⟨ Real Time Optimal Control Midterm ⟩ 2020124149 Seokhun Choi

Consider a process described by

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma\mathbf{u}(k) + \Gamma_w w(k)$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$ is the state, $\mathbf{u} \in \mathbb{R}^{n_u}$ is the input, $w \in \mathbb{R}^{n_w}$ is the disturbance, $\Phi \in \mathbb{R}^{n_x \times n_x}$, $\Gamma \in \mathbb{R}^{n_x \times n_u}$, $\Gamma_w \in \mathbb{R}^{n_x \times n_w}$ are the system, input, and disturbance input matrices, respectively.

Assuming that the disturbance $w(\cdot)$ is known, develop an MPC formulation to consider the disturbance $w(\cdot)$.

a) Develop an output prediction model

$$\tilde{\mathbf{y}}(k) = \Psi\mathbf{y}(k) + \Upsilon\mathbf{u}(k-1) + \Theta\Delta\tilde{\mathbf{u}}(k) + \Omega\tilde{w}(k).$$

Give specific descriptions of Ψ, Υ, Θ, and Ω.

Sol)

$u(k) = \Delta u(k) + u(k-1)$

$$x(k) = x(k)$$

$$x(k+1) = \Phi x(k) + \underline{\Gamma u(k)} + \Gamma_w w(k) = \Phi x(k) + \underline{\Gamma u(k-1) + \Gamma \Delta u(k)} + \Gamma_w w(k)$$

$$x(k+2) = \Phi x(k+1) + \Gamma u(k+1) + \Gamma_w w(k+1)$$

$u(k+1) = \Delta u(k+1) + \Delta u(k) + u(k-1)$

$$= \Phi\left(\Phi x(k) + \Gamma u(k-1) + \Gamma \Delta u(k) + \Gamma_w w(k)\right)$$

$$+ \Gamma\left(\underline{\Delta u(k+1) + \Delta u(k) + u(k-1)}\right) + \Gamma_w w(k+1)$$

$$= \Phi^2 x(k) + (\Phi\Gamma + \Gamma)\, u(k-1) + (\Phi\Gamma + \Gamma)\Delta u(k) + \Gamma \Delta u(k+1)$$

$$+ \Phi\Gamma_w w(k) + \Gamma_w w(k+1)$$

$u(k+2) = \Delta u(k+2) + \Delta u(k+1) + \Delta u(k) + u(k-1)$

$$x(k+3) = \Phi x(k+2) + \Gamma \underline{u(k+2)} + \Gamma_w w(k+2)$$

$$= \Phi^3 x(k) + (\Phi^2\Gamma + \Phi\Gamma + \Gamma)\, u(k-1)$$

$$+ (\Phi^2\Gamma + \Phi\Gamma + \Gamma)\Delta u(k) + (\Phi\Gamma + \Gamma)\Delta u(k+1) + \Gamma \Delta u(k+2)$$

$$+ \Phi^2\Gamma_w w(k) + \Phi\Gamma_w w(k+1) + \Gamma_w w(k+2)$$

$\vdots$

$$x(k+N_p-1) = \Phi^{N_p-1} x(k) + \left(\sum_{i=0}^{N_p-2} \Phi^i \Gamma\right) u(k-1) + \left(\sum_{i=0}^{N_p-2} \Phi^i \Gamma\right)\Delta u(k)$$

$$+ \left(\sum_{i=0}^{N_p-3} \Phi^i \Gamma\right)\Delta u(k+1) + \cdots + \left(\sum_{i=0}^{N_p-N_c-1} \Phi^i \Gamma\right)\Delta u(k+N_c-1)$$

$$+ \Phi^{N_p-1}\Gamma_w w(k) + \Phi^{N_p-2}\Gamma_w w(k+1) + \cdots + \Phi\Gamma_w w(k+N_p-3) + \Gamma_w w(k+N_p-2)$$

$$x(k) = x(k)$$

$$x(k+1) = \Phi x(k) + \Gamma u(k-1) + \Gamma \Delta u(k) + \Gamma_w w(k)$$

$$x(k+2) = \Phi^2 x(k) + (\Phi\Gamma + \Gamma) u(k-1) + (\Phi\Gamma + \Gamma)\Delta u(k) + \Gamma \Delta u(k+1) + \Phi\Gamma_w w(k) + \Gamma_w w(k+1)$$

$$\vdots$$

$$x(k+N_p-1) = \Phi^{N_p-1} x(k) + \left(\sum_{i=0}^{N_p-2} \Phi^i \Gamma\right) u(k-1) + \left(\sum_{i=0}^{N_p-2} \Phi^i \Gamma\right)\Delta u(k) + \left(\sum_{i=0}^{N_p-3} \Phi^i \Gamma\right)\Delta u(k+1) + \cdots + \left(\sum_{i=0}^{N_p-N_c-1} \Phi^i \Gamma\right)\Delta u(k+N_c-1) + \Phi^{N_p-2}\Gamma_w w(k) + \Phi^{N_p-3}\Gamma_w w(k+1) + \cdots + \Phi\Gamma_w w(k+N_p-3) + \Gamma_w w(k+N_p-2)$$

$$\underbrace{\begin{bmatrix} x(k) \\ x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N_p-1) \end{bmatrix}}_{\tilde{X}(k)} = \underbrace{\begin{bmatrix} I \\ \Phi \\ \Phi^2 \\ \vdots \\ \Phi^{N_p-1} \end{bmatrix}}_{\Psi_x} x(k) + \underbrace{\begin{bmatrix} 0\Gamma \\ I\Gamma = \sum_{i=0}^{0}\Phi^i\Gamma \\ \Phi\Gamma + \Gamma = \sum_{i=0}^{1}\Phi^i\Gamma \\ \vdots \\ \sum_{i=0}^{N_p-2}\Phi^i\Gamma \end{bmatrix}}_{\gamma_x} u(k-1)$$

$$+ \; n_xN_p \overbrace{\underbrace{\begin{bmatrix} 0 & & & & \\ \Gamma & 0 & & & \\ \sum_{i=0}^{1}\Phi^i\Gamma & \Gamma & 0 & & \\ \sum_{i=0}^{2}\Phi^i\Gamma & \sum_{i=0}^{1}\Phi^i\Gamma & \Gamma & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sum_{i=0}^{N_c-2}\Phi^i\Gamma & \sum_{i=0}^{N_c-3}\Phi^i\Gamma & \cdots & & \Gamma \\ \vdots & \vdots & \vdots & & \vdots \\ \sum_{i=0}^{N_p-2}\Phi^i\Gamma & \sum_{i=0}^{N_p-3}\Phi^i\Gamma & \sum_{i=0}^{N_p-4}\Phi^i\Gamma & \cdots & \sum_{i=0}^{N_p-N_c-1}\Phi^i\Gamma \end{bmatrix}}_{n_uN_c}}^{\Theta_x} \underbrace{\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \Delta u(k+2) \\ \vdots \\ \Delta u(k+N_c-1) \end{bmatrix}}_{\Delta\tilde{U}(k)}$$

