Game Theorey: Nash Existence Theorem and Minimax Theorem

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Basic Notation

- N = {1,2,$\cdots$,n}: set of players
- Upper case letter refers a set, lower case letter is an element of the set.
- Subscript refer to something related to a particular player.
- a := $(a_1, a_2, \cdot, a_n)$

Game Theory

Analysis of strategies in which multiple rational players make strategically interdependent decisions.

Game theory has a long history beginning with the 1600s. Cardano, Pascal, Huygens, and other many researchers formulated basic ideas of game theory. One of the famous economic oligopoly models, Cournot competition, is also based on game theory.

After that, John von Neumann established a unique field of game theory in "On the Theory of Games of Strategy" in 1928. Then, game theory has a heyday in the 1950s with many fields scholars. John Nash is one of the reasons for the development of game theory. John Nash developed the nash equilibrium and proved that that always exists.

Pure Strategy Nash Equilibrium

A_i is a set of actions that player i can choose
u_i is a function as $A \to \mathbb{R}$
$a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$

$$\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

a_i^* is a best response.

$$\text{Nash Equilbrium:} \quad a^* = (a_1^*, \cdots, a_n^*)$$

In order words, pure strategy nash equilibrium means that the best response to the other players selects such strategies.

		Player Y	
		A	B
Player X	A	$(1,-1)$	$(-1,1)$
	B	$(-1,1)$	$(1,-1)$

But, pure strategy nash equilibrium is not always exist.

Mixed Strategy Nash Equilibrium

Pure strategy nash equilibrium can not explain probability. Moreover, sometimes pure strategy nash equilibrium does not exist.
Mixed strategy nash eqilibrium can explain repeate game, confuse oppinent and uncertaun other's action.

S_i is all probability distribution of mixed strategies on A.
S := $S_1 \times S_2 \times \cdots \times S_n$

$$u_i(s) = \sum u_i(a)P(a|s)$$

where

$$P(a|s) = \prod_{j \in N} s_j(a_j)$$

$$\forall s_i \in S_i,\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

s_i^* is best response.

$$\text{Nash Equilbrium:} \quad s^* = (s_1^*, \cdots, s_n^*)$$

Nash's Existence Theorem

Theorem 1. Every finite game has an equilibrium point.

Proof of Theorem 1.

$$\varphi_i(s,a) = max\{0, u_i(a, s_{-i}) - u_i(s_i, s_{-i})\}$$

$$g_i(s)(a) = s_i(a) + \varphi_i(s, a)$$

$$\sum_{a \in A_i} g_i(s)(a) = 1 + \sum_{a \in A_i} \varphi_i(s, a) > 0$$

Let $f = (f_1, \cdots, f_N) : S \to S$ and

$$f_i(s)(a) = \frac{g_i(s)(a)}{\sum_{b \in A_i} g_i(s)(b)}$$

Now, we need to show that $\forall i \in 1, \cdots, N$, $\forall a \in A_i : \quad \varphi_i(s^*, a) = 0$

Then, assume that the gains are not all zero, $\varphi_i(s^*, a) > 0$

$$\sum_{a \in A_i} g_i(s^*, a) = 1 + \sum_{a \in A_i} \varphi_i(s^*, a) > 1$$

So let $C = \sum_{a \in A_i} g_i(s^*, a)$.

Also, we shall denote Gain(i,.) as the gain vector indexed by actions in A_i. Since s^* is the fixe point we have:

$$\begin{aligned}
s^* = f(s^*) &\Rightarrow s_i^* = f_i(s^*) \\
&\Rightarrow s_i^* = \frac{g_i(s^*)}{\sum_{a \in A_i} g_i(s^*)(a)} \\
&\Rightarrow s_i^* = \frac{1}{C}(s_i^* + Gain_i(s^*, .)) \\
&\to s_i^* = (\frac{1}{C-1})Gain_i(s^*, .)
\end{aligned}$$

Since C ¿ 1 we have that s_i^* is some positive scaling of the vector $\varphi_i(s^*, .)$

$$\forall a \in A_i : \quad s_i^*(a)(u_i(a_i, s_{-i}^*) - u_i(s_i^*, s_{-i}^*)) = s_i^*(a)\varphi_i(s^*, a)$$

Now we assume that $\varphi_i(s^*, a) = 0$. By our previous statements we have that $s_i^*(a) = 0$

So we finally have that

$$\begin{aligned}
0 &= u_i(a_i, s_{-i}^*) - u_i(s_i^*, s_{-i}^*) \\
&= \sum_{a \in A_i} s_i^*(a) u_i(a_i, s_{-i}^*) - u_i(s_i^*, s_{-i}^*) \\
&= \sum_{a \in A_i} s_i^*(a)\varphi_i(s^*, a) \\
&= \sum_{a \in A_i} (C-1) s_i^*(a)^2 > 0
\end{aligned}$$

where the last inequality follows since s_i^* is a non-zero vectore. But this is a clear contradiction, so all the gains must indeed be zero. Therefore, s^* is a nash equilibrium.

Q.E.D.

Zeros-Sum Game

		Player Y	
		A	B
Player X	A	$(A, -A)$	$(B, -B)$
	B	$(C, -C)$	$(D, -D)$

Zero-sum game has property of someone gains then other must loses and the sum of total gains and total loses equal to 0. Therefore, any result of zero-sum game is pareto optimal.

Max-Min & Min-Max strategy

Max-min strategy is player i select maximize his payoff when worst case. In order words, player -i play strategies that involve the biggest harmful effect to i.

