Partition of Unity

Definition 1. Let X be a set. A collection $\mathcal{T}$ of subsets of X is a topology of X if it satisfies the followings.

I. $\emptyset, X \in \mathcal{T}$

II. $U_\alpha \in \mathcal{T}\ for\ \alpha \in A \implies \bigcup_\alpha U_\alpha \in \mathcal{T}$

III. $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$

(X, $\mathcal{T}$) is called a 'topological space' and subsets of X contained in $\mathcal{T}$ are 'open' in X.

Definition 2. Let X be a topological space.

(a) A set E is 'closed' in X $:\Leftrightarrow$ $E^C \in \mathcal{T}$

(b) The 'closure' $\bar{E}$ of a set E is the smallest closed set in X which contains E.

(c) A set $K \subseteq X$ is 'compact' $:\Leftrightarrow$ every open covering of K has a finite sub-covering.

(d) A 'neighborhood' of a point $p \in X$ is any open subset of X which contains p.

(e) X is a 'Hausdorff space' $:\Leftrightarrow$ $\forall p, q \in X, p \neq q$ $\exists$neighborhoods U of p, and V of q such that $U \cap V = \emptyset$

(f) X is 'locally compact' $:\Leftrightarrow$ $\forall p \in X$ $\exists$a neighborhood U of p such that $\bar{U}$ is compact.

Theorem 1. Let X be a topological space, K is compact, and F is closed in X.

If $F \subseteq K$, then F is compact.

Proof. Let $\{V_\alpha\}$ be an open covering of F. Then $F \subseteq \bigcup_\alpha V_\alpha$ implies $X = F^C \cup (\bigcup_\alpha V_\alpha)$. Since $\{V_\alpha\} \cup \{F^C\}$ is an open covering of a compact set K, there exists a finite collection $\{V_1, \cdots, V_n\}$ such that

$$K \subseteq F^C \cup (\bigcup_{i=1}^{n} V_i)$$

Hence, we get $F \subseteq \bigcup_{i=1}^{n} V_i$. i.e there is a finite sub-covering $\{V_1, \cdots, V_n\}$ of $\{V_\alpha\}$ for F

Therefore, F is compact. ■

Corollary 1. If $A \subseteq B$ and $\bar{B}$ is compact, then $\bar{A}$ is also compact.

Theorem 2. Let X is a Hausdorff space, $K \subseteq X$ where K is compact, and $p \in K^C$.

Then there exist open sets U and V such that $p \in U$, $K \subseteq V$, and $U \cap V = \emptyset$.

Proof. For each $q \in K$, there are open sets U_q and V_q such that $p \in U_q$, $q \in V_q$ and $U_q \cap V_q = \emptyset$.

Since $K \subseteq \bigcup_{q\in K} V_q$, $\{V_q\}$ is an open covering of compact K. Hence there are points $q_1,\cdots,q_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_{q_i}$$

Set $U = \bigcap_{i=1}^{n} U_{q_i}$ and $V = \bigcup_{i=1}^{n} V_{q_i}$. Then

$$p \in U, K \subseteq V \text{ and } U\cap V = (\bigcap_{i=1}^{n} U_{q_i})\cap(\bigcup_{j=1}^{n} V_{q_j}) = \bigcup_{j=1}^{n}[(\bigcap_{i=1}^{n} U_{q_i})\cap V_{q_j}] = \bigcup_{j=1}^{n}\emptyset = \emptyset$$

■

Corollary 2.

(a) Compact subsets of Hausdorff spaces are closed.

(b) If F is closed and K is compact in a Hausdorff space, then $F\cap K$ is compact.

Proof. (a): For Theorem2, let U_p and V_p be such sets for each $p \in K^{\mathsf{C}}$. Since $U_p\cap V_p = \emptyset \Rightarrow U_p \subseteq V_p^{\mathsf{C}}$ and $K \subseteq V_p \Rightarrow V_p^{\mathsf{C}} \subseteq K^{\mathsf{C}}$ for each p, $K^{\mathsf{C}} \subseteq \bigcup_{p\in K^{\mathsf{C}}} U_p \subseteq K^{\mathsf{C}}$. And $\bigcup_{p\in K^{\mathsf{C}}} U_p$ is open. Thus $K^{\mathsf{C}} = \bigcup_{p\in K^{\mathsf{C}}} U_p$ is open.

