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Construction of Lebesgue Measure

Motivation. What are length, area and volume?What is a quantity of 4 or more dimensional object?

Question. How can we define quantity measurement of some sets in Euclidean space?

Definition1. Let $a, b \in \mathbb{R}$ with $a \leq b$.

Define naturally the lengthℓ of each interval such that

$\ell\big((a,b]\big) = b - a$, $\ell\big((a,+\infty)\big) = +\infty$, $\ell\big((-\infty,b]\big) = +\infty$, $\ell\big((-\infty,+\infty)\big) = +\infty$,

and $\ell(\bigcup_{i=1}^{n}(a_i,b_i]) = \sum_{i=1}^{n}(b_i - a_i)$ where $(a_i, b_i]$ are disjoint.

Definition 2.

(1). A family $\mathcal{A}$ of subsets of a set X is said to be an 'algebra'if it satisfies followings.

I. $\emptyset, X \in \mathcal{A}$

II. $E \in \mathcal{A} \Rightarrow X \setminus E \in \mathcal{A}$

III. $E_1, E_2, \cdots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} E_i \in \mathcal{A}$

(2).A family $\sum$ of subset of a set X is said to be a 'σ-algebra' if it satisfies followings.

I. $\emptyset, X \in \sum$

II. $E \in \sum \Rightarrow X \setminus E \in \sum$

III. $E_1, E_2, \cdots, E_{n,} \cdots \in \sum \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \sum$

Definition 3. Let $\mathcal{A}$ be an algebra and $\sum$be aσ-algebra of a set X.

(1). A 'measure' on $\mathcal{A}$ is an extended real-valued function $\mu: \mathcal{A} \to \overline{\mathbb{R}}$ defined on $\mathcal{A}$ such that

I. $\mu(\emptyset) = 0$

II. $\mu(E) \geq 0 \; for \; \forall E \in \mathcal{A}$

III. If $(E_n)$is a disjoint sequence in $\mathcal{A}$ with$\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

(2). A 'measure' on $\sum$is an extended real-valued function $\mu: \sum \to \overline{\mathbb{R}}$ defined on $\sum$ such that

I. $\mu(\emptyset) = 0$

II. $\mu(E) \geq 0 \; for \; \forall E \in \sum$

III. $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for any disjoint sequence (E_n)in $\sum$

Lemma 1. Let $\mathcal{F}$ be a collection of all finite union of set of the form $(a,b], (a,+\infty), (-\infty,b], (-\infty,+\infty)$.

Then $\mathcal{F}$ is an algebra of subsets of $\mathbb{R}$ and length is a measure on $\mathcal{F}$.

Proof) By construction of $\mathcal{F}$, it is readily seen that $\mathcal{F}$ is an algebra. And ℓ satisfies the condition I and II of Definition 3 trivially.

It is enough to show that $(a,b] = \bigcup_{n=1}^{\infty}(a_n,b_n]$ where $(a_n,b_n]$ are disjoint implies $\ell\big((a,b]\big) = \sum_{n=1}^{\infty}\ell((a_n,b_n])$ to check the condition IIIof Definition 3.

Let $(a,b] = \bigcup_{n=1}^{\infty}(a_n,b_n]$ where $(a_n,b_n]$ are disjoint and consider any finite collection $(a_1,b_1],(a_2,b_2],\cdots,(a_m,b_m]$ of such intervals.

Suppose that

$$a \le a_1 < b_1 \le a_2 < \cdots < b_{n-1} \le a_n < b_n \le b$$

by renumbering the finite indices.

Then

$$\sum_{n=1}^{m}\ell((a_n,b_n]) = \sum_{n=1}^{m}(b_n - a_n) \le b_n - a_1 \le b - a = \ell\big((a,b]\big)$$

Since m is arbitrary, we infer that

$$\sum_{n=1}^{\infty}\ell((a_n,b_n]) \le \ell\big((a,b]\big)$$

Conversely, let $\varepsilon > 0$ and (ε_n) be a sequence of positive numbers with $\sum \varepsilon_n < \frac{\varepsilon}{4}$.

