# Chapter 17

# Likelihood Ratio Tests (LRT)

**Detailed Solutions**

## 17.1 Intermediate (Neyman-Pearson Lemma)

### 17.1 Neyman-Pearson for Exponential Distribution

**Problem:** $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$.
***Solution:***

(a) **Likelihood Ratio** $\Lambda$**:** The likelihood function for $n$ i.i.d. samples is $L(\theta) = \theta^n e^{-\theta \sum x_i}$.

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\theta_0^n e^{-\theta_0 \sum x_i}}{\theta_1^n e^{-\theta_1 \sum x_i}}$$
$$= \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left(-(\theta_0 - \theta_1)\sum_{i=1}^{n} x_i\right)$$

(b) **Derivation:** We reject $H_0$ if $\Lambda < k$. Taking the natural log:

$$\ln \Lambda < \ln k$$
$$n \ln\left(\frac{\theta_0}{\theta_1}\right) - (\theta_0 - \theta_1)\sum x_i < \ln k$$
$$-(\theta_0 - \theta_1)\sum x_i < \ln k - n \ln\left(\frac{\theta_0}{\theta_1}\right)$$

Since $\theta_1 > \theta_0$, the term $(\theta_0 - \theta_1)$ is **negative**. Let $A = \theta_0 - \theta_1 < 0$.

$$-A \sum x_i < C' \quad \text{(Note: } -A \text{ is positive)}$$

Wait, let's substitute directly:

$$(\theta_1 - \theta_0)\sum x_i < \text{Constant}'$$

Actually, let's rearrange carefully:

$$\exp\left((\theta_1 - \theta_0)\sum x_i\right) < k\left(\frac{\theta_1}{\theta_0}\right)^n$$
$$(\theta_1 - \theta_0)\sum x_i < \ln(K')$$

Since $\theta_1 > \theta_0$, the coefficient $(\theta_1 - \theta_0)$ is positive. Thus, we divide by a positive number, preserving the inequality sign.

$$\sum x_i < \frac{\ln(K')}{\theta_1 - \theta_0} = c$$

(c) **Result:** The rejection region is $\sum_{i=1}^{n} X_i < c$. *Intuition:* For exponential distributions, a smaller sum implies a larger rate $\theta$. Thus, small observed values support $H_1$ $(\theta_1 > \theta_0)$.

$\boxed{\text{Reject } H_0 \text{ if } \sum X_i < c.}$

## 17.2 Neyman-Pearson for Power Function

**Problem:** $f(x) = \theta x^{\theta-1}$. Test $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ $(n = 1)$.
***Solution:***

(a) **Likelihood Ratio:**

$$\begin{aligned} L(1) &= 1 \cdot x^{1-1} = 1 \\ L(2) &= 2 \cdot x^{2-1} = 2x \\ \Lambda(x) &= \frac{1}{2x} \end{aligned}$$

(b) **Rejection Region:** Reject $H_0$ if $\Lambda(x) < k$:

$$\begin{aligned} \frac{1}{2x} &< k \\ 2x &> \frac{1}{k} \\ x &> c \end{aligned}$$

Find $c$ for size $\alpha = 0.05$ under $H_0$ $(\theta = 1, f(x) = 1)$:

$$P(X > c|\theta = 1) = \int_c^1 1\,dx = 1 - c$$
$$1 - c = 0.05 \implies c = 0.95$$

Thus, reject if $x > 0.95$. (Note: If the prompt suggested $\sqrt{0.95}$, it would correspond to $H_0 : \theta = 2$. For $H_0 : \theta = 1$, the answer is 0.95).

