Van der Waerden's Theorem

Definition)

arithmetic progression: Given $k, r, a \in \mathbb{Z}$ with $k \geq 1$, we denote $a + [0,k) \cdot r = \{a + ri \mid 0 \leq i \leq k-1\}$. We call k the **length**, and a the **basepoint** of the arithmetic progression.

We say a function $c: \{1,...,N\} \to \{1,...,m\}$ an m **coloring** of $\{1,...,N\}$.

Given an m coloring $c: \{1,...,N\} \to \{1,...,m\}$ of $\{1,...,N\}$, we say an arithmetic progression $a + [0,k) \cdot r$ in $\{1,...,N\}$ to be **monochromatic** if all elements of the progression are in the same color.

Van der Waerden's Theorem)

Let $k, m \geq 1$. Then there is an $N \in \mathbb{N}$ such that any m coloring of $\{1,2,...,N\}$ contains a monochromatic progression of length k.

Proof)

We prove the theorem using induction on k. It is obvious that the theorem holds for $k=1$. Suppose that $k \geq 2$ and that the theorem holds for $k-1$.

We define the notion of 'polychromatic fan' and then we proceed to prove the theorem.

Let $c: \{1,...,N\} \to \{1,...,m\}$ be a coloring.

Let $k \geq 1, d \geq 0$, and $a \in \{1,...,N\}$.

We call a $d-$tuple $(a + [0,k) \cdot r_1, \dots, a + [0,k) \cdot r_d)$ a **fan** of progressions in $\{1,...,N\}$ with $r_1, \dots, r_d > 0$. Let us call k the **radius**, d the **degree**, and a the **basepoint** of the fan.

We call each progression $a + [1,k) \cdot r_i$ a **spoke** of the fan, for $1 \leq i \leq d$.

We call the fan $(a + [0,k) \cdot r_1, \dots, a + [0,k) \cdot r_d)$ to be **polychromatic** if the base point a and the d spokes are all monochromatic with distinct colors, i.e. if

(1) $c(a + j_1 r_i) = c(a + j_2 r_i)$ for each fixed i in $\{1,...,d\}$ and each j_1, j_2 in $\{1,...,k-1\}$.

(2) $c(a) \neq c(a + j r_i)$ for each $j \in \{1,...,k-1\}$ and $i \in \{1,...,d\}$.

and (3) $c(a + j_1 r_{i_1}) \neq c(a + j_2 r_{i_2})$ whenever $i_1 \neq i_2$.

With these definitions, our main claim is what follows:
For each $d \geq 0$, there is an N such that any m coloring of $\{1,...,N\}$ contains a monochromatic progression of length k or a polychromatic progression of radius k and degree d. This actually shows our theorem. (Why? Suppose that we have shown this claim and consider the case where $d > m$.)

We prove the claim using the induction on d. It is obvious that the claim holds if $d=0$. The claim also holds for $d=1$ by the induction hypothesis (Why? let N be any positive integer which guarantees the existence of a monochromatic progression of length $k-1$ and consider an m coloring of $\{1,...,2N\}$.).
Suppose that $d > 1$ and that the claim holds for $d-1$.
Let N_1 be a positive integer which guarantees that any m coloring of $\{1,...,N_1\}$ contains a monochromatic progression of length k or a polychromatic fan of radius k and degree $d-1$.
Let N_2 be a positive integer which guarantees that any $m^d N_1^d$ coloring of $\{1,...,N_2\}$ contains a monochromatic progression of length $k-1$.
Let $N = 4kN_1N_2$. Let $c:\{1,...,N\} \to \{1,...,m\}$ be a m coloring of $\{1,...,N\}$. If $b \in \{1,...,N_2\}$, then $\{bkN_1+1, \ldots, bkN_1+N_1\}$ is a subset of $\{1,...,N\}$ with cardinality N_1, because we have
$bkN_1 + N_1 \leq kN_1N_2 + N_1 \leq kN_1N_2 + kN_1N_2 = 2kN_1N_2 < 4kN_1N_2 = N$
By our definition of N_1, for each $b \in \{1,...,N_2\}$, the set $\{bkN_1+1, \ldots, bkN_1+N_1\}$ contains a monochromatic progression of length k or a polychromatic fan of radius k and degree $d-1$.
If there is at least one b in $\{1,...,N_2\}$ such that the set $\{bkN_1+1, \ldots, bkN_1+N_1\}$ contains a monochromatic progression of length k, then our proof is done, so we may assume that for each $b \in \{1,...,N_2\}$, there is no monochromatic progression of length k in $\{bkN_1+1, \ldots, bkN_1+N_1\}$. Then for each $b \in \{1,...,N_2\}$, there are $a(b), r_1(b), \ldots, r_{d-1}(b) \in \{1,...,N_1\}$ and distinct colors $c_0(b), \ldots, c_{d-1}(b) \in \{1,...,m\}$ such that $c(bkN_1 + a(b)) = c_0(b)$ and $c(bkN_1 + a(b) + jr_i) = c_i(b)$ for each $1 \leq i \leq d-1, 1 \leq j \leq k-1$. Then we can consider the map $b \mapsto (a(b), r_1(b), \ldots, r_{d-1}(b), c_0(b), \ldots, c_{d-1}(b))$ as an $m^d N_1^d$ coloring of $\{1,...,N_2\}$. Then, by our definition of N_2, there is a monochromatic progression $b+[0,k-1) \cdot s$ for some $b \in \{1,...,N_2\}$ and $s > 0$, with some color $(a, r_1, \ldots, r_{d-1}, c_0, \ldots, c_{d-1})$ among those $m^d N_1^d$ colors. To ease the proof we may assume that $s < 0$ by reversing the original monochromatic progression $b+[0,k-1) \cdot s$.
Let $b_0 = (b-s)kN_1 + a$. Note that b_0 lies in $\{1,...,N\}$ since we have
$b_0 = (b-s)kN_1 + a \leq (N_2+N_2)kN_1 + N_1 \leq 2kN_1N_2 + 2kN_1N_2 = 4kN_1N_2 = N.$
Consider the fan
$(b_0 + [0,k) \cdot skN_1, b_0 + [0,k) \cdot (skN_1 + r_1), \ldots, b_0 + [0,k) \cdot (skN_1 + r_{d-1}))$
of radius k, degree d, basepoint b_0.

