Differential Geometry and Topology in Physics: Spring-2021

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Abstract

These are lecture notes of the course at Fudan in Spring 2021, and I write these by wishing that more students will be interested in the relationship between geometry and physics. Homework sets can be found in the website. These notes are written by referring to various sources without mentioning them. Comments are welcome. If you find typos, please email me.

Contents

1 Prologue

1.1 Euler characteristics

Let P be a polyhedron with V vertices, E edges, and F faces. The Euler characteristic of P is defined as

$$\chi(P) = V - E + F \,.$$

For example, regular polyhedra provide **cell decomposition** of a sphere as follows, which provide $\chi(S^2) = 2$. Note that the Euler characteristics are independent of a choice of cell decomposition. On the other hand, the Euler characteristic of a torus is equal to zero. The Euler characteristic is the most important **topological invariant**. For a mathematical formulation, we need to learn the notion of **homology**.

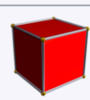
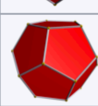
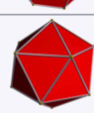
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Figure 1: Euler characteristic of the sphere from Wikipedia:Euler_characteristic

There are many ways to approach the Euler characteristic. For instance, let us consider a vector field on a surface Σ. A physics student is familiar with vector electric and magnetic fields. These are vector fields on $\mathbb{R}^3$, and we can generalize them to a **smooth**

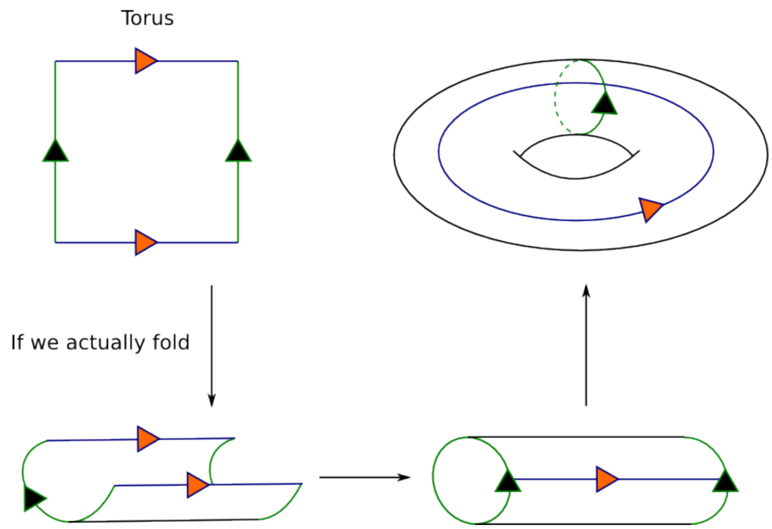


Figure 2: Cell decomposition of a torus

vector field on a surface Σ. For a zero of a vector field, one can introduce the notion of an **index** which is illustrated in Figure 3. (The definition will be given in §7.5) Then, the **Poincaré-Hopf** theorem states that the sum of indices at zeros of a vector field X is the Euler characteristic:

$$\sum_p \mathrm{ind}_p(X) = \chi(\Sigma) \ .$$

To formulate the Poincaré-Hopf theorem, we need to introduce the notion of **vector fields** and **tangent bundles**.

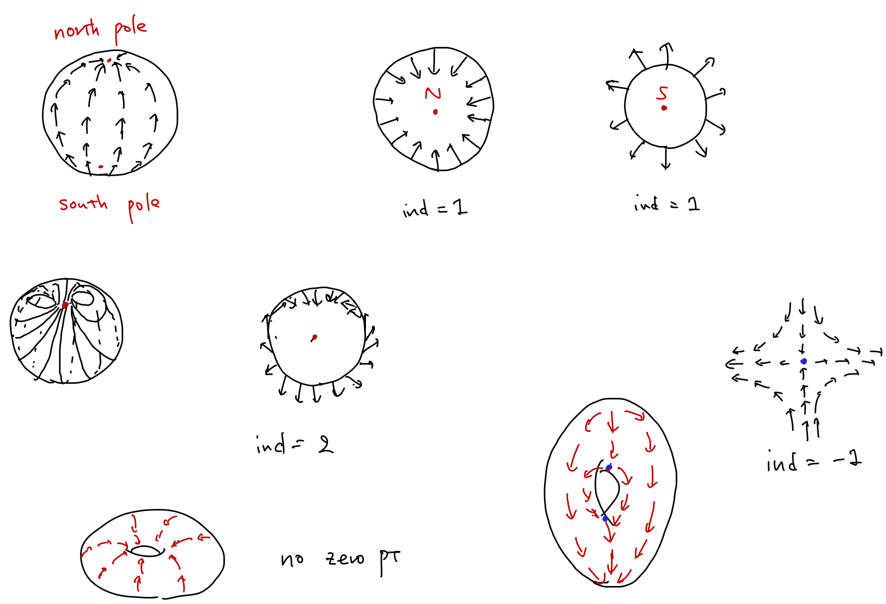


Figure 3: Vector fields on surfaces and index at a zero.

Another important theorem is the **Gauss-Bonnet theorem**:

$$\chi(\Sigma) = \int_M \frac{\kappa}{2\pi} dA$$

where κ is called the Gauss curvature. The Gauss-Bonnet theorem was later generalized to the celebrated Hirzebruch-Riemann-Roch theorem and index theorem, which can be regarded as one of the milestones in mathematics of the twentieth century. The motto of these theorems is "from local to global". We will glimpse these theorems in §12.

The geometric study of surfaces was preceded by Gauss, which indicates the geometry beyond Euclid. In the middle of the 19th century, Riemann proposed a concept of manifolds of arbitrary dimensions [Rie54]. Furthermore, he clearly envisioned **Riemannian geometry** which deals with metrics, connections and curvatures on a manifold. Remarkably, his idea on "holomorphic function on a Riemann surface" [Rie57] also has later led to the theory of **complex geometry** and **algebraic geometry**. Around 1900, Poincaré has opened up a new area of mathematics, nowadays called **algebraic topology** [Poi95]. He has illustrated his idea of studying topology by homology groups and fundamental groups.

The large part of the ideas of Riemann and Poincaré is not furnished with mathematical rigorous techniques of their time, so the necessary tools had to be invented. As a result, their revolutionary ideas and methods are to be a source of inspiration for a century, and the later development justified their intuitions.[1] One can see similar phenomena in the current relation between physics and mathematics.

In this course, we will learn the basic concepts of geometry and topology developed after Gauss, Riemann and Poincaré. More concretely, we will deal with the following subjects

- manifolds and tangent, cotangent bundles,
- homology, cohomology and fundamental groups
- metric, connections, curvatures (Riemannian geometry)
- Lie group and Lie algebras
- vector bundles and principal G-bundles
- topological invariants, characteristic classes

It turns out that these notions are indispensable to a description of physics. Hamilton's formulation [Ham34] of classical mechanics is the birth of symplectic geometry. Maxwell's equation [Max65] is described by differential forms. We need to learn Riemannian geometry for Einstein's equations [Ein15]. Yang-Mills theory [YM54] is constructed based on the theory of vector bundles. The non-perturbative effects in quantum field theories are often formulated in terms of characteristic classes. Supersymmetry and quantum anomaly require the index theorem.

1.2 Disclaimer, textbooks and references

In this course, we will omit proofs of theorems, delegating them to the mathematical literature. We rather learn the meanings of theorems and how to use them. Moreover, we will learn how geometric and topological methods are indispensable in physics through examples and homework sets.

There are many textbooks and you can pick what suits you best. However, I do not recommend you to stick only to one book, and it is often illuminating to compare books since they are written from different perspectives. (Don't try to read through all the books

[1] For a history of algebraic and differential topology, we refer to [Die09].

because it's impossible. What's important is not reading all the books, but understanding the subject.) For basics of differential geometry, one can refer to [ST67, Spi70, KN63, Mor01, War13, BT82b]. There are many books [Arn74, Fra11, NS88, Nak03] that explain the connections to physics. It should be noted that Milnor's books [Mil65, Mil63] would be wonderful-read.

1.3 Localization and toy model of TQFT

The relation between physics and topology has a long history. However, one of the most important steps in the modern interaction between physics and topology has been made by Witten [Wit82b]. Here we provide an essence of [Wit82b].

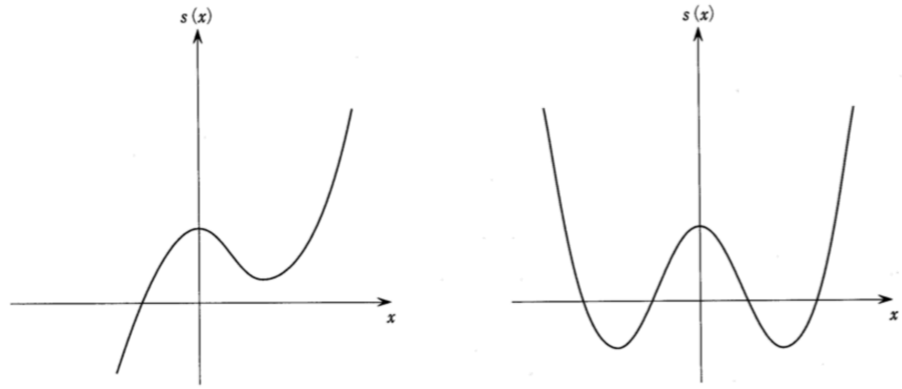


Figure 4: The partition function depends only on the asymptotic behavior of $s(x)$ at $x \to \pm\infty$.

Let us consider the following integral

$$Z = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}s(x)^2\right) \cdot \frac{ds}{dx} . \tag{1.1}$$

Recalling the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} ,$$

if the function $s(x)$ takes the form in the left of Figure 4, then we have

$$Z = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}s(x)^2\right) \cdot \frac{ds}{\sqrt{2\pi}} = 1 .$$

On the other hand, if the function $s(x)$ takes the form in the left of Figure 4, then we have

$$Z = \int_{\infty}^{\infty} \exp\left(-\frac{1}{2}s(x)^2\right) \cdot \frac{ds}{\sqrt{2\pi}} = 0 .$$

Therefore, if $s(x)$ is a polynomial of x

$$s(x) = ax^n + \text{lower orders}$$

then $Z = 1, 0$ for $n =$ odd,even. Importantly, the partition function Z does not depend on the detail of the function $s(x)$, but it depends only on the asymptotic behavior at $\pm\infty$. Thus, the scaling $s(x) \to ts(x)$ does not change Z so that we can consider $t \to \infty$ of

$$Z = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}s(x)^2\right) \cdot \frac{tds}{dx} .$$

Then, the partition function receives contribution only when $s(x) = 0$ so that

$$Z = \sum_{s(x_i)=0} \operatorname{sign}\left(\frac{ds}{dx}\right)\bigg|_{x_i} = \sum_i \pm 1 .$$

Hence, the integral (1.1) localizes at the saddle points x_i where $s(x_i) = 0$.

This can be formulated in terms of topological quantum field theory. To this end, we introduce Grassmann variables ρ, χ which obey

$$\rho\chi = -\chi\rho .$$

The integration rules of Grassmann variables are given by

$$\int d\chi d\rho = \int d\chi d\rho\rho = \int d\chi d\rho\chi = 0 \qquad \int d\chi d\rho\rho\chi = 1$$

so that the partition function (1.1) can be written as

$$Z = \int \frac{dx}{\sqrt{2\pi}} d\chi d\rho \exp\left(-\frac{1}{2}s(x)^2 + \rho\frac{ds}{dx}\chi\right) .$$

The exponent in the integrand can be regarded as a Lagrangian

$$\mathcal{L} = -\frac{1}{2}s(x)^2 + \rho\frac{ds}{dx}\chi .$$

Remarkably, this Lagrangian has BRST symmetry

$$\delta x = \chi, \quad \delta\rho = s(x), \quad \delta\chi = 0 . \tag{1.2}$$

For, we can check

$$\begin{aligned}\delta\mathcal{L} &= -s(x)\delta s(x) + \delta\rho\frac{ds}{dx}\chi + \rho\delta\frac{ds}{dx}\chi \\ &= -s(x)s'(x)\chi + s(x)s'(x)\chi \\ &= 0\end{aligned}$$

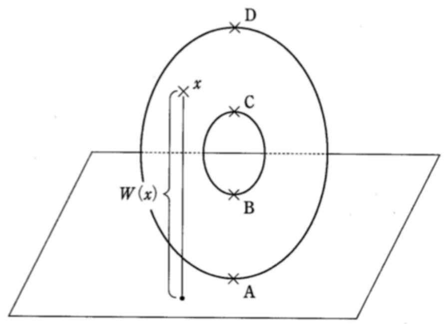
The BRST symmetry has the property that its square becomes zero

$$\begin{aligned}&\delta^2 x = \delta\chi = 0 \\ &\delta^2\rho = \delta s(x) = s'(x)\chi = 0\end{aligned}$$

where we impose the equation of motion $s'(x)\chi$. The saddle points $s(x) = 0$ can be understood as the fixed points of the BRST transformation (1.2) of ρ. In general, a partition function of TQFT localizes the BRST fixed points.

Let us generalize this one-dimensional model to the n-dimensional case.

$$Z = \int \prod_{i=1}^{n} \frac{dx_i}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\partial_i W(x)\right)^2\right) \det \partial_i\partial_j W(x)$$



Again, we can introduce the fermionic degrees ρ_i, χ_i of freedom, and write the partition function as

$$Z = \int \prod_i \frac{dx_i}{\sqrt{2\pi}} d\rho_i d\chi_i \exp(\mathcal{L}) , \qquad \mathcal{L} = -\frac{1}{2}\sum_{i=1}^{n} (\partial_i W(x))^2 + \sum_{i,j=1}^{n} \rho_i \partial_i \partial_j W(x) \chi_i .$$

We again take $W \to tW$ and the partition function is localized at critical point of W as $t \to \infty$

$$\begin{aligned} Z &= \sum_{dW\left(x^{(a)}\right)=0} \operatorname{sign}\left(\det \partial_i \partial_j W\right)\big|_{x^{(a)}} \\ &= \sum_a (-1)^{n_-^{(a)}} = \chi(M) \end{aligned} \tag{1.3}$$

Here the matrix $\partial_i \partial_j W|_{x^{(a)}}$ is called the Hessian of W at $x^{(a)}$, and we denote the number of its positive and negative eigenvalues by $n_+^{(a)}$ and $n_-^{(a)}$ respectively. In fact, W is called a **Morse function** $W : M \to \mathbb{R}$ of M if all the critical points are non-degenerate. There is the classic [Mil63] that wonderfully explains the relation between Morse theory and topology. The last equality follows from the Morse fundamental theorem, and the partition function is given by the Euler characteristics of M. The partition function is independent of a choice of Morse functions W and it only depends on the topology of M. Therefore, it is called a **topological quantum field theory**.

2 Manifolds

2.1 Manifolds

The modern concept of **manifolds** has been first introduced by Riemann [Rie54] in his inaugural lecture at Göttingen University where he defines a manifold by gluing local patches. Furthermore, he has introduced a Riemann metric and curvature on a manifold.

Definition 2.1 (Manifold). Let M be a Hausdorff space (See Definition A.37). M is called an n-dimensional smooth (differentiable) manifold if it has the following structure:

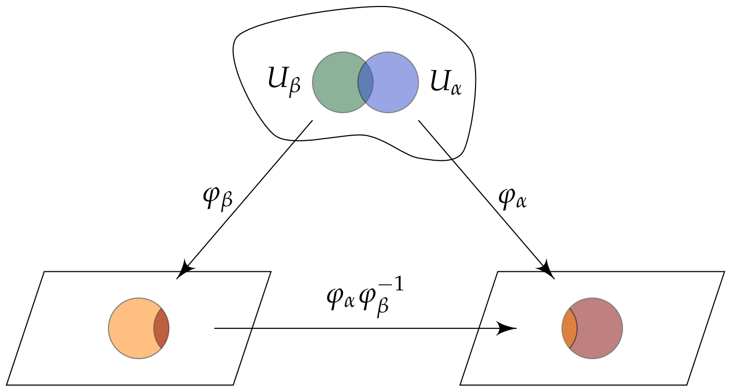
1. Let $M = \bigcup_\alpha U_\alpha$ be an open covering.

2. There is a continuous and invertible map $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, where $\varphi(U_\alpha)$ is open in $\mathbb{R}^n$.

3. For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$, and the transition function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth (C^∞-function).



$(U_\alpha, \varphi_\alpha)$ is called a **coordinate chart** and $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called an **atlas**. We can write

$$\varphi_\alpha = (x^1, \cdots, x^n)$$

where each $x^i : U_\alpha \to \mathbb{R}$. We call these the **local coordinates**.

An important point of the definition of a smooth manifold is the following. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in some atlas, and $f : M \to \mathbb{R}$, then $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(p)$ if and only if $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$.

Example 2.2. Let us consider the n-dimensional sphere

$$S^n = \{u = (u^0, \cdots, u^n) \in \mathbb{R}^{n+1} : |u|^2 = 1\}.$$

We define an atlas as follows. We define an open covering

$$U^+ = S^n \backslash \{\text{south pole}\}, \quad U^- = S^n \backslash \{\text{north pole}\}\ .$$

where the south pole is $u^0 = -1$ and the north pole is $u^0 = 1$, and continuous maps

$$\varphi^+ : U^+ \to \mathbb{R}^n; (u^0, \cdots, u^n) \mapsto \frac{1}{1+u^0}(u^1, \cdots, u^n)$$
$$\varphi^- : U^- \to \mathbb{R}^n; (u^0, \cdots, u^n) \mapsto \frac{1}{1-u^0}(u^1, \cdots, u^n)\ .$$

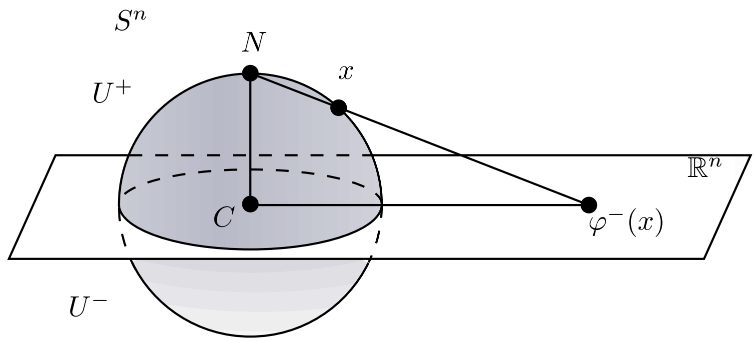
This is called the **stereographic projection**. Then, their inverse maps are

$$(\varphi^\pm)^{-1} : \mathbb{R}^n \to U^\pm; (x^1, \ldots, x^n) \mapsto \frac{1}{1+|x|^2}(\pm(1-|x|^2), 2x^1, \ldots, 2x^n)\ .$$

The transition functions are given by

$$\varphi^+ \circ (\varphi^-)^{-1}(x) = \frac{x}{|x|^2} \quad \varphi^- \circ (\varphi^+)^{-1}(x) = \frac{x}{|x|^2}\ .$$

In fact, the n-dimensional sphere is a compact manifold.



Definition 2.3. Let M be an n-dimensional manifold. A subset $M' \subset M$ is called a **submanifold** if it satisfies the following property: for each $p \in M'$, there exists a local coordinate $(U; x^1, \ldots, x^n)$ such that

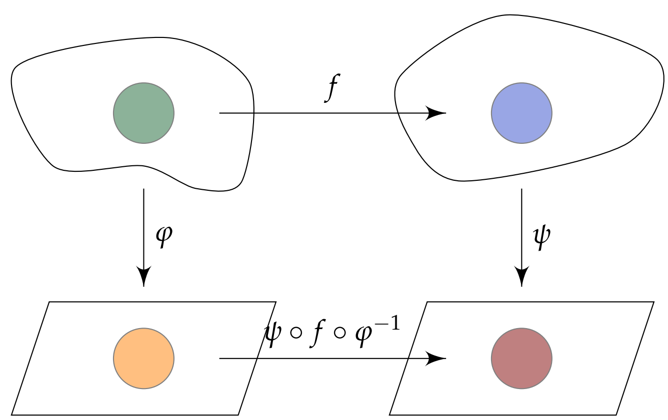
$$M' \bigcap U = \{q \in U | x^{n'+1}(q) = \cdots = x^n(q) = 0\} .$$

In fact, M' is itself an n'-dimensional manifold because we can take a local coordinate $M' \bigcap U; x^1, \ldots, x^{n'})$ around $p \in M'$. Sometimes M' is called a submanifold of **codimension** $n - n'$ in M.

Example 2.4. $S^n \supset S^{n'} = \{x = (x^0, \cdots, x^n) \in \mathbb{R}^{n+1} : |x|^2 = 1\,, \quad x^{n'+1} = \cdots = x^n = 0\}$ is a submanifold of S^n.

Let M and N be smooth manifolds, and let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ be their atlas. We usually consider smooth maps between manifolds.

Definition 2.5 (Smooth map). Let M and N be m- and n-dimensional manifolds, respectively. A map $f : M \to N$ is **smooth** if, for a chart $(U_\alpha, \varphi_\alpha)$ of $p \in M$ and (V_β, ψ_β) of $f(p) \in N$, a map $\psi_\beta \circ f \circ (\varphi_\alpha)^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta)$ is smooth. If it has the smooth inverse (in that case $m = n$), it is called a **diffeomorphism**.



Equivalently, f is smooth at p if $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$ for **any** such charts (U, φ) and (V, ψ).

Example 2.6. Let M and N be smooth manifolds. Then, a projection $M \times N \to N$ is a smooth map.

Example 2.7. A rotation of S^n is a diffeomorphism.

Example 2.8. Let M and N be be smooth manifolds, and $f : M \to N$ be a smooth map. Then, $\Gamma_f = \{(x, f(x)) \in M \times N\}$ is a submanifold of $M \times N$. It is called the **graph** of f.

Manifolds with boundary

In a similar fashion, one can define a manifold with boundary. To this end, we introduce the upper half space $\mathbf{H}^n = \left\{x = \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n; x^n \geq 0\right\}$ and its boundary $\partial \mathbf{H}^n = \left\{x \in \mathbf{H}^m; x^n = 0\right\}$. The definition of a manifold with boundary is given by just replacing $\mathbb{R}^n$ by $\mathbf{H}^n$ in Definition 2.1. For a Hausdorff space M, let $\{U_\alpha\}$ be an open covering of M. There is a homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ from U_α onto an open set $\varphi_\alpha(U_\alpha)$ of $\mathbf{H}^n$. For all α, β,

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \cap U_\beta\right) \to \varphi_\beta \left(U_\alpha \cap U_\beta\right)$$

is a C^∞-function. We denote by ∂M the set of all the points p in U_α that are mapped by φ_α to $\partial \mathbf{H}^n$ for any α. If $\partial M \neq \emptyset$, then M is called a manifold with boundary ∂M. The boundary ∂M itself is an $(n-1)$-dimensional manifold. A compact smooth manifold without boundary is called a **closed** manifold.

Example 2.9. The n-dimensional disk $D^n = \{x \in \mathbb{R}^n | \; |x| < 1$ is a manifold with boundary $\partial D^n = S^{n-1}$.

2.2 Tangent space

Definition 2.10 (Tangent vector). A **tangent vector** X_p at $p \in M$ is a map $X_p : C^\infty(M) \to \mathbb{R}$, which is subject to
(1) **linearity** $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$, for $\alpha, \beta \in \mathbb{R}$
(2) **Leibniz rule** $X_p(fg) = f(p) X_p(g) + X_p(f) g(p)$

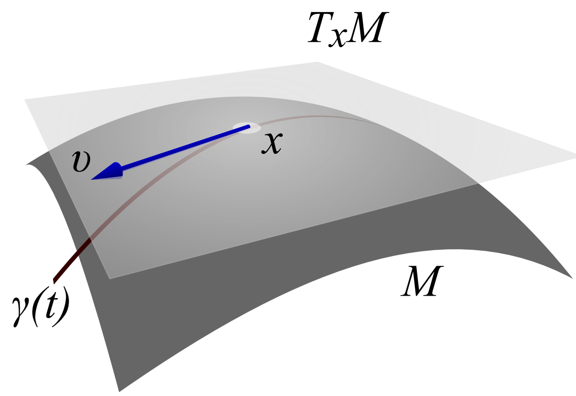
Namely, a **tangent vector** X_p behaves like a first derivative on $C^\infty(M)$. Then, the set of tangent vectors at p become a vector space

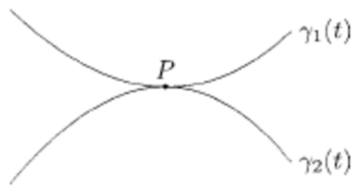
$$(X_p + Y_p)(f) = X_p(f) + Y_p(f) \qquad (\alpha X_p)(f) = \alpha(X_p(f)) \,,$$

and we call it the **tangent space** $T_p M$ at $p \in M$.

There is another way to think about tangent vectors. Let us consider a **curve**, which is a smooth map $\gamma : I \to M$ with $\gamma(0) = p$ where $I = (-1, 1)$ is a non-empty open interval. Two curves γ_1, γ_2 are **tangent** at p if

$$\gamma_1(0) = p = \gamma_2(0) \,, \qquad \frac{d}{dt} \varphi(\gamma_1(t)) \Big|_{t=0} = \frac{d}{dt} \varphi(\gamma_2(t)) \Big|_{t=0}$$





where (U, φ) is a chart around p. We write two tangent curves $\gamma_1 \sim \gamma_2$, which forms an equivalence class (See Definition A.1.). For $^\forall f \in C^\infty(M)$, We can then take the derivative of f along γ

$$X_p(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) \,.$$

It is easy to see that X_p satisfies the definition of a tangent vector. So we can provide the definition of the tangent space at p

$$T_pM = \{\gamma : I \to M | \gamma(0) = p\}/\sim$$

Given a local coordinate $\varphi = (x^1, \cdots, x^n)$, the tangent vector along a curve γ can be written as

$$X_p = \sum_{i=1}^{n} X_p^i \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{where} \quad X_p^i = \frac{d}{dt} x^i(\gamma(t))\Big|_{t=0} \,.$$

Therefore, $(\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p)$ can be considered as a basis of T_pM. Suppose that we also have coordinates $y^1, \cdots, y^n$ near p given by some other chart. Then, we can write

$$\left.\frac{\partial}{\partial y^i}\right|_p = \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^i}(p) \left.\frac{\partial}{\partial x^j}\right|_p \,,$$

where $\frac{\partial x^j}{\partial y^i}(p)$ is called the **Jacobian** at p.

2.3 Tangent bundles

Let us consider a collection of tangent spaces over every point p on M

$$TM = \bigcup_{p \in M} T_pM = \{(p, X_p) | p \in M \,, X_p \in T_pM\} \,.$$

into a manifold. There is then a natural map $\pi : TM \to M$ sending $X_p \in T_pM$ to p for each $p \in M$, and this is smooth.

We can consider TM as a manifold of dimension $2\dim M$, which is called the **tangent bundle** of M. Let $x^1, \cdots, x^n$ be coordinates on a chart (U, φ). Then for any $p \in U$ and $X_p \in T_pM$, there are some $\alpha^1, \cdots, \alpha^n \in \mathbb{R}$ such that

$$X_p = \sum_{i=1}^n \alpha^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

This gives a bijection

$$\begin{aligned} \widetilde{\varphi} : \pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^n \\ X_p &\mapsto (x^1(p), \cdots, x^n(p), \alpha^1, \cdots, \alpha^n), \end{aligned} \tag{2.1}$$

If (V, ψ) is another chart on M with coordinates $y^1, \cdots, y^n$, then

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_p .$$

So we have $\widetilde{\psi} \circ \widetilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ given by

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(x^1, \cdots, x^n, \alpha^1, \cdots, \alpha^n) = \left(y^1, \cdots, y^n, \sum_{i=1}^n \alpha^i \frac{\partial y^1}{\partial x^i}, \cdots, \sum_{i=1}^n \alpha^i \frac{\partial y^n}{\partial x^i} \right),$$

and is smooth (and in fact fiberwise linear).

2.4 Vector fields

A smooth map $X : M \to TM$ is called a **section** of $\pi : TM \to M$ if $\pi \cdot X = id_M$:

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \arrow[u, bend left, "X"] \end{tikzcd}$$

Since it is actually a smooth assignment $X : p \mapsto X(p)$, it is called a **vector field**.

Example 2.11. Let $S^n \subset \mathbb{R}^{n+1}$ be an n-sphere. We have a vector field

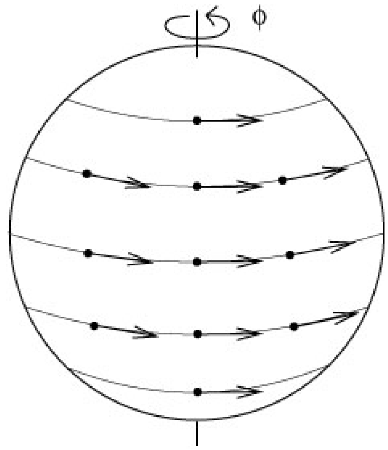
$$Y = \sum Y^i \frac{\partial}{\partial x^i}$$

where

$$Y^i = \begin{cases} (-x^1, x^0, -x^3, x^2, \cdots, -x^{2k+1}, x^{2k}) & n = 2k+1 \\ (-x^1, x^0, -x^3, x^2, \cdots, -x^{2k+1}, x^{2k}, 0) & n = 2k+2 \end{cases} .$$

We write a set of vector fields by $\mathfrak{X}(M) = \Gamma(TM)$. In fact, a vector field $X \in \mathfrak{X}(M)$ is a map $C^\infty(M) \to C^\infty(M)$, which satisfies

$$\begin{aligned} &X(\alpha f + \beta g) = \alpha X(f) + \beta X(g), \quad \text{for} \quad \alpha, \beta \in \mathbb{R} \\ &X(fg) = fX(g) + X(f)g . \end{aligned} \tag{2.2}$$



Let $X, Y \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$. Then we have $X+Y, fX \in \mathfrak{X}(M)$. Therefore, $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

Note that the product of two vector fields $X, Y \in \mathfrak{X}(M)$ is not a vector field:

$$\begin{aligned} XY(fg) &= X(Y(fg)) \\ &= X(fY(g) + gY(f)) \\ &= X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f). \end{aligned} \tag{2.3}$$

However, that $XY - YX$ is a vector field and we denote it as $[X, Y]$, called the **Lie bracket**. In fact, $\mathfrak{X}(M)$ with Lie bracket satisfies

1. $[\cdot, \cdot]$ is bilinear.
2. $[\cdot, \cdot]$ is antisymmetric, i.e. $[X, Y] = -[Y, X]$.
3. The **Jacobi identity** holds

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

2.5 Flows

An **integral curve** of a vector field $X \in \mathfrak{X}(M)$ is a smooth $\gamma : I \to M$ such that I is an open interval in $\mathbb{R}$ and

$$\dot{\gamma}(t) = X_{\gamma(t)}. \tag{2.4}$$

Example 2.12. A vector field $X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ in $\mathbb{R}^2$. Then, the integral curve is a translation

$$\gamma_t(x, y) = (x + t\alpha, y + t\beta) .$$

Suppose that the vector field can be written in terms of a local coordinate $(x^1, \ldots, x^N)$ on an open neighborhood U around $p \in M$

$$X = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i} .$$

Then, (2.4) can be expressed as

$$\frac{dx^i}{dt} = \alpha^i(x^1(t), \ldots, x^n(t)) .$$

These are merely differential equations and $\gamma(t)$ is their integral curve. Moreover, there is a theorem stating that there exists a unique solution $\gamma : (-\epsilon, \epsilon) \to M$ for a sufficiently small $\epsilon > 0$ such that $\gamma(0) = p$.

Theorem 2.13 (Existence of integral curves). Let $X \in \mathfrak{X}(M)$ and $p \in M$. Then there exists some open interval $I \subseteq \mathbb{R}$ with $0 \in I$ and an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.
Moreover, if $\widetilde{\gamma} : \widetilde{I} \to M$ is another integral curve for X, and $\widetilde{\gamma}(0) = p$, then $\widetilde{\gamma} = \gamma$ on $I \cap \widetilde{I}$.

If $\gamma(t)$ is defined for any $t \in \mathbb{R}$ on M, then X is called **complete**. If M be a compact smooth manifold, any vector field is complete. In this situation, a smooth map

$$\gamma : \mathbb{R} \times M \to M; (t, p) \mapsto \gamma_t(p)$$

is called **flow** (or one-parameter group of diffeomorphisms) generated by a vector X which satisfies

$$\gamma_{t+s}(p) = \gamma_t(\gamma_s(p)) \ , \qquad \frac{d\gamma_t(p)}{dt} = X_{\gamma_t(p)} \ .$$

2.6 Orientation

Suppose that we pick two ordered bases $(e_1, \cdots, e_n)$ and $(\widetilde{e}_1, \cdots, \widetilde{e}_n)$ of T_pM. Then, we define an equivalence class $(e_1, \cdots, e_n) \sim (\widetilde{e}_1, \cdots, \widetilde{e}_n)$ iff

$$e_i = \sum_j B_{ij} \widetilde{e}_j \qquad \det B > 0 \ .$$

We call the two ordered bases have the **same orientation** if they are the same equivalence class. Therefore, for $p \in M$, we can assign an orientation $\mathcal{O}_p$. If we can give continuous assignment $p \mapsto \mathcal{O}_p$, then M is called **orientable**.

A smooth manifold M is orientable iff there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of M such that the determinant of the Jacobian is positive on any $U_\alpha \cap U_\beta$.

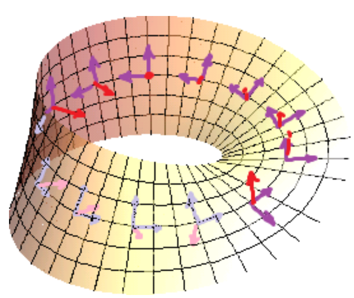


Figure 5: The Möbius strip is not orientable.

3 Differential forms

3.1 Cotangent bundles

Given a vector space V on $\mathbb{R}$, one can take its dual space

$$V^* = \{\omega : V \to \mathbb{R} | \omega(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 \omega(X_1) + \alpha_2 \omega(X_2) \text{ for } X_i \in V \text{ and } \alpha_i \in \mathbb{R}\}$$

The dual space V^* is also a vector space: $\beta_1\omega_1 + \beta_2\omega_2 \in V^*$ for $\omega_1, \omega_2 \in V^*$ and $\beta_1, \beta_2 \in \mathbb{R}$. The dual vector space T_p^*M of the tangent space T_pM is called the **cotangent space**. In fact, given $f \in C^\infty(M)$, we can define its differential df_p at p

$$df_p : T_pM \to \mathbb{R}; X_p \mapsto X_p(f) \ . \tag{3.1}$$

For a local coordinate $(U, \varphi = (x^1, \cdots, x^n))$, we have seen that $(\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p)$ is a basis of T_pM. On the other hand, we can take $(dx^1|_p, \cdots, dx^n|_p)$ as a basis of T_p^*M so that

$$dx^i|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta_i^j \ .$$

Therefore, in this basis, we can write

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i|_p$$

Like the tangent bundle, we can consider a collection of the cotangent spaces

$$T^*M = \cup_{p\in M} T_p^*M \tag{3.2}$$

which has a manifold structure. We call T^*M the **cotangent bundle** of M. Moreover, the section of the cotangent bundle is called one-form, and we denote the set of one-form by $\Omega^1(M) = \Gamma(T^*M)$. For example, if f is a smooth function on M, then $df \in \Omega^1(M)$, which can take a pairing with ${}^\forall X \in \mathfrak{X}(M)$

$$df(X) = X(f) \ .$$

Push-forward and pull-back

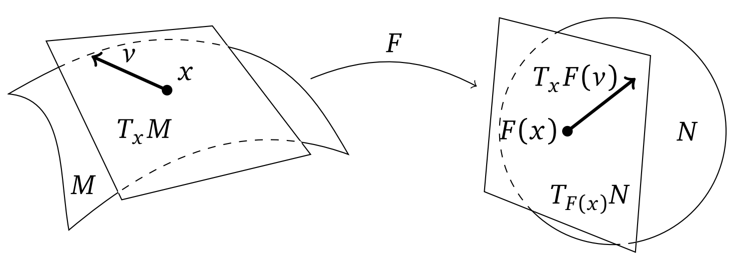
Let $f : M \to N$ be a smooth map between smooth manifolds M and N. It induces a **push-forward** of tangent vectors

$$f_* : T_pM \to T_{f(p)}N$$

which is defined by

$$f_*(X_p)(g) = X_p(g \circ f)$$

for $g \in C^\infty(N)$. As in 3.1, f_* is often denoted by df_p in the literature.



On the other hand, it induces a **pull-back** of the cotangent space

$$f^* : T_{f(p)}^*N \to T_p^*M$$

which is defined by

$$\langle f^*(\omega_{f(p)}), X_p\rangle = \langle \omega_{f(p)}, f_*(X_p)\rangle$$

for $\omega_{f(p)} \in T_{f(p)}N$ where $\langle \cdot, \cdot \rangle$ is a natural pairing between the tangent and cotangent space.

