Chapter 12

Analysis of Variance (ANOVA)

Detailed Solutions

12.1 Basic Concept

12.1 Why not multiple T-Tests?

Problem: Comparing $k = 5$ means.
Solution:

(a) **Type I Error Inflation:** If we perform $\binom{5}{2} = 10$ independent t-tests, each with $\alpha = 0.05$. The probability of making **at least one** Type I error (Family-wise Error Rate) is:

$$P(\text{At least 1 error}) = 1 - (1 - \alpha)^{10} = 1 - (0.95)^{10} \approx 1 - 0.5987 = 0.4013$$

The error rate inflates from 5% to **40%**. This is unacceptably high.

(b) **ANOVA Solution:** ANOVA performs a single global F-test to check if **any** of the means differ. It controls the overall Type I error at exactly $\alpha = 0.05$. If (and only if) the F-test is significant, we proceed to post-hoc tests (like Tukey's) which are designed to control the error rate for multiple comparisons.

(a) Error inflates to $\approx 40\%$ (b) Single F-test controls overall α.

12.2 Assumptions and Decomposition

Problem: Assumptions and $SST = SSTR + SSE$.
Solution:

(a) **Assumptions:**

(a) **Normality:** The residuals (errors) are normally distributed.

(b) **Homoscedasticity:** The variance of the errors is constant across all treatment groups ($\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$).

(c) **Independence:** The observations are independent of each other.

(b) **Decomposition:**

- **SST (Total):** Total variation in the data (Total "Noise" if no model was used).
- **SSTR (Treatments):** Variation explained by the difference in group means ("Signal").
- **SSE (Error):** Variation within the groups that cannot be explained by the treatments ("Random Noise").

Identity: Total Variation = Explained Variation + Unexplained Variation.

(a) Normality, Equal Variance, Independence (b) Total = Signal (Between) + Noise (Within)

12.2 Intermediate

12.3 Fill in the Blanks (One-Way ANOVA)

Problem: $k = 4, n = 5, N = 20.$
Solution:

(a) **Calculations:**

- **A (DF Treatments):** $k - 1 = 4 - 1 = 3.$
- **B (DF Error):** $N - k = 20 - 4 = 16.$
- **C (SS Error):** $SST - SSTR = 230 - 150 = 80.$
- **D (MS Treatments):** $SSTR/DF_{Trt} = 150/3 = 50.$
- **Verify MS Error:** $SSE/DF_{Err} = 80/16 = 5.$ (Matches given).
- **F (F-statistic):** $MSTR/MSE = 50/5 = 10.$

(b) **Conclusion:** Given P-value < 0.05 (or $F = 10$ which is likely $> F_{crit}$), there is a **significant difference** between the process means.

A=3, B=16, C=80, D=50, F=10. Significant Difference.

12.4 Randomized Block Design (Tire Wear)

Problem: 3 Tires (A,B,C), 4 Cars (Blocks).
Solution:

(a) **Sum of Squares:** Grand Mean $\bar{y}_{..} = \frac{132}{12} = 11.$

- **SST (Total):**

$$\sum (y_{ij} - 11)^2 = (8 - 11)^2 + \cdots + (11 - 11)^2 = 126.$$

- **SSTR (Tire):**
  Tire means:

$$A = 11, \quad B = 8.25, \quad C = 13.75.$$

$$\begin{aligned} SSTR &= 4\big[(11-11)^2 + (8.25-11)^2 + (13.75-11)^2\big] \\ &= 4\big[0 + 7.5625 + 7.5625\big] \\ &= 60.5. \end{aligned}$$

- **SSB (Car):**
  Car means:

$$\text{Sedan} = 7.67,\ \text{SUV} = 11.67,\ \text{Truck} = 14.67,\ \text{Sports} = 9.00.$$

$$SSB = 3\big[(7.67 - 11)^2 + \cdots\big] \approx 60.67.$$

- **SSE (Error):**

$$SSE = SST - SSTR - SSB = 126 - 60.5 - 60.67 = 4.83.$$

(b) **ANOVA Table:**

Source	DF	SS	MS	F
Tire	2	60.5	30.25	37.58
Car	3	60.67	20.22	25.12
Error	6	4.83	0.805	
Total	11	126		

(c) **Test Tires:** $F_{Tire} = 37.58$. $F_{crit}(2, 6) = 5.14$. $37.58 > 5.14 \implies$ **Significant Difference**.

(d) **Importance of Blocking:**

With blocking, the error mean square is very small:

$$MS_{\text{Error}} = 0.805.$$

Without blocking, the variation due to *Car* would be absorbed into the error term. Thus,

$$SS_{\text{Error,new}} = SSE + SSB = 4.83 + 60.67 = 65.5, \quad DF = 9,$$

and

$$MS_{\text{Error,new}} = \frac{65.5}{9} = 7.28.$$

As a result, the F-statistic for Tire becomes

$$F = \frac{30.25}{7.28} = 4.15,$$

which is **not significant**.

Conclusion: Blocking removes the large variability due to Car types, substantially reducing experimental noise and revealing the true effect of Tire.

