not true for v_1 in G_1 .

Graphs

Example

- Are these two graphs G_1 and G_2 are isomorphic?
- Do the graphs G_1 and G_2 have the same number of vertices and edges?
- Do the corresponding vertices have the same degree i.e., $\deg(V_1) = \deg(V_2)$.
- Label the vertices of the graphs.



Paths in Graphs

Connectivity

- Many problems can be modeled with the help of paths traveling along the edges of graphs.
- Problems can be studied by using the concept of paths.

  i. Paths can determine weather a message can be delivered from one computer to another computer using intermediate links.

  ii. Models involving paths in graphs can be used to solve the problems related to
    - routes for mail delivery,
    - diagnostics in computer networks.

Paths in Graphs

Paths

- Informally, a **path** is a sequence of edges, begins at one vertex traverses from vertex to vertex along edges of the graph.

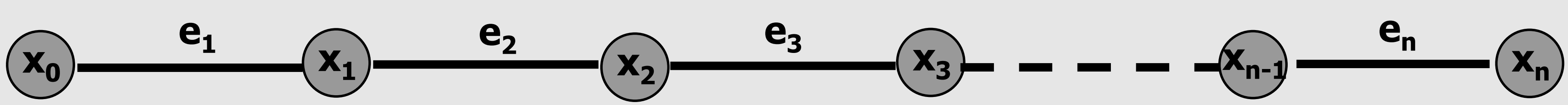
OR

- More formally, let n be a non-negative integer and G an undirected [directed] graph. A path of length n from vertex u to v in G is a sequence of n edges $e_1, e_2, e_3 \cdots, e_n$ of G, for which there exists a sequence $u = x_0, x_1, x_2 \cdots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, 2, 3, \ldots, n$, the endpoints x_{i-1} and x_i. A $\boldsymbol{u - v}$ **path** is written as $u = x_1 e_1 x_2 e_2 x_3 e_3 \cdots x_n e_n = v$.
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, x_2 \cdots, x_{n-1}, x_n$.

Paths in Graphs

Path

- Let n be a nonnegative integer and G a directed graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2) and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$.
- When there are no multiple edges in the directed graph, this path is denoted by vertex sequence $x_0, x_1, x_2, \ldots, x_n$.
  - **Way to represent Graphically**



Paths in Graphs

Simple Path

- A path is simple if it does not include
    - repeated vertices,
    - loops,
    - parallel edges.

