DEPARTMENT OF MATHEMATICS

MTL 106 (Introduction to Probability Theory and Stochastic Processes) 4 Credits (3-1-0)

II Semester 2015-16

Lecture Classes: Tuesday, Wednesday and Friday between 9:00 AM and 9:50 AM in LH 121.

Tutorial Classes: MZ 163 on Mon, Tue, Thu and Fri days between 2:00 PM and 2:50 PM.

For the students who registered I Semester 2015-16, Re Major (for both I grade and E Grade) is scheduled on Jan. 8th, 2016 (Friday) between 12 noon and 2 PM. Venue will be MZ 158 (opposite to my office).

INFORMATION SHEET

Probability Theory: Axioms of probability, Probability space, Conditional probability, Independence, Baye's rule, Random variable, Some common discrete and continuous distributions, Distribution of Functions of Random Variable, Moments, Generating functions, Two and higher dimensional distributions, Functions of random variables, Order statistics, Conditional distributions, Covariance, correlation coefficient, conditional expectation, Modes of convergences, Law of large numbers, Central limit theorem.

(No. of Lectures - 28)

Stochastic Processes: Definition of Stochastic process, Classification and properties of stochastic processes, Simple stochastic processes, Stationary processes, Discrete and continuous time Markov chains, Classification of states, Limiting distribution, Birth and death process, Poisson process, Steady state and transient distributions, Simple Markovian queuing models (M/M/1, M/M/1/N, M/M/c/N, M/M/N/N).

(No. of Lectures - 14)

Main Text Books

1. **Introduction to Probability and Stochastic Processes with Applications, Liliana Blanco Castaneda, Viswanathan Arunachalam, Selvamuthu Dharmaraja, Wiley, New Jersey, June 2012.**

2. **Probability and Statistics with Reliability, Queueing and Computer Science Applications, Kishor S. Trivedi, John Wiley, second edition, 2001.**

3. **Introduction to Probability Models, Sheldon M. Ross, Academic Press, tenth edition, 2009.**

http://web.iitd.ac.in/~dharmar/mtl106/main.html

Course Web Page

06/01/2016 $\boxed{\text{LECTURE 2}}$

Definition : Probability Space

Let E be a random experiment
Let Ω be the collection of all possible outcomes.
Let F be a σ-field (σ-algebra) on Ω

σ-field :
(i) $\phi \in F$
(ii) If $A \in F$ $\Rightarrow A^c \in F$
(iii) If $A_i \in F$ $\cup A_i \in F$

The set function P defined on F satisfying :
i) $P(A) \geq 0 \quad \forall A \in F$
ii) $P(\Omega) = 1$
iii) If A_i, $i = 1, 2, \ldots$ disjoint events, then $P(\cup A_i) = \Sigma P(A_i)$

Then the function P is called probability. (kolmogrov axiomitic def^n of probability)
(Ω, F, P) is called a probability space.

Ex:- ①
$\Omega = \{a, b, c\}$
$F_1 = \{\phi, \{a\}, \{b,c\}, \Omega\}$
$F_2 = \{\phi, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \Omega\}$

Both are σ field on Ω

② $\Omega = [0,1]$
$F_1 = \{\phi, \{0\}, (0,1], \Omega\}$

Largest σ-field on R – "Borel" σ field [all subsets of R]
$(-\infty, \infty)$

ex: E : Drawing a card from a deck of 52 cards
③ $\Omega = \{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ $\quad P(a) = 1/13 \quad \forall a \in \Omega$

$F_1 = \{\phi, \Omega, \{A,K,Q,J\}, \{10,9,8,7,6,5,4,3,2\}\}$

$P(\phi) = 0 \quad P(\Omega) = 13 \times \frac{1}{13} = 1 \quad P(\{A,K,Q,J\}) = \frac{4}{13}$ & $P(\{10,9,8,7,6,5,4,3,2\}) = \frac{9}{13}$

$\Rightarrow$ Probability Space

08/01/2016 $\boxed{\text{LECTURE 3}}$

Conditional Probability

Definition : (Ω, F, P) , $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (\text{P of A given B occurs})$$

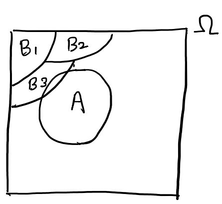
It satisfies : (i) $P(A|B) \geq 0$ (ii) $P(B|B) = 1$ (iii) $P\left(\bigcup_i P(A_i|B)\right) = \sum_i P(A_i|B)$

A_i are disjoint

Probability Space : (Ω_B, F_B, P_B)

Remark : (i) When A & B are independent events , $P(A|B) = P(A)$

Definition : Total Probability Rule



For any event $A \in F$

$$P(A) = \sum P(A|B_i) . P(B_i)$$

Condition :

$B_i \cap B_j = \phi$

$\bigcup_i B_i = \Omega$

Definition : Baye's Rule

$$P(B_i|A) = \frac{P(A|B_i)\, P(B_i)}{\sum_j P(A|B_j)\, P(B_j)}$$

Example :

② $\Omega = (-\infty, \infty)$ $\quad P([a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} . dx$

$\mathcal{F}$: Borel σ-field

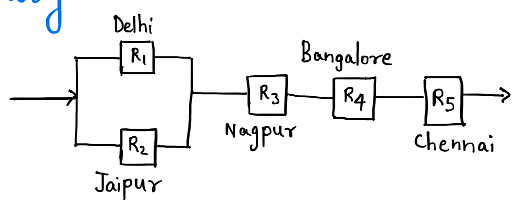
$P([a,b]) = 0$ if $a = b$ $\quad P([a,b]) = 1$ if $a = -\infty$ & $b = \infty$

i.e. $[a,b] = \Omega$

12/01/2016 $\boxed{\text{LECTURE 4}}$

Reliability

eg:-



$R_1 = P(A_1)$

Reliability of system

$= P((A_1 \cup A_2) \cap A_3 \cap A_4 \cap A_5)$

$= [R_1 + R_2 - R_1R_2] * R_3 * R_4 * R_5$

A_1, A_2, A_3, A_4 & A_5 are mutually independent.

$\boxed{\text{THEOREM}}$

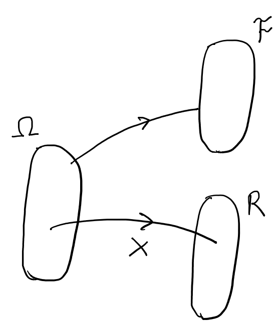
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n, n=1,2....\}$ be an increasing sequence of elements in $\mathcal{F}$ i.e. $A_n \in \mathcal{F}$ & $A_n \subseteq A_{n+1}$ for $n=1,2....$

Then $P(\lim_{n\to\infty} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Similarly, if $\{A_n, n=1,2,...\}$ is a decreasing seq. of elements in $\mathcal{F}$ i.e. $A_n \in \mathcal{F}$ & $A_n \supseteq A_{n+1}$ for $n=1,2....$, then

$P(\lim_{n\to\infty} A_n) = P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Random Variable



X is a real valued function such that $X^{-1}\{(-\infty, x]\} \in F \;\; \forall \; x \in R$. Then x is a random variable w.r.t. $\mathcal{F}$.

eg:- $\Omega = \{a,b,c\}$, $\mathcal{F} = \{\phi, \{a\}, \{b,c\}, \Omega\}$

Define $X(\omega) = \begin{cases} -10 & \omega = \{a\} \\ 5 & \omega = \{b\} \\ 5 & \omega = \{c\} \end{cases} \Rightarrow X^{-1}\{(-\infty, x]\} = \begin{cases} \phi & -\infty < x < -10 \\ \{a\} & -10 \le x < 5 \\ \Omega & 5 \le x < \infty \end{cases}$

$\Rightarrow$ Random variable.

Similary for not a random variable, (for some $\mathcal{F}$)

Define $\quad X(w) = \begin{cases} 5 & w = \{a\} \\ -10 & w = \{b\} \\ 5 & w = \{c\} \end{cases} \quad \Rightarrow \quad X^{-1}\{(-\infty, x]\} = \begin{cases} \phi & -\infty < x < -10 \\ \{b\} & -10 \le x < 5 \\ \Omega & 5 \le x < \infty \end{cases}$

$\not\Rightarrow$ Random variable.

Remark: Any real function will become a random variable if we take Power set as the σ-field

13/01/2016 $\boxed{\text{LECTURE 5}}$

Distribution Function

Any real valued function satisfying:

i) $0 \le F(x) \le 1 \quad \forall \ -\infty < x < \infty$

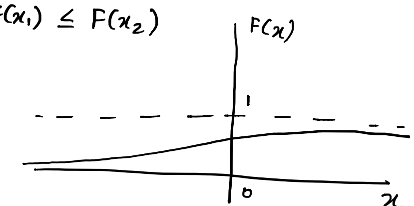
ii) Monotonically increasing in x, i.e., if $x_1 < x_2$ then $F(x_1) \le F(x_2)$

iii) $\lim\limits_{x \to -\infty} F(x) = 0 \quad \& \quad \lim\limits_{x \to \infty} F(x) = 1$

iv) F is (right) continuous function,

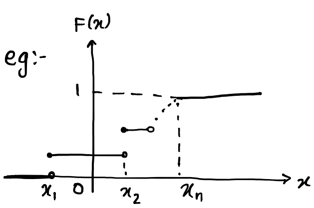
then F is called a distribution function.

eg:-

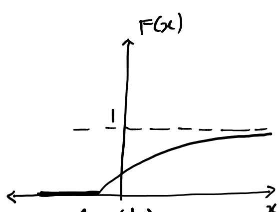


$\dfrac{\tan^{-1}x}{\pi} + \dfrac{1}{2}$

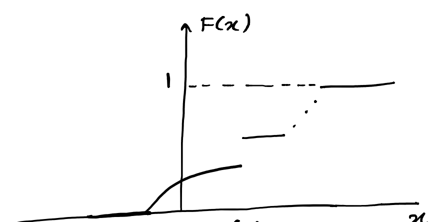
eg:-



1(a)



1 (b)



1 (c)

Cummulative Distribution Function (C.D.F.)

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$$F(x) = P(X \le x) \qquad -\infty < x < \infty$$

is said to be C.D.F of the random variable X.

$$\underset{\text{short hand notation}}{P\{X \le x\}} = P\{\underset{w \in \Omega}{w \mid X(w) \le x}\}$$

Classification of Random Variable

1.) Discrete Type Random Variable Fig. 1(a) above

C.D.F. has countable no. of discontinuities

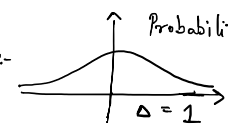
2.) Continuous Type Random Variable Fig 1(b)

If its C.D.F. is a continuous fn. in x.

$$F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{\text{probability density fn. (pdf) of r.v. } X}.dt$$

$$\left.\begin{array}{l} f(x) \geq 0 \quad \forall\, x \\ \int_{-\infty}^{\infty} f(t).dt = 1 \end{array}\right] \text{ Properties of probability density function}$$

eg:- Probability density fn.



3) Mixed Type Random Variable. Fig 1(c)

If it's C.D.F. has countable discontinuities as well as continuous function in some subintervals.

i.e. $$\sum_i P\,|\,x = x_i) + \int_{-\infty}^{\infty} f(x).dx = 1$$

15/01/2016 $\boxed{\text{LECTURE 6}}$

Example :

① Discrete Type Random Variable

a) $$P(x) = \begin{cases} 0 & x < 0 \\ \dfrac{[x]}{10} & 0 \leq x < 11 \\ 1 & 11 \leq x \end{cases}$$

b) $$F(x) = \sum_{n=1}^{x} \frac{1}{2^n} = 1$$

$$P(x) = \begin{cases} \dfrac{1}{2^x} & x = 1,2,3\ldots \\ 0 & \text{others} \end{cases}$$

② Continuous Type Random Variable

a) $$F(x) = \frac{\tan^{-1} x}{\pi} + \frac{1}{2}$$

$$f(x) = \frac{d\,F(x)}{dx} = \frac{1}{\pi(1+x^2)}$$

b) $$\int_0^{\infty} e^{-x}.dx = 1$$

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{, others} \end{cases}$$

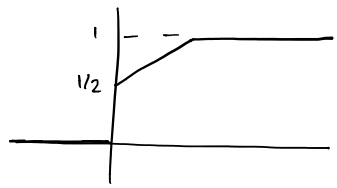
c) $\sum_{k=0}^{n} {}^nC_k \left(1/2\right)^n = 1$

$$P(x) = \begin{cases} {}^nC_k \left(1/2\right)^n & k=0,1,\ldots n \\ 0 & \text{others.} \end{cases}$$

c) $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

(Mixed Type Random Variable)

a) $$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{2} + \frac{x}{2} & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$



$$P(x) = \begin{cases} \frac{1}{2} & x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_k P(X=k) + \int_0^1 1/2 \, dx$$

$$1/2 + 1/2 = 1$$

19/01/2016 $\boxed{\text{LECTURE 7}}$

Example : ϵ - Random Experiment
Ω - Collection of all possible outcomes
$\mathcal{F}$ - largest σ- field on Ω
$X : \Omega \to R$ s.t. $X^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall x \in R$

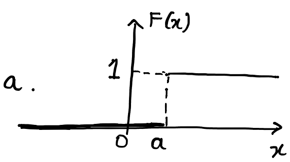
Standard Distributions

- **Discrete Type Random Variable**

① Constant Random Variable

X takes only one value . suppose the point is a.

