Chapter 2: Probability Distributions & Theorems

Detailed Solutions

2.1 Basic Concept

2.1 Axioms of Probability

Problem: State Kolmogorov's axioms and prove $P(A^c) = 1 - P(A)$.
Solution:

Kolmogorov's Three Axioms:

1. **Non-negativity:** For any event E, $P(E) \geq 0$.
2. **Normalization:** The probability of the sample space S is $P(S) = 1$.
3. **Additivity:** For any countable sequence of disjoint (mutually exclusive) events $E_1, E_2, \ldots,$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Proof that $P(A^c) = 1 - P(A)$:

1. By definition, A and its complement A^c are disjoint ($A \cap A^c = \emptyset$) and their union is the entire sample space ($A \cup A^c = S$).
2. By Axiom 3 (Additivity for two events):

$$P(A \cup A^c) = P(A) + P(A^c)$$

3. By step 1, $P(A \cup A^c) = P(S)$.
4. By Axiom 2 (Normalization), $P(S) = 1$.
5. Therefore, $1 = P(A) + P(A^c)$.
6. Rearranging gives:

$$P(A^c) = 1 - P(A)$$

Since $S = A \cup A^c$ and $A \cap A^c = \emptyset$,
$P(S) = P(A) + P(A^c)$
$1 = P(A) + P(A^c)$
$P(A^c) = 1 - P(A) \quad$ (Q.E.D.)

2.2 Independence vs. Mutually Exclusive

Problem: Define Independent and Mutually Exclusive events. Can they overlap?
Solution:

1. **Independent Events:** Two events A and B are independent if the occurrence of one does not affect the probability of the other. Mathematically:

$$P(A \cap B) = P(A)P(B)$$

   Alternatively, $P(A|B) = P(A)$.

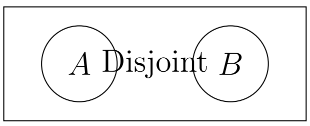
2. **Mutually Exclusive (Disjoint) Events:** Two events cannot occur at the same time. Mathematically:

$$P(A \cap B) = 0$$

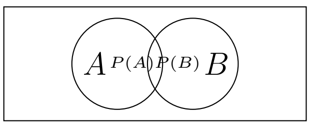
3. **Can they be both?** Suppose non-impossible events A and B ($P(A) > 0, P(B) > 0$).
   - If they are **Mutually Exclusive**, then $P(A \cap B) = 0$.
   - If they are **Independent**, then $P(A \cap B) = P(A)P(B)$.
   - Since $P(A) > 0$ and $P(B) > 0$, their product $P(A)P(B) > 0$.
   - Thus, $P(A \cap B)$ cannot be both 0 and > 0 simultaneously.

No. If two events have non-zero probabilities, they cannot be both independent and mutually exclusive.
If they are disjoint, the occurrence of A implies B cannot happen (total dependence).

Mutually Exclusive



Independent (Typically)



2.3 Set Theory for Engineers

Problem: $A =$ Capacitor fails, $B =$ Resistor fails. Express in set notation.
Solution:

(a) **Both components fail:** Intersection of A and B.

$$A \cap B$$

(b) **At least one component fails:** Union of A and B.

$$A \cup B$$

(c) **Only the Capacitor fails:** Capacitor fails AND Resistor does NOT fail.

$$A \cap B^c \quad (\text{or } A \setminus B)$$

(d) **Neither component fails:** Not A AND Not B. By De Morgan's Law, this is the complement of the union.

$$A^c \cap B^c \quad (\text{or } (A \cup B)^c)$$

(a) $A \cap B$
(b) $A \cup B$
(c) $A \cap B^c$
(d) $A^c \cap B^c$

2.2 Intermediate

2.4 Conditional Probability Calculation

Problem: $P(A) = 0.5, P(B) = 0.6, P(A \cup B) = 0.8$.
Solution:

(a) **Find $P(A \cap B)$:** Using the Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.8 = 0.5 + 0.6 - P(A \cap B)$$

$$P(A \cap B) = 1.1 - 0.8 = 0.3$$

(b) **Find $P(A|B)$:**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.6} = 0.5$$

(c) **Find $P(B|A)$:**

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.5} = 0.6$$

(d) **Independence Check:** Check if $P(A|B) = P(A)$ or $P(A \cap B) = P(A)P(B)$.

$$P(A|B) = 0.5 \quad \text{and} \quad P(A) = 0.5$$

Since they are equal, A and B are independent. Alternatively: $P(A)P(B) = 0.5 \times 0.6 = 0.3$, which equals $P(A \cap B)$.

