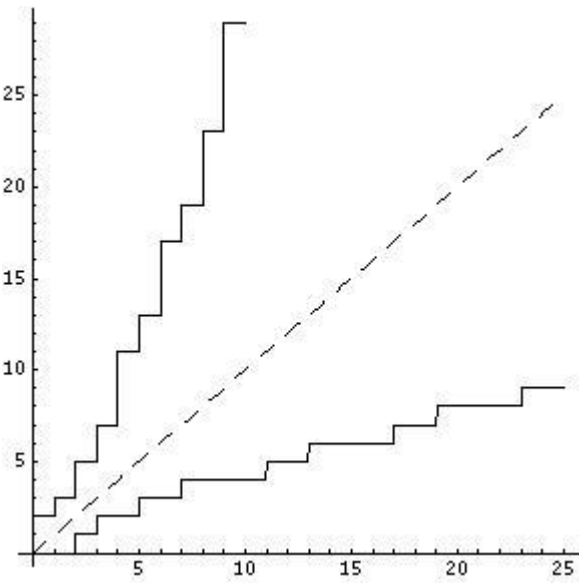
Introduction to Inverse Sequences

In this seminar, we only deal with non-decreasing (but constant only finite intervals) and non-negative integer sequence a_n.

1. Define $a_0 = 0$ and point (n, a_n) in x-y plane.
2. Draw horizontal segments connecting the points (n-1, a_n) and (n, a_n) for n>0.
3. Also, draw vertical segments connecting the points (n, a_n) and (n, a_{n+1}) for n≥ 0.
4. Flip symmetrically this drawing with respect to y=x.

Then we get another drawing and a new interesting sequence.



Definition. Let a_n be a non-decreasing and non-negative integer sequence. Then an (geometric) inverse sequence is a sequence such that defined by the step in introduced at the beginning. Let $a^{-1}{}_n$ denote that the (geometric) inverse sequence of a_n.

Remark. Let a_n be a non-decreasing and non-negative integer sequence. Then an (geometric) inverse sequence uniquely exists.

Theorem 1. Let a_n be a non-decreasing and non-negative integer sequence. Then $a^{-1}{}_k$ is the number of integers less than k in a_n for $\forall k \in \mathbb{N}$

Proof. Let $a_t = k_1$ and $a_{t+1} = k_2$ where $k_1 < k \leq k_2$. Then

$$a^{-1}{}_{k_1+1} = a^{-1}{}_{k_1+2} = \cdots a^{-1}{}_k = \cdots = a^{-1}{}_{k_2} = t$$

Since $a_t = k_1$, there are t elements less than $k_1 + 1$ in a_n, and $a_t = k_1 < k_2 = a_{t+1}$ implies there is no element such that greater than or equal to $k_1 + 1$($\Leftrightarrow$ greater than k_1) and less than k_2 in a_n.

∴ There are t element less than k in a_n . ■

Examples. (1) i_n=1, 2, 3, 4, 5, 6, 7, 8, ⋯

⇒ $i^{-1}{}_n$=0, 1, 2, 3, 4, 5, 6, 7,⋯

(2) p_n=2, 3, 5, 7, 11, 13, 17, 19, ⋯

⇒ $p^{-1}{}_n$=0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6,⋯

(3) f_n=1, 1, 2, 3, 5, 8, 13, 21, 34, ⋯

⇒ $f^{-1}{}_n$=0, 2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, ⋯

Remark. We can guess enough about $(a^{-1})^{-1}{}_n = a_n$.

Examples. (1)$i^{-1}{}_n$=0, 1, 2, 3, 4, 5, 6, 7, ⋯

⇒ $(i^{-1})^{-1}{}_n$=1, 2, 3, 4, 5, 6, 7, 8, ⋯

(2)$p^{-1}{}_n$=0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, ⋯

⇒ $(p^{-1})^{-1}{}_n$=2, 3, 5, 7, 11, 13, ⋯

(3) $f^{-1}{}_n$=0, 2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, ⋯

⇒ $(f^{-1})^{-1}{}_n$=1, 1, 2, 3, 5, 8, 13, ⋯

Theorem 2. Let a_n be a non-decreasing and non-negative integer sequence. Then the inverse sequence of ${a^{-1}}_n$ is a_n.

Proof. Let $n \in \mathbb{N}$ satisfy that $a_n < a_{n+1}$ and arrange the sequence a_n.

$$\cdots = a_{n-t} < a_{n-t+1} = \cdots = a_{n-1} = a_n < a_{n+1} = a_{n+2} \cdots = a_{n+m} < a_{n+m+1} = \cdots$$

Suppose there are t elements equal to a_n and m elements equal to a_{n+1}.

By ***Theorem 1***, we can also arrange ${a^{-1}}_n$.

$$\cdots = {a^{-1}}_{a_{n-t}} < {a^{-1}}_{a_{n-t}+1} = \cdots = {a^{-1}}_{a_n} < {a^{-1}}_{a_n+1} = \cdots = {a^{-1}}_{a_{n+1}} < {a^{-1}}_{a_{n+1}+1} = \cdots = {a^{-1}}_{a_{n+m+1}} < \cdots$$

Note that ${a^{-1}}_{a_{n-t}} < n-t$,

${a^{-1}}_{a_{n-t}+1} = \cdots = {a^{-1}}_{a_n} = n-t$,

${a^{-1}}_{a_n+1} = \cdots = {a^{-1}}_{a_{n+1}} = n$,

and ${a^{-1}}_{a_{n+1}+1} = \cdots = {a^{-1}}_{a_{n+m+1}} = n+m$.

$\therefore {(a^{-1})^{-1}}_{n-t} = a_{n-t}$,

${(a^{-1})^{-1}}_{n-t+1} = \cdots = {(a^{-1})^{-1}}_{n} = a_n$,

${(a^{-1})^{-1}}_{n+1} = \cdots = {(a^{-1})^{-1}}_{n+m} = a_{n+1}$,

and ${(a^{-1})^{-1}}_{n+m+1} = a_{n+m+1}$. ∎

Reference.

Tanya Khovanova, How to Create a New Integer Sequence, (2007)
http://www.tanyakhovanova.com/Sequences/CreatingNewSequences.html#inverse