If $f_* : T_pM \to T_{f(p)}N$ is surjection, *i.e.* $\mathrm{rank}(f_*) = \dim N$, then f is **regular** at the point $p \in M$. Otherwise, $p \in M$ is called a **critical point** of f and $f(p) \in N$ is called the **critical value** of f.

Example 3.1. Let $f : S^n \to \mathbb{R}$ be a map defined by $f(x^0, \ldots, x^n) = x^n$ where $S^n = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} | \, |x| = 1\}$. Then, the north and south pole $x^n = \pm 1$ are critical points of f and a generic point is regular.

Theorem 3.2 (Sard). The set of critical values of a smooth $f : M \to N$ has Lebesgue measure zero.

A proof is given in [Mil65].

Definition 3.3. Let M, N be smooth manifolds and $f : M \to N$ be a smooth map. If $f_* : T_pM \to T_{f(p)}N$ is injective for $^\forall p \in M$, f is called an **immersion**. If an immersion $f : M \to N$ is a homeomorphism onto $f(M) \subset M$, then it is called an **embedding**.

Example 3.4. Let us define a map $\gamma : \mathbb{R} \to S^1 \times S^1$ by $t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$. Since its derivative is given by $\frac{d\gamma}{dt} = (2\pi i e^{2\pi i t}, 2\pi i \alpha e^{2\pi i \alpha t}) \neq 0$, γ is an immersion. When α is irrational, γ is one-to-one, but not embedding. Although $\mathbb{Z}$ are isolated in $\mathbb{R}$, $\gamma(\mathbb{Z})$ is not isolated in $\gamma(\mathbb{R})$.

Definition 3.5. If f_* is regular for $^\forall p \in M$, f is called a **submersion**. For $q \in f(M)$, $f^{-1}(q)$ is a submanifold of codimension $\dim N$ in M.

Example 3.6. The projection $f : M \times N \to M$ is a submersion.

3.2 Differential forms

It is natural to consider an algebra generated by the basis of T_p^*M over $\mathbb{R}$ with unit 1 that satisfies

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

Here $\wedge$, called the **wedge product**, can be understood as a multiplication of this algebra, and we call $\wedge$ the **exterior algebra**. For an n-dimensional manifold, we have a direct sum decomposition

$$\wedge^\bullet T_p^*M = \bigoplus_{k=0}^{n} \wedge^k T_p^*M\,.$$

An element $\omega \in \wedge^k T_p^*M$ defines an alternating k-linear map

$$T_pM \times \cdots \times T_pM \to \mathbb{R} \tag{3.3}$$

with

$$\omega(X_{\sigma(1)} \cdots X_{\sigma(k)}) = \mathrm{sign}(\sigma)\, \omega\,(X_1, \cdots, X_k) \quad (X_i \in V)$$

for $\sigma \in S_k$. Moreover, for an element $\omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k$, we define

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k(X_1, X_2, \cdots, X_k) = \frac{1}{k!} \det(\omega_a(X_b)) \ .$$

More generally, if α is a k-form and β is an ℓ-form,

$$(\omega \wedge \eta)(X_1, \cdots, X_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sign}(\sigma)\, \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) \ .$$

Like (3.2), we can consider a family of the vector spaces over the manifold M

$$\wedge^k T^*M = \bigcup_p \wedge^k T_p^* M \ ,$$

which satisfies Definition 2.1 of a manifold. Given two local coordinates $(U; x^1, \ldots, x^n)$ and $(V; y^1, \ldots, y^n)$, the transformation is given by

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{j,<\cdots<j_k} \frac{D\left(x^{i_1}, \cdots, x^{i_k}\right)}{D\left(y^{j_1}, \cdots, y^{j_k}\right)} dy^{j_1} \wedge \cdots \wedge dy^{j_k}$$

where $\frac{D\left(x^{i_1}, \cdots, x^{i_k}\right)}{D\left(y^{j_1}, \cdots, y^{j_k}\right)}$ denotes the Jacobian. Moreover, we write the set of all sections as

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M) \ .$$

In particular, we have $\Omega^0(M) = C^\infty(M)$. An element of $\Omega^k(M)$ is known as a **differential k-form**. As a result, a k-form is expressed as

$$\begin{aligned} \omega &= \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}\left(x^1, \cdots, x^n\right) dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_1 < \cdots < i_k} g_{i_1 \cdots i_k}\left(y^1, \cdots, y^n\right) dy^{i_2} \wedge \cdots \wedge dy^{i_k} \end{aligned} \tag{3.4}$$

on the intersection of two local charts $(U; x^1, \ldots, x^n)$ and $(V; y^1, \ldots, y^n)$, and f and g are related by the Jacobian. Putting all p together in (3.3), an element $\omega \in \Omega^k(M)$ defines a alternating k-linear map

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M) \ .$$

Moreover, there exists a unique linear map called **exterior derivative**

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \ ,$$

such that

1. On $\Omega^0(M)$ this is as previously defined, i.e.

$$df(X) = X(f) \text{ for all } X \in \mathfrak{X}(M).$$

2. We have

$$d \circ d = 0 : \Omega^k(M) \to \Omega^{k+2}(M).$$

3. It satisfies the **Leibniz rule**

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

For $\omega \in \Omega^k(M)$, the exterior derivative can be defined as

$$\begin{aligned}(d\omega)\left(X_1, \cdots, X_{k+1}\right) = \frac{1}{k+1}\Bigg\{ &\sum_{i=1}^{k+1}(-1)^{i+1} X_i \left(\omega\left(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1}\right)\right) \\ &+ \sum_{i<j}(-1)^{i+j}\omega\left(\left[X_i, X_j\right], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{k+1}\right)\Bigg\}, \end{aligned} \tag{3.5}$$

where $^\forall X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$ and $\widehat{}$ means omitting.

In term of local coordinates $x^1, \cdots, x^n$, we can define the exterior derivative as

$$d\left(\sum_{i_1<\ldots<i_k} \omega_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum d\omega_{i_1,\ldots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Let $f : M \to N$ be a smooth map between smooth manifolds M and N. This induces an algebra homomorphism

$$f^* : \Omega^\bullet(N) \to \Omega^\bullet(M).$$

The pull-back of differential forms associated to f can be defined as

$$(f^*\omega)(X_1, \cdots, X_k) = \omega(f_*(X_1), \cdots, f_*(X_k)).$$

for $\omega \in \Omega^k(N)$ and $X_1, \cdots, X_k \in \mathfrak{X}(M)$. Note that the pull-back f^* has the following property

1. $f^* : \Omega^k(N) \to \Omega^k(M)$ is a linear map.
2. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.
3. If $g : N \to L$ is a smooth map between two manifolds N and L, then $(g \circ f)^* = f^* \circ g^*$.
4. It commutes with exterior derivative: $df^* = f^*d$.

Let us introduce some operations on differential forms. For $X \in \mathfrak{X}(M)$, a linear map

$$i(X) : \Omega^k(M) \to \Omega^{k-1}(M)$$

is defined by

$$\left(i(X)\omega\right)\left(X_1, \cdots, X_{k-1}\right) = k\omega\left(X, X_1, \cdots, X_{k-1}\right) \tag{3.6}$$

for $\omega \in \Omega^k(M), X_1, \cdots, X_{k-1} \in \mathfrak{X}(M)$. Note that if $k = 0$, we define $i(X) = 0$. We call $i(X)\omega$ the **interior product** of ω by X. By definition, $i(X)$ is obviously linear with respect to functions. Next, we shall define a linear operator

$$L_X : \Omega^k(M) \longrightarrow \Omega^k(M)$$

called the **Lie derivative**, also involving the vector field $X \in \mathfrak{X}(M)$. This is defined by

$$\left(L_X\omega\right)\left(X_1, \cdots, X_k\right) \quad = X\omega\left(X_1, \cdots, X_k\right) - \sum_{i=1}^{k}\omega\left(X_1, \cdots, \left[X, X_i\right], \cdots, X_k\right). \tag{3.7}$$

The interior products and Lie derivatives provide many identities. We refer to [Mor01, §2.2] for them.

3.3 Integrals of differential forms

Let M be n-dimensional orientable smooth manifold and $\omega \in \Omega^n(M)$. We shall define the integral of ω over M. To this end, we define a partition of unity.

Definition 3.7. A cover $\{U_\alpha\}_{\alpha\in A}$ of M is called **locally finite** if for all points $p \in M$, there exists a neighborhood $U_p \ni p$ such that it intersects only finitely many elements of the cover, namely $U_p \bigcap U_\alpha \neq \emptyset$ for only a finite number of $\alpha \in A$. Another cover $\{V_\beta\}_{\beta\in B}$ is called a **refinement** of $\{U_\alpha\}_{\alpha\in A}$ when for $^\forall\beta \in B$, there exists $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$.

Example 3.8. The collection of all subsets of $\mathbb{R}$ of the form $(n, n+2)$ with integer n is a locally finite open covering of $\mathbb{R}$. The collection of all subsets the form $(n+\frac{1}{3}, n+\frac{5}{3})$ with integer n is its refinement.

Let us accept the following theorem:

Theorem 3.9. Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of M. There exists a refinement $\{V_\alpha\}$ of $\{U_\alpha\}_{\alpha\in A}$, which is locally finite.

Definition 3.10 (Partition of unity). Let $\{U_\alpha\}$ be a locally-finite open cover of a manifold M. A **partition of unity** associated to $\{U_\alpha\}$ is a collection $\chi_\alpha \in C^\infty(M,\mathbb{R})$ such that

1. $0 \le \chi_\alpha \le 1$
2. $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$
3. $\sum_\alpha \chi_\alpha = 1$.

If we use local coordinate $(x^1, \cdots, x^n)$ on a chart U_α, we can write

$$\chi_\alpha\omega = f_\alpha(x)x^1 \wedge \cdots \wedge x^n .$$

Therefore, we define its integral

$$\int_M \chi_\alpha\omega = \int \cdots \int f_\alpha(x)x^1 \cdots x^n .$$

Then, we can define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha\omega .$$

One can show that this is independent of the choice of a locally-finite open covering $\{U_\alpha\}$ and a partition of unity on $\{U_\alpha\}$.

Theorem 3.11 (Stokes' theorem). Let M be an oriented manifold with boundary of dimension n. Then if $\omega \in \Omega^{n-1}(M)$ has compact support, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, if M has no boundary, then

$$\int_M d\omega = 0 \,.$$

4 de Rham cohomology

4.1 de Rham cohomology

Given a differential operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \,,$$

$\omega \in \Omega^k(M)$ is called a **closed form** if $d\omega = 0$, and an **exact form** if there exists $(k-1)$-form such that $\omega = d\eta$. Let us denote the set of all closed k-forms on M by $Z^k(M)$ and the set of all exact k-forms by $B^k(M)$.

$$\begin{aligned} Z^k(M) &= \mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M)) \\ B^k(M) &= \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) \end{aligned}$$

The **de Rham cohomology group** of M is defined as the quotient space

$$H^k_{dR}(M) = Z^k(M)/B^k(M) \,, \tag{4.1}$$

and its dimension is called the k-th **Betti number** of M. In other words, the de Rham cohomology group of M is the cohomology group of **de Rham complex** $(\Omega^\bullet, d)$

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0 \,. \tag{4.2}$$

It is an abelian group as a vector space over $\mathbb{R}$.

If $x \in H^k_{dR}(M), y \in H^\ell_{dR}(M)$ are represented by closed forms $\omega \in Z^k(M), \eta \in Z^\ell(M)$ respectively, then we set $x \wedge y = [\omega \wedge \eta] \in H^{k+\ell}_{dR}(M)$. Obviously, we have $y \wedge x = (-1)^{k\ell} x \wedge y$. $(H^*_{dR}(M), \wedge)$ equipped with the product structure is called the **de Rham cohomology ring**.

Lemma 4.1 (Poincaré Lemma). The de Rham cohomology of $\mathbb{R}^n$ is

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

where $H^0_{dR}(\mathbb{R}^n) \cong \mathbb{R}$ are represented by constant functions.

This also holds when M is contractible, namely when one can smoothly shrink M to a point. However, this is not true on a general manifold although it is true on each coordinate chart. The issue is how these charts are patched together globally. The de Rham cohomology is a topological invariant of a manifold as we will see in §7.

4.2 Riemannian metrics

So far, a manifold we have considered is endowed with the notions of derivatives and integrals. We need to introduce a metric for the concept of length.

Definition 4.2 (Riemannian metric). A Riemannian metric g on a smooth manifold M is a bilinear form $g : T_pM \times T_pM \to \mathbb{R}$ for any $p \in M$ with the following properties
1. (symmetric) $g(X_p, Y_p) = g(Y_p, X_p)$ for all $X_p, Y_p \in T_pM$
2. (positive-definite) $g(X_p, X_p) \geq 0$ for all $X_p \in T_pM$
3. (non-degenerate) $g(X_p, X_p) = 0$ if and only if $X_p = 0$.
Furthermore, g is smooth in the sense that for any smooth vector fields X and Y, the function $x \mapsto g_p\left(X_p, Y_p\right)$ is smooth.

A pair (M, g) with a Riemannian metric g on M is called a Riemannian manifold. In physics, a metric is often not positive-definite, and the number of negative eigenvalues of a metric is called **signature**. In a chart $(U; x^1 \cdots, x^n)$, it can be written locally as

$$ds^2 = g_{ij}(x) dx^i \otimes dx^j \ .$$

On an arbitrary smooth manifold, one can show there exists a Riemannian metric by using a partition of unity. On a Riemannian manifold, one can introduce the notion of the length of a tangent vector $X_p \in T_pM$

$$|X_p| = \sqrt{g(X_p, X_p)} \ .$$

For a curve $\gamma : [a, b] \to M$, the length $L(\gamma)$ of a curve can be defined by

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt = \int_a^b dt \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \ .$$

Therefore, very roughly speaking, Riemannian geometry studies a geometry in which the "length" of a one-dimensional curve γ is defined.

For a smooth map $f : M \to N$, the metric g on a smooth manifold N can be pull-back to the metric f^*g on a smooth manifold M in such a way that for $X_p, Y_p \in T_pM$

$$f^*g(X_p, Y_p) = g(f_*X_p, f_*Y_p) \ .$$

Definition 4.3 (Isometry). Let (M, g) and (N, h) be Riemannian manifolds. We say $f : M \to N$ is an *isometry* if it is a diffeomorphism and $f^*h = g$. In other words, for any $p \in M$ and $X_p, Y_p \in T_pM$, we need

$$h\big(f_*X_p, f_*Y_p\big) = g(X_p, Y_p).$$

In fact, given isometries f_1, f_2, its product $f_1 \circ f_2$ is also an isometry so that the set of isometries forms a group, called the **isometry group**.

Example 4.4. Any reflection, translation and rotation is a global isometry of $\mathbb{R}^n$. The isometry group $E(n)$ of $\mathbb{R}^n$ is the Euclidian group, and therefore it has as subgroups the translational group $T(n)$, and the orthogonal group $O(n)$. Any element of $E(n)$ is a translation followed by an orthogonal transformation (the linear part of the isometry), in a unique way:

$$x \mapsto A(x + b)$$

where $A \in O(n)$ is an orthogonal matrix.

A vector field X on a Riemannian metric (M, g) is called a **Killing vector field** if its flow becomes an isometry $\varphi_t^* g = g$. Or its Lie derivative of the metric g vanishes

$$L_X g = 0 \,.$$

If we write the metric $ds^2 = g_{ij} dx^i dx^j$ and the Killing vector field $X = X^k \partial_k$ in terms of a local coordinate $(U; x^1, \ldots, x^n)$, the Killing equation can be written as

$$X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ik} \partial_j X^k = 0 \,.$$

Using a metric g, one have an isomorphism between the vector field and one form

$$\hat{g} : \mathfrak{X}(M) \cong \Omega^1(M)$$

in such a way that for vector fields $X, Y \in \mathfrak{X}(M)$ on M,

$$\hat{g}(X)(Y) = g(X, Y)$$

By using the isomorphism $T_p M \cong T_p^* M$, we can introduce an inner product on $T_p^* M$.

Definition 4.5 (gradient vector field). Let $f : M \to \mathbb{R}$ be a smooth function. The dual of the one-form df under this isomorphism is called the **gradient vector field** of f denoted by $\operatorname{grad} f$ (or ∇f)

$$g(\operatorname{grad} f, X) = df(X) = Xf \,,$$

for $^\forall X \in \mathfrak{X}(M)$. Given a local coordinate $(U; x^1, \ldots, x^n)$, it is expressed as

$$\operatorname{grad} f = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \,.$$

For simplicity, we will assume that the metric g_{ij} is positive definite. For a metric with more general signature, we just have to introduce appropriate sign factors to some of the formulae below. Since the metric g_{ij} is symmetric, we can find a basis $\{e_i^a\}$ $(a = 1, \cdots, n)$ at $p \in M$ so that

$$g_{ij} = \sum_{a=1}^{n} e_i^a e_j^a \,.$$

This basis is called an **orthonormal frame** at $p \in M$. In fact, the orthonormal frame can be found on a local chart $(U; x^1, \ldots, x^n)$ by the standard Gram-Schmidt process. For a given metric, an orthonormal frame is defined modulo $O(n)$. The frame e^a's are called **vielbeins** where viel means many in German, and bein is a leg. (In 4 dimensions, they are also called vierbeins or tetrads. In dimensions other than 4, words like fünfbein, etc. have been used. Vielbein covers all dimensions.)

4.3 Hodge theorem and Hodge decomposition

On an oriented compact Riemannian manifold (M, g), we can find the one distinguished differential form among the set of all closed forms representing a de Rham cohomology class. Such a form is called a **harmonic form**, and it can be characterized by using a differential operator called the **Laplacian**. This is the theory due to Hodge. For the details, the reader is referred to [War13].

In this subsection, we assume M is an oriented compact Riemannian manifold. Given a metric g, there is an isomorphism

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

defined by

$$*(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n \ .$$

This is called **Hodge $*$-operator**. In particular, $*1 \in \Omega^n(M)$ is called the **volume form**

$$\mathrm{vol} = *1 = e^1 \wedge e^2 \wedge \cdots \wedge e^n \ .$$

On a local coordinate $(U; x^1, \ldots, x^n)$, we can express the volume form as

$$\mathrm{vol} = \sqrt{g} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \ ,$$

where $g = \det g_{ij}$ (we are assuming that the metric is positive definite).

For a k-form ω, it is defined as

$$(*\omega)_{i_{k+1}\cdots i_n} = \frac{1}{k!} \frac{\epsilon^{j_1\cdots j_k j_{k+1}\cdots j_n}}{\sqrt{g}} \omega_{j_1\cdots j_k} g_{j_{k+1}i_{k+1}} \cdots g_{j_n i_n} \ ,$$

where the anti-symmetric tensor $\epsilon_{i_1\cdots i_n}$ and $\epsilon^{i_1\cdots i_n}$ is normalized as

$$\epsilon_{12\cdots n} = \epsilon^{12\cdots n} = 1 \ .$$

The important point is that the Hodge star depends on the metric g. Under coordinate transformations, $\epsilon_{i_1\cdots i_n}$ does not transform as a tensor. However, we can remedy this by multiplying $\sqrt{g}$ to make it into the volume form. The volume form transforms as a tensor if coordinate transformations preserve the orientation. If we change the orientation, we get an extra (-1).

The adjoint operator δ on Ω^k of the exterior differential d is defined by

$$\delta\omega = (-1)^{nk+n+1} * d * \omega \ .$$

We have commutative diagram

$$\begin{array}{ccc}
\Omega^k(M) & \xrightarrow{\ *\ } & \Omega^{n-k}(M) \\
\Big\downarrow \delta & & \Big\downarrow d \\
\Omega^{k-1}(M) & \xrightarrow{(-1)^k *} & \Omega^{n-k+1}(M)
\end{array}$$

We can easily verify the following properties,

$$\delta^2 = 0 \ , \quad *\delta d = d\delta * \ , \quad d * \delta = \delta * d = 0 \ .$$

For a closed manifold M, one can define the positive definite inner product on $\Omega^k(M)$ by

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega$$

for $\omega, \eta \in \Omega^k(M)$. The operator δ is an adjoint operator of d in the sense that

$$(d\omega, \eta) = (\omega, \delta\eta) \ .$$

On a Riemannian manifold M, an operator defined by

$$\Delta = \delta d + d\delta : \Omega^k(M) \to \Omega^k(M)$$

is called **Laplace-Beltrami operator**. A form $\omega \in \Omega^*(M)$ such that $\Delta\omega = 0$ is called a **harmonic form**. In particular, a function f such that $\Delta f = 0$ is called a **harmonic function**. The Laplace-Beltrami operator is self-adjoint in the sense that

$$(\Delta\omega, \eta) = (\omega, \Delta\eta)$$

for $^\forall\omega, \eta \in \Omega^k(M)$. In addition, one can show that a necessary and sufficient condition of harmonic form:

$$\Delta\omega = 0 \quad \text{iff} \quad d\omega = 0 = \delta\omega\ . \tag{4.3}$$

Let us denote the set of all harmonic k-forms on M

$$\mathbb{H}^k(M) = \{\omega \in \Omega^k(M) | \Delta\omega = 0\}\ .$$

Theorem 4.6 (Hodge decomposition). For an oriented compact Riemannian manifold, we have the orthogonal decomposition

$$\Omega^k(M) = \mathbb{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M)\ .$$

Theorem 4.7 (Hodge theorem). On an oriented compact Riemannian manifold, the natural map $\mathbb{H}^k(M) \to H^k_{dR}(M)$ is an isomorphism.

In fact, a proof of Hodge theorem requires advanced techniques in analysis, which is given in [War13, §6].

The Hodge theorem tells us that for a nontrivial $\omega \in H^k_{dR}(M)$, one can choose a harmonic k-form as a representative. Due to $\Delta* = *\Delta$, $\eta = *\omega$ is a harmonic $(n-k)$-form. Since $\omega \neq 0$, we have

$$\int_M \omega \wedge \eta = ||\omega||^2 \neq 0\ .$$

Theorem 4.8 (Poincaré duality). For a connected, oriented, closed n-dimensional Riemannian manifold, the bilinear map

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R};\ (\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is non-degenerate and hence induces an isomorphism

$$H^{n-k}_{dR}(M) \cong H^k_{dR}(M)^*\ .$$

5 Riemannian geometry

On a Riemann manifold (M, g), one can introduce the concept of curvature of a manifold, and the idea dates back to the Riemann's epoch-making augural speech [Rie54]. Einstein's theory of gravity has been constructed based on these concepts. In this section, we study the basics of Riemannian geometry following [Mil63, Part II]. In this section, we use the Einstein summation convention.

5.1 Covariant derivative and parallel transport

Definition 5.1 (Connection). Let X and Y be vector fields on M. The symbol $\nabla_X Y$ denotes the derivative of the vector field Y along the flow of the vector field X. In fact, it is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M); (X, Y) \mapsto \nabla_X Y$$

which satisfies the following conditions:
1. ∇ is linear in the first variable and additive in the second:

$$\nabla_{fX+hY} Z = f \nabla_X Z + h \nabla_Y Z$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$$

where $f, h \in \mathbb{C}^\infty(M)$ are functions and $X, Y, Z \in \mathfrak{X}(M)$ are vector fields.
2. ∇ obeys the Leibniz rule in the second variable:

$$\nabla_X (fY) = X(f)Y + f \nabla_X Y \, .$$

The operator ∇ is called a **connection**.

Let $(U, \{x^i\})$ be a local coordinate on M. Since $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}$ is a vector field, it can be expressed as a linear combination of the coordinate fields on U:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} := \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

where Γ^i_{jk} are called the **Christoffel symbols**. If we write the vector fields $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ in terms of these basics, then the covariant derivative is expressed as

$$\nabla_x Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + \Gamma^k_{ij} Y^j \right) \frac{\partial}{\partial x^k} \, .$$

Let $\gamma : I \to M$ be a curve on M and we define

$$J(\gamma) = \{\text{vector fields along } \gamma\}.$$

Then, given a connection ∇, we define the **covariant derivative** along γ, which is a map $\frac{D}{dt} : J(\gamma) \to J(\gamma)$ such that

- $\frac{D(X+Y)}{dt} = \frac{DX}{dt} + \frac{DY}{dt}$
- $\frac{D(fX)}{dt} = \frac{df}{dt} X + f \frac{DX}{dt}$ for all $f \in C^\infty(I)$
- If $X \in J(\gamma)$ is induced by a vector field $\tilde{X}$ ($\tilde{X}|_{\gamma(t)} = X(t)$ for all $t \in I$), then

$$\left. \frac{DX}{dt} \right|_{t=0} = \nabla_{\dot{\gamma}(0)} \tilde{X}.$$

In terms of the local coordinate, we can write it as

$$\frac{DX}{dt} = \left(\frac{dX^k}{dt} + \frac{dx^i}{dt} X^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} \, .$$

where $\gamma(t) = (x^1(t), \ldots, x^n(t))$.

Given a path $\gamma(t)$ (not necessarily a geodesic), a vector field X is called a **parallel vector field** along γ if

$$\frac{DX}{dt} = 0 \, .$$

The Christoffel symbols Γ^k_{ij}, the path γ, and the derivatives $\frac{dx^i}{dt}$ are known. Therefore it is a system of n first order linear differential equations in terms of a local coordinate:

$$\frac{dX^k}{dt} + X^j \frac{dx^i}{dt} \Gamma^k_{ij} = 0 \quad \text{for} k = 1, \ldots, n \, .$$

Since this is a set of ordinary differential equations, there exists a unique solution $X(t)$ given an initial condition $X(0) \in T_{\gamma(0)}M$. The vector field $X(t)$ is said to be obtained from $X(0)$ by **parallel transportation** along the curve γ.

Levi-Civita Connection

There are an infinitely many connections on a manifold M. However, given a Riemannian metric g on M, there is the natural choice for the connection, which is called the **Levi-Civita connection**.

Definition 5.2 (Levi-Civita connection). Let (M, g) be a Riemannian manifold. The Levi-Civita connection is the connection ∇ on M satisfying the following conditions:

- Compatibility with metric:
$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$
- Symmetry/torsion-free:
$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{5.1}$$

The condition of the compatibility can be expressed as

$$d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y) \, .$$

The second property can be expressed in a local coordinate by

$$\Gamma^i_{jk} = \Gamma^i_{kj} \, .$$

On a Riemannian manifold (M, g), there exists the **unique** Levi-Civita connection ∇. Using a local coordinate, the metric compatibility condition is

$$\frac{\partial}{\partial x^i} g_{jk} = g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \right) \, ,$$

which leads to

$$\Gamma^k_{ij} = \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) g^{lk}. \tag{5.2}$$

Definition 5.3 (Geodesic). A curve $\gamma(t)$ on a Riemannian manifold (M, g) is called a **geodesic** if its tangent vectors are parallel transported along the curve itself with respect to the Levi-Civita connection:

$$\frac{D\dot{\gamma}}{dt} = 0 \,. \tag{5.3}$$

In terms of a local coordinate, it can be written as

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \text{ for } i = 1, \ldots, n \,.$$

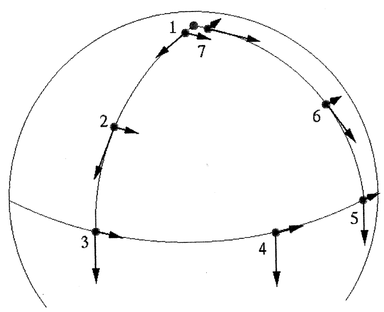
This equation is indeed the Euler-Lagrange equation of the action

$$S = \int dt \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} \,.$$

Given an initial conditions $p = \gamma(0)$ and $v = \dot{\gamma}(0)$, there exists a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ on a sufficiently small open neighborhood $U \ni p$.

5.2 Riemann curvature

We can consider doing some parallel transports on S^n along the loop counterclockwise: We see that after the parallel transport around the loop, we get a different vector.



The Riemann curvature is introduced to measure this difference.

Definition 5.4 (Curvature). Let ∇ is a connection on M. Then, the curvature of ∇ is a map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M); (X, Y, Z) \mapsto R(X, Y)\, Z$$

where $R(X, Y)\, Z$ is defined as follows:

$$R(X, Y)\, Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,, \tag{5.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

The curvature satisfies the following properties:

$$\begin{gathered} R(X, Y)\, Z = -R(Y, X)\, Z \,, \\ R(fX, gY)(hZ) = fghR(X, Y)(Z) \end{gathered} \tag{5.5}$$

for ${}^\forall X, Y, Z \in \mathfrak{X}(M)$ and $f, g, h \in C^\infty(M)$. The Riemann tensor is also called the **curvature tensor**.

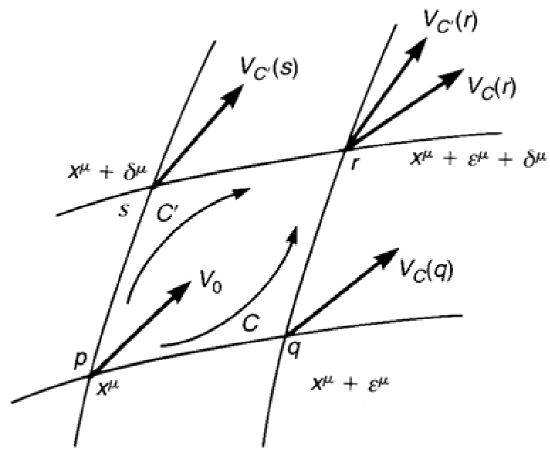
In terms of a local coordinate, the curvature is written as

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l{}_{kij}\frac{\partial}{\partial x^l}.$$

where R is expressed by the Christoffel symbols

$$R^l{}_{kij} = \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js} + \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} \,. \tag{5.6}$$

Roughly speaking, a curvature measures the failure of commutativity of two derivatives when applied to vector fields. Let us take infinitesimal parallelogram as in the



figure, which is parametrized as $(t, s) \mapsto \gamma(t, s)$. Then, the difference between the two vectors is indeed given by the curvature

$$\frac{D}{dt}\frac{D}{ds}X - \frac{D}{ds}\frac{D}{dt}X = R\left(\frac{d\gamma}{dt}, \frac{d\gamma}{ds}\right)X \,.$$

Let (M, g) be a Riemann manifold and ∇ is the Levi-Civita connection. The corresponding curvature is called the **Riemann curvature tensor**, and we assume that the curvature in the following is always the Riemann curvature tensor. If we introduce the notation

$$\begin{aligned} R(X,\, Y,\, Z,\, W) &:= g(R(X, Y)Z,\, W) \\ R(\partial_k,\, \partial_\ell,\, \partial_j,\, \partial_i) = R_{ijk\ell} &:= R^s{}_{jk\ell}\, g_{si} \,, \end{aligned} \tag{5.7}$$

then the Riemann curvature tensor satisfies the following identities:

$$\begin{aligned} &R(X,\, Y,\, Z,\, W) = -R(Y,\, X,\, Z,\, W) \\ &R(X,\, Y,\, Z,\, W) = -R(X,\, Y,\, W,\, Z) \\ &R(X,\, Y,\, Z,\, W) = R(Z,\, W,\, X,\, Y) \\ &R(X,\, Y)Z + R(Y,\, Z)X + R(Z,\, X)Y = 0 \qquad \text{(first Bianchi identity)} \\ &\nabla_W R(X,\, Y)Z + \nabla_X R(Y,\, W)Z + \nabla_Y R(W,\, X)Z = 0 \quad \text{(second Bianchi identity)} \end{aligned} \tag{5.8}$$

In terms of a local coordinate, they can be read off

$$\begin{aligned}
R_{ijkl} &= -R_{ijlk} \\
R_{ijkl} &= -R_{jikl} \\
R_{ijkl} &= R_{klij} \\
R_{lijk} \, + \, R_{ljki} \, + \, R_{lkij} &= 0 \\
R_{lijk;s} \, + \, R_{ljsk;i} \, + \, R_{lsik;j} &= 0 \, .
\end{aligned} \tag{5.9}$$

Contracting the first and final indices of the Riemann curvature tensor gives the **Ricci curvature tensor**

$$R_{ij} = \mathrm{Ric}_{ij} := R^s{}_{isj} = R_{kijl} g^{kj} . \tag{5.10}$$

If there is a constant Λ such that $R_{ij} = \Lambda g_{ij}$, then (M, g) is called an **Einstein manifold**. Contracting the indices again, we get the **scalar curvature**

$$R := \mathrm{Ric}_{ij} g^{ij} . \tag{5.11}$$

5.3 Gauss-Bonnet theorem

Let (M, g) be a two-dimensional oriented closed manifold and let vol be the volume form. Then, the Gauss curvature κ is defined by the Riemann curvature tensor

$$\kappa := \frac{R_{1212}}{\det g} \, .$$

Then, the Gauss-Bonnet theorem is

$$\frac{1}{2\pi} \int_M \kappa \ \mathrm{vol} = \chi(M) \, . \tag{5.12}$$

It is remarkable in the sense that the RHS is independent of the choice of a metric g although the Gauss curvature κ depends on the metric g. This formula connects differential geometry to topology.

5.4 Einstein equations

The celebrated Einstein equations [Ein15] can be expressed

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \, , \tag{5.13}$$

where Λ is called the cosmological constant, G is the Newton constant, c is the speed of light and $T_{\mu\nu}$ is the stress-energy tensor. These are the Euler-Lagrange equations for the Einstein-Hilbert action

$$S = \int \left[\frac{c^4}{16\pi G} \left(R - 2\Lambda \right) + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g} \, \mathrm{d}^4 x \, .$$

Note that the metric g has a Minkowski signature in physics so that a coordinate transformation can bring $g = \mathrm{diag}(-1, 1, 1, 1)$ at a point. In this equation, a metric becomes a dynamical variable and, roughly speaking, the curvature tensor of the spacetime is determined by the stress-energy tensor of matter. The equation is highly non-linear so that

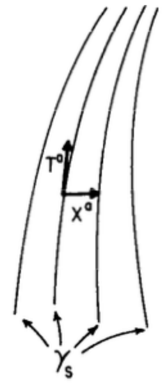


Figure 6: A one-parameter family of geodesics $\gamma(t;s)$, with tangent T^a and deviation vector X^a

it can be solved analytically only in very special situations. Most of the cases require numerical simulations to solve the Einstein equations. Remarkably, this equation describes nature at a terrestrial scale, and predicts the gravitational wave, which was detected by LIGO. It also describes the history of the universe [Wei77, Wei72].

For example, the physical meaning of the Riemann curvature can be seen in the gravitational tidal force. Let us consider a one-parameter family of geodesics $\gamma(t;s)$, so that for each $s = s_0$ =const, it satisfies (5.3). In a local coordinate $(x^1, \ldots, x^n)$, we write tangent vectors as

$$T^i = \frac{\partial x^i(t;s)}{\partial t} , \qquad X^i = \frac{\partial x^i(t;s)}{\partial s} ,$$

respectively. We consider the tangent vector T^i is timelike and X^i is spacelike. Moreover, we parametrize the family in such a way that

$$\frac{d}{ds} g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = \frac{d}{ds}(g_{ij} T^i T^j) = 0 .$$

From the definition of T^i and X^i, it follows that they satisfy the relation $\partial T^i/\partial s = \partial X^i/\partial t$. Then it can be easily shown from (5.1) that

$$X^j \nabla_j T^i = T^j \nabla_j X^i .$$

Then, the relative acceleration of an infinitesimally nearby geodesic in the family is measured by the Riemann curvature tensor (5.6)

$$\begin{aligned}
\frac{D^2 X^i}{dt^2} &= T^k \nabla_k \left(T^j \nabla_j X^i\right) \\
&= T^k \nabla_k \left(X^j \nabla_j T^i\right) \\
&= \left(T^k \nabla_k X^j\right) \left(\nabla_j T^i\right) + X^j T^k \nabla_k \nabla_j T^i \\
&= \left(X^k \nabla_k T^j\right) \left(\nabla_j T^i\right) + X^j T^k \nabla_j \nabla_k T^i + R^i{}_{lkj} X^j T^k T^l \\
&= X^k \nabla_k \left(T^j \nabla_j T^i\right) + R^i{}_{lkj} X^j T^k T^l \\
&= R^i{}_{lkj} X^j T^k T^l .
\end{aligned}$$

This is the gravitational tidal force.

6 Symplectic geometry

In this lecture, we will introduce symplectic geometry and learn the relation to classical mechanics. I highly recommend a great classic [Arn74] on this subject to you, and I guarantee that it is a wonderful-read. The relation between symplectic geometry and classical mechanics dates back to the work of Lagrange and Poisson in 1808 [dL08, dL09, Poi08, Poi09]. Their works are formulated in a clear language, so-called Hamiltonian formalism [Ham34, Ham35, Cau37], and it becomes deeper and deeper, involving dynamical systems, Lie groups, integrable systems and deformation quantizations in the 20th century. In the 1980s, the seminal works by Gromov [Gro85] and Floer [Flo88a, Flo88b, Flo89] open up a new area that studies symplectic manifolds globally, and it gives one aspect of "quantum geometry". Although Gauss-Bonnet theorem in Riemannian geometry and Riemann-Roch theorem in complex geometry preceded long before, we need more sophisticated techniques to study global topology in symplectic geometry simply because symplectic geometry is difficult. Hence, we just open the door to such a profound subject in this lecture.