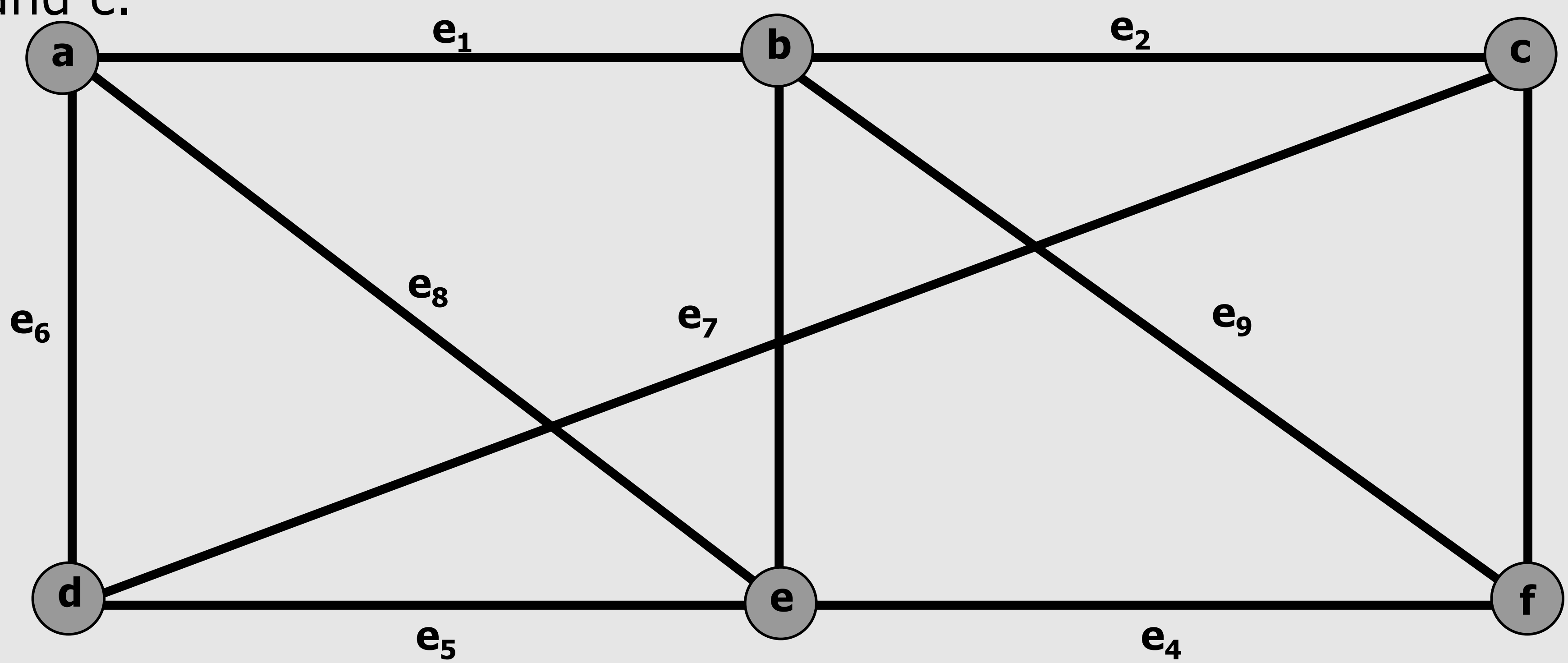
Circuits

- A path of length greater than zero is a **circuit/cycle** if it begins and ends at the same vertex i.e., $u = v$.
- A circuit is a **closed path** that doesn't contain a repeated edge.
- A **simple circuit** is a circuit which does not have a repeated vertex except first and last.

Paths in Graphs

Example

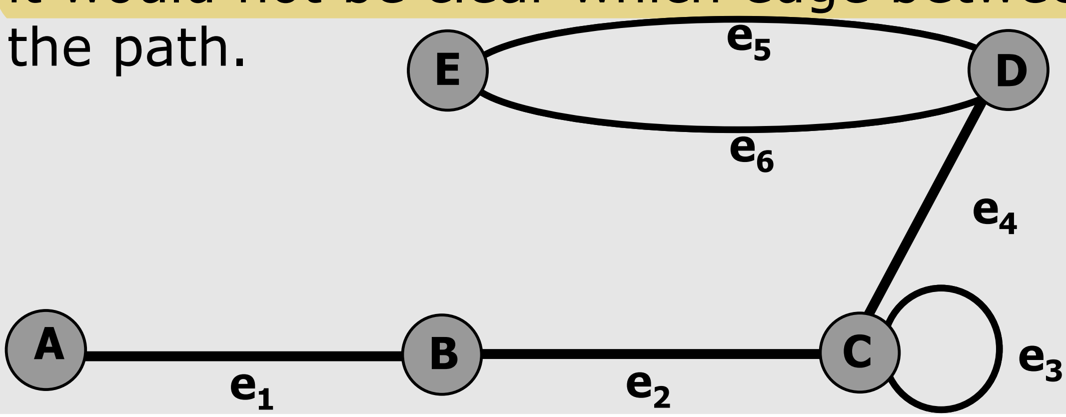
- In the simple graph shown in figure, adcfe is a simple path of length 4 which can also be written as $ae_6de_7ce_3fe_4e$.
- However, deca is not a path, because there is no edge between e and c.