$$+ \; n_xN_p \underbrace{\overbrace{\begin{bmatrix} 0 & & & & & \\ \Gamma_w & 0 & & & & \\ \Phi\Gamma_w & \Gamma_w & 0 & & & \\ \Phi^2\Gamma_w & \Phi\Gamma_w & \Gamma_w & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \Phi^{N_p-2}\Gamma_w & \Phi^{N_p-3}\Gamma_w & \Phi^{N_p-4}\Gamma_w & \cdots & \Gamma_w & 0 \end{bmatrix}}^{n_wN_p}}_{\Omega_x} \underbrace{\begin{bmatrix} w(k) \\ w(k+1) \\ w(k+2) \\ \vdots \\ w(k+N_p-2) \\ w(k+N_p-1) \end{bmatrix}}_{\tilde{w}(k)}$$

$$\therefore \quad \tilde{x}(k) = \Psi_x x(k) + \gamma_x u(k-1) + \Theta_x \Delta\tilde{u}(k) + \Omega_x \tilde{\omega}(k)$$

$$\text{Since} \quad \tilde{y}(k) = \begin{bmatrix} y(k) \\ \vdots \\ y(k+N_p-1) \end{bmatrix} = \begin{bmatrix} C & & & \\ & C & & \\ & & \ddots & \\ & & & C \end{bmatrix} \begin{bmatrix} x(k) \\ \vdots \\ x(k+N_p-1) \end{bmatrix} = \tilde{C}\tilde{x}(k)$$

$$\tilde{y}(k) = \tilde{C}\tilde{x}(k) = \tilde{C}\Psi_x x(k) + \tilde{C}\gamma_x u(k-1) + \tilde{C}\Theta_x \Delta\tilde{u}(k) + \tilde{C}\Omega_x \tilde{\omega}(k)$$

since $y(k) = Cx(k)$ and (n_y, n_x) matrix C is full row rank because of observability, right inverse matrix of C, that is, C^{+}_{right} exists as follows:

$$C^{+}_{right} = C^{T}(CC^{T})^{-1}$$

Thus $x(k) = C^{+}_{right}\, y(k) = C^{T}(CC^{T})^{-1} y(k)$

$$\therefore \quad \tilde{y}(k) = \tilde{C}\tilde{x}(k) = \underbrace{\tilde{C}\Psi_x C^{T}(CC^{T})^{-1}}_{\Psi = \tilde{C}\Psi_x C^{T}(CC^{T})^{-1}} y(k) + \underbrace{\tilde{C}\gamma_x}_{\gamma = \tilde{C}\gamma_x} u(k-1) + \underbrace{\tilde{C}\Theta_x}_{\Theta = \tilde{C}\Theta_x} \Delta\tilde{u}(k) + \underbrace{\tilde{C}\Omega_x}_{\Omega = \tilde{C}\Omega_x} \tilde{\omega}(k)$$

$$\text{where} \quad \Psi_x = \begin{bmatrix} I \\ \Phi \\ \Phi^2 \\ \vdots \\ \Phi^{N_p-1} \end{bmatrix}, \quad \gamma_x = \begin{bmatrix} 0\Gamma \\ I\Gamma \\ \Phi\Gamma + \Gamma \\ \vdots \\ \sum_{i=0}^{N_p-2} \Phi^i \Gamma \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & & & \\ & C & & \\ & & \ddots & \\ & & & C \end{bmatrix}$$

$$\Theta_x = \begin{bmatrix} 0 & & & & \\ \Gamma & 0 & & & \\ \sum_{i=0}^{1}\Phi^i\Gamma & \Gamma & 0 & & \\ \sum_{i=0}^{2}\Phi^i\Gamma & \sum_{i=0}^{1}\Phi^i\Gamma & \Gamma & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sum_{i=0}^{N_c-2}\Phi^i\Gamma & \sum_{i=0}^{N_c-3}\Phi^i\Gamma & \cdots & & \Gamma \\ \vdots & \vdots & \vdots & & \vdots \\ \sum_{i=0}^{N_p-2}\Phi^i\Gamma & \sum_{i=0}^{N_p-3}\Phi^i\Gamma & \sum_{i=0}^{N_p-4}\Phi^i\Gamma & \cdots & \sum_{i=0}^{N_p-N_c-1}\Phi^i\Gamma \end{bmatrix}, \quad \Omega_x = \begin{bmatrix} 0 & & & & & \\ \Gamma_\omega & 0 & & & & \\ \Phi\Gamma_\omega & \Gamma_\omega & 0 & & & \\ \Phi^2\Gamma_\omega & \Phi\Gamma_\omega & \Gamma_\omega & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \Phi^{N_p-2}\Gamma_\omega & \Phi^{N_p-3}\Gamma_\omega & \Phi^{N_p-4}\Gamma_\omega & \cdots & \Gamma_\omega & 0 \end{bmatrix}$$

b) Considering the disturbance $w(\cdot)$, give a quadratic cost function for the finite prediction horizon N_p and N_c. Develop a constrained QP in terms of a decision variable $\Delta\tilde{\mathbf{u}}$. Explain how the QP can be solved by describing H, $f(\cdot)$ and constraints to consider the disturbance $w(\cdot)$. What is your assumption, if applied?

Sol

we derive $\tilde{y}(k) = \Psi y(k) + \Gamma u(k-1) + \Theta \Delta\tilde{u}(k) + \Omega \tilde{\omega}(k)$ in problem (a)

Also let define $\tilde{C}(k) = \tilde{r}(k) - \Psi y(k) - \Gamma u(k-1)$ then by above expression,

$\tilde{C}(k) = \tilde{r}(k) - \tilde{y}(k) + \Theta \Delta\tilde{u}(k) + \Omega \tilde{\omega}(k)$

Then we have $\tilde{y}(k) - \tilde{r}(k) = \Theta \Delta\tilde{u}(k) - \tilde{C}(k) + \Omega \tilde{\omega}(k)$

which leads to

$$(\tilde{y}(k) - \tilde{r}(k))^T \tilde{Q} (\tilde{y}(k) - \tilde{r}(k)) = (\Theta \Delta\tilde{u}(k) - \tilde{C}(k) + \Omega \tilde{\omega}(k))^T \tilde{Q} (\Theta \Delta\tilde{u}(k) - \tilde{C}(k) + \Omega \tilde{\omega}(k))$$

$$= \{\Theta \Delta\tilde{u}(k) - (\tilde{C}(k) - \Omega \tilde{\omega}(k))\}^T \tilde{Q} \{\Theta \Delta\tilde{u}(k) - (\tilde{C}(k) - \Omega \tilde{\omega}(k))\}$$

$$= \Delta\tilde{u}(k)^T \Theta^T \tilde{Q} \Theta \Delta\tilde{u}(k) - 2 \Delta\tilde{u}(k)^T \Theta^T \tilde{Q} (\tilde{C}(k) - \Omega \tilde{\omega}(k))$$
$$+ (\tilde{C}(k) - \Omega \tilde{\omega}(k))^T \tilde{Q} (\tilde{C}(k) - \Omega \tilde{\omega}(k))$$

and Since $J_N(\Delta\tilde{u}(k), y(k), u(k-1)) = (\tilde{y}(k) - \tilde{r}(k))^T \tilde{Q} (\tilde{y}(k) - \tilde{r}(k)) + \Delta\tilde{u}(k)^T \tilde{R} \Delta\tilde{u}(k)$,

$$J_N(\Delta\tilde{u}(k), y(k), u(k-1)) = \frac{1}{2} \Delta\tilde{u}(k)^T H \Delta\tilde{u}(k) + \Delta\tilde{u}(k)^T f(k) + f_0(k)$$

where $H = 2(\Theta^T \tilde{Q} \Theta + \tilde{R})$, $f(k) = -2 \Theta^T \tilde{Q} (\tilde{C}(k) - \Omega \tilde{\omega}(k))$

and $f_0(k) = (\tilde{C}(k) - \Omega \tilde{\omega}(k))^T \tilde{Q} (\tilde{C}(k) - \Omega \tilde{\omega}(k))$

Here since $\tilde{\omega}(k) = \begin{bmatrix} \omega(k) \\ \omega(k+1) \\ \omega(k+2) \\ \vdots \\ \omega(k+N_p-1) \end{bmatrix}$, we need to make assumptions that we can know the disturbance ahead as much as Prediction horizon N_p which will be applied to the plant.