6.1 Hamiltonian formulation of classical mechanics

Let us review Hamiltonian dynamical systems to connect symplectic geometry. The action of a point particle with mass m move in a potential $V(q)$ is written as

$$S = \int L(q,\dot{q})dt\,, \qquad L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)\,,$$

where $q = (q_1, \ldots q_n)$ is a vector of n-dimensional space. Its Euler-Lagrange equation is described by the Newton equations

$$m\ddot{q}_i = -\frac{\partial U}{\partial q_i}$$

Introducing the momentum $p = (p_1, \ldots, p_n)$, where $p_i = m\dot{q}_i$, the Hamiltonian can be written as

$$H = p\dot{q} - L = \frac{1}{2m}p^2 + V(q)$$

The Hamiltonian is conserved

$$\frac{dH}{dt} = \frac{1}{m}p_i\dot{p}_i + \dot{q}_i\frac{\partial U}{\partial q_i} = \frac{1}{m}m^2\dot{q}_i\ddot{q}_i + \dot{q}_i\frac{\partial U}{\partial q_i} = 0$$

due to the Newton equations of motion. Moreover, the equations of motion can be rewritten in Hamilton's canonical form

$$\dot{q}_j = \frac{\partial H}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}\,. \tag{6.1}$$

These can be understood as the Euler-Lagrange equation of the action

$$S = \int (p \cdot dq - H(q,p)dt)\,.$$

Their importance lies in the fact that they are valid for arbitrary dependence of $H \equiv H(p,q)$ on the dynamical variables p and q. To write it more elegant way, we consider the phase space $\mathbb{R}^{2n}$ with coordinate $x = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ with the gradient vector field grad$X = (\frac{\partial H}{\partial p_j}, \frac{\partial H}{\partial q_j})$ of H. Using $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} ,$$

the Hamilton's canonical equations of motion can be written in the form

$$\dot{x} = J \cdot \nabla H, \quad \text{or} \quad J \cdot \dot{x} = -\nabla H$$

This form of equations were written for the first time by Lagrange in 1808 [dL08]. Furthermore, we introduce the Poisson bracket

$$\{f, g\} = J^{ij} \partial_i f \partial_j g = \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} . \tag{6.2}$$

The Hamiltonian equations can be now rephrased in the form

$$\dot{x} = \{H, x\} . \tag{6.3}$$

This is the birth of symplectic geometry!

6.2 Symplectic manifolds

A symplectic form on a smooth manifold M is non-degenerate closed 2-form ω. "Non-degenerate" means that the mapping $\omega : TM \to T^*M$; $X \mapsto \omega(X, -)$ is an isomorphism. We denote the 1-form $\omega(X, -)$ by $i(X)\omega$.

The couple (M, ω) of a smooth manifold M and a symplectic form ω is called a **symplectic manifold**. Any symplectic manifold is even-dimensional and if M is closed with $\dim(M) = 2n$, ω^n is a volume-form.

Therefore, very roughly speaking, symplectic geometry studies a geometry in which the "area" of a two-dimensional surface Σ is defined. Since $d\omega = 0$, the "area" $\int_\Sigma \omega$ of a closed surface Σ does not change even if Σ is continuously deformed.

Example 6.1. Let $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ be the standard coordinate of $\mathbb{R}^{2n}$. Then,

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

is the symplectic form.

A symplectic manifold (M, ω) is called **exact** if there exists one form θ such that $\omega = d\theta$ where θ is called **Liouville 1-form**. The vector field Z dual to the Liouville 1-form with respect to ω

$$i(Z)\omega = \theta$$

is called **Liouville vector field**.

Example 6.2 (cotangent bundle). The most important example of exact symplectic manifolds is a cotangent bundle $M = T^*N$ of a smooth manifold N. Given a local coordinate $(q_1, \cdots, q_n)$, they induces the coordinate $(p_1, \cdots, p_n)$ on the fiber T_q^*N.

Then, the Liouville 1-form is written in terms of the local coordinate of T^*N

$$\theta = \sum_{i=1}^{n} p_i dq_i$$

so that the symplectic form can be locally written as

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \ .$$

Example 6.3. The complex projective space

$$\mathbb{C}\mathbf{P}^n = \left(\mathbb{C}^{n+1} \backslash \{0\}\right) / \mathbb{C}^\times$$

has a non-denerate 2-form called the **Fubini-Study form**

$$\begin{aligned}\omega_{\mathrm{FS}} =& \sqrt{-1}\partial\bar{\partial} \log\left(|z_1|^2 + \cdots + |z_{n+1}|^2\right) \\ =& \sqrt{-1}\frac{\sum_{i=1}^{n+1} dz_i \wedge d\bar{z}_i - \left(\sum_{i=1}^{n+1} \bar{z}_i \wedge dz_i\right) \wedge \left(\sum_{i=1}^{n+1} z_i \wedge d\bar{z}_i\right)}{\left(|z_1|^2 + \cdots + |z_{n+1}|^2\right)^2}\end{aligned} \tag{6.4}$$

$(\mathbb{C}\mathbf{P}^n, \omega_{\mathrm{FS}})$ is a symplectic manifold, and moreorver it is a Kähler manifold.

Theorem 6.4 (Darboux's theorem). Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(U, q_1, \cdots, q_n, p_1, \cdots, p_n)$ centered at p such that on U

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \ . \tag{6.5}$$

This theorem states that any symplectic manifold is locally equivalent to a Euclidean space with its standard symplectic structure. As a result, the most important questions in symplectic geometry are the global ones.

Symplectomorphisms

The important maps in symplectic topology are the **symplectomorphisms**. A symplectomorphism between two symplectic manifolds (M, ω_M) and (N, ω_N) is a diffeomorphism $\psi : M \to N$ such that

$$\psi^* \omega_N = \omega_M \ .$$

This condition is pretty strong. The necessary condition for the existence of symplectomorphisms is $\dim M \leq \dim N$.

Example 6.5 (symplectic group). Let $(\mathbb{R}^{2n}, \omega)$ be as in Example 6.1. Then, examples of symplectomorphisms are

$$\mathrm{Sp}(n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) \ | g^T J g = J \ , \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\}$$

which is called the **symplectic group**.

Theorem 6.6 (Moser Stability Theorem). Let (M, ω_t) be a closed manifold with a family of cohomologous symplectic forms. Then there is a family of symplectomorphisms $\psi_t : M \to M$ such that

$$\psi_0 = 1 \, , \psi_t^* \omega_t = \omega_0 \, .$$

Moreover, if $\omega_t(q) = \omega_0(q)$ for all points q on a compact submanifold Q of M, we may assume ψ_t is the identity on Q.

This theorem means that one cannot change the symplectic form in any important way by deforming it, provided that the cohomology class is unchanged.

Lagrangian submanifolds

Definition 6.7 (Lagrangian submanifold). A Lagrangian submanifold of a symplectic manifold (M, ω) is a submanifold where the restriction of the symplectic form ω to $L \subset M$ is vanishing, i.e. $\omega|_L = 0$ and $\dim L = 1/2 \cdot \dim M$.

If we perturb a Lagrangian submanifold, then it is no longer Lagrangian in general. Thus, they are "rigid" objects in symplectic geometry. Since Lagrangian submanifolds play a significant role in symplectic geometry, Weinstein advertised the slogan that everything is a Lagrangian submanifold. (A. Weinstein's Lagrangian creed.)

Example 6.8. The zero section N of the cotangent bundle $M = T^*N$ is a Lagrangian submanifold. Let $f : Z \hookrightarrow N$ is an embedding. Then, the conormal bundle to Z in T^*N defined as

$$L_Z := \{(x, \alpha) \in T^*N | x \in Z \, , \quad \alpha(v) = 0 \text{ for all } v \in T_x Z\} \subset T^*N \, ,$$

is a Lagrangian submanifold.

It is known that the neighborhood of a Lagrangian also has the standard symplectic structure as its cotangent bundle.

Theorem 6.9 (Weinstein's tubular neighborhood theorem). Every lagrangian submanifold L in a symplectic manifold (M, ω) has a neighborhood U which is symplectomorphic to the cotangent bundle T^*L.

6.3 Hamiltonian system

Any smooth function $H \in C^\infty(M)$ gives rise to a vector field X_H defined uniquely by the equation

$$i(X_H)\omega = -dH \, ,$$

where $i(X)\omega$ is the interior product in (3.6). This vector field is called the **Hamiltonian vector field** with Hamiltonian H. For $f, g \in C^\infty(M)$, we define the Poisson bracket by

$$\{f, g\} = \omega(X_f, X_g) = -X_f(g) = X_g(f) \, .$$

In the phase space $(\mathbb{R}^{2n}, \omega)$ of Example 6.1, the Poisson bracket of $f, g \in C^\infty(M)$ is expressed as (6.2). ALso, the flow of the Hamiltonian vector field X_H associated to a Hamiltonian H can be described by the Hamilton's canonical equations (6.1) and (6.3).

The Poisson bracket satisfies the following properties

Skew symmetry: $\{f, g\} = -\{g, f\}$.

Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Leibniz's Rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

Using these properties, one can show that

$$[X_f, X_g] = X_{\{f,g\}} \,.$$

For $f \in C^\infty(M)$, the differentiation with respect to the Hamiltonian flow can be expressed by

$$\frac{df}{dt} = \{H, f\} \,.$$

If $\{f, H\} = 0$, the flow φ_s generated by X_f is the symmetry of the Hamiltonian system with H. Namely, we have

$$\frac{dH}{ds} = \{H, f\} = 0 \,,$$

or equivalently,

$$\frac{df}{dt} = \{H, f\} = 0 \,.$$

This is called **Noether theorem**.

Example 6.10. If the system has spherical symmetry, the potential $V(r)$ is independent of θ and ϕ. Then, the momenta p_θ and p_ϕ are conserved.

6.4 Arnold-Liouville theorem

Let us consider Poisson commuting functions (Hamiltonians) $H_1, \cdots, H_k$ with $\{H_i, H_j\} = 0$ for all i, j. To insure that we are not discussing a degenerate situation, we assume that $dH_1 \wedge \cdots \wedge dH_k(x) \neq 0$ for $\forall x \in M$, in which H_i are called **analytically independent**. Then, the maximal number of Poisson commuting functions is a half of dimension of M which is n. A $2n$-dimensional symplectic manifold (M, ω) with n Poisson commuting Hamiltonians is called **completely integrable system**.

Theorem 6.11 (Arnold-Liouville theorem). Let $H_1, \cdots, H_n$ be a completely integrable system on a $2n$-dimensional symplectic manifold (M, ω). Namely, $\pi = (H_1, \cdots, H_n) : M \to \mathbb{R}^n$ are analytically independent Poisson commuting functions with $\{H_i, H_j\} = 0$ for all i, j. Then, a compact connected component of the preimage of a non-singular point of π is a Lagrangian submanifold that is diffeomorphic to a torus T^n.

Let us denote the image of π by B. Then, (compact connected component of) the preimage $L = \pi^{-1}(b)$ of $b \in B$ is diffeomorphic to T^n so that we can take the angle coordinate

$$(\phi_1 \cdots \phi_n) : L \to T^n$$

We can further take local coordinate $(I_1, \cdots, I_n)$ of B that the symplectic form can be locally expressed as

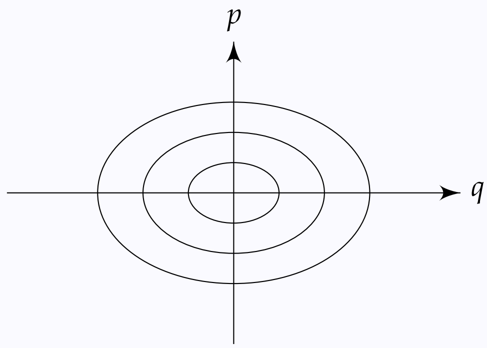
$$\omega = \sum_{i=1}^{n} dI_i \wedge d\phi_i \, ,$$

where (ϕ_i, I_i) are called **angle-action** coordinate.

Example 6.12. Let us consider the harmonic oscillator with Hamiltonian

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \, ,$$

as an example. Since it is a 2-dimensional system, we only need a single integral. Since H is a first integral for trivial reasons, this is an integrable Hamiltonian system.
We can actually draw the lines on which H is constant — they are just ellipses:



We note that the ellipses are each homeomorphic to S^1. Now we introduce the coordinate transformation $(q, p) \mapsto (\phi, I)$, defined by

$$q = \sqrt{\frac{2I}{\omega}} \sin\phi, \quad p = \sqrt{2I\omega} \cos\phi,$$

Hence, ϕ is the angle coordinate that parametrizes S^1 in the Arnold-Liouville theorem. We can manually show that this transformation is canonical. However, it is merely a computation, and we will not waste time doing that. In these new coordinates, the Hamiltonian looks like

$$\tilde{H}(\phi, I) = H(q(\phi, I), p(\phi, I)) = \omega I \, .$$

The Hamiltonian is independent of ϕ! Therefore, the Hamilton equations become

$$\dot{\phi} = -\frac{\partial \tilde{H}}{\partial I} = \omega \, , \qquad \dot{I} = \frac{\partial \tilde{H}}{\partial \phi} = 0 \, ,$$

yielding

$$\phi(t) = \phi_0 - \omega t \, , \quad I(t) = I_0 \, .$$

It is interesting to consider the integral of the Liouville 1-form along a path of constant

H:

$$\begin{aligned}
\frac{1}{2\pi}\oint p\,\mathrm{d}q &= \frac{1}{2\pi}\int_0^{2\pi} p(\phi,I)\left(\frac{\partial q}{\partial\phi}\,\mathrm{d}\phi + \frac{\partial q}{\partial I}\,\mathrm{d}I\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi} p(\phi,I)\left(\frac{\partial q}{\partial\phi}\,\mathrm{d}\phi\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \sqrt{\frac{2I}{\omega}}\sqrt{2I\omega}\cos^2\phi\,\mathrm{d}\phi\\
&= I\,.
\end{aligned}$$

We can always perform the integral of the Liouville 1-form along paths of constant H without knowing anything about I and ϕ, which magically gives us the new coordinate I. This is the reason why it is called **integrable system**.

7 Homology and cohomology groups

Next, we will learn the basics of **algebraic topology**, which studies the topology of a manifold by groups and algebras. The fundamental idea was introduced by Poincaré in the seminal paper [Poi95] where he has conveyed the concept of homology groups and fundamental groups. His idea has been put on a mathematical rigorous footing in the early twentieth century. Let us begin with homology groups.

7.1 Simplicial homology

Algebraic information can be extracted from a manifold via **complexes**. Modern algebraic topology is established based on **singular complexes**. Since the technical apparatus of singular homology is somewhat complicated, we will use a more primitive version called **simplicial homology**, which is friendly to intuitive understanding. For singular complexes, one can refer to the comprehensive textbook [Hat05].

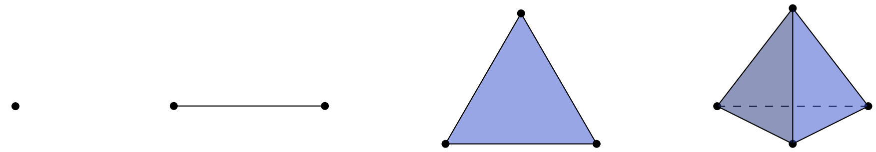
The relevance is that these can be used to define simplices (which are simple, as opposed to complexes).

Definition 7.1 (q-simplex). A **q-simplex** is the convex hull of $(q+1)$ affinely independent (or general position) points $a_0,\cdots,a_q\in\mathbb{R}^m$, i.e. the set

$$\sigma = \langle a_0,\cdots,a_q\rangle = \left\{\sum_{i=0} t_i a_i : \sum_{i=0}^{q} t_i = 1, t_i\geq 0\right\}.$$

The points $a_0,\cdots,a_q$ are the **vertices**, and are said to **span** σ. The $(q+1)$ tuples $(t_0,\cdots,t_q)$ are called the **barycentric coordinates** for the point $\sum t_i a_i$.
We often denote an oriented simplex as σ, and then $\bar{\sigma}$ denotes the same simplex with the opposite orientation.

Some examples are as follows. When $q=0$, it is a point. $q=1$, it is a line.



$q = 0$ $\quad$ $q = 1$ $\quad$ $q = 2$ $\quad$ $q = 3$

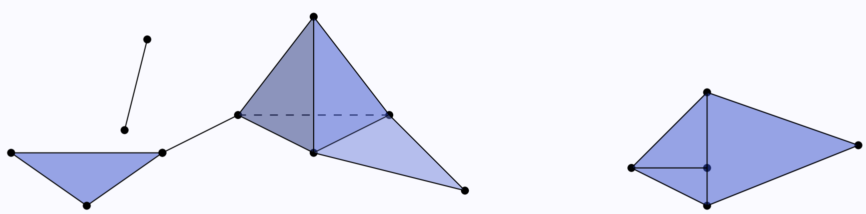
A **face** of a simplex σ is a subset (or subsimplex) spanned by a subset of the vertices. The **boundary** $\partial\sigma$ is the union of the proper faces, and the **interior** $\mathring{\sigma}$ is the complement of the boundary. We write $\tau \leq \sigma$ when τ is a face of σ. In particular, the interior of a vertex is the vertex itself. Note that this notion of interior and boundary is distinct from the topological notion of boundary.

We will now glue simplices together to build **complexes**, or **simplicial complexes** (oxymoron!).

Definition 7.2. A **simplicial complex** is a finite set K of simplices in $\mathbb{R}^n$ such that

1. If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.
2. If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a face of both σ and τ.

Example 7.3. the left is a simplicial complex, but the right is not:



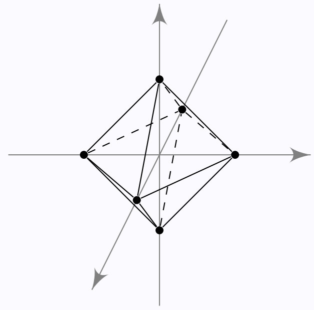
Technically, a simplicial complex is defined to be a set of simplices, which is just collections of points. It is not a subspace of $\mathbb{R}^n$. The **polyhedron** defined by K is the union of the simplices in K, and denoted by $|K|$. The **dimension** of K is the highest dimension of a simplex of K. The d**-skeleton** $K^{(d)}$ of K is the union of the n-simplices in K for $n \leq d$. A **triangulation** of a space M is a homeomorphism $h : |K| \to M$, where K is some simplicial complex.

Example 7.4. Let σ be the standard $(n+1)$-simplex. The boundary $\partial\sigma$ is homeomorphic to S^n (e.g. the boundary of a (solid) triangle is the boundary of the triangle, which is also a circle) . This is called the **simplicial** n**-sphere**.

We can also triangulate our S^n in a different way:

Example 7.5. In $\mathbb{R}^{n+1}$, consider the simplices $\langle \pm\mathbf{e}_0, \cdots, \pm\mathbf{e}_n \rangle$ for each possible combination of signs. So we have 2^{n+1} simplices in total. Then their union defines a simplicial complex K, and

$$|K| \cong S^n.$$



The nice thing about this triangulation is that the simplicial complex is invariant under the antipodal map. So not only can we think of this as a triangulation of the sphere, but a triangulation of $\mathbb{RP}^n$ as well.

An **oriented** q**-simplex** in a simplicial complex K is $\langle a_0, \cdots, a_q\rangle \in K$ with ordering of vertices where two $(q+1)$-simplices $\langle a_0, \cdots, a_q\rangle$ and $\langle a_{\pi(0)}, \cdots, a_{\pi(q)}\rangle$ are the same **oriented** simplex if $\pi \in \mathfrak{S}_{q+1}$ is an **even** permutation.

Definition 7.6 (Chain group $C_q(K)$). Let K be a simplicial complex and let $\{\sigma_1, \cdots, \sigma_\ell\}$ span $K^{(q)}$ with orientation. Then we define $C_q(K)$ be the free abelian group (free $\mathbb{Z}$-module) with basis $\{\sigma_1, \cdots, \sigma_\ell\}$, i.e.

$$C_q(K) \cong \mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_\ell\ .$$

(See Definition A.21 for a $\mathbb{Z}$-module.) We define **boundary homomorphisms**

$$\partial_q : C_q(K) \to C_{q-1}(K);\ \langle a_0, \cdots, a_q\rangle \mapsto \sum_{i=0}^{q} (-1)^i \langle a_0, \cdots, \widehat{a}_i, \cdots, a_q\rangle\ ,$$

where $\langle a_0, \cdots, \widehat{a}_i, \cdots, a_q\rangle = \langle a_0, \cdots, a_{i-1}, a_{i+1}, a_q\rangle$ is the simplex with a_i removed. We can show that $\partial_{q-1} \cdot \partial_q = 0$. Hence, given a complex K, one obtains a chain complex $(C_\bullet(K), \partial)$.

Definition 7.7 (Chain complex). A **chain complex** $(C_\bullet, \partial)$ is a sequence of chain groups $C_0, C_1, C_2, \cdots$ equipped with boundary maps $\partial_q : C_q \to C_{q-1}$ such that $\partial_{q-1} \circ \partial_q = 0$ for all q:

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Notice that it is very similar to de Rham complex (4.2), but degrees of complexes are decreasing by ∂ here. As in the de Rham cohomology (4.1), we define the q**-cycles** and the q**-boundaries** of the chain complex $C_\bullet(K)$ of a complex K

$$Z_q(K) = \operatorname{Ker} \partial_q\ , \quad B_q(K) = \operatorname{Im} \partial_{q+1}\ .$$

Then, the q**-th homology group** of $C_\bullet$ is defined to be

$$H_q(K; \mathbb{Z}) = \frac{\operatorname{Ker} \partial_q}{\operatorname{Im} \partial_{q+1}} = \frac{Z_q(K)}{B_q(K)}.$$

Given a triangulated manifold $|K| \to M$, it is obvious that a chain complex depends on a complex K. However, as we will see briefly below, the homology group is independent of a triangulation of M. Namely, for two triangulation $|K| \to M$, $|K'| \to M$, we have

$$H_q(K; \mathbb{Z}) \cong H_q(K'; \mathbb{Z})\ ,$$

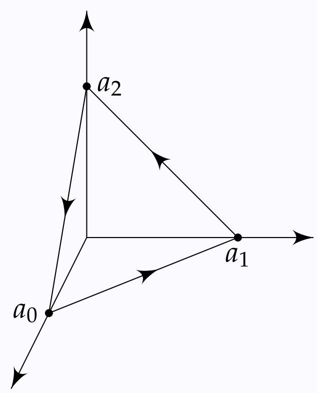
so that we can write it as $H_q(M;\mathbb{Z})$. As we will see below, homology groups are homotopy-invariant, and therefore **topological invariant**. It is a $\mathbb{Z}$-module so that it may contain a torsion subgroup like $\mathbb{Z}_m$ according to Theorem A.23.

In Definition 7.6, one can consider a free $\mathbb{R}$-module $C_q(K;\mathbb{R})$ instead of $\mathbb{Z}$-module. Then, the corresponding homology group is denoted by $H_q(K;\mathbb{R})$. Given a triangulation $h : |K| \to M$ of a manifold M, the dimension of $H_q(K;\mathbb{R})$ is called the q-th **Betti number**. The **Euler characteristic** of a triangulated space $h : |K| \to M$ is the alternative sum of dimensions of real-valued homology groups

$$\chi(M) = \sum_{i\geq 0}(-1)^i \dim H_i(M;\mathbb{R}).$$

Let $|K| \to M$ be a triangulation of an n-dimensional oriented closed connected manifold M. Then, the n-th homology group is $H_n(K,\mathbb{Z}) \cong \mathbb{Z}$ and its generator is denoted by $[M]$ and called the **fundamental class**.

Example 7.8. Let K be the standard simplicial 1-sphere, ie, we have the following in $\mathbb{R}^3$.



Our simplices are thus

$$K = \{\langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_0, a_1\rangle, \langle a_1, a_2\rangle, \langle a_2, a_0\rangle\}.$$

Our chain groups are

$$\begin{aligned} C_0\langle K\rangle &= \{\langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle\} \cong \mathbb{Z}^3 \\ C_1\langle K\rangle &= \{\langle a_0, a_1\rangle, \langle a_1, a_2\rangle, \langle a_2, a_0\rangle\} \cong \mathbb{Z}^3. \end{aligned}$$

All other chain groups are zero.
Hence, the only non-zero boundary map is

$$\partial_1 : C_1(K) \to C_0(K).$$

We can write down its matrix with respect to the given basis.

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

We now have everything we need to know about the homology groups, and we just need to do some linear algebra to figure out the image and kernel, and thus the homology groups. We have

$$H_0(K;\mathbb{Z}) = \frac{\mathrm{Ker}(\partial_0 : C_0(K) \to C_{-1}(K))}{\mathrm{Im}(\partial_1 : C_1(K) \to C_0(K))} \cong \frac{C_0(K)}{\mathrm{Im}\,\partial_1} \cong \frac{\mathbb{Z}^3}{\mathrm{Im}\,\partial_1}.$$

After doing some row operations with our matrix, we see that the image of ∂_1 is a two-dimensional subspace generated by the image of two of the edges. Hence we have

$$H_0(K;\mathbb{Z}) = \mathbb{Z}.$$

We interpret this to mean K has one path-connected component. In fact, generators of $H_0(K;\mathbb{Z})$ represent path-connected components of K. If K has r path-connected components, then we expect $H_0(K;\mathbb{Z}) \cong \mathbb{Z}^r$.
Similarly, we have

$$H_1(K;\mathbb{Z}) = \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} \cong \operatorname{Ker}\partial_1.$$

It is easy to see that we have

$$\operatorname{Ker}\partial_1 = \mathbb{Z}\{\langle a_0, a_1\rangle + \langle a_1, a_2\rangle + \langle a_2, a_0\rangle\} \cong \mathbb{Z}.$$

So we also have

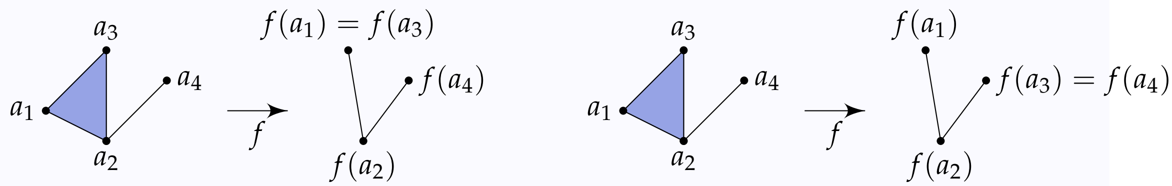
$$H_1(K;\mathbb{Z}) \cong \mathbb{Z}.$$

We see that this $H_1(K;\mathbb{Z})$ is generated by precisely the single loop in the triangle, which is the fundamental class $[S^1]$. The fact that $H_1(K;\mathbb{Z})$ is non-trivial means that we do indeed have a hole in the middle of the circle.

We can also consider a map between two simplicial complexes.

Definition 7.9 (Simplicial map). A **simplicial map** $f : K \to K'$ is a function $f : V_K \to V_{K'}$ such that if $\langle a_0, \cdots, a_q\rangle$ is a simplex in K, then $\{f(a_0), \cdots, f(a_q)\}$ spans a simplex of K'.

Example 7.10. the left is a simplicial map, but the right is not:



A simplicial map f induces a homomorphism among chain groups

$$f_* : C_\bullet(K) \to C_\bullet(K')$$

by

$$f_*(\langle a_0, a_1, \ldots, a_q\rangle) = \begin{cases} \langle f(a_0), f(a_1), \ldots, f(a_q)\rangle & \text{if } f(a_i) \neq f(a_j) \text{ for all } i \neq j \\ 0 & \text{if } f(a_i) = f(a_j) \text{ for some } i \neq j \end{cases}.$$

for a q-simplex $\sigma = \langle a_0, a_1, \ldots, a_q\rangle \in C_q(K)$. This induced homomorphism satisfies

$$\partial \cdot f_* = f_* \cdot \partial$$

Therefore, we have the chain map

Definition 7.11 (Chain map). A chain map $f_\bullet : C_\bullet(K) \to C_\bullet(K')$ is a sequence of homomorphisms $f_q : C_q(K) \to C_q(K')$ such that

$$f_{q-1} \circ \partial_q = \partial_q \circ f_q$$

for all n. In other words, the following diagram commutes:

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{\partial_{q+1}} & C_q(K) & \xrightarrow{\partial_q} & C_{q-1}(K) & \xrightarrow{\partial_{q-1}} & \cdots & \xrightarrow{\partial_2} & C_1(K) & \xrightarrow{\partial_1} & C_0(K) & \xrightarrow{\partial_0} & 0 \\ & & \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_{q-1}} & & \\ \cdots & \xrightarrow{\partial_{q+1}} & C_q(K') & \xrightarrow{\partial_q} & C_{q-1}(K') & \xrightarrow{\partial_{q-1}} & \cdots & \xrightarrow{\partial_2} & C_1(K') & \xrightarrow{\partial_1} & C_0(K') & \xrightarrow{\partial_0} & 0 \end{array}$$

Hence it also induces $f_* : H_n(K;\mathbb{Z}) \to H_n(K';\mathbb{Z}); [z] \mapsto [f(z)]$. The following theorem is very important for homotopy-invariance of homology groups. Using this theorem, it is easy to prove that the homology group of a topological space M is independent of the choice of triangulations.

Theorem 7.12. Let K, K' be simplicial complexes and $F : K \times I \to K'$ be a simplicial map. If we denote the restricted maps as $f_0 = F|_{K\times\{0\}}$ and $f_1 = F|_{K\times\{1\}}$, then the two induced homomorphisms of homology groups are identical

$$f_{0*} = f_{1*} : H_*(K;\mathbb{Z}) \to H_*(K';\mathbb{Z}) .$$

Since this is an important theorem, we give a sketch of a proof. Let us denote vertices $a_i \times \{0\}$ and $a_i \times \{1\}$ of a simplicial complex $K \times I$ by $\underline{a}_i$ and $\overline{a}_i$. Let us define a homomorphism $D : C_q(K) \to C_{q+1}(K \times I)$ by

$$\langle a_0, a_1, \ldots, a_q \rangle \mapsto \sum_{i=0}^{q} (-1)^i \left\langle \underline{a}_0, \underline{a}_1, \ldots, \underline{a}_i, \overline{a}_i, \ldots, \overline{a}_q \right\rangle .$$

Then, one can show that

$$(\partial_{q+1} D_q + D_{q-1}\partial_q)\left(\langle a_0, a_1, \cdots, a_q \rangle\right) = \langle \overline{a}_0, \overline{a}_1, \cdots, \overline{a}_q \rangle - \langle \underline{a}_0, \underline{a}_1, \cdots, \underline{a}_q \rangle .$$

If we denote the elements corresponding to $z \in Z_q(K)$ by $\underline{z} \in Z_q(K \times \{0\})$ and $\overline{z} \in Z_q(K \times \{1\})$, then we have

$$\left(\partial_{q+1} D_q + D_{q-1}\partial_q\right)(z) = \overline{z} - \underline{z} .$$

Therefore, the difference of $(f_0)_*$ and $(f_1)_*$ is exact

$$F(\partial_{q+1} D_q)(z) = (f_1)_*(z) - (f_0)_*(z) .$$

Hence, $(f_0)_* = (f_1)_*$.

7.2 Mayer-Vietoris exact sequence

We just briefly explain a powerful theorem called Mayer-Vietoris sequence to compute homology. For more detail, I refer to [Hat05, p.149]. A sequence of homomorphisms $\{f_q\}$ between modules M_q

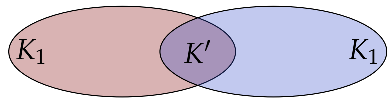
$$\cdots \xrightarrow{f_{q+2}} M_{q+1} \xrightarrow{f_{q+1}} M_q \xrightarrow{f_q} M_{q-1} \xrightarrow{f_{q-1}} \cdots$$

is called an **exact sequence** if

$$\operatorname{Im} f_{q+1} = \operatorname{Ker} f_q \qquad \text{for } \forall q\ .$$

Notice the difference from a Chain complex where $\operatorname{Im}\partial_{q+1} \in \operatorname{Ker}\partial_q$, so that a Chain complex is **not** an exact sequence.

Let K be a simplicial complex, and K_1, K_2 are subcomplexes such that $K = K_1 \cup K_2$. Moreover, $K' = K_1 \cap K_2$ is also a subcomplex of K.



Then, we have the following inclusion maps:

$$\begin{tikzcd}
K' \arrow[r, hook, "i"] \arrow[d, hook, "j"] & K_1 \arrow[d, hook, "k"] \\
K_2 \arrow[r, hook, "\ell"] & K.
\end{tikzcd}$$

Then, for $[z'] \in H_q(K')$, $[z_1] \in H_q(K_1)$, $[z_2] \in H_q(K_2)$, we define homomorphisms

$$\begin{aligned}
&\varphi : H_q(K') \to H_d(K_1) \oplus H_q(K_2);\ [x'] \mapsto (i_*[x'], -j_*[x']) \\
&\psi : H_q(K_1) \oplus H_q(K_2) \to H_q(K);\ ([x_1],[x_2]) \mapsto k_*[x_1] + \ell_*[x_2]\ .
\end{aligned}$$

Furthermore, for $[z] \in H_q(K)$, we decompose it as

$$z = c_1 + c_2 \qquad (c_1 \in C_q(K_1), c_2 \in C_q(K_2))\ .$$

Then, we define

$$\Delta : H_q(K) \to H_{q-1}(K');\ [z] \mapsto [\partial c_1] = -[\partial c_2]\ .$$

One can show that this map is independent of the choice of a representative of $[z]$, and the way of the decomposition $z = c_1 + c_2$. The theorem tells us how to compute the homology of the union $K = K_1 \cup K_2$ in terms of those of K_1, K_2 and K'.

Theorem 7.13 (Mayer-Vietoris). The following sequence obtained from φ, ψ, Δ is a **exact sequence**:

$$\begin{array}{cccccccc}
\cdots & \xrightarrow{\Delta} & H_{q+1}(K') & \xrightarrow{\varphi} & H_{q+1}(K_1) \oplus H_{q+1}(K_2) & \xrightarrow{\psi} & H_{q+1}(K) & \\
 & \xrightarrow{\Delta} & H_q(K') & \xrightarrow{\varphi} & H_q(K_1) \oplus H_q(K_2) & \xrightarrow{\psi} & H_q(K) & \to \cdots \\
\cdots & \xrightarrow{\Delta} & H_1(K') & \xrightarrow{\varphi} & H_1(K_1) \oplus H_1(K_2) & \xrightarrow{\psi} & H_1(K) & \\
 & \xrightarrow{\Delta} & H_0(K') & \xrightarrow{\varphi} & H_0(K_1) \oplus H_0(K_2) & \xrightarrow{\psi} & H_0(K) & \to 0
\end{array}$$

Example 7.14. Let us compute the homology group of n-sphere by the Mayer-Vietoris sequence. We have obtained the homology group of S^1 in Example 7.8, and we will compute it by an induction on n. Suppose that the homology group of S^{n-1} is

$$H_q(S^{n-1}) = \begin{cases} \mathbb{Z} & q = 0, n-1 \\ 0 & \text{otherwise} \end{cases}\ .$$

We decompose an n-sphere into the northern and southern hemisphere

$$K_1 = \left\{(x^1, x^2, \cdots, x^{n+1}) \in S^n | x^{n+1} \geq 0\right\}$$
$$K_2 = \left\{(x^1, x^2, \cdots, x^{n+1}) \in S^n | x^{n+1} \leqq 0\right\}$$

Then, the Poincare lemma (7.23) is

$$H_q(K_1) = H_q(K_2) = \begin{cases} \mathbb{Z} & (q = 0 \\ 0 & (q \neq 0 \end{cases}$$

By the assumption of the induction, we have

$$H_q(K') = H_q(S^{n-1}) = \begin{cases} \mathbb{Z} \\ 0 \end{cases}$$

Then, the Mayer-Vietoris exact sequence leads to

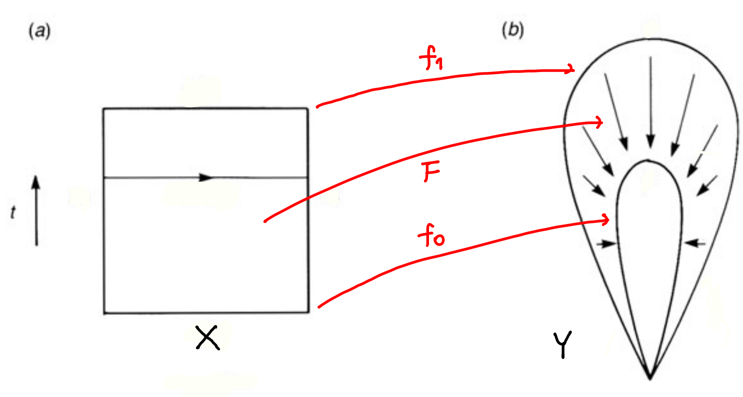
$$0 \to H_q(S^n) \to H_{q-1}(S^{n-1}) \to 0$$

for $q \geq 2$. For the $q = 0$ degree, φ is an injection so that $H_1(S^n; \mathbb{Z}) = 0$. In addition, ψ is surjective so that $H_0(S^n; \mathbb{Z}) = \mathbb{Z}$. Therefore, we obtain

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise} \end{cases} .$$

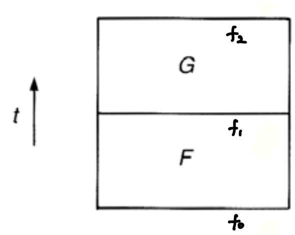
7.3 Homotopy invariance of homology groups

In topology, we study spaces up to "continuous deformation". Famously, a coffee mug can be continuously deformed into a doughnut, and thus they are considered to be topologically the same. Now we also talk about maps between topological spaces. So a natural question is if it makes sense to talk about the continuous deformations of maps. It turns out we can, and the definition is sort-of the obvious one:



Definition 7.15 (Homotopy). Let M, N be a topological space. A **homotopy** between $f_0, f_1 : M \to N$ is a map $F : [0,1] \times M \to N$ such that $F(0, x) = f_0(x)$ and $F(1, x) =$

$f_1(x)$. If such an F exists, we say f_0 is **homotopic** to f_1, and write $f_0 \simeq f_1$.



