Notation

WLOG: Without Loss Of Generality

$\mathbb{N}$: The set of natural numbers.

$\mathbb{Z}$: The set of integers.

$\mathbb{R}$: The set of real numbers.

$\lfloor x \rfloor$: The largest integer not bigger than x.

$frac(x) = x - \lfloor x \rfloor$

s.t.: such that

Preliminary

• Formula and property of Trigonometric Function

1) $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$,

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

2) $\cos(x+y) = \cos x\cos y - \sin x\sin y$, $\sin(x+y) = \sin x\cos y + \cos x\sin y$,

$\cos x\cos y = \dfrac{\cos(x-y)+\cos(x+y)}{2}$, $\sin x\sin y = \dfrac{\cos(x-y)-\cos(x+y)}{2}$,

$\sin x\cos y = \dfrac{\sin(x+y)+\sin(x-y)}{2}$,

$\cos x + \cos y = \dfrac{\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2}$, $\sin x + \sin y = \dfrac{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2}$

3) $\forall x \geq 0$, $\sin x \leq x$.

4) $\forall x \in \mathbb{R}$, $\sin(-x) = -\sin x$, $|\sin x| \leq |x|$.

5) $\forall x \in \mathbb{R}$, $\sin(k\pi + x) = \begin{cases} \sin x & (k\ is\ even) \\ -\sin x & (k\ is\ odd) \end{cases}$ for $k \in \mathbb{Z}$.

6) $y = \sin x$ and $y = \cos x$ are continuous on real numbers.

7) $\forall x \in \left[0, \frac{\pi}{2}\right]$, $\sin x \geq \frac{2}{\pi}x$.

• π is irrational.

• Def1) Let non-empty set $X \subseteq Y \subseteq \mathbb{R}$.

X is dense in Y if and only if $\forall y \in Y$ $\forall \varepsilon > 0$, $\exists x \in X$ s.t. $x \in B_\varepsilon(x)$.

• prop1) Let non-empty set $X \subseteq Y \subseteq \mathbb{R}$.

If X is dense in Y, then $\forall y \in Y$, $\exists$ sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ s.t. $\lim_{n\to\infty} x_n = y$.

Thm1) $\{\sin n\}_{n\in\mathbb{N}}$ is divergent.

proof) Suppose $\{\sin n\}_{n\in\mathbb{N}}$ is convergent and let $\alpha=\lim_{n\to\infty}\sin n$.

Note that $\lim_{n\to\infty}\sin 2n=\lim_{n\to\infty}\sin 3n=\alpha$.

$\lim_{n\to\infty}\sin^2 2n=\lim_{n\to\infty}4\sin^2 n(1-\sin^2 n)\ \Rightarrow\ \alpha^2=4\alpha^2(1-\alpha^2)\ \Rightarrow\ \alpha=0$ or $\pm\dfrac{\sqrt{3}}{2}$.

$\lim_{n\to\infty}\sin 3n=\lim_{n\to\infty}\left(3\sin n-4\sin^3 n\right)\ \Rightarrow\ \alpha=3a-4\alpha^3\ \Rightarrow\ \alpha=0$ or $\pm\dfrac{1}{\sqrt{2}}$.

Thus from above, we get $\alpha=0$ and $\lim_{n\to\infty}\cos n=\pm 1$.

But $\alpha=\lim_{n\to\infty}\sin(n+1)=\lim_{n\to\infty}(\sin n\cos 1+\cos n\sin 1)=\pm\sin 1\neq 0$.

This is contradiction. Thus $\{\sin n\}_{n\in\mathbb{N}}$ is divergent. ■

Cor1) $\sum_{n=1}^{\infty}\sin n$ is divergent.

※ $\sum_{k=1}^{n}\sin k$ is bounded. Since $\sin k\sin 1=\dfrac{1}{2}\left\{\cos\left(k-\dfrac{1}{2}\right)-\cos\left(k+\dfrac{1}{2}\right)\right\}$,

$$\sum_{k=1}^{n}\sin k=\frac{1}{\sin 1}\sum_{k=1}^{n}\sin k\sin 1=\frac{1}{2\sin 1}\sum_{k=1}^{n}\left\{\cos\left(k-\frac{1}{2}\right)-\cos\left(k+\frac{1}{2}\right)\right\}$$
$$=\frac{\cos\frac{1}{2}-\cos\left(n+\frac{1}{2}\right)}{2\sin 1}.$$

Since cosine function is bounded, we get the fact that $\sum_{k=1}^{n}\sin k$ is bounded.

