1. Banach Space

A Banach space is a complete normed space. The definition of a complete normed space is as follows:

(1) Vector Spaces

A *vector space*(or a *linear space*) over $\mathbb{R}$ is a set of objects(called *vectors*) which can be added together or multiplied by scalars in $\mathbb{R}$.

More formally, a vector space V over $\mathbb{R}$ is a set V with two operations: *vector addition* and *scalar multiplication*. With these operations given, the vector space V must satisfy the following axioms:

For each $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{R}$

A1 (Associativity of addition): $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

A2 (Commutativity of addition): $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

A3 (Identity element of addition): There exists an element $\mathbf{0} \in V$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$

A4 (Inverse elements of addition): For every $\mathbf{u} \in V$, there exists an element $-\mathbf{u} \in V$, called the additive inverse of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

A5 (Compatibility of scalar multiplication with field multiplication): $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$

A6 (Identity element of scalar multiplication): $1\mathbf{u} = \mathbf{u}$, where 1 denotes the multiplicative identity in $\mathbb{R}$

A7 (Distributivity of scalar multiplication with respect to vector addition): $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

A8 (Distributivity of scalar multiplication with respect to field addition): $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$

(2) Norms on vector spaces

Let V be a vector space over $\mathbb{R}$. A *norm* $\|\cdot\|: V \to \mathbb{R}$ on the vector space V is a function satisfying the following.

Given $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$,

(i) $\| \mathbf{u} \| \geq 0$, the equality holds if and only if $\mathbf{u} = \mathbf{0}$

(ii) $\| \alpha\mathbf{u} \| = |\alpha| \| \mathbf{u} \|$

(iii) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$

A vector space with norm is called a *normed vector space*.

(3) Convergent sequences, Cauchy sequences

Let (u_n) be a sequence in a vector space V over $\mathbb{R}$.

Then,

u_n is $convergent$: $\Leftrightarrow$ there exists some $u \in V$ such that for each $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $\| u_n - u \| < \varepsilon$

In this case, we denote $\lim\limits_{n\to\infty} u_n = u$

u_n is $Cauchy$: $\Leftrightarrow$ for each $\varepsilon > 0$, there exist some $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\| u_m - u_n \| < \varepsilon$

(4) Banach Space

A vector space V over $\mathbb{R}$ is called to be *complete* if every Cauchy sequence is convergent.

A *complete normed vector space* is also called a *Banach space*.

2. Closed sets

Given $\mathbf{x} \in V$ and $\varepsilon > 0$, the set $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in V \mid \| \mathbf{y} - \mathbf{x} \| < \varepsilon\}$ is called an *open ball* centered at $\mathbf{x}$ with radius ε.

A subset U of a vector space V is called to be *open*

if for every $\mathbf{x} \in U$, there is an open ball $B(\mathbf{x}, \varepsilon)$ centered at $\mathbf{x}$, which is contained in U.

That is, $\mathbf{x} \in B(\mathbf{x}, \varepsilon) \subset U$

A subset M of V is called to be *closed* if $M^C := V - M$ is open.

Each closed subset M of V satisfies the following property:

If $(\mathbf{u}_n)$ is a convergent sequence in M and $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$, then $\mathbf{u} \in M$

3. Operators

Let M and Y be sets. An *operator* $A: M \to Y$ associates to each point u in M to a point v in Y denoted by v=Au.

The set M is called the *domain* of A, we also write M=D(A).

The set $A(M) := \{v \in Y \mid v = Au \text{ for some } u \in M\}$ is called the *range* of A.

Operators are also called functions.

4. The Banach Fixed Point Theorem

We assume that:

(a) M is a closed nonempty set in the Banach space X over $\mathbb{R}$, and

(b) the operator $A: M \to M$ is $k-$contractive, i.e., by definition,

$\| Au - Av \| \leq k \| u - v \|$ for all $u, v \in M$, and fixed $k, 0 \leq k < 1$.

Then the following hold true:

(i) **Existence and uniqueness.** The equation $Au = u$ has exactly one solution u,

i.e., the operator A has exactly one fixed point u on the set M.

(ii) **Convergence of the iteration method.**

For each given $u_0 \in M$, the sequence (u_n) defined recursively

by $u_{n+1} = Au_n, n = 0,1, \ldots$ converges to the unique solution u of the equation $Au = u$

Proof)

Step 1: We show first that (u_n) is a Cauchy sequence.

Let n=1,2,… . Since $\| Au - Av \| \leq k \| u - v \|$ for all $u, v \in M$, we have

$\| u_{n+1} - u_n \| = \| Au_n - Au_{n-1} \| \leq k \| u_n - u_{n-1} \| = k \| Au_{n-1} - Au_{n-2} \|$

$\leq k^2 \| u_{n-1} - u_{n-2} \| \leq \cdots \leq k^n \| u_1 - u_0 \|$

Now let n=0,1,… and m=1,2,… The triangular inequality and the sum formula for the geometric series yield

$\| u_n - u_{n+m} \| = \| (u_n - u_{n+1}) + (u_{n+1} - u_{n+2}) + \cdots + (u_{n+m-1} - u_{n+m}) \|$

$\leq \| u_n - u_{n+1} \| + \| u_{n+1} - u_{n+2} \| + \cdots + \| u_{n+m-1} - u_{n+m} \|$

$\leq (k^n + k^{n+1} + \cdots + k^{n+m-1}) \| u_1 - u_0 \|$

$\leq k^n(1 + k + k^2 + \cdots) \| u_1 - u_0 \|$

$= k^n(1 - k)^{-1} \| u_1 - u_0 \|$

It follows from $0 \leq k < 1$ that $k^n \to 0$ as $n \to \infty$. Hence the sequence (u_n) is Cauchy.

Since X is a Banach space, the Cauchy sequence (u_n) converges, i.e.,

$u_n \to u$ as $n \to \infty$

Step 2: We show that the limit point u is a solution of the equation $Au = u$.

From $u_0 \in M$ and $u_1 = Au_1$ with $A: M \to M$, we get $u_1 \in M$. Similarly, by induction,

$u_n \in M$ for all $n = 0,1, \ldots$.

Since the set M is closed, we obtain $u \in M$

,and hence $Au \in M$. By $\| Au - Av \| \leq k \| u - v \|$ for all $u, v \in M$, we have

$$\| u - Au \| = \| \lim_{n\to\infty} u_{n+1} - Au \| = \| \lim_{n\to\infty} Au_n - Au \| = \lim_{n\to\infty} \| Au_n - Au \|$$

$$\leq \lim_{n\to\infty} k \| u_n - u \| = 0$$

Hence $Au = u$, proving (2).

Step 3: We show the uniqueness of the solution u of $Au = u$. Suppose u and v are solutions.

Then $Au = u$ and $Av = v$. It follows that $\| u - v \| = \| Au - Av \| \leq k \| u - v \|$.

If $\| u - v \| \neq 0$, then $1 \leq k$, a contradiction. Hence $\| u - v \| = 0$.

Therefore $u = v$, proving (1).