Max-min strategy

$$arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

Max-min value

$$\max_{i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

Min-max strategy is player i select case that minimize -i's best case payoff. In order words, player i want to decrease by other players payoff.

Min-max strategy

$$arg \min_{s_i} \max_{s_{-i}} u_i(s_i, s_{-i})$$

Min-max value

$$\min_i \max_{s_{-i}} u_i(s_i, s_{-i})$$

Minimax Theorem

Theorem 2. Let x $= (x_1, x_2, \cdots, x_n)$ is player 1 mixed strategies and y $= (y_1, y_2, \cdots, y_n)$ is player 2 mixed strategies. Moreover, A is $m \times n$ payoff(utility) matrix A_i denote of i row of A and A^j denote j row of A. Then, fixed strategy x for player 1

$$\min_{y \in Y} xAy^T = \min_{1 \leq j \leq n} xA^j$$

fixe strategy y for player 2

$$\max x \in X xAy^T = \max_{1 \leq i \leq m} A_i y^T$$

Proof of Theorem 2

Let xA' is a real number. Let $w = \min_{1 \leq j \leq n} xA'$

$$\begin{aligned} xAy^T &= \sum_{1 \leq j \leq n} xA^j y_j \\ &\geq \sum_{1 \leq j \leq n} w y_j \\ &= w \sum_{1 \leq j \leq n} y_j = w \end{aligned}$$

Q.E.D.

Theorem 3(Minimax Theorem). In any finite, two-player, zero-sum game, in any Nash Equilibrium each player receives a payoff is equl to both his maxmin value and his minmax value.

Proof ot Theorem 3

v_1 and v_2 is defined by

$$v_1 = \max_{x \in X} \min_{1 \leq j \leq n} xA^j$$

$$v_2 = \min_{y \in Y} \max_{1 \leq i \leq m} A_i y^T$$

Then, minimax theorem can be written by $v_1 = v_2$

Case1. All $a_{ij} > 0$

we can xA' rewrite by

$$
\begin{aligned}
&a_{11}x_1 + a_{21}x_2 + \dots a_{m1}x_m \\
&a_{12}x_1 + a_{22}x_2 + \dots a_{m2}x_m \\
&\vdots \\
&a_{1n}x_1 + a_{2n}x_2 + \dots a_{mn}x_n
\end{aligned}
$$

Let, w be minimum of these equations.

$$
\begin{aligned}
&a_{11}x_1 + a_{21}x_2 + \dots a_{m1}x_m \geq w \\
&a_{12}x_1 + a_{22}x_2 + \dots a_{m2}x_m \geq w \\
&\vdots \\
&a_{1n}x_1 + a_{2n}x_2 + \dots a_{mn}x_n \geq w
\end{aligned}
$$

We know that $a_{ij} > 0$, then $v_1 > 0$ and $w > 0$. Therefore, we can before inequailities.

$$\text{maximize } w$$

subject to

$$
\begin{aligned}
&a_{11}\frac{x_1}{w} + \dots + a_{m1}\frac{x_m}{w} \geq 1 \\
&a_{1n}\frac{x_1}{w} + \dots + a_{mn}\frac{x_m}{w} \geq 1 \\
&\frac{x_1}{w} + \dots + \frac{x_m}{w} = \frac{1}{w} \\
&\frac{x_i}{w} \geq 0, 1 \leq i \leq m
\end{aligned}
$$

Let $x_i^{'} = \frac{x_i}{w}$, $\forall i = (1, 2, ..., m)$ Then, we can be rewritten as

$$\text{minimize } x_1^{'} + x_2^{'} + \dots + x_m^{'}$$

subject to

$$
\begin{aligned}
&a_{11}x_1^{'} + \dots + a_{m1}x_m^{'} \geq 1 \\
&\vdots \\
&a_{1n}x_1^{'} + \dots + a_{mn}x_m^{'} \geq 1 \\
&x_i^{'} \geq 0, 1 \leq i \leq m
\end{aligned}
$$

Let x' = $(x_1^{'}, x_2^{'}, \cdots, x_m^{'})$, b = $(1, 1, ..., 1)^T$, and c = $(1, 1, ..., 1)^T$. Then, we can express this more concisely in vector notation.

minimize b * x'

subject to

$A^T x' \geq c$, $x' \geq 0$

Then, minimum value of b * x' is equal to v_1.
v_2 is similar method, maximum value of c * y' is equal to v_2.

maximize c*y'

subject to

$Ay' \leq b$, $y' \geq 0$

Now, we have two dual problem. Moreover, dual problem must have a finite optimal solution, because the objective function b*x' $= x_1^{'} + \cdots x_m^{'}$ is bouned below by zero, and since all a_{ij} are positive, there exist feasible solutions to the system of constraints $A^T x' \geq c$. Thus the primal problem also has a finite optimal solution. Therefore, by the Duality Theorem, both problems have finite solutions obtaining the same optimal value. Hence, there exist strategies, the maximin strategy and minimax strategy, for player 1 and player 2, respectively, and these values are equal.

Case2. some $a_{ij} \leq 0$

Let any constant r with the property that $a_{ij} + r > 0$ for all i,j. Let, E be the m $\times$ n matrix with all entries equal to 1. Then, our expected payoff of (x,y) is

$$x(A + rE)y^T = xAy^T + rxEy^T$$

Then, xAy^T and $x(a + rE)y^T$ is only iffer by the constant r, then they are have same strategies. Therefore, we can apply the results of Case 1. Then, maximin an minimax strategies exist and $v_1 = v_2$

Q.E.D.

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