Karger's Min Cut Algorithm

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Min Cut

- Assume you are given an undirected, connected weighted graph $G = (V, E)$, with weights of all edges positive reals.
- A *cut* $T = (X, Y) \in G$ is any partition of the set of vertices V into two non empty disjoint subsets X and Y such that $V = X \cup Y$.
- The capacity of a *cut* $T = (X, Y) \in G$ is the total sum of weights of all edges which have ends in X and Y.
- A *cut* $T = (X, Y) \in G$ is a *minimal cut* if it has the lowest capacity among all cuts in G.
- We say that an edge $e(u, v)$ belongs to a cut $T = (X, Y)$ if one of its vertices belongs to X and the other belongs to Y.

Karger's Min Cut Algorithm
<u>Idea</u>
We can contract edges by fusing two vertices into a single vertex, summing weights of edges where necessary.
<u>Initial Heuristics</u>

1. Claim: If u and v belong to the same side of a minimal cut (X, Y), then after collapsing u and v into a single vertex, the capacity of the minimal cut in G_{uv} is the same as the capacity of the minimal cut in G.
2. If u and v belong to opposite sides of $(X, Y) \in G$, then after collapsing u and v into a single vertex, the capacity of the minimal cut in G_{uv} is larger or equal to the capacity of the minimal cut in G.

Proof:

1. Let $T_1 = (X_1, Y_1)$ be a minimal cut in $G_{u,v}$.
2. Split vertex $[u, v]$ back into two vertices u and v but keep them on the same side of the minimal cut T_1.

3. This produces a cut T_2 in G of the same capacity as the minimal cut $T_1 \in G_{uv}$.

4. Thus, the capacity of the minimal cut in G can only be smaller than the capacity of the minimal cut T_1 in G_{uv}.

Algorithm 1:

1. Pick an edge to contract with a probability proportional to the weight of that edge:
$$P(e(u,v)) = \frac{w(u,v)}{\sum_{e(p,q)\in E} w(p,q)}$$

2. Continue until only one edge is left.

3. Take the capcity of that last edge to be the estimate of the capcity of the minimal cut in G.

Theorem 1:
Let G_{uv} be the graph obtained from G with n vertices by contracting an edge $e(u,v) \in E$. Then the probability that the capcity of a minimal cut in G_{uv} is larger than the capacity of a minimal cut in G is smaller than $\frac{2}{n}$:

$$P(\text{MIN-CUT-CAPACITY}(G_{uv}) > \text{MIN-CUT-CAPACITY}(G)) < \frac{2}{n} \tag{1}$$

Proof:

1. As we have shown, the capacity of the min cut can increase only if the vertices collapsed are on the opposite sides of every min cut in G.

2. Let also $M = \{e(x,y) : x \in X, y \in Y\}$ be a minimum cut in G; then
$$P(\text{MIN-CUT-CAPACITY}(G_{uv}) > \text{MIN-CUT-CAPACITY}(G)) \leq P(e(u,v) \in M) \tag{2}$$

3. Note that
$$P(e(u,v) \in M) = \frac{\sum\{w(p,q) : e(p,q) \in M\}}{\sum\{w(u,v) : e(u,v) \in E\}} \tag{3}$$

Claim:

$$2\sum_{e\in E} w(e) = \sum_{v\in V}\sum_{u:e(v,u)\in E} w(v,u) \tag{4}$$

Proof:
In the sum on the right, every edge is counted twice, once for each of its vertices.