Consider the intervals $I_n = (a_n - \varepsilon_n, b_n + \varepsilon_n)$ for $n \in \mathbb{N}$.

Since $[a+\frac{\varepsilon}{2}, b] \subset (a,b] \subset \bigcup_{n=1}^{\infty} I_n$ for sufficiently small ε, (I_n) is an open covering of the compact set $[a+\frac{\varepsilon}{2}, b]$. It implies that there is a finite sub-covering $I_1, I_2, \cdots, I_k$ of $[a+\frac{\varepsilon}{2}, b]$. By renumbering and discarding some extra intervals, we may assume that

$$a_1 - \varepsilon_1 < a + \frac{\varepsilon}{2}, \qquad b < b_k + \varepsilon_k, \qquad a_j - \varepsilon_j < b_{j-1} + \varepsilon_{j-1}\, for\; j = 2,3,\cdots,k$$

Then

$$b - (a + \frac{\varepsilon}{2}) \le (b_k + \varepsilon_k) - (a_1 - \varepsilon_1) \le \sum_{j=1}^{k}\big[(b_j + \varepsilon_j) - (a_j - \varepsilon_j)\big] \le \sum_{j=1}^{k}(b_j - a_j) + \frac{\varepsilon}{2} \le \sum_{n=1}^{\infty}(b_n - a_n) + \frac{\varepsilon}{2}$$

Thus we get

$$b - a \le \sum_{n=1}^{\infty}(b_n - a_n) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\ell\big((a,b]\big) \le \sum_{n=1}^{\infty}\ell((a_n,b_n])$.

Therefore, we get $\ell\big((a,b]\big) = \sum_{n=1}^{\infty}\ell((a_n,b_n])$. ■

Definition 4. Let $B \subseteq X$ be an arbitrary subset. Define

$$\mu^*(B) := inf\{\sum\mu(E_j) \big| B \subseteq \cup E_j, E_j \in \mathcal{A}\}$$

the 'outer measure' generated by μwithan algebra $\mathcal{A}$.

Lemma 2. The function $\mu^*: \wp(X) \rightarrow \overline{\mathbb{R}}$ of Definition 4 satisfies the followings.

(a) $\mu^*(\emptyset) = 0$

(b) $\mu^*(B) \geq 0 \, for \, \forall B \subseteq X$

(c) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$

(d) If $B \in \mathcal{A}$, then $\mu^*(B) = \mu(B)$

(e) If (B_n) is a sequence of subsets of X, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$$

Proof)(a), (b) and (c) are trivial.

(d): Since $\{B, \emptyset, \emptyset, \cdots\}$ is a countable cover of B in $\mathcal{A}$, it follows that

$$\mu^*(B) \leq \mu(B) + \sum \mu(\emptyset) = \mu(B)$$

Conversely, if (E_n) is a sequence in $\mathcal{A}$ whose union contains B, then

$$B = B \cap \left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty}(B \cap E_n)$$

Since μ is a measure on $\mathcal{A}$, by II andIIIof definition 3, we have known that

$$\mu(B) \leq \sum_{n=1}^{\infty} \mu(B \cap E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Since (E_n) is arbitrary chosen, it follows that $\mu(B) \leq \mu^*(B)$.

(e): Let (B_n) is a sequence of subsets of X and $\varepsilon > 0$.Choose a sequence $(E_{nk})_{k\in\mathbb{N}}$ in $\mathcal{A}$for each nsuch that

$$B_n \subseteq \bigcup_{k=1}^{\infty} E_{nk} \quad and \quad \sum_{k=1}^{\infty} \mu(E_{nk}) \leq \mu^*(B_n) + \frac{\varepsilon}{2^n}$$

Since $\{E_{nk} | n, k \in \mathbb{N}\}$ is a countable collection in $\mathcal{A}$ such that $\bigcup_n B \subseteq \bigcup_{n,k} E_{nk}$, it follows from the definition of μ^* that

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(B_n) + \varepsilon$$

Since ε is arbitrary chosen, we get $\mu^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$. ■

Definition 5. [*Carathéodory Condition*]

A subset E of X is said to be 'μ^*-measurable' if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)\ for\ any\ A \subseteq X$$

Define$\mathcal{A}^* \coloneqq \{E \subseteq X | E\ is\ \mu^* - \text{measurable}\}$.