(c) **Power $(1 - \beta)$:** Calculate probability of rejection under $H_1$ $(\theta = 2, f(x) = 2x)$:

$$\begin{aligned} \text{Power} &= P(X > 0.95|\theta = 2) \\ &= \int_{0.95}^1 2x\,dx = \left[x^2\right]_{0.95}^1 \\ &= 1^2 - (0.95)^2 = 1 - 0.9025 = 0.0975 \end{aligned}$$

(a) $\Lambda = 1/(2x)$
(b) Reject if $x > 0.95$
(c) Power $= 0.0975$

## 17.3 Neyman-Pearson for Normal Mean

**Problem:** $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ ($\mu_1 > \mu_0$), known $\sigma^2$.
***Solution:***

$$\begin{aligned}\Lambda &= \frac{\exp\left(-\frac{1}{2\sigma^2}\sum(x_i-\mu_0)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum(x_i-\mu_1)^2\right)} \\ &= \exp\left(-\frac{1}{2\sigma^2}\left[\sum(x_i-\mu_0)^2 - \sum(x_i-\mu_1)^2\right]\right)\end{aligned}$$

Simplify the exponent term in brackets:

$$\begin{aligned}[...] &= \sum(x_i^2 - 2x_i\mu_0 + \mu_0^2) - \sum(x_i^2 - 2x_i\mu_1 + \mu_1^2) \\ &= -2\mu_0\sum x_i + n\mu_0^2 + 2\mu_1\sum x_i - n\mu_1^2 \\ &= 2(\mu_1-\mu_0)\sum x_i + n(\mu_0^2 - \mu_1^2)\end{aligned}$$

The rejection condition $\Lambda < k$ implies $\ln\Lambda < \ln k$:

$$\begin{aligned}-\frac{1}{2\sigma^2}\left(2(\mu_1-\mu_0)\sum x_i + C_1\right) &< \ln k \\ 2(\mu_1-\mu_0)\sum x_i &> C_2 \quad \text{(Multiplying by negative flips sign)}\end{aligned}$$

Since $\mu_1 > \mu_0$, dividing by $2(\mu_1 - \mu_0)$ maintains the sign $>$.

$$\begin{aligned}\sum x_i &> C_3 \\ \bar{X} &> c\end{aligned}$$

Standardizing gives the critical region $\bar{X} > \mu_0 + z_\alpha\sigma/\sqrt{n}$.

| Proven. |
|---|

## 17.4 Neyman-Pearson for Uniform Distribution

**Problem:** $U(0,\theta)$. Test $\theta_0$ vs $\theta_1$ ($\theta_1 < \theta_0$).
***Solution:***

(a) **Likelihood:** $L(\theta) = \theta^{-n}\mathbb{I}(x_{(n)} < \theta)$.

(b) **Ratio Evaluation:**

- **Case 1 ($x_{(n)} > \theta_1$):** Since $x_{(n)} > \theta_1$, the data is impossible under $H_1$. $L(\theta_1) = 0$. $\Lambda = L(\theta_0)/0 = \infty$. We never reject $H_0$ here (data supports $H_0$).
- **Case 2 ($x_{(n)} \le \theta_1$):** Both likelihoods are non-zero.

$$\Lambda = \frac{\theta_0^{-n}}{\theta_1^{-n}} = \left(\frac{\theta_1}{\theta_0}\right)^n$$

Since $\theta_1 < \theta_0$, this ratio is a constant $< 1$.

(c) **Rejection Region:** To minimize Type II error (maximize power), we reject $H_0$ when $\Lambda$ is small. In Case 2, $\Lambda$ is small (constant $< 1$). In Case 1, $\Lambda$ is infinite. So we reject $H_0$ whenever we are in Case 2: $X_{(n)} \leq \theta_1$. To satisfy size $\alpha$: Reject if $X_{(n)} \leq C$.

$$P(X_{(n)} \leq C|\theta_0) = \left(\frac{C}{\theta_0}\right)^n = \alpha$$
$$C = \theta_0 \cdot \alpha^{1/n}$$