Paths in Graphs

Example

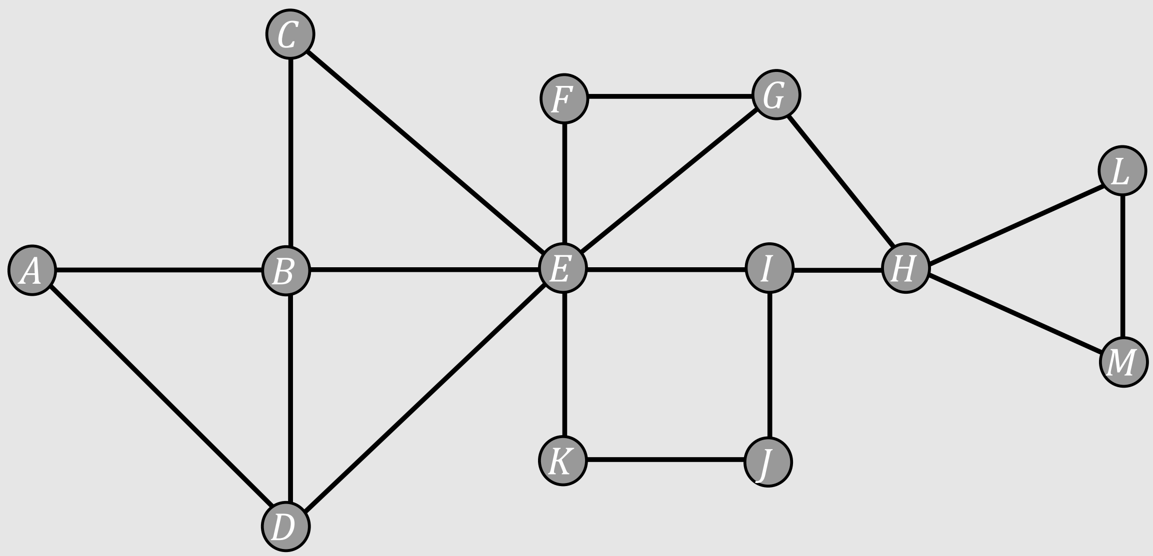
- The below diagram represents a multigraph but not a simple graph because there are two parallel edges e_5 and e_6 connecting the vertices D and E and loop at vertex c.
- In graph Ae_1Be_2C is a path of length 2 from A to C, also be written as e_1e_2. Similarly, $e_1e_2e_3$ is a path of length 3 from A to C, and Ae_1Be_1A is a path of length 2 from A to A.
- The path Ee_6D cannot be described by just listing the vertices ED since it would not be clear which edge between E and D, e_5 or e_6 is part of the path.



Paths in Graphs

Example

- What is the type of graph is?
- Find the simple paths in the graph?
- Find the circuits?
- Find the simple circuits?
- Find a circuit in the graph having maximum length?



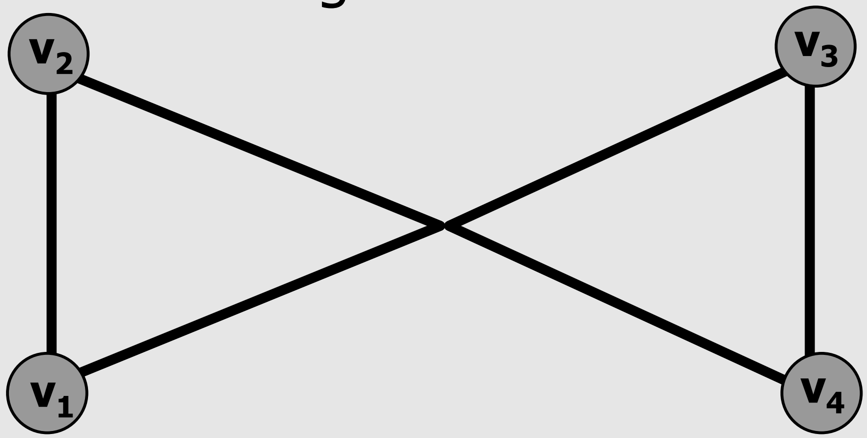
Paths in Graphs

Counting Paths between vertices

- Let G be graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of A^r.