(c) Significant ($F = 37.58$).
(d) Blocking reduced MSE from 7.28 to 0.8, increasing Power.

12.5 Two-Way ANOVA (Interaction)

Problem: Table Analysis.
Solution:

(a) **Critical Value:** $F_{0.05,1,16} \approx 4.49$.

(b) **Interaction Test:** $F_{AxB} = 4.0$. Since $4.0 < 4.49$, the Interaction is **Not Significant**. The effect of Material does not depend on Temperature.

(c) **Main Effects:** $F_{Mat} = 10.0 > 4.49$ (Significant). $F_{Temp} = 9.0 > 4.49$ (Significant). Both main factors affect battery life independently.

(d) **Interaction Plot:** If lines cross (e.g., M1 is better at Low Temp, but M2 is better at High Temp), it implies a significant interaction. Since our interaction is not significant, the lines should be roughly parallel.

(b) Interaction Not Significant. (c) Main Effects Significant.

12.6 Visual Interpretation (Engine Efficiency)

Problem: Boxplot and ANOVA Table.
Solution:

(a) **Visual:** Engine B is clearly higher than A and C. The separation is large compared to the box width (variance). Expect a **Large F-statistic**.

(b) **Table Completion:**

- **A (DF Between):** $k - 1 = 3 - 1 = 2$.
- **B (DF Error):** $Total - Between = 17 - 2 = 15$.
- **C (MS Between):** $350/2 = 175$.
- **D (MS Error):** $75/15 = 5$.
- **E (F-stat):** $175/5 = 35$.

(c) **Decision:** $F = 35$. Critical $F \approx 3.68$. $35 \gg 3.68$. **Highly Significant**. Matches visual intuition perfectly.

A=2, B=15, C=175, D=5, E=35. Significant.

12.7 Missing Values: Randomized Block

Problem: 4 Fertilizers, 5 Soils.
Solution:

(a) **DFs:**

- A (Fert): $4 - 1 = 3$.
- B (Soil): $5 - 1 = 4$.
- Total DF $= 20 - 1 = 19$.
- C (Error): $19 - 3 - 4 = 12$.

(b) **SSE (E):** $360 = 180 + 120 + SSE \implies SSE = 60$.

(c) **MS and F:**

- D (MS Fert): $180/3 = 60$.
- MS Error check: $60/12 = 5$. (Matches given).
- F (F-stat): $60/5 = 12$.

(d) **Decision:** $12 > 3.49$. **Significant**. Fertilizer affects yield.

A=3, B=4, C=12.
E=60.
D=60, F=12.
Significant.

12.3 Applications

12.8 Visualizing ANOVA

Problem: Python Simulation.
Solution:

Answers:

(a) **Boxplot:** Method C's box should be shifted significantly higher (Mean $\approx$ 91) compared to A ($\approx$ 84) and B ($\approx$ 77). There is little to no overlap.

(b) **P-value:** Extremely small (e.g., 10^{-10}) confirms that the observed differences are real and not due to chance. H_0 is rejected.

(c) **Post-hoc:** ANOVA only tells us "at least one differs". We need **Tukey's HSD** (Honestly Significant Difference) test to perform pairwise comparisons (A vs B, B vs C, A vs C) while maintaining the family-wise error rate at 0.05.

12.9 Interpreting Excel Output

Problem: Marketing Strategy A, B, C.
Solution:

(a) **Manual Calculation:** $F = MS_{Between}/MS_{Within} = 52.50/4.00 = 13.125$. Matches Excel exactly.

(b) **Hypothesis Test:**

- $H_0 : \mu_A = \mu_B = \mu_C$. H_1: At least one differs.
- P-value $= 0.00011 < 0.05$. **Reject** H_0.
- $F = 13.125 > F_{crit} = 3.354$. **Reject** H_0. Both methods agree.

(c) **Variation:**

- Caused by Strategy: $SS_{Between}$ (105.0).
- Random Noise: SS_{Within} (108.0).

(d) **Conclusion:** Strategy B has the highest average (18.0). Since the ANOVA is significant, Strategy B likely outperforms at least one of the others (probably C). We trust this result.

(a) 13.125 (Matches)
(b) Reject H_0.
(d) Strategy B is best.

12.10 Two-Way ANOVA Output

Problem: Glass Type x Temperature.
Solution:

(a) **Interaction Check:** P-value for Interaction is 0.4895. Since $0.48 > 0.05$, the Interaction is **NOT significant**. This implies Glass Type and Temperature act independently. (e.g., Higher temp increases light output similarly for both glass types).

(b) **Main Effects:** Glass Type: $P = 0.0001 < 0.05$ (Significant). Temperature: $P = 0.0012 < 0.05$ (Significant).

(c) **Decision Strategy: Yes**. Because the interaction is not significant, we can interpret the main effects directly and perform pairwise comparisons on the main factors without worrying about the other factor.

$$\boxed{\begin{array}{l}\text{(a) No Interaction } (P \approx 0.49). \\ \text{(b) Main Effects Significant.} \\ \text{(c) Yes, safe to interpret.}\end{array}}$$