(b): It follows from Theorem 1 and (a). ■

Theorem 3. Let $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space such that $\bigcap_\alpha K_\alpha = \emptyset$.

Then there exists a finite sub-collection $\{K_1,\cdots,K_n\}$ such that $\bigcap_{i=1}^{n} K_i = \emptyset$.

Proof. Put $V_\alpha = K_\alpha^{\mathsf{C}}$. By Corollary 2 (a), since K_α is compact, V_α is open for each α. Since $\bigcap_\alpha K_\alpha = \emptyset$, foe each $p \in K_1$, $\exists \alpha_p$ such that $p \notin K_{\alpha_p}$. Thus, $\{V_\alpha\}$ is an open covering of K_1. Since K_1 is compact, there is a finite sub-covering $\{V_{\alpha_1},\cdots,V_{\alpha_k}\}$ of K_1 i.e $K_1 \subseteq \bigcup_{i=1}^{k} V_{\alpha_k}$. This implies that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_k} = \emptyset$$

■

Theorem 4. Suppose U is open in a locally compact Hausdorff space X, $K \subseteq U$ and K is compact.

Then there is an open set V such that

$$\bar{V} \text{ is compact and } K \subseteq V \subseteq \bar{V} \subseteq U$$

Proof. Since X is locally compact, $\exists G_q$: a neighborhood of q where $\overline{G_q}$ is compact for each $q \in K$. Since $K \subseteq \bigcup_{q\in K} G_q$ and K is compact, there are points $q_1,\cdots q_m \in K$ such that $\{G_{q_1},\cdots,G_{q_m}\}$ covers K.

Note that a finite union of sets with compact closure has a compact closure. Let $G = \bigcup_{j=1}^{m} G_{q_m}$. Then G is open and has a compact closure. If $U = X$, take $V = G$.

Suppose $U \neq X$. Theorem 2 shows that to each $p \in U^{\mathsf{C}}$ there corresponds an open set W_p such that $K \subseteq W_p$ and $p \notin \overline{W_p}$. Hence $\{U^{\mathsf{C}} \cap \bar{G} \cap \overline{W_p}\}_{p\in U^{\mathsf{C}}}$ is a collection of compact sets by Corollary 2 (b). Indeed,

$$\bigcap_{p\in U^{\mathsf{C}}} U^{\mathsf{C}} \cap \bar{G} \cap \overline{W_p} = U^{\mathsf{C}} \cap \bar{G} \cap \bigcap_{p\in U^{\mathsf{C}}} \overline{W_p} = \emptyset$$

By Theorem 3, there are points $p_1,\cdots p_n \in U^{\mathsf{C}}$ such that

$$U^{C} \cap \bar{G} \cap \bigcap_{i=1}^{n} \overline{W_{p_i}} = \emptyset$$

Let $V = G \cap \bigcap_{i=1}^{n} W_{p_i}$. Then V is open and contains K, and $\bar{V} = \overline{G \cap \bigcap_{i=1}^{n} W_{p_i}} \subseteq \bar{G} \cap \bigcap_{i=1}^{n} \overline{W_{p_i}} \subseteq U$. Since $V \subseteq G$ and $\bar{G}$ is compact, $\bar{V}$ is compact by Corollary 1. Therefore, V has the required properties. ■

Definition 3. Let X be a topological space and $f: X \to \mathbb{R}$ be a function.

I. f is lower semi-continuous $:\Leftrightarrow \{x \in X | f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$

II. f is upper semi-continuous $:\Leftrightarrow \{x \in X | f(x) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$

III. Let $f: X \to Y$ is a function between two topological spaces X and Y.

f is continuous $:\Leftrightarrow f^{-1}(V)$ is open for every open subset $V \subseteq Y$

IV. χ_E is said to be a 'characteristic function' of each subset E of X defined by

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

Remark.

(a) A real valued function is continuous if and only if it is both upper and lower semi-continuous.

(b) χ_E is lower semi-continuous if and only if E is open.

(c) χ_E is upper semi-continuous if and only if E is closed.

(d) $f \coloneqq sup\,\{f_\alpha\}$ is lower semi-continuous if every f_α is lower semi-continuous.