  i. **Example:**

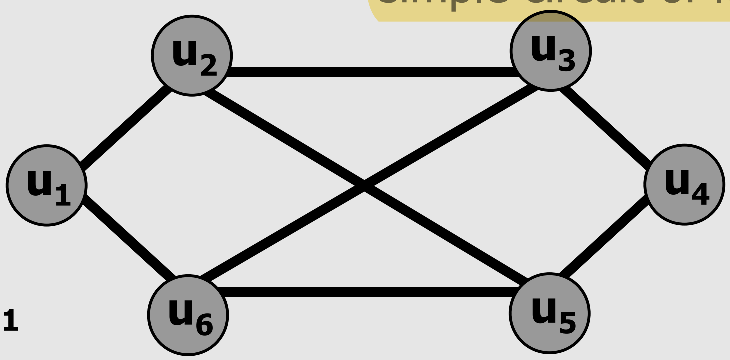
  - How many paths of length four are there from v_1 to v_4?



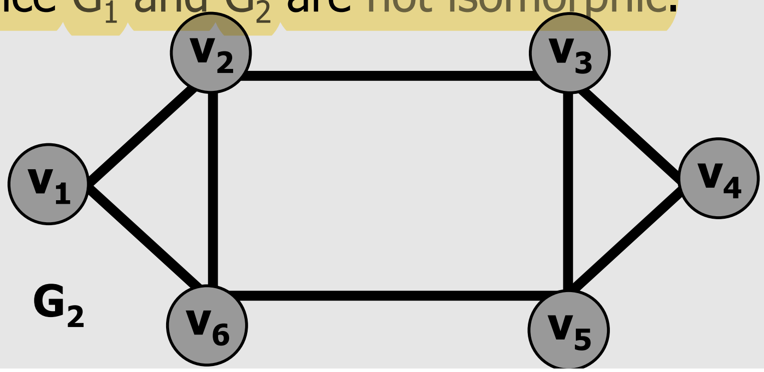
Paths in Graphs

Paths and Isomorphism

- In different ways both paths and circuits help to determine whether two graphs are isomorphic or not.
  - ➢ **Example**
    - Determine whether the graphs G_1 and G_2 are isomorphic?
      - ➢ **Solution:**
        - Both graphs G_1 and G_2 have six vertices and eight edges. Each has four vertices of degree three and two vertices of degree two. So, the necessary conditions, number of edges, no. of vertices and degrees of vertices all are agreed for two graphs.
        - G_2 has a simple circuit of length three, namely $v_1v_2v_6v_1$ whereas G_1 has no simple circuit of length three, hence G_1 and G_2 are not isomorphic.



