Chapter 18

Miscellaneous Problems

Detailed Solutions

18.1 Application

18.1 Thermodynamic Stability of Nano-Coating Process

Problem: Joint PDF $f(x, y) = ke^{-2x+3y}$ on $0 < y < x < 1$.
Solution:

Part I: Mathematical Modeling

1. **Derivation from ODEs:** Given $\frac{g'(x)}{g(x)} = -2 \implies \ln g(x) = -2x + C_1 \implies g(x) \propto e^{-2x}$. Given $\frac{h'(y)}{h(y)} = 3 \implies \ln h(y) = 3y + C_2 \implies h(y) \propto e^{3y}$. Thus, $f(x, y) = g(x)h(y) = ke^{-2x}e^{3y} = ke^{-2x+3y}$.

2. **Normalization (Finding k):** We integrate over the triangle $0 < x < 1$ and $0 < y < x$:

$$
\begin{aligned}
1 &= \int_0^1 \int_0^x ke^{-2x+3y}\,dy\,dx \\
&= k \int_0^1 e^{-2x} \left[\frac{e^{3y}}{3}\right]_0^x dx \\
&= \frac{k}{3} \int_0^1 e^{-2x}(e^{3x} - 1)\,dx \\
&= \frac{k}{3} \int_0^1 (e^x - e^{-2x})\,dx \\
&= \frac{k}{3} \left[e^x - \frac{e^{-2x}}{-2}\right]_0^1 \\
&= \frac{k}{3} \left[(e^1 + 0.5e^{-2}) - (1 + 0.5)\right] \\
&= \frac{k}{3}(e + 0.5e^{-2} - 1.5)
\end{aligned}
$$

Thus, $k = \frac{3}{e+0.5e^{-2}-1.5}$. (Numeric: $k \approx 2.33$).

3. **Marginal PDF $f_X(x)$:** For a fixed x, y ranges from 0 to x.

$$\begin{aligned} f_X(x) &= \int_0^x ke^{-2x+3y}\,dy = ke^{-2x}\left[\frac{e^{3y}}{3}\right]_0^x \\ &= \frac{k}{3}e^{-2x}(e^{3x}-1) = \frac{k}{3}(e^x - e^{-2x}), \quad 0 < x < 1 \end{aligned}$$

4. **Conditional Probability $P(Y > 0.8|X = 1)$:** At $X = 1$, the conditional density is proportional to e^{3y} on $0 < y < 1$. $f_{Y|X}(y|1) = \frac{f(1,y)}{f_X(1)} = \frac{ke^{-2}e^{3y}}{\frac{k}{3}(e^1-e^{-2})} = \frac{3e^{3y}}{e^3-1}$ (Adjusted for range). Actually simpler: $f_{Y|X}(y|x) = \frac{3e^{3y}}{e^{3x}-1}$ for $0 < y < x$. At $x = 1$: $f(y|1) = \frac{3e^{3y}}{e^3-1}$.

$$\begin{aligned} P(Y > 0.8|X = 1) &= \int_{0.8}^1 \frac{3e^{3y}}{e^3-1}\,dy = \frac{1}{e^3-1}[e^{3y}]_{0.8}^1 \\ &= \frac{e^3 - e^{2.4}}{e^3-1} \approx \frac{20.08 - 11.02}{19.08} \approx 0.47 \end{aligned}$$

Yes, probability is high (nearly 50%).

5. **Conditional Expectation $E[Y|X = x]$:** We need $\int_0^x y f_{Y|X}(y|x)\,dy$. PDF term: $C \cdot e^{3y}$ where $C = \frac{3}{e^{3x}-1}$.

$$E[Y|x] = C\int_0^x ye^{3y}\,dy$$

Integration by parts ($u = y, dv = e^{3y}dy$):

$$\begin{aligned} \int ye^{3y}dy &= \frac{ye^{3y}}{3} - \frac{e^{3y}}{9} \\ \int_0^x \dots &= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right] - \left[0 - \frac{1}{9}\right] = \frac{e^{3x}(3x-1)+1}{9} \end{aligned}$$

Multiply by $C = \frac{3}{e^{3x}-1}$:

$$g(x) = \frac{3}{e^{3x}-1} \cdot \frac{e^{3x}(3x-1)+1}{9} = \frac{e^{3x}(3x-1)+1}{3(e^{3x}-1)}$$

This is non-linear. As x increases, the average Y increases (curved).