※ Using Dirichlet Test, we get the fact that $\sum_{n=1}^{\infty}\dfrac{\sin n}{n}$ is bounded.

※ Using contour integral for complex analysis,

we can calculate the value $\sum_{n=1}^{\infty}\dfrac{\sin n}{n}=\dfrac{\pi-1}{2}$.

※ In the same way as above, we can also know about $\sum_{n=1}^{\infty}\cos n$, $\sum_{n=1}^{\infty}\dfrac{\cos n}{n}$.

Thm2) (**Pigeon-Hall Principle**)

: If n pigeons are put into m halls, with $n > m$, then at least one hall must contain more than one pigeon.

proof) Trivial..

ex1) (Putnam 1958)

: Let X be the set $\{1, 2, 3, \dots, 2n\}$, take $Y \subseteq X$ with $|Y| = n+1$.
Show that we can find $a, b \in Y$ with a dividing b.

sol) Let all elements in X be the form of $2^{a_i} b_i$ with $a_i \in \mathbb{N} \cup \{0\}$ and b_i is odd.

Since X has n odd integers, possible different number $b_i{}'s$ are at most n.

So by Pigeon-Hall Principle, Y has two elements $2^{a_i} b_i, 2^{a_j} b_j$ satisfying $b_i = b_j$.

WLOG, $a_i > a_j$. Then $\dfrac{2^{a_i} b_i}{2^{a_j} b_j} = 2^{a_i - a_j} \in \mathbb{N}$. Thus the proof is completed. ■

ex2)

: $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|\sin n| < \varepsilon$.

sol) By Archimedean Principle, we can choose $m \in \mathbb{N}$ such that $\dfrac{1}{m} < \varepsilon$.

Let a_n be the fractional part of $n\pi$, that is, $a_n = n\pi - \lfloor n\pi \rfloor$.

By irrationality of π, $n \neq m \Rightarrow a_n \neq a_m$. ($\because$ $a_n - a_m = (n-m)\pi + $some integer)

Partition the interval $[0, 1)$ $m's$ piece such that $[0, \dfrac{1}{m}), [\dfrac{1}{m}, \dfrac{2}{m}), \dots, [\dfrac{m-1}{m}, 1)$.

Then by Pigeon-Hall Principle, there exist two a_i, a_j with $1 \leq j < i \leq m+1$ such that they are contained above some an interval.

So $0 < a_i - a_j \leq \dfrac{1}{m}$ and we get $\sin(a_i - a_j) \leq \dfrac{1}{m}$. Let $K = \lfloor i\pi \rfloor - \lfloor j\pi \rfloor \in \mathbb{N}$.

Then $|\sin K| = |\sin\{(i-j)\pi - (a_i - a_j)\}| = |\sin(a_i - a_j)| \leq \dfrac{1}{m}$.

Therefore choosing $n = K$ completes the proof. ■

Cor1) There exists an subsequence of $\{\sin n\}_{n \in \mathbb{N}}$ that converse 0.

proof) It is directive by 'Preliminary-prop1' (by ex2), $\{\sin n\}_{n \in \mathbb{N}}$ is dense in $\{0\}$). ■

Lemma)

: Let $\alpha > 0$ be an irrational number. Then $\{a_n := frac(\alpha n)\}_{n\in\mathbb{N}}$ is dense in $[0,1]$.

proof) Obviously $0 < a_n$. We claim that a_n can be enough to small.

Since a_1 is irrational lower than 1, $\exists\, n_1 \in \mathbb{N}$ s.t. $n_1 a_1 < 1 < (n_1+1)a_1$.

Note that at this time, $a_{(n_1+1)} = (n_1+1)a_1 - 1$ with $a_1 > a_{(n_1+1)}$. Similarly,

$\exists\, n_2 \in \mathbb{N}$ s.t. $n_2 a_{(n_1+1)} < 1 < (n_2+1)a_{(n_1+1)}$ and $a_{(n_2+1)(n_1+1)} = (n_2+1)a_{(n_1+1)} - 1$.

