Space Filling Curves

Definition

Let I be an interval of the real numbers and let X be a topological space. A "curve" is a continuous map $f: I \to X$ or its image $f(I)$.

Examples

$f: \mathrm{R} \to \mathrm{R}^2$ given by

$f(x) = (x, x^2), f(x) = (\cos x, \sin x), \dots$ are curves but not onto R^2.

Question

Is there a curve $f: \mathrm{R} \to \mathrm{R}^2$ which is onto?

$\Leftrightarrow$ Is there a map which is continuous and maps R onto R^2?

It suffices to show that there exists a continuous, onto map $f: [0,1] \to [0,1]^2$

Proof) 생략

Furthermore, if there is a curve $f: \mathrm{R} \to \mathrm{R}^2$ which is onto, then there is a curve $g: \mathrm{R} \to \mathrm{R}^n$ for each $n \in \mathrm{N}$.

Proof)

We prove by using induction.

For $n=1$, take $f(t) = t$. For n=2, it is already assumed that there is such a curve $s: \mathrm{R} \to \mathrm{R}^2$. Suppose that the statement holds for some $n \in \mathrm{N}$. Consider the composition of functions

$\mathrm{R} \ \to_{\mathrm{s}} \ \mathrm{R} \times \mathrm{R} \ \to_{\mathrm{g}} \ \mathrm{R}^{\mathrm{n}} \times \mathrm{R} \ \to_{\mathrm{h}} \ \mathrm{R}^{\mathrm{n}+1}$

, where s is as given above, h is the natural mapping given by $h((x_1, x_2, \dots, x_n), x_{n+1}) = (x_1, x_2, \dots, x_n)$. g is defined by $g(x,y) = (u(x), y)$ where $u: \mathrm{R} \to \mathrm{R}^{\mathrm{n}}$ is a continuous onto map whose existence is assumed in the induction hypothesis. Then the composition is a continuous onto map.

It remains to show the existence of a continuous, onto map $f: I \to I^2$ where $I = [0,1]$.

Theorem

There exists a continuous map $f: I \to I^2$ which is onto.

Proof)

We first consider the following operation.

Consider $g: [a,b] \to [x, x+\epsilon] \times [y, y+\epsilon]$ whose graph is in Figure 1:

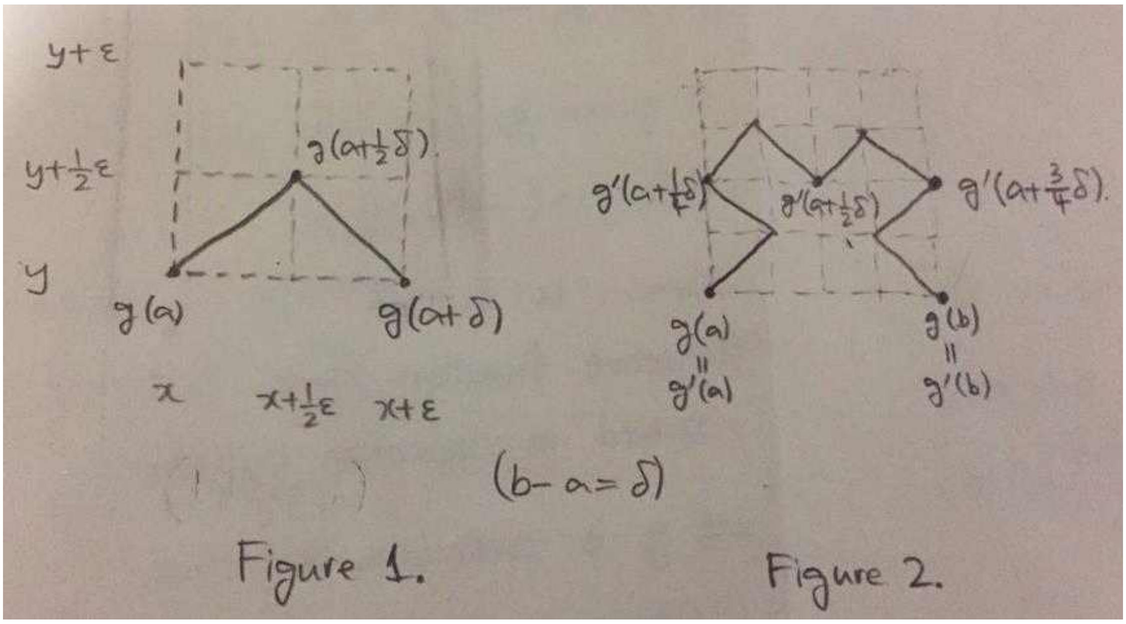

Figure 1. Figure 2.