$\Rightarrow P\{X=a\} = 1$

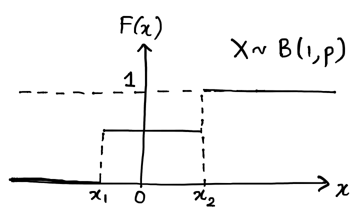


② Bernaulli Distributed Random Variable $X \sim B(1,p)$

X takes values x_1 & x_2

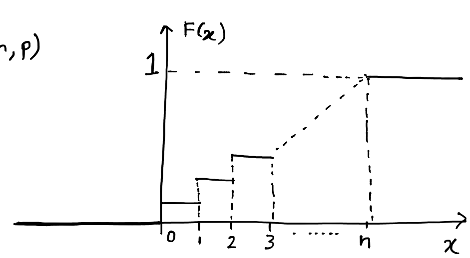
$P\{x=x_1\} = 1 - P\{x=x_2\} = P$

$0 < P < 1$



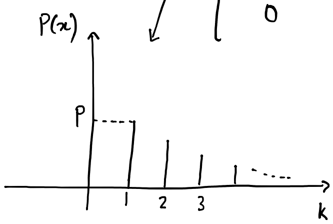
③ Binomial Distributed Random Variable $X \sim B(n,p)$

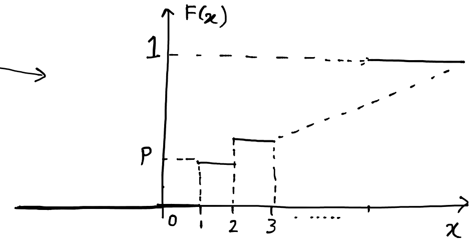
$$P\{X=k\} = \begin{cases} {}^nC_k \, p^k (1-p)^{n-k} & , k=0,1,2,\ldots n \\ 0 & , \text{otherwise} \end{cases}$$



4 Geometric Distribution

$$P\{x=k\} = \begin{cases} (1-p)^{k-1}\,p & k=1,2,3,\dots \\ 0 & \text{otherwise} \end{cases}$$





5 Negative Binomial or Pascal $X \sim NB(r,p)$

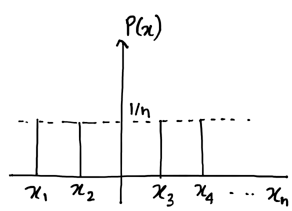
$$P\{x=n\} = \begin{cases} {}^{n-1}C_{r-1}\cdot p^{r}(1-p)^{n-r} & n=r,r+1,\dots \\ 0 & \text{otherwise} \end{cases}$$

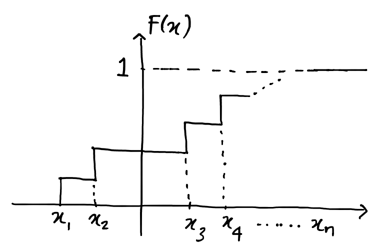
if $r=1 \longrightarrow$ Geometric Distribution

20/01/2016 $\boxed{\text{LECTURE 8}}$

6 Discrete Uniform

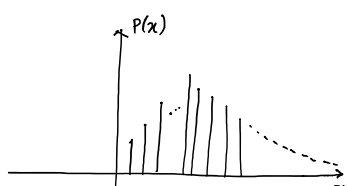
$$P\{x=k\} = \begin{cases} \frac{1}{n} & k=x_1,x_2\dots x_n \\ 0 & \text{otherwise} \end{cases}$$





7 Poisson Distribution

$$P\{x=k\} = \begin{cases} \dfrac{e^{-\lambda}\lambda^{k}}{k!} & k=0,1,2\dots \\ 0 & \text{otherwise} \end{cases}$$



For some λ

8 Poisson Process

$\{X(t),\ t\in T\}$ – <u>Stochastic Process</u>

For fixed t, $X(t) \sim$ Poisson distributed $X(t) \sim P(\lambda t)$

Assume
1.) $X(0)=0$
2.) $P\{X(t+\Delta t)=1 / X(t)=0\} = \lambda\Delta t + o(\Delta t)$
3.) $P\{X(t+\Delta t)=0 / X(t)=0\} = 1-\lambda\Delta t + o(\Delta t)$
4.) $P\{X(t+\Delta t)>1 / X(t)=0\} = o(\Delta t)$
5.) Non-overlapping intervals are independent.

Divide the interval $[0,t]$ into n equal parts.

$$P\{X(t)=k\} = \begin{cases} {}^{n}C_k \left(\lambda \cdot \frac{t}{n}\right)^k \left(1-\frac{\lambda t}{n}\right)^{n-k} & k=0,1\ldots n \\ 0 & \text{otherwise} \end{cases}$$

Binomial

$$\Rightarrow \lim_{n\to\infty} P\{X(t)=k\} = \begin{cases} e^{-\lambda t}\frac{(\lambda t)^k}{k!} & k=0,1,2,\ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

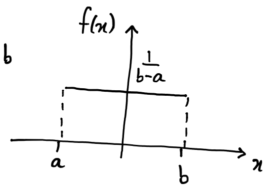
Poisson Distribution

22/01/2016 $\boxed{\text{LECTURE 9}}$

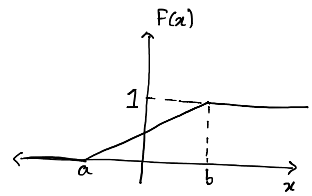
- **Continuous Type Random Variable**

① Uniform Distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{, others} \end{cases}$$

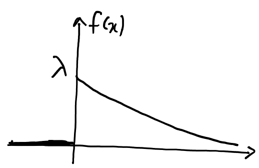


$$F(x) = \begin{cases} 0 & -\infty < x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x < \infty \end{cases}$$

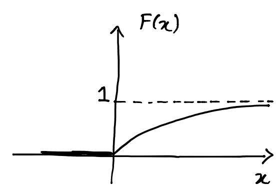


② Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$



$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1-e^{-\lambda x} & 0 \le x < \infty \end{cases}$$

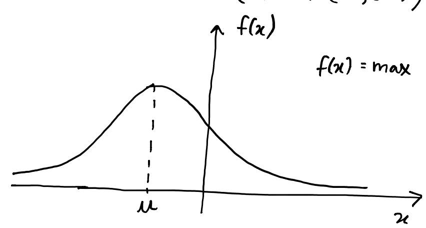


$$P(X > x+t \,/\, X > t) = \frac{P(X > x+t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x}$$ (independent of $t \Rightarrow$ Memoryless / Marcov Property)

as $P(x>t) = 1-F(x=t) = e^{-\lambda t}$

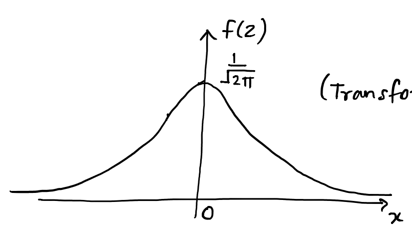
③ Normal Distribution or Gaussian Distribution $(X \sim N(\mu, \sigma^2))$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \begin{matrix} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{matrix}$$

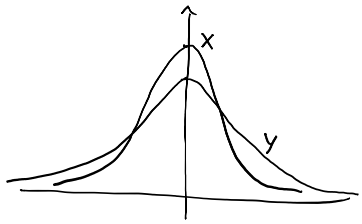


If we take random variable to be $z = \frac{x-\mu}{\sigma}$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad -\infty < z < \infty$$



$X \sim N(0,1)$
$Y \sim N(0,2)$



$f(y)\big|_{y=0} = \frac{1}{2\sqrt{2\pi}}$ $\quad f(x)\big|_{x=0} = \frac{1}{\sqrt{2\pi}}$

X will have higher peak than Y.

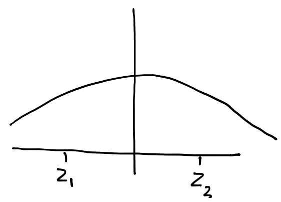
X will interset Y since area = 1 for both.

27/01/2016 $\boxed{\text{LECTURE 10}}$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2}\, dt$$

$$P(a < x \le b) = F_x(b) - F_x(a)$$

$$= P\left\{\frac{a-u}{\sigma} < \frac{x-u}{\sigma} \le \frac{b-u}{\sigma}\right\}$$



$$= P\{z_1 < z \le z_2\}$$

$$= P(z \le z_2) - P(z \le z_1)$$

$$= F_z(z_2) - F_z(z_1) = 0.5 + \phi(z_2) - [1 - \phi(z_1) - 0.5]$$

④ $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$$

29/01/2016 $\boxed{\text{LECTURE 11}}$

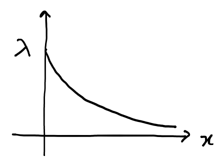
Definition: Let X be a continuous type r.v. with p.d.f $f(x)$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,.dx \quad \text{provided} \int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$$

i.e., $E(|x|) < \infty$

eg:- $X \sim \text{Exp}(\lambda)$

① $E(x) = \int_{-\infty}^{\infty} x f(x)\,.dx = \frac{1}{\lambda}$ $\qquad \left(f(x) = \lambda e^{-\lambda x}\right)$ $\quad 0 < x < \infty$



② $X \sim U(x_1, \dots x_n)$ [Uniformly Distributed]

$$E(x) = \sum_i x_i P(x = x_i) = \sum x_i \frac{1}{n} = \frac{x_1 + x_2 + \dots + x_n}{n} \quad \text{(Average)}$$

③ $X \sim P(\lambda)$

$$E(x) = \sum x P(x = x_i) = \lambda$$

④ $X \sim \text{Geometric}(p)$

$$P(x = k) = (1-p)^{k-1} p \qquad k = 1, 2, \dots$$

$$E(x) = \sum k P(x = k)$$

$$= 1. P(x=1) + 2. P(x=2) + 3\, P(x=3) + \dots$$

$$\begin{aligned} = P\{x=1\} + P\{x=2\} &+ P(x=3) + \dots \\ + P\{x=2\} &+ P(x=3) + \dots \\ &+ P(x=3) + \dots \end{aligned} \quad = \sum_{k=1}^{\infty} P\{x \geq k\} = \sum_{0}^{\infty} P\{x > k\}$$

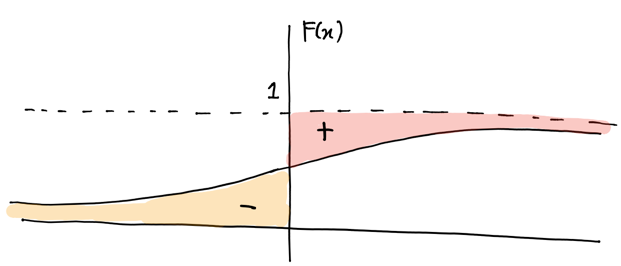
$$= \sum_{k=0}^{\infty} (1 - F_x(k))$$

In general,

if X is a continuous type r.v.

$$E(x) = \int_0^{\infty} (1 - F_x(x))\, dx - \int_{-\infty}^{0} F_x(x)\,.dx \qquad \text{(provided R.H.S. exist)}$$

(● − ● Area wise)



Definition : 2^{nd} order moment

$$\sigma^2 = E[(x-\mu)^2]$$ about the mean (i.e. $\mu = E(x)$)

(Variance of r.v. x)

Remark ① $\sigma^2 \geq 0$

② $\sigma^2 = 0 \Rightarrow Var(X) = 0 \Rightarrow P(X = \mu) = 1$

③ $Var(AX + B) = A^2 \, Var(X)$

④ $\sigma^2 = E[(x-\mu)^2] = E(x^2) - (E(x))^2$

⑤ $E(g(x)) = \begin{cases} \sum g(x) \, P\{x = x_i\} \\ \int_{-\infty}^{\infty} g(x) \, f_x(x) \, dx \end{cases}$

If $g(x)$ is Borel measurable function then $g(x)$ will become a random variable from R to R

For $g(x)$ to be Borel measurable, every piece wise continuous fn. should be Borel Measurable.