And let's consider the constraints as follows.

$$\begin{cases} G_y \tilde{y}(k) \le g_y \\ G_u \tilde{u}(k) \le g_u \\ G_{\Delta u} \Delta\tilde{u}(k) \le g_{\Delta u} \end{cases} \longrightarrow \begin{aligned} & G_y (\Psi y(k) + \Gamma u(k-1) + \Theta \Delta\tilde{u}(k) + \underline{\Omega \tilde{\omega}(k)}) \le g_y \\ & F_d \Delta\tilde{u}(k) + F_b u(k-1) \le g_u \\ & G_{\Delta u} \Delta\tilde{u}(k) \le g_{\Delta u} \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} G_y \Theta \\ F_d \\ G_{\Delta u} \end{bmatrix}}_{G \in \mathbb{R}^{n_b \times n_u N_c}} \Delta\tilde{u}(k) \le \underbrace{\begin{bmatrix} g_y - G_y \Omega \tilde{\omega}(k) \\ g_u \\ g_{\Delta u} \end{bmatrix}}_{b_0 \in \mathbb{R}^{n_b}} + \underbrace{\begin{bmatrix} -G_y \Psi & -G_y \Gamma \\ 0 & -F_b \\ 0 & 0 \end{bmatrix}}_{B_p \in \mathbb{R}^{n_b \times (n_y + n_u)}} \underbrace{\begin{bmatrix} y(k) \\ u(k-1) \end{bmatrix}}_{p(k) \in \mathbb{R}^{n_y + n_u}} = b_0 + B_p p(k)$$

$$\therefore G \Delta\tilde{u}(k) \le b_0 + B_p p(k)$$

The difference in contrast by adding disturbance to the plant model is that the changed expression on output constraints must be considered.

c) Develop an LQ state feedback controller. Using MATLAB/Simulink, conduct a computational experiment with the constraints on $\mathbf{u}(\cdot)$ being considered. Plot your results: e.g. $\mathbf{x}^r(k)$, $\mathbf{x}(k)$, and $\mathbf{u}(k)$ according to the simulation time. Explain your results.

(Sol) Let the plant model be as follows

$$X(k+1) = \Phi X(k) + \Gamma u(k) + \Gamma_\omega \omega(k)$$

$$Z(k) = C_{zx} X(k) + D_{zu} u(k)$$

and let reference be r to which the controlled output Z wants to converge

Then at equilibrium point $X(\infty) = X_{eq}$, $u(\infty) = u_{eq}$, $Z(\infty) = Z_{eq} = r$ and $\omega(\infty) = \omega_{eq}$

Thus,
$$\begin{cases} X_{eq} = \Phi X_{eq} + \Gamma u_{eq} + \Gamma_\omega \omega_{eq} \\ Z_{eq} = r = C_{zx} X_{eq} + D_{zu} u_{eq} \end{cases}$$

which is the same expression as follows.

$$\begin{bmatrix} I - \Phi & \Gamma \\ -C_{zx} & D_{zu} \end{bmatrix} \begin{bmatrix} -X_{eq} \\ u_{eq} \end{bmatrix} = \begin{bmatrix} -\Gamma_\omega \omega_{eq} \\ r \end{bmatrix} \Leftrightarrow \quad AP = q$$

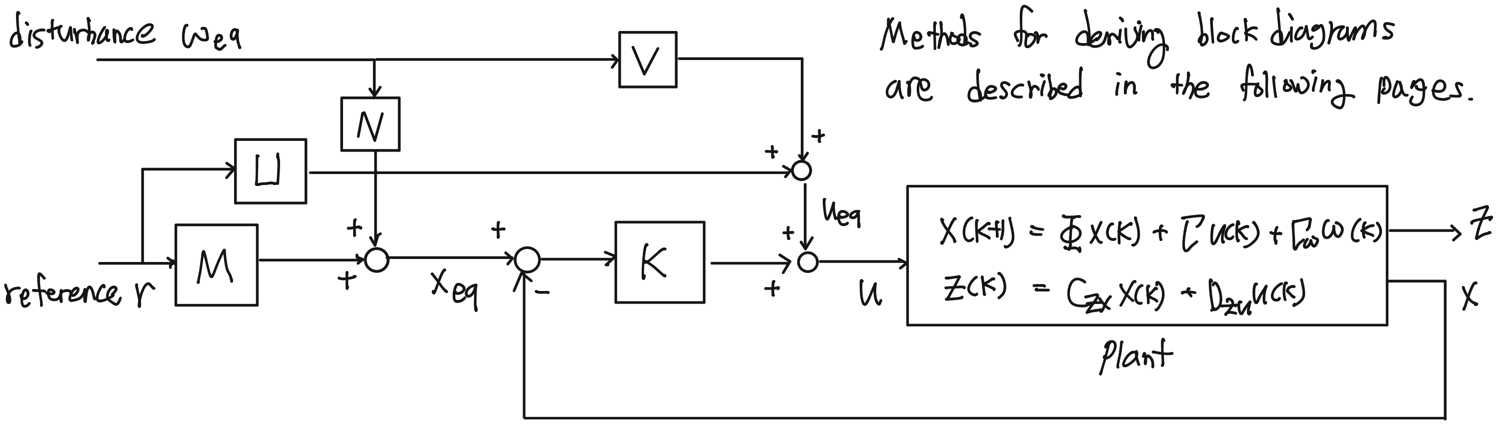
since matrix A is full column rank, left inverse, that is, A^{-1}_{left} exists as follows

$$A^{-1}_{left} = (A^T A)^{-1} A^T$$

Then we can find the expression about X_{eq} and u_{eq} as follows.

$$P = \begin{bmatrix} -X_{eq} \\ u_{eq} \end{bmatrix} = A^{-1}_{left}\, q \quad \Leftrightarrow \quad \begin{cases} X_{eq} = Mr + N\omega_{eq} \\ u_{eq} = Ur + V\omega_{eq} \end{cases}$$

which can be expressed as below block diagram



Methods for deriving block diagrams are described in the following pages.

$$x(k+1) = \Phi x(k) + \Gamma u(k) + \Gamma_\omega \omega(k) \quad \text{——— (i)}$$

$$z(k) = C_{zx} x(k) + D_{zu} u(k) \quad \text{——— (ii)}$$

Let's define $\bar{x}(k+1) = x(k+1) - x_{eq}$ and $\bar{z}(k) = z(k) - r(k)$

Since
$$\begin{cases} x_{eq} = \Phi x_{eq} + \Gamma u_{eq} + \Gamma_\omega \omega_{eq} & \text{——— (iii)} \\ z_{eq} = r = C_{zx} x_{eq} + D_{zu} u_{eq} & \text{——— (iv)} \end{cases}$$