Inductively, for all $m \in \mathbb{N}$ we get $a_{k_m} = \dfrac{k_m}{k_{(m-1)}} a_{k_{m-1}} - 1$

where $k_m = (n_m+1)\cdots(n_1+1)$ and $\dfrac{k_{m+1}}{k_m} a_{k_m} > 1 > (\dfrac{k_{m+1}}{k_m} - 1)a_{k_m}$.

Note that $\forall\, i = 1,2,\ldots,$ $\dfrac{1}{(n_{i+1})+1} < a_{k_i} < \dfrac{1}{n_{i+1}}$ and $\dfrac{1}{k_1 - 1} > a_1 > \dfrac{1}{k_1}$ $\cdots (a)$.

Suppose that $\forall\, i \in \mathbb{N}$, All n_i's are same, that is, $n_1 = n_2 = \cdots = n_m$ $\cdots (b)$.

Substituting $\dfrac{a_{k_i}}{k_i} = b_i$ for $i = 1,\ldots,m$, we get $b_1 = a_1 - \dfrac{1}{k_1}$,

$$b_2 = b_1 - \frac{1}{k_2},$$

$$\vdots$$

$$b_m = b_{m-1} - \frac{1}{k_m}.$$

Then adding left and right side, we get $b_m = a_1 - \displaystyle\sum_{i=1}^{m} \frac{1}{k_i} = a_1 - \sum_{i=1}^{m} \frac{1}{(k_1)^i}$. $(\because (b))$

Since last term is geometric series, $b_m = a_1 - \dfrac{1}{k_1 - 1}\left(1 - \left(\dfrac{1}{k_1}\right)^m\right)$ for all $m \in \mathbb{N}$.

But by (a) this is contradiction that $b_m \geq 0$. (considering $m \to \infty$)

Thus (b) isn't hold and we can choose $n_1 < n_{m_1} < \cdots$ $(1 < m_1 < \cdots)$.

With (a) this implies that claim is true. So fix any $x \in (0,1)$ and $\varepsilon > 0$.

By claim, $\exists\, m,n \in \mathbb{N}$ s.t. $0 < a_m < \varepsilon$ and $a_m + x < 1$. Then $\exists\, k \in \mathbb{N}$ s.t.

$(k-1)a_m < x < k a_m < 1 \Rightarrow 0 < k a_m - x < a_m < \varepsilon$. This completes the proof. ■

Thm3) (**Density of Sin**n) The sequence $\{\sin n\}_{n\in\mathbb{N}}$ is dense in $[-1,1]$.

proof) Fix any $\alpha\in[-1,1]$. By surjectivity of sine function, $\exists\,\beta\geq 0$ s.t. $\alpha=\sin\beta$.
Let $\varepsilon>0$ and $b=\beta-\lfloor\beta\rfloor$. By 'Lemma)' $\exists\, a_{n_k}$ of $\{a_n=frac(2n\pi)\}_{n\in\mathbb{N}}$ $(1\leq n_1<n_2<\cdots)$ s.t. $\lim\limits_{k\to\infty}a_{n_k}=b$. Thus $\exists\, k_0\in\mathbb{N}$ such that $\left|b-a_{n_{k_0}}\right|<\varepsilon$.

$$\begin{aligned}\text{So } \left|\sin\beta-\sin(\lfloor 2k_0\pi\rfloor-\lfloor\beta\rfloor)\right| &= \left|2\cos\frac{\lfloor 2k_0\pi\rfloor}{2}\sin\left(\frac{b-\lfloor 2k_0\pi\rfloor}{2}\right)\right| \\ &= \left|2\cos\frac{\lfloor 2k_0\pi\rfloor}{2}\sin\left(\frac{b-a_{n_{k_0}}}{2}\right)\right| \\ &\leq 2\cdot\left|\frac{b-a_{k_0}}{2}\right|<\varepsilon.\end{aligned}$$

Thus choosing $m=\lfloor 2k_0\pi\rfloor-\lfloor\beta\rfloor$ implies $|\sin\beta-\sin m|<\varepsilon$.
At this time, WLOG we can set $m>0$ and this complete the proof. ■