⑥ $E\left(\frac{1}{X}\right) \neq \frac{1}{E(X)}$

⑦ n^{th} order moment about the mean

$$\mu'_n = E((x-\mu)^n)$$

provided R.H.S. exist

⑧ n^{th} order moment exist $\not\Rightarrow$ $(n+1)^{th}$ order moment exist

02/02/2016 $\boxed{\text{LECTURE 12}}$

Given a random variable X, $x^{-1}\{(-\infty, x]\} \in F \;\; \forall \; x \in R$

g is a function $R \to R$ such that $y = g(x)$ is a random variable.

i.e. $Y^{-1}\{(-\infty, y]\} \in F \;\; \forall \; y \in R$

What is the distribution of Y?

g is a Borel Measurable Function in $R \longrightarrow R$

(Special case : Every piecewise continuous function is Borel Measurable)

eg:- $y = x^2$, $\sin x$, $2x$, $\begin{cases} x & x<0 \\ 3 & x>0 \end{cases}$

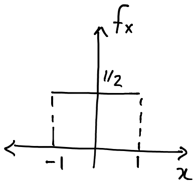
2.) $X \sim B(n,p)$, $Y = n - X$

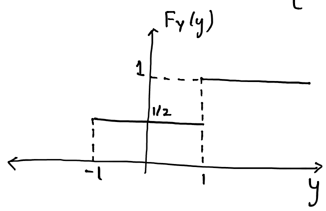
$$P\{Y = k\} = P\{n - X = k\} = P\{X = n-k\} = {}^nC_{n-k} \, (p)^{n-k} (1-p)^k \Rightarrow Y \sim B(n, 1-p)$$
$$= {}^nC_k \, (1-p)^k \, p^{n-k}$$

3.)

X	$Y = g(X)$ (Also r.v.) ☆☆
discrete	discrete
continuous	continuous
mixed	mixed

4.) $X \sim U(-1,1)$ $\qquad Y = \begin{cases} 1 & X<0 \\ -1 & X \geq 0 \end{cases} \Rightarrow F_Y(y) = \begin{cases} 0 & y<-1 \\ P(X\geq 0) & -1\leq y<1 \\ 1 & 1\leq y \end{cases} = \begin{cases} 0 & y<-1 \\ \int_0^1 \frac{1}{2}\,dx & -1\leq y<1 \\ 1 & 1\leq y \end{cases}$

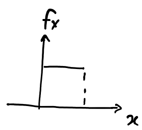




$$= \begin{cases} 0 & y<-1 \\ \frac{1}{2} & -1\leq y<1 \\ 1 & 1\leq y \end{cases}$$

5.) $X \sim U(0,1)$

$Y = -\frac{1}{\lambda} \ln(1-X)$



$$F_Y(y) = P(Y\leq y) = P(X \leq 1-e^{-\lambda y}) = \begin{cases} 0 & y<0 \\ \int_0^{1-e^{-\lambda y}} 1\,.dx & y\geq 0 \end{cases} = \begin{cases} 0 & y<0 \\ 1-e^{-\lambda y} & y\geq 0 \end{cases}$$

03/02/2016 $\boxed{\text{LECTURE 13}}$

Theorem : ☆

Let X be a continuous type random variable with p.d.f. $f_X(x)$. If h is strictly monotonic function & differentiable
Then the p.d.f. of $Y = h(x)$ is given by

$$\boxed{f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d\, h^{-1}(y)}{dy} \right| & , \text{ if } y = h(x) \\ 0 & y \neq h(x) \end{cases}}$$

eg:- ① $X \sim U(0,1)$, $Y = -\frac{1}{\lambda}\ln(1-X)$

$$f_Y(y) = f_X(1-e^{-\lambda y}) \left| \frac{d}{dy}(1-e^{-\lambda y}) \right| = \begin{cases} 1.\lambda.e^{-\lambda y} & , 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

Proof :

$$F_Y(y) = P(Y\leq y) = P(h(X)\leq y) = P(X \leq h^{-1}(y)) \quad (\text{if } h(x) \text{ is monotonically increasing})$$
$$= F_X(h^{-1}(y))$$

On differentiating, $f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y)$

eg:- ② $X \sim N(\mu, \sigma^2)$ $\qquad Z = \frac{X-\mu}{\sigma}$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty<x<\infty$

$$h(x) = \frac{x-\mu}{\sigma} \Rightarrow h^{-1}(z) = \sigma z + \mu \Rightarrow f_Y(y) = f_X(h^{-1}(y))\left|\frac{d}{dz}h^{-1}(y)\right| = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} \times \sigma \quad \forall z \in R$$

$$= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

③ $X \sim U(-1,1) \qquad Y = |X|$

$$F_Y(y) = P(\underset{(y<0)}{|X| \le y}) = P(-y \le X \le y) = F_X(y) - F_X(-y) \Rightarrow f_Y(y) = f_X(y) + f_X(-y)$$

$$\Rightarrow f_Y(y) = \begin{cases} 1 & y>0 \\ 0 & y<0 \end{cases}$$

Alternate :

Corollary : Let h be a piecewise strictly montonic function & differentiable. Further there exists intervals $I_1, I_2, \dots I_n$

Then the pdf of $Y = h(x)$ is given by

$$f_Y(y) = \begin{cases} \sum_{k=1}^{n} f_X(h_k^{-1}(y)) \left|\frac{d}{dy}(h_k^{-1}(y))\right| \\ 0 & \text{otherwise} \end{cases}$$

$I_1 \to (-\infty, 0) \qquad I_2 \to [0,\infty)$

$y = -x \quad h^{-1}(y) = -y \qquad y = x \quad h^{-1}(y) = y$

$$f_Y(y) = \begin{cases} f_X(-y)\,|-1| + f_X(y)\,|1| & y>0 \\ 0 & y<0 \end{cases} = \begin{cases} 1 & y \ge 0 \\ 0 & y<0 \end{cases}$$

④ $X \sim N(\mu, \sigma^2)$, $Y = e^X$ Log normal

⑤ $X \sim Exp(1)$, $Y = \begin{cases} 1 & X \le 1 \\ \frac{1}{X} & X > 1 \end{cases}$

$f_X(x) = e^{-x}$

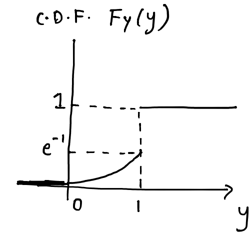
$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{x} \le y\right) \quad \text{If } (y<1) \quad / \quad 1 \quad \text{If } (y>1)$$

$$= P\left(x \ge \frac{1}{y}\right) \quad \text{If } (0<y<1)$$

$$= 0 \quad \text{If } y<0$$

$$= \begin{cases} 1 & y>1 \\ \int_{1/y}^{\infty} e^{-x}\,dx & 0<y<1 \\ 0 & y<0 \end{cases} = \begin{cases} 1 & y>1 \\ e^{-1/y} & 0<y<1 \\ 0 & y<0 \end{cases}$$

C.D.F. $F_Y(y)$



$$\Rightarrow f_Y(y) = \begin{cases} \frac{e^{-1/y}}{y^2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases} \qquad P_Y(y) = \begin{cases} 1-e^{-1} & y=1 \\ 0 & \text{otherwise} \end{cases}$$

05/02/2016 $\boxed{\text{LECTURE 14}}$

GENERATING FUNCTIONS

Uniqueness Theorem :

1. P.G.F. $G_X(t)$ Probability Generating Function
2. M.G.F. $M_X(t)$ Moment Generating Function
3. C.F. $\phi_X(t)$ Characteristic Function

If $G/M/\phi_X(t) = G/M/\phi_Y(t) \quad \forall\, t \quad \Rightarrow \quad X \stackrel{d}{=} Y$

Probability Generating Function (P.G.F.)

Let X be a non-negative integer values r.v. with p.m.f. $P_k = P\{X=k\}, k=0,1,2,\ldots$

Then the prb. gen. func. (pgf) be defined as

$$G_X(t) = \sum_{k=0}^{\infty} P\{X=k\}\, t^k \qquad |t| \le 1$$

e.g.:- $X \sim B(n,p) \Rightarrow G_X(t) = (pt + 1-p)^n$

Remarks : (1) $G_X(1) = 1$

(2) $G_X(t) = E(t^k)$

(3) $P_k = P\{X=k\} = \dfrac{1}{k!} \left.\dfrac{d^k G_X(t)}{(dt)^k}\right|_{t=0}$

(4) $\underbrace{E(X(X-1))}_{\text{Factorial moments}} = \left.\dfrac{d^2 G_X(t)}{dt^2}\right|_{t=1}$

Moment Generating Function (M.G.F.)

Let X be a r.v. s.t. $E(e^{tX})$ is finite for some interval in $(-\infty,\infty)$ including the point 0. Then the M.G.F. of X is defined as

$$M_X(t) = E(e^{tX}) = E\left(1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \cdots + \frac{t^nX^n}{n!} + \cdots\right)$$

eg:- $X \sim P(\lambda)$

1. $M_X(t) = E(e^{tX}) = \displaystyle\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} e^{tk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(1-e^t)}$ $\quad t \in (-\infty,\infty)$

② $X \sim Exp(\lambda)$

$$M_X(t) = \int_0^{\infty} e^{xt} \lambda e^{-\lambda x} \cdot dx = \frac{\lambda}{t-\lambda}(-1) = \frac{\lambda}{\lambda - t} \quad \forall \; t \in (-\infty, \lambda)$$

Remarks : (1.) $E(X^k) = \left.\frac{d^k M_X(t)}{(dt)^k}\right|_{t=0}$

③ Find distribution of X if $M_X(t) = \frac{e^t}{2} + \frac{e^{-t}}{3} + \frac{1}{6}$

X	-1	0	1
$P\{X=k\}$	$1/3$	$1/6$	$1/2$

$M_X(t) = E(e^{tX})$

Characteristic Function (C.F.)

$$\psi_X(t) = E(e^{itX}) \qquad i = \sqrt{-1}$$

Remarks : (1) $\psi_X(0) = 1$

(2.) $|\psi_X(t)| \leq 1$

(3.) If $E(X^k)$ exist,

$$E(X^k) = \frac{1}{i^k} \left.\frac{d^k \psi_X(t)}{dt^k}\right|_{t=0}$$

Ex :- ① $X \sim N(\mu, \sigma^2)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2} \cdot dx \quad \cdots\cdots = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

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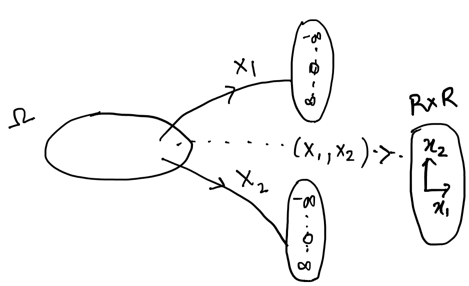
MINOR 1 SYLLABUS

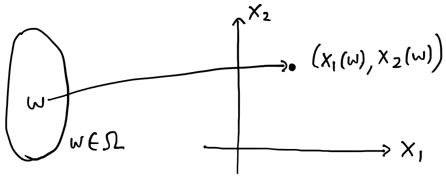
09/02/2016 $\boxed{\text{LECTURE 15}}$

Two & Higher Dimensional Random Variables



If $X^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall \; x \in R$

$X = (X_1, X_2)$ random vector of 2 dimension



In general,

$(x_1, x_2, \ldots\ldots x_n)$ n-dim random variable

C.D.F. (x_1, x_2)

$$F_{x_1,x_2}(x_1, x_2) = \text{Prob}\{X_1 \le x_1, X_2 \le x_2\}$$
$$= \text{Prob}\{w \mid x_1(w) \le x_1, x_2(w) \le x_2\}$$
$$w \in \Omega \qquad -\infty < x_1 < \infty, \; -\infty < x_2 < \infty$$

It satisfies

1.) $0 \le F(x_1, x_2) \le 1 \quad \forall \; x_1, x_2$

2.) $\lim\limits_{x_1 \to -\infty,\, x_2 \to -\infty} F(x_1, x_2) = 0$

3.) $\lim\limits_{x_1 \to \infty,\, x_2 \to \infty} F(x_1, x_2) = 1$

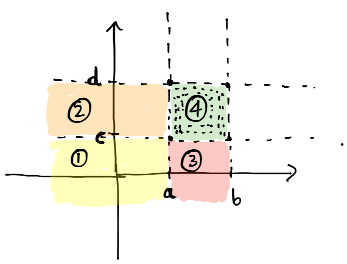
4) $F(x_1, x_2)$ is montonically increasing in both x_1 & x_2.