By (i) and (iii),

$$\begin{aligned}\bar{x}(k+1) = x(k+1) - x_{eq} &= \Phi x(k) + \Gamma u(k) + \Gamma_\omega \omega(k) - (\Phi x_{eq} + \Gamma u_{eq} + \Gamma_\omega \omega_{eq}) \\ &= \Phi(x(k) - x_{eq}) + \Gamma(u(k) - u_{eq}) + \Gamma_\omega(\omega(k) - \omega_{eq}) \\ &= \Phi \bar{x}(k) + \Gamma \bar{u}(k) + \Gamma_\omega \bar{\omega}(k)\end{aligned}$$

Also by (ii) and (iv),

$$\begin{aligned}\bar{z}(k) = z(k) - r(k) &= C_{zx} x(k) + D_{zu} u(k) - (C_{zx} x_{eq} + D_{zu} u_{eq}) \\ &= C_{zx}(x(k) - x_{eq}) + D_{zu}(u(k) - u_{eq}) \\ &= C_{zx} \bar{x}(k) + D_{zu} \bar{u}(k)\end{aligned}$$

So we obtain $\bar{x}(k+1) = \Phi \bar{x}(k) + \Gamma \bar{u}(k) + \Gamma_\omega \bar{\omega}(k)$ and $\bar{z}(k) = C_{zx} \bar{x}(k) + D_{zu} \bar{u}(k)$

By LQR problem, optimal solution is $\bar{u}(k) = -K \bar{x}(k)$

which leads to $u(k) - u_{eq} = -K(x(k) - x_{eq})$

$$\begin{aligned}\Rightarrow u(k) &= -K(x(k) - x_{eq}) + u_{eq} \\ &= -K(x - Mr - N\omega_{eq}) + Ur + V\omega_{eq} \\ &= -Kx + (KM + U)r + (KN + V)\omega_{eq}\end{aligned}$$

And let $z = \begin{bmatrix} a x_1 \\ b x_2 \end{bmatrix} = C_{zx} x + D_{zu} u$ where $C_{zx} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $D_{zu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

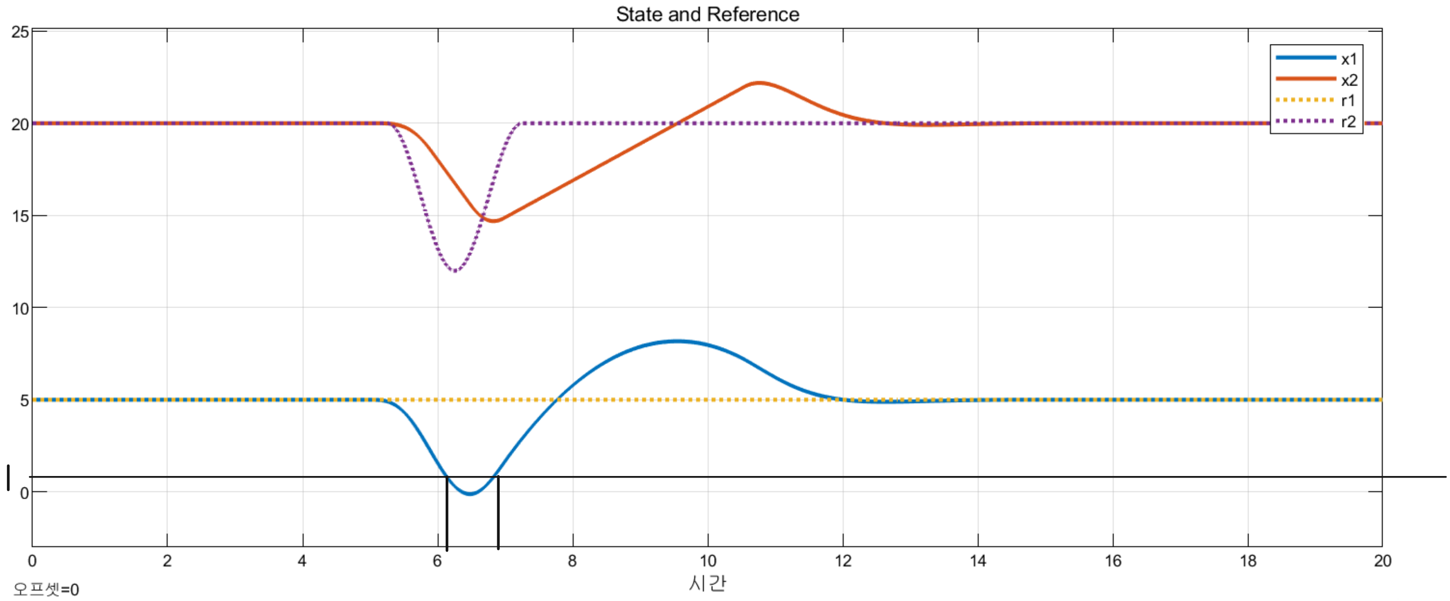
and we want $x_1 \to r_1$ and $x_2 \to r_2$ so, $r = \begin{bmatrix} a r_1 \\ b r_2 \end{bmatrix}$

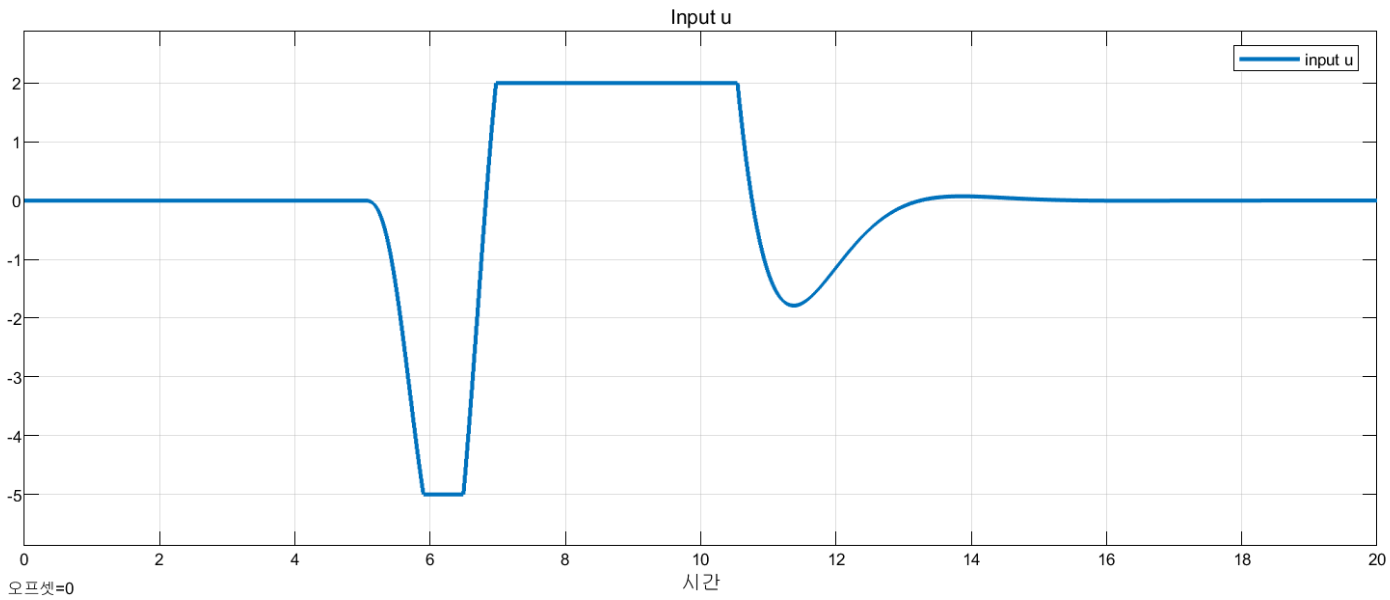
and let $Q = C_{zx}^T C_{zx}$, $R = D_{zu}^T D_{zu} + \rho I$ and $N = C_{zx}^T D_{zu}$

then statefeedback LQ gain K can be obtained as follows.