Thm4) (**Diophantine's Approximation Theorem**)
: Given any irrational number α, there exist infinitely many integers p,q such that $\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^2}\ (\Leftrightarrow|q\alpha-p|<\frac{1}{q})\ \cdots(a)$.
(For convenience sake, we will call this 'DAT')

proof) Substituting $p=\lfloor\alpha\rfloor, q=1$, check there exists at least one pair p,q.
Suppose there exists finitely many integers satisfying above.
Denote those $(p_1,q_1),\ldots,(p_n,q_n)$. Let $M:=\min\{q_k\alpha-p_k\}_{1\leq k\leq n}$, $K:=\left\lfloor\frac{1}{M}\right\rfloor+1$.
Partition the interval $[0,1)$ $K's$ piece s.t. $[0,\frac{1}{K}),\ldots,[\frac{K-1}{K},1)$.
Let $a_n=frac(n\alpha)$. Then By Pigeon-Hall Principal, $\exists\, i,j\,(1\leq j<i\leq K+1)$ such that $a_i-a_j<\frac{1}{K}$. Since $a_i-a_j=(i-j)\alpha-(\lfloor i\alpha\rfloor-\lfloor i\beta\rfloor)$,
choosing $p_0=\lfloor i\alpha\rfloor-\lfloor i\beta\rfloor$, $q_0=i-j$ implies that $|q_0\alpha-p_0|<\frac{1}{K}\leq\frac{1}{q_0}$.
Note that (p_0,q_0) satisfies (a), but since $|q_0\alpha-p_0|<M$, $\forall\, k(1\leq k\leq n)$ $(p_0,q_0)\neq(p_k,q_k)$. This is contradiction. And this complete the proof. ■

Thm5) $\left\{\frac{1}{n\sin n}\right\}_{n\in\mathbb{N}}$ is divergent.

proof) Suppose the given sequence is convergent. Then by Thm3), $\exists\{n_k\}_{k\in\mathbb{N}}\subseteq\mathbb{N}$ s.t. $1\le n_1<n_2<\cdots$ and $\lim_{k\to\infty}|\sin n_k|=1$. So $\lim_{n\to\infty}\frac{1}{n\sin n}=0$. $\cdots(a)$

By Diophantine's Approximation Thm, there exists infinitely many integers such that $|q\pi-p|<\frac{1}{q}$ and WLOG $p,q>0$. Those rearrange same below $|q_k\pi-p_k|<\frac{1}{q_k}$ with $1\le q_1\le q_2\le\cdots$, $1\le p_1\le p_2\le\cdots$.

Then $|q_k\pi-p_k|<\frac{1}{q_k} \Rightarrow |\sin(q_k\pi-p_k)|<\frac{1}{q_k}$

$$\Rightarrow q_k<\frac{1}{\sin p_k}$$

$$\Rightarrow \frac{q_k}{p_k}<\frac{1}{p_k\sin p_k}.$$

Take $k\to\infty$. Since $\left|\pi-\frac{p_k}{q_k}\right|<\frac{1}{q_k^2}$, from above we get $\frac{1}{\pi}<\lim_{k\to\infty}\frac{1}{p_k\sin p_k}$.

This is contradiction of (a). Thus the given sequence is divergent. ■

Def1) (**Irrationality Measure**)

: Fix $x \in \mathbb{R}$. Let $M(x) \subseteq \mathbb{R}$ be the set defining $r \in M(x)$ if and only if there exist infinitely many integer pairs satisfying $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^r}$ with $q > 0$.

At this time, $sup M(x)$ is called 'irrationality measure of x' denote it $\mu(x)$.

※ Note that if $\alpha > \mu(x)$, there exists at most finitely many integers (p, q) $(q > 0)$ satisfying $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^r}$. That is, for having large absolute values integers every (n, m), this inequality is holds $\left| x - \frac{n}{m} \right| \geq \frac{1}{m^\alpha}$.

ex1)

: Let α be a rational number. Then irrationality measure of α is 1.

sol) Denote $\alpha = \frac{b}{a}$ where $a, b \in \mathbb{Z}$, $a > 0$ and $\gcd(a, b) = 1$.

$i)$ $r > 1$, $|q\alpha - p| < \frac{1}{q^{r-1}} < 1 \Rightarrow |qb - pa| < a$.

Choosing $q = ka, p = kb - 1$, we get $|qx - p| = 1$. So $r < 1 \Rightarrow r \in M(x)$.

$ii)$ $r < 1$, $|q\alpha - p| < \frac{1}{q^{r-1}} \Rightarrow |qb - pa| < \frac{a}{q^{r-1}}$ $\cdots (a)$.

If there exists many integers pair (p, q), both p and q exist infinitely many because if only one side holds, $\left| \alpha - \frac{p}{q} \right|$ converge zero or diverge infinity.

