Calculus of Variations and Symmetry Supplements

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Notations

A dot always denotes the Euclidean scalar product in $\mathbb{R}^d$, i.e. if

$$x = \left(x^1, \ldots, x^d\right), y = \left(y^1, \ldots, y^d\right) \in \mathbb{R}^d$$

then

$$x \cdot y = \sum_{i=1}^{d} x^i y^i = x^i y^i$$

and

$$|x|^2 = x \cdot x$$

For a function $u(t)$, we write

$$\dot{u}(t) = \frac{d}{dt} u(t)$$

We use the standard notation

$$C^k(\Omega)$$

for the space of k-times continuously differentiable functions on some open set $\Omega \subset \mathbb{R}^d$, for $k = 0, 1, 2, \ldots, \infty$. For vector valued functions, with values in $\mathbb{R}^d$, we write

$$C^k\left(\Omega, \mathbb{R}^d\right)$$

for the corresponding spaces.

We use the notation

$$C_0^k(\Omega)$$

for C^k functions of Ω that again vanish outside some compact subset $K \subset \Omega$ (where K may depend on the function).

1 The Euler-Lagrange Equations

Lemma 1.1. *Fundamental Lemma of the Calculus of Variations If* $h \in C^0\left((a,b),\mathbb{R}^d\right)$ *satisfies*

$$\int_a^b h(t)\phi(t)dt = 0$$

for all $\phi \in C_0^\infty\left((a,b),\mathbb{R}^d\right)$ *then* $h \equiv 0$ *on* (a,b).

Proof. 그렇지 않다면 적당한 $t_0 \in (a,b)$가 존재하여 다음을 만족한다.

$$h(t_0) \neq 0$$

그러므로 어떤 첨자 $i_0 \in \{1,\ldots,d\}$에 대하여 $h^{i_0} \neq 0$이다. h가 연속이므로 적당한 $\delta > 0$이 존재하여

$$a < t_0 - \delta < t_0 + \delta < b$$

와 다음을 만족한다.

$$|t_0 - t| \implies \left|h^{i_0}(t)\right| > \frac{1}{2}\left|h^{i_0}(t_0)\right|$$

그렇다면 다음을 만족하는 $\phi \in C_0^\infty\left((a,b),\mathbb{R}^d\right)$를 고를 수 있다.

$$\phi(t) = 0 \text{ if } |t_0 - t| \geq \delta$$

$$\phi^{i_0}(t) > 0 \text{ if } |t_0 - t| < \delta$$

$$\phi^{i_0}(t) = 0 \text{ for } i \neq i_0, i \in \{1,\ldots,d\}$$

그런데 이러한 ϕ는 다음을 만족하는데 이는 가정에 모순이다.

$$\int_a^b h(t)\phi(t)dt = \int_{t_0-\delta}^{t_0+\delta} h(t)\phi(t)dt \neq 0$$

따라서 모든 $t_0 \in (a,b)$에 대해서 $h(t_0) = 0$이어야 한다. □

Theorem 1.1. *Let* $F \in C^2([a,b]\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R})$*, and let* $u \in C^2([a,b],\mathbb{R}^d)$ *be a minimizer of*

$$I(u) = \int_a^b F(t,u(t),\dot{u}(t)dt$$

among all functions with prescribed boundary values $u(a)$ *and* $u(b)$*. Then* u *is a solution of the following system of second order ordinary differential equations, the Euler-Lagrange equations*

$$\frac{d}{dt}(F_p(t,u(t),\dot{u}(t))) - F_u(t,u(t),\dot{u}(t)) = 0$$

Proof. 발표자료를 참고하라. □

1.1 Some Exercises and Remarks

Remark 1.1.1. Let the functional under consideration have the form

$$\int_a^b F(t, \dot{u}(t))dt$$

where F does not contain $u(t)$ explicitly. In this case, Euler's equation becomes

$$\frac{d}{dt}F_{\dot{u}} = 0$$

which obviously has the first integral[1]

$$F_{\dot{u}} = C$$

where C is a constant.

