Database Access

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Database Access Problem:

- Assume that n processes want to acces a database, and that the time is discrete, $t = 1, 2, 3, \ldots$
- Simultaneous access requests $\rightarrow$ conflict and all proccesses are locked out of access.
- We assume that the processes cannot communicate with each other to agree on a joint startegy.

Solution:
One possible approach in such a situation is that each process at each instant t requests with probability p and does not request with probability $1 - p$. How should we choose p to maximise the probability of a successful access to the database for a process at any instant t?

1. Access Success Probability: $P(S(i,t)) = p(1-p)^{n-1}$
2. Extremal points found by:

$$\frac{d}{dp}P(S(i,t)) = (1-p)^{n-1} - p(n-1)(1-p)^{n-2} = 0$$

3. Dividing both sides by $(1-p)^{n-2}$, we get that the above equality holds just in case $1 - p - p(n-1) = 0$ which is equivalent to $p = \frac{1}{n}$.
4. Hence we get the the optimal $p = \frac{1}{n}$. Hence

$$P(S(i,t)) = p(1-p)^{n-1} = \frac{1}{n}(1 - \frac{1}{n})^{n-1}$$

5. The following two facts are useful:
   - $(1 - \frac{1}{n})^n$ increases monotonomically from $\frac{1}{4}$ up to $\frac{1}{e}$ as n increases from 2 to ∞.
   - $(1 - \frac{1}{n})^{n-1}$ decreases monotonically from $\frac{1}{2}$ down to $\frac{1}{e}$ as n increases from 2 to ∞.

6. Since we had
$$P(S(i,t)) = \frac{1}{n}(1-\frac{1}{n})^{n-1}$$
we obtain
$$\frac{1}{n}\cdot\frac{1}{e} \le P(S(i,t)) \le \frac{1}{n}\cdot\frac{1}{2}$$

7. Thus $P(S(i,t)) = \Theta(\frac{1}{n})$.

8. $P(\text{failure after } t \text{ instants}) = (1 - \frac{1}{n}(1-\frac{1}{n})^{n-1})^t$.

9. Using the second inequality we get $P(\text{failure after } t \text{ instants}) \approx (1 - \frac{1}{en})^t$.

10. Strange phenomenon: If we choose $t = en$ many consecutive instants, then the probability of failure is quite large, because
$$P(\text{failure after } t = en \text{ instants} \approx (1-\frac{1}{en})^{en}\frac{1}{e})$$

11. However, if we increase the number of instants only slightly , by taking $t = en \cdot 2\ln(n)$, then
$$\begin{aligned} P(\text{failure after } t = en2\ln(n) \text{ instants}) &\approx (1-\frac{1}{en})^{2n2\ln(n)} \\ &= ((1-\frac{1}{en})^{en})^{\ln(n^2)} \\ &\approx (\frac{1}{e})^{\ln(n^2)} \\ &= \frac{1}{n^2} \end{aligned}$$

12. Thus a slight increase in the number of time instants from en to $2dn\ln(n)$ caused a dramatic reduction in the probability of failure.

13. If failure probability is less than $\frac{1}{n^2}$ and there are n processes, then probability that at least one process failed cannot be larger than $n\cdot\frac{1}{n^2} = \frac{1}{n}$.

14. Thus after $2en\ln(n)$ instants, all processes succeeded to access the db with probability of at least $1-\frac{1}{n}$.