Part II: Parameter Estimation (Triangular $f(w) = 2w/\theta^2$)

6. **MOM:**

$$\begin{aligned} E[W] &= \int_0^\theta w\frac{2w}{\theta^2}\,dw = \frac{2}{\theta^2}\int_0^\theta w^2\,dw = \frac{2}{\theta^2}\frac{\theta^3}{3} = \frac{2\theta}{3} \\ \bar{W} &= \frac{2\theta}{3} \implies \hat{\theta}_{MOM} = \frac{3\bar{W}}{2} \end{aligned}$$

7. **MLE:** $L(\theta) = \prod \frac{2w_i}{\theta^2}\mathbb{I}(w_i \le \theta) = \frac{2^n(\prod w_i)}{\theta^{2n}}\mathbb{I}(w_{(n)} \le \theta)$. To maximize L, we minimize θ^{2n} (denominator) subject to $\theta \ge w_{(n)}$.

$$\hat{\theta}_{MLE} = W_{(n)} = \max(W_i)$$

Differentiation fails because the support depends on θ.

8. **Distribution of $Y = W_{(n)}$:** CDF of W: $F_W(w) = \int_0^w \frac{2t}{\theta^2}dt = \frac{w^2}{\theta^2}$. CDF of Y: $F_Y(y) = (y^2/\theta^2)^n = y^{2n}/\theta^{2n}$. PDF of Y: $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{2ny^{2n-1}}{\theta^{2n}}$.

9. **Bias MSE:**

$$E[\hat{\theta}_{MLE}] = \int_0^\theta y\frac{2ny^{2n-1}}{\theta^{2n}}dy = \frac{2n}{\theta^{2n}}\int_0^\theta y^{2n}dy$$
$$= \frac{2n}{\theta^{2n}}\frac{\theta^{2n+1}}{2n+1} = \frac{2n}{2n+1}\theta$$

It is Biased (underestimates). Correction: $\hat{\theta}_{Unbiased} = \frac{2n+1}{2n}W_{(n)}$. **MSE Trade-off:** MLE has bias but lower variance than MOM. Often preferred.

10. **Invariance:** $\hat{\eta}_{MLE} = \ln(\hat{\theta}^2_{MLE} + 1) = \ln(W^2_{(n)} + 1)$.

Part III: Statistical Inference

11. **Hypothesis Testing:** $H_0 : \mu = 0.35, H_1 : \mu \neq 0.35$. $\bar{y} = 0.38, s = 0.08, n = 25$.

$$T = \frac{0.38 - 0.35}{0.08/\sqrt{25}} = \frac{0.03}{0.016} = 1.875$$

$df = 24$. Two-tailed critical $t_{0.025,24} = 2.064$. $|1.875| < 2.064 \implies$ **Fail to Reject**. Recommendation: Do not shut down. Variation is within chance.

12. **Confidence Interval:** $0.38 \pm 2.064(0.016) = 0.38 \pm 0.033 = [0.347, 0.413]$. Includes 0.35. Consistent with test.

Part IV: Predictive Modeling

13. **Interpretation:** $\beta_1 = 0.45$. For every 1 unit increase in Pressure, Film Thickness increases by 0.45 units on average.

14. **Slope Test:** $t = \beta_1/SE = 0.45/0.05 = 9.0$. $t = 9$ is huge ($P \approx 0$). Significant.

15. **Defect Probability ($P = 35$):** $\hat{T} = 12.5 + 0.45(35) = 12.5 + 15.75 = 28.25$. $\hat{\sigma} = \sqrt{MSE} = \sqrt{4} = 2$. $P(T > 30) = P(Z > \frac{30-28.25}{2}) = P(Z > 0.875) \approx 0.19$. 19% defect rate.

16. **ANOVA:** $R^2 = SSR/SST \implies 0.85 = SSR/1000 \implies SSR = 850$. $SSE = SST - SSR = 1000 - 850 = 150$.

End of Problem 18.1

18.2 The Aerospace Composite Wing Project

Problem: Method A vs B.
Solution:

Part I: Visuals

- **Median:** Method B (≈ 2800) is clearly higher than A (≈ 2480).
- **IQR:** Method A is shorter (more consistent). Method B is taller (more variable).
- **Outlier:** Method B has a dot at 3300.

Part II: Inference

- **Assumption:** $s_A = 100, s_B = 250$. Ratio $2.5^2 = 6.25$. Variances are likely unequal. Use Welch's T-test.
- **Test:**

$$T = \frac{2800 - 2480}{\sqrt{\frac{100^2}{15} + \frac{250^2}{15}}} = \frac{320}{\sqrt{666.7 + 4166.7}}$$
$$= \frac{320}{\sqrt{4833.4}} = \frac{320}{69.5} \approx 4.60$$

$T = 4.60 > t_{crit} \approx 2.5$. **Reject** H_0. Switch to Method B (Stronger).