G_1

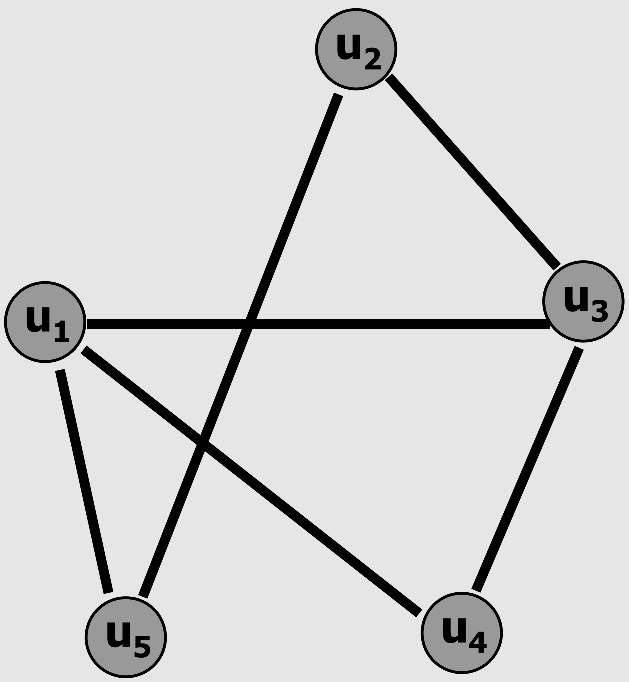


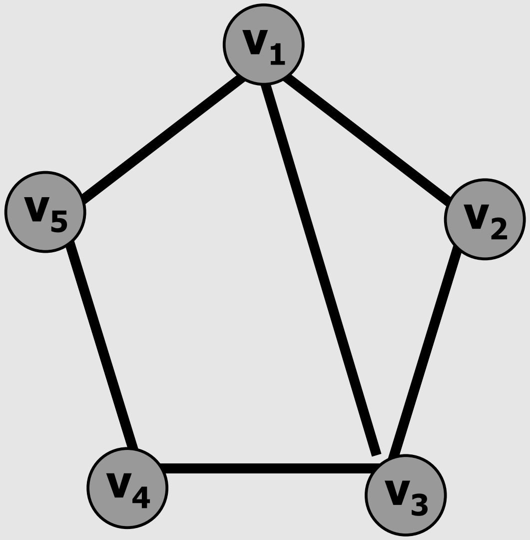
G_2

Paths in Graphs

Example

- Determine whether the graphs are isomorphic?

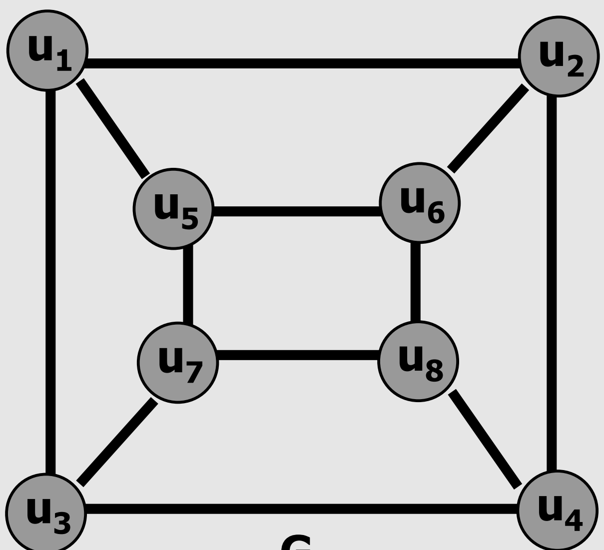




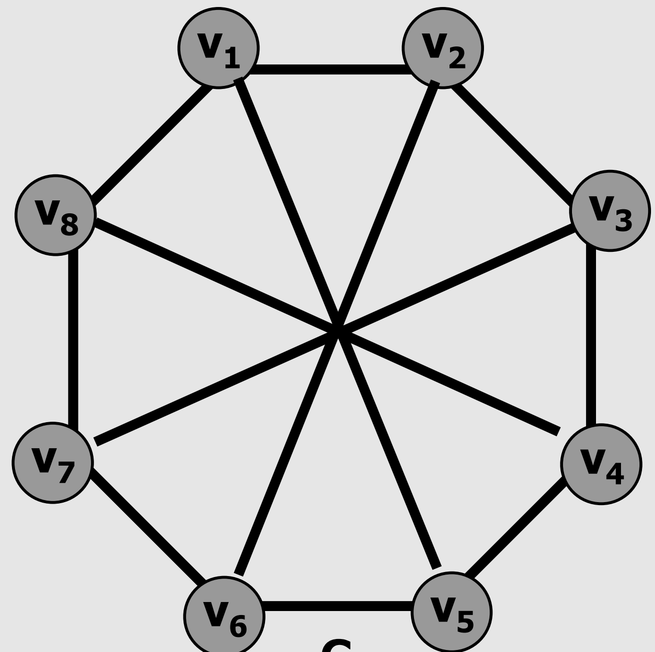
Paths in Graphs

Example

- Determine whether the graphs G_1 and G_2 are isomorphic?