5.) $\forall \; a < b, \; c < d$

$F_{x_1 x_2}(b, d) - F(a, d) - F(b, c) + F(a, c) \ge 0$ (similar for multi-dimension)

i.e. $(1+2+3+4) - (1+2) - (1+3) + 1$
$= 4$
$F(4) \ge 0$

eg:- ① $F(x_1, x_2) = \begin{cases} 1 - e^{-(x+y)} & 0 < x < \infty, \; 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$



② $F(x_1, x_2) = \dfrac{\tan^{-1}(x_1 + x_2)}{\pi} + \dfrac{1}{2} \qquad x \in \mathbb{R}, \; y \in \mathbb{R}$

- **Two Dimensional Discrete Type Random Variable**

$$P_{x_1,x_2}(x_1, x_2) = \text{Prob}\{X_1 = x_1, X_2 = x_2\} \qquad \text{(Mass function)}$$
$$= \text{Prob}\{w \mid X_1(w) = x_1, X_2(w) = x_2\} \quad w \in \Omega$$

It satisfies

1.) $0 \le P(x_1, x_2) \le 1 \quad \forall \; x_1, x_2$

2.) $\sum\limits_{x_1} \sum\limits_{x_2} P(x_1, x_2) = 1$

$$P_{x_1}(x_1) = \sum_{x_2} P(x_1, x_2) \; ; \quad P_{x_2}(x_2) = \sum_{x_1} P(x_1, x_2)$$

10/02/2016 $\boxed{\text{LECTURE 16}}$

- **Two Dimensional continuous type random vector** (X_1, X_2)

$$F_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \underbrace{f_{X_1,X_2}(t,s)}_{\text{joint pdf}}\, ds\, dt$$

Joint pdf satisfies

1.) $f(x_1,x_2) \geq \ \forall\ x_1, x_2$

2.) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)\, ds\, dt = 1$

eg :- 1.) $f(x,y) = \begin{cases} e^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

2.) $f(x,y) = \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}[x^2+y^2]} & -\infty < x < \infty \\ & -\infty < y < \infty \end{cases}$

eg :-

E : Tossing a coin Thrice

X : # of heads

Y : Difference (in absolute) of # of heads & # of tails

$\Omega = \{(HHH), \ldots\ldots, (T,T,T)\}$ $N_q(\Omega) = 8$

joint p.m.f. of (X,Y)

X ↓ \ Y →	1	3	$P(X=x)$
0	0	1/8	1/8
1	3/8	0	3/8
2	3/8	0	3/8
3	0	1/8	1/8
$P(Y=y)$	6/8	2/8	

Individually : (Marginal Distrubution of X,Y)

Y	P
1	2/8
3	6/8

X	P
0	1/8
1	3/8
2	3/8
3	1/8

- **Independent Random Variable**

We say that two random variables are independent iff

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y) \quad \forall\ x,y$$

i.e. if (X,Y) - discrete type $\Rightarrow P_{X,Y}(x,y) = P_X(x)\, P_Y(y) \quad \forall\ x,y$

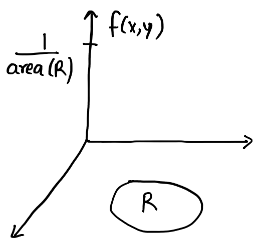
if (X,Y) - continuous type $\Rightarrow f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \quad \forall (x,y)$

eg:- Multivariable Normal Distribution (x,y,z)

$$f_{X,Y,Z}(x,y,z) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}(x^2+y^2+z^2)} \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{matrix}$$ (Independent)

eg:- (x,y) - joint pdf 'R' in X-Y plane
Uniform Distribution.

(Independent)



3.) $f_{X,Y}(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Not independent.

16/02/2016 LECTURE 17 Password - Markov

CONDITIONAL DISTRIBUTION

$P(A|B)$; $P(B) > 0$

Defn: Let (X,Y) be a 2 dimension discrete type R.V. with joint pmf $P_{X,Y}(x,y)$.
Then the conditional dist. of X given $y = y_j$ is defined as

$$\boxed{P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}}} \quad \forall i$$

provided $P\{Y = y_j\} > 0$

Note:- $\sum_i P(X = x_i / Y = y_j) = \frac{\sum_i P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}} = 1$

Defn: Let (X,Y) be a 2 dimension continuous type R.V. with joint pdf $f_{X,Y}(x,y)$.
Then the conditional dist. of X given $y = y_j$ is defined as

$$\boxed{f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}} \qquad \forall x \text{ provided } f_Y(y) > 0$$

C.D.F. of $X/Y = y_j$ $= \int_{-\infty}^{x} f_{X/Y}(t/y_j)\,.dt$

eg:- $f_{X,Y}(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$

p.d.f. of

(i) $X/Y \Rightarrow f_{X/Y}(x,y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2}{2(1-y)} & y < x < 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X/y \sim U(y,1)$

$f_Y(y) = \int_y^1 2.dx = 2(1-y)$

ii) $f_X(x) = \int_0^x 2.dy = 2x$

$Y/x = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{2}{2x} & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad Y \sim U(0,x)$

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17/02/2016 LECTURE 18

FUNCTIONS OF RANDOM VARIABLES

eg:- Let $X \sim B(n,p)$, $Y \sim B(n,p)$ Assume that X & Y are independent r.v.'s.
Define $Z = X+Y$, dist of Z = ?

$$P(Z=k) = \sum_i P(X=i, Y=k-i) = \sum_i P(x=i).P(y=k-i)$$

$$= \sum_i {}^nC_i\, p^i (1-p)^{n-i} \,.\, {}^nC_{k-i}\, p^{k-i} .\, (1-p)^{n-k+i}$$

$$= \sum_i {}^nC_i\, {}^nC_{k-i}\, p^k (1-p)^{2n-k}$$

$$= \left(\sum_i {}^nC_i\, {}^nC_{k-i}\right) p^k (1-p)^{2n-k}$$

$$= {}^{2n}C_k\, p^k\, p^{2n-k}$$

In general, $X_i \sim B(n_i, p)$, $i = 1,2,3,\dots,k$ & X_i's are independent r.v.

$$Z = \sum_{i=1}^{k} X_i \sim B\left(\sum_{i=1}^{k} n_i, p\right)$$

iid - independent + identically distributed

② $X_i \sim B(1,p)$, $i = 1,\dots,n$ X_i - iid's

$$Z = \sum_{i=1}^{n} X_i \sim B(n,p)$$

$\sum X_i$'s - same distribution

$\Rightarrow$ Reproductive Property

③ $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

X & Y are independent

$$Z = X + Y \sim N(\mu_1 + \mu_2, \dots)$$

④ $X \sim P(\lambda_1)$, $Y \sim P(\lambda_2)$

$Z = X + Y$

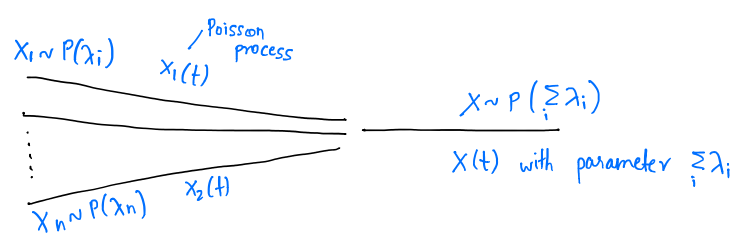
$$P(Z=k) = \sum_i P(X=i, Y=k-i)$$

$$= \sum_i P(X=i)\,P(Y=k-i) = \sum_i \frac{e^{-\lambda_1}(\lambda_1)^i}{i!} \times \frac{e^{-\lambda_2}(\lambda_2)^{k-i}}{(k-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!} \quad \Rightarrow \quad Z \sim P(\lambda_1+\lambda_2) \quad \text{(Reproductive)}$$

In general, $X_i \sim (\lambda_i)$ $i = 1,2,\dots n$ X_i's are idd.

$$Z = \sum_i X_i \sim P\left(\sum_i \lambda_i\right)$$



MOMENTS OF RANDOM VARIABLE

Defn: Let $x, x_2, \ldots x_n$ be R.V.'s

$$E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i)$$

$$Var\left(\sum x_i\right) = \sum_{i=1}^{n} Var(x_i) + 2 \sum_{i}\sum_{j}^{i<j} Cov(x_i, x_j)$$

Defn: $Cov(x,y)$

$$Cov(x,y) = E\left((x-E(x))(y-E(y))\right)$$

1^{st} order cross moment about the mean.

$$\mu_{i,j} = E\left((x-E(x))^i(y-E(y))^j\right) \quad (i,j)^{th} \text{ order cross moment about the mean.}$$

$E(x_i, x_j) \longrightarrow$ about the zero.

$E\left((x-E(x))^2\right) \longrightarrow Var(x) \quad [\mu_{(2,0)}]$

Variance Covariance Matrix of n-variables

$$\begin{bmatrix} Var(x_1) & Cov(x_1,x_2) & \cdots & Cov(x_1,x_n) \\ Cov(x_2,x_1) & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \vdots & \cdots & \cdots & Cov(x_{n-1},x_n) \\ Cov(x_n,x_1) & \cdots & \cdots & Var(x_n) \end{bmatrix} \qquad Cov(x,y) = Cov(y,x)$$

Remark: If X & Y are independent r.v.'s

$$Cov(x,y) = E\left((\underbrace{x-E(x)}_{\mu_1})(\underbrace{y-E(y)}_{\mu_2})\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_1)(y-\mu_2)\, f_{x,y}(t,s)\, dt.\, ds$$

$$= E(x-E(x))\; E(y-E(y))$$

$$= [E(x)-E(x)][E(y)-E(y)]$$

$$= 0$$

$\Rightarrow$ If X & Y are independant $\Longrightarrow Cov(x,y) = 0$

$\not\Longleftarrow$

Vice-versa is not true.

Defn :- Correlation Coefficient

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$ provided R.H.S. exist.

X & Y are independent, then $\rho_{X,Y} = 0$
$\not\Leftarrow$

Remark:-

(i) $|\rho_{X,Y}| \leq 1$

(ii) If $\rho_{X,Y} = \pm 1$

$\Rightarrow Y = aX + b$

$a > 0 \Rightarrow \rho_{X,Y} = 1$

$a < 0 \Rightarrow \rho_{X,Y} = -1$

23/02/2016 $\boxed{\text{LECTURE 19}}$

eg:- ① $X \sim Exp(\lambda)$ Game overs whenever X or Y reaches 1.

$Y \sim Exp(\mu)$ Assume X & Y are independent.

Define $Z = \min\{x, y\}$

$$P(Z > z) = P(\min\{x,y\} > z) = P(x, y > z)$$

$$= P(x > z) \cdot P(y > z) = \int_z^\infty \lambda e^{-\lambda x} \cdot dx \int_z^\infty \mu e^{-\mu y} dy = e^{-(\lambda+\mu)z}$$

$$P(Z \leq z) = 1 - e^{-(\lambda+\mu)z}$$

$$F_Z(z) = \begin{cases} 1 - e^{-(\lambda+\mu)z} & z \geq 0 \\ 0 & z < 0 \end{cases} \Rightarrow Z \sim Exp(\lambda + \mu)$$

In general, $X_i \sim Exp(\lambda_i)$, $i = 1, 2, \dots, n$, X_i independent r.v.'s

$$Z = \min\{X_i, i = 1, 2, \dots n\} \sim Exp\left(\sum_{i=1}^n \lambda_i\right)$$

$$W = \max\{X_i \dots\} \not\sim Exp$$

$$(1 - e^{-\lambda w})(1 - e^{-\mu w})$$

② A camera runs on 2 batteries. We have 6 batteries (Battery life $X_i \sim Exp(\lambda)$)

a) What is average usage time

Suppose battery 1 & 2 are running. For one to fail, $Y = \min\{X_1, X_2\} \sim Exp(2\lambda)$

$\Rightarrow$ Avg Time for one battery to fade $= \frac{1}{2\lambda}$. Suppose battery 2 fails. Then put battery 3, then using memoryless property on battery 1, again $1/2\lambda$.

$\Rightarrow$ Total avg. time $= \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{5}{2\lambda}$ (till 5 batteries fail)

(b) C.D.F. of $Z = \underset{\sim Exp(2\lambda)}{Y_1} + Y_2 + Y_3 + Y_4 + Y_5 = Gamma(5, 2\lambda)$

Theorem: Let (X,Y) be a two dimensional cont. type random variable with a joint p.d.f. $f_{X,Y}(x,y)$. Define $Z = H_1(x,y)$ & $W = H_2(x,y)$

Assume (i) $z = h_1(x,y)$ & $w = h_2(x,y)$ can be solved uniquely for x,y in terms of z & w, i.e. $x = g_1(z,w)$, $y = g_2(z,w)$

(ii) $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous function

Then (Z,W) is 2-dim. cont. type r.v. with joint p.d.f.

$$f_{Z,W}(z,w) = \begin{cases} f_{X,Y}(g_1(z,w), g_2(z,w))\, |J(z,w)| & x = g_1(z,w),\ y = g_2(z,w) \\ 0 & \text{otherwise} \end{cases}$$

where $$J(z,w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} \neq 0$$

eg:- 1.) Let (X,Y) - 2 dim cont. type. X & Y are idd r.v.'s

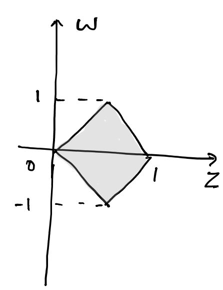
$$f_X = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$Z = X + Y$, $W = X - Y$ $\Rightarrow$ $X = \frac{Z+W}{2}$ & $Y = \frac{Z-W}{2}$

$X \in (0,1)$ & $Y \in (0,1)$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

$$f_{Z,W}(z,w) = \begin{cases} 1 \cdot 1 \cdot \frac{1}{2} & 0 < \frac{z+w}{2} < 1 \ \& \ 0 < \frac{z-w}{2} < 1 \\ 0 & \text{otherwise} \end{cases}$$



Note: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

But $f_{Z,W}(z,w) \neq f_Z(z) \cdot f_W(w) \Rightarrow Z$ & W are not independent

24/02/2016 **LECTURE 20**

ex: 2) $X_i \sim N(0,1)$, $i = 1,2,\ldots n$, iid r.v.'s

$Z = \sum_{i=1}^{n} X_i$

Assume that M.G.F. of Z exists $\quad M_X(t)$

$$M_Z(t) = M_{\sum_{i=1}^{n} X_i}(t) = E\left(e^{\sum_{i=1}^{n} x_i t}\right) = \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^n$$

$$= e^{\frac{1}{2} n t^2}$$

$\Rightarrow \quad Z \sim N(0, n) \qquad$ (Other alternatives lengthy !!)