As easily can be seen from the figure above, homotopy $\simeq$ defines an equivalence relation on the set of maps from M to N. In algebraic topology, we study quantities invariant under homotopy.

Definition 7.16 (Homotopy equivalence). A map $f : M \to N$ is a **homotopy equivalence** if there is some $g : N \to M$ such that $f \circ g \simeq \mathrm{id}_M$ and $g \circ f \simeq \mathrm{id}_N$. We call g the **homotopy inverse** to f.

If $f_0 \simeq f_1 : M \to N$ and $g_0 \simeq g_1 : N \to L$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : M \to L$.

$$\begin{tikzcd} M \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & N \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & L \end{tikzcd}$$

Therefore, homotopy equivalence is an equivalence relation.

A **deformation retraction** of a space M onto a subspace $N \subset M$ is a family of maps $f_t : M \to M$, $t \in I$, such that $f_0 = \mathrm{id}$, $f_1(X) = A$, and $f_t \mid A = \mathrm{id}$ for all t. A deformation retraction $f_t : X \to X$ is a special case of a homotopy.

Example 7.17. Let $i : \{0\} \to \mathbb{R}^n$ be the inclusion map. A homotopy equivalence can be constructed by $F : [0,1] \to \mathbb{R}^n \to \mathbb{R}^n$ be

$$F(t, \mathbf{v}) = t\mathbf{v}.$$

We have $F(0, \mathbf{v}) = 0$ and $F(1, \mathbf{v}) = \mathbf{v}$. From the point of view of homotopy, the point $\{0\}$ is equivalent to $\mathbb{R}^n$, so that dimension is irrelevant.

Example 7.18. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere, and $i : S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$. We show that this is a homotopy equivalence. We define $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ by

$$r(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

It is easy to see $r \circ i = \mathrm{id}_{S^n}$. In the other direction, we need to construct a path from each $\mathbf{v}$ to $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ in a continuous way. We could do so by

$$\begin{aligned} H : [0,1] \times (\mathbb{R}^{n+1} \setminus \{0\}) &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (t, \mathbf{v}) &\mapsto (1-t)\mathbf{v} + t\frac{\mathbf{v}}{\|\mathbf{v}\|}. \end{aligned}$$

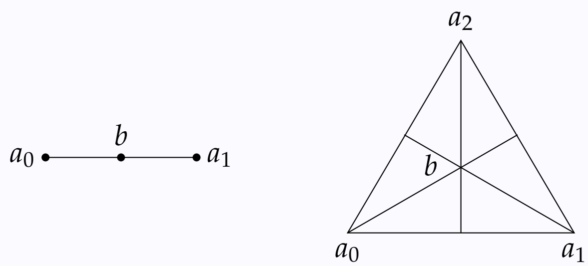
We can easily check that this is a homotopy from $\mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ to $i \circ r$.

Even though the dimensions are different, homotopy equivalence does not lose information about "hole".

By using the homotopy equivalence, a continuous map $f : |K| \to |K'|$ between simplicial complexes K, K' can be "approximated" f by a simplicial map $g : K \to K'$. (See [Hat05, p.177] for more detail.)

The **barycentric subdivision** $\mathrm{Sd}(\sigma)$ of a n-simplex $\sigma = \langle a_0, \cdots, a_q\rangle$ is the decomposition of $\langle a_0, \cdots, a_q\rangle$ into the n-simplices $\langle b, c_0, \cdots, c_{q-1}\rangle$ where, inductively, $\langle c_0, \cdots, c_{q-1}\rangle$ is an $(n-1)$-simplex in the barycentric subdivision of a face $\langle a_0, \cdots, \hat{a}_i, \cdots, a_q\rangle$. The induction starts with the case $n = 0$ when the barycentric subdivision of $\langle a_0\rangle$ is defined to be just $\langle a_0\rangle$ itself. Note that $\mathrm{Sd}(\sigma)$ is a simplicial complex.

Example 7.19. The cases $q = 1, 2$ are illustrated as follows:



We can also define the barycentric subdivision of a simplicial complex K as

$$\mathrm{Sd}(K) = \{\tau \,;\, \tau \in \mathrm{Sd}(\sigma)\,,\, \sigma \in K\}\,.$$

Theorem 7.20 (Simplicial approximation). Let K, K' be simplicial complexes and $f : |K| \to |K'|$ be a continuous map. Then, there exists a simplicial map $\varphi : \mathrm{Sd}^n(K) \to K'$ for a sufficiently large positive integer $n \in \mathbb{Z}_{\geq 0}$ that is homotopic to f.

Theorem 7.21. Let K, K' be simplicial complexes and $f_0, f_1 : |K| \to |K'|$ be a continuous map. If $f_0 \simeq f_1$, then the induced homomorphisms $(f_0)_*, (f_1)_* : H_\bullet(|K|, \mathbb{Z}) \to H_\bullet(|K'|, \mathbb{Z})$ are identical: $(f_0)_* = (f_1)_*$.

This theorem can be proven by a combination of a simplicial approximation of a homotopy $F : |K| \times I \to |K'|$ and Theorem 7.12. Using this theorem, it is easy to prove that the homology group of a topological space M is independent of the choice of triangulations.

Theorem 7.22. Let K, K' be homotopy-equivalent simplicial complexes. Then $H_\bullet(K; \mathbb{Z}) \cong H_\bullet(K'; \mathbb{Z})$.

Lemma 7.23 (Poincare's Lemma). The homology groups of a contractible space are isomorphic to those of a point. In particular, we have

$$H_q(\mathbb{R}^n) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \neq 0 \end{cases} \quad .$$

7.4 Cohomology groups

Given a simplicial complex K and its chain group $C_q(K)$, we can define the **cochain group**

$$C^q(K) = \{f : C_q(K) \to \mathbb{Z}, \text{ homomorphism}\}$$

In addition, we can define a **coboundary map** $\delta : C^q(K) \to C^{q+1}(K)$ as $\delta f(c) = f(\partial c)$ were $c \in C_{q+1}(K)$. It is easy to see $\delta \cdot \delta = 0$, so that we obtain the cochain complex

$$0 \xleftarrow{\delta^{n+1}} C^n \xleftarrow{\delta^n} C^{n-1} \xleftarrow{\delta^{n-1}} \cdots \xleftarrow{\delta^2} C^1 \xleftarrow{\delta^1} C^0 \xleftarrow{\delta^0} 0$$

We can define **cocycle** $Z^q(K) = \operatorname{Ker} \delta^q$ and **coboundary** $B^q(K) = \operatorname{Im} \delta^{q-1}$. The cohomology group of K is defined by

$$H^q(K;\mathbb{Z}) = Z^q(K)/B^q(K) .$$

There exists bilinear map

$$H_q(K;\mathbb{Z}) \otimes H^q(K;\mathbb{Z}) \to \mathbb{Z}; ([c],[f]) \mapsto f(c)$$

which is well-defined.

In a similar fashion, one can also define the cohomology group with $\mathbb{R}$-coefficient. In fact, the cohomology group with $\mathbb{R}$-coefficient is dual to the homology group

$$H^q(C^\bullet, \mathbb{R}) = \left(H_q\left(C_\bullet, \mathbb{R}\right)\right)^* .$$

The case of the $\mathbb{Z}$-coefficient is not so straightforward. (Can you find an example in which the cohomology groups are not dual of the homology groups?) We need the universal coefficient theorem for $H^q(C^*, \mathbb{Z})$, for which we refer to [Hat05, Theorem 3.2].

Theorem 7.24 (de Rham). Let M be a smooth manifold and $|K| \to M$ be its triangulation. Then, we have an isomorphism

$$H^\bullet_{dR}(M) \cong H^\bullet(K;\mathbb{R}) .$$

A proof requires Čech cohomology, which is given in [War13, Mor01]. As in the wedge product of the de Rham cohomology group, the simplicial cohomology group is endowed with product structure, called **cup product**. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(K;R)$ with R-coefficient ($R = \mathbb{Z}_m, \mathbb{Z}, \mathbb{R}$), the cup product $\varphi \cup \psi \in C^{k+\ell}(K;R)$ is the cochain whose paring with a simplex $\sigma = \langle a_0, a_1, \ldots, a_{k+\ell}\langle$ is given by the formula

$$(\varphi \cup \psi)(\sigma) = \varphi(\langle a_0, \ldots, a_k\rangle)\psi(\langle a_k, \ldots, a_{k+\ell}\rangle)$$

where the RHS is given by the product in the ring R. As a result, $(H^\bullet(K;R), \cup)$ is callled the cohomology ring of K.

In fact, there is an important relation between homology group and cohomology group of an oriented connected closed n-dimensional manifold M. A k-dimensional oriented submanifold $N \subset M$ without boundary indeed represents a generator $[N] \in H_k(M;\mathbb{R})$. There exists $\eta \in H^{n-k}(M;\mathbb{R})$ such that

$$\int_N \omega = \int_M \omega \wedge \eta$$

for any $\omega \in H^k(M;\mathbb{R})$.

Theorem 7.25 (Poincaré duality). Let M be an n-dimensional oriented connected closed manifold. Then, we have an isomorphism

$$\vartheta : H_k(M;\mathbb{R}) \cong H^{n-k}(M;\mathbb{R}); [N] \mapsto \eta \ .$$

In fact, the Poincaré duality holds any ring coefficient $R = \mathbb{Z}_m, \mathbb{Z}, \mathbb{R}$. Using Theorem 4.8, one can show that

$$H_k(M;\mathbb{R}) \cong H^{n-k}(M;\mathbb{R}) \cong H^k(M;\mathbb{R}) \cong H_{n-k}(M;\mathbb{R})$$

for an oriented connected closed n-dimensional manifold M. Using this isomorphism, one can define an **intersection number** $[N_1]\cdot[N_2]$ for $[N_1] \in H_k(M;\mathbb{R})$ and $[N_2] \in H_{n-k}(M;\mathbb{R})$.

$$[N_1]\cdot[N_2] := \int_M \eta_1 \wedge \eta_2$$

where $\vartheta(N_i) = \eta_i$.

7.5 Lefschetz fixed point theorem and Poincaré-Hopf theorem

Let M and N be n-dimensional smooth oriented connected closed manifolds. A smooth map $f : M \to N$ induces an homomorphism $f_* : H_\bullet(M;\mathbb{Z}) \to H_\bullet(N;\mathbb{Z})$. In particular, we define **mapping degree** of f by

$$f_*([M]) := (\deg f)[N] \ .$$

In the language of cohomology group, we can define

$$\deg f := \frac{\int_M f^*\omega}{\int_N \omega}$$

where ω is the volume form of N.

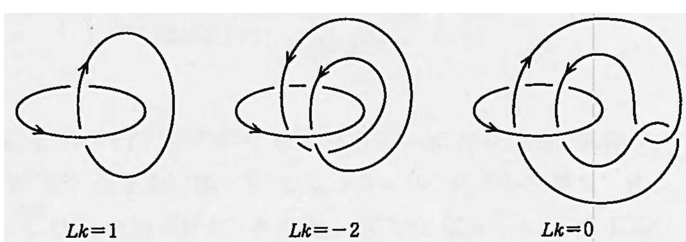
Given two knots $K_1, K_2 : S^1 \to \mathbb{R}^3$, we can define $F : K_1 \times K_2 \to S^2$ by

$$F(p,q) = \frac{x_1 - x_2}{|x_1 - x_2|}$$

Then, we define the **linking number** of K_1 and K_2 as

$$Lk(K_1,K_2) := \deg F = \frac{1}{4\pi}\int_{K_1\times K_2} F^*\omega = \frac{1}{4\pi}\int_{K_1} dx_1^i \int_{K_2} dx_2^j \frac{(x_1-x_2)^k}{|x_1-x_2|^3}\epsilon_{ijk} \tag{7.1}$$

where ω is the volume form of S^2.



This number was first introduced by Gauss in the study of electromagnetism. Biot-Savart law tells us that the magnetic field generated by electric current running on K_1 with unit strength is

$$\vec{B}(\vec{x}) = \frac{1}{4\pi} \int_{K_1} \frac{(\vec{x} - \vec{x}_1(t_1)) \times \frac{d\vec{x}_1}{dt_1}}{|\vec{x} - \vec{x}_1|^3} dt_1$$

Therefore, $Lk(K_1, K_2)$ is the energy required to move a magnetic monopole along K_2 under this magnetic field. Gauss has noticed that the integral always provides an integer however K_1 and K_2 are drawn. Moreover, this number stays invariant even though you deform K_1 and K_2!

Let M be an n-dimensional smooth closed oriented manifold. We assume that a map $f : M \to M$ has isolated fixed points $f(p) = p$. We define a map $h : S^{n-1}_\epsilon \to S^{n-1}$ by

$$h(q) = \frac{q - f(q)}{|q - f(q)|}$$

The index of f at the fixed point p is defined by

$$\mathrm{ind}_p f := \deg h$$

We define the **Lefschetz number** of f as

$$L(f) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(f_* : H_i(M; \mathbb{R}) \to H_i(M; \mathbb{R})).$$

Theorem 7.26 (Lefschetz fixed point theorem). The Lefschetz number is the sum of indices over isolated fixed points

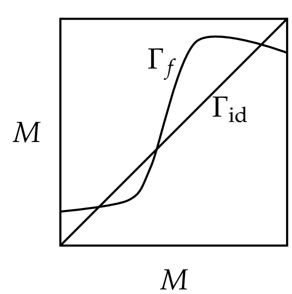
$$L(f) = \sum_p \mathrm{ind}_p(f)$$

The Lefschetz theorem can be interpreted from the viewpoint of intersection numbers. Given a smooth map $f : M \to M$, we can define a submanifold Γ_f called the **graph** of f

$$\Gamma_f = \{(x, f(x)) \in M \times M\} .$$

For the identity map, the corresponding graph Γ_{id} is called the diagonal submanifold of $M \times M$. Then, the Lefschetz number is given by an intersection number of $[\Gamma_f]$ and $[\Gamma_{\mathrm{id}}]$ in $M \times M$

$$L(f) = [\Gamma_f] \cdot [\Gamma_{\mathrm{id}}] .$$



Let X is a smooth vector field on M and φ_t is a flow generated by X. Then, the index of a zero point p of X is defined by

$$\operatorname{ind}_p(X) = \operatorname{ind}_p(\varphi_t) \ .$$

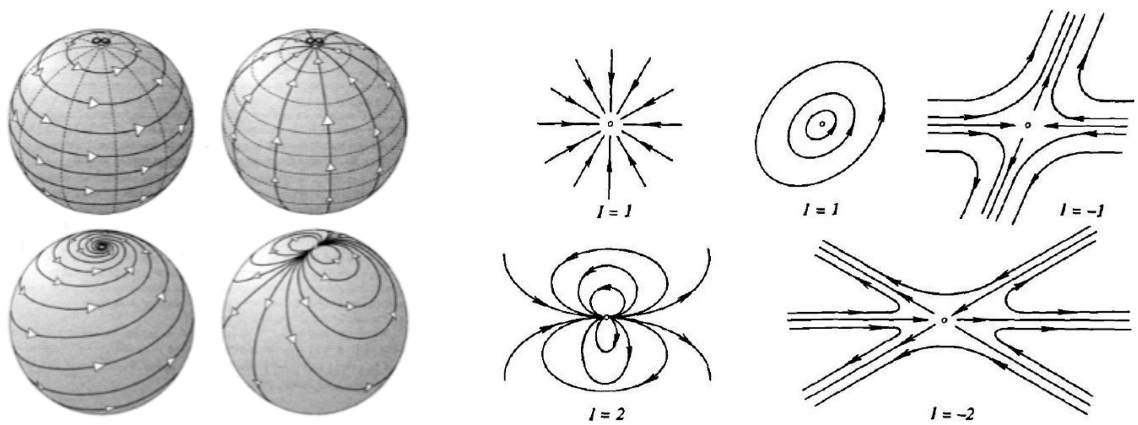
Since the flow φ_t is homotopic to the identity $\varphi_t \simeq \mathrm{id}$, $L(\varphi_t)$ is equal to the Euler characteristic.

$$L(\varphi_t) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\mathrm{id} : H_i(M; \mathbb{R}) \to H_i(M; \mathbb{R})). = \sum_{i \geq 0} (-1)^i \dim H_i(M; \mathbb{R}) = \chi(M)$$

Therefore, we obtain:

Theorem 7.27 (Poincaré-Hopf theorem). The Euler characteristics is the sum of indices over isolated fixed points

$$\chi(M) = \sum_p \operatorname{ind}_p(X)$$



This theorem was introduced in §1.1. There is another important "fixed point theorem" due to Brouwer, which we delegate to [Mil65].

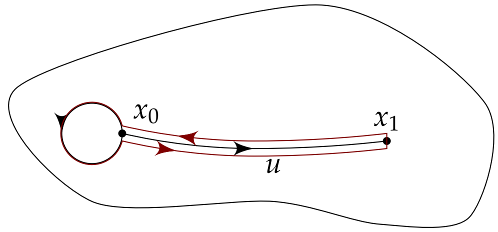
8 Fundamental groups and Homotopy groups

8.1 Fundamental groups

A **path** in a topological space M is a map $\gamma : I \to M$. If $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we say γ is a path from x_0 to x_1. If $\gamma(0) = \gamma(1)$, then γ is called a **loop** (based at x_0). The idea of fundamental groups is to consider homotopy equivalence of loops at a fixed base point x_0

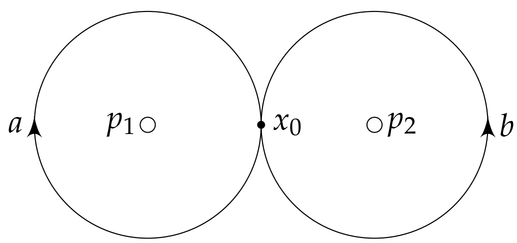
Definition 8.1 (Fundamental group). Let M be a topological space and $x_0 \in M$. The **fundamental group** of M (based at x_0), denoted $\pi_1(M, x_0)$, is the set of homotopy classes of loops in M based at x_0 (i.e. $\gamma(0) = \gamma(1) = x_0$). The group operations are defined as follows:
We define a group multiplication by $[\gamma_0][\gamma_1] = [\gamma_0 \cdot \gamma_1]$; inverses by $[\gamma]^{-1} = [\gamma^{-1}]$; and the identity as the constant map $e = [c_{x_0}]$.

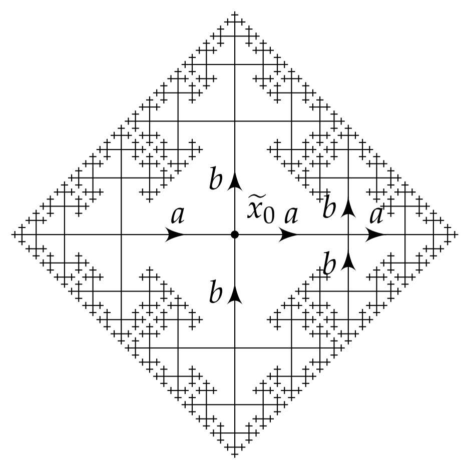
It is easy to see from the figure below that fundamental groups are independent of the choice of a base point. Therefore, it is often written as $\pi_1(M)$.



Unlike homology groups, the fundamental groups are non-abelian groups in general.

Example 8.2. Let us consider the space $M = \mathbb{R}^2 \backslash \{p_1, p_2\}$ which is obtained by removing two points from the plane. The fundamental group of M is the free group $\pi_1(M) = F(\{a, b\})$ with the two generators. See Definition A.27. It is easy to understand the free group on two generators by the **Cayley graph** below.





Usually, a fundamental group is presented as $\langle S \mid R \rangle$ by generators S and the relations R. See Definition A.28. As it can be easily read off from Figure 2, the fundamental group of a torus is

$$\pi_1(T^2) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \ .$$

A **based space** is a pair (M, x_0) of a topological space M and a **base point** $x_0 \in M$. A **map of based spaces**

$$f : (M, x_0) \to (N, y_0)$$

is a continuous map $f : M \to N$ such that $f(x_0) = y_0$. A based map f induces the group homomorphism

$$f_* : \pi_1(M, x_0) \to \pi_1(N, y_0),$$

defined by $[\gamma] \mapsto [f \circ \gamma]$. Thus, the induced homomorphisms are natural under compositions: For $(M, x_0) \xrightarrow{h} (N, y_0) \xrightarrow{h} (L, z_0)$, we have $(k \circ h)_* = k_* \circ h_*$ Moreover, if two based maps f_0, f_1 are homotopic $f_0 \simeq f_1$, then $(f_0)_* = (f_1)_*$.

Theorem 8.3. If $f : M \to N$ is a homotopy equivalence, and $x_0 \in M$, then the induced map

$$f_* : \pi_1(M, x_0) \to \pi_1(N, f(x_0)).$$

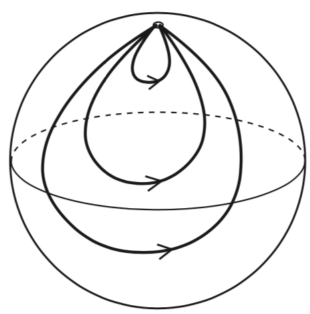
is an isomorphism.

Example 8.4. Let us denote the space obtained by joining two copies of S^1 at one point by $S^1 \vee S^1$. Then, it is homotopic to $\mathbb{R}^2 \backslash \{p_1, p_2\}$ so that the fundamental group is $\pi_1(S^1 \vee S^1) \cong F(\{a, b\})$.

Definition 8.5 (simply-connected space). A topological space M is **simply-connected** if it is path connected and $\pi_1(M, x_0) \cong 1$ for any choice of $x_0 \in M$.

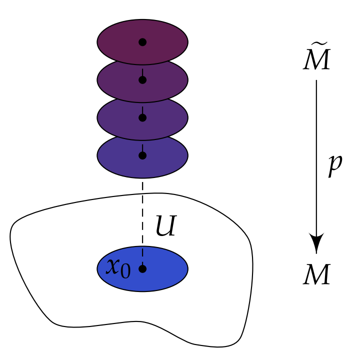
Example 8.6. Clearly, a point is simply-connected. Hence, any contractible space is simply-connected since it is homotopic to a point. For example, $\mathbb{R}^n$ is simply-connected for any n.

Example 8.7. An n-sphere S^n ($n > 1$) is simply-connected. If we remove a point $\{\infty\}$ from S^n, it is homeomorphic to $\mathbb{R}^n$ which is contractible. (Therefore, S^n is often written as $S^n = \mathbb{R}^n \cup \{\infty\}$.) given a loop $\gamma : I \to S^n$, it is easy to find a homotopy $F : [0,1] \times I \to S^n$ such that $F|_{0 \times I} = \gamma$ and $F|_{1 \times I}$ is a constant map.



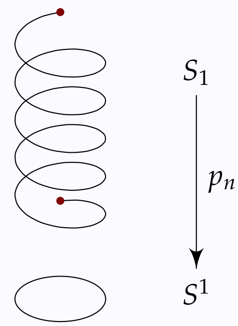
Covering space

Intuitively, a **covering space** of M is a pair $(\widetilde{M}, p : \widetilde{M} \to M)$, such that if we take any $x_0 \in M$, there is some neighborhood U of x such that the pre-image of the neighborhood is "many copies" of U.



Definition 8.8. A pair $(\widetilde{M}, p : \widetilde{M} \to M)$ is a **covering space** of M if each $x \in M$ has a path-connected open neighborhood U such that the restriction of p to a path-connected component V_α of $p^{-1}(U)$ is $p|_{V_\alpha} : V_\alpha \xrightarrow{\sim} U$.

Example 8.9. Consider $p_n : S^1 \to S^1$ (for any $n \in \mathbb{Z} \setminus \{0\}$) defined by $z \mapsto z^n$. We can consider this as "winding" the circle n times, or as the following covering map:



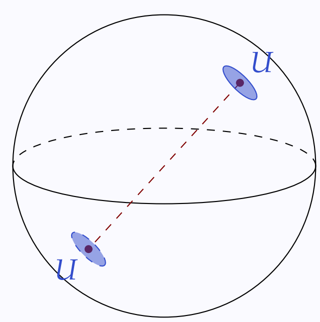
where we join the two red dots together. The preimage of 1 would be n copies of 1.

Definition 8.10 (Universal cover). A covering map $p : \widetilde{M} \to M$ is a **universal cover** if $\widetilde{M}$ is simply-connected.

If $p : \widetilde{M} \to M$ is a universal cover, then there is a bijection $\ell : \pi_1(X, x_0) \to p^{-1}(x_0)$. More precisely, the set of covering transformations $\{f : \widetilde{M} \to \widetilde{M} \mid p \circ f = p\}$ forms a group and it is isomorphic to $\pi_1(M)$.

Example 8.11. $\exp : \mathbb{R} \to S^1 \,; x \mapsto \exp(2\pi i x)$ is a universal covering of S^1, and the preimage of a point 1 consist of $\mathbb{Z}$ so that $\pi_1(S^1) \cong \mathbb{Z}$.

Example 8.12. The real projective space $\mathbb{R}P^2$ is defined by $S^2/\sim$, where we identify every $x \sim -x$, i.e. every pair of antipodal point. In fact, the quotient map $p : S^2 \to \mathbb{R}P^2$ is indeed a universal covering.



Example 8.13. In fact, the Cayley graph is a universal covering of $S^1 \vee S^1$.

In §7.2, we have seen that Mayer-Vietoris exact sequence allow us to compute the homology group of a manifold by decomposing it into two parts. Even for fundamental groups, there is a similar theorem, called the van Kampen theorem. In the following, you will find the simple version of the theorem. Due to the time constraint, I have to omit how to use this theorem. If you are interested in it, you can read the book of Hatcher [Hat05, §1.2].

Theorem 8.14 (van Kampen theorem). Let a manifold $M = U_1 \cup U_2$ be a union of open path-connected sets U_i, and let $U_1 \cap U_2$ be path-connected. We write the normal subgroup $N \triangleleft (\pi_1(U_1) * \pi_1(U_2))$ generated by $\{i_{1*}(\omega) i_{2*}(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2)\}$ where $i_i : U_1 \cap U_2 \hookrightarrow U_i$ are the inclusion maps. Then there is an isomorphism

$$\pi_1(M) \cong (\pi_1(U_1) * \pi_1(U_2))/N\,.$$

Theorem 8.15 (Simple version of van Kampen theorem). If $M = U_1 \cup U_2$ with U_i open and path-connected, and $U_1 \cap U_2$ path-connected and simply-connected, then there is an isomorphism $\pi_1(U_1) * \pi_1(U_2) \cong \pi_1(M)$.

Example 8.16. Let us take $M = S^1 \vee S^1$ and $U_i = S^1$ so that $U_1 \cap U_2$ is the point, which is simply-connected. Therefore, the free group $F(\{a,b\})$ is the free product $F(\{a,b\}) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$.

Now we can understand the statement of the famous Poincaré "conjecture":

Conjecture 8.17. Every simply-connected, closed n-dimensional manifold is homeomorphic to the n-sphere S^n.

In fact, the Poincaré "conjecture" is no longer a conjecture. For $n \geq 5$, it was proven by Smale [Sma61]. Then, the case of $n = 4$ was proven by Freedman [Fre82]. Perelman has proven the case of $n = 3$ [Per02] by solving the geometrization conjecture of Thurston.

8.2 Homotopy groups

Let I^n ($n \geq 1$) denote the unit-cube $I \times \cdots \times I$, The boundary ∂I^n is the geometrical boundary of I^n. As in the fundamental group, we assume here that we shall be concerned with continuous maps $\alpha : I^n \to M$, which map the boundary ∂I^n to a point $x_0 \in M$. Since the boundary is mapped to a single point x_0. The map α is called an n**-loop** at x_0.

Definition 8.18 (Homotopy class). Let M be a topological space. The set of homotopy classes of n-loops ($n \geq 1$) at $x_0 \in M$ is denoted by $\pi_n(M, x_0)$ and called the n-th homotopy group at x_0. $\pi_n(M, x_0)$ is called the higher homotopy group if $n \geq 2$.

The product $\alpha * \beta$ just defined naturally induces a product of homotopy classes defined by

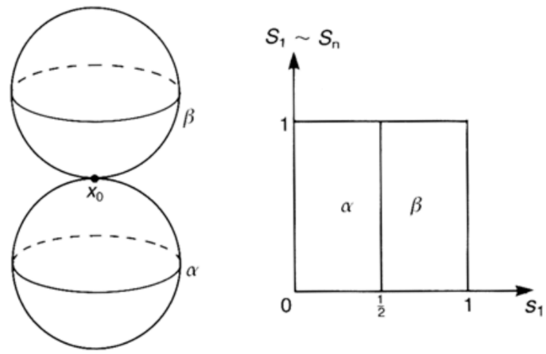
$$[\alpha] * [\beta] = [\alpha * \beta]$$

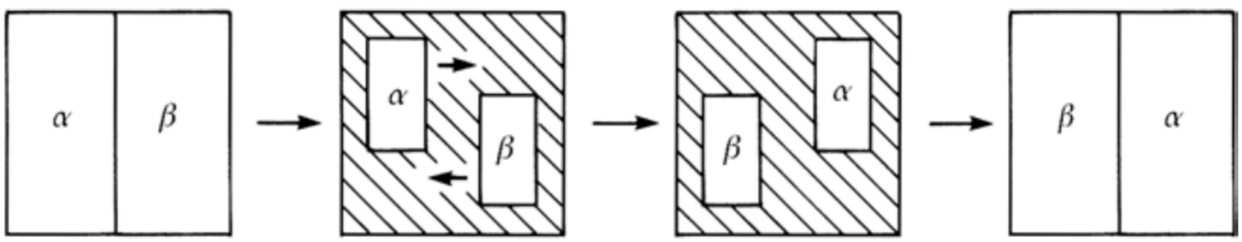
Higher homotopy groups are always Abelian; for any n-loops α and β at $x_0 \in M$, $[\alpha]$ and $[\beta]$ satisfy

$$[\alpha] * [\beta] = [\beta] * [\alpha].$$

Global $SU(2)$ anomaly

There are various applications of homotopy groups to physics, but only one example is presented here.





Let us consider the path integral of a non-Abelian gauge theory with chiral fermion ψ in a representation R of G.

$$Z[A_\mu] = \int [\mathcal{D}\psi][\mathcal{D}\overline{\psi}] e^{-\int \overline{\psi}\gamma^\mu D_\mu \psi} \tag{8.1}$$

To perform functional integral over gauge fields A_μ consistently, we need to impose the equivalence

$$Z[A_\mu] = Z[A_\mu^g], \quad A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g. \tag{8.2}$$

for any gauge transformation $g : \mathbb{R}^4 \to G$. We will learn gauge transformations of non-Abelian gauge theory in §10.

When we consider an odd number of chiral fermions in the doublet of $\mathrm{SU}(2)$ gauge group, there is a global anomaly [Wit82a]. Since continuous deformations of g do not change the phase of the measure $[\mathcal{D}\psi][\mathcal{D}\overline{\psi}]$, we need to consider maps $g : \mathbb{R}^4 \to \mathrm{SU}(2)$ up to continuous deformations. Upon one-point compactifiction of $\mathbb{R}^4$ to S^4, they are characterized by $\pi_4(\mathrm{SU}(2))$, which is known

$$\pi_4(\mathrm{SU}(2)) = \pi_4(S^3) = \mathbb{Z}_2\,. \tag{8.3}$$

Let $g_0 : \mathbb{R}^4 \to \mathrm{SU}(2)$ be the gauge transformation corresponding to the nontrivial element in this $\mathbb{Z}_2$. It is known that the measure $[\mathcal{D}\psi][\mathcal{D}\overline{\psi}]$ gets a minus sign under this gauge transformation. Thus, one cannot have an odd number of Weyl fermions in the doublet representation of gauge group $\mathrm{SU}(2)$.

9 Lie groups and Lie algebras

Definition 9.1 (Lie group). A **Lie group** is a manifold G with a group structure such that multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are smooth maps. The **dimension** of a Lie group G is the dimension of the underlying manifold.

For each $h \in G$, we define the **left and right translation maps**

$$\begin{aligned} L_h &: G \to G; g \mapsto hg\,, \\ R_h &: G \to G; g \mapsto gh\,. \end{aligned} \tag{9.1}$$

These maps are indeed diffeomorphisms because they have smooth inverse $L_{h^{-1}}$ and $R_{h^{-1}}$ respectively.

Some of Lie groups will be given by subsets of the space $M_n(\mathbb{F})$ of $n \times n$ matrices where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ specified by certain algebraic equations. For example,

- General linear group: $\mathrm{GL}(n, \mathbb{F}) = \{A \in \mathrm{M}_n(\mathbb{F}) | \det A \neq 0\}$
- Special linear group: $\mathrm{SL}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) | \det A = 1\}$
- Symplectic group $\mathrm{Sp}(n, \mathbb{F}) = \{A \in \mathrm{GL}(2n, \mathbb{F}) | A^{\mathrm{T}} J A = J \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$
- Orthogonal group $\mathrm{O}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) | A^T A = I\}$
- Special orthogonal group $\mathrm{SO}(n, \mathbb{F}) = \{A \in \mathrm{O}(n, \mathbb{F}) | \det A = 1\}$
- Unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | A^\dagger A = I\}$
- Special unitary group $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) | \det A = 1\}$

Definition 9.2 (Lie algebra). A **Lie algebra** $\mathfrak{g}$ is a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with a **bracket**

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ for all $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$ (bilinear)
2. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (antisymmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$. (Jacobi identity)

Note that linearity in the second argument follows from linearity in the first argument and antisymmetry.

We now try to get a Lie algebra from a Lie group G, by considering $T_e(G)$. The tangent space of a Lie group G at the identity naturally admits a Lie bracket

$$[\,\cdot\,,\,\cdot\,] : T_e G \times T_e G \to T_e G; (X, Y) \mapsto [X, Y] = XY - YX$$

such that

$$\mathfrak{g} = (T_e(G), [\,\cdot\,,\,\cdot\,])$$

is a Lie algebra.

Definition 9.3 (Lie algebra of a Lie group). Let G be a Lie group. The **Lie algebra** of G, written $\mathfrak{g}$, is the tangent space T_eG under the natural Lie bracket.

The general convention is that if the name of a Lie group is denoted by capital letters, then the corresponding Lie algebra is the same name with fraktur font. For example, the Lie group of $\mathrm{SL}(n,\mathbb{C})$ is $\mathfrak{sl}(n,\mathbb{C})$. The semisimple Lie algebras over $\mathbb{C}$ have been classified by Wilhelm Killing and Élie Cartan around 1890. The classification assigns the types

$$\begin{aligned} A_n &= \mathfrak{sl}(n+1,\mathbb{C}) \\ B_n &= \mathfrak{so}(2n+1,\mathbb{C}) \\ C_n &= \mathfrak{sp}(n,\mathbb{C}) \\ D_n &= \mathfrak{so}(2n,\mathbb{C}) \end{aligned}.$$

Contrasting with the classical Lie algebras listed above are the exceptional Lie algebras, E_6, E_7, E_8, F_4, and G_2, which share their abstract properties. For the classification of the simple Lie algebras, we refer the reader to [Kir08].

Given a finite-dimensional Lie algebra, we can pick a basis B for $\mathfrak{g}$.

$$B = \{T_a : a = 1, \cdots, \dim \mathfrak{g}\}. \tag{9.2}$$

Then any $X \in \mathfrak{g}$ can be written as

$$X = X^a T_a = \sum_{a=1}^{n} X^a T_a,$$

where $X^a \in \mathbb{F}$.