Exercise 1.1.1. If the integrand does not depend on t, i.e., if

$$\int_a^b F(u(t), \dot{u}(t))dt$$

then Euler's equation has the first integral

$$F - \dot{u}F_{\dot{u}} = C$$

where C is a constant.

Hint: $F_{\dot{u}}\ddot{u}$를 더하고 빼보라!

Exercise 1.1.2. In a variety of problems, one encounters functionals of the form

$$\int_a^b f(x, y)\sqrt{1 + y'^2}dx$$

representing the integral of a function $f(x, y)$ with respect to the arc length[2]. In this case, Euler's equation can be transformed into

$$f_y - f_x y' - f\frac{y''}{1 + y'^2} = 0$$

Exercise 1.1.3. 발표 자료에 있는 최단 거리 곡선 문제와 최단 강하 곡선 문제를 풀어보라.

Hint: 첫번째는 exercise 1.1.2를 두번째는 exercise 1.1.1를 이용하는 것이 쉽다.

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[1] First integral이란 매개변수 t에 대한 불변량을 말한다.

[2] $ds = \sqrt{1 + y'^2}dx$

2 Symmetries and the Theorem of E. Noether

Theorem 2.1. *We consider the variational intergral*

$$I(u) = \int_a^b F(t, u(t), \dot{u}(t))dt$$

with $F \in C^2([a,b] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$*. We suppose that there exists a smooth one-parameter family of differentiable maps*

$$h_s : \mathbb{R}^d \to \mathbb{R}^d$$

(the precise smoothness requirement is that

$$h(s,z) := h_s(z)$$

is of class $C^2((-\epsilon_0, \epsilon_0) \times \mathbb{R}^d, \mathbb{R})$ *for some* $\epsilon_0 > 0$*), with*

$$h_0(z) = z \text{ for all } z \in \mathbb{R}^d$$

and satisfying

$$\int_a^b F(t, h_s(u(t)), \frac{d}{dt}h_s(u(t)))dt = \int_a^b F(t, u(t), \frac{d}{dt}u(t))dt \tag{1}$$

for all $s \in (-\epsilon_0, \epsilon_0)$ *and all* $u \in C^2([a,b], \mathbb{R}^d)$*. Then, for any solution* $u(t)$ *of the Euler-Lagrange equation for* I*,*

$$F_p(t, u(t), \dot{u}(t))\frac{d}{ds}h_s(u(t))|_{s=0} \tag{2}$$

is constant in $t \in [a,b]$*.*

Proof. Equation (1) yields for any $t_0 \in [a,b]$, using $h_0(z) = z$,

$$0 = \frac{d}{ds}\int_a^{t_0} F\left(t, h_s(u(t)), \frac{d}{dt}h_s(u(t))\right) dt|_{s=0} \tag{3}$$

$$= \int_a^{t_0} \left\{F_u(t, u(t), \dot{u}(t))\frac{d}{ds}h_s(u(t)) + F_p(t, u(t), \dot{u}(t))\frac{d}{dt}\frac{d}{ds}h_s(u(t))\right\} dt|_{s=0}$$

We recall the Euler-Lagrange equations for u:

$$0 = \frac{d}{dt}F_p(t, u(t), \dot{u}(t)) - F_u(t, u(t), \dot{u}(t)) \tag{4}$$

Using (4) in (3) to replace F_u, we obtain

$$0 = \int_a^{t_0} \left\{F_u(t, u(t), \dot{u}(t))\frac{d}{ds}h_s(u(t)) + F_p(t, u(t), \dot{u}(t))\frac{d}{dt}\frac{d}{ds}h_s(u(t))\right\} dt|_{s=0}$$

$$= \int_a^{t_0} \frac{d}{dt}\left(F_p(t,u(t),\dot u(t))\frac{d}{ds}h_s(u(t))|_{s=0}\right) dt \tag{5}$$