G_1



G_2

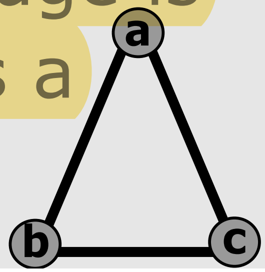
Paths in Graphs

Connectedness in Undirected Graphs

- An undirected graph is connected *if and only if* every pair of its vertices is connected in the graph.

- If undirected graph is connected, then any pair of its vertices is connected by a simple path.

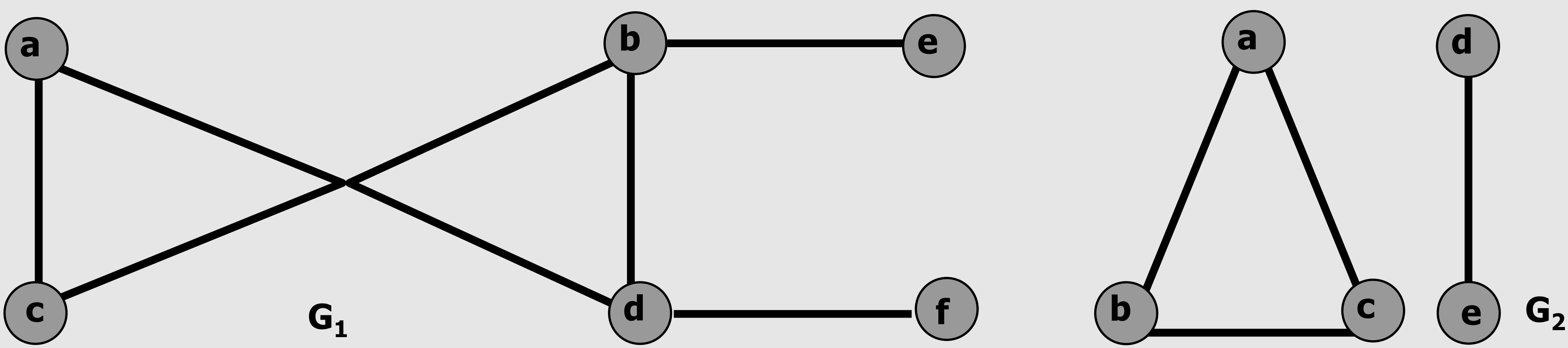
- If two vertices are a part of a circuit and one edge is removed from the circuit, then there still exists a path between these two vertices.



Paths in Graphs

Example

- Two graphs G_1 and G_2 are given, the graph G_1 is connected, because for every pair of distinct vertices there is path between them.
- The graph G_2 is disconnected because there is no path between c and e.



Paths in Graphs

Connected Components

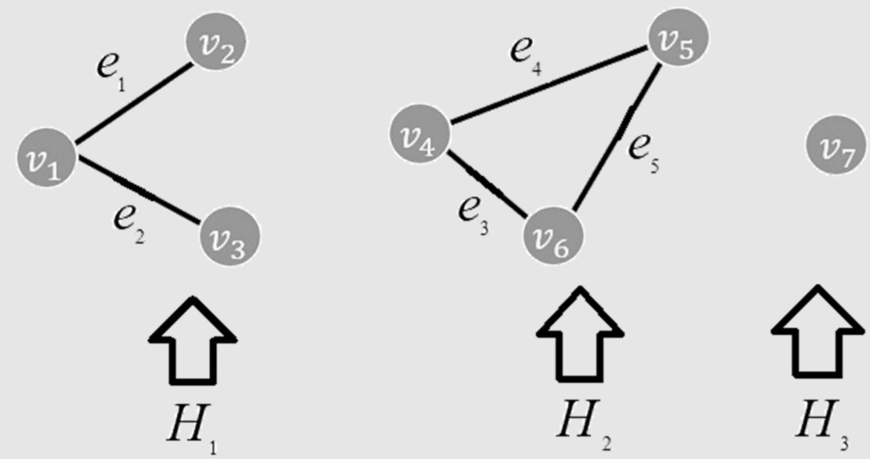
- A Graph H is called a connected component of graph G if, and only if,
  i. when H is a subgraph of G;
  ii. H is connected; and
  iii. H is not be a subgraph of any larger connected subgraph of G and contains the vertices or edges that are not in H.