By linearity, the bracket of elements $X, Y \in \mathfrak{g}$ can be computed via

$$[X,Y] = X^a Y^b [T_a, T_b].$$

In other words, the whole structure of the Lie algebra can be given by the bracket of basis vectors. We know that $[T_a, T_b]$ is again an element of $\mathfrak{g}$. So we can write

$$[T_a, T_b] = f_{ab}{}^c T_c,$$

where $f_{ab}{}^c \in \mathbb{F}$ are called the **structure constants**. By the antisymmetry of the bracket, we know

$$f_{ba}{}^c = -f_{ab}{}^c.$$

The Jacobi identity amounts to

$$f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e = 0.$$

Example 9.4. Take $G = \mathrm{SO}(3,\mathbb{R})$. Then $\mathfrak{so}(3,\mathbb{R})$ is the space of 3×3 real anti-symmetric matrices, which one can manually check are generated by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We then have

$$(T_a)_{bc} = -\varepsilon_{abc}.$$

Then the structure constants are $f_{ab}{}^c = \varepsilon_{abc}$.

Given a vector $X \in \mathfrak{g}$ in the tangent space of the identity e, one can generate the vector field by push-forward by the left translation L_g. Let us denote the corresponding vector field by X too. Since $(L_g)_* X = X$, it is called a **left-invariant vector field**. The flow generated by the vector field X is called **exponential map**, which can be expressed as a matrix

$$\exp(tX) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(tX)^\ell .$$

Therefore, for any matrix Lie group G, the exponential map defines a map $\exp : \mathfrak{g} \to G$.

Maurer-Cartan form

Given a Lie algebra $\mathfrak{g}$, it is also natural to think about its dual space $\mathfrak{g}^*$. This can be identified with the set of all left invariant 1-forms ω on G such that $L_g^* \omega = \omega$. Note that $\omega(X)$, $\omega(Y)$ are constant over G for $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$. Therefore, we have $Y(\omega(X)) = 0 = X(\omega(Y))$ so that (3.5) reduces to

$$d\omega(X, Y) = -\frac{1}{2}\omega([X, Y]) .$$

Therefore, if we take the basis $\omega^1, \cdots, \omega^{\dim \mathfrak{g}}$ dual to (9.2), we can write it as

$$d\omega^c = -\frac{1}{2}\sum_{a,b} f_{ab}{}^c\, \omega^a \wedge \omega^b .$$

Moreover, let $\omega \in \Omega^1(G; \mathfrak{g})$ be $\mathfrak{g}$-valued 1-form on G such that $\omega(X) = X$ for $X \in \mathfrak{g}$. Using the above basis, it is describe as

$$\omega = \sum_a \omega^a T_a , \tag{9.3}$$

which is called **Maurer-Cartan form**. Then, the equation above has the following form

$$d\omega = -\frac{1}{2}[\omega, \omega] ,$$

which is called the **Maurer-Cartan equation**.

Adjoint action

The **adjoint action** $G \circlearrowright G$ is defined by

$$\mathrm{Ad} : G \times G \to G; \quad (g, h) \mapsto ghg^{-1} . \tag{9.4}$$

The **adjoint action** $G \circlearrowright \mathfrak{g}$ is defined by

$$\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}; \quad (g, X) \mapsto gXg^{-1} . \tag{9.5}$$

This has the dual representation $G \circlearrowright \mathfrak{g}^*$ called the **coadjoint action**

$$\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^* \tag{9.6}$$

defined by

$$(\mathrm{Ad}^*_g \omega)(X) = \omega(\mathrm{Ad}_{g^{-1}} X) \,. \tag{9.7}$$

These representations have infinitesimal versions (derivations)

$$\begin{aligned} &\mathrm{ad}\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X,Y) \mapsto \mathrm{ad}_X(Y) = [X,Y] \\ &\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \qquad \mathrm{ad}^*_Y(\omega)(X) = \omega(-\mathrm{ad}_Y X)\,, \quad X,Y \in \mathfrak{g}, \quad \omega \in \mathfrak{g}^* \end{aligned} \tag{9.8}$$

G-action

In general, a left G-action on a space M is defined by

$$G \times M \to M;\ (g,p) \mapsto gp$$

such that $g \cdot (h \cdot p) = (gh) \cdot p$, $e \cdot p = p$. A right G-action

$$M \times G \to M;\ (p,g) \mapsto pg$$

is similarly defined as $(ph)g = p(hg)$. When $M = G$, (9.1) define the left and right G-action on itself.

For a closed subgroup $H \subset G$, the quotient space G/H becomes a manifold called the **homogenous space** and it receives the left G action. The projection $\pi : G \to G/H$ is a smooth map between manifolds. Moreover, the tangent space at H is identified with

$$T_H(G/H) \cong \mathfrak{g}/\mathfrak{h}\,.$$

When a Lie group G acts on a manifold M, $O_G(p) := Gp = \{gp \mid g \in G\} \subset M$ is called the G**-orbit** of $p \in M$. If $O_G(p) = M$, the G-action on M is **transitive**. A subgroup $G_p := \{g \in G \mid gp = p\}$ is indeed a closed subgroup called the **stabilizer (isotropy) subgroup** of p. As a result, we can a diffeomorphism

$$G/G_p \simeq M\,.$$

Example 9.5. The $\mathrm{SO}(n+1)$-orbit of a unit vector $\mathbf{e}_1 = {}^t(1,0,\ldots,0) \in \mathbb{R}^{n+1}$ is an n-sphere S^n and its stabilizer subgroup is $O_{\mathrm{SO}(n+1)}(\mathbf{e}_1) = \mathrm{SO}(n)$. Therefore,

$$\mathrm{O}(n+1)/\mathrm{O}(n) \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n) \simeq S^n$$

Note that $\mathrm{O}(n) = \mathrm{SO}(n) \sqcup \mathrm{diag}(-1,1,\ldots,1)\mathrm{SO}(n)$.

Example 9.6. The n-dimensional real projective space is $\mathbb{RP}^n \cong S^n/\mathbb{Z}_2$, and $\mathrm{O}(1) \cong \mathbb{Z}_2$.

$$\mathbb{RP}^n \simeq \mathrm{O}(n+1)/(\mathrm{O}(1) \times \mathrm{O}(n))$$

In fact, $\mathrm{O}(1) \times \mathrm{O}(n)$ is the stabilizer subgroup of a line $\ell \in \mathbb{RP}^n$. Similarly, the n-dimensional complex projective space is

$$\mathbb{CP}^n \simeq \mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n))$$

10 Vector bundles and Principal G-bundles

We have learned tangent bundles, cotangent bundles and their tensor products. Generalizing these leads to a notion called vector bundles where a fiber is a vector space, namely $\mathbb{R}^r$. Further generalizations will lead to fiber bundles where a fiber is a general manifold. Among them, principal G-bundles play a distinctive role where a fiber is a Lie group G and transition funtions take the value on G. Remarkably, the notion of vector bundles and principal G-bundles is indispensable for the description of non-Abelian gauge theories. I would recomend you to [Mor01, BT82a] for this subject.

10.1 Vector bundles

The notion of vector bundles was introduced by Whitney.

Definition 10.1 (Vector bundle). A **vector bundle** of rank r on M is a smooth manifold E with a smooth **projection** $\pi : E \to M$ such that

1. For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an r-dimensional vector space,
2. For all $p \in M$, there is an open $U \subseteq M$ containing p and a diffeomorphism

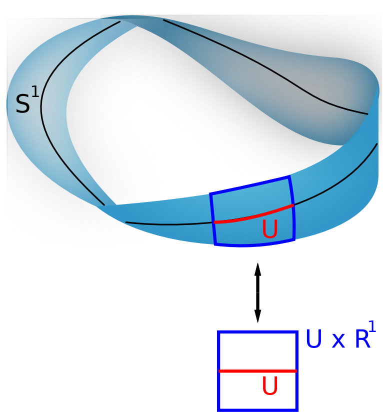
$$t : E_U = \pi^{-1}(U) \to U \times \mathbb{R}^r$$

such that

$$\begin{tikzcd} E_U \arrow[r, "t"] \arrow[d, "\pi"] & U \times \mathbb{R}^r \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

commutes, and the induced map $E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism for all $q \in U$.

We call t a **trivialization** of E over U; call E the **total space**; call M the **base space**. Also, for each $q \in M$, the vector space $E_q = \pi^{-1}(\{q\})$ is called the **fiber** over q. If $r = 1$, it is called **line bundle**.



Definition 10.2 (Transition function). Suppose that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ and t_β :

$E|_{U_\beta} \to U_\beta \times \mathbb{R}^r$ are trivializations of E. Then

$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is fiberwise linear, i.e.

$$t_\alpha \circ t_\beta^{-1}(q, v) = (q, g_{\alpha\beta}(q)v),$$

where $g_{\alpha\beta}(q)$ is in $\mathrm{GL}(r, \mathbb{R})$.
In fact, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ is smooth. Then $g_{\alpha\beta}$ is known as the **transition function** from β to α.

We have the following equalities for transition functions:

$$\begin{aligned} &(1) \quad g_{\alpha\alpha} = \mathrm{id} \\ &(2) \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1} \\ &(3) \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \text{ , which is called } \textbf{cocycle condition}. \end{aligned} \tag{10.1}$$

On the other hand, given an open cover $\{U_\alpha\}$ of open sets of M, suppose we have transition functions $g_{\alpha\beta}$ which satisfy all the above properties. Then, we can glue $U_\alpha \times \mathbb{R}^n$ and $U_\beta \times \mathbb{R}^n$ by the transition functions $g_{\alpha\beta}$ and construct a bundle $E \to M$.

Example 10.3. We construct a line bundle L over the real projective space $\mathbb{R}P^n$ as follows. First we consider the direct product $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. An arbitrary point ℓ in $\mathbb{R}P^n$ can be regarded as a line through the origin in $\mathbb{R}^{n+1}$. We construct a line bundle by

$$\begin{aligned} L &= \{(\ell, y) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid y \in \ell\} \\ &= \{([x^0; \cdots; x^n], y) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} | \, y = \lambda \cdot (x^0, \cdots, x^n) \,, \quad \lambda \in \mathbb{R}\} \,. \end{aligned} \tag{10.2}$$

This line bundle is called **Hopf line bundle** or **tautological line bundle**. For an open subset $U_i = \{\ell = [x^0; x^1; \cdots; x^n] | x_i \neq 0\}$, a trivialization map is given by

$$t_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R} \,; \; (\ell, y) \mapsto y^i \,.$$

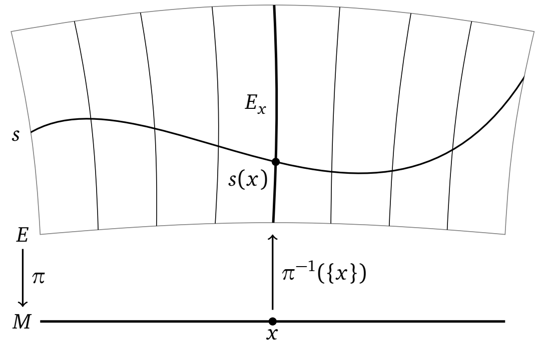
Then, the transition map is given by

$$t_j \circ t_i^{-1} : (U_i \cap U_j) \times \mathbb{R} \to (U_i \cap U_j) \times \mathbb{R} \,; \; (\ell, \eta) \mapsto \left(\ell, \frac{x^j}{x^i} \eta\right)$$

It is easy to see that the transition functions obey (10.1) above.

Definition 10.4 (Section). A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$. In other words, $s(p) \in E_p$ for each $p \in M$. We denote a set of sections by $\Gamma(M, E)$.

Definition 10.5 (Bundle map). We can consider about maps between vector bundles. Let $E \to M$ and $E' \to M'$ be vector bundles. A **bundle map** from E to E' is a pair of



smooth maps $(F : E \to E', f : M \to M')$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd}.$$

i.e. such that $F_p : E_p \to E'_{f(p)}$ is linear for each p.

Two vector bundles $(E_i, \pi_i, M)\,, i = 1, 2$, over the same base M are said to be **isomorphic** if there is a bundle map $E_1 \to E_2$ over the identity map of M. In this case, we write $E_1 \cong E_2$.

In fact, every operation on vector spaces can be performed on vector bundles, by doing it on each fiber.

Definition 10.6 (Whitney sum of vector bundles). Let $\pi : E \to M$ and $\rho : F \to M$ be vector bundles. The **Whitney sum** is given by

$$E \oplus F = \{(e, f) \in E \times F : \pi(e) = \rho(f)\}.$$

This has a natural map $\pi \oplus \rho : E \oplus F \to M$ given by $(\pi \oplus \rho)(e, f) = \pi(e) = \rho(f)$. This is again a vector bundle, with $(E \oplus F)_p = E_p \oplus F_p$ and again local trivializations of E and F induce one for $E \oplus F$.

Tensor products can be defined similarly.

Definition 10.7 (Tensor product of vector bundles). Given two vector bundles E, F over M, we can construct $E \otimes F$ similarly with fibers $(E \otimes F)|_p = E|_p \otimes F|_p$.

Similarly, we can construct the alternating product of vector bundles $\Lambda^n E$. Finally, we have the **dual** vector bundle.

Definition 10.8 (Dual vector bundle). Given a vector bundle $E \to M$, we define the **dual vector bundle** by

$$E^* = \bigcup_{p \in M} (E_p)^*.$$

Suppose again that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ is a local trivialization. Taking the dual of this map gives

$$t_\alpha^* : U_\alpha \times (\mathbb{R}^n)^* \to E|_{U_\alpha}^*.$$

since taking the dual reverses the direction of the map. We pick an isomorphism $(\mathbb{R}^n)^* \to \mathbb{R}$ once and for all, and then reverse the above isomorphism to get a map

$$E|_{U_\alpha}^* \to U_\alpha \times \mathbb{R}^n.$$

This gives a local trivialization.

One important operation we can do on vector bundles is **pullback**:

Definition 10.9 (Pullback of vector bundles). Let $\pi : E \to M$ be a vector bundle, and $f : N \to M$ a map. We define the **pullback**

$$f^*E = \{(y, e) \in N \times E : f(y) = \pi(e)\}.$$

This has a map $f^*\pi : f^*E \to N$ given by projecting to the first coordinate. The vector space structure on each fiber is given by the identification $(f^*E)_y = E_{f(y)}$. It is a little exercise in topology to show that the local trivializations of $\pi : E \to M$ induce local trivializations of $f^*\pi : f^*E \to N$.

Definition 10.10 (Subbundles and quotient bundles). Let $\pi : E \to M$ be a vector bundle over M. A vector bundle $\pi : F \to M$ is called a **subbundle** if F is a submanifold of E such that, for each point $p \in M$, the fiber F_p is a vector subspace of the fiber E_p of E.
Given a subbundle F of E, consider the quotient subspace E_p/F_p for each point $p \in M$, and set

$$E/F = \bigcup_{p \in M} E_p/F_p$$

We can verify that the natural projection $\pi : E/F \to M$ is a vector bundle, called the **quotient bundle** of E by F.

Example 10.11. If $f : M \hookrightarrow N$ is an embedding, $TM \to M$ is a subbundle of the pullback bundle $f^*TN \to M$. The quotient bundle $f^*TN/TM = NM$ is a vector bundle over M called the **normal bundle** of M.

One can introduce a metric g on fibers of a vector bundle E. Namely, we have a non-degenerate symmetric positive-definite 2-form on a fiber E_p

$$g_p : E_p \times E_p \to \mathbb{R}\ ,$$

and g_p is differentiable in terms of x. If it is the tangent bundle TM, it is a Riemannian metric. Given a trivialization $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$, we can take an orthonormal frame $(e_1, \cdots, e_r)$ for $e_i \in \Gamma(U, E)$ with respect to g. Then, the transition function takes the value at $\mathrm{O}(r)$

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(r)\ .$$

We can further generalize that the transition function takes the value at an arbitrary Lie group G with a representation $\rho : G \to \mathrm{GL}(V)$.

Definition 10.12 (G-bundle). Let V be a vector space, G a Lie group, and $\rho : G \to \mathrm{GL}(V)$ a representation. Then a G-bundle $\pi : E \to M$ consists of the following data:

1. For each $p \in M$, the fiber is $\pi^{-1}(p) \cong V$.
2. One can take a trivializing cover $\{U_\alpha\}$ with transition functions $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V$.
3. The transition functions are constructed by maps $g_{\alpha\beta} : U_\alpha \cap U_\beta :\to G$ satisfying the cocycle conditions with the representation ρ such that $g_{\alpha\beta} = \rho \circ g_{\alpha\beta}$.

10.2 Principal G-bundles

We have studied vector bundles where a fiber is a vector space. We can further generalize it to a **fiber bundle** where a fiber is a general manifold F and a transition function is given by a diffeomorphism of F. Among fiber bundles, principal G-bundles play an important role in physics.

Definition 10.13 (Principal G-bundle). Let G be a Lie group, and M a manifold. A **principal G-bundle** is a smooth manifold P with a projection $\pi : P \to M$ such that a fiber is $\pi^{-1}(\{x\}) \cong G$ for each $x \in M$. More precisely, we are given an open cover $\{U_\alpha\}$ of M and diffeomorphisms

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "t"] \arrow[d, "\pi"] & U \times G \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

such that the transition functions

$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$$

is of the form

$$(q, g) \mapsto (q, g_{\alpha\beta}(q) \cdot g)$$

for some $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ where G is called the **structure group**. Transition funcitons $\{g_{\alpha\beta}\}$ obey the three conditions (10.1).

A bundle map for two principal G-bundles is defined in a similar manner to Definition 10.5, and a bundle map over the identity map of M defines the isomorphism between two principal G-bundles.

For $g \in G$ we can define the right action R_g on the total space P

$$R_g P \to P; u \mapsto ug$$

where each fiber onto itself. For a principal bundle to be trivial it is necessary and sufficient that it admits a section. Namely, if there is a section $s : M \to P$, one can have a trivialization by setting $s(p) = e$ and the other points can be specified in $M \times G$ by the right G-action.

Given a principle G-bundle and a representation $\rho : G \to \mathrm{GL}(V)$ for a vector space V, we can construct **associated vector bundle**

$$E = P \times_\rho V$$

as follow. Let us consider the direct product $P \times V$ and G action

$$(u, y) \mapsto (us, \rho(s)^{-1}y) \qquad \text{for} \quad s \in G$$

We define the associated bundle as the quotient space $P \times_\rho V := (P \times V)/G$. Conversely, given a G-bundle $E \to M$ with fiber V, there is a canonical way of producing a principal G-bundle by using transition functions.

10.3 Connections and curvatures

In Riemannian geometry §5, we have learned Levi-Civita connections and Riemann curvature. Even in vector bundles and principal G-bundles, we can introduce connections and curvatures.

Definition 10.14 (Connection). A **connection** in a vector bundle $\pi : E \to M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E); (X, s) \mapsto \nabla_X s$$

satisfying

1. $\nabla_{fX}(s) = f\nabla_X(s)$
2. Leibnitz property: $\nabla_X(fs) = (Xf)s + f(\nabla_X s)$

for all $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

If a vector bundle E has a metric g, a connection ∇ is **compatible** with the metric if

$$Xg(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2)$$

for any $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in \Gamma(E)$.

Definition 10.15 (Curvature). A **curvature** in a vector bundle $\pi : E \to M$ is a trilinear map

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E); (X, Y, s) \mapsto F(X, Y)s$$

defined by

$$F(X, Y)s = \left[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right] s \, .$$

It has the following properties

1. $F(X, Y)s = -F(Y, X)s$
2. $F(fX, gY)(hs) = fghF(X, Y)s$ for $f, g, h \in C^\infty(M)$

for all $s \in \Gamma(E)$.

For a certain open subset of M, we can take a frame $s_1, \cdots, s_r \in \Gamma(\pi^{-1}(U))$. For any vector field X on U, the connection can be locally written as

$$\nabla_X s_j = \sum_{i=1}^{r} s_i A^i{}_j(X)$$

where $A^i{}_j \in \Omega^1(U, \mathfrak{gl}(r,\mathbb{R}))$ (1-form on U taking its value on $\mathfrak{gl}(r,\mathbb{R})$) is called **connection form**. We now look at the curvature R from differential forms.

$$F(X,Y)s_j = \sum_{i=1}^{r} s_i F^i{}_j(X,Y)$$

where $F^i{}_j \in \Omega^2(U, \mathfrak{gl}(r,\mathbb{R}))$ (2-form on U taking its value on $\mathfrak{gl}(r,\mathbb{R})$) is called **curvature form**. They are related by the following equation:

$$F = dA + A \wedge A \ .$$

The curvature form satisfies the Bianchi identity

$$dF - F \wedge A + A \wedge F = 0 \ . \tag{10.3}$$

More explicitly, in physics, we write a section $s = \sum_{j=1}^{r} v^j(x) s_j$ on U so that

$$\nabla_{\frac{\partial}{\partial x^\mu}} s = \sum_{i=1}^{r} s_i \left[\frac{\partial}{\partial x^\mu} v^i(x) + (A_\mu)^i{}_j v^j(x) \right] \ .$$

Also, the curvature can be written in terms of local coordinates

$$F\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu] \ .$$

In the case of Maxwell $\mathrm{U}(1)$ theory, the last term vanishes because it is a commutative group.

It is useful to know how the connection transforms under a change of local trivialization. Given a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{R})$, the gauge fields on U_α and U_β are related by

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d(g_{\alpha\beta}) \ , \tag{10.4}$$

and the curvature forms are related by

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta} \tag{10.5}$$

In a similar manner, one can construct connections and curvatures for G-bundle where we have to replace the first term of (10.4) by the adjoint representation of G on $\mathfrak{g}$, and the second of (10.4) by the Maurer–Cartan form (9.3).

In particular, for the Maxwell theory, the gauge transformation can be written as $g_{\alpha\beta} = e^{i\lambda_{\alpha\beta}(x)}$ so that

$$A_\beta = A_\alpha + i d\lambda_{\alpha\beta}$$

and the curvature form stays invariant.

Parallel transport and holonomy group

Given a connection ∇ on a vector bundle E, on can define **horizontal** directions in the space $\Gamma(E)$ of sections. A section $s \in \Gamma(E)$ is **parallel** along a path $\gamma : I \to (M)$

$$\nabla_{\dot\gamma(t)} s = 0 \quad \text{for} \quad t \in I \ .$$

In terms of local coordinates, it can be written as

$$\frac{ds_i}{dt} + \sum_{j=1}^{r} s_j (A_\mu)^j{}_i \frac{dx^\mu}{dt} = 0\,.$$

A theorem of ordinary differential equations tells us that given an initial data $s(t=0) \in E_{\gamma(0)}$, one can do parallel transform along $\gamma(t)$ so that we have a map

$$E_{\gamma(0)} \ni s(t=0) \to s(t=1) \in E_{\gamma(1)}\,.$$

In particular, if we consider a curve $p = \gamma(0) = \gamma(1)$, we obtain a map $\tau_\gamma : E_p \to E_p$. For given two curves γ_1 and γ_2, we can have a multiplication

$$\tau_{\gamma_1 \circ \gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2}$$

and the inverse is defined by

$$\tau_{\gamma^{-1}} = \tau_\gamma^{-1}\,,$$

so that it forms a group called **holonomy group**. In the case of $\mathrm{U}(1)$, this is the origin of the **Aharonov-Bohm effect**.

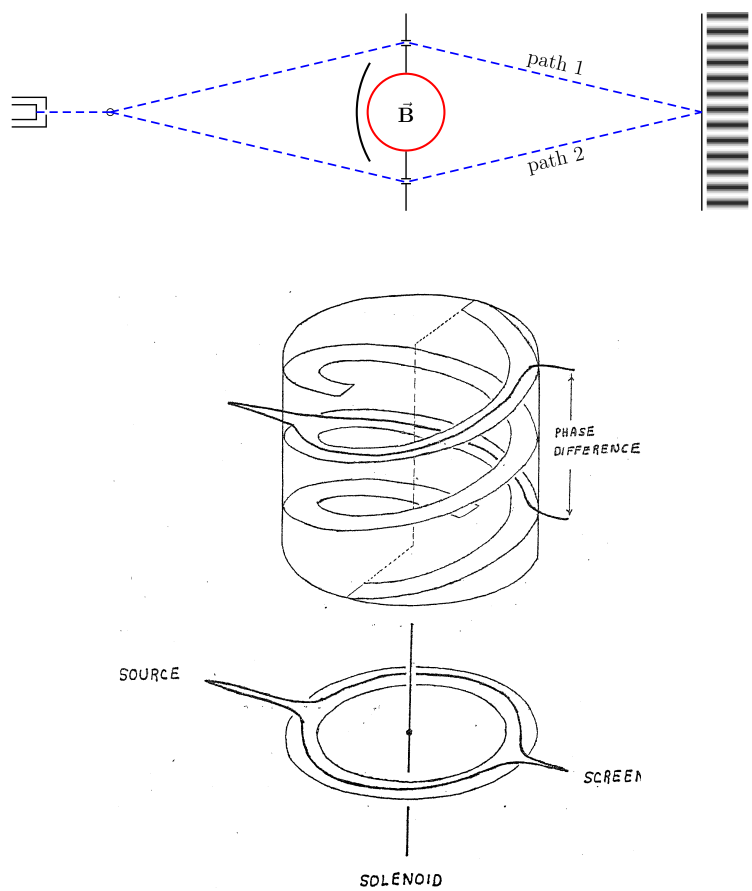


Figure 7: The Aharonov-Bohm solenoid effect takes place when the wave function of a charged particle passing around a long solenoid experiences a phase shift as a result of the enclosed magnetic field, despite the magnetic field being negligible in particle trajectories.

Levi-Civita connections

We can consider the tangent bundle TM as a vector bundle and its metric g is indeed a Riemannian metric. As we have seen in §5, there is the unique natural connection called **Levi-Civita connections** in TM. Let $X_1, \cdots, X_n$ be an orthonormal frame vector field on an open set $U \subset M$ and their dual $e^1, \cdots, e^n \in \Omega^1(U)$. For the Levi-Civita connection ∇, we can write

$$\begin{aligned} \nabla_{X_j} X_i &= \sum_k \mathbf{\Gamma}^k_{ij} X_k \ , \\ R(X_i, X_j) X_k &= \sum_l \mathbf{R}^l{}_{kij} X_l \ . \end{aligned}$$

We can define connection one-form and curvature two-form taking their values on $\mathfrak{so}(n, \mathbb{R})$:

$$\omega^k{}_j = \sum_k \mathbf{\Gamma}^k_{ij} e^i \ , \quad \Omega^l{}_k = \sum_l \mathbf{R}^l{}_{kij} e^i \wedge e^j \ . \tag{10.6}$$

They satisfy the following conditions

$$\begin{aligned} de^i &= -\sum_j \omega^i{}_j \wedge e^j \ , \\ \Omega^i{}_j &= d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \ . \end{aligned}$$

Ehresmann connections

Similarly, one can construct the theory of connections on principal G-bundles, which are called **Ehresmann connections**. In principle, an Ehresmann connection determines the horizontal direction of a principal G-bundle.

Definition 10.16 (Ehresmann connection). A **Ehresmann connection** $A \in \Omega^1(P, \mathfrak{g})$ on a principle G-bundle $\pi : P \to M$ is a one-form taking its value on $\mathfrak{g}$, which satisfies the following conditions:

1. Given $X \in \mathfrak{g}$, there is the corresponding vector field $\overline{X}$ on P

$$\overline{X}_u = \frac{d}{dt}(u \cdot \exp tX)\Big|_u \quad \text{for} \quad u \in P \ .$$

   Then, A is subject to

$$A(\overline{X}) = X \in \mathfrak{g} \ .$$

2. For $^\forall g \in G$, under the right action R_g, it behaves as

$$R_g^*(A) = \mathrm{ad}(g^{-1}) A \ .$$

   In other words,

$$A_{ug}(R_g(Y)) = \mathrm{ad}(g^{-1})(A_u(Y)) \ , \quad \text{for} \quad Y \in \mathfrak{g} \ .$$

For an open set $U_\alpha \subset M$, there is a local trivialization $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. Then, we have a section on $\pi^{-1}(U_\alpha)$

$$\sigma_{U_\alpha} : U_\alpha \to P; x \mapsto t_\alpha^{-1}(x, e) \ .$$

Then, if we define $A_\alpha = \sigma^*_{U_\alpha}(A)$, we have cocycle condition

$$A_\beta = g^{-1}_{\alpha\beta} A_\alpha g_{\alpha\beta} + g^{-1}_{\alpha\beta} d(g_{\alpha\beta}) , \quad \text{for} \quad U_\alpha \cap U_\beta ,$$

where $g_{\alpha\beta}$ is the transition function on $U_\alpha \cap U_\beta$.

We denote the space of Ehresmann connections on a principal G-bundle P by $\mathscr{A}_P$. The space $\mathscr{G}_P$ of gauge transformations is the section $\Gamma(M, G_P)$ of the bundle $G_P := P \times_{Ad} G$. A gauge transformation $g \in \mathscr{G}_P$ acts on the space of connection $\mathscr{A}_P$ via

$$\mathscr{G}_P \times \mathscr{A}_P \to \mathscr{A}_P; (g, A) \mapsto g^*(A) = \{g^*(A)_U, g^*(A)_V, \cdots\}$$

where

$$g^*(A)_U = g_U^{-1} d(g_U) + g_U^{-1} A g_U .$$

If $A' = g^*(A)$ for $A, A' \in \mathscr{A}_P$, they are physically identical, and we denote the space of physically different connections by

$$\mathscr{B}_P := \mathscr{A}_P / \mathscr{G}_P .$$

10.4 Yang-Mills theory

Now we can describe non-Abelian gauge theory called **Yang-Mills theory** [YM54]. Let us consider a principal G-bundle or its associated G-bundle. In addition, let A be a connection on it and F be its curvature. The classical **Yang-Mills action** can be written as

$$\begin{aligned} S_{YM}[A] &= \frac{1}{2g_{YM}^2} \int_M \operatorname{Tr} F \wedge *F \\ &= \frac{1}{2g_{YM}^2} \int_M \operatorname{Tr} \left(F_{\mu\nu} F^{\mu\nu}\right) \sqrt{g} d^d x, \end{aligned} \tag{10.7}$$

where the curvature is 2-form taking its value on the Lie algebra $\mathfrak{g}$. The action is almost the same as that of the Maxwell theory. However, due to the gauge transformation (10.5) of the curvature form, we need to take Tr over Lie algebra $\mathfrak{g}$ in order for the action to be gauge-invariant. The parameter g_{YM} is called the Yang-Mills coupling constant. For the flat space, we have $\sqrt{g} = 1$ so that we will drop this term.

For example, if $G = \mathrm{SU}(N)$, we can choose the basis of the Lie algebra $\mathfrak{su}(N)$

$$\operatorname{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab},$$

and on a local $U \subseteq M$, we have

$$S_{YM}[A] = \frac{1}{4g_{YM}^2} \int F^a_{\mu\nu} F^{b,\mu\nu} \delta_{ab} \, d^d x,$$

with $F_{\mu\nu} = \sum_a F^a_{\mu\nu} T_a$ and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu .$$

Thus, Yang-Mills theory is the natural generalization of Maxwell theory to the non-Abelian case.

At the level of the classical field equations, if we vary our connection by $A_\mu \mapsto A_\mu + \delta a_\mu$, where δa is a matrix-valued 1-form, then we have

$$\delta F_{\mu\nu} = \partial_{[\mu}\delta a_{\nu]} + [A_\mu, \delta a_\nu].$$

The equation of motion can be obtained by taking the variation of the Yang-Mills action

$$\delta S_{YM}[A] = \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\big(\delta F_{\mu\nu}, F^{\mu\nu}\big)\ d^d x = \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\big(\nabla_\mu \delta a_\nu, F^{\mu\nu}\big)\ d^d x = 0.$$

Therefore, the **Yang–Mills equation** is

$$\nabla^\mu F_{\mu\nu} := \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0\ ,$$

or we can write it without coordinates

$$\delta_A F := *d_A * F = *(d+A) * F = 0\ .$$

Recall we also have the Bianchi identity (10.3) which can be expressed in terms of a local coordinate

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0\ .$$

Unlike Maxwell's equations, these are non-linear PDE's for A. We no longer have the principle of superposition, which is similar to Einstein's equation (5.13).

The Yang-Mills path integral is expressed by

$$Z_{YM} = \int_{\mathscr{A}/\mathscr{G}} DA\ \exp(iS_{YM}[A])\ .$$

You can try to solve one of the seven Millennium Prize Problems: quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap [JW06].

11 Characteristic classes

As we (will) see in homework, principal $U(1)$-bundles on S^2 are classified by the monopole number n which tells us how the $U(1)$ fibers over the upper hemisphere and the lower hemisphere are glued together. The generalization of this notion for a vector bundle $\pi : E \to M$ leads to a **characteristic class** associated to a cohomology class of M. Characteristic classes were introduced to extract topological information of a base manifold of vector bundles or principal G-bundles from curvature forms. This is called **Chern-Weil theory**, on which [Mor01, §6] is a wonderful exposition.

Characteristic classes are constructed as **invariant polynomials** of the curvature $F = dA + A \wedge A$. As we see in (10.5), F transforms as $F \to g^{-1}Fg$ under the gauge transformation where $g \in \mathscr{G}_E$. To construct characteristic classes, we need to introduce invariant polynomials $P(X)$ of matrixes, that is invariant under the conjugation, $P(g^{-1}Fg) = P(F)$. Examples of invariant polynomials are $\mathrm{Tr}\, F^k$ $(k = 1, 2, \cdots)$ and $\det F$. In fact, we can use these to construct nice bases of invariant polynomials as follows:

(1) $\sigma_k(F)$ defined by

$$\det(1 + tF) = 1 + t\sigma_1(F) + t^2\sigma_2(F) + \cdots + t^r\sigma_r(F)\ .$$

(2) $s_k(F)$ defined by

$$s_k(F) = \mathrm{Tr}\, F^k\,, \qquad (k = 1, \cdots, r)\,.$$

They are related to each other by Newton's formula,

$$s_1 = \sigma_1\,, \quad s_2 = \sigma_1^2 - 2\sigma_2\,, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3\,, \cdots$$

We can also express the invariant polynomials in terms of eigenvalues. If F is a hermitian matrix, we can diagonalize it with eigenvalues $x_1, \cdots, x_k$. Then,

$$\prod_{k=1}^{r}(1 + tx_k) = 1 + t\sigma_1(x) + \cdots + t^r\sigma_r(x).$$

Similarly, $s_k(F) = \sum_{j=1}^{r}(x_j)^k$.

If $P_k(F)$ is an invariant polynomial of degree k, we can use the curvature 2-form F to define a $2k$-form $P_k(F)$ so that it is invariant under the gauge transformation, $F \to g^{-1}Fg$. It is also a closed-form because the Bianchi identity

$$dF + [A \wedge F] = 0$$

tells us

$$\begin{aligned}
d\ \mathrm{Tr}\, F^k &= \mathrm{Tr}\Big(dF\ F^{k-1} + F\ dF\ F^{k-1} + \cdots + F^{k-1}dF\Big) \\
&= -\,\mathrm{Tr}\Big(([A \wedge F])F^{k-1} + \cdots + F^{k-1}([A \wedge F])\Big) \\
&= -\,\mathrm{Tr}\Big(A \wedge F^k - F^k \wedge A\Big) = 0
\end{aligned}$$

Thus, we find $P_k(F) \in H^{2k}(M)$.

Moreover, $P_k(F)$ is invariant under continuous deformation of the gauge field A as an element of $H^{2k}(M)$. Suppose that we change $A \to A + \eta$ with η being an infinitesimal one-form. Under this deformation, F changes by $\delta F = d\eta + A \wedge \eta + \eta \wedge A$. Therefore,

$$\begin{aligned}
\delta\, \mathrm{Tr}\, F^k &= \mathrm{Tr}\Big((d\eta + A \wedge \eta + \eta \wedge A)F^{k-1} + \cdots + F^{k-1}(d\eta + A \wedge \eta + \eta \wedge A)\Big) \\
&= k\, \mathrm{Tr}\Big((d\eta + A \wedge \eta + \eta \wedge A)F^{k-1}\Big) \\
&= k\, \mathrm{Tr}\Big(d\eta F^{k-1} - \eta dF F^{k-2} - \cdots - \eta F^{k-2}dF\Big) \\
&= k\ d\, \mathrm{Tr}\Big(\eta F^{k-1}\Big)\,.
\end{aligned}$$

Since both η and F transform homogeneously under the gauge transformation, $\mathrm{Tr}\big(\eta F^{k-1}\big)$ is a well-defined $(2k-1)$-form. Thus, under any infinitesimal deformation, $P_k(F)$ changes by an exact form. Thus, $P_k(F)$ depends only on the type of the bundle E and not on a choice of the connection A on E. For a more complete proof, please refer to [Mor01, Proposition 5.28].