We make an operation on g to obtain $g':[a,b]\to[x,x+\epsilon]\times[y,y+\epsilon]$ whose graph is in Figure 2. Notice that $g(a)=g'(a)$ and $g(b)=g'(b)$

We define a sequence of functions $f_n : I \to I^2$ as follows.
Let $f_1 = g$ in the case $a=0, b=1, x=y=0, \epsilon=1$ where g is as defined above. The two line segments in the graph of f_1 is the images of $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$ under f_1 respectively. Let f_2 be defined to be a map whose graph is obtained by applying our operation on the graph of f_1. Hence the graph of f_2 looks like the one in Figure 2. In this case, the 8 line segments in the graph of f_2 are images of $[0,\frac{1}{8}]$, $[\frac{1}{8},\frac{2}{8}]$, $\ldots$, $[\frac{7}{8},1]$ respectively. Notice that given $t\in[0,1]$, we have $\|f_1(t)-f_2(t)\| \le \sqrt{2}$ since the graphs of f_1, f_2 lie in the unit square. Define f_3 by characterizing its graph; for each triangle appearing in the graph of f_2, apply our operation. The graph of f_3 is obtained from the graph of f_2 as in the Figure 3.

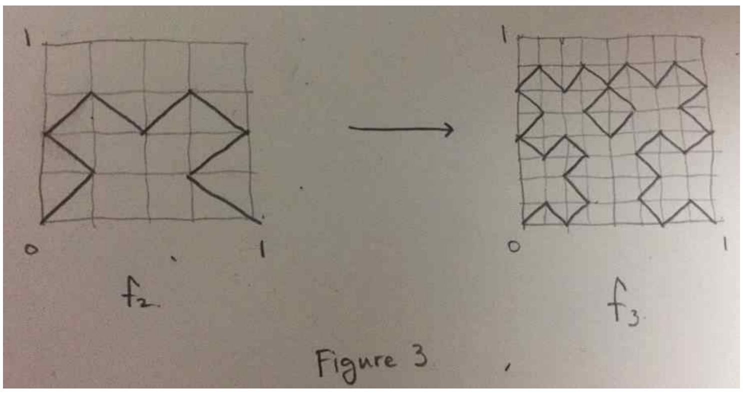

Figure 3

In this case, the 32 line segments are images of $[0,\frac{1}{32}]$, $[\frac{1}{32},\frac{2}{32}],\ldots,[\frac{31}{32},1]$ under f_3 respectively. Notice that given any $t\in[0,1]$, both $f_2(t)$ and $f_3(t)$ lie in a square with sidelength $\frac{1}{2}$, so $\|f_2(t)-f_3(t)\| \leq \frac{\sqrt{2}}{2}$ Define f_n recursively by characterizing its graph as follows; given the graph of f_{n-1}, apply the operation on each triangle to obtain the graph of f_n. Then we have $\|f_n(t)-f_{n+1}(t)\| \leq \frac{\sqrt{2}}{2^{n-1}}$ for each $t\in[0,1]$ and $n\in\mathrm{N}$.

Let $C(I,I^2)$ be the set of all continuous functions $f:I\rightarrow I^2$. Then $f_n\in C(I,I^2)$ for each $n\in\mathrm{N}$. Define a metric ρ on $C(I,I^2)$ by

$\rho(f,g)=$ sup{ $\|f(t)-g(t)\|:t\in[0,1]$ }

Then $C(I,I^2)$ with the metric ρ is a complete metric space (A metric space where every Cauchy sequence converges). We will prove this fact in the next theorem.