$\boxed{\text{Reject } H_0 \text{ if } X_{(n)} \leq \theta_0\alpha^{1/n}.}$

## 17.5 Neyman-Pearson for Geometric Distribution

**Problem:** $P(x) = (1-p)^{x-1}p$. Test $p_0$ vs $p_1$ ($p_1 > p_0$).
***Solution:***

(a) **Derivation:** $L(p) = (1-p)^{\sum x_i - n}p^n$.

$$\Lambda = \frac{(1-p_0)^{\sum x - n}p_0^n}{(1-p_1)^{\sum x - n}p_1^n}$$
$$\ln \Lambda = (\sum x - n)\ln\frac{1-p_0}{1-p_1} + n\ln\frac{p_0}{p_1}$$

Rejection condition $\ln \Lambda < k$:

$$(\sum x - n)\,(\ln(1-p_0) - \ln(1-p_1)) < K'$$

(b) **Direction:** Since $p_1 > p_0 \implies 1-p_1 < 1-p_0$. Thus $\ln(1-p_0) > \ln(1-p_1)$, so the term $[\ln(1-p_0) - \ln(1-p_1)]$ is **Positive**. Dividing by a positive constant preserves the inequality:

$$\sum x - n < C$$
$$\bar{X} < c'$$

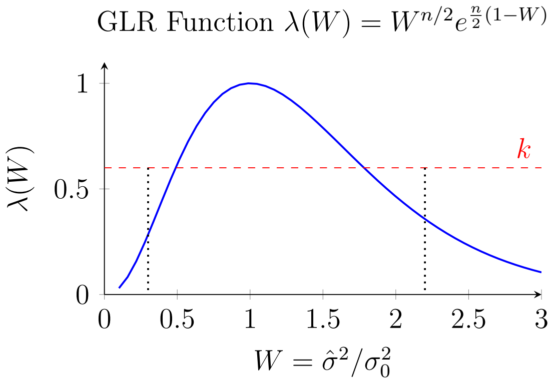
This makes sense: for Geometric distribution, a higher $p$ (probability of success) leads to faster success (smaller $X$). So small $\bar{X}$ supports $H_1$.

$\boxed{\text{Reject if } \bar{X} \text{ is small.}}$

# 17.2 Challenge (GLRT)

## 17.6 GLRT for Normal Variance

**Problem:** Testing $\sigma^2 = \sigma_0^2$ vs $\neq$.




***Solution:***

(a) **MLEs:** Unrestricted MLE for Normal variance is $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$. Restricted ($H_0$): The parameter is fixed at $\sigma_0^2$, so the "estimator" is just $\sigma_0^2$.

(b) **Likelihood Ratio:**

$$L(\sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{n\hat{\sigma}^2}{2\sigma^2}\right)$$

$$\lambda = \frac{L(\sigma_0^2)}{L(\hat{\sigma}^2)} = \frac{(\sigma_0^2)^{-n/2} e^{-n\hat{\sigma}^2/2\sigma_0^2}}{(\hat{\sigma}^2)^{-n/2} e^{-n\hat{\sigma}^2/2\hat{\sigma}^2}}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left(-\frac{n}{2}\frac{\hat{\sigma}^2}{\sigma_0^2} + \frac{n}{2}\right)$$

$$= W^{n/2} \exp\left(\frac{n}{2}(1 - W)\right)$$

(c) **Rejection Region:** From the graph, the function $\lambda(W)$ is unimodal (peaks at $W = 1$). The condition $\lambda(W) < k$ corresponds to the tails of the distribution. Hence, we reject if $W$ is very small ($< c_1$) OR very large ($> c_2$). This aligns with intuition: reject if the observed variance is drastically smaller or larger than the hypothesized $\sigma_0^2$.

$\boxed{\text{Reject if } W < c_1 \text{ or } W > c_2.}$

## 17.7 GLRT for Poisson

**Problem:** $\lambda_0$ vs $\lambda \neq \lambda_0$.
***Solution:***

(a) **MLEs:** Restricted $\hat{\lambda}_0 = \lambda_0$. Unrestricted $\hat{\lambda} = \bar{X}$.