11.1 Pontryagin classes

Let us consider a vector bundle $\pi : E \to M$ of rank r. One can always put a metric g on E and consider a connection A which is compatible with the metric g. Therefore, we can consider the curvature form F takes its value on $\mathfrak{so}(r)$. Namely it is an anti-symmetric $(F^i{}_j + F^j{}_i = 0)$ 2-form so that $\operatorname{Tr} F^k = 0$ if k is odd. Therefore, for the invariant polynomial P_k of odd degree, we have

$$P_k(F) = 0\ .$$

As a result, the **Pointryagin class** is defined as

$$p_k(E) := \frac{1}{(2\pi)^{2k}} \sigma_{2k}(F) \in H^{4k}(M; \mathbb{R})$$

They may be written as

$$p(E) = \det\left(1 + \frac{1}{2\pi} F\right) = 1 + p_1(E) + p_2(E) + \cdots + p_{[r/2]}(E)\ .$$

Theorem 11.1 (Hirzebruch signature theorem). Let M be an oriented compact 4-dimensional manifold. Since the Hodge star $* : H^2(M) \to H^2(M)$ satisfies $*^2 = 1$ on $H^2(M)$, we can decompose it into $H^2(M) = H^2_+(M) \oplus H^2_-(M)$ with eigenvalues ± 1 of $*$. Let us define the signature of M by

$$\tau(M) = \dim H^2_+(M) - \dim H^2_-(M)\ .$$

Then, the signature can be expressed by

$$\tau(M) = \frac{1}{3} \int_M p_1(TM)\ .$$

11.2 Chern classes

Now let us consider complex vector bundle $\pi : E \to M$ of rank r. Similarly, one can always put a Hermitian metric g on E and consider a connection A which is compatible with the metric g. Then, the curvature form F takes its value $\mathfrak{u}(r)$. Namely it is a skew-Hermitian $(F^i{}_j + \overline{F}^j{}_i = 0)$ 2-form so that $\operatorname{Tr} \left(\frac{F}{2\pi i}\right)^k$ is a real $2k$-form.

Then, **Chern class** is defined as

$$c_k(E) := \left(\frac{-1}{2\pi i}\right)^k \sigma_k(F) \in H^{2k}(M; \mathbb{R})\ .$$

This can be written as

$$c(E) = \det\left(1 - \frac{1}{2\pi i} F\right) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)\ .$$

For example, we can explicitly write

$$c_1 = \frac{-1}{2\pi i} \operatorname{Tr} F\ , \quad c_2 = -\frac{1}{8\pi^2}(\operatorname{Tr} F \wedge \operatorname{Tr} F - \operatorname{Tr} F \wedge F), \cdots$$

If the structure group is in $\mathrm{SU}(r) \in \mathrm{U}(r)$, we have a trivial first Chern class $c_1 = 0$ because $\mathfrak{su}(r)$ is traceless.

Instead of σ_k, we can use s_k for invariant polynomials, which defines the **Chern characters**

$$ch_k(E) = \frac{1}{k!}\operatorname{Tr}\left(-\frac{F}{2\pi i}\right)^k \in H^{2k}(M)\ . \tag{11.1}$$

We can also write it as

$$ch(E) = 1 + ch_1(E) + \cdots = \operatorname{Tr}\exp\left(-\frac{F}{2\pi i}\right)\ .$$

Some properties

The Pontryagin class and Chern class are related by

$$p_k(E) = (-1)^k c_{2k}(E\otimes\mathbb{C}) \in H^{4k}(M;\mathbb{R}) \tag{11.2}$$

where $E\otimes\mathbb{C}$ is the complexification of a real bundle $E \to M$. Since $\sigma_k = 0$, we have

$$c_k(E\otimes\mathbb{C}) = 0\ , \quad \text{for } k \text{ odd} \tag{11.3}$$

One of the important properties of the Pontryagin and Chern classes is that it behaves nicely when we take a direct sum $E_1 \oplus E_2$ of vector bundles E_1, E_2 as,

$$\begin{aligned} p(E_1\oplus E_2) &= p(E_1)\wedge p(E_2)\ ,\\ c(E_1\oplus E_2) &= c(E_1)\wedge c(E_2)\ . \end{aligned}$$

On the other hand, it does not behave nicely under the direct product $E_1 \otimes E_2$.

The Chern characters behave nicely under both the direct sum and direct product as,

$$\begin{aligned} ch(E_1\oplus E_2) &= ch(E_1) + ch(E_2)\ ,\\ ch(E_1\otimes E_2) &= ch(E_1)\wedge ch(E_2)\ . \end{aligned}$$

This property plays an important role in **K-theory**.

11.3 Euler class

Let us turn to a real vector bundle $\pi : E \to M$ of even rank $2r$. In this case, in addition to Tr and det, we can consider one more way to construct an invariant polynomial, which is called the **Pfaffian**,

$$Pf(F) = \frac{1}{2^r r!}\sum_{\sigma\in S_{2r}} \operatorname{sgn}(\sigma)\ F_{\sigma(1)\sigma(2)}F_{\sigma(3)\sigma(4)}\cdots F_{\sigma(2r-1)\sigma(2r)}\ .$$

Note that, for antisymmetric matrices, the Pfaffian is a square root of the determinant, $\det F = Pf(F)^2$. If F is real and anti-symmetric, we can block diagonalize it by $\mathrm{SO}(2r)$ as

$$F = \begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 & 0\\ -x_1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & x_2 & & & \\ 0 & 0 & -x_2 & 0 & & & \\ \cdot & \cdot & & & \cdots & \cdot & \cdot\\ 0 & 0 & & & \cdots & 0 & x_r\\ 0 & 0 & & & \cdots & -x_r & 0 \end{pmatrix} \tag{11.4}$$

We can then write the Pfaffian as

$$Pf(F) = \prod_{i=1}^{r} x_k.$$

Under the conjugation $F \to g^T F g$, the Pfaffian transforms as $Pf(g^T F g) = \det g \cdot Pf(F)$.

Thus, if $g \in \mathrm{SO}(2r)$, the Pfaffian is invariant. We can now define the **Euler class** by

$$e(E) = \frac{1}{(2\pi)^r} Pf(E) \in H^{2r}(M;\mathbb{R}) \ .$$

In fact, the Euler class can be understood as the square root of the highest Pontryagin class $p_r(E)$

$$p_r(E) = e(E)^2 \ .$$

In particular, let us consider the tangent bundle TM of an closed orientable Riemannian manifold (M, g) of dimensions $n = 2r$. Then TM is an $\mathrm{SO}(2r)$ bundle, and its Euler form can be written in terms of the Riemann curvatures

$$\begin{aligned} n = 2: \quad e(TM) &= \frac{1}{4\pi} \epsilon_{ab} \Omega^{ab} \ , \\ n = 4: \quad e(TM) &= \frac{1}{32\pi^2} \epsilon_{abcd} \Omega^{ab} \wedge \Omega^{cd} \ , \end{aligned}$$

where the Riemann curvatures are regarded as the 2-forms as in (10.6)

$$\Omega^a{}_b = \mathbf{R}^a{}_{bcd} e^c \wedge e^d = R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu \ .$$

The integral of the Euler class $e(TM)$ over M is indeed equal to the Euler characteristic

$$\chi(M) = \int_M e(TM) \ ,$$

which can be considered as the higher dimensional version of the Gauss-Bonnet theorem (5.12).

11.4 Todd, L- and $\widehat{A}$-classes

Here we introduce other characteristic classes, such as **Todd classes, Hirzebruch L-class**, and **$\widehat{A}$-class** [EGH80]. These classes are just defined by a different basis of invariant polynomials. However, there are very important since these classes will show up in the index theorem.

To describe Todd class, let $x_1, \cdots, x_k$ be eigenvalues of curvature form $\frac{-F}{2\pi i}$ of a complex vector bundle. For example, the total Chern classes can be expressed as

$$c(F) = \det\left(1 - \frac{F}{2\pi i}\right) = \prod_{i=1}^{r} (1 + x_k).$$

It is worth mentioning that the right-hand side takes the form $\prod_k c(L_k)$, where L_k is a line bundle with a curvature given by x_k and $c(L_k) = 1 + x_k$. Thus, as far as the Chern classes are concerned, the vector bundle E behaves like a Whitney (direct) sum of the line bundles $L_1 \oplus L_2 \oplus \cdots \oplus L_k$ although they are not isomorphic as bundles. This phenomenon is called the **splitting principle**, and x_i are called **Chern roots**. Using this notation, the Todd class is defined by,

$$Td(E) = \prod_k \frac{x_k}{1 - e^{-x_k}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \cdots . \tag{11.5}$$

Furthermore, for a real vector bundle E, (11.3) tells you that the Chern roots of its complexification $E \otimes \mathbb{C}$ come in opposite pairs $x, -x$. Moreover, the relation (11.2) tells us that Pontryagin classes of E are written as

$$p(E) = \prod_j \left(1 + x_j^2\right)$$

where x_j are Chern roots of $E \otimes \mathbb{C}$ with the positive sign in each pair. With this notation, the Hirzebruch L-classes for the real vector bundle E are defined by

$$L(E) = \prod_k \frac{x_k}{\tanh x_k} = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots .$$

Indeed Hirzebruch has introduced this class for the signature theorem 11.1 of a $4k$-dimensional closed oriented manifolds. The $\widehat{A}$-classes for the real vector bundle E are defined by,

$$\widehat{A}(E) = \prod_k \frac{x_k/2}{\sinh x_k/2} = 1 + \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots ,$$

which appears in the index theorem of a Dirac operator.

12 Index Theorem

Finally, we will study celebrated **Atiyah-Singer index theorem**. The index theorem states the equivalence between the index of an elliptic operator D, which is an analytic object, and the characteristic classes, which are a topological object. This theorem is one of the milestones in mathematics of the 20th century. The index theorem plays a crucial role in physics like an anomaly, fermion zero modes and supersymmetry. In fact, in the tasteful book [Yau09b], you can feel excitement when the index theorem has been formulated. I highly recommend you to read this book. Another good reference is [Hit10], in which the essense of the index theorem is concisely summarized.

12.1 Symbol, elliptic operator, analytic index

Let E and F be vector bundles of rank r over an n-dimensional M. In this section, we assume that M is closed and oriented. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator. Namely, on local trivializations $\pi_E^{-1}(U) \cong U \times \mathbb{R}^r$ and $\pi_F^{-1}(U) \cong U \times \mathbb{R}^r$ of E and F, it can be written as

$$D = \sum_{|\alpha| \le k} a_\alpha^{ij}(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $a_\alpha^{ij}(x)$ is an $M_r(\mathbb{R})$-valued function on U. It is said to be of **elliptic type** if for ${}^\forall\xi = (\xi_1, \cdots, \xi_n) \neq 0 \in \mathbb{R}^n$, the matrix

$$\sigma(D)(\xi) = \sum_{|\alpha|=k} a_\alpha^{ij}(x)\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

is non-singular. In other words, $\sigma(D)(\xi) : E_x \to F_x$ is an isomorphism for $\xi \neq 0$. Let us note that $\sigma(D)$ depends only on the highest order portion of D. It is called the **symbol** of D. For example, the Laplacian

$$\sum_i \frac{\partial^2}{\partial x_i^2}$$

is clearly of elliptic type. If M is compact and D is elliptic, it is of **Fredholm type**, meaning that $\operatorname{Ker} D$ and $\operatorname{Coker} D = \Gamma(F)/D(\Gamma(E))$ are finite-dimensional. We can define the **analytic index**

$$\operatorname{ind}(D) := \dim \operatorname{Ker}(D) - \dim \operatorname{Coker}(D) \ .$$

12.2 de Rham complex

Let M be a $2n$-dimensional closed oriented manifold. Let us define

$$\Omega^{\text{even}}(M) = \Omega^0(M) \oplus \Omega^2(M) \oplus \cdots$$
$$\Omega^{\text{odd}}(M) = \Omega^1(M) \oplus \Omega^3(M) \oplus \cdots$$

Then, the operator

$$d + \delta : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$$

is of elliptic type so that we can define its analytic index from (4.3) as the difference of dimensions of harmonic forms of even and odd degrees

$$\begin{aligned} \operatorname{ind}(d+\delta) &= \sum_{k;\text{even}} \dim \mathbb{H}^k(M) - \sum_{k;\text{odd}} \dim \mathbb{H}^k(M) \\ &= \sum_i (-1)^i \dim H^i_{dR}(M) = \chi(M) \ . \end{aligned} \tag{12.1}$$

Therefore, the index theorem for de Rham complex is

$$\operatorname{ind}(d+\delta) = \int_M e(TM) \ .$$

12.3 Dolbeault complex

In this course, we have not introduced to complex manifolds yet. However, one can complexify the definition of manifolds by taking care of "holomorphicity", and the basics is laid out in Appendix B. In the following, I just explain the index theorem for a complex vector bundle over a complex manifold, which is called **Hirzebruch-Riemann-Roch** theorem.

Let $E \to M$ be a holomorphic vector bundle over an n-dimensional complex manifold M and we consider Cauchy-Riemann operator

$$\begin{aligned} \overline{\partial} : \Gamma(E) \otimes \Omega^{0,p}(M) &\to \Gamma(E) \otimes \Omega^{0,p+1}(M) \\ \phi \, d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p} &\mapsto \sum_k \frac{\partial \phi}{\partial \overline{z}_k} d\overline{z}_k \wedge d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p} \end{aligned}$$

where $\Omega^{0,p}(M)$ is the set of anti-holomorphic p-forms locally spanned by $d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}$. Then, the following long exact sequence is called **Dolbeault complex**

$$0 \to \Gamma(E) \xrightarrow{\bar{\partial}} \Gamma(E) \otimes \Omega^{0,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Gamma(E) \otimes \Omega^{0,n}(M) \xrightarrow{\bar{\partial}} 0 \ .$$

In a similar fashion to de Rham cohomology, one can define Dolbeault cohomology

$$H^{0,*}(M, E) := \mathrm{Ker}(\bar{\partial}) / \mathrm{Im}(\bar{\partial}) \ .$$

If we put Hermitian metric on M and E, we obtain the adjoint operator ϑ of $\bar{\partial}$

$$\vartheta = * \cdot \bar{\partial} \cdot * : \Gamma(E) \otimes \Omega^{0,p}(M) \to \Gamma(E) \otimes \Omega^{0,p-1}(M) \ .$$

Then, the operator

$$\bar{\partial} + \vartheta : \bigoplus_{p:\text{even}} \Gamma(E) \otimes \Omega^{0,p}(M) \to \bigoplus_{p:\text{odd}} \Gamma(E) \otimes \Omega^{0,p}(M)$$

turns out to be elliptic, and its analytic index is

$$\mathrm{ind}(\bar{\partial} + \vartheta) = \sum_{p:\text{even}} \dim H^{0,p}(M, E) - \sum_{p:\text{odd}} \dim H^{0,p}(M, E) \ .$$

Then, the Hirzebruch-Riemann-Roch theorem states that the index can be expressed as the Chern character (11.1) of E and the Todd class (11.5) of $TM \otimes \mathbb{C}$

$$\mathrm{ind}(\bar{\partial} + \vartheta) = \int_M ch(E) \cdot Td(TM \otimes \mathbb{C}) \ .$$

Let us apply this theorem to the case that M is a Riemann surface Σ, and E is a complex line bundle L over Σ. Then, we have

$$\begin{aligned} \mathrm{ch}(L) &= 1 + c_1(L) \\ Td(T\Sigma \otimes \mathbb{C}) &= 1 + \frac{1}{2} c_1(T\Sigma) \end{aligned} \tag{12.2}$$

Now $c_1(T\Sigma) = -K_\Sigma$ where K_Σ is the canonical class (the first Chern-class $c_1(T^*\Sigma)$ of the cotangent bundle). Then, the theorem says

$$\begin{aligned} \dim H^0(\Sigma, L) - \dim H^1(\Sigma, L) &= \deg \left((1 + c_1(L)) \left(1 - \frac{1}{2} K_\Sigma \right) \right)_1 \\ &= \deg \left(-\frac{1}{2} K_\Sigma + c_1(L) \right) \\ &= \deg L - g + 1 \end{aligned}$$

This is the **Riemann-Roch theorem**. Note that the degree of a line bundle over a Riemann surface Σ is

$$\deg L = \int_\Sigma c_1(L)$$

12.4 Dirac operator

Another important example is the index of a Dirac operator. Unfortunately, due to time constraints, I cannot explain mathematics of Dirac equations, which involves Clifford algebra, spin group, spin representations, spinor bundle and Dirac operators. Instead, I will use an example in 4-dimension. I refer to [BGV03] for detail.

On a certain basis, the gamma matrices in $\mathbb{R}^4$ with Euclidean signature can be written as

$$\gamma_\mu = i\begin{pmatrix} 0 & \sigma_\mu \\ -\overline{\sigma}_\mu & 0 \end{pmatrix}\ , \quad \sigma_\mu = (I, -i\vec{\sigma})\ , \quad \overline{\sigma}_\mu = (I, i\vec{\sigma})\ .$$

In fact, one can check they satisfy the following algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\ .$$

Generalization of this anti-commutation relation leads to Clifford algebra. The massive Dirac equation on $\mathbb{R}^4$ can be written as

$$(i\gamma^\mu\partial_\mu - m)\psi = 0\ ,$$

which is the Euler-Lagrange equation for the action

$$S = \int d^4x\ \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\ .$$

The Dirac equation can be interpreted as the square root of the Klein-Gordon equation

$$(\Box + m^2)\phi = 0\ .$$

The solutions ψ to the massless Dirac equation

$$i\gamma^\mu\partial_\mu\psi = 0\ ,$$

are called **zero modes**. Defining

$$\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

the Dirac spinor ψ can be decomposed into eigenspaces $\mathcal{S}^\pm$ of γ_5 with eigenvalues $\pm$

$$\frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \mathcal{S}^+\ , \qquad \frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \mathcal{S}^-$$

which can be understood as left-handed and right-handed fermions, respectively. Now let us consider the Dirac equation on an oriented closed 4-manifold M. Then, the derivative is replaced by the covariant derivative

$$\partial_\mu \to \nabla_\mu = \partial_\mu + \omega_\mu$$

where ω_μ is taking its value on $\mathrm{Spin}(4) = \mathrm{SU}(2)\times\mathrm{SU}(2)$, which is called **spin connection**. The Dirac operator is defined

$$\not{D} = \begin{pmatrix} 0 & \not{\nabla}^\dagger \\ \not{\nabla} & 0 \end{pmatrix}$$

where
$$\nabla\!\!\!/ = -i\bar{\sigma}^{\mu}\nabla_{\mu}\,, \quad \nabla\!\!\!/^{\dagger} = i\sigma^{\mu}\nabla_{\mu}\,.$$
Then, we have the **spin complex**

$$\mathcal{S}^{+} \underset{\nabla\!\!\!/}{\overset{\nabla\!\!\!/^{\dagger}}{\rightleftarrows}} \mathcal{S}^{-}\,.$$

where $D\!\!\!/$ is of elliptic type. Then, the analytic index of the Dirac operator is
$$\mathrm{ind}(D\!\!\!/) = \dim \mathrm{Ker}\, \nabla\!\!\!/^{\dagger} - \dim \mathrm{Ker}\, \nabla\!\!\!/ = n_{+} - n_{-}$$
which is the difference between right-handed and left-handed zero modes. Then, the index theorem for the Dirac operator states that the index can be expressed as $\widehat{A}$-genus
$$\mathrm{ind}(D\!\!\!/) = \int_M \widehat{A}(TM)\,.$$

Usually (like QCD), we consider fermion interacting with non-Abelian gauge fields. To couple fermion to gauge field, we tensor the spinor bundle to a vector bundle E
$$\mathcal{S}^{\pm} \to \mathcal{S}^{\pm} \otimes E\,, \qquad \nabla_{\mu} \to \nabla_{\mu} + A_{\mu}\,,$$
Then, one can modify the spin complex accordingly and, in this case, the index theorem is
$$\mathrm{ind}(D\!\!\!/) = \int_M \widehat{A}(TM) \cdot ch(E)\,.$$

12.5 Anomaly

Although the classical Lagrangian of quantum chromodynamics (QCD) preserves a chiral $U(1)$ symmetry, it is not realized in QCD. The phenomena that the classical symmetry is broken at a quantum level is called **anomaly**.

The massless Dirac Lagrangian has a chiral symmetry which can be stated by
$$\partial_{\mu} j^{\mu 5} = 0\,, \qquad j^{\mu 5} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi\,.$$
If we integrate it out
$$\int d^4x\, \partial_{\mu} j^{\mu 5} = n_{+} - n_{-} = 0\,,$$
the difference in the number of right-handed fermions and left-handed fermions is conserved at a classical level. However, if we take quantum effect into account in QCD, this is no longer true.

As we have seen above, the difference between right-handed and left-handed fermion zero modes is given by
$$n_{+} - n_{-} = \int_{\mathbb{R}^{1,3}} \widehat{A}(\mathbb{R}^{1,3}) \cdot ch(E) = \frac{1}{8\pi^2} \int_{\mathbb{R}^{1,3}} \mathrm{Tr}(F \wedge F)\,.$$
In the presence of instanton effect, the right hand side is no longer zero so that the chiral symmetry is broken. This is called **Adler-Bell-Jackiw**, **chiral** or **triangle anomaly** [Pes18, §19].

The pion π^0 can be considered as a Goldstone boson for chiral symmetry breaking. The decay rate of the pion into two photons $\pi^0 \to 2\gamma$ can be computed by the index theorem and it is experimentally checked to an accuracy of a few percent.

12.6 Supersymmetric quantum mechanics

Supersymmetry is a symmetry between bosons and fermions. Let us consider the simplest supersymmetric theory where there is only one fermionic field. Hence, we describe

$$\begin{pmatrix}1\\0\end{pmatrix} : \text{ fermionic state} \qquad \begin{pmatrix}0\\1\end{pmatrix} : \text{bosonic state} .$$

The Hilbert space $\mathcal{H}$ of states consists of wave functions of the form

$$\begin{pmatrix}\phi_F(x)\\ \phi_B(x)\end{pmatrix} = \phi_F(x)\begin{pmatrix}1\\0\end{pmatrix} + \phi_B(x)\begin{pmatrix}0\\1\end{pmatrix} ,$$

where it obeys the normalizable (L^2-norm) condition

$$\int_{-\infty}^{\infty}(|\phi_B(x)|^2 + |\phi_F(x)|^2)dx < \infty . \tag{12.3}$$

Let us denote the Pauli matrices as

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} , \quad \sigma_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} , \quad \sigma_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} .$$

Then, $\sigma_\pm$ are the fermion creation $(+)$ and annihilation $(-)$ operators and the fermion number is defined by $F = \frac{1}{2}(1+\sigma_3)$.

Now we define supercharges

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}i}\sigma_+\left(\frac{d}{dx} + W'(x)\right) \\ Q^\dagger &= \frac{1}{\sqrt{2}i}\sigma_-\left(\frac{d}{dx} - W'(x)\right) , \end{aligned} \tag{12.4}$$

where $W(x)$ is called a **superpotential**. It is easy to see that

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 .$$

Since the Hilbert space is decomposed into bosonic states and fermionic states $\mathcal{H}_B \oplus \mathcal{H}_F$, the supercharges are indeed linear maps

$$\mathcal{H}_B \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} \mathcal{H}_F .$$

The Hamiltonian of a supersymmetric theory is written as the anti-commutator of supercharges

$$H := \{Q, Q^\dagger\} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + W'(x)^2\right) + \frac{1}{2}\sigma_3 W''(x)$$

In other words, the supercharge is a "square-root" of Hamiltonian. In addition one can check that the supercharges commute with the Hamiltonian

$$[H, Q] = [H, Q^\dagger] = 0 . \tag{12.5}$$

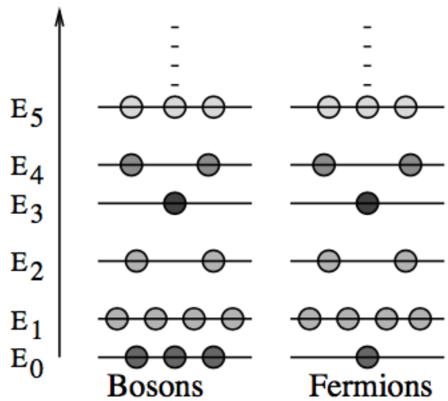
Let us consider a bosonic eigenstate $|b\rangle \in \mathcal{H}_B$ of the Hamiltonian

$$H|b\rangle = \{QQ^\dagger + Q^\dagger Q\}|b\rangle = Q^\dagger Q|b\rangle = E|b\rangle \,,$$

for $E > 0$. Now its **superpartner** $|f\rangle = Q|b\rangle$ has the same energy

$$H|f\rangle = \{QQ^\dagger + Q^\dagger Q\}Q|b\rangle = QQ^\dagger Q|b\rangle = E|f\rangle \,.$$

In a similar fashion, for a fermionic eigenstate $|f\rangle \in \mathcal{H}_F$ with $E > 0$, one can show that its superpartner $|b\rangle = Q^\dagger|f\rangle$ has the same energy. In fact, the commutation relation (12.5) is responsible for the degeneracy.



On the other hand, the situation drastically changes for states with $E = 0$. For a bosonic state $|b\rangle$ with zero energy $H|b\rangle = 0$, we have $\langle b|Q^\dagger Q|b\rangle = 0$ so that the supercharge annihilates the state $Q|b\rangle = 0$ and therefore there is no fermionic partner. Similarly, for a fermionic state $|f\rangle$, one can show that $Q^\dagger|f\rangle$ so that there is no bosonic partner. Since these zero energy states $|z\rangle$ are annihilated by both supercharges Q and $Q^\dagger$, they are invariant under supersymmetric transformation

$$\exp(i\epsilon Q)|z\rangle = |z\rangle \,, \qquad \exp\left(i\epsilon Q^\dagger\right)|z\rangle = |z\rangle \,.$$

Therefore, they are also called **supersymmetric states**.

Witten considered the difference between bosonic and fermionic zero energy states [Wit82b]. To this end, he has introduced the index $\mathrm{Tr}(-1)^F$. Since the excited states do not contribute to the index because there are always boson-fermion pairs. Therefore, the index is equal to

$$\mathrm{Tr}(-1)^F = \dim \operatorname{Ker} Q - \dim \operatorname{Ker} Q^\dagger$$

Note that the space of bosonic zero energy states can be expressed as $\operatorname{Ker} Q$ whereas the space of fermionic zero energy states can be written as $\operatorname{Ker} Q^\dagger$.

Since zero energy sates obey

$$\left(\frac{d}{dx} \pm W'(x)\right)\phi(x) = 0 \,,$$

we have

$$\phi(x) = \mathrm{const} \times \exp(\mp W(x)) \,.$$

The normalizable condition (12.3) requires $\phi(x) \to 0$ as $|x| \to \infty$. Suppose that the superpotential $W(x)$ is subject to polynomial growth $W(x) \sim \lambda x^n$ in the region $|x| \gg R$.

- When n is even, there exists either bosonic or fermionic zero energy states depending on the sign of λ. Therefore, the index is equal to the sign of λ.
- When n is odd, there is no zero energy state because $W(x)$ changes the sign when x moves from $-\infty$ to ∞ so that the index is equal to zero. Although the Hamiltonian is supersymmetric, there is no supersymmetric state for a potential of this type. Therefore, supersymmetry is **spontaneously broken**.

As we have seen, the index is independent of the detail of the superpotential $W(x)$ and it depends only on its asymptotic behavior. This is what we have seen in §1.3.

13 Chern-Simons theory

13.1 Flat connections and holonomy homomorphisms

Let $P \to M$ be a principal G-bundle over a manifold M. An Ehresmann connection A on P is called a **flat connection** If its curvature form F is identically zero $F = 0$. A principal G-bundle equipped with a fiat connection is called a flat G-bundle. As an example of a flat bundle, the product bundle $M \times G$ has a trivial connection, which is flat. This example is too trivial, and there are much richer stories for flat G-bundles.

A connection A determines the horizontal direction of a principal G-bundle. Namely, at each point $u \in P$, we define

$$\mathcal{H}_u = \{X \in T_u P | A(X) = 0\} .$$

The collection $\mathcal{H} = \cup_u \mathcal{H}_u$ is called **distribution**. The distribution is called **completely integrable** if

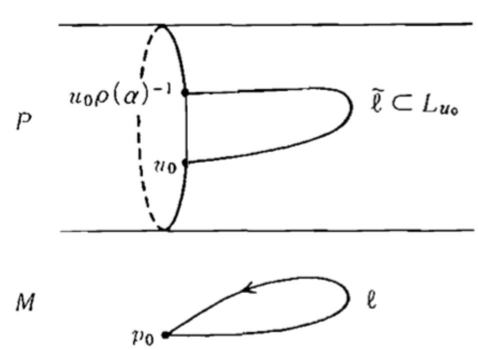
$$\text{for any two vector fields } X, Y \in \mathcal{H} \longrightarrow [X,Y] \in \mathcal{H} .$$

Another way to describe completely integrable distribution $\mathcal{H}$ is that for any point on M there exists an integral manifold containing it. (A submanifold N of M is called an **integral manifold** of $\mathcal{H}$ if $T_u N = \mathcal{H}_u$ for $^\forall u \in N$.)

For horizontal vector field X, Y, we have $A(X) = 0 = A(Y)$.

$$F(X,Y) = dA(X,Y) = \frac{1}{2}\left\{X(A(Y)) + Y(A(X)) - A([X,Y])\right\} = -\frac{1}{2}A([X,Y])$$

Therefore, $F = 0 \leftrightarrow [X,Y]$ horizontal vector field. A connection A on a principal G-bundle is fiat if and only if the corresponding distribution $\mathcal{H}$ is completely integrable.



Then, for each point $u \in P$ if we denote by L_u the maximal integral manifold passing through it, then the projection $\pi : L_u \to M$ becomes a covering map. Now let $p_0 \in M$ be a base point of the base space and choose $u_0 \in \pi^{-1}(p_0)$. Then a homomorphism

$$\rho : \pi_1(M) \to G ,$$

called the **holonomy homomorphism**, is defined as follows. For each element $\alpha \in \pi_1(M)$ of the fundamental group, we choose a closed curve γ with initial point p_0 which represents α. Let $\tilde{\gamma}$ be the lift of γ to L_{u_0} with initial point u_0. Then we can write the endpoint of $\tilde{\gamma} = u_0 \cdot g$. Since $\pi : L_{u_0} \to M$ is the covering map. we see that the endpoint of $\tilde{\gamma}$ is determined uniquely by α, and it is independent of the choice of $\tilde{\gamma}$. Then, we set

$$\rho(\gamma) = g^{-1} .$$

Theorem 13.1. Via the holonomy homomorphism, the set of flat connections on P is in a one-to-one correspondence with the set of conjugacy classes of homomorphisms $\rho : \pi_1(M) \to G$.

Therefore, if we specify an element of

$$\mathcal{M}_{\text{flat}}(M, G) = \text{Hom}(\pi_1(M), G)/G , \tag{13.1}$$

there is the corresponding flat connection on P. To the contrary, (13.1) actually parametrizes a family of flat connections on M, and this space $\mathcal{M}_{\text{flat}}(M, G)$ is called the **moduli space of G-flat connections over M.** Let us look at very simple examples.

Example 13.2. If $M = T^d = S^1 \times \cdots \times S^1$ is a d-torus, then its fundamental group is Abelian

$$\pi_1(T^d) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} . \tag{13.2}$$

For the sake of brevity, let us consider $G = \text{SU}(2)$. Then, a holonomy homomorphism is essentially a map from each generator $[S^1_i]$ of $\pi_1(S^1_i) \cong \mathbb{Z}$ into a diagonal matrix in SU(2)

$$[S^1_i] \mapsto \begin{pmatrix} x_i & 0 \\ 0 & 1/x_i \end{pmatrix} \qquad \text{where} \quad x_i \in \text{U}(1) \tag{13.3}$$

Note that the adjoint action identifies

$$\begin{pmatrix} 1/x_i & 0 \\ 0 & x_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_i & 0 \\ 0 & 1/x_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

As a result, we have

$$\mathcal{M}_{\text{flat}}(T^d, G) \cong \frac{S^1 \times \cdots \times S^1}{\mathbb{Z}_2} \tag{13.4}$$

In particular, when M is a 2-torus $T^{d=2}$, it becomes

$$\mathcal{M}_{\text{flat}}(T^2, G) \cong \frac{S^1 \times S^1}{\mathbb{Z}_2} \tag{13.5}$$

which is called a **pillow case**.

Example 13.3. The next simplest example appears when the gauge group is Abelian $G = \mathrm{U}(1)$. Moreover, we assume that M is a Riemann surface $M = \Sigma_g$ of genus g. For $g > 1$, the fundamental group $\pi_1(\Sigma_g)$ is non-commutative.

$$\pi_1\left(\Sigma_g\right) \cong \left\langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \right\rangle \tag{13.6}$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$. The image of the holonomomy homomorphisym $\rho : \pi_1(\Sigma_g) \to \mathrm{U}(1)$ is Abelian. Therefore, its Kernel is the **commutator subgroup**, which is the normal subgroup of $\pi_1(\Sigma_g)$ generated by $[a_i, b_i]$. In fact, the Abelianization of the fundamental group is isomorphic to the first homology group

$$\pi_1(\Sigma_g)/[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \cong H_1(\Sigma_g, \mathbb{Z})\ . \tag{13.7}$$

Consequently, the moduli space of $\mathrm{U}(1)$-flat connections over Σ_g is

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{U}(1)) \cong \mathrm{Hom}(H_1(\Sigma_g), \mathrm{U}(1)) \cong \mathrm{U}(1)^{2g}\ .$$

Namely, it become a T^{2g} torus. This space has another name, the **Jacobian variety** $\mathrm{Jac}(\Sigma_g)$.

For a general gauge group G, the moduli space $\mathcal{M}_{\text{flat}}(\Sigma_g, G)$ has complicated topology, but for $g > 2$ and a semisimple Lie group G, it become a Kähler manifold of real dimension $(2g-2)\dim G$. A moduli space of flat connections is a fun playground on which physicists and mathematicians interact.

13.2 Chern-Simons theory

Let M be a compact 3-manifold. We will consider a particular physical theory called **Chern-Simons** theory on 3-dimension. Let $P = M \times G$ be a trivial principal G-bundle, and we denote a connection on P by A.

The Chern-Simons action for A can be written as

$$\begin{aligned} S_{CS}[M, A] &= \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \\ &= \frac{k}{8\pi} \int_M \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3} A_\mu [A_\nu, A_\rho]\right) \end{aligned}$$

The action is independent of a metric of M so that it gives a topological invariant of M. In fact, the Chern-Simons 3-form is another kind of characteristic class of flat G-bundle. The parameter k of the theory (inverse of the coupling constant) is called **level**. If G is compact and semi-simple, the level k has to be an integer in order for the action to be gauge invariant. Classically the equations of motion which are the extrema of the action are flat connections:

$$\frac{\delta S}{\delta A} = \frac{k}{2\pi} F = 0\ .$$

Abelian Chern-Simons theory

Let us first consider the case when $G = \mathrm{U}(1)$, namely the Abelian Chern-Simons theory. It has the action,

$$S_{U(1)}[M, A] = \frac{k}{4\pi} \int_M A \wedge dA\ ,$$

Since U(1) is not a semi-simple group, the level k is not necessarily an integer in this case. The Abelian Chern-Simons theory describes the fractional quantum Hall effect.

Given an orientable close 3-manifold M, we can consider the path integral of U(1) Chern-Simons theory

$$Z[M] = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A e^{iS_{U(1)}(M,A)} .$$

This can be evaluated by so-called **one-loop determinant** and $Z[M]$ turns out to be also topological invariant, called **analytic (Reidemeister) torsion**. This means that Chern-Simons theory provides topological invariants even at a quantum level! This was first shown by A. Schwarz in 1978, giving the first construction of what we now call a **topological quantum field theory** [Sch78].

Furthermore, we can consider the holonomy group in Chern-Simons theory on $M = S^3$. Given a loop $\gamma : I \to S^3$ with $I(0) = I(1) = p_0$, the parallel transport along K with respect to A provides a holonomy group, and it is expressed as

$$u_0 \to u_0 \cdot \exp\left(i \oint_K A\right) .$$

We denote it by

$$W_K = \exp\left(i \oint_K A\right)$$

This is called **Wilson loop operator**, which plays an important role in physics. The expectation value of the Wilson loop operator can be expressed by the Feynman path integral

$$\langle W_K \rangle = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A \, W_K(A) \, e^{iS_{U(1)}(A)} .$$

Let us evaluate the expectation value of two loops K_1 and K_2 in $\mathbb{R}^3$.

$$\langle W_{K_1} W_{K_2} \rangle = \left\langle \exp\left(\oint_{K_1} dx_1^\mu A_\mu \oint_{K_2} dx_2^\nu A_\nu \right) \right\rangle . \tag{13.8}$$

Clearly, this expression can be easily evaluated in terms of the two-point correlator (propagator) in S^3

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i}{k} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} .$$

Plugging it into (13.8), the expectation value can be written in terms of the linking number that appear in (7.1)

$$\langle W_{K_1}(A) W_{K_2}(A) \rangle = \exp\left(\frac{4\pi i}{k} Lk(K_1, K_2) \right) .$$

Non-Abelian Chern-Simons theory

The generalization to non-Abelian Chern-Simons theory has been made by the seminal paper of Witten [Wit89]. Let us consider Wilson loop in non-Abelian Chern-Simons theory where the connections no longer commute. Therefore, the holonomy group should be written

$$u_0 \to u_0 \cdot P \exp\left(i \oint_K A\right)$$

where P is the path-ordered integral due to non-commutativity:

$$P\exp\left(i\oint_K A\right) = \prod_{t=0}^{1} e^{A(\gamma(t'))\,dt'} \equiv \lim_{N\to\infty}\left(e^{A(\gamma(t_N))\Delta t}e^{A(\gamma(t_{N-1}))\Delta t}\cdots e^{A(\gamma(t_1))\Delta t}e^{A(\gamma(t_0))\Delta t}\right)$$

where we subdivide $1 \geq t_N \geq \cdots \geq t_0 \geq 0$ by $\Delta t = \frac{1}{N}$. If the starting point is different $u_0 \to u_0' = u_0 \cdot g$, then the holonomy group is

$$P\exp\left(i\oint_K A\right) \to g\cdot P\exp\left(i\oint_K A\right)\cdot g^{-1}\,.$$

Therefore, we can define the operator that is independent of a starting point by taking trace

$$W_K := \mathrm{Tr}\, P\exp\left(i\oint_K A\right)\,.$$

When $G = \mathrm{SU}(2)$, the expectation value of the Wilson loops

$$\langle W_K\rangle = \int_{\mathscr{A}/\mathscr{G}} \mathcal{D}A\; W_K(A)\; e^{iS_{CS}(A)}\,. \tag{13.9}$$

gives the Jones polynomial which are invariants of knots and links [Wit89].

