1 Multiplicative Function

1.1 Dirichlet Convolution

Definition: The Dirichlet Convolution f*g (f and g are arithmetic functions - whose domains are the natural numbers) is a new arithmetic functions defined by:

$$(f * g)(n) = \sum_{d|n} f(d) * g(\frac{n}{d})$$

Theorem: Let f and g be arithmetic functions. If f and g are multiplicative then $h = f * g$ is multiplicative.
(Notice) Corollary: Let f and g be arithmetic functions. If f and g are **completely multiplicative** then the convolution of f and g, $f * g$ is **multiplicative**.
Property:
permutation $f * g = g * f$
association $f * g * h = f * (g * h)$

1.2 Common Multiplicative Functions

$I(n) = 1$
$id_k(n) = n^k$ $(k >= 0)$
$f(n) = gcd(n, k)$ (k is constant)
$phi(n)$
$mobius(n)$:

1. n have a divisor is a square number, mobius(n) = 0
2. n have the odd number of prime divisors, mobius(n) = -1
3. n have the even number of prime divisors, mobius(n) = 1
4. mobius(1) = 1

$f_k(n) = \sum_{d|n} d^k$

1.3 Mobius and application

1.3.1 Dirichlet identity function e(n):

$e(n) = 1$ if $n = 1$
$e(n) = 0$ if $n > 1$

1.3.2 Mobius inversion formular:

Let f is a MF

$S_f(n) = (f * I)(n) = \sum_{d|n} f(d)$ ($S_f(n)$ is a MF)

Fomular:

$f(n) = (S_f * mobius)(n) = \sum_{d|n} S_f(d) * mobius(\frac{n}{d}) = (f * I * mobius)(n) = (f * e)(n) = f(n)$

Notice: $S_\varphi(n) = \sum_{d|n} \varphi(d) = id(n)$

+ $\mu * 1 = \varepsilon$ (the Mobius inversion formula)

+ $\varphi * 1 = \mathrm{Id}$

+ $d = 1 * 1$

+ $\sigma = \mathrm{Id} * 1 = \varphi * d$

+ $\sigma_k = \mathrm{Id}_k * 1$

+ $\mathrm{Id} = \varphi * 1 = \sigma * \mu$

+ $\mathrm{Id}_k = \sigma_k * \mu$

+ $\sigma_k(n)$: the divisor function, which is the sum of the k-th powers of all the positive divisors of n.