(b) **Ratio:**

$$\lambda_{GLR} = \frac{\frac{e^{-n\lambda_0}\lambda_0^{\sum x}}{\prod x!}}{\frac{e^{-n\bar{x}}\bar{x}^{\sum x}}{\prod x!}} = e^{-n(\lambda_0 - \bar{x})}\left(\frac{\lambda_0}{\bar{x}}\right)^{n\bar{x}}$$

(c) **Visualization:** Taking logs: $\ln \lambda = -n(\lambda_0 - \bar{x}) + n\bar{x}(\ln \lambda_0 - \ln \bar{x})$. Let $t = \bar{x}/\lambda_0$. This simplifies to a function that drops as $t$ moves away from 1. Rejection occurs for $\bar{X} \ll \lambda_0$ or $\bar{X} \gg \lambda_0$.

(d) **Wilks:** $-2\ln\lambda \sim \chi^2_1$. Reject if $-2\ln\lambda > \chi^2_{1,\alpha}$.

## 17.8 GLRT for Rayleigh

**Problem:** $H_0 : v = 1$ vs $H_1 : v \neq 1$ $(v = \sigma^2)$.
***Solution:***

(a) **Unrestricted MLE:** Log-likelihood $\ell(v) = \sum \ln x_i - n \ln v - \frac{1}{2v}\sum x_i^2$.

$$\frac{\partial \ell}{\partial v} = -\frac{n}{v} + \frac{1}{2v^2}\sum x_i^2 = 0$$
$$v = \frac{\sum x_i^2}{2n} \implies \hat{v} = \frac{\sum x_i^2}{2n}$$

(b) **Ratio:**

$$\lambda = \frac{L(1)}{L(\hat{v})} = \frac{v^{-n}e^{-\sum x^2/2(1)}|_{v=1}}{v^{-n}e^{-\sum x^2/2v}|_{v=\hat{v}}}$$
$$= \frac{1^{-n}e^{-\sum x^2/2}}{\hat{v}^{-n}e^{-n}} = \hat{v}^n e^{n-\frac{1}{2}\sum x_i^2}$$

Substituting $\hat{v} = \frac{\sum x_i^2}{2n}$, we get the form dependent on the sufficient statistic $\sum x_i^2$.

## 17.9 GLRT for Shifted Exponential

**Problem:** $f(x) = e^{-(x-\theta)}, x \geq \theta$. Test $\theta = 0$.
***Solution:***

(a) **Likelihood:** $L(\theta) = \prod e^{-(x_i - \theta)} = e^{-\sum x_i + n\theta}$ for $\theta \leq \min(x_i)$.

(b) **Unrestricted MLE:** $L(\theta)$ is increasing in $\theta$ (since coefficient $n$ is positive). We want to maximize $\theta$ subject to the constraint $\theta \leq x_{(1)}$. Thus, $\hat{\theta} = x_{(1)}$.

(c) **Restricted Likelihood:** Under $H_0 : \theta = 0$, $L(0) = e^{-\sum x_i}$.

(d) **Ratio:**

$$\lambda = \frac{L(0)}{L(\hat{\theta})} = \frac{e^{-\sum x_i}}{e^{-\sum x_i + nx_{(1)}}}$$
$$= e^{-nx_{(1)}}$$

(e) **Rejection:** Reject if $\lambda < k$:

$$e^{-nx_{(1)}} < k$$
$$-nx_{(1)} < \ln k$$
$$x_{(1)} > -\frac{1}{n} \ln k = c$$

**Intuition:** Under $H_0(\theta = 0)$, the minimum value $X_{(1)}$ should be close to 0 (Exponential distribution starts at 0). If $X_{(1)}$ is large (far from 0), it suggests the distribution has shifted, so we reject $H_0$.

(d) $\lambda = e^{-nX_{(1)}}$
(e) Reject if $X_{(1)} > c$. Makes sense.