CONDITIONAL EXPECTATION

Defn. Let (X,Y) be a 2 dim. r.v.

$$E(X/y) = \begin{cases} \sum_i x_i \, P\{X = x_i / Y = y\} & (X,Y) \text{ - disc.} \\ \int_{-\infty}^{\infty} x \, f_{X/y}(x/y) \, dx & (X,Y) \text{ - cont.} \end{cases}$$

provided in absolute sense, R.H.S. exist.

Remark : ① If X & Y are independent random variables, $E(X/y) = E(x)$; $E(Y/x) = E(y)$

② $$E(E(X/y)) = \int_{-\infty}^{\infty} E(X/y) \cdot f_Y(y) \cdot dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X/y}(x/y) \cdot dx \cdot f_Y(y) \cdot dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{X/y}(x/y) \cdot f_Y(y) \cdot dy}_{} \, dx = \int_{-\infty}^{\infty} x \, f_X(x) \cdot dx = E(X)$$

$$\Rightarrow \quad E(E(X/y)) = E(X)$$

③ $E(X/y)$ is a function of y ; $E(X/y)$ is a r.v.

④ $E(XY/y) = y E(X/y)$

⑤ $\{X_1, X_2, \ldots, X_n, \ldots\}$

Martingale Property

$$E(X_{n+1} / X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) = \begin{cases} > x_n \\ < x_n \\ = x_n \quad \text{Fair game} \end{cases}$$

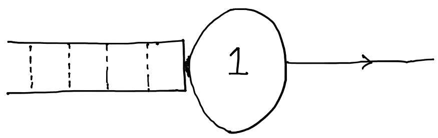
where

$X_i = X_{i-1} + Y_i$

$\qquad\qquad \downarrow$ i^{th} game

Y_i's are iid r.v.'s

26/02/2016 $\boxed{\text{LECTURE 21}}$



queing system

Random Sum

X : total time spent in the system

$= \underbrace{X_1}_{\text{Residual service time}} + X_2 + \dots + \underbrace{X_n}_{\text{his own service time}}$

$X_i \sim Exp(\lambda)$ iid r.v.'s

where even n is a random variable.

$$E(X) = E\left(E(X/N=n)\right)$$

$$= \sum_n E(X/n)\, P(N=n)$$

(But X_i are independent with N)

$\Rightarrow E(X/n) = n\,E(X_i)$

$$= \sum_n n\,E(X_i)\, P(N=n)$$

$$E(X) = E(X_i) \cdot E(N)$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = E\left(E(X^2/N)\right)$$

dist of X, given dist of X_i, N

$P(N=n) = (1-p)p^n$

$n = 0, 1, 2, \dots$

$$f_X(x) = \sum_n \underbrace{f_{X/N=n}(x/n)}_{\substack{\text{cont.} \\ \downarrow \\ \text{Gamma}}} \cdot \underbrace{P(N=n)}_{\text{discrete}}$$

$$f_{(X,N)}(x,n) = f_{X/N}(x/n) \cdot P(N=n)$$

$$\boxed{f_{X,Y}(x,y) = f_{X/Y}(x/y) \cdot f_Y(y)} \quad \text{Always } \forall x,y$$

08/03/2016 $\boxed{\text{LECTURE 22}}$

1.) Inequalities

2.) Law of Large Numbers

3.) Limiting Distribution

4.) Central Limit Theorem

Defn. : Markov Inequality

Let X be non-negative random variable with E(X) exists and is known.

For fixed $t > 0$

$$\boxed{P(X > t) \le \frac{E(X)}{t}}$$

event $\{w \mid X(w) > t \quad w \in \Omega\}$

Define $Y = \begin{cases} 0 & , x \le t \\ t & x > t \end{cases}$ $P\{Y=0\} = P\{x \le t\}$ $P\{Y=t\} = P\{x > t\}$ Y is a discontinuous type r.v.

$E(Y) = t \cdot P\{x > t\}$ — (1)

Now $X \ge Y$

$\therefore$ $E(X) \ge E(Y)$ — (2)

$$\Rightarrow \quad P\{x > t\} \le \frac{E(x)}{t}$$

Defn. : Chebyshev's Inequality

Let X be a r.v. with $E(X) = \mu$, $Var(X) = \sigma^2$ exist & are known.

Then for any positive number t,

$$P\{|X-\mu| > t\} \le \frac{\sigma^2}{t^2}$$

or

$$P\{|X-\mu| \le t\} > 1 - \frac{\sigma^2}{t^2}$$

Proof :- X is a r.v. $\Rightarrow$ $(X-\mu)^2$ is a non neg. r.v.

where $E((X-\mu)^2) = \sigma^2$

Apply Markov Inequality. Hence Proved.

eg:- Let X be a r.v. with $E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{12}$. Find the lower bound for

$P\left\{|X - \frac{1}{2}| < 2\sqrt{\frac{1}{12}}\right\} \Rightarrow \frac{3}{4}$

Let X be a uniform distribution in interval $[0, 1]$.

$$\frac{1}{2} + 2\sqrt{\frac{1}{12}} > 1 \quad \& \quad \frac{1}{2} - 2\sqrt{\frac{1}{12}} < 0 \quad \Rightarrow \quad P\left\{ |X - 1/2| < 2\sqrt{\frac{1}{12}} \right\} = 1$$

Remark: In Chebyshev's Inequality.
μ can be replaced by any number c.
In that case, $\sigma^2 \longrightarrow E((X-c)^2)$

09/03/2016 $\boxed{\text{LECTURE 23}}$

LIMITING DISTRIBUTIONS / PROBABILITIES / THEOREMS

$X_1, X_2, X_3, \ldots\ldots \longrightarrow X \qquad (\Omega, \mathcal{F}, P)$

① $X_i \sim U(-1, 1)$, $i = 1, 2, \ldots$ iid's

Define $Y_1 = X_1$, $Y_2 = X_1 + X_2$, ... $Y_n = \sum_{i=1}^{n} X_i$

What is the convergence of Y series.

Y's will be polygons. $\Rightarrow$ Increasing no. of sides $\Rightarrow$ Normal distribution.

② $X_i \sim B(1, p)$ $i = 1, 2, \ldots$ iid r.v.s

$Y_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$

For fixed n, $Y_n \sim B(n, p)$

For large n, $n \longrightarrow \infty$
normal distribution

Theorem ☆☆: Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables defined on $(\Omega, \mathcal{F}, P)$ with $E(X_i) = \mu_i$, $i = 1, 2, \ldots$ and $Var(X_i) = \sigma_i^2 > 0$, $i = 1, 2, \ldots$

Define $$\boxed{Z_n = \frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)}{\sqrt{Var\left(\sum_{i=1}^{n} X_i\right)}}} \qquad n = 1, 2, \ldots$$

Then for larger n, Z_n approaches standard normal distribution.
(approximately)

i.e. $$P(Z_n \leq x) \underset{n \to \infty}{\simeq} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \qquad x \in \mathbb{R}$$

CENTRAL LIMIT THEOREM

11/03/2016 $\boxed{\text{LECTURE 24}}$

MODES OF CONVERGENCE

1) IN PROBABILITY

for given $\epsilon > 0$, $X_n \xrightarrow{P} X$

if $\quad P\{|X_n - X| > \epsilon\} \longrightarrow 0 \quad$ as $n \to \infty$

2.) IN DISTRIBUTION

$$X_n \xrightarrow{d} X$$

if $\quad F_{X_n}(x) \longrightarrow F(x) \qquad \forall\ x \in R$

3.) IN RTH MOMENT

$$X_n \xrightarrow{r^{th}\text{ moment}} X \qquad r = 1, 2, 3, \ldots$$

$$E\left(|X_n - X|^r\right) \longrightarrow 0 \quad \text{as } n \to \infty$$

4.) IN ALMOST SURELY

$$X_n \xrightarrow{a.s.} X$$

if $\quad P\{\lim_{n\to\infty} X_n = X\} = 1$

eg:- Let $\{X_n, n = 1, 2, 3, \ldots\}$ be a sequence of r.v.'s

s.t. $P\{X_n = 0\} = 1 - \frac{1}{n}$; $P\{X_n = n\} = \frac{1}{n}$ $\quad n = 1, 2, 3, \ldots$

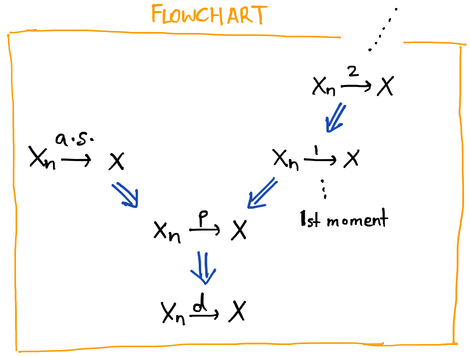
check $X_n \xrightarrow{P} 0$

$$\lim_{n\to\infty} P\{|X_n - 0| > \epsilon\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

check $X_n \xrightarrow{d} 0$

H.W.

FLOWCHART



eg:- $\{X_n, n=1,2,3,\dots\}$ be a seq. of iid r.v.'s with $E(X_i)=\mu$; $Var(X_i)=\sigma^2$, $i=1,2,\dots$

Define $S_n = X_1 + X_2 + \dots + X_n$

Check (1) $\frac{S_n}{n} \xrightarrow{\text{2nd order}} \mu$ (2) $\frac{S_n}{n} \xrightarrow{P} \mu$

$$\lim_{n\to\infty} E\left(\left|\frac{S_n}{n}-\mu\right|^2\right) = \lim_{n\to\infty}\left[\frac{1}{n^2}E(S_n^2) - \frac{2\mu}{n}E(S_n) + \mu^2\right]$$

$$= \sigma^2$$

15/03/2016 $\boxed{\text{LECTURE 25}}$

• LAW OF LARGE NUMBERS (Bernoulli Law)

Let ϵ be a random experiment and A be an event. Consider n independent trials.

Define n_A = # of times event A occurs in n-trials

Let $f_A = \frac{n_A}{n}$; $p(A) = p$, constant.

Then $f_A \xrightarrow{P} p$

Proof:

$n_A = B(n,p)$

$$E(n_A) = \sum n_A \cdot {}^nC_{n_A}\, p^{n_A} \cdot (1-p)^{n-n_A} = np \sum {}^{n-1}C_{n_A-1}\, p_A^{n_A-1}(1-p)^{n-n_A}$$
$$= np$$

$$E(f_A) = E\left(\frac{n_A}{n}\right) = \frac{1}{n}\cdot np = p$$

$$Var(f_A) = Var\left(\frac{n_A}{n}\right) = \frac{1}{n^2}Var(n_A) = \frac{p(1-p)}{n}$$

Apply Chebyshev's Inequality,

$$P\{|f_A-p| > \epsilon\} \le \frac{p(1-p)}{n\epsilon^2}$$

$$\lim_{n\to\infty} P\{|f_A-p|>\epsilon\} = 0 \quad \Rightarrow \quad f_A \xrightarrow{P} p$$

eg:- ϵ : die what is min. n for given $\epsilon = 0.01$ s.t.