We see that each spoke of this fan is monochromatic. For the first spoke we have $c(b_0 + jskN_1) = c((b-s)kN_1 + a + jskN_1) = c((b-s+js)kN_1 + a) = c((b+(j-1)s)kN_1 + a)$
$= c_0(b+(j-1)s) = c_0$
,for $1 \le j \le k-1$.
For $1 \le t \le d-1$ we have
$c(b_0 + j(skN_1 + r_t)) = c((b-s)kN_1 + a + j(skN_1 + r_t))$
$= c((b-s+js)kN_1 + a + jr_t) = c((b+(j-1)s)kN_1 + a + jr_t)$
$= c_t(b+(j-1)s) = c_t$
, for $1 \le j \le k-1$. Hence, each spoke of the fan is monochromatic. If $c(b_0)$ is identical with the color of some spoke, then we have found a monochromatic progression of length k. Otherwise, we have found a polychromatic fan of radius k, degree d, so this completes the proof of our claim on d, and this in turn completes the proof of the theorem.

Lemma)
If $c: \mathbb{Z} \to \{1,...,m\}$ is an m coloring of $\mathbb{Z}$, then there is a color i in $\{1,...,m\}$ such that there is a monochromatic progression of color i with arbitrary length.
Proof)
Let $m \ge 1$ and let $c: \mathbb{Z} \to \{1,...,m\}$ be an m coloring of $\mathbb{Z}$. Suppose the contrary that for each i in $\{1,...,m\}$, there is an $k_i \in \mathbb{N}$ such that there is no monochromatic progression in $\mathbb{Z}$ of length k_i with the color i. Let $k = \max\{k_1,...,k_m\}$. By the Van der Waerden's Theorem, there is some N such that there is a monochromatic progression in $\{1,...,N\}$ of length k with some color i. Since $k_i \le k$, there is a monochromatic progression of length k_i with the color i in $\{1,...,N\}$ so the monochromatic progression lies in $\mathbb{Z}$. This contradicts to our definition of k_i.

Corollary)
Let α be a real number and let $\epsilon > 0$. Then $\|\alpha r^2\|_{\mathbb{R}/\mathbb{Z}} < \epsilon$ for infinitely many $r \in \mathbb{N}$, where $\|x\|_{\mathbb{R}/\mathbb{Z}}$ denote the distance from x to the closest integer of x.
Proof)
Partition the interval $[0,1)$ with some finite half open subintervals $I_1,...,I_m$, in such a way that the lengths are all less than $\epsilon/4$. For each $i \in \{1,...,m\}$, let $N_i = \{n \in \mathbb{Z} \mid (\alpha n^2/2) \bmod 1 \in I_i\}$. Then $\{N_1,...,N_m\}$ induces an m coloring of $\mathbb{Z}$. By the previous lemma, one of the m colors has a monochromatic progression $n, n+r, n+2r$ of length 3 for arbitrarily large spacing r. This means that $\alpha n^2/2, \alpha(n+r)^2/2, \alpha(n+2r)^2/2$ belongs to some I_i mod 1. It follows that $\alpha r^2 = \alpha n^2/2 - 2\alpha(n+r)^2/2 + \alpha(n+2r)^2/2$ satisfies $\|\alpha r^2\|_{\mathbb{R}/\mathbb{Z}} < \epsilon$ for such large r's.