(a) 0.3 (b) 0.5 (c) 0.6 (d) Yes, independent ($P(A \cap B) = P(A)P(B)$).

2.5 The Night Shift Crew (Combinatorics)

Problem: 20 machinists, choose 3.
Solution:

(a) **Total Crews:** Order does not matter (Crew A,B,C is same as C,B,A). Use Combination.

$$\binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 20 \times 19 \times 3 = 1140$$

(b) **Exclude Rank #1:** We must choose 3 machinists from the remaining 19.

$$\binom{19}{3} = \frac{19 \times 18 \times 17}{3 \times 2 \times 1} = 19 \times 3 \times 17 = 969$$

(c) **At least one of Top 5:** Using the complement rule: Total Crews - Crews with NONE of Top 5. None of Top 5 means choosing 3 from the bottom 15 (20 - 5).

$$\text{None Top 5} = \binom{15}{3} = \frac{15 \times 14 \times 13}{3 \times 2 \times 1} = 5 \times 7 \times 13 = 455$$

$$\text{At least one} = \binom{20}{3} - \binom{15}{3} = 1140 - 455 = 685$$

(a) 1,140 crews
(b) 969 crews
(c) 685 crews

2.6 Concrete Strength (Conditional)

Problem: A = Curing $\leq$ 14 days (Rows 1+2). B = Above Standard (Col 2). Total = 204.
Solution:

Marginal Totals:

- Total $\leq$ 14 days (Row 1 + Row 2): $(12 + 40) + (44 + 16) = 52 + 60 = 112$.
- Total Above Standard (Col 2): $40 + 16 + 36 = 92$.
- Intersection ($A \cap B$): Curing $\leq$ 14 AND Above Standard $= 40 + 16 = 56$.

(a) $P(A)$ **and** $P(B)$**:**

$$P(A) = \frac{112}{204} \approx 0.549$$
$$P(B) = \frac{92}{204} \approx 0.451$$

(b) $P(A|B)$**:**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{56/204}{92/204} = \frac{56}{92} \approx 0.609$$

(c) $P(B|A)$**:**

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{56/204}{112/204} = \frac{56}{112} = 0.5$$

(d) **Independence:** Check if $P(B|A) = P(B)$. $0.5 \neq 0.451$. The probability of being Above Standard changes if we know the Curing Time is short. Therefore, they are **dependent**.

(a) $P(A) = 112/204, \quad P(B) = 92/204$
(b) $P(A|B) = 56/92 \approx 0.61$
(c) $P(B|A) = 56/112 = 0.5$
(d) Not independent ($0.5 \neq 0.45$).

2.7 Rare Disease (Bayes' Theorem)

Problem: $P(D) = 0.002$. $P(+|D) = 0.95$ (Sensitivity). $P(+|D^c) = 0.01$ (False Positive).
Solution:

(a) **Calculate $P(D|+)$:** Using Bayes' Theorem:

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

Note that $P(D^c) = 1 - 0.002 = 0.998$.

$$P(D|+) = \frac{0.95 \times 0.002}{(0.95 \times 0.002) + (0.01 \times 0.998)}$$

$$P(D|+) = \frac{0.0019}{0.0019 + 0.00998} = \frac{0.0019}{0.01188} \approx 0.1599$$

(b) **Intuition:** Even though the test is accurate, the disease is extremely rare (1 in 500). In a group of 1000 people:

- ≈ 2 have disease $\rightarrow \approx 1.9$ test positive.
- ≈ 998 healthy $\rightarrow \approx 9.98$ test positive (false alarms).

So out of ≈ 12 positive tests, only ≈ 2 are real. The false positives drown out the true positives.

(a) $P(D|+) \approx 16\%$
(b) The "Base Rate Fallacy". The low prior probability (0.002) dominates the posterior result.