A : getting #6 $P\{|f_A-p|<\epsilon\} \geq 0.95$

Sol^n : $\quad p = \frac{1}{6} \qquad P\{|f_A - p| > \epsilon\} < 0.05$

$$\Rightarrow \quad n = \frac{p(1-p)}{(0.05)\,\epsilon^2} \approx 27{,}778$$

Proof of CENTRAL LIMIT THEOREM

Assume : (i) X_i's are iid r.v.'s

(ii) M.G.F. of X_i's exist

$$M_{Z_n}(t) = M_{\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}}(t) = E\left(e^{\left(\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}\right)t}\right) = e^{-\frac{\sqrt{n}\mu}{\sigma}t}\, E\left(e^{\frac{\Sigma X_i}{\sigma\sqrt{n}}t}\right)$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} \left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$M_X(t) = 1 + \underbrace{\frac{\mu t}{1!} + \frac{E(X^2)t^2}{2!} + \dots}_{x}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \qquad |x|<1$$

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\mu t}{\sigma} + n \ln\Big[1 + \underbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2!\, n\sigma^2} \dots}_{x}\Big]$$

$$= -\frac{\sqrt{n}\mu t}{\sigma} + n\Big[\underbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2n\sigma^2}}_{x} \underbrace{- \frac{1}{2}\left(\frac{\mu^2 t^2}{n\sigma^2} + \dots\right)}_{-x^2/2}\Big]$$

$$= \frac{(\sigma^2+\mu^2)t^2}{2\sigma^2} - \frac{1}{2}\frac{\mu^2 t^2}{\sigma^2} = \frac{t^2}{2}$$

$$\Rightarrow \lim_{n\to\infty} \ln M_{Z_n}(t) = \frac{t^2}{2} \quad \text{or} \quad \lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2}$$

$$Z_n \sim N(0,1)$$

16/03/2016 $\boxed{\text{LECTURE 26}}$

eg:- Let $X_1, X_2 \dots X_n$ be a sequence of iid r.v. with common p.m.f.

$$P(x_1) = \begin{cases} p & x_1 = 1 \\ 1-p & x_1 = 0 \end{cases} \qquad p = 1/2$$

Let $X = X_1 + X_2 + \dots + X_{10}$

(a) Find $P(X \le 8)$ exactly (distribution)

(b) $P\{X \le 8\}$ lower bound (Inequalities)

(c) $P\{X \le 8\}$ approximately (CLT)

Solⁿ (a) $P(X \le 4) = \sum_{0}^{8} {}^{10}C_i\, p^i (1-p)^{10-i}$

(b) Markov's inequality

$$P\{X \le 8\} \ge 1 - \frac{E(X)}{8} = 1 - \frac{5}{8} = \frac{3}{8}$$

(c) $$P\{X \le 8\} = P\left\{\frac{X - \overset{\mu = np}{5}}{\underset{\text{Variance } np(1-p)}{\sqrt{2.5}}} \le \frac{8-5}{\sqrt{2.5}}\right\} = P\left\{\underset{\text{normal}}{Z} \le \underset{1.89}{\frac{3}{\sqrt{2.5}}}\right\} = F_Z(1.89) = \underset{\text{from table}}{0.9698}$$

eg :- Find $\lim_{n \to \infty} \sum_{k=0}^{n} e^{-n} \frac{n^k}{k!}$ approximately using C.L.T.

Solⁿ $X_i \sim P(1)$

$X = \sum_{i=1}^{n} X_i \sim P(n)$ $\qquad P(X \le n) = P\left(\frac{X-n}{\sqrt{n}\sigma} \le \frac{n-n}{\sqrt{n}\sigma}\right) = P(Z \le 0) = \frac{1}{2}$ //

$Z = \frac{X - n}{\sqrt{n}\,\sigma}$

18/03/2016 **LECTURE 27**

TUTORIAL 5

28/03/2016 **LECTURE 28**

Defn :

Let $(\Omega, \mathcal{F}, P)$ be a prob space. The collection of random variables $\{x(t), t \in T\}$ be fixed on the probability space $(\Omega, \mathcal{F}, P)$ is called a stochastic process.

Parameter Space, T

state space, S

		T: disc	T: cont
S	disc		
	cont		

eg:- Temp at time t in a city

① $\{X(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ cont. state, cont. time stoch. process

$S = \{x \mid a \leq x \leq b\}$

② # of vehicles parked in the main gate upto t

$\{Y(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ $S = \{0, 1, 2, \ldots\}$ disc. state cont. time stoch. process

Time Series :

- Observed information / data over the time.

Sample Path is right continuous.

29/03/2016 LECTURE 29

PROPERTIES OF STOCHASTIC PROCESS

1) Independent Increments $\{X(t), t \geq 0\}$

for arbitrarily $0 < t_0 < t_1 < \ldots < t_n < \ldots$ $\forall n$

of the rvs

$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are mutually independent.

2) Wide sense stationary / Covariance stationary

If (i) $E(X(t))$ is not a function of time t

(ii) $E(X^2(t)) < \infty$

(iii) $Cov(X(t), X(s))$ depends only on $|t-s|$.

then process is wide sense stationary

3) Strict sense stationary

$t_1 < t_2 < t_3 \ldots < t_n$

$$(X(t_1), X(t_2), \ldots, X(t_n)) \stackrel{d}{=} (X(t_1+h), X(t_2+h), \ldots, X(t_n+h)) \quad \forall n \quad \forall h > 0$$

Then it is time variant.

4.) Markov Property

$0 \leq t_0 < t_1 < \ldots < t_n < t$ $\forall n$

If $P\{X(t) \leq x \mid X(t_0) = x_0, X(t_1) = x_1, \ldots, X(t_n) = x_n\} = P\{X(t) \leq x \mid X(t_n) = x_n\}$

Then it is Markov Property.

Ex:- Let $\{X_1, X_2, \ldots\}$ be a sequence of iid r.v.s with common p.m.f's.

$P\{X_i = 0\} = 1 - p = 1 - P\{X_i = 1\}$ $\quad 0 < p < 1$

Define $S_0 = 0$

$$S_n = \sum_{i=1}^{n} X_i$$

$\{S_n, n = 0, 1, 2, \ldots\}$ is a stochastic process

For fixed n, $S_n \sim B(n,p)$

S_n will satisfy only Independent Increment & Markov Property.

$$P\{S_{n+m} = k / S_0 = 0, S_1 = i_1, S_2 = i_2, \ldots S_n = i_n\}$$

$$= \frac{P\{S_{n+m} = k, S_0 = 0, \ldots S_n = i_n\}}{P\{S_0 = 0, S_1 = i_1, \ldots S_n = i_n\}} = \frac{P\{S_{n+m} - S_n = k - i_n,\ S_n - S_{n-1} = i_n - i_{n-1}, \ldots S_1 - S_0 = i_1 - 0\}}{P\{S_n - S_{n-1} = i_n - i_{n-1}, \ldots S_1 - S_0 = i_1 - 0\}}$$

But $S_n - S_{n-1}, \ldots S_1 - S_0$ are independent, terms will cancel out to

$$P\{S_{n+m} = k / S_n = i_n\}$$

$\boxed{S_n = \underbrace{S_{n-1} + X_n}_{\text{independent increment}}}$ First order dependent or Markov process Auto regressive AR(1)

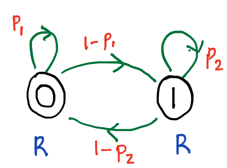
30/03/2016 LECTURE 30

01/04/2016 $\boxed{\text{LECTURE 31}}$

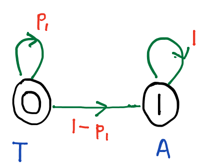
05/04/2016 $\boxed{\text{LECTURE 32}}$

CLASSIFICATION OF STATE

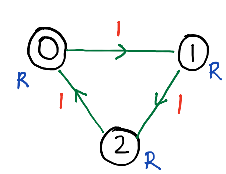
1.)



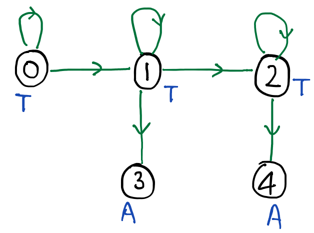
2.)



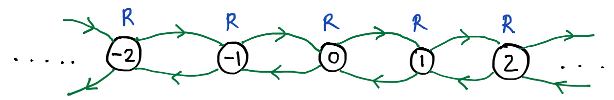
3.)



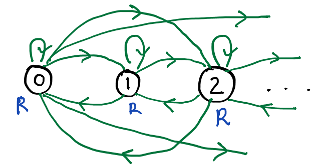
4.)



5.)



6.)



Definitions :

1.) Visit $i \rightarrow j$, if $P_{ij}^{(n)} > 0$ for some n.

2.) Communicate $i \rightarrow j$ if $P_{i,j}^{(m)} > 0$ for some m

& $P_{j,i}^{(m)} > 0$ for some m

3.) Periodicity of state i

$$d_i = g.c.d \{x \geq 1 : P_{ii}^{(n)} > 0\}$$

4.) First visit

$f_{i,j}^{(n)}$ - conditional probability that the system first visit to state j in exactly n steps (with initial state i)

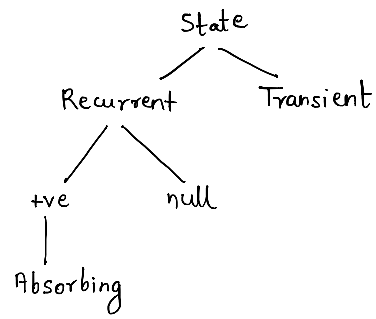
$$P\{X_n = j / X_0 = i, X_k \neq j \quad k = 1, 2, \dots, n-1\}$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$ ever visiting state j starting from state i

$$P_{ij}^{(n)} = \sum_{k} f_{ij}^{(k)} P_{jj}^{(n-k)}$$

5.) Recurrent State

iff $f_{ii} = 1$



6.) Transient State iff $f_{ii} < 1$

7.) Absorbing state iff $P_{ii} = 1$

8.) Mean recurrence time $\mu_i = \sum n f_{ii}$

9.) +ve recurrent state if $\mu_i < \infty$

10.) null recurrent state if $\mu_i = \infty$

06/04/2016 $\boxed{\text{LECTURE 33}}$

Definition : 1.) Closed Communicating Class (C.C.C.)

A set of communicating states, say $C \subseteq S$

If no state outside C can be reached from any state in C, we say the set C is C.C.C.

2.) Aperiodic state

$d_i = 1$

3.) Irreducable

if $S = C_1$ where C_1 is only one C.C.C.

Defn : Stationary Distribution

Let P be the one-step transition probability matrix of a (time-homogeneous) DTMC.

A probability distribution $\{\pi_i\}$ $i \in S$ is said to be stationary distribution (or time invariant) for the given DTMC or steady state distribution or equillibrium distribution

if $\pi_j = \sum_i \pi_i P_{ij}$, $j \in S$ $\quad P = [P_{ij}]_{i,j \in S}$

$$\underbrace{\pi_j}_{1\times n}^{\text{matrix}} = \underbrace{\pi_i}_{1\times n}^{\text{matrix}} \times \underbrace{P_{ij}}_{n\times n}^{\text{matrix}} = 1\times n$$

s.t. $\pi_i \geq 0$ & $\sum_i \pi_i = 1$

Defn : Limiting Distribution

Let $\pi_j = \lim_{n\to\infty} \text{Prob}\{X_n = j\}$, $j \in S$ if it exist

$\{\pi_j\}$, $j \in S$

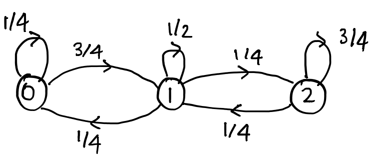
Theorem :

For an irreducable, aperiodic, +ve recurrent DTMC, the limiting distribution exist and is independent of initial distribution.
This is same as the stationary distribution and is given by

$$\pi P = \pi \quad \text{with} \quad \sum_{j\in S} \pi_j = 1$$

eg:-

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 1/4 & 3/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 1/4 & 3/4 \end{array}$$



$S = \{0, 1, 2\} = C_1 \Rightarrow$ Irreducable

$$f_{00} = f_{00}^{(1)} + f_{00}^{(2)} + \dots$$
$$= 1/4 + \frac{3}{4}\times\frac{1}{4} + \frac{3}{4}\times\frac{1}{2}\times\frac{1}{4} + \dots\dots$$
$$= 1$$

$f_{11} = 1$ & $f_{22} = 1$

Check for aperiodic, +ve recurrent, irreducable to apply the theorem.

$$\Rightarrow \quad (\pi_0 \;\; \pi_1 \;\; \pi_2) = (\pi_0 \;\; \pi_1 \;\; \pi_2)\begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \quad \& \quad \sum \pi_j = 1$$

Limiting dist. same as stationary distribution

$$\pi_0 = \frac{1}{4}(\pi_0 + \pi_1) \quad \text{—①}$$
$$\pi_1 = \frac{3}{4}\pi_0 + \frac{\pi_1}{2} + \frac{\pi_2}{4} \quad \text{—②}$$
$$\pi_2 = \frac{\pi_1}{4} + \frac{3\pi_2}{4} \quad \text{—③}$$

$\Rightarrow \quad \pi_0 = \frac{\pi_1}{3}$, $\pi_1 = \pi_2$

$\frac{\pi_1}{3} + \pi_1 + \pi_1 = 1 \Rightarrow \pi_1 = \frac{3}{7} = \pi_2$, $\pi_0 = \frac{1}{7}$

$(1/7, 3/7, 3/7)$ //

08/04/2016 $\boxed{\text{LECTURE 34}}$

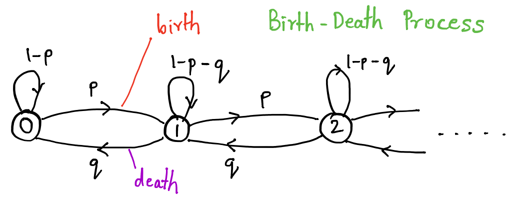
FEW IMPORTANT RESULTS

1.) For a finite state space DTMC, if the state is recurrent, then it has to be +ve recurrent.