(e) $g \coloneqq inf\,\{g_\alpha\}$ is upper semi-continuous if every g_α is upper semi-continuous.

Definition 4. Let X be a topological space and $f: X \to \mathbb{R}$ be a function. The support of f is defined by

$$spt(f) \coloneqq \overline{\{x \in X | f(x) \neq 0\}}$$

The collection of all continuous real valued function on X whose support is compact is denoted by $C_c(X)$.

Note: $C_c(X)$ is a vector space over $\mathbb{R}$ since sum of two continuous functions and a scalar multiplication of a continuous function are continuous, $spt(f+g) \subseteq spt(f) \cup spt(g)$ and $spt(\alpha f) = spt(f)$ if $\alpha \neq 0$ and $spt(\alpha f) = \emptyset$ if $\alpha = 0$.

Theorem 5. Let $f: X \to Y$ is a continuous function between two topological spaces X and Y.

If K is a compact subset of X, then $f(K)$ is compact in Y.

Proof. Let $\{V_\alpha\}$ be an open covering of $f(K)$. Then $\{f^{-1}(V_\alpha)\}$ is an open covering of K since f is continuous. Hence $K \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$ for some $\alpha_1, \cdots, \alpha_n$ and therefore

$$f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} V_{\alpha_i}$$

■

Notation.

i. $K \prec f \;:\Leftrightarrow\; K$ is compact subset of X, $f \in C_c(X)$, $0 \le f \le 1$, and $f(x) = 1$ for all $x \in K$.

ii. $f \prec V \;:\Leftrightarrow\; V$ is open in X, $f \in C_c(X)$, $0 \le f \le 1$, and $spt(f) \subseteq V$.

iii. $K \prec f \prec V \;:\Leftrightarrow\; K \prec f$ and $f \prec V$

Urysohn's Lemma. Suppose X is a locally compact Hausdorff space, V is open in X, $K \subseteq V$, and K is compact. Then there exists a function $f \in C_c(X)$ such that

$$K \prec f \prec V$$

Proof. Put $r_1 = 0, r_2 = 1$, and let $r_3, r_4, r_5, \cdots$ be an enumeration of the rationals in $(0,1)$. Applying Theorem 4 twice, we can choose open sets V_0 and V_1 such that they have compact closures and

$$K \subseteq V_1 \subseteq \overline{V_1} \subseteq V_0 \subseteq \overline{V_0} \subseteq V$$

Suppose $n \ge 2$ and $V_{r_1}, \cdots, V_{r_n}$ have been chosen in such a manner that $r_i < r_j$ implies $\overline{V_{r_j}} \subseteq V_{r_i}$. Then one of the numbers $r_1, \cdots, r_n$, say r_i will be the largest one which is smaller than r_{n+1}, and another, say r_j, will be the smallest one larger than r_{n+1}. Using Theorem 4 again, we can find $V_{r_{n+1}}$ so that

$$\overline{V_{r_j}} \subseteq V_{r_{n+1}} \subseteq \overline{V_{r_{n+1}}} \subseteq V_{r_i}$$

By mathematical induction, we obtain a collection $\{V_r\}_{r\in\mathbb{Q}\cap[0,1]}$ of open sets with the following properties: $K \subseteq V_1, \overline{V_0} \subseteq V$, each $\overline{V_r}$ is compact, and

$$s > r \text{ implies } \overline{V_s} \subseteq V_r$$

Define

$$f_r(x) = \begin{cases} r & \text{if } x \in V_r \\ 0 & \text{otherwise} \end{cases}, \qquad g_s(x) = \begin{cases} 1 & \text{if } x \in \overline{V_s} \\ s & \text{otherwise} \end{cases}$$

on X for each $r, s \in \mathbb{Q} \cap [0,1]$, and

$$f(x) = sup_r\{f_r(x)\}, \qquad g(x) = inf_s\{g_s(x)\}$$

on X for each $x \in X$.

Since $f_r = r\chi_{V_r}$ and V_r is open for each r, f_r is lower semi-continuous. Since $\overline{V_s}$ is compact in a Hausdorff space, $\overline{V_s}$ is closed by Corollary 1 and so g_s is upper semi- continuous for each s. Thus f is lower semi-continuous and g is upper semi- continuous by Remark.