$K = \text{dlqr}(\Phi, \Gamma, Q, R, N)$

Result plots are as belows. I use disturbance parameter $\rho = 0.4$





Undershoot phenomenon has occured in both x_1 and x_2 about at $t = 12 \sim 13s$

Constraints on input u are satisfied by using saturation function but constraint on x_1 ($x_1 \geq 1$) is not satisfied about at $t = 6 \sim 7s$ because current LQ state feedback controller cannot impose constraints on output (in this case x_1) which is the one of the drawbacks of LQ state feedback controller in comparison to MPC.

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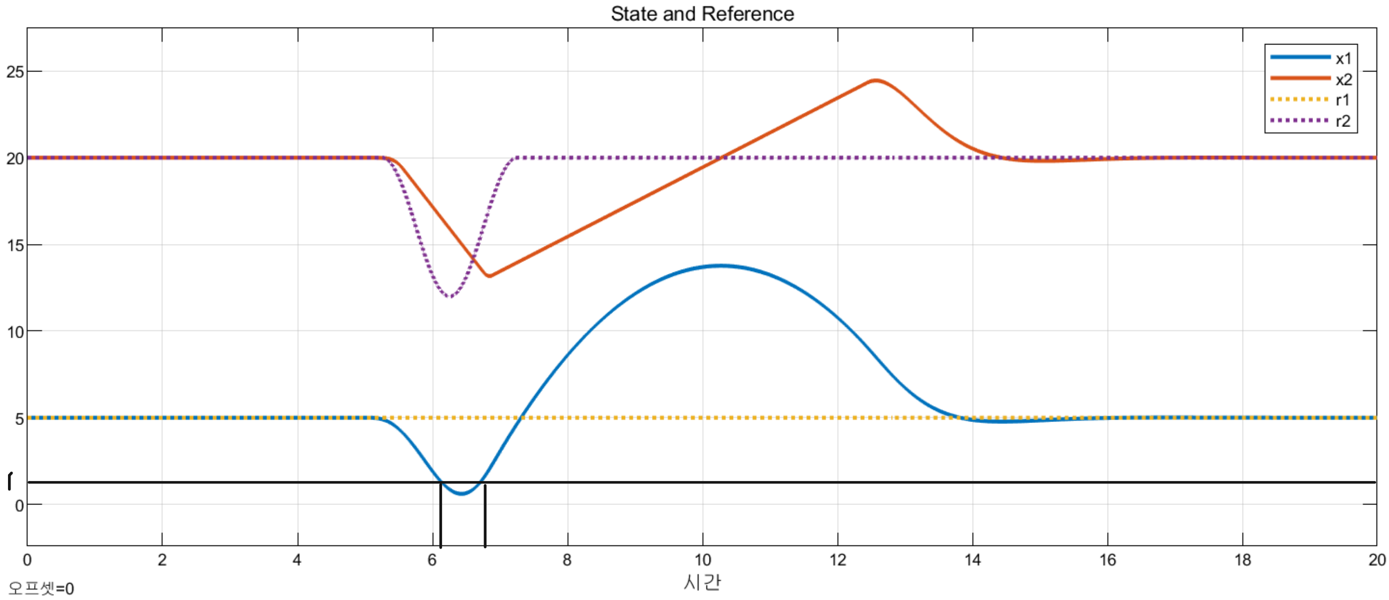