Therefore

$$F_p(t_0,u(t_0),\dot u(t_0))\frac{d}{ds}h_s(u(t_0))|_{s=0} = F_p(a,u(a),\dot u(a))\frac{d}{ds}h_s(u(a))|_{s=0} \tag{6}$$

for any $t_0 \in [a,b]$. This means that (2) is constant on [a, b]. $\square$

Theorem 2.2. *Theorem of E. Noether We consider the variational integral*

$$I(u) = \int_a^b F(t,u(t),\dot u(t))dt$$

with $F \in C^2([a,b]\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R})$*. We suppose that there exists a smooth one-parameter family of differentiable maps*

$$\bar h_s = (h_s^0, h_s) : [a,b]\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d$$

($s \in (-\epsilon_0,\epsilon_0)$ *as before) with*

$$\bar h_0(t,z) = (t,z) \textit{ for all } (t,z)\in[a,b]\times\mathbb{R}^d$$

and satisfying

$$\int_{h_s^0(a)}^{h_s^0(b)} F(t_s, h_s(u(t_s)), \frac{d}{dt_s}h_s(u(t_s)))dt_s = \int_a^b F(t,u(t),\frac{d}{dt}u(t))dt \tag{7}$$

for all $s\in(-\epsilon_0,\epsilon_0)$ *and all* $u \in C^2([a,b],\mathbb{R}^d)$*. Then, for any solution* $u(t)$ *of the Euler-Lagrange equations for* I*,*

$$F_p(t,u(t),\dot u(t))\frac{d}{ds}h_s(u(t))|_{s=0}+(F(t,u(t),\dot u(t))-F_p(t,u(t),\dot u(t))\dot u(t))\frac{d}{ds}h_s^0(t)|_{s=0} \tag{8}$$

is constant in $t\in[a,b]$*.*

Proof. We consider the integrand

$$\bar F\left(t(\tau),u(t(\tau)),\frac{dt}{d\tau},\frac{d}{d\tau}u(t(\tau))\right)$$

$$:= F\left(t,u(t),\frac{\frac{d}{d\tau}u(t(\tau))}{\frac{dt}{d\tau}}\right)\frac{dt}{d\tau} \tag{9}$$

$$F\left(t,u(t),\dot u(t)\right)\frac{dt}{d\tau}$$

Then

$$\bar I(t,u) := \int_{\tau_a}^{\tau_b} \bar F\left(t(\tau),u(t(\tau)),\frac{dt}{d\tau},\frac{d}{d\tau}u(t(\tau))\right)d\tau$$

$$= \int_a^b F(t, u(t), \dot{u}(t))dt, \quad \text{if } t(\tau_a) = a, t(\tau_b) = b \tag{10}$$

$$= I(u)$$

By our assumption, $\bar{F}$ remains invariant under replacing (t, u) by $\bar{h}_s(t, u)$. Consequently, 2.1 applied to $\bar{I}$ yields that

$$\bar{F}_p(t, u(t), \dot{u}(t))\frac{d}{ds}h_s(u(t))|_{s=0} + \bar{F}_{p^0}(t, u(t), \dot{u}(t))\frac{d}{ds}h_s^0(t)|_{s=0}$$

with p^0 standing for the place of the argument $\frac{dt}{d\tau}$ of $\bar{F}$ (while p stands as before for the arguments $\dot{u}$), is invariant. Since, by 9,

$$\bar{F}_p = F_p$$

$$\bar{F}_{p^0} = F - F_p\dot{u}$$

at $s = 0$ (note $\frac{dt}{d\tau} = 1$ for $s = 0$ since $h_0^0(t) = t$), this implies the invariance of 8. □

2.1 Some Exercises and Remarks

Exercise 2.1.1 (Conservation of Angular Momentum). We know that physics never change under space rotation. Consider the below Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

Prove that the z-component of angular momentum of the particle is conserved under rotation aboout z-axis [3].

Hint: 각운동량은 $M_z = xp_y - yp_x = m(x\dot{y} - y\dot{x})$로 주어진다.

[3] $h_s(s, \vec{r}) = (x\cos(s) - y\sin(s), x\sin(s) + y\cos(s), z)$