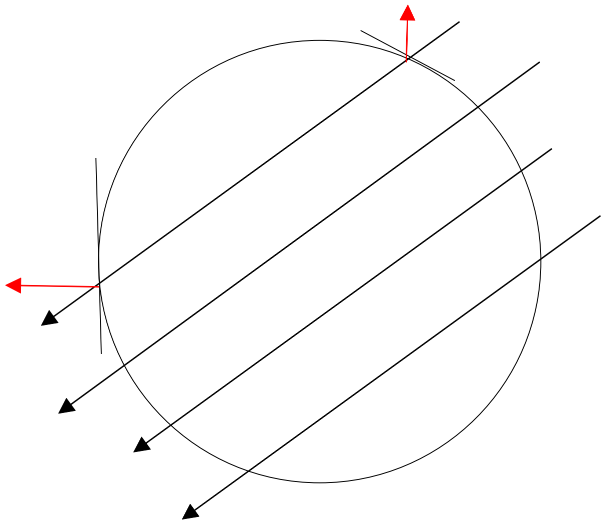
Heat Equation

1. physical interpretation



Let Ω is region in R^3 where physical quantity flow and let $u(x,t)$ is concentration at position x at time t .

Then it satisfy the 'balance principle' such that $\frac{d}{dt}\int_B u(x,t)dV = -\int_{\partial B} u(x,t)v \cdot ndS$ where v is velocity of physical quantity and $B \subseteq \Omega$.

In heat case,
$u(x,t)$ is temperature at position x at time t and uv is net flux through ∂B.
Thus the balance principle forms $\frac{d}{dt}\int_B c\rho u(x,t)dV = -\int_{\partial B}(-k)\nabla u(x,t) \cdot ndS$ where c,ρ,k is some physical constant.
By divergence thm, we can obtain $c\rho u_t(x,t) - k\triangle u(x,t) = 0$ and let $c=\rho=k=1$, then there is heat equation $u_t - \triangle u = 0$.

2. Fundamental Solution

$u_t - \triangle u = 0, t > 0$ $\cdots$ (2.1)
we observe that this equation involves one derivative w.r.t t, bet two derivative w.r.t x. we see that if $u(x,t)$ solves (2.1), then $u(\lambda x, \lambda^2 t)$ is also solve. and we let $u(\lambda x, \lambda^2 t) = u_\lambda(x,t)$. This process is called scale. Now we will reduce the time variant using scale invariance.

Which scale factor is invariant under the scaling $(x,t) \rightarrow (\lambda x, \lambda^2 t)$?

Let $u(x,t) = t^{\beta} v(t^{\alpha} x)$ for some function v on R^n as not yet determined. Then we can get α using scaling and get β using energy conservation.

So $u(x,t) = t^{-\frac{n}{2}} v(t^{-\frac{1}{2}} x)$ ⋯ (2.2)

Let us now insert (2.2) into (2.1) using the chain rule. Then the result is

$\triangle v(y) + \frac{1}{2} y \cdot \nabla v(y) + \frac{n}{2} v(y) = 0$ where $y = t^{-\frac{1}{2}} x$ ⋯ (2.3)

and we convert this equation into an ODE.

Since v is radial, we can write $v(y) = w(|y|)$ for some function w on R.

Then we have an ODE $w''(r) + \frac{n-1}{r} w'(r) + \frac{1}{2} w'(r) + \frac{n}{2} w(r) = 0$ where $r = |y|$ ⋯ (2.4)

we solve the (2.4), the $w(r) = Ce^{-\frac{r^2}{4}}$ where C is some constant. and we convert w to u. Then $u(x,t) = Ct^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. Determine C with normalization, finally, we can get special solution of heat eqn $u(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ ⋯ (2.5)

The special solution is called the heat kernel (fundamental solution) to the heat equation and will be denoted by $K(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$, $t > 0$ ⋯ (2.6).

Indeed the heat kernel is radial.

3. Cauchy Problem (Initial Value Problem)

Consider the PDE: $\begin{cases} u_t - \triangle u = 0 \\ u(x,0) = f(x) \end{cases}$ where $f(x) \in R^n$

Ley us first observe that the function $K(x,t)$ solves the heat equation and so does $K(x-y,t)$ for each fixed $y \in R^n$.

By superposition principle, we may expect that

$u(x,t) = \int_{R^n} K(x-y,t) f(y) dy = (4\pi t)^{-\frac{n}{2}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy$ is solution of IVP.

(Theorem)

Let f be continuous and bounded. Then $u(x,t) = \int_{R^n} K(x-y,t) f(y)$ is the solution of IVP and $u \in C^{\infty}(R^n \times (0, \infty))$.

We obtain a solution of heat equation using heat kernel. Now we consider the some interesting remarks that is related to heat equation.

(Remark 1. Smoothness)

since $u \in C^{\infty}(R^n \times (0,\infty))$, the solution of heat equation is smooth

(Remark 2. Dissipation)

Consider the integrable $f \geq 0$. Then by monotone convergence thm,

$$\lim_{t \to \infty} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy = \int_{R^n} \lim_{t \to \infty} e^{-\frac{|x-y|^2}{4t}} f(y)dy = \int_{R^n} f(y)dy < +\infty$$

and $(4\pi t)^{-\frac{n}{2}} \to 0$ as $t \to \infty$.

Thus $\lim_{t \to \infty} u(x,t) = 0$. It means the heat will be dissipation.

(Remark 3. Infinite propagation speed)

Consider the $f \geq 0$ with continuity and boundedness.

Then $u(x,t)$ has positive value at any position $x \in R^n$ and time $t > 0$.

So the heat has infinite propagation speed.