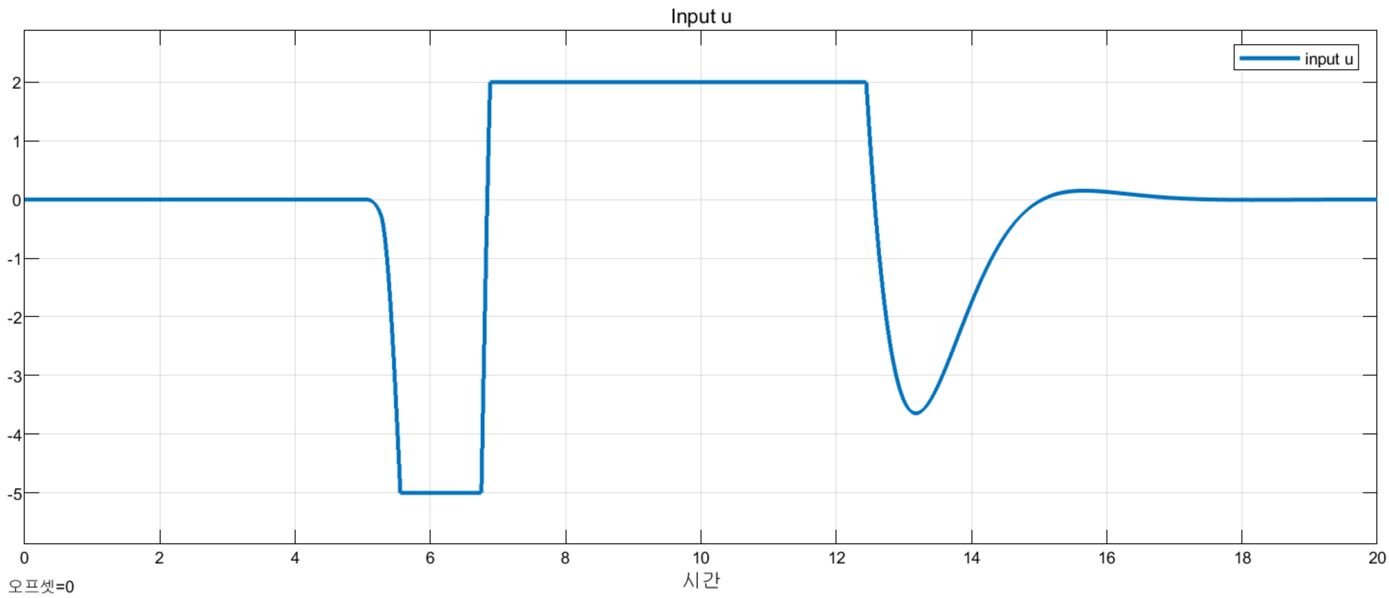

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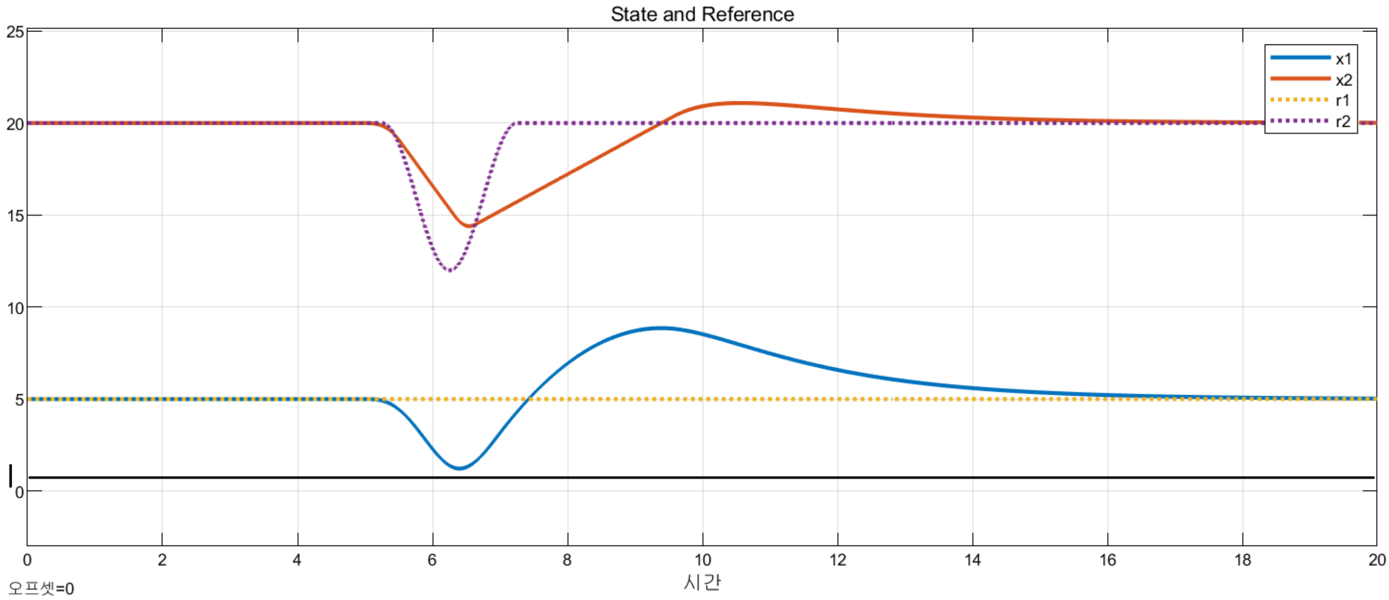
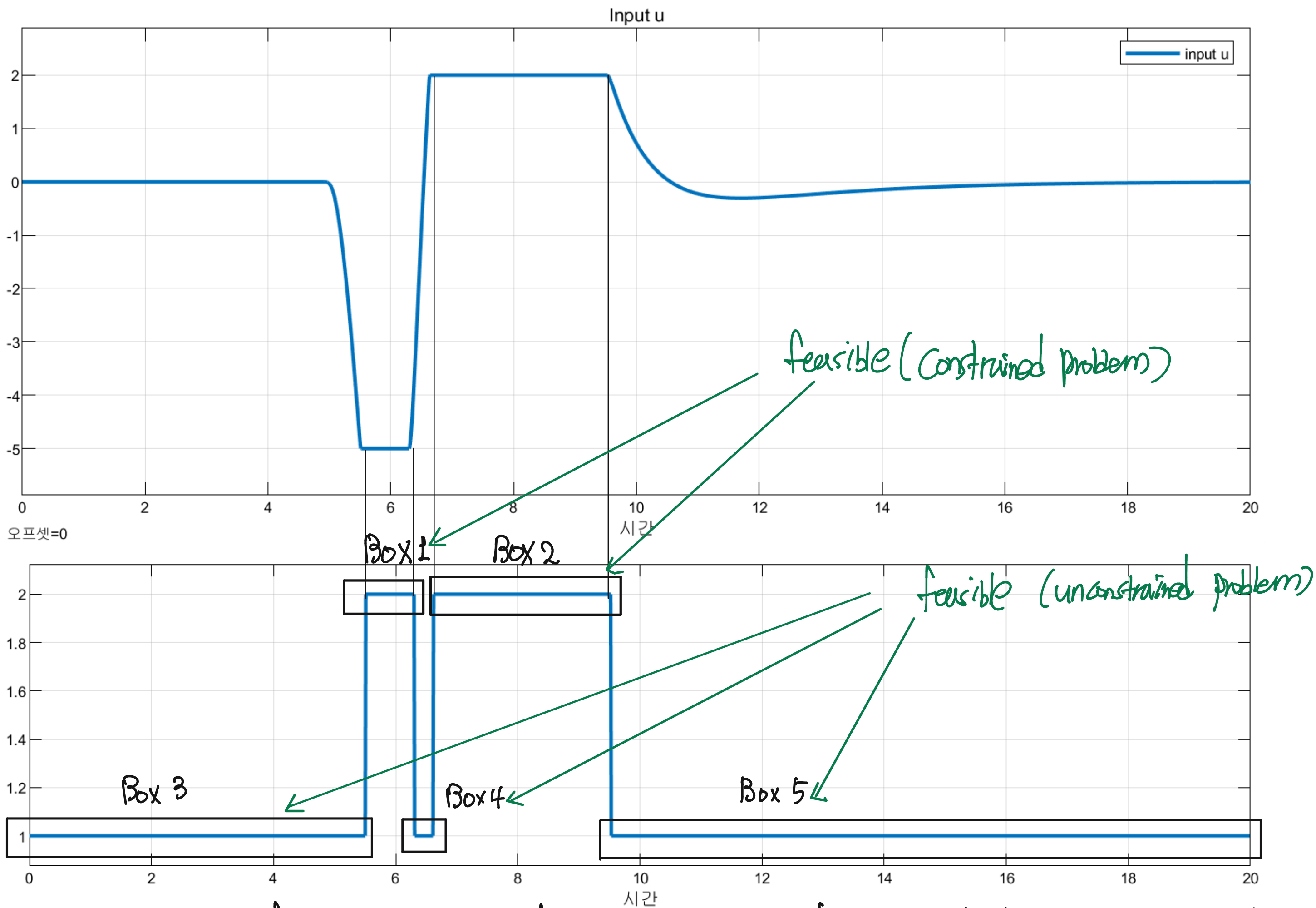
Constraints on input u are satisfied by using saturation function but constraint on x_1 ($x_1 \geq 1$) is not satisfied about at $t = 6 \sim 7s$ because current LQ state feedback controller cannot impose constraints on output (in this case x_1) which is the one of the drawbacks of LQ state feedback controller in comparison to MPC.

d) Develop an infinite horizon MPC considering the aforementioned constrains. Using MATLAB/Simulink, conduct a computational experiment with the constraints on $\mathbf{u}(\cdot)$ being considered. Check the feasibility of the optimization problem. Plot your results: e.g. $\mathbf{x}^r(k)$, $\mathbf{x}(k)$, and $\mathbf{u}(k)$ according to the simulation time. Explain your results.

Sol) I use reference error parameter $\alpha = 0.005$ to get following results.