2.) In an irreducible MC, all the states are either +ve recurrent or null recurrent. All states are having the same period.

3.) For an irreducible finite state space DTMC, all the states are +ve recurrent.

4.) For an irreducible aperiodic DTMC, if the states are null recurrent, then $\pi_j = 0 \ , j \in S$

eg:- 

$\{X_n : n = 0, 1, 2, \dots\}$ with $S = \{0, 1, 2, \dots\}$

<u>Case (i)</u> $q = 0, \ p > 0$

$T = \{0, 1, 2, \dots\}$

$$\pi_j = \lim_{n \to \infty} \text{Prob}\{X_n = j\} \ , j = 0, 1, 2, \dots$$
$$= 0$$

<u>Case (ii)</u> $p = 0, \ q > 0$

$C_1 = \{0\} \ , T = \{1, 2, \dots\}$

$\pi_0 = 1 \ , \ \pi_j = 0 \ , j \in S \setminus \{0\}$

<u>Case (iii)</u> $0 < p, q < 1$

$C = \{0, 1, 2, \dots\} = S$

irreducible, recurrent, aperiodic

It can be +ve/null recurrent

Assume that all states are +ve recurrent.

$\Rightarrow$ Stationary dist. exist

$$\pi = \pi P \quad \& \quad \sum_{i \in S} \pi_i = 1$$

$$(\pi_0 \ \pi_1 \ \dots) = (\pi_0 \ \pi_1 \ \dots) \begin{pmatrix} 1-p & p & 0 & \dots\dots \\ q & 1-p-q & p & \dots\dots \\ 0 & q & 1-p-q & p \ \dots \\ & & \vdots & \end{pmatrix}$$

$$\pi_0 = (1-p)\pi_0 + q\pi_1 \quad \Rightarrow \quad \pi_1 = \frac{p}{q}\pi_0$$

$$\pi_1 = p\pi_0 + (1-p-q)\pi_1 + q\pi_2 \quad \Rightarrow \quad \pi_2 = \frac{p}{q}\pi_1 = \frac{p^2}{q^2}\pi_0$$

$\vdots$

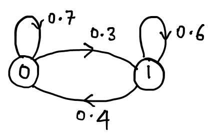
$|||^{ly}$ $\pi_n = \frac{p}{q}\pi_{n-1} = \dots = \left[\frac{p}{q}\right]^n \pi_0 \quad \Rightarrow$

Now, $\sum \pi_i = 1$

For this to occur, $\frac{p}{q} < 1$

or $\boxed{p < q}$ Condition for +ve recurrent

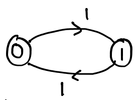
$$\& \quad \pi_0 \times \frac{1}{1 - \frac{p}{q}} = 1 \quad \text{or} \quad \pi_0 = \frac{q-p}{q}$$

eg:- ① 

Limiting Dist: $(4/7, 3/7)$
Stationary Dist: $(4/7, 3/7)$

$C = \{0, 1\} = S$ Irreducable
Aperiodic
$\Rightarrow$ Stationary = Limiting
Finite DTMC recurrent $\Rightarrow$ +ve recurrent

$$(\pi_0 \quad \pi_1) = (\pi_0 \quad \pi_1)\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \Rightarrow \pi_0 = \frac{4}{7}, \pi_1 = \frac{3}{7}$$

② 

Not aperiodic
$\Rightarrow$ Theorem won't hold

Limiting Dist: D.N.E. (Check $P, P^2, \ldots$)
Stationary Dist: $(1/2, 1/2)$

③

Remarks : When the stationary distribution exists,

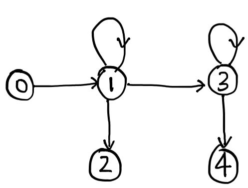
1.) $X_n \xrightarrow{d} \pi$

2.) $\pi P^n = \pi P \cdot P^{n-1} = \pi \cdot P^{n-1} = \ldots = \pi \quad (n \geq 2)$

3.) When $\sum_i P_{ij} = 1$, $P = [P_{ij}]$ is doubly stochastic matrix.
When the DTMC is irreducable finite state space & +ve recurrent & P is doubly stochastic, then
$\pi_j = \frac{1}{N}$, $j \in S$ π is uniformly distributed.

12/04/2016 $\boxed{\text{LECTURE 35}}$

REDUCIBLE MARKOV CHAIN

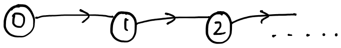
① 

$S = T \cup C_1 \cup C_2$
$T = \{0, 1, 3\}$, $C_1 = \{2\}$, $C_2 = \{4\}$

② 

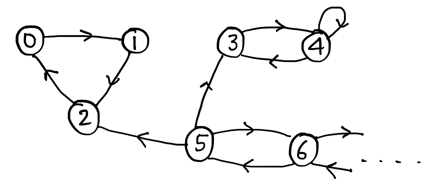
$S = T \cup C_1$
$T = \{1, 2, \ldots n\}$, $C_1 = \{0\}$

3



$S = \underbrace{T}_{\{0,1,2,\dots\}}$

4



$S = T \cup C_1 \cup C_2$, $T = \{5,6,7,\dots\}$, $C_1 = \{0,1,2\}$, $C_2 = \{3,4\}$

FINITE STATE SPACE (say n) REDUCIBLE MARKOV CHAIN

$|S| = n$

$$P = \begin{array}{c} \\ \text{Absorbing or Recurrent} \\ \text{Transient} \end{array} \begin{array}{c} \begin{array}{cc} \text{Ab/Rec} & \text{Transient} \end{array} \\ \begin{pmatrix} R_{r\times r} & O_{r\times n-r} \\ A_{n-r\times r} & B_{n-r\times n-r} \end{pmatrix} \end{array}$$

If all recurrent states are absorbing state

$R_r = I_r$

$$S = \underbrace{C_1 \cup C_2 \cup \dots \dots C_k}_{\text{Recurrent state}} \cup T$$

Define $M = (I-B)^{-1}$ (fundamental matrix)

$= I + B + B^2 + B^3 + \dots$

Theorem : Define

1.) μ_{ij} - Mean # of visits of the system to the state j before reaching an absorbing state given that $X_0 = i$, $i \in T$, $j \in T$

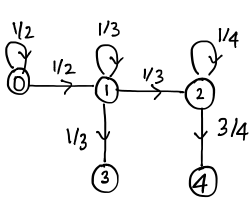
2.) g_{ij} - Conditional property that the system in absorbing state j given that $X_0 = i$, $i \in T$, $j \in S \setminus T$

Then

$$G = (g_{ij}) = (I-B)^{-1} A$$

$$M = (\mu_{ij}) = (I-B)^{-1}$$

Verification :



$$P = \begin{array}{c|c} & \begin{array}{ccccc} 3 & 4 & 0 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 3/4 & 0 & 0 & 1/4 \end{bmatrix}_{5\times 5} \end{array}$$

$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix}$ $B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 \end{bmatrix}$

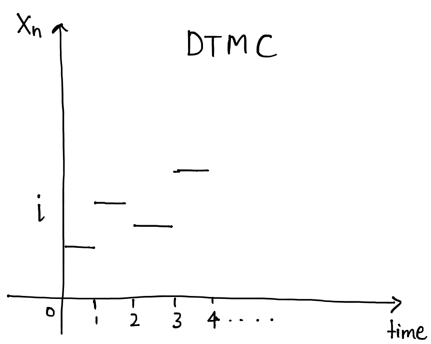
$$I - B = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 2/3 & -1/3 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$$(I-B)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 3/8 & 3/8 & 0 \\ 1/6 & 1/6 & 1/3 \end{bmatrix}^T * 4 = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \end{array}$$

Similarly solve for M.

→ CTMC

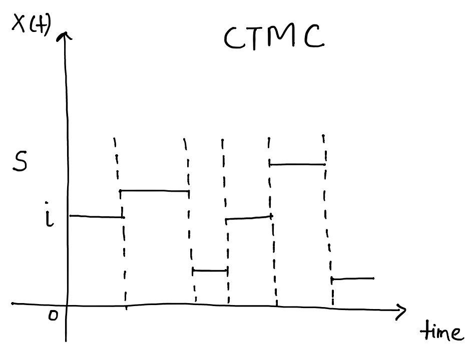


$\{X_n, n = 0, 1, \ldots\}$

Given $\pi(0)$ – initial dist.

$\pi(0) = (\pi_0(0) \quad \pi_1(0) \ \ldots)$

$P = [P_{ij}]$



$\{X(t), t \geq 0\}$

$\pi(0) = (\pi_0(0) \quad \pi_1(0) \ \ldots)$

$Q = [q_{ij}] \quad i \in S$

$\uparrow$ rate

13/04/2016 $\boxed{\text{LECTURE 36}}$

Definitions :

1.) Initial Distribution

$\pi(0) = [\pi_0(0) \quad \pi_1(0) \ \ldots]$ where $\pi_i(t) = \text{Prob}\{X(t) = i\}, t \geq 0$

2.) Transmission Probability

$P_{ij}(s,t) = \text{Prob}\{X(t+s) = j \,/\, X(s) = i\} \quad i,j \in S$

Since system is time homogeneous,

$P_{ij}(t) = \text{Prob}\{X(t) = j \,/\, X(0) = i\} \quad i,j \in S$

$= \text{Prob}\{X(t+s) = j \,/\, X(s) = i\} \quad \forall s > 0$

3.) Transmission Probability Matrix

$P(t) = [P_{ij}(t)] \ , \ t \geq 0$

$$\pi_j(t) = \text{Prob}\{X(t)=j\} = \sum_{i\in S} \pi_i(0)\, P_{ij}(t)$$

$\pi_j(t)$ ↓ unconditional $\qquad$ $P_{ij}(t)$ ↓ conditional

$$\boxed{\pi(t) = \pi(0)\, P(t)}$$

4.) Generator Matrix

$$Q = [q_{ij}]$$

for $i \neq j$ $\qquad q_{ij} = \frac{d}{dt} P_{ij}(t) \Big|_{t=0}$

$i = j$ $\qquad q_{ii} = -\sum_{i\neq j} q_{ij}$

$P_{ij}(\Delta t) = q_{ij}\,\Delta t + o(\Delta t) \quad , \; j \neq i$

$P_{ii}(\Delta t) = 1 + q_{ii}\,\Delta t + o(\Delta t) \quad , \; j = i$

q_{ij} satisfies

(1) $q_{ij} \geq 0$

(2) $q_{ii} \leq 0$

(3) $\sum_j q_{ij} = 0$

We know that

$$\boxed{P_{ij}(t+T) = \sum_{k\in S} P_{ik}(t) \cdot P_{kj}(T)}$$

CHAPMAN KOLMOGOROV EQUATION

Differenciate w.r.t. T & put T=0

$$P_{ij}'(t) = \sum_{k\in S} P_{ik}(t)\, q_{kj}$$

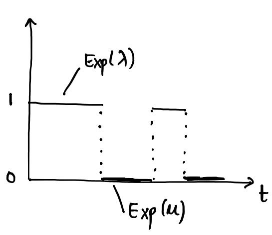
$$\Rightarrow \boxed{P'(t) = P(t) \cdot Q}$$

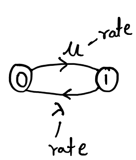
KOLMOGOROV FORWARD Eq^n

Since it is a differential equation, initial cond^n $P(0)$ is required.

$$\pi'(t) = \pi(t)\, Q$$

eg:-





$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \end{array} \qquad \pi(0) = [0 \quad 1]$$

$\pi'(t) = \pi(t) \cdot Q \qquad \pi_0(t) + \pi_1(t) = 1$

$\Rightarrow \pi_0'(t) = -\mu\pi_0(t) + \lambda\pi_1(t)$

$\Rightarrow \pi_0'(t) = (-\mu - \lambda)\pi_0(t) + \lambda$

$$\Rightarrow \frac{\pi_0'(t)}{\pi_0(t) - \frac{\lambda}{\mu+\lambda}} = -(\mu+\lambda)$$

$$\Rightarrow \ln\left(\pi_0(t) - \frac{\lambda}{\mu+\lambda}\right) + c = -(\mu+\lambda)t$$

$$\Rightarrow \pi_0(t) = e^{-(\mu+\lambda)t} \cdot c' + \frac{\lambda}{\mu+\lambda} \;\Rightarrow\; \pi_0(t) = \frac{\lambda}{\mu+\lambda}\left[1 - e^{-(\mu+\lambda)t}\right]$$

At $t=0$, $\pi_0(0) = 0 \;\Rightarrow\; c' = -\frac{\lambda}{\mu+\lambda}$

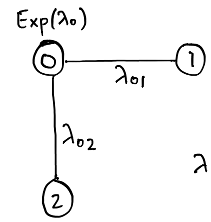
19/04/2016 $\boxed{\text{LECTURE 37}}$

Theorem: Time spent in any state i before moving to another state for (time homogeneous) CTMC is always independent exponential distribution with parameter λ_i, $i \in S$