Theorem 1. [*Carathéodory Extension Theorem*]

The collection $\mathcal{A}^*$ is a σ-algebra containing $\mathcal{A}$.Moreover, if (E_n) is a disjoint sequence in $\mathcal{A}^*$, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

Proof) Clearly, $\emptyset, X \in \mathcal{A}^*$ and $E \in \mathcal{A}^*$ implies $X \setminus E \in \mathcal{A}^*$.

Claim: $\mathcal{A}^*$ is closed under intersections

Let $E, F \in \mathcal{A}^*$. Then for any $A \subseteq X$ and $E \in \mathcal{A}^*$, we have

$$\mu^*(A \cap F) = \mu^*(A \cap F \cap E) + \mu^*((A \cap F) \setminus E)$$

Since $F \in \mathcal{A}^*$, $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F)$.

Let $B = A \setminus (E \cap F)$. Then it is readily seen that

$$B \cap F = (A \cap F) \setminus E \quad and \quad B \setminus F = A \setminus F$$

It follows that

$$\mu^*(A \setminus (E \cap F)) = \mu^*\big((A \cap F) \setminus E\big) + \mu^*(A \setminus F)$$

Thus, we get

$$\begin{aligned}\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F) &= \big[\mu^*(A \cap F \cap E) + \mu^*\big((A \cap F) \setminus E\big)\big] + \mu^*(A \setminus F) \\ &= \mu^*(A \cap F \cap E) + \mu^*\big(A \setminus (E \cap F)\big)\end{aligned}$$

Which shows that $E \cap F \in \mathcal{A}^*$.

Since $\mathcal{A}^*$ is closed under intersections and complementation, $\mathcal{A}^*$ is an algebra.

Let $E, F \in \mathcal{A}^*$ with $E \cap F = \emptyset$. Since $A \cap (E \cup F) \subseteq X$ and $E \in \mathcal{A}^*$, we obtain

$$\mu^*\big(A \cap (E \cup F)\big) = \mu^*(A \cap (E \cup F) \cap E) + \mu^*(A \cap (E \cup F) \setminus E) = \mu^*(A \cap E) + \mu^*(A \cap F)\ for\ any\ A \subseteq X$$

For $A = X$, it shows that μ^* is finitely additive on $\mathcal{A}^*$.

Let (E_k) be a disjoint sequence in $\mathcal{A}^*$ and $E = \bigcup E_k$. Since $\mathcal{A}^*$ is an algebra, we know that$F_n \coloneqq \bigcup_{k=1}^{n} E_k \in \mathcal{A}^*$ and finite additivity implies

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \setminus F_n) = \sum_{k=1}^{n} \mu^*(E_k) + \mu^*(A \setminus F_n)$$

for any $A \subseteq X$ and each $n \in \mathbb{N}$.

Since $F_n \subseteq E$, $A \setminus E \subseteq A \setminus F_n$ and letting $n \to \infty$implies

$$\sum_{k=1}^{\infty} \mu^*(E_k) + \mu^*(A \setminus E) \leq \mu^*(A)$$

Conversely, it follows from Lemma 2. (e) that

$$\mu^*(A \cap E) \leq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) \quad and \quad \mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E)$$

From the last three inequalities, we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \setminus E)$$

This shows that $E = \bigcup E_k \in \mathcal{A}^*$. For $A = E$, we get $\mu^*(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu^*(E_k)$.