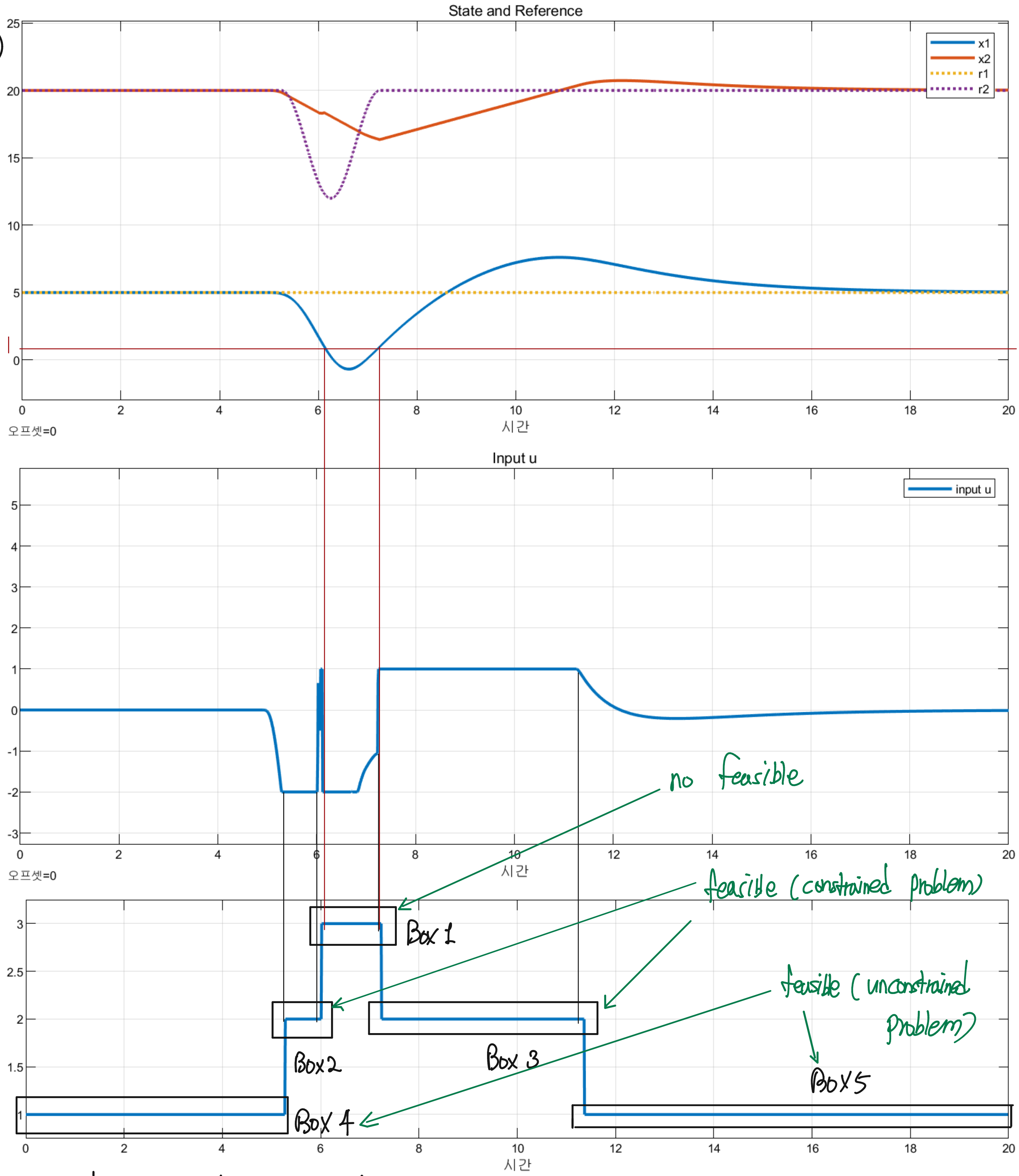




During the simulation, feasibility is satisfied. We have one decision variable u at current time. At Box 1, active constraint is "$u \geq -5$" and at Box 2 active constraint is "$u \leq 5$". So "# active constraints $\leq$ # decision variables" is satisfied. And at Box 3, 4, 5 there are no active constraints so they are unconstrained problem.

e) You may be tempted to introduce stricter constraints on $\mathbf{u}(\cdot)$. Develop infinite horizon MPCs with the new constraints. Check the feasibility of the optimization problem. Conduct a computational simulation with the constraints on $\mathbf{u}(\cdot)$ being considered. Explain your results.

Sol



I changed constraints on u from $-5 \le u \le 2$ to $-2 \le u \le 1$. In addition to ability of MPC to try to satisfy the constraints on u, I add saturation function on u to follow the capacity of actuator. At about $t = 6 \sim 7.5$ s, feasibility is not satisfied as we can see in Box 1 ($x_1 \ge 1$ is violated). I use Hildreth iterative method as QP solver so even though there is no feasible solution, the solver find out any plausible u. So Algorithm does not suffer from program termination At other times, they are unconstrained problem.

f) You may also be tempted to try different observation matrices. Develop infinite horizon MPCs for the observation matrices, if possible. Explain your results.

Sol (i) when $C = [1 \ \ 0]$,

Then observability matrix $\mathcal{O} = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -0.01 \end{bmatrix}$ which has rank 2

so if $C = [1 \ \ 0]$, this system is observable.

(ii) when $C = [0 \ \ 1]$,

Then observability matrix $\mathcal{O} = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ which has rank 1

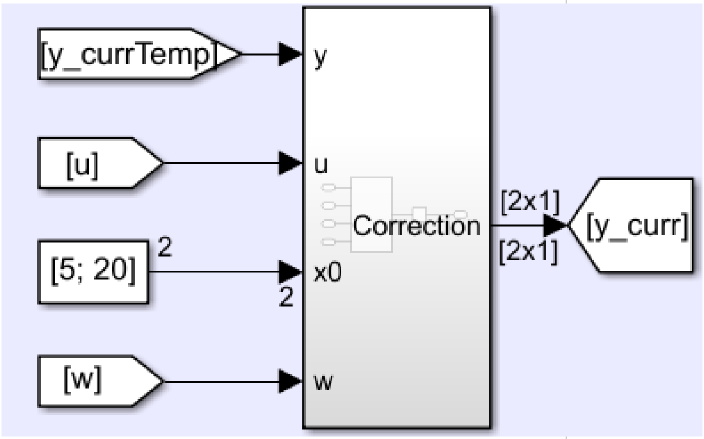
so if $C = [0 \ \ 1]$, this system is unobservable.

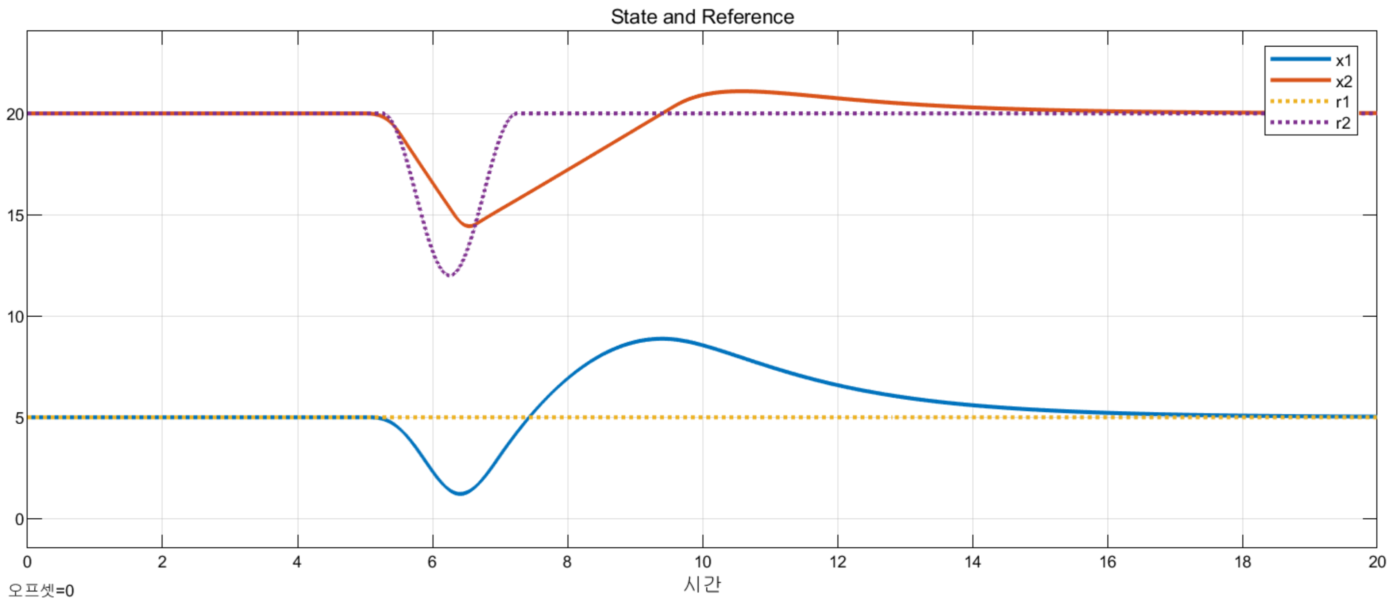
By (i),(ii), Possible observation matrix is only $C = [0 \ \ 1]$