Proof :

τ - r.v.

eg:-



$\lambda_0 = \lambda_{01} + \lambda_{02}$

$$P(\tau > s+t \,/\, X(0)=i)$$
$$= P(\tau > s+t \,/\, X(s)=i) \,.\, P(\tau > s \,/\, X(0)=i)$$
$$= P(\tau > t \,/\, X(0)=i) \,.\, P(\tau > s \,/\, X(0)=i)$$

$$F_\tau^c(s+t) = F_\tau^c(t) \,.\, F_\tau^c(s) \quad , \; s,t > 0$$

$$\Rightarrow \quad e^{-\lambda_i(s+t)} = e^{-\lambda_i s} \,.\, e^{-\lambda_i t}$$

$$\Rightarrow \quad F_\tau(s) = 1 - e^{-\lambda_i s} \quad \Rightarrow \quad \tau \sim Exp(\lambda_i)$$
$$s > 0$$

eg:- $\{X(t),\, t \geq 0\}$, $S = \{0,1,2,\ldots\}$ $\qquad \pi_0(0) = 1$

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots \qquad \pi'(t) = \pi(t) \,.\, Q$$

$$\pi_j(t) = \text{Prob}\{X(t) = j\}$$

$$\Rightarrow \quad (\pi_0'(t) \;\; \pi_1'(t) \;\; \pi_2'(t) \;\ldots) = (\pi_0(t) \;\; \pi_1(t) \;\; \pi_2(t) \;\ldots) \begin{array}{c|cccc} & 0 & 1 & 2 & \ldots \\ \hline 0 & -\lambda & \lambda & 0 & \ldots \\ 1 & 0 & -\lambda & \lambda & \ldots \\ 2 & 0 & 0 & -\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{array}$$

$$\pi_0'(t) = -\lambda \pi_0(t)$$

$$\ln(c\,\pi_0(t)) = -\lambda t$$

At $t = 0$, $\pi_0(0) = 1 \Rightarrow c = 1 \Rightarrow \pi_0(t) = e^{-\lambda t}$

$$\pi_1'(t) = \lambda \pi_0(t) - \lambda \pi_1(t) \quad \Rightarrow \quad \pi_1(t) = \lambda t e^{-\lambda t}$$

$$\vdots$$

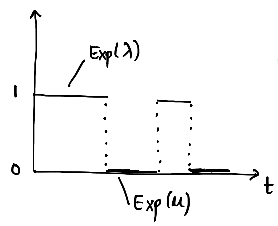
$$\pi_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad , \; n = 1, 2, \ldots$$

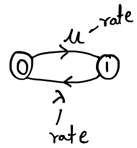
For fixed t, the dist of $X(t)$ is Poisson with parameter λt

$\{X(t),\, t \geq 0\} \longrightarrow$ Poisson Process (C.T.M.C.)

Consider the previous example

eg:-





$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \end{array} \qquad \pi(0) = [0 \quad 1]$$

$$\pi'(t) = \pi(t) \cdot Q \qquad \pi_0(t) + \pi_1(t) = 1$$

$$\Rightarrow \pi_0'(t) = -\mu \pi_0(t) + \lambda \pi_1(t)$$

$$\Rightarrow \pi_0'(t) = (-\mu - \lambda)\pi_0(t) + \lambda$$

$$\Rightarrow \frac{\pi_0'(t)}{\pi_0(t) - \frac{\lambda}{\mu+\lambda}} = -(\mu+\lambda)$$

$$\Rightarrow \pi_0(t) = e^{-(\mu+\lambda)t} \cdot c' + \frac{\lambda}{\mu+\lambda} \Rightarrow \pi_0(t) = \frac{\lambda}{\mu+\lambda}\left[1 - e^{-(\mu+\lambda)t}\right]$$

$$\Rightarrow \ln\left(\pi_0(t) - \frac{\lambda}{\mu+\lambda}\right) + c = -(\mu+\lambda)t$$

At $t=0$, $\pi_0(0) = 0 \Rightarrow c' = -\frac{\lambda}{\mu+\lambda}$

Availability $= \pi_1(t)$

Steady state availability $\underset{t\to\infty}{=} \pi_1 = \frac{\mu}{\lambda+\mu} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{\text{Mean Failure Time}}{\text{Mean F Time} + \text{Mean Repair Time}}$

22/04/2016 $\boxed{\text{LECTURE 38}}$

BIRTH DEATH PROCESS

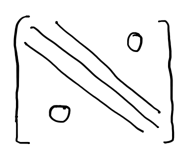
A B.D.P. is a CTMC $\{X(t), t \geq 0\}$ with a state space S $(= \{0,1,2,\dots\})$ s.t.

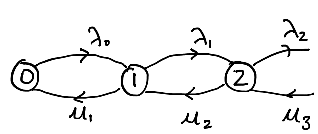
$$q_{i,i+1} = \lambda_i, \quad i = 0,1,2,\dots$$

$$q_{i,i-1} = \mu_i, \quad i = 1,2,\dots$$

$$q_{i,j} = 0 \quad |i-j| > 1$$

$$q_{i,i} = \begin{cases} -(\lambda_i + \mu_i) & i = 1,2,\dots \\ -\lambda_i & i = 0 \end{cases}$$

$Q =$ 



$\lambda_i \longrightarrow$ birth rate

$\mu_i \longrightarrow$ death rate

eg:- ① $\{X(t), t \geq 0\}$ — P.P.(λ) is a BDP $\begin{pmatrix} \lambda_i = \lambda & i = 0,1,2,\dots \\ \mu_i = 0 & i = 1,2,\dots \end{pmatrix}$

Poison Process

Pure birth process

② Pure death process

$\lambda_i = 0 \quad , \ i = 0,1,2,\dots n$

$\mu_i = \alpha \quad , \ i = 1,2,\dots n$

It should be finite (Has to start from somewhere)

→ Steady state Probability

$\pi'(t) = \pi(t) . Q$ F.K.E.

For steady state, $\pi'(t) = 0$ & $\pi(t) = \pi$

$\Rightarrow \quad 0 = \pi . Q \quad ; \quad \sum_{i \in S} \pi_i = 1$

$\pi = (\pi_i)_{i \in S}$

$$\Rightarrow \quad 0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

$$0 = \lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2$$

$$\vdots$$

$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$

$\pi_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} \pi_0$

$\vdots$

$\pi_n = \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n . \mu_{n-1} . \dots \mu_1} \pi_0$

$\pi_1, \pi_2, \dots$ in terms of π_0

$\Sigma \pi_i = 1$ Solve for π_0

$$\pi_0 \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} + \dots \right) = 1 \quad \Rightarrow \quad \pi_0 = \frac{1}{1 + \dots} > 0$$

Special case:

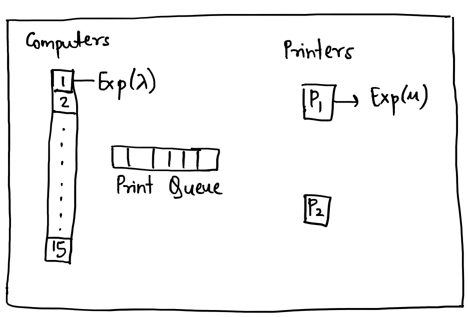
$\lambda_i \to \lambda$

$\mu_i \to \mu$

Condition for steady state to exist:

$\frac{\lambda}{\mu} < 1$ or $\lambda < \mu$ (common ratio of G.P.)

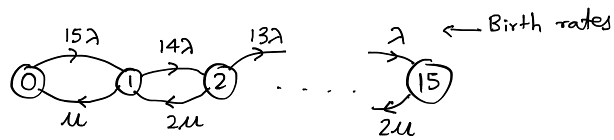
eg:-



$\{X(t), t \geq 0\}$

# of print tasks in system at time t

$S = \{0, 1, \dots, 15\}$



Since there are finite terms, steady state will always exist.

Probability that system is always busy $= \sum_{i=2}^{15} \pi_i$

26/04/2016 **LECTURE 39**

QUEUEING MODELS

Parameters

1.) Inter arrival — deterministic
— probabilistic

2.) Service

3.) # of servers

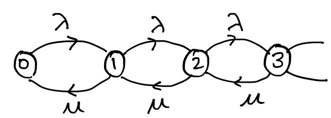
4.) Capacity

$X(t)$ = no. of people in system (queue + service)

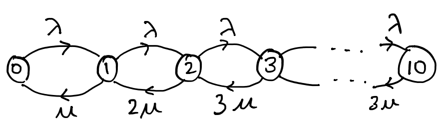
eg:- 1.) $M/M/1/\infty$ — infinite capacity

$M \to Exp(\lambda)$, $M \to Exp(\mu)$, $1 \to$ one server

For this to be B.D.P., at a small interval of time, maximum of 1 person can enter / leave the system

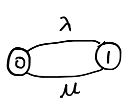


2.) $M/M/3/10$



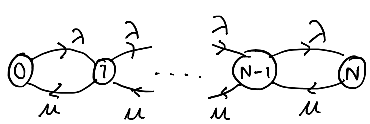
27/04/2016 **LECTURE 40**

3.)



$\Rightarrow M/M/1/1$

4.)



$\Rightarrow M/M/1/N$

Note: For a B.D.P.,

$$\boxed{\lambda \cdot E(R) = E(N)}$$ Little's Formula

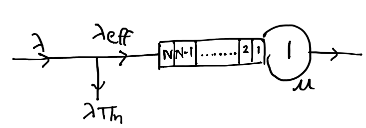
λ — mean arrival rate; $E(R)$ — mean spending time; $E(N)$ — mean # of customers

$\Rightarrow$ Mean spending time in queue $= E(Q) = E(R) - \frac{1}{\mu}$

$\frac{1}{\mu}$ — mean service time

☆ For a N capacity system, at capacity N people are not allowed to enter the system.

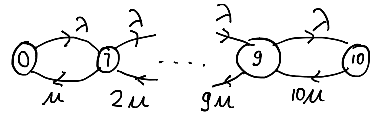
$\Rightarrow$ Effective Arrival Rate $\lambda_{eff} = \lambda(1 - \pi_n)$



eg:- What is average queue & spending time for M/M/10/10

Sol^n: $E(Q) = 0$

$E(R) = \frac{1}{\mu}$

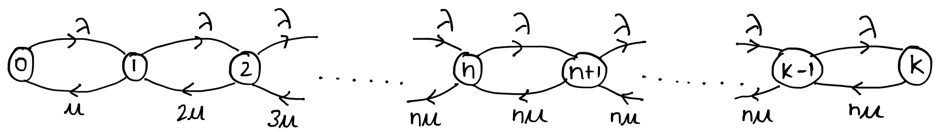


$\pi' = \pi Q = 0$

29/04/2016 $\boxed{\text{LECTURE 41}}$

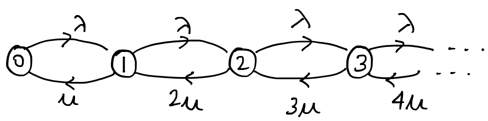
M/M/n/k

$k > n$



$$\pi_n = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\Pi\lambda_{k-1}}{\Pi\mu_k}} \times \frac{\Pi\lambda_{n-1}}{\Pi\mu_n}$$

M/M/∞



$$\pi_0 = \frac{1}{1 + \rho + \frac{\rho^2}{2!} + \cdots} = e^{-\rho} \Rightarrow \pi_n = \frac{e^{-\rho} \times \rho^n}{n!}$$

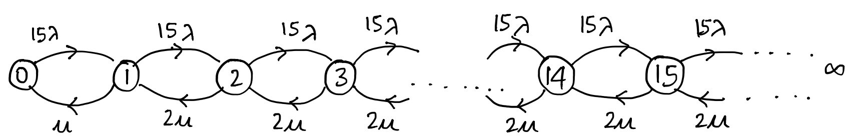
$\Rightarrow$ Poisson distribution with parameter ρ

$\pi_n \sim P(\rho)$

M/M/2/∞ with population size 15

$\sim$ M/M/2/∞ (M → Exp(15λ))

M → Exp(λ) of individual P.C.

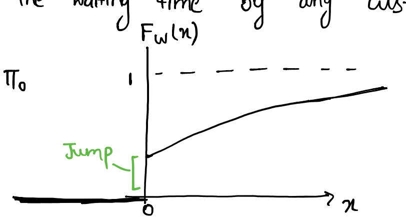


15 Printers each giving print command with rate 15 with any P.C. can give subsequently multiple print command.

M/M/1/∞

Let W be the waiting time by any customer. $\Rightarrow$ Mixed Type R.V. (Jump at 0 when no one in service)

$P\{W=0\} = \pi_0$



$P\{0 < W \leq t\}$

$= \sum_{n=1}^{\infty} P\{0 < W \leq t / N=n\} . P\{N=n\}$ → Erlang distribution

$= \rho\,(1 - e^{-(\mu-\lambda)t})$