Claim:
For every $v \in V$,

$$\sum_{u:e(v,u)\in E} w(v,u) \geq \text{MIN-CUT-CAPACITY}(G) \tag{5}$$

Proof:

1. If we let $X = \{v\}$ and $Y = V\ \{v\}$, we get a cut $T = (X, Y)$ whose capacity must be larger or equal to the capcity of the minimal cut M.
2. Since $|V| = n$, summing the inequalities (5) over all $v \in V$ and using (4) we now obtain

$$\sum_{w(e)} w(e) \geq \frac{n}{2} \cdot \text{MIN-CUT-CAPACITY}(G) \tag{6}$$

3. From (3) and (6) we now obtain

$$\begin{aligned} P(e(u,v) \in M) &= \frac{\sum\{w(p,q) : e(p,q) \in M\}}{\sum w(u,v) : e(u,v) \in E} \\ &\leq \frac{\text{MIN-CUT-CABACITY}(G)}{\frac{n}{2} \cdot \text{MIN-CUT-CAPACITY}(G)} \\ &= \frac{2}{n} \end{aligned}$$

Thus, we obtain

$$P(\text{MIN-CUT-CAPACITY}(G_{uv}) > \text{MIN-CUT-CAPACITY}(G) \leq P(e(u,v) \in M) \leq \frac{2}{n}) \tag{7}$$

Theorem 2:
If we run edge contraction procedure until we get a single edge, then the probability π that the capacity of that final edge is equal to the capacity of a minimal cut in G is $\Omega(\frac{1}{n^2})$.
Proof:

1. Let G_i for $0 \leq i \leq n-2$, be the sequence of graphs obtained by successive edge contractions, starting from $G_0 = G$.

2. The probability π that the capacity of the final edge is equal to the capacity of a minimal cut in G is greater or equal to the probability that we never contracted an edge belonging to M.

3. Thus, (7) implies

$$\begin{aligned}
\pi &= P(\text{MIN-CUT-CAPACITY}(G) = \text{MIN-CUT-CAPACITY}(G_{n-2})) \\
&= \prod_{i=1}^{n-2} P(\text{MIN-CUT-CAPACITY}(G_i) = \text{MIN-CUT-CAPACITY}(G_{i-1})) \\
&\geq (1-\frac{2}{n})(1-\frac{2}{n-1})(1-2\frac{2}{n-2})\ldots(1-\frac{2}{3}) \\
&= \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdot\ldots\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} \\
&= \frac{2}{n(n-1)}
\end{aligned}$$

   which implies the claim of the theorem.

Problem:
$\pi = \Omega(\frac{1}{n^2})$ is a very small probability for large n.
Heuristic:
Let us run our contraction algorithm only until the number of vertices is $\lfloor \frac{n}{2} \rfloor$.

Then such an algorithm runs in time $O(n^2)$ and we have

$$
\begin{aligned}
&P(\text{MIN-CUT-CAPACITY}(G) = \text{MINN-CUT-CAPACITY}(G_{\frac{n}{2}})\\
&= \prod_{i=1}^{\frac{n}{2}} P(\text{MIN-CUT-CAPACITY}(G) = \text{MINN-CUT-CAPACITY}(G_{i-1}))\\
&\geq (1-\frac{2}{n})(1-\frac{2}{n-1})(1-\frac{2}{n-2})\dots(1-\frac{2}{\frac{n}{2}+1})(1-\frac{2}{\frac{n}{2}})\\
&= \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdot\frac{n-5}{n-3}\cdot\dots\cdot\frac{\frac{n}{2}}{\frac{n}{2}+2}\cdot\frac{\frac{n}{2}-1}{\frac{n}{2}+1}\cdot\frac{\frac{n}{2}-2}{\frac{n}{2}}\\
&= \frac{(\frac{2}{2}-1)(\frac{n}{2}-2)}{n(n-1)}\\
&\approx \frac{1}{4}
\end{aligned}
$$

This shows that the probability of not picking an edge which belongs to a min cut M is fairly large after $\frac{n}{2}$ many contractions, but drops quickly afterwards. The following algorithm is therfore made.