For $m,n\in\mathrm{N}$ and $t\in[0,1]$, We have

$\|f_n(t)-f_{n+m}(t)\|$

$\leq \|f_n(t)-f_{n+1}(t)+f_{n+1}(t)-f_{n+2}(t)+\ldots+f_{n+m-1}(t)-f_{n+m}(t)\|$

$\leq \|f_n(t)-f_{n+1}(t)\|+\|f_{n+1}(t)-f_{n+2}(t)\|+\ldots+\|f_{n+m-1}(t)-f_{n+m}(t)\|$

$\leq \frac{\sqrt{2}}{2^{n-1}}+\frac{\sqrt{2}}{2^n}+\ldots+\frac{\sqrt{2}}{2^{n+m-2}}$

$$=\frac{\frac{\sqrt{2}}{2^{n-1}}(1-\frac{1}{2^m})}{1-\frac{1}{2}}<\frac{\sqrt{2}}{2^n}$$

It follows that $\rho(f_n,f_{n+m})\leq\frac{\sqrt{2}}{2^n}$ for each $n,m\in\mathrm{N}$. Hence the sequence (f_n) is Cauchy in $C(I,I^2)$ with respect to the metric ρ. Since $C(I,I^2)$ with the metric ρ is complete, $f_n\rightarrow f$ for some $f\in C(I,I^2)$. By definition of $C(I,I^2)$, f is continuous. We claim that f is onto.

$f\,is\,onto\Leftrightarrow f(I)=I^2$

Since $f\in C(I,I^2)$, we have $f(I)\subseteq I^2$, so it remains to show $I^2\subseteq f(I)$. Suppose $x\in I^2$, and let $\epsilon>0$. Since the graph of f_n passes all the squares of sidelengths $\frac{1}{2^n}$, there exists some $t_0\in[0,1]$ such that $\|x-f_n(t_0)\|\leq\frac{\sqrt{2}}{2^n}$. Choose $n\in\mathrm{N}$ sufficiently larch such that $\frac{\sqrt{2}}{2^n}<\frac{\epsilon}{2}$ and $\rho(f_n,f)<\frac{\epsilon}{2}$. Then we have

$\|x-f(t_0)\| \le \|x-f_n(t_0)+f_n(t_0)-f(t_0)\| \le \|x-f_n(t_0)\|+\|f_n(t_0)-f(t_0)\| \le \dfrac{\sqrt{2}}{2^n}+\rho(f_n,f)$

$< \dfrac{\epsilon}{2}+\dfrac{\epsilon}{2}=\epsilon$

This implies $x\in\overline{f(I)}$. Since f is continuous and I is compact, $f(I)$ is compact. It follows that $f(I)$ is a closed subset of R^2, so $f(I)=\overline{f(I)}$ and hence $x\in f(I)$. Therefore, $f(I)=I^2$ so f is onto. ////

We call $f:I\to I^2$ a "Space Filling Curve".

Theorem

$C(I,I^2)$ with the metric $\rho(f,g)=\{\|f(t)-g(t)\|:t\in[0,1]\}$ is complete.

Proof)

Suppose (f_n) is a Cauchy sequence in $C(I,I^2)$ with respect to ρ. Then, given $\epsilon>0$, there is some $N\in\mathrm{N}$ such that $m,n\ge N$ implies $\rho(f_m,f_n)<\dfrac{\epsilon}{2}$.

Then, for each $t\in[0,1]$, we have

$\|f_n(t)-f_m(t)\| \le \rho(f_n,f_m)<\dfrac{\epsilon}{2}$ -------(1)

, so $(f_n(t))$ is a Cauchy sequence in I^2. Since I^2 is complete, $f_n(t)$ converges to some $f(t)$ as $n\to\infty$. Letting $m\to\infty$ in (1), we obtain $\|f_n(t)-f(t)\|\le\dfrac{\epsilon}{2}<\epsilon$ for each $t\in[0,1]$ and each $n\ge N$. Hence f_n converges to f uniformly. Since each f_n is continuous, f is continuous. Finally, since $f(t)=\lim_{n\to\infty}f_n(t)$, $f_n(t)\in I^2$ and I^2 is closed in R^2, we have $f(t)\in I^2$. Hence, $f\in C(I,I^2)$.