Jones polynomials are knot invariants which can be computed by the following skein relation

$$q^2 J\left(\times\right) - q^{-2} J\left(\times\right) = (q-q^{-1}) J\left(\,)(\,\right)\,. \tag{13.10}$$

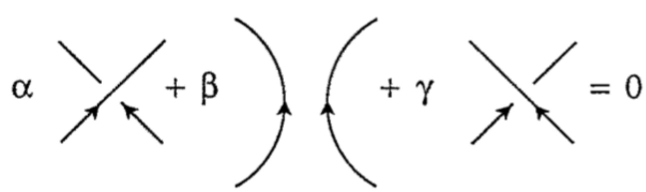
where the “quantum” parameter q is expressed as the Chern-Simons level

$$q = \exp\left(\frac{2\pi i}{k+2}\right)\,.$$

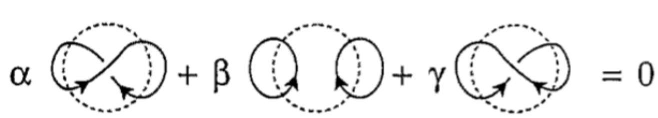
The reason why we obtain such a simplification from an infinite-dimensional Feynman path integral (13.9) is that we can use the relation between Chern-Simons theory M_3 and Wess-Zumino-Witten model on $\partial M_3 = \Sigma$. The action of WZW model is given by

$$S_{\mathrm{WZW}} = \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1}dg\wedge *g^{-1}dg\right) + \frac{k}{12\pi}\int_{M_3}\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)$$

where $g: M_3 \to G$. In fact, the Chern-Simons partition function on M_3 defines a vector in the WZW Hilbert space at the boundary Σ, which is finite-dimensional. In particular, for $G = \mathrm{SU}(2)$, the Hilbert space on 4-punctured sphere S^2 is 2-dimensional. Thus, the following three configurations are linearly dependent



The unknot invariant can be evaluated by



so that

$$\left\langle \bigcirc \right\rangle = -\frac{\alpha+\gamma}{\beta} \,. \tag{13.11}$$

To determine the coefficients α, β, γ, we need to use the braiding operator B. It is easy to see that

$$BL_+ = L_0 \,, \qquad B^2 L_+ = L_- \,.$$

Also, the braiding matrix is a 2-by-2 matrix so that it is

$$B^2 - (\mathrm{Tr}\, B)B + (\det B) = 0 \,.$$

Therefore, we have

$$L_- - (\mathrm{Tr}\, B)L_0 + (\det B)L_+ = 0 \,.$$

The braiding matrix can be computed by representation theory

$$B = \begin{pmatrix} q & 0 \\ 0 & -q^3 \end{pmatrix}$$

which gives the skein relation (13.10). Therefore, the Jones polynomial of the unknot (13.11) is

$$\left\langle \bigcirc \right\rangle = \frac{q^2 - q^{-2}}{q - q^{-1}} = [2]_q \,. \tag{13.12}$$

TQFT axiom

Motivated by the work of Witten, Atiyah gives the axiom of topological quantum field theory [Ati88]. To each compact oriented d-dimensional smooth manifold Σ without boundary one associates a finite-dimensional vector space $\mathcal{H}_\Sigma$. A compact oriented $(d+1)$-dimensional smooth manifold M with $\partial M = \Sigma$ determines a vector $Z(M) \in \mathcal{H}_\Sigma$.

1. If $-\Sigma$ is the surface with orientation reversed, then we associate the dual vector space $\mathcal{H}_{-\Sigma} = \mathcal{H}_\Sigma^*$.
2. For disjoint union $\Sigma_1 \cup \Sigma_2$, we have $\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$. Therefore, for $\partial M = (-\Sigma_1) \cup \Sigma_2$, a vector $Z(M) \in \mathcal{H}_{\Sigma_1}^* \otimes \mathcal{H}_{\Sigma_2}$ is indeed a linear map $Z(M) : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$.
3. For the composition of cobordisms $\partial M_1 = (-\Sigma_1) \cup \Sigma_2, \partial M_2 = (-\Sigma_2) \cup \Sigma_3$, we have

   $$Z\left(M_1 \cup M_2\right) = Z\left(M_2\right) \cdot Z\left(M_1\right)$$

   where $Z\left(M_1\right) : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$ and $Z\left(M_2\right) : \mathcal{H}_{\Sigma_2} \to \mathcal{H}_{\Sigma_3}$
4. For an empty set, we have $\mathcal{H}_\emptyset = \mathbb{C}$.
5. $I = [0.1]$ an interval, then $Z(\Sigma \times I)$ is the identity map as a linear transformation $\mathcal{H}_\Sigma$. Consequently, we have

   $$\mathrm{Tr}\, \mathrm{Id} = Z(\Sigma \times S^1) = \dim \mathcal{H}_\Sigma \,.$$

14 Moduli spaces

In mathematics, one of the most important problems is a classification problem. A moduli space arises from a classification problem in Algebraic Geometry [Rie57]. Classification of certain objects with some properties leads to the study of the entire space $\mathcal{A}$ of objects. Very roughly speaking, a moduli space $\mathcal{M}$ is a space (or a family) of objects $\in \mathcal{A}$ up to equivalence relations $\sim$:

$$\mathcal{M} = \mathcal{A}/\sim \ .$$

Often (almost always if properly formulated), a moduli space is naturally endowed with topology and geometry reflecting properties of the objects in $\mathcal{A}$ under the equivalence relations $\sim$. Namely, it encodes information about

- how 'close' two classes are to each other; how they vary
- how they degenerate
- the existence of special objects

Consequently, the study of a moduli space leads to a deeper and more intrinsic understanding of the objects in $\mathcal{A}$ under the equivalence relations $\sim$. One example has already appeared in (13.1), moduli space of flat connections. In Example 13.2, we have seen that the moduli space of SU(2)-flat connections over a torus T^2 is a pillowcase where there are four singular points. At these four singular points, the generators A, B of the fundamental group are mapped to $\pm$ Id

$$A, B \mapsto \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \ .$$

14.1 Toy examples

Since the moduli problem can be difficult and intimidating to first learners, let us begin with the simplest examples.

Example 14.1. Finite-dimensional vector spaces up to linear isomorphisms are classified by dimensions $d \in \mathbb{N}$.

Example 14.2. Closed oriented surfaces up to homeomorphisms are classified by genera $g \in \mathbb{N}$.

Example 14.3. The moduli space of lines up to affine transformations is a point.

These moduli spaces consist of points, which means that there is no continuous deformation. Or, in other words, these are boring examples. If equivalence classes admit continuous deformations, the problem becomes more interesting.

Example 14.4. A sphere up to congruence (isometry) is classified by radius $r > 0$ so that the moduli space is $\mathcal{M} \cong \mathbb{R}^+$. $\mathcal{M} \cong \mathbb{R}^+$ is a connected, smooth, non-compact manifold, encoding the continuous variations of spheres up to congruence:

- connectedness means that any sphere can be continuously deformed to any other sphere.
- smoothness means that there are no special spheres (with extra symmetries): all

spheres are equally symmetric.

- At $r = 0 \in \overline{\mathcal{M}} \backslash \mathcal{M}$, a sphere degenerates to a point.
- If we enlarge the ambient space of the moduli space to $\mathbb{R}^+ \cup \{\infty\}$, then at $r = \infty \in \overline{\mathcal{M}} \backslash \mathcal{M}$, 'a plane at infinity' shows up as degenerate sphere of radius ∞.

All these are indeed general principles of moduli spaces.

Example 14.5. In Example 14.3, we have seen that the moduli space of lines up to affine transformations is a point. What about if we consider lines up to translations. Then, we can always bring a line that passes through the origin. Hence, the moduli space of one-dimensional subspaces in $\mathbb{R}^n$ is indeed $\mathbb{R}\mathbf{P}^{n-1}$. We can complexify the story and the moduli space of complex lines $\mathbb{C}$ in $\mathbb{C}^n$ up to translations is $\mathbb{C}\mathbf{P}^{n-1}$. Therefore, the projective spaces can be interpreted as the moduli space of lines. As we have studied in the previous lectures, the moduli space can be endowed with geometric structures such as metric and symplectic forms.

Example 14.6. Now let us consider lines in $\mathbb{R}^n$ with no equivalence relations. Then, we can translate a line $\ell \in \mathbb{R}\mathbf{P}^{n-1}$ along a plane orthogonal to a line $\ell \in \mathbb{R}\mathbf{P}^{n-1}$. In fact, a plane orthogonal to a line $\ell \in \mathbb{R}\mathbf{P}^{n-1}$ is the tangent space $T_\ell \mathbb{R}\mathbf{P}^{n-1}$. Therefore, the moduli space of lines is the tangent bundle $T\mathbb{R}\mathbf{P}^{n-1}$.

We can easily generalize the moduli space of r-dimensional planes $\mathbb{R}^r$ in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) (up to affine transformations, up to translations, or with no equivalent relations). If the equivalence relation is up to affine transformations, then the moduli space is again a point. up to translations, we can always bring a plane that passes through the origin. Then, the moduli space of r-dimensional subspaces in $\mathbb{R}^n$ is called a **Grassmanniann** $\mathrm{Gr}_{\mathbb{R}}(r,n)$. In a similar manner to Example 9.6, it can be realized as a homogeneous space

$$\mathrm{Gr}_{\mathbb{R}}(r,n) \cong \mathrm{O}(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r)) \,. \tag{14.1}$$

Its complex version is

$$\mathrm{Gr}_{\mathbb{C}}(r,n) \cong \mathrm{U}(n)/(\mathrm{U}(r) \times \mathrm{U}(n-r)) \,. \tag{14.2}$$

14.2 Moduli space of triangles

Let us consider the moduli space of triangles up to similarity. The reader can refer to the wonderful lecture note [Beh14] for more details, and the Figures are taken from there. Let us fix the perimeter of a triangle to be (say) 2. Then, the moduli space of triangles up to similarity are determined by the length of their sides, so

$$\mathcal{M}_{\mathrm{tri}} = \left\{ (a,b,c) \in \mathbb{R}^3 \mid a \le b \le c, a+b+c = 2, c < a+b \right\}$$

This is a right triangle in $\mathbb{R}^3$ minus one of its edges, which has the following properties. (Figure 8.)

- the moduli space $\mathcal{M}_{\mathrm{tri}}$ is not compact. To compactify $\mathcal{M}$, we need to add line segments, as degenerate triangles.
- at boundary of $\mathcal{M}_{\mathrm{tri}}$, special triangles, namely isosceles, show up where symmetry of a triangle is enhanced to $\mathbb{Z}_2$

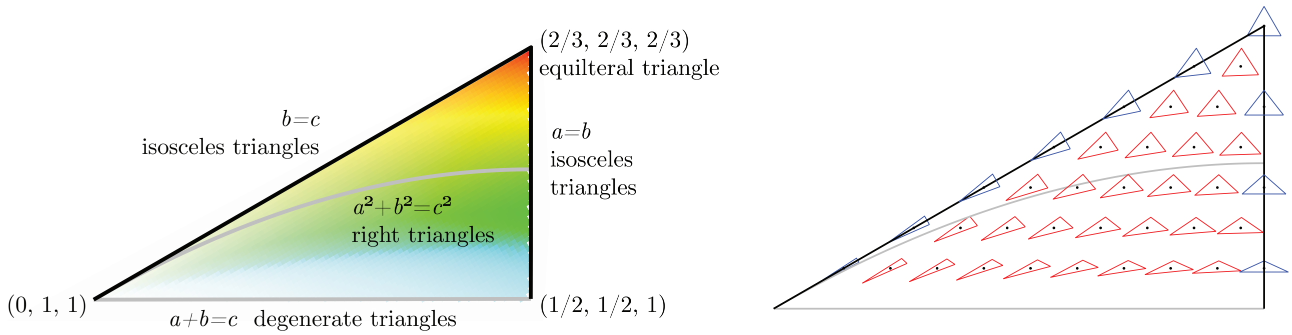


Figure 8:

- at the singular point, equilateral triangle will show up and symmetries is enhanced to the dihedral group $D_3 = \langle r, s \mid r^3 = s^2 = (sr)^2 = 1 \rangle$.

Again, the geometry and topology of the moduli space itself give information on the individual triangles being parameterized!

Now we slightly change the moduli problem by considering oriented triangles. This means that similarity transformations between triangles are only rotations, translations, and scalings, but not reflections. The corresponding moduli space is

$$\widetilde{\mathcal{M}}_{\text{tri}} = \left\{ (a,b,c) \in \mathbb{R}^3 \mid a \le c, b \le c, a+b+c = 2, c < a+b \right\}$$

The moduli space can be obtained by "doubling" the previous one and identifying the edges. (Figure 9.)

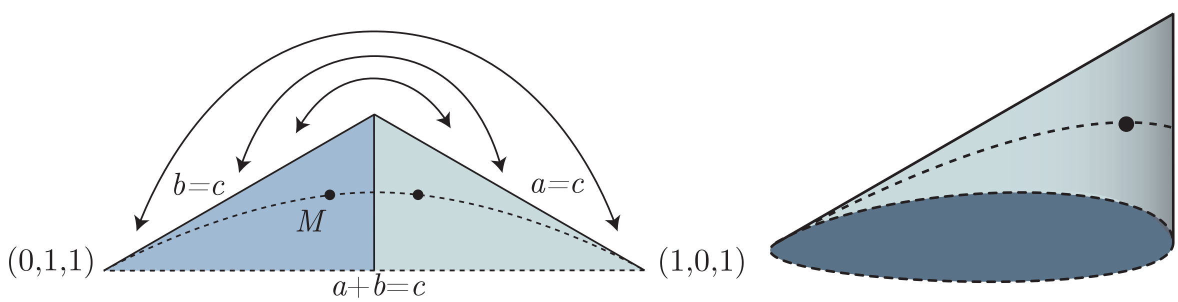


Figure 9:

Let us take a look at it from slightly different viewpoint. Let us place an oriented triangle to the upper half plane $\mathbf{H}$ where we scale one edge to be located at 1. If we adjust $\mathrm{C}-\mathrm{B}$ to be 1 in the upper half plane $\mathbf{H}$, then $\mathrm{A}-\mathrm{B}$ is located at a complex number z:

$$z = \frac{(\mathrm{A}-\mathrm{B})}{(\mathrm{C}-\mathrm{B})}$$

On the other hand, if we normalize $\mathrm{A}-\mathrm{C}$ to be 1, then $\mathrm{B}-\mathrm{C}$ is located at

$$\frac{1}{1-z} = \frac{(\mathrm{B}-\mathrm{C})}{(\mathrm{A}-\mathrm{C})} \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})\,. \tag{14.3}$$

Note that $\mathrm{PSL}(2,\mathbb{Z})$ acts on the upper half plane $\mathbf{H}$ as

$$z \mapsto \frac{az+b}{cz+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})\,, \tag{14.4}$$

and we assign the corresponding matrix element in (14.3). We can continue along this line, and the remaining is when we normalize $\mathrm{B}-\mathrm{A}$ to 1. Then, $\mathrm{C}-\mathrm{A}$ sits at

$$\frac{z-1}{z} = \frac{\mathrm{C}-\mathrm{A}}{\mathrm{B}-\mathrm{A}} \longleftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})\,.$$

Therefore, we identify the subgroup $\mathbb{Z}_3 \in \mathrm{PSL}(2,\mathbb{Z})$ generated by

$$\omega_3 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\,,$$

and the moduli space of oriented triangles up to similarity can be identified with

$$\widetilde{\mathcal{M}}_{\mathrm{tri}} \cong \mathbf{H}/\mathbb{Z}_3\,. \tag{14.5}$$

The moduli space is indeed the right of Figure 10. Note that the fixed point of $\mathbb{Z}_3 \in \mathrm{PSL}(2,\mathbb{Z})$ is

$$\rho = e^{2\pi i/6} = \frac{1+\sqrt{3}i}{2}\,.$$

where the equilateral triangle shows up.

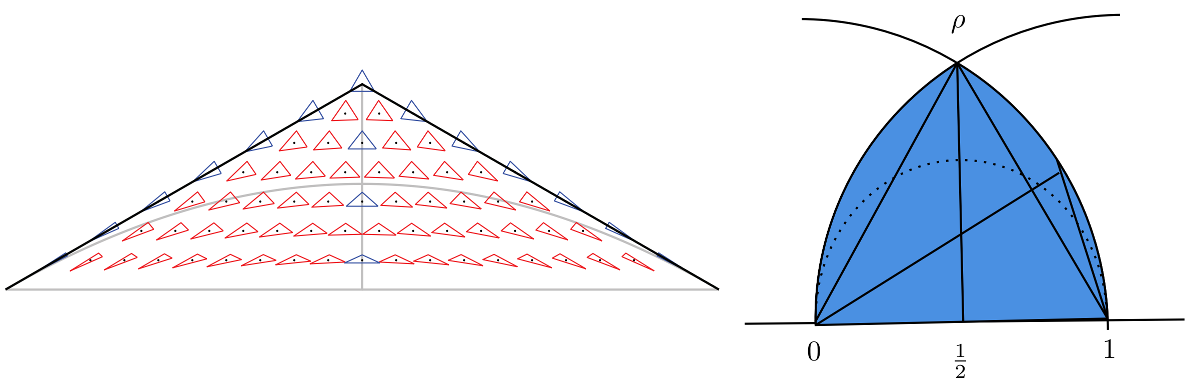


Figure 10: The 0ρ arc is identified with the 1ρ arc.

To identify the two arcs in the right of Figure 10 in a smarter way, we map the moduli space by a fractional linear transformation

$$z \mapsto \frac{\rho^2 - \rho z}{\rho^2 + z}$$

where the corner points $\rho, 0, 1$ are mapped to $0, 1, \rho^2$. (Left to middle in Figure 11.) Then, a map $z \mapsto z^3$ brings the moduli space to a unit open disk where the locus of right triangles is a (dotted) curve in the right of Figure 11 parametrized by

$$r = \left(2\cos\left(\frac{\theta-\pi}{3}\right) - \sqrt{4\cos^2\left(\frac{\theta-\pi}{3}\right)-1}\right)^3\,. \tag{14.6}$$

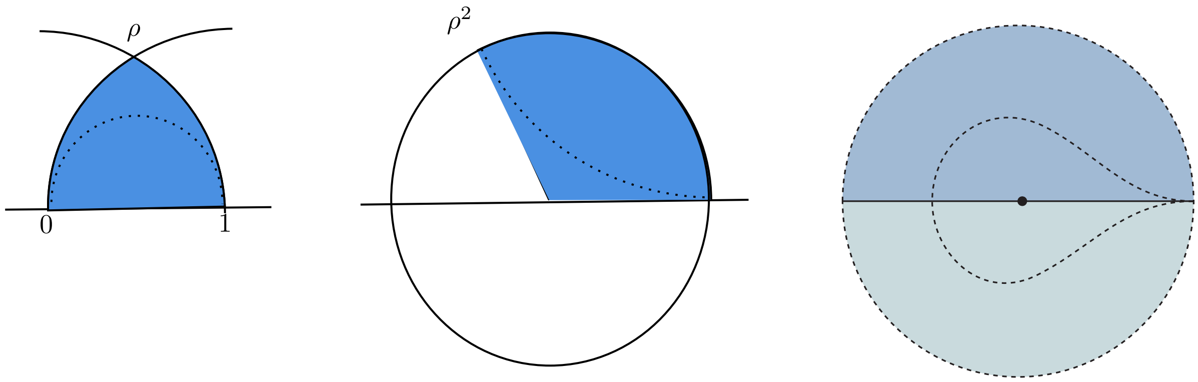


Figure 11: From left to middle, $z \mapsto (\rho^2 - \rho z)/(\rho^2 + z)$. From middle to right, $z \mapsto z^3$.

14.3 Moduli space of Riemann surfaces

These are very simple examples to get used to the notion of moduli spaces. Now we introduce the genuine moduli problem originated from Riemann []. A 2-dimensional oriented manifold can be constructed as a complex manifold in Appendix B. The moduli space of Riemann surfaces classifies Riemann surfaces up to biholomorphic maps.

Let us first consider the simplest example: $\mathcal{M}_{0,n}$ is the moduli space of n-pointed Riemann spheres ($n \geq 3$). The moduli space $\mathcal{M}_{0,n}$ parameterizes n distinct points on $\mathbb{CP}^1$ up to biholomorphic maps

$$\left[\mathbb{CP}^1, z_1, \ldots, z_n\right] \in \mathcal{M}_{0,n}$$

Let $z_1, z_2, z_3 \in \mathbb{CP}^1$ be three distinct points. There exists a unique linear fractional transformation

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

satisfying $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$. Therefore, $\mathcal{M}_{0,3}$ is a single point.

Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{CP}^1$, the first three can be moved via linear fractional transformation to $0, 1, \infty \in \mathbb{CP}^1$. Therefore, the moduli space parametrizes the position of z_4 so that

$$\mathcal{M}_{0,4} \cong \mathbb{CP}^1 \backslash \{0, 1, \infty\}$$

The statement may also be approached via the classical cross-ratio (which goes back to Pappus of Alexandria 300 AD). More generally, we have

$$\mathcal{M}_{0,n} \cong \left(\mathbb{CP}^1 \backslash \{0, 1, \infty\}\right)^{n-3} \backslash \text{ Diagonals.}$$

The next elementary example is the moduli space $\mathcal{M}_{1,0}$ of a torus. As we have seen, a torus is constructed as a complex plane quotient by a two-dimensional lattice Λ, $T^2 \cong \mathbb{C}/\Lambda$. In fact, a two-dimensional lattice Λ is in the one-to-one correspondence with a torus up to biholomorphic maps. Roughly speaking, Λ defines the "shape" of a torus.

A lattice Λ is spanned by two complex vector ω_1, ω_2 such that ω_2/ω_1 has a positive imaginary number.

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{aligned} \tag{14.7}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \ .$$

We can normalize one of the vectors to be 1 and we write $\tau = \omega_2/\omega_1$ and $\tau' = \omega_2'/\omega_1'$. Then, (14.7) can be rewritten as

$$\tau' = \frac{a\tau + b}{c\tau + d} \ . \tag{14.8}$$

The action of $\mathrm{PSL}(2,\mathbb{Z})$ is called the **modular transformation**, and any element of $\mathrm{PSL}(2,\mathbb{Z})$ can be written as a product of the following elements

$$\begin{gathered} T : \tau \to \tau + 1 \ , \quad S : \tau \to -\frac{1}{\tau} \\ \mathrm{PSL}(2,\mathbb{Z}) = \langle S, T : S^2 = (ST)^3 = \mathrm{Id} \rangle \end{gathered} \tag{14.9}$$

Hence, the modular transformations do not change Λ, namely the "shape" of a torus. Since τ is a point of the upper half plane $\mathbf{H}$ and $\mathrm{PSL}(2,\mathbb{Z})$ acts on the Teichmüller space $\mathbf{H}$ as in 12, the moduli space of tori is

$$\mathcal{M}_{1,0} = \mathbf{H}/\mathrm{PSL}(2,\mathbb{Z}) \ , \tag{14.10}$$

which is the shaded region in Figure 12, called the **fundamental region**. There are three special points $i\infty, i, e^{2\pi i/3}$ at the fundamental region. At $i\infty$, a torus degenerates to a circle. At i, Λ is a square so that it has a symmetry of $\mathbb{Z}_4$ generated by a rotation of $\pi i/2$. At $e^{2\pi i/3}$, Λ has a symmetry of $\mathbb{Z}_3$ generated by a rotation of $2\pi i/3$.

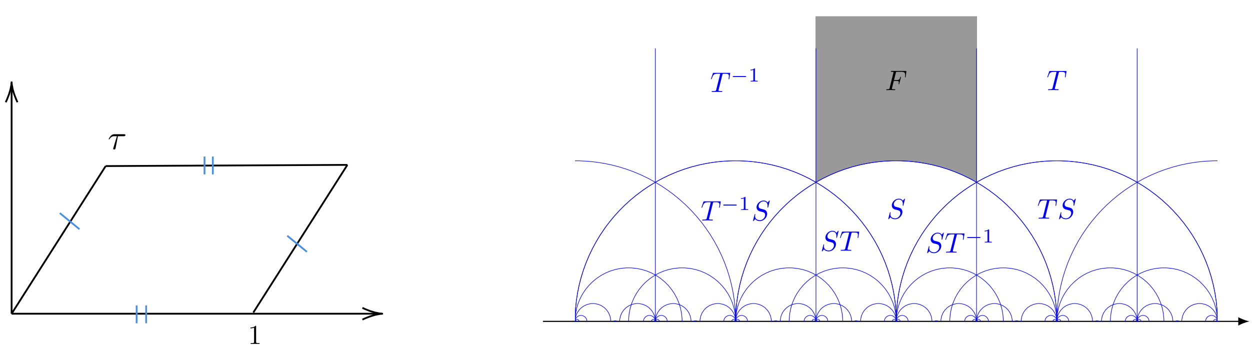


Figure 12: The Teichmüller space of a torus is the upper half plane, and the mapping class group $\mathrm{PSL}(2,\mathbb{Z})$ acts on it. The moduli space is the fundamental region F (the shaded region).

Riemann has introduced the notion of the moduli space $\mathcal{M}_g$ of Riemann surfaces, and he has noticed that its complex dimension of $\mathcal{M}_{g>0}$ is $3g-3$ [Rie57]. However, it requires sophisticated techniques to construct the moduli space of Riemann surfaces in a mathematically rigorous way. This was done mainly by Mumford in the 1960s from the viewpoint of algebraic geometry. Thurston gave another construction by using hyperbolic geometry:

$$\mathcal{M}_{g>2,0} = \mathbb{R}^{6g-6}/\mathrm{MCG}_g\ , \tag{14.11}$$

where MCG_g is a discrete group called the **mapping class group** of Riemann surface of genus g.

14.4 Nilpotent orbits

Now let us consider the moduli problem in linear algebra. Let us consider the moduli space of matrices $A \in$ that satisfy $A^m = 0$ for $n \in \mathbb{N}$.

Definition 14.7. $A \in M_n(\mathbb{C})$ is called **nilpotent** if $A^m = 0$. The **nilpotent orbit** of A is the set of all matrices conjugate to A. The nilpotent cone $\mathcal{N}$ is the set of all nilpotent matrices.

Every nilpotent matrices has its Jordan form $\mathrm{diag}\left(J_{d_1}, \cdots, J_{d_k}\right)$, where J_{d_i} is the Jordan matrix of size d_i with zeros on diagonal. Hence nilpotent orbits in $M_n(\mathbb{C})$ are parametrized by partitions of n.

Example 14.8. Let $A \in \mathfrak{sl}(2, \mathbb{C})$ be as

$$\begin{pmatrix} a & -b \\ c & -a \end{pmatrix}$$

A is nilpotent iff $A^2 = 0$, namely

$$a^2 - bc = 0\ . \tag{14.12}$$

The nilpotent cone of $\mathfrak{sl}(2, \mathbb{C})$ consists of two orbits $[2], [1, 1]$, corresponding to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We want to understand geometry of the nilpotent orbit. It turns out that the nilpotent orbit is $\mathbb{C}^2/\mathbb{Z}_2$. The ring of holomorphic functions on $\mathbb{C}^2$ is $\mathbb{C}[z_1, z_2]$. The ring of holomorphic functions on $\mathbb{C}^2/\mathbb{Z}_2$ is $R = \mathbb{C}\left[z_1^2, z_1z_2, z_2^2\right] \subset \mathbb{C}[z_1, z_2]$. After change of variable, this is isomorphic to $R \cong \mathbb{C}[a, b, c]/\left(a^2 - bc\right)$. Thus, the nilpotent orbit of $\mathfrak{sl}(2, \mathbb{C})$ is geometrically $\mathbb{C}^2/\mathbb{Z}_2$. If we stare at it a bit more, we can notice that the ring of holmorphic functions $\mathbb{C}^2/\mathbb{Z}_2$ can be expressed as

$$R \cong \mathbb{C}[x, y, z]/\left(x^2 + y^2 + z^2\right)\ .$$

The ideal $x^2 + y^2 + z^2 = 0$ can be deformed as

$$x^2 + y^2 + z^2 = \xi^2\ , \tag{14.13}$$

which is $T^*\mathbb{C}\mathbf{P}^1$ where ξ is the "complex volume" of $\mathbb{C}\mathbf{P}^1$. In other words, the nilpotent orbit $\mathbb{C}^2/\mathbb{Z}_2$ is the zero volume limit of $T^*\mathbb{C}\mathbf{P}^1$. The cotangent bundle $T^*\mathbb{C}\mathbf{P}^1$ is a symplectic manifold as seen in Example 6.2. How about general nilpotent orbits?

A coadjoint orbit $\mathcal{O}_\mu$ for $\mu \in \mathfrak{g}^*$ of $\mathfrak{g}$ is defined as the orbit $\mathrm{Ad}_G^*\, \mu$ of μ under the coadjoint action (9.6). In other words, it can be identified with

$$\mathcal{O}_\mu \cong G/G_\mu$$

where G_μ is the stabilizer of μ with respect to the coadjoint action (9.6). The coadjoint orbits are submanifolds of $\mathfrak{g}^*$ and carry a natural symplectic structure. On each orbit $\mathcal{O}_\mu$, there is a G-invariant symplectic form $\omega \in \Omega^2(\mathcal{O}_\mu)$

$$\omega_\nu(\mathrm{ad}^*_X \nu, \mathrm{ad}^*_Y \nu) := \langle \nu, [X,Y] \rangle, \nu \in \mathcal{O}_\mu, X, Y \in \mathfrak{g}\,.$$

The well-definedness, non-degeneracy, and G-invariance of ω follow from the following facts:

- The tangent space $T_\nu \mathcal{O}_\mu = \{-\mathrm{ad}^*_X \nu : X \in \mathfrak{g}\}$ may be identified with $\mathfrak{g}/\mathfrak{g}_\nu$, where $\mathfrak{g}_\nu$ is the Lie algebra of G_ν.
- The kernel of the map $X \mapsto \langle \nu, [X, \cdot] \rangle$ is exactly $\mathfrak{g}_\nu$.
- The bilinear form $\langle \nu, [\cdot, \cdot] \rangle$ on $\mathfrak{g}$ is invariant under G_ν.
- ω is also closed.

The symplectic form ω is referred to as the **Kirillov-Kostant-Souriau** symplectic form on the coadjoint orbit. So any nilpotent orbit is a symplectic manifold.

The nilpotent orbits appear in physics as the moduli space of vacua in 3d $\mathcal{N} = 4$ theories. For instance, let us consider 3d $\mathcal{N} = 4$ SQED with two flavors in which the superpotential is given by

$$W = \widetilde{Q}_{1\times 2}\phi_{1\times 1}Q_{2\times 1}\,. \tag{14.14}$$

Then, the moduli space of vacua is given by the critical point of the superpotential

$$\frac{\partial W}{\partial \phi} = \widetilde{Q}_{1\times 2}Q_{2\times 1} = 0\,, \qquad \frac{\partial W}{\partial \widetilde{Q}} = \phi_{1\times 1}Q_{2\times 1} = 0 \quad \frac{\partial W}{\partial Q} = \widetilde{Q}_{1\times 2}\phi_{1\times 1} = 0\,.$$

If we assume $\langle \phi \rangle = 0$, then the vacua are parametrized by the gauge invariant operator $A = Q_{2\times 1}\widetilde{Q}_{1\times 2}$, $\mathrm{rank}(A) \le 1$, which is subject to

$$\left\{ A \in M_2(\mathbb{C}) \mid A^2 = 0, \mathrm{Tr}\, A = 0 \right\}\,.$$

Therefore, the moduli space of vacua is the nilpotent orbit of $\mathfrak{sl}(2,\mathbb{C})$. The deformation parameter ζ in (14.13) can be interpreted as FI and mass parameter of the 3d $\mathcal{N} = 4$ SQED.

A Mathematical preliminary

A.1 Basic definitions

Definition A.1 (Equivalence relation). We say $\sim$ is an equivalence relation on a set A if it satisfies the following three properties:
(1) reflexivity: for all $a \in A, a \sim a$
(2) symmetry: for all $a, b \in A$, if $a \sim b$, then $b \sim a$
(3) transitivity: for all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$

Example A.2. Let V be a vector space and $W \subset V$ is a subspace. For $u, v \in V$, we define an equivalence relation $u \sim v$ by $u - v \in W$:
(1) reflexivity: $v \sim v$ since $v - v = 0 \in W$.

(2) symmetry: if $u \sim v$, then $u - v \in W$ so that $v - u \in W$. Hence $v \sim u$.
(3) transitivity: if $u \sim v$ and $v \sim w$, then $u - v \in W$ and $v - w \in W$ so that $u - w \in W$. Hence $u \sim w$.

Definition A.3. The **quotient set** of S with respect to the equivalence relation $\sim$ is the collection of all equivalence classes: $S/{\sim} = \{[a] : a \in S\}$ In Example A.2, the addition $[u] + [v] = [u + v]$ and scalar multiplication $\alpha[v] = [\alpha v]$ are well-defined in $V/{\sim}$. We denote $V/{\sim}$ by V/W, which is called the **quotient (vector) space** of V by W.

Definition A.4 (Group). A group is a set, G, together with an operation $*$ such that for $^\forall a, b \in G$, the group operation combines them into another element $a * b \in G$ or ab. Moreover, $(G, *)$ satisfies the following properties
(1) associativity: For $^\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$.
(2) identity element: There exists an element e in G such that, for $^\forall a \in G$, $e * a = a * e = a$. Such an element is unique, and thus one speaks of the **identity element**.
(3) inverse element: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$, where e is the identity element.

If $a * b = b * a$ for $^\forall a, b \in G$, then G is called an **abelian group**.

Example A.5. $(\mathbb{Z}, +)$ is an abelian group.

Example A.6. A **finite cyclic group** with order n is an abelian group $\mathbb{Z}_n = \{e, g, g^2, \ldots, g^{n-1}\}$, where e is the identity element and $g^j = g^k$ whenever $j \equiv k$ modulo n. It is isomorphic to the abelian group $(\mathbb{Z}/n\mathbb{Z}, +)$ or simply $(\mathbb{Z}_n, +)$, formed by integers modulo n with addition $+$.

Example A.7. The **symmetric group** S_n on a finite set $X = \{1, \ldots, n\}$ is the group whose elements are all bijective maps from X to X, and its group operation is a composition of maps. $S_{n>2}$ is non-abelian group.

Definition A.8 (Quotient group). A subgroup $N \triangleleft G$ of a group G is called a **normal subgroup** if it is invariant under conjugation:

$$N \triangleleft G \Leftrightarrow \forall n \in N, \forall g \in G \colon gng^{-1} \in N.$$

For a normal subgroup $N \triangleleft G$, the set $G/N = \{aN \mid a \in G\}$ is endowed with the natural group operation $(aN)(bN) = (ab)N$, which does not depend on the choice of the representatives. We call G/N the **quotient group** of G by N.

Definition A.9. A **group homomorphism** from $(G, *)$ to $(H, \cdot)$ is a map $h : G \to H$ such that for all $u, v \in G$ it holds that

$$h(u * v) = h(u) \cdot h(v)\ .$$

From this property, the identify e_G in G is mapped to the identity e_H in H, $h(e_G) = e_H$. The **kernel** is defined as $\mathrm{Ker}(h) = \{g \in G |\ h(g) = e_H\}$, which is a normal subgroup of G.

Example A.10. Given an integer $n \in \mathbb{Z}$, $\times n : \mathbb{Z} \to \mathbb{Z}$; $x \mapsto nx$ is a group homomorphism of $(\mathbb{Z}, +)$.