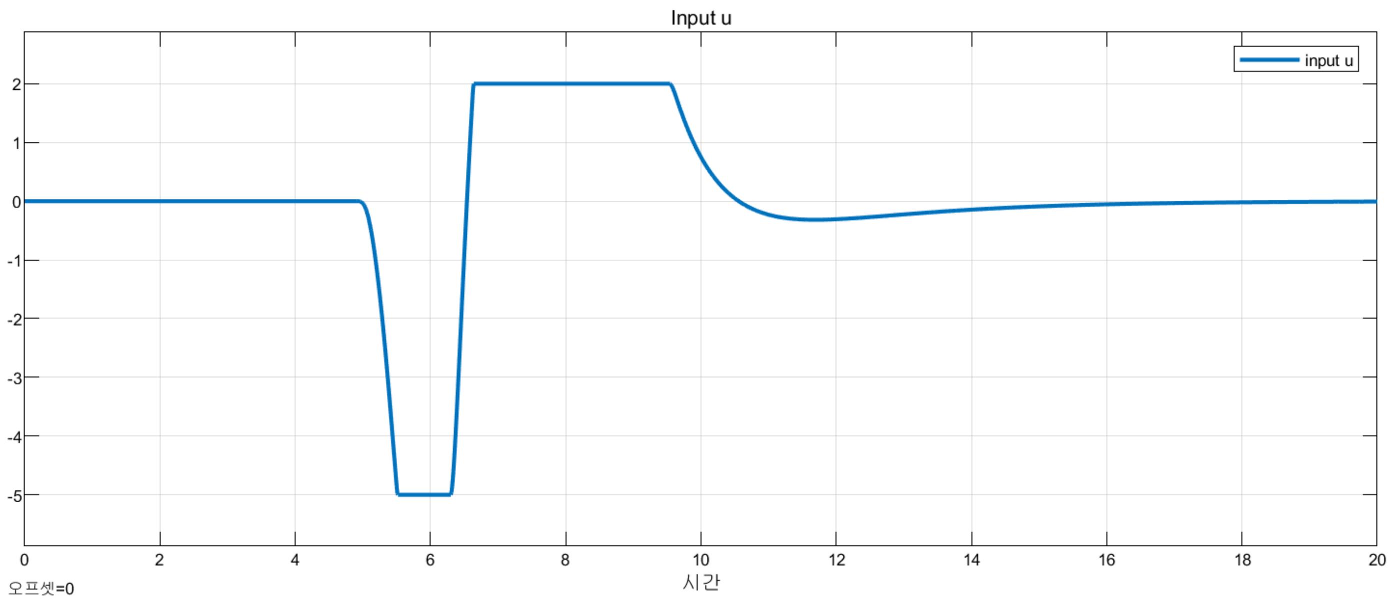
Since x_1 is only measurable, state estimation is needed.

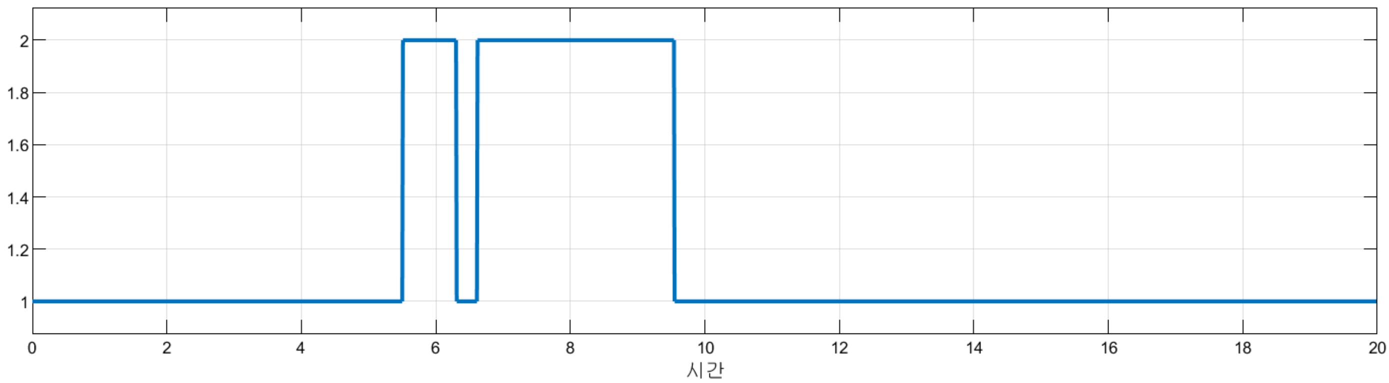
So I use Kalman Filter to estimate state x, especially x_1

```
function correction = KalmanFilter(y, u, x_0, w)
persistent first
persistent Phi Gamma Gamma_w C Q R P x

if isempty(first)
    first = 1;
    Phi = [1 -0.01; 0 1];
    Gamma = [-0.0001; 0.01];
    Gamma_w = [0.01; 0];
    C = [1 0];
    Q = 0.5*eye(2); R = 0.01;
    P = 2*eye(2);
    x = x_0;
end


xp = Phi*x + Gamma*u + Gamma_w*w;
Pp = Phi*P*Phi' - Phi*P*C'*(C*P*C'+R)^-1*C*P*Phi' + Q;
K = Pp*C'*(C*Pp*C' + R)^-1;
x = xp + K*(y - C*xp);
P = (eye(2)-K*C)*Pp;

correction = x;
end
```









Since Kalman Filter estimation on x_2 works well, result plots are the same as ones in problem (d)

g) Did you get and enjoy fabulous results? What's your favorite? Give specific reasons for wanting to work for the controller.

Sol) Among MPC and LQ state feedback controller, my favorite controller is MPC. MPC has a significant advantage of effectively handling linear systems with various constraints on states and control, in this case, $-2 \le u \le 5$ and $x_1 \ge 1$. In LQ state feedback control, we can impose constraints on u but it might cause unpredictable results. Also LQ state feedback control cannot impose constraints on output y as we can see in problem (c). Further MPC design is simple because we just introduce appropriate weights for states and control. But I know that this attractive MPC also has disadvantages. Long and uncertain computation time limits practical applications of MPC to fast system since fast system requires larger N_p to attain good performance which results in increasing size of decision variable where as a result MPC may not complete the computation of optimization at each sample time.