So in (a) if q is enough to big, right hand enough to closely zero.

But since left hand not smaller than 1, this is contradiction.

Therefore, from $i), ii)$ irrationality measure of α is 1.

Thm1)

: Any irrational number has irrationality measure not smaller than 2.

proof) This is directive by 'Diophantine's Approximation Thm'.

Thm2) (Roth)

: Any irrational algebraic number has irrationality measure 2.

ex2) $\mu(\sqrt{2}) = 2, \mu(e) = 2$

Thm3) (Sondow)

: Let $x=[a_0, a_1, a_2, \ldots]$ be a simple continued fraction of x and (p_n/q_n) be n-th convergent. Then,

$$\mu(x) = 1 + \lim_{n\to\infty} sup \frac{\ln q_{n+1}}{\ln q_n} = 2 + \lim_{n\to\infty} sup \frac{\ln a_{n+1}}{\ln q_n}.$$

Thm4) (Note that this is generalization for page6-Thm5.)

: For positive real numbers u and v,

1) If $\mu(\pi) < 1 + \frac{u}{v}$, the sequence $\frac{1}{n^u |\sin n|^v}$ converges to zero,

2) If $\mu(\pi) > 1 + \frac{u}{v}$, the sequence $\frac{1}{n^u |\sin n|^v}$ diverges.

proof)

: 1) Let n be natural number and $m = \left\lfloor \frac{n}{\pi} \right\rfloor$. And $\exists\, \varepsilon > 0$ s.t. $\mu(\pi) < 1 + \frac{u}{v} - \varepsilon$.

Then since $|n - m\pi| \in \left[0, \frac{\pi}{2}\right]$, $|\sin n| = |\sin(n - m\pi)| \ge \frac{2|n - m\pi|}{\pi} = \frac{2m}{\pi}\left|\pi - \frac{n}{m}\right|$.

Since $\mu(\pi) < 1 + \frac{u}{v} - \varepsilon$, $\left|\pi - \frac{n}{m}\right| \ge \frac{1}{m^{1+\frac{u}{v}-\varepsilon}}$ for large $n \in \mathbb{N}$.

So for large n, $|\sin n| \ge \frac{2}{\pi}\frac{1}{m^{\frac{u}{v}-\varepsilon}} \Leftrightarrow m^{-\varepsilon}|\sin n| \ge \frac{2}{\pi}\left(\frac{n}{m}\right)^{\frac{u}{v}}\frac{1}{n^{\frac{u}{v}}}$

$$\Leftrightarrow m^{-\frac{\varepsilon}{v}} \ge \frac{2}{\pi}\left(\frac{n}{m}\right)^{u}\frac{1}{n^u|\sin n|^v}.$$

Because $\lim_{n\to\infty}\frac{n}{m} = \frac{1}{\pi}$, for large n $\exists$ constant K s.t. $m^{-\frac{\varepsilon}{v}} \ge \frac{K}{n^u|\sin n|^v}$.

Therefore taking $n \to \infty$, we get the result $\lim_{n\to\infty}\frac{1}{n^u|\sin n|^v} = 0$. ■

2) There exists some $\varepsilon > 0$ such that $\mu(\pi) > 1 + \frac{u}{v} + \varepsilon$.

Thus there exists infinitely many integers satisfying $\left|\pi - \frac{p_k}{q_k}\right| \le \frac{1}{q_k^{1+\frac{u}{v}+\varepsilon}}$.

WLOG, we can assume $1 \le q_1 < q_2 < \cdots$, $1 \le p_1 < p_2 \le \cdots$.

Since $\left|\pi - \frac{p_k}{q_k}\right| \le \frac{1}{q_k^{1+\frac{u}{v}+\varepsilon}} \Leftrightarrow |q_k\pi - p_k| \le \frac{1}{q_k^{\frac{u}{v}+\varepsilon}}$ for all $k \in \mathbb{N}$,

we get $|\sin p_k| \le \frac{1}{q_k^{\frac{u}{v}+\varepsilon}} \Leftrightarrow \frac{q_k^{u+v\varepsilon}}{p_k^u} \le \frac{1}{p_k^u|\sin p_k|^v}$. Note that $\lim_{k\to\infty}\frac{q_k}{p_k} = \frac{1}{\pi}$.

Therefore taking $k \to \infty$, right hand is divergent to infinity. ■