Definition A.11 (Ring). A ring is a set R equipped with two binary operations $+$ and $*$ satisfying the following three sets of axioms
(1) $(R, +)$ is an abelian group under addition: the identity element under addition is denoted by 0.
(2) $(R, *)$ is a monoid under multiplication; $(a * b) * c = a * (b * c)$ for $^\forall a, b, c \in R$ and $^\exists 1 \in R$ such that $a * 1 = a = 1 * a$ for $^\forall a \in R$
(3) distributivity of multiplication over addition: $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$ for $^\forall a, b, c \in R$

Example A.12. $(\mathbb{Z}, +, \times)$ are $(\mathbb{Z}_n, +, \times)$ are rings.

Example A.13. A set $\mathbb{R}[x] = \{a_0 + a_1 x + \cdots + a_n x^n | a_i \in \mathbb{R}\}$ of polynomials with one variable x over $\mathbb{R}$ is a ring under natural addition $+$ and multiplication $\times$.

Definition A.14 (Ideals and quotient ring). A subset I is called a (two-sided) ideal of a ring R if it satisfies the following two conditions:
(1) $(I, +)$ is a subgroup of $(R, +)$,
(2) For every $r \in R$ and every $x \in I$, the product $rx, xr \in I$.
For an idel I of R, we can define an equivalent relation by $a \sim b$ if and only if $a - b \in I$. The set of all such equivalence classes is denoted by R/I becomes a ring by

$$\begin{aligned}
(a + I) + (b + I) &= (a + b) + I; \\
(a + I)(b + I) &= (ab) + I \,,
\end{aligned}$$

and it is called the **quotient ring** of R modulo I

Definition A.15. For rings R and S, a **ring homomorphism** is a function $f : R \to S$ which satisfies the following properties:
(1) $f(a + b) = f(a) + f(b)$ for all $a, b \in R$
(2) $f(ab) = f(a)f(b)$ for all $a, b \in R$
(3) $f(1_R) = 1_S$.
The kernel of f, defined as $\mathrm{Ker}(f) = \{a \in R \mid f(a) = 0_S\}$, is an ideal in R.

Example A.16. mod $n : \mathbb{Z} \to \mathbb{Z}_n$; $x \mapsto [x] = x \bmod n$ is a ring homomorphism. The kernel of this map is an ideal $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ of a ring $\mathbb{Z}$.

Definition A.17 (Field). A field is a set F equipped with two binary operations $+$ and $*$ satisfying the following two sets of axioms
(1) $(F, +, *)$ is a ring.
(2) Multiplicative inverses: for $^\forall a \neq 0$ in F, $^\exists a^{-1}$ such that $a * a^{-1} = 1 = a^{-1} * a$.

Example A.18. $(\mathbb{R}, +, \times)$ is a field.

In this lecture, we consider $\mathbb{R}$ (or $\mathbb{C}$) for the ground field.

Definition A.19 (Algebra over a field). Let F be a field, and let $(A, +)$ be a vector space over F equipped with an additional binary operation $*$ satisfying the following axioms
(1) distributivity: $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$ for $^\forall a, b, c \in A$
(2) Compatibility with scalars: $(\xi a) * (\eta b) = (\xi\eta)(a * b)$ for $^\forall a, b \in R$ and $^\forall \xi, \eta \in F$.

An algebra is **unital** or **unitary** if it has an identity element 1 with respect to the multiplication $*$. An algebra is **associative** if $(a * b) * c = a * (b * c)$ for $^\forall a, b, c \in A$.

Definition A.20. A homomorphism between two unital associative algebras, A and B, over a field F, is a map $f : A \to B$ such that for all $\xi \in F$ and $x, y \in A$

$$\begin{aligned} f(1_A) &= 1_B \\ f(kx) &= kf(x) \\ f(x+y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \end{aligned} \tag{A.1}$$

A module over a ring is a generalization of the notion of vector space over a field.

Definition A.21 (Module). Suppose that R is a ring and 1_R is its multiplicative identity. A left R-module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$, we have:
(1) $r \cdot (x + y) = r \cdot x + r \cdot y$
(2) $(r + s) \cdot x = r \cdot x + s \cdot x$
(3) $(rs) \cdot x = r \cdot (s \cdot x)$
(4) $1_R \cdot x = x$.

A right R-module M is defined in a similar fashion.

Example A.22. $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ with positive integers $n_i > 1$ $(i = 1, \ldots, k)$ is a $\mathbb{Z}$-module.

Theorem A.23. A finitely generated $\mathbb{Z}$-module M is isomorphic to a direct sum

$$M \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$$

where m_i is a divisor of m_{i+1}. Moreover, this representation is unique. The part $\mathbb{Z}_{m_1} \oplus \cdots \mathbb{Z}_{m_k}$ of elements of finite order is called the **torsion submodule** of M.

A free R-module is a module that is isomorphic to a direct sum of copies of the ring R. These are the modules that behave very much like vector spaces. More precisely,

Definition A.24 (Free module). A free R-module is a module with a basis E
(1) E is a generating set for M: every element of$M$ is a finite sum of elements of E multiplied by coefficients in R
(2) E is linearly independent: $r_1e_1 + r_2e_2 + \cdots + r_ne_n = 0_M$ for distinct elements $e_1, e_2, \ldots, e_n$ of E implies $r_1 = r_2 = \cdots = r_n = 0_R$.

A free $\mathbb{Z}$-module is a $\mathbb{Z}$-module without a torsion submodule so that Example A.22 is not a free $\mathbb{Z}$-module.

Definition A.25 (Words). Let $S = \{s_\alpha : \alpha \in \Lambda\}$ be a set, and we have an extra set of symbols $S^{-1} = \{s_\alpha^{-1} : \alpha \in \Lambda\}$. We assume that $S \cap S^{-1} = \emptyset$. We define S^* to be the set of **words** over $S \cup S^{-1}$, i.e. it contains n-tuples $x_1 \cdots x_n$ for any $0 \le n < \infty$, where each $x_i \in S \cup S^{-1}$.

Example A.26. Let $S = \{a, b\}$. Then words could be the empty word $\emptyset$, or a, or $aba^{-1}b^{-1}$, or $aa^{-1}aaaaabbbb = aa^{-1}a^5b^4$.

When we see things like $s_\alpha s_\alpha^{-1}$, we would want to cancel them. An **elementary reduction** takes a word $us_\alpha s_\alpha^{-1}v$ and gives uv, or turn $us_\alpha^{-1}s_\alpha v$ into uv. A word is **reduced** if it does not admit an elementary reduction. For example, $\emptyset$, a, $aba^{-1}b^{-1}$ are reduced words, while $aa^{-1}aaaaabbbb$ is not.

Definition A.27 (Free group). The **free group** on the set S, written $F(S)$, is the set of reduced words on S^* together with some operations:
(1) A multiplication $X * Y$ of $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ is given by the reduced word of $x_1 \cdots x_n y_1 \cdots y_m$ after elementary reductions.
(2) The identity is the empty word $\emptyset$.
(3) The inverse of $x_1 \cdots x_n$ is $x_n^{-1} \cdots x_1^{-1}$.

Definition A.28 (Presentation of a group). Let S be a set, and let $R \subseteq F(S)$ be a subset of the free group $F(S$. We denote by $\langle\langle R \rangle\rangle$ the normal subgroup generated by R:

$$\langle\langle R \rangle\rangle = \left\{ \prod_{i=1}^{n} x_i r_i x_i^{-1} \middle| r_i \in R,\ x_i \in F(S) \right\}.$$

Then we write the quotient subgroup by $\langle\langle R \rangle\rangle$ as

$$\langle S \mid R \rangle = F(S) / \langle\langle R \rangle\rangle .$$

Example A.29. $\langle a, b \mid b \rangle \cong \langle a \rangle \cong \mathbb{Z}$.

Note that a presentation of a group G is not unique.

Example A.30. For the finite cyclic group of order 5, we can have two presentations $\langle a \mid a^5 \rangle \cong (\mathbb{Z}_5, +)$, and $\left\langle ab \mid ab^{-3}, ba^{-2} \right\rangle \cong (\mathbb{Z}_5, +)$

Definition A.31 (Free product). Given the presentation $G_1 = \langle S_1 \mid R_1 \rangle, G_2 = \langle S_2 \mid R_2 \rangle$ of two groups, where we assume $S_1 \cap S_2 = \emptyset$. The **free product** $G_1 * G_2$ is defined as

$$G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle.$$

The free product is well-defined though the groups G_1 and G_2 can have many different presentations.

A.2 Topological spaces

The notion of manifolds is constructed based on **topological spaces**. Learning the basics of topological spaces requires several sets of lectures, but we will avoid it in this lecture. Here, basics definitions and their rough meanings will be given. The rigorous introduction is given in [Mun75], and brief explanation is given in [Nak03, §2].

The notation of topological spaces essentially provides open and closed sets that allow for the definition of concepts such as continuity, connectedness, and convergence.

Definition A.32. Let X be a set and $\mathcal{T}$ be a collection of subsets of X which satisfies the following properties
(1) $\varnothing$ and X are in $\mathcal{T}$
(2) For a subcollection $\{U_i \in \mathcal{T} | i \in I\}$, $\bigcup U_i \in \mathcal{T}$.
(3) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \bigcap U_2 \in \mathcal{T}$.
Then, $\mathcal{T}$ is called **topology** and $(X, \mathcal{T})$ is called a **topological space**. Often we denote a topological space by X for the sake of brevity. An element of $\mathcal{T}$ is called an **open set**.

Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. Given a map $f : X \to X'$, it is called a **continuous map** if $f^{-1}(V) \in \mathcal{T}$ for $V \in \mathcal{T}'$. If f is a bijection, and both f and f^{-1} are continuous, then f is called a **homeomorphism**.

Let $(X, \mathcal{T})$ be a topological space. A subset V of X is **closed** if its complement in X is an open set, that is $X - V \in \mathcal{T}$. According to the definition, X and $\varnothing$ are closed. For a collection $\{V_i | i \in I\}$ of closed subsets, $\bigcap_i V_i$ is also closed. For V_1, V_2 closed sets, $V_1 \bigcup V_2$ is closed.

A family $\{U_i\}$ of open subsets of X is called an open covering of X, if $\bigcup_{i \in I} U_i = X$.

Definition A.33. A topological space X is **compact** if every open cover of X has a finite subcover, i.e. if $X = \bigcup_{i \in I} U_i$, for a collection of open sets $\{U_i | i \in I\}$, then we can find finitely many U_{i_k} $(k = 1, \ldots, n)$ such that $X = \bigcup_{k=1}^{n} U_{i_k}$.

The compactness is the notation of closed and bounded subsets in the Euclidean space $\mathbb{R}^n$.

Definition A.34. A topological space X is **connected** if it cannot be written as $X = X_1 \cup X_2$, where X_1 and X_2 are both open and $X_1 \cap X_2 = \varnothing$. Otherwise X is called disconnected.

Definition A.35. A topological space X is **path-connected** if, for every pair of points x and x' in X, there is a path in X from x to x': there's a continuous function $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = x'$.

Every path-connected space is connected, but the converse is not true. (Can you find an example that is connected, but not path-connected?)

Definition A.36. Let $(X, \mathcal{T})$ be a topological space and $\sim$ is an equivalence relation in X. We can define a topology on the quotient set $X/\sim$ by $\mathcal{T}' = \{U | U \subset X/\sim\ , \quad p^{-1}(U) \in \mathcal{T}\}$ where $p : X \to X/\sim$ is the natural projection. Then, $(X/\sim, \mathcal{T}')$ is the quotient space of $(X, \mathcal{T})$.

Suppose $\mathcal{T}$ gives a topology to X. N is a neighborhood of a point $x \in X$ if N is a subset of X and N contains some (at least one) open set U_i to which x belongs.

Definition A.37. A topological space X is called a **Hausdorff space** if for each pair $x_1, x_2 \in M$ of distinct points of X, there exist neighborhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.

Roughly speaking, two distinguished points can be "separated" in a Hausdorff space. It may sound obvious to you, but mathematicians are smart enough to find examples of topological spaces that cannot "separate" two points.

B Complex manifolds

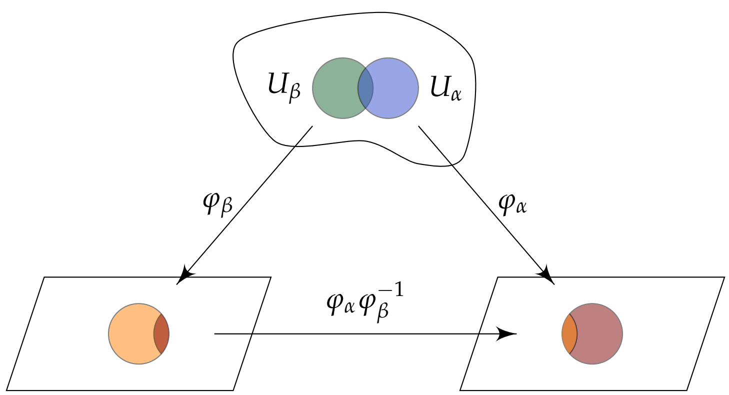
In this appendix, we briefly introduce complex manifolds. For more details, we refer to [Bou07].

Definition B.1 (Complex manifold). Let M be an $2n$-dimensional manifold. M is a **complex manifold** of complex dimension n if

1. Let $M = \bigcup_\alpha U_\alpha$ be an open covering.
2. There is a homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$, where $\varphi(U_\alpha)$ is open in $\mathbb{C}^n$.
3. For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{C}^n$, and the transition function

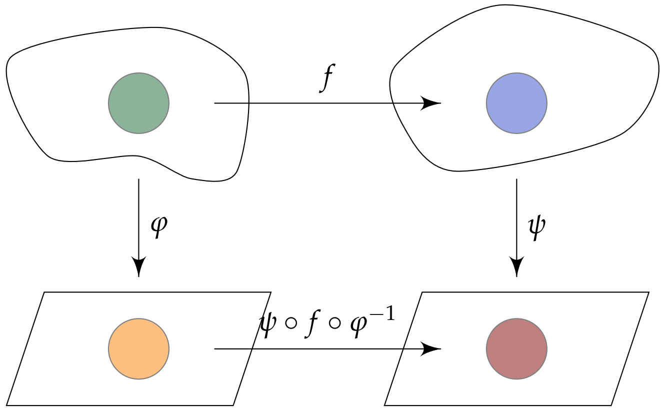
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

   is holomorphic (i.e. they depend only on the z_μ but not on their complex conjugate).



As we consider differentiable (smooth) maps between two smooth manifolds M and N, we can consider holomorphic maps between two complex manifolds M and N. Let M and N be smooth manifolds, and let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ be their atlas.

Definition B.2 (holomorphic map). A map $f : M \to N$ is **holomorphic** if, for a chart $(U_\alpha, \varphi_\alpha)$ of $p \in M$ and (V_β, ψ_β) of $f(p) \in N$, $\psi_\beta \circ f \circ (\varphi_\alpha)^{-1} : \varphi(U_\alpha \cap f^{-1}(V_\beta)) \to \xi(V_\beta)$ is holomorphic.
If there is the inverse map that is holomorphic, it is called a **biholomorphic** map, or M and N are **biholomorphic equivalent**.



Example B.3. Complex projective space $\mathbb{C}\mathbf{P}^n$ is a complex manifold. (Check it satisfies the definition.)

Example B.4. Let us now consider compact complex manifolds. It turns out that submanifolds of $\mathbb{C}^n$ are not interesting, since a connected compact analytic submanifold of $\mathbb{C}^n$ is a point. However, many compact complex manifolds can be constructed as submanifolds of projective spaces $\mathbb{C}\mathbf{P}^n$. We saw that $\mathbb{C}\mathbf{P}^n$ is compact; all its closed complex submanifolds are also compact. In fact, there is a theorem by Chow stating that any compact submanifold of $\mathbb{C}\mathbf{P}^n$ can be realized as the zero loci $F_i(z) = 0$ of a finite number of homogeneous polynomial equations in the homogeneous coordinates z_μ. These compact complex manifolds are called **complex projective variety**. One important example is the **Fermat quintic** in $\mathbb{C}\mathbf{P}^4$, given as the zero locus of the equation

$$5\psi z_0 z_1 z_2 z_3 z_4 - \sum_{\mu=0}^{4} (z_\mu)^5 = 0. \tag{B.1}$$

This three-dimensional compact complex manifold turns out to be Calabi–Yau, probably the most studied Calabi–Yau threefold because of mirror symmetry [CDLOGP91].

B.1 Holomorphic vector bundles

Definition B.5 (Holomorphic vector bundle). Let M be a complex manifold and let $\pi : E \to M$ be a vector bundle of real rank $2r$

1. For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an r-dimensional complex vector space,

2. For all $p \in M$, there is an open $U \subseteq M$ containing p and a diffeomorphism

$$t : E_U = \pi^{-1}(U) \to U \times \mathbb{C}^r$$

such that

$$\begin{tikzcd} E_U \arrow[r, "t"] \arrow[d, "\pi"] & U \times \mathbb{C}^r \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

commutes, and the induced map $E_q \to \{q\} \times \mathbb{C}^r$ is a linear isomorphism for all $q \in U$.

3. Suppose that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ and $t_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{C}^r$ are trivializations of E. Then
$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$
is fiberwise linear, i.e.
$$t_\alpha \circ t_\beta^{-1}(q, v) = (q, g_{\alpha\beta}(z)v),$$
where $g_{\alpha\beta}(z) : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ is a holomorphic function. The transition functions have the following properties
   (a) $g_{\alpha\alpha} = \mathrm{id}$
   (b) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
   (c) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

B.2 Holomorphic tangent and cotangent bundle

Let $(U, (z_1, \cdots, z_n))$ be a local coordinate of a complex manifold M and we decompose the complex coordinate $z_j = x_j + iy_j$ into the real and imaginary part. Then, the basis of the tangent bundle TM on an open set $U \subset M$ can be taken by $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ $(j = 1, \cdots, n)$. Now let us complexify the tangent bundle $T_\mathbb{C} M := TM \otimes \mathbb{C}$ where we can take the basis on U as
$$\frac{\partial}{\partial z_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big)\,, \qquad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)\,.$$
Each fiber of $T_p M \otimes \mathbb{C}$ is a vector space with basis $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ on $\mathbb{C}$. Similarly, one can complexify the cotangent bundle $T^*_\mathbb{C} M := T^*M \otimes \mathbb{C}$ where we can take basis dz_j, $d\bar{z}_j$ on U. At each point $p \in M$, we can take the subspace $T_p^{(1,0)}M$ of $T_p M \otimes \mathbb{C}$ spanned by $\{\frac{\partial}{\partial z_j}\}$. A family of the subspace is denoted by $T^{(1,0)}M = \cup_p T_p^{(1,0)}M$ and we can also define $T^{*(1,0)}M = \cup_p T_p^{*(1,0)}M$ in a similar fashion. Given another chart $(V(w_1, \cdots, w_n))$, they transform
$$\frac{\partial}{\partial z_j} = \frac{\partial w_k}{\partial z_j}\frac{\partial}{\partial w_k}\,, \quad dz_j = \frac{\partial z_j}{\partial w_k} dw_k$$
where the transition functions $\frac{\partial w_k}{\partial z_j}$ and $\frac{\partial z_j}{\partial w_k}$ are holomorphic so that they are holomorphic vector bundles. Indeed, we can decompose complxified (co)tangent bundle as
$$T_\mathbb{C} M = T^{(1,0)}M \oplus T^{(0,1)}M\,, \qquad T^*_\mathbb{C} M = T^{*(1,0)}M \oplus T^{*(0,1)}M\,.$$
We call $T^{(1,0)}M$ (resp. $T^{(0,1)}M$) the **holomorphic (resp. anti-holomorphic) (co)tangent bundle**.

As a real bundle. $T^{(1,0)}M$ is isomorphic to TM. In fact, We can define a map $\mathrm{Re} : T_\mathbb{C} M \to TM$ by taking real part
$$2\mathrm{Re}\, \frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j}\,,$$

which is an isomorphism. Since $T^{(1,0)}M$ is a complex vector bundle, we have the multiplication by i, which induces an isomorphism $J : TM \to TM$:

$$2\mathrm{Re}\,\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j}\,, \quad 2\mathrm{Re}\,i\frac{\partial}{\partial z_j} = \frac{\partial}{\partial y_j}\,,$$

so that

$$J\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j}\,, \quad J\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}\,.$$

It is easy to check $J^2 = -\mathrm{id}_{TM}$, and the eigenvalues of J in $T_pM \otimes \mathbb{C}$ are $\pm i$. Hence, we consider $T_p^{(1,0)}M$ (resp. $T_p^{(0,1)}M$) as the eigenspace of J_p with eigenvalue i (resp. $-i$). J is called **almost complex structure**.

An almost complex structure $J_M : TM \to TM$ is a complex structure if for any $p \in m$ there exist an open neighborhood $p \in U \subset M$, $V \subset \mathbb{C}^n$, and a continuous map $\varphi : U \to V$ such that $d\varphi \cdot J_M = J_{\mathbb{C}^n} \cdot d\varphi$. If an almost complex structure J_M is a complex structure, J_M is called to be **integrable**. The Newlander-Nirenberg theorem states that an almost complex structure J is integrable if and only if the Nijenhuis tensor vanishes:

$$N_J(X,Y) = -J^2[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0$$

for ${}^\forall X, Y \in \mathfrak{X}(M)$.

The only spheres which admit almost complex structures are S^2 and S^6. In particular, S^4 cannot be given an almost complex structure. In the case of S^2, the almost complex structure comes from the complex structure of the Riemann sphere. The 6-sphere S^6 inherits an almost complex structure from the octonion multiplication;. However, whether S^6 has a complex structure is an open question.

B.3 Differential forms

Locally, any differential forms can be expressed as a linear combination of the exterior products of dz_j and $d\bar{z}_k$. In particular, a differential form which can be expanded in terms of

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$$

called (p,q)-form, which is a section of $\bigwedge^p T^{*(1,0)}M \otimes \bigwedge^q T^{*(0,1)}M$. We denote the vector space of (p,q)-forms by $\Omega^{p,q}(M)$. In fact, it is easy to show that the following complexified bundles decompose as

$$\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{j=0}^{k} \bigwedge^j T^{*(1,0)}M \otimes \bigwedge^{k-j} T^{*(0,1)}M,$$

On a complex manifold, the exterior derivative also admits a simple decomposition: $d = \partial + \bar{\partial}$, where we defined the operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)\,, \qquad \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\,.$$

The identity $d^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

A $(p,0)$-form is an element of $\Omega^{p,0}(M)$ and it can be locally written as

$$\varphi = \sum \varphi_{i_1,\cdots,i_p}(z) dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

where $\varphi_{i_1,\cdots,i_p}(z)$ is holomorphic. Therefore, ϕ is called **a holomorphic p-form**. In particular, when $p = n$,

$$K_M = \bigwedge^n T^{*(1,0)}M$$

is called **the canonical line bundle**. Since TM is isomorphic to $T^{(1,0)}M$, the first Chern class is

$$c_1(M) = c_1(TM) = c_1(T^{(1,0)}M) = c_1(\bigwedge^n T^{(1,0)}M) = -c_1(K_M)$$

Now we define the analog of the de Rham cohomology groups for complex manifolds:

Definition B.6. Let M be a complex manifold of complex dimension m. As $\overline{\partial}^2 = 0$, we can form the complex

$$0 \xrightarrow{\overline{\partial}} \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n}(M) \xrightarrow{\overline{\partial}} 0\,.$$

We define the **Dolbeault cohomology groups** $H^{p,q}_{\overline{\partial}}(M)$ of M by

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\mathrm{Ker}(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Remark that the Dolbeault cohomology groups depend on the complex structure of M. Note also that we could have defined the cohomology groups using ∂ instead of $\overline{\partial}$, this is just a matter of convention since they are complex conjugate.

Theorem B.7 (Hodge decomposition theorem for Kähler manifolds). For a compact Kähler manifold M, the complex cohomology satisfies

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M)\,,$$
$$H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)} \tag{B.2}$$

where $H^{p,q}_{\overline{\partial}}$ is the Dolbeault cohomology restricted to forms of type (p,q). Further, we could instead choose the de Rham cohomology to obtain the same result, and the cohomology classes contain unique harmonic representatives. Consequently, holomorphic forms are therefore harmonic for any Kähler metric on a compact manifold, and the odd Betti numbers are even.

We now define the **Hodge numbers** to be $h^{p,q} = \dim H^{p,q}_{\overline{\partial}}(M)$. The Hodge numbers of a complex manifold are summarized in what is commonly called the **Hodge diamond**:

$$\begin{array}{ccccc}
 & & h^{n,n} & & \\
 & h^{n,n-1} & \vdots & h^{n-1,n} & \\
h^{n,0} & \cdots & & \cdots & h^{0,n} \\
 & h^{1,0} & \vdots & h^{1,0} & \\
 & & h^{0,0} & &
\end{array}$$

Theorem B.8 (Hard Lefschetz Theorem). Let (M, ω) be a Kähler manifold. Let L be exterior multiplication by the Kähler form ω. Then

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for $1 \leq k \leq n$. Further, if we define the primitive cohomology $P^j(M)$ by

$$P^j(M) = \ker L^{n-j+1} : H^j \to H^{2n-j+2} ,$$

then we have the Lefschetz decomposition

$$H^m(M) = \bigoplus_k L^k P^{m-2k}(M) .$$

Thus, the Betti numbers of a compact Kähler manifold have a pyramid structure, called the **Hodge pyramid**.

B.4 Kähler manifolds

The condition for a Riemannian metric g to be Hermitian is that for $^\forall X, Y \in \mathfrak{X}(M)$, and $^\forall \alpha, \beta \in \mathbb{C}$, it satisfies

$$g(\alpha X, \beta y) = \alpha\overline{\beta} g(X, Y) .$$

This condition amounts to the case when $\alpha = \beta = i$. Since the multiplication by i corresponds to J, the Hermitian metric can be defined as follows.

Definition B.9. Let (M, J) be a complex manifold, and let g be a Riemannian metric on M. We call g a **Hermitian metric** if $g(v, w) = g(Jv, Jw)$ for all vector fields v, w on M

In other words, a Hermitian metric is a positive-definite inner product $T^{(1,0)}M \otimes T^{(0,1)}M \to \mathbb{C}$ at every point on a complex manifold M. In local coordinate, we can express g on $T_{\mathbb{C}}M$ by

$$g_{j\bar{k}} = g\Big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_k}\Big) ,$$

so that $g_{j\bar{k}}$ is a Hermitian metric.

Using this Hermitian metric g, we can define a two-form ω on M called the **Hermitian form** by $\omega(v, w) = g(Jv, w)$ for all vector fields on v, w on M. In local coordinate, it can be written as

$$\omega = i g_{j\bar{k}} dz_j \wedge d\overline{z}_k$$

Therefore, ω is a $(1, 1)$-form.

Definition B.10. Let (M, J) be a complex manifold, and g a Hermitian metric on M, with Hermitian form ω. g is a **Kähler metric** if $d\omega = 0$. In this case we call ω a **Kähler form**, and we call a complex manifold (M, J) endowed with a Kähler metric a **Kähler manifold**.

Theorem B.11. Any complex submanifold $N \subset M$ of a Kähler manifold (M, ω) is Kähler where the Kähler form is its restriction $\omega_N = \omega\Big|_N$.

For a Kähler manifold M, the holonomy group is $U(n) \subset SO(2n)$. The condition that a Riemannian metric g can be Kähler is equivalent to $\nabla J = 0$ with respect to Levi-Civita connection of g. In addition, it can be shown that locally, the Kähler condition $d\omega = 0$ is equivalent to the condition $\partial_\ell g_{j\bar{k}} = \partial_j g_{\ell\bar{k}}$ and its conjugate equation $\bar{\partial}_{\bar{\ell}} g_{j\bar{k}} = \bar{\partial}_{\bar{k}} g_{j\bar{\ell}}$. Locally, we can always integrate these equations as,

$$g_{j\bar{k}} = \partial_j \bar{\partial}_{\bar{k}} K(z, \bar{z}),$$

for some function $K(z, \bar{z})$, which is known as the **Kähler potential**. It is unique up to Kähler transformation, $K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + f(\bar{z})$ for any holomorphic function $f(z)$. We should note that the Kähler potential cannot be a globally defined smooth function on a compact manifold M.

Since ω is closed, it defines a Dolbeault cohomology class $[\omega] \in H^{1,1}_{\bar{\partial}}(M)$, or a de Rham cohomology class $[\omega] \in H^2_{dR}(M, \mathbb{R})$. Further, the wedge product of n copies of ω, denoted by ω^n, is proportional to the volume form of g. Therefore it defines a non-trivial element in both $H^{n,n}_{\bar{\partial}}(M)$ and $H^{2n}_{dR}(M, \mathbb{R})$ so that $[\omega]$ must be non-trivial. However, $\partial_j \bar{\partial}_{\bar{k}} K(z, \bar{z})$ is exact and therefore zero as a cohomology class. It follows that on a compact Kähler manifold it is impossible to define a Kähler potential globally.

Example B.12. Complex projective space $\mathbb{C}\mathbf{P}^n$ is a Kähler manifold. Consider the function $u(z_0, \cdots, z_m) = \sum_{\mu=0}^n |z_\mu|^2$ where z_μ, $\mu = 0, \cdots, n$ are homogeneous coordinates on $\mathbb{C}^{n+1} \backslash \{0\}$. Define a $(1,1)$-form α by $\alpha = \partial\bar{\partial}(\log u)$. α cannot be the Kähler form of any metric on $\mathbb{C}^{n+1} \backslash \{0\}$, since it is not positive. However, if we consider the projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}\mathbf{P}^n$ defined by $\pi : (z_0, \cdots, z_n) \mapsto [z_0; \cdots; z_n]$, one can show that there exists a unique positive $(1,1)$-form ω on $\mathbb{C}\mathbf{P}^n$ such that $\alpha = \pi^*(\omega)$. ω is a Kähler form on $\mathbb{C}\mathbf{P}^n$; its associated Kähler metric is called the **Fubini-Study metric**, and is given in components by $g_{\mu\bar{\nu}} = \partial_j \bar{\partial}_{\bar{k}} \log u$.

$$\omega_{\mathrm{FS}} = i \frac{\sum_j dz_j \wedge d\bar{z}_j - \sum_{k,j} (\bar{z}_j dz_j) \wedge (z_k d\bar{z}_k)}{(|z_0|^2 + \cdots + |z_n|^2)^2} .$$

Given the metric in the above form, we can compute the Riemann curvature. Straightforward calculations show that the only non-zero components are $\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}}$, and its complex conjugate. Thus, the only non-zero components of the curvature tensor are $R^\ell{}_{i\bar{j}k} = \partial_{\bar{j}} \Gamma^\ell_{ik}$. Furthermore, the Ricci tensor turns out to be

$$R_{j\bar{k}} = R^\ell{}_{\ell j\bar{k}} = i\partial_j \bar{\partial}_{\bar{k}} \log \det(g) .$$

For a Kähler manifold, the complexified de Rham cohomology decomposes into the Dolbeault cohomology (note that the de Rham cohomology groups are complex, as we are now considering complexified k-forms)

$$H^k_{dR}(M, \mathbb{C}) = \bigoplus_{j=0}^{k} H^{j,k-j}_{\bar{\partial}}(M).$$

This can be shown by using the Hodge theorem of a Kähler manifold.

B.5 Calabi–Yau manifolds

We are now ready to study Calabi–Yau manifolds, which are a particular kind of Kähler manifolds.

In 1954, Calabi conjectured that a compact Kähler manifold M with $c_1 = 0$ admits Ricci-flat Kähler metric. Yau proved the conjecture in 1976 by solving the complex Monge-Ampére equation. Therefore, such a manifold is called a **Calabi–Yau manifold**.

Calabi–Yau manifolds have been studied extensively in recent decades, particularly because of their importance in string theory [Yau09a]. While the mathematical study of Calabi–Yau manifolds has helped us understand compactifications of string theory, the study of string theory has led to fascinating insights in the geometry of Calabi–Yau manifolds, for example, the study of the Calabi–Yau moduli space and mirror symmetry. Calabi–Yau manifolds are thus a very good example of the fruitful interactions between mathematics and physics that have been taking place in the recent decades.

Let us first list some of the most common definitions of Calabi–Yau manifolds. A Calabi–Yau manifold of real dimension $2n$ is a compact Kähler manifold (M, J, g):

1. with zero Ricci form,
2. with vanishing first Chern class,
3. with $\mathrm{Hol}(g) = SU(n)$ (or $\mathrm{Hol}(g) \subseteq SU(n)$),
4. with trivial canonical bundle,
5. that admits a globally defined and nowhere vanishing holomorphic n-form.

The Hodge numbers of a Calabi–Yau manifold satisfy a few more properties, which drastically decrease the number of undetermined Hodge numbers. We will now focus on Calabi–Yau three-folds, that is Calabi–Yau manifolds with complex dimension 3, for the sake of brevity. These are the most important Calabi–Yau manifolds in string theory applications. But most results extend straightforwardly to higher dimensional Calabi–Yau manifolds.

The Hodge numbers of Kähler manifolds satisfy a Hodge star duality $h^{p,q} = h^{3-q,3-p}$ and a complex conjugation duality $h^{p,q} = h^{q,p}$. For Calabi–Yau manifolds, there is a further duality, sometimes called **holomorphic duality**. The triviality of the canonical bundle of a Calabi–Yau manifold M implies that $h^{3,0} = 1$, i.e. the existence of a unique holomorphic volume form Ω. Given a $(0, q)$ cohomology class $[\alpha]$, there is a unique $(0, 3-q)$ cohomology class $[\beta]$ such that $\int_M \alpha \wedge \beta \wedge \Omega = 1$ (using Stoke's theorem). Thus $h^{0,q} = h^{0,3-q}$. Therefore, for a Calabi–Yau manifold we have that $h^{3,0} = h^{0,3} = h^{0,0} = h^{3,3} = 1$.

Moreover, one can show that $h^{1,0} = 0$. Thus, $h^{1,0} = h^{0,1} = h^{0,2} = h^{2,0} = h^{2,3} = h^{3,2} = h^{3,1} = h^{1,3} = 0$. Therefore, the only remaining independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$, and the Hodge diamond takes the form:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The Euler characteristic of a Calabi–Yau manifold accordingly simplifies. Recall that $\chi = \sum_{k=0}^{2m}(-1)^k b^k$, so we now have that $\chi = 2b^0 - 2b^1 + 2b^2 - b^3 = 2 - 0 + 2h^{1,1} - 2 - 2h^{2,1}$, that is

$$\chi = 2(h^{1,1} - h^{2,1}).$$

Therefore, if the Euler characteristic is easily computed, we only have to compute one of the two independent Hodge numbers to get all the topological information. In fact, the Euler characteristic is given by the integral over M of the top Chern class of M, which is $c_3(M)$ for a Calabi–Yau threefold:

$$\chi = \int_M c_3(M).$$

This formula can be used to compute the Euler characteristic of M.

The Hodge number $h^{1,1}$ classifies infinitesimal deformations of the Kähler structure that, roughly speaking, parametrizes the volume of M. For a Calabi–Yau threefold, $h^{2,1}$ classifies infinitesimal deformations of the complex structure that, roughly speaking, parametrizes the shape of M.

One of the fascinating property of Calabi–Yau three-folds is that they come in mirror pairs, (M, W), such that $H^{2,1}(W) \cong H^{1,1}(M)$ and $H^{1,1}(W) \cong H^{2,1}(M)$. Roughly speaking, the complex structure moduli is exchanged with the Kähler structure moduli. This is the basic idea behind **mirror symmetry**.

B.6 Examples

Example B.13. A torus T^2 is a compact Calabi–Yau manifold one-fold. The Hodge diamond of T^2 is

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

The moduli space of the complex structure is studied in §14.3.

Example B.14. T^4 is a compact Calabi–Yau manifold 2-fold. The Hodge diamond of T^4 is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Example B.15. The other compact Calabi–Yau 2-fold is a K3 surface, which is constructed as follows. Let us consider the quotient space $T^4/\mathbb{Z}_2$. There are 16 singular points and the neighborhood around a singular point is a cone of $\mathbb{R}\mathbf{P}^3$. If we consider the set $V = \{p \in TS^2 |\, |v| \leq 1\}$ of points with length ≤ 1 in a fiber of TS^2, its boundary is $\partial V \cong \mathbb{R}\mathbf{P}^3$. Therefore, we can replace the neighborhood of each singular point by V. Then, the resulting space is smooth and it is a K3 surface. The Hodge diamond of a K3 surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example B.16. As mentioned before, a submanifold $\mathbb{C}\mathbf{P}^4$ defined by

$$M = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0\}$$

is a compact Calabi–Yau 3-fold. Let us define the group

$$G = \{(a_0, \cdots, a_4) \in (\mathbb{Z}_5)^5 | \sum_i a_i = 0\} / (\mathbb{Z}_5 = \{(a, a, a, a, a)\}).$$

Then, the mirror manifold W is constructed by the resolution of singularities of the orbifold M/G where G acts on M by $(z_j) \to (z_j \omega^{a_j})$ with $\omega = e^{2\pi i/5}$. The Hodge diamonds of M and W are given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Hodge diamonds of M Hodge diamonds of W

where you can see the mirror symmetry.

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