- An undirected connected graph is a union of connected components.

Paths in Graphs

Example

- Graph G is disconnected graph because we can not walk to vertex v_7.
- The graph G is decomposed into three **connected components** H_1, H_2 and H_3.
- All three graphs H_1, H_2 and H_3 are **connected subgraphs** of the graph G.

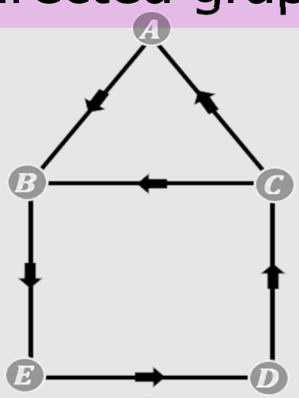


- $V(H_1) = \{v_1, v_2, v_3\}$, $V(H_2) = \{v_4, v_5, v_6\}$, $V(H_3) = \{v_7\}$
- $E(H_1) = \{e_1, e_2\}$, $E(H_2) = \{e_3, e_4, e_5\}$, $E(H_3) = \{\ \}$
- Graph G is the union of three connected subgraphs H_1, H_2 and H_3.

Paths in Graphs

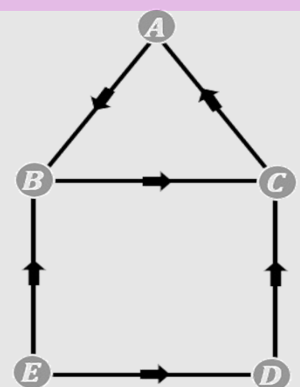
Directed Connectedness

- A **directed graph** G is **strongly connected** if there is a **directed path** for any u to any other vertices v of graph.
- A **directed graph** become **weakly connected** if the underlying undirected graph (*if edge directions are removed*) is connected.



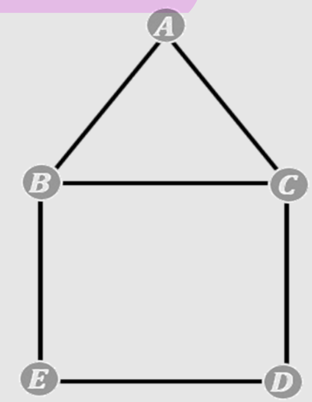
G_1

Strongly Connected



G_2

Weakly Connected



Connected Graph

- Draw a directed graph with six vertices that is weakly connected but not strongly connected.
- Draw an undirected graph with two connected components.

UNITY CHART

Connecting The Parts Of Knowledge With The Wholeness Of Knowledge

Understanding Graphs

1. Graphs contain knowledge about how objects (vertices) are related to one another (edges).

2. A graph isomorphism can show that two graphs that appear to be different are essentially the same.

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3. **Transcendental Consciousness** is a field of pure knowledge and complete harmony.

4. **Impulses within the transcendental field** structure the first relationships and the first differences between objects.

5. **Wholeness moving within itself:** In Unity Consciousness, when we see all objects connected to our Self, we are successful in every activity.

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[2]Dillbeck, M.C. (1982). Meditation and flexibility of visual perception and verbal problem solving. Memory and Cognition 10(3): 207-215; Alexander, C.N., Davies, J.L., Newman, R.I., and Chandler, H.M. (1983). The effects of Transcendental Meditation on cognitive and behavioral flexibility, health, and longevity in the elderly: An experimental comparison of the Transcendental Meditation, program, mindfulness training, and relaxation. Department of Psychology and Social Relations and Graduate School of Education, Harvard University, Cambridge, Massachusetts, USA, and Macquarie University, North Ryde, New South Wales.