Note that $0 \le f \le 1$, that $f(x) = 1$ if $x \in K$, and that $spt(f) \subseteq \overline{V_0} \subseteq V$. It remains to show that $f = g$ and then it implies f is continuous by Remark (a).

Suppose $f_r(x) > g_s(x)$ for some $x \in X$, r and s. Then $x \in V_r$, $x \notin V_s$, and $r > s$. It contradicts to construction of $\{V_r\}_{r\in\mathbb{Q}\cap[0,1]}$. Thus $f_r \le g_s$ for all r and s, so $f \le g$.

Suppose $f(x) < g(x)$ for some x. Then there are rationals r and s such that $f(x) < r < s < g(x)$. Since $sup_r\{f_r(x)\} = f(x) < r$, we have $x \notin V_r$. Since $s < g(x) = inf_s\{g_s(x)\}$, we have $x \in V_s$. It contradicts to construction of $\{V_r\}_{r\in\mathbb{Q}\cap[0,1]}$. Hence $f = g$, so f is continuous.

Therefore, $f \in C_c(X)$ with $spt(f) \subseteq \overline{V_0} \subseteq V$ ($\because$ Since $\overline{V_0}$ is compact and $spt(f)$ is a closure of some set, by corollary 1), $0 \le f \le 1$, $f(x) = 1$ for all $x \in K$, we conclude that $K \prec f \prec V$. ■

Partition of Unity for Locally Compact Hausdorff Space.

Suppose $V_1, \cdots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and

$$K \subseteq \bigcup_{i=1}^{n} V_i$$

Then there exist functions $h_i \prec V_i$ $(i = 1, \cdots, n)$ such that

$$\sum_{i=1}^{n} h_i(x) = 1 \text{ for all } x \in K$$

The collection $\{h_1, \cdots, h_n\}$ is called a 'partition of unity' on K, subordinate to the cover $\{V_1, \cdots, V_n\}$.

Proof. By Theorem 4, for each $x \in K$, taking $\{x\}(\subseteq V_i)$ as the compact set, $\exists$a neighborhood of x, W_x with compact closure $\overline{W_x} \subseteq V_i$ for some i(depending on x). Since $K \subseteq \bigcup_{x \in K} W_x$, there are points $x_1, \cdots, x_m$ such that $K \subseteq \bigcup_{j=1}^{m} W_{x_j}$. If $1 \le i \le n$, let

$$H_i \coloneqq \bigcup_{\overline{W_{x_j}} \subseteq V_i} \overline{W_{x_j}}$$

be the finite union of those $\overline{W_{x_j}}$ which lie in V_i. Then H_i is compact and $K \subseteq H_i \subseteq V_i$ for each $i = 1, \cdots, n$. By Urysohn's Lemma, there are functions g_i such that $H_i \prec g_i \prec V_i$. Define

$$h_1 = g_1$$

$$h_2 = (1 - g_1) g_2$$

$$\cdots\cdots\cdots$$

$$h_n = (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1}) g_n$$

Suppose $h_i(x) \neq 0$ for some $x \in X$. Then $g_i(x) \neq 0$. This means that $spt(h_i) \subseteq spt(g_i) \subseteq V_i$. Since $0 \le g_i \le 1$, $0 \le h_i \le 1$. And $g_i \in C_c(X)$ implies $h_i \in C_c(X)$ ($\because f(x)g(x) = 0 \Leftrightarrow f(x) = 0$ or $g(x) = 0$ and union of closure is closure of union. Thus $spt(fg) = spt(f) \cup spt(g)$). Thus $h_i \prec V_i$ for each $i = 1, \cdots, n$. By mathematical induction, we easily get

$$h_1 + h_2 + \cdots + h_n = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_n)$$

Since $K \subseteq \bigcup_{i=1}^{n} H_i$, $x \in K$ implies that at least one H_i contains x. Since $H_i \prec g_i$, at least one $g_i(x) = 1$ at each point $x \in K$. Therefore, we conclude that

$$\sum_{i=1}^{n} h_i(x) = 1 \text{ for all } x \in K$$

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Reference. Rudin, *Real and Complex Analysis*, 1987