4-Contract(G):

1. $G_0 = (V_0, E_0) \leftarrow G = (V, E)$
2. while $|V_0| > 2$:
   (a) for $i = 1 \dots 4$:
      i. run the randomised edge contraction algorithm on G_0 until you get $G_i = (V_i, E_i)$ such that $|V_i| = \frac{|V_0|}{2}$ many vertices.
   (b) 4-Contract(G_1)
   (c) 4-Contract(G_2)
   (d) 4-Contract(G_3)
   (e) 4-Contract(G_4)
3. return the smallest capacity among the capacities of all produced single edges.

Analysis: Run time: $T(n) = 4T(\frac{n}{2}) + O(n^2)$
By the master theorem (case 2), $T(n) = O(n^2\log(n))$.

What is the probability that at least one of the edges will have the capacity of the min cut of G, and thus that the algorithm will produce the correct value of MIN-CUT-CAPACITY(G)?

$$\begin{aligned}
&P(\text{success for a graph of size } n) = 1 - P(\text{failure on all 4 branches}) \\
&= 1 - P(\text{failure on one branch})^4 \\
&= 1 - (1 - P(\text{success on one branch}))^4 \\
&= 1 - (1 - \frac{1}{4}P(\text{success for a graph of size } \frac{n}{2}))^4
\end{aligned}$$

Let $p(n) = P(\text{success for a graph of size } n)$; then

$$p(n) = 1 - (1 - \frac{1}{4}p(\frac{n}{2}))^4$$

Note that

$$\begin{aligned}
p(n) &= 1 - (1 - \frac{1}{4}p(\frac{n}{2}))^4 && (8)\\
&= p(\frac{n}{2}) - \frac{3}{8}p(\frac{n}{2})^2 = \frac{1}{16}p(\frac{n}{2})^3 - \frac{1}{256}p(\frac{n}{2})^4 && (9)\\
&> p(\frac{n}{2}) - \frac{3}{8}p(\frac{n}{2})^2 && (10)
\end{aligned}$$

We now use an induction of type

$$\phi(1) \wedge \forall n(\phi(\lfloor n/2 \rfloor) \to \phi(n)) \to \forall n \phi(n)$$

and prove that the assumption $p(\frac{n}{2}) > \frac{1}{\log(\frac{n}{2})} \to p(n) > \frac{1}{\log(n)}$.

Using the fact that function $f(x) = x - \frac{3}{8} \cdot x^2$ is monotonically increasing on $[0, 1]$, we ontain from the induction hypothesis and (8)

$$\begin{aligned}
p(n) &> p(\frac{n}{2}) - \frac{3}{8}p(\frac{n}{2})^2 \\
&> \frac{1}{\log(\frac{n}{2})} - \frac{3}{8}\frac{1}{(\log(\frac{n}{2}))^2} \\
&= \frac{1}{\log(n) - 1} - \frac{3}{8}\frac{1}{(\log(n) - 1)^2}
\end{aligned}$$

We now use the fact that

$$\frac{1}{x-1} - \frac{3}{8(x-1)^2} \geq \frac{1}{x}$$

for all $x \geq \frac{8}{5}$ to finally obtain $p(n) > \frac{1}{\log(n)}$, which proves the induction hypothesis and we conclude that $p(n) > \frac{1}{\log(n)}$ for all $n \geq 4$.

Thus if we run 4-CONTRACT(G) algorithm $(\log(n))^2$ many times and take the smallest capacity estimate produced, probability π that this estimate will be correct is

$$\pi = 1 - (\frac{1}{\log(n)})^{(\log(n))^2}$$

We now use the fact that for all large k, $(1 - \frac{1}{k})^k \approx e^{-1}$.Thus,

$$\pi \approx 1 - e^{-\log(n)} = 1 - \frac{1}{n}$$

So for large n we get the correct value with probability $1 - \frac{1}{n} \approx 1$.

Furthermore, to run our algorithm $(\log(n))^2$ times, it takes the total number of steps of only $O(n^2)\log(n) \cdot (\log(n)^2) = O(n^2 \log(n)^3) < O(n^4)$.

Hence our randomised algorithm runs much faster than the deterministic algorithm but also succeeds with high probability for large n.