Part III: Quality (Chi-Square)

- **Risk:** Mold Z Failure Rate = $35/100 = 35\%$ (Highest).
- **Expected Z Fail:** Total Fail Rate = $60/300 = 20\%$. $E = 100 \times 0.20 = 20$.
- **Test:** $\chi^2 = 14.6 > 5.99$. Significant. Mold type affects quality. Replace Mold Z.

Part IV: Process Optimization (ANOVA)

- $DF_{Total} = 27, DF_{Trt} = 3 \implies DF_{Error} = 24$ (Value B).
- $SS_{Total} = 7200, SS_{Error} = 4800 \implies SS_{Trt} = 2400$ (Value A).
- $MS_{Error} = 4800/24 = 200$ (Value D).
- $F = 800/200 = 4.0$ (Value C).
- $P = 0.004 < 0.01$. Significant.

$\boxed{\text{A=2400, B=24, C=4.0, D=200. Significant.}}$

18.3 The Autonomous Drone Safety System

Problem: Reliability Calculation.
Solution:

1. **Components ($t = 10$):** $R_A = e^{-10/50} = e^{-0.2} \approx 0.8187$. $R_B = e^{-10/100} = e^{-0.1} \approx 0.9048$. $R_C = e^{-10/20} = e^{-0.5} \approx 0.6065$.

2. **Power System (Parallel):** $R_P = 1 - (1 - R_A)(1 - R_C) = 1 - (0.1813)(0.3935) = 1 - 0.0713 = 0.9287$.

3. **System (Series):** $R_{Sys} = R_B \times R_P = 0.9048 \times 0.9287 \approx 0.8403$. (Current reliability is 84%, below 85

4. **Improvement:**
   - **Option X (New B, MTTF=200):** $R_{B'} = e^{-10/200} = e^{-0.05} \approx 0.9512$. $R_{Sys} = 0.9512 \times 0.9287 \approx \mathbf{0.883}$. (Success!)
   - **Option Y (New C = A):** $R_{P'} = 1 - (1 - 0.8187)^2 = 1 - 0.0329 = 0.9671$. $R_{Sys} = 0.9048 \times 0.9671 \approx 0.875$. (Also Success).

   Both work, but improving the single point of failure (Series component B) yields higher gain (88.3% vs 87.5

   $\boxed{\text{Choose Option X.}}$

18.4 Conceptual Cloze Test

Solution:

Theme A: Deterministic Model, Stochastic Process, Probability Density Function, Differential Equation.
Theme B: Point Estimator, Unbiased, Bias-Variance Tradeoff, Maximum Likelihood.
Theme C: Type I Error, Significance Level, P-value, Null Hypothesis, Power.
Theme D: Residuals, Homoscedasticity, Extrapolation, Normal Probability Plot.
Theme E: Skewness, IQR, Robust, Outlier, Statistic, Parameter, Central Limit Theorem, Standard Error.

18.5 True/False Gauntlet

Solution:

Part I:
1. False ($n - 1$). 2. True. 3. True. 4. True. 5. False (c^2). 6. False (Histogram for peaks). 7. True. 8. False (z=0 means equal to Mean).
Part II:
9. False (Cannot happen together $\neq$ independent). 10. False ($Var(c) = 0$). 11. True. 12. False ($Var(X) + Var(Y)$). 13. False (Mean = Variance). 14. False ($P(X \leq x)$). 15. True.

Part III:

16. False (Sample Mean becomes Normal). 17. True. 18. True. 19. True. 20. True. 21. True. 22. False (F-distribution).

Part IV:

23. False (Expected value equals parameter). 24. True. 25. True. 26. False (Asymptotic). 27. False. 28. False (Method Confidence). 29. False (Narrower). 30. True. 31. True. 32. True.

Part V:

33. True. 34. False ($P(Data|H_0)$). 35. False (Prob of rejecting false H0). 36. True. 37. False (Evidence insufficient). 38. False (Fail to reject). 39. False (Both tails).

Part VI:

40. True. 41. True. 42. False (Linear association only). 43. True. 44. True ($t^2 = F$). 45. False (Assumption violation). 46. True. 47. True. 48. True. 49. False (Small F means means are equal). 50. False (X correlated with another X).