It remains to show that $\mathcal{A} \subseteq \mathcal{A}^*$. Let $E \in \mathcal{A}$ and $A \subseteq X$ be arbitrary. By Lemma 2. (d) and (e), we know that

$$\mu^*(E) = \mu(E) and \ \ \mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E)$$

Let $\varepsilon > 0$ and (F_n) be a sequence such that $A \subseteq \bigcup F_n$ and

$$\sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(A) + \varepsilon$$

by definition of outer measure.

Since $A \cap E \subseteq \bigcup(F_n \cap E)$ and $A \setminus E \subseteq \bigcup(F_n \setminus E)$, it follows from Lemma 2. (e) that

$$\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu(F_n \cap E) \quad and \quad \mu^*(A \setminus E) \leq \sum_{n=1}^{\infty} \mu(F_n \setminus E)$$

Hence we have

$$\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \sum_{n=1}^{\infty} [\mu(F_n \cap E) + \mu(F_n \setminus E)] = \sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(A) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \mu^*(A)$. Therefore, we conclude that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ for any $A \subseteq X$, that means$E \in \mathcal{A}^*$. ■

Conclusion. Suppose the algebra $\mathcal{F}$ defined in Lemma 1 and the 'length' ℓ that is a measure on $\mathcal{F}$.

Consider

$$m \coloneqq \ell^*$$

,the outer measure generated by ℓ and

$$\mathcal{L} \coloneqq \mathcal{F}^* = \{E \subseteq \mathbb{R} | E \ is\ a\ m - measurable\}$$

,the collection of subsets of $\mathbb{R}$ that satisfies the Carathéodory Condition.

By Theorem 1, $\mathcal{L}$ is a σ-algebra containing $\mathcal{F}$ and m is a measure on $\mathcal{L}$.

A subset of $\mathbb{R}$ contained in $\mathcal{L}$ is called a 'Lebesgue measurable set' and m is called the 'Lebesgue measure'.

Theorem 2. [*Hahn Extension Theorem*]

If μ is a measure on $\mathcal{L}$ such that $\mu(I) = l(I)$ for all open intervals $I \subseteq \mathbb{R}$, then $\mu = m$.In other word, the

Lebesgue measure is unique.

Proof) For $n \in \mathbb{N}$, let $I_n = (-n, n)$. Let $E \in \mathcal{L}$ be any set with $E \subseteq I_n$ for some n and (J_k) be a sequence of open intervals such that $E \subseteq \bigcup_{k=1}^{\infty} J_k$. Since μ is a measure and $\mu(J_k) = m(J_k)$ for all $k \in \mathbb{N}$, we have

$$\mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} J_k\right) \leq \sum_{k=1}^{\infty} \mu(J_k) = \sum_{k=1}^{\infty} l(J_k)$$

Therefore, $\mu(E) \leq m^*(E) = m(E)$ for all Lebesgue measurable sets $E \subseteq I_n$.

Since μ and m are additive,

$$\mu(E) + \mu(I_n \setminus E) = \mu(I_n) = l(I_n) = m(I_n) = m(E) + m(I_n \setminus E)$$

Since all of these terms are finite and $\mu(E) \leq m(E)$ and $\mu(I_n \setminus E) \leq m(I_n \setminus E)$, it follows that $\mu(E) = m(E)$ for all Lebesgue measurable sets $E \subseteq I_n$.

Let E be an arbitrary Lebesgue measurable set. Then it can be written as the union of disjoint Lebesgue measurable sets E_n, defined by

$$E_1 \coloneqq E \cap I_1,\ \ E_n \coloneqq E \cap (I_n \setminus I_{n-1})\ for\ n > 1$$

Since $\mu(E_n) = m(E_n)$ for all $n \in \mathbb{N}$ by previous step, it follows that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} m(E_n) = m(E)$$

Therefore, $\mu = m$ on $\mathcal{L}$. ■

Reference: *Bartle, The Elements of Integration and Lebesgue Measure, 1995*