2.8 System Reliability

Problem: Components $R_1 = 0.9, R_2 = 0.8, R_3 = 0.95$.
Solution:

(a) **Series:** System works only if ALL work.

$$R_{sys} = R_1 \times R_2 \times R_3$$

$$R_{sys} = 0.9 \times 0.8 \times 0.95 = 0.72 \times 0.95 = 0.684$$

(b) **Parallel:** System works if AT LEAST ONE works. Calculate probability of failure for each: $F_i = 1 - R_i$. $F_1 = 0.1, F_2 = 0.2, F_3 = 0.05$. System fails only if ALL fail:

$$F_{sys} = F_1 \times F_2 \times F_3 = 0.1 \times 0.2 \times 0.05 = 0.001$$

$$R_{sys} = 1 - F_{sys} = 1 - 0.001 = 0.999$$

(a) Series Reliability = 0.684
(b) Parallel Reliability = 0.999

2.3 Challenge

2.9 Inclusion-Exclusion Principle

Problem: Prove formula for $P(A \cup B \cup C)$.
Solution:

Let $D = B \cup C$. Then $P(A \cup B \cup C) = P(A \cup D)$. Using the 2-set formula:

$$P(A \cup D) = P(A) + P(D) - P(A \cap D)$$

Expand $P(D) = P(B \cup C) = P(B) + P(C) - P(B \cap C)$. Expand $P(A \cap D) = P(A \cap (B \cup C))$. Using distributive property: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Apply 2-set formula to this union:

$$P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

The last term simplifies to $P(A \cap B \cap C)$. Substitute everything back:

$$P(A \cup B \cup C) = P(A) + [P(B) + P(C) - P(B \cap C)] - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proven.

2.10 Bonferroni's Inequality

Problem: Prove $P(A \cap B) \geq P(A) + P(B) - 1$.
Solution:

Start with the basic addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Rearrange to solve for intersection:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

We know that any probability is at most 1, so $P(A \cup B) \leq 1$. Therefore, subtracting $P(A \cup B)$ subtracts something ≤ 1. This means subtracting 1 subtracts *more or equal* to subtracting $P(A \cup B)$.

$$P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

Thus:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Proven.

2.11 Conditional Independence Proof

Problem: If $A \perp B$, prove $A^c \perp B^c$.
Solution:

We need to show $P(A^c \cap B^c) = P(A^c)P(B^c)$. From De Morgan's Law: $A^c \cap B^c = (A \cup B)^c$.

$$P(A^c \cap B^c) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

Since A, B are independent, substitute $P(A \cap B) = P(A)P(B)$:

$$= 1 - P(A) - P(B) + P(A)P(B)$$

Factor the expression:

$$= (1 - P(A)) - P(B)(1 - P(A))$$
$$= (1 - P(A))(1 - P(B))$$
$$= P(A^c)P(B^c)$$

$$\boxed{\text{Proven.}}$$

2.4 Application

2.12 Birthday Paradox Simulation

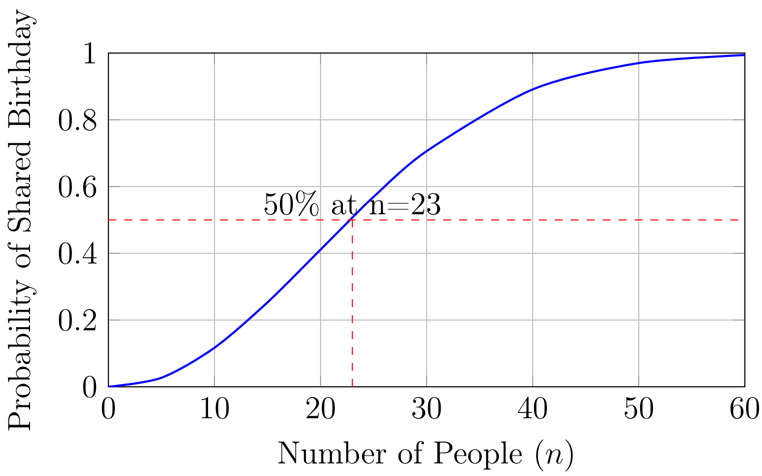
Problem: Python simulation results.
Solution:

(a) **Code Analysis:** The provided function generates n random integers between 1 and 365 and checks for duplicates using a Set data structure (which removes duplicates). If 'len(list) != len(set)', a collision occurred.

(b) **Expected Outcome:** Running the simulation for $n = 10$ to 50 will produce a probability curve.

- At $n = 10$, probability is low ($\approx 12\%$).
- At $n = 23$, probability crosses the 50% threshold ($\approx 50.7\%$).
- At $n = 50$, probability is extremely high ($\approx 97\%$).

The "Paradox" is that only 23 people are needed for a 50-50 chance, which is much lower than 365/2.



The simulation confirms that with just 23 people, there is a >50% chance of a shared birthday.