

Chapter 4

Continuous Random Variables & Distributions

Detailed Solutions

4.1 Basic Concept

4.1 PDF Properties

Problem: Validating PDF properties (True/False).

Solution:

- (a) **True.** The probability density $f(x)$ represents height, not probability. For example, a Uniform distribution on $[0, 0.1]$ has height $f(x) = \frac{1}{0.1} = 10$.
- (b) **True.** For continuous variables, the probability of any single exact point is zero because the area under a single point is zero ($\int_c^c f(x)dx = 0$).
- (c) **True.** The total area under the PDF curve must equal 1 (Normalization axiom).
- (d) **True.** Since $f(x) = F'(x)$, if $f(x)$ is increasing ($f'(x) > 0$), then $F''(x) > 0$. A function with a positive second derivative is convex.

- (a) True ($f(x)$ is density, not probability)
 - (b) True ($P(X = c) = 0$)
 - (c) True (Total Prob = 1)
 - (d) True ($F''(x) > 0$)

4.2 Fundamental of Continuous RV

Problem: $f(x) = k(1 - x^2)$ for $-1 < x < 1$.

Solution:

(a) **Find k:** Integrate over the support and set to 1.

$$\begin{aligned} \int_{-1}^1 k(1-x^2)dx &= 1 \\ k \left[x - \frac{x^3}{3} \right]_{-1}^1 &= 1 \\ k \left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) \right] &= 1 \\ k \left[\frac{2}{3} - \left(-\frac{2}{3}\right) \right] &= 1 \implies k \left(\frac{4}{3}\right) = 1 \implies k = \frac{3}{4} \end{aligned}$$

(b) **Find CDF $F(x)$:** For $-1 < x < 1$:

$$\begin{aligned} F(x) &= \int_{-1}^x \frac{3}{4}(1-t^2)dt = \frac{3}{4} \left[t - \frac{t^3}{3} \right]_{-1}^x \\ &= \frac{3}{4} \left[\left(x - \frac{x^3}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] \\ &= \frac{3}{4} \left(x - \frac{x^3}{3} + \frac{2}{3} \right) = \frac{3x - x^3 + 2}{4} \end{aligned}$$

(c) **Calculate Probability:**

$$\begin{aligned} P(-0.5 < X < 0.5) &= \int_{-0.5}^{0.5} \frac{3}{4}(1-x^2)dx \\ &= 2 \times \int_0^{0.5} \frac{3}{4}(1-x^2)dx \quad (\text{Symmetry}) \\ &= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_0^{0.5} = \frac{3}{2} \left[0.5 - \frac{0.125}{3} \right] \\ &= \frac{3}{2} \left[\frac{1.5 - 0.125}{3} \right] = \frac{1.375}{2} = 0.6875 \end{aligned}$$

(d) **Expectation and Variance:** $E[X] = \int_{-1}^1 x \cdot k(1-x^2)dx = 0$ (Odd function over symmetric interval).

$$Var(3X) = 3^2 Var(X) = 9E[X^2] \text{ (since } E[X] = 0\text{).}$$

$$\begin{aligned} E[X^2] &= \int_{-1}^1 x^2 \cdot \frac{3}{4}(1-x^2)dx = \frac{3}{4} \int_{-1}^1 (x^2 - x^4)dx \\ &= \frac{3}{4} \cdot 2 \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{3}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{3}{2} \left(\frac{2}{15} \right) = \frac{1}{5} = 0.2 \end{aligned}$$

$$Var(3X) = 9(0.2) = 1.8.$$

- (a) $k = 3/4$
- (b) $F(x) = \frac{-x^3+3x+2}{4}$
- (c) 0.6875
- (d) $E[X] = 0, Var(3X) = 1.8$

4.2 Intermediate

4.3 Uniform Distribution: Rounding Error

Problem: $X \sim U(-0.5, 0.5)$.

Solution:

- (a) **PDF:** Since length $b - a = 0.5 - (-0.5) = 1$. Height $= 1/(b - a) = 1$.

$$f(x) = \begin{cases} 1 & -0.5 < x < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

- (b) **Probability:** Area of rectangle from -0.2 to 0.2. Width = 0.4, Height = 1.

$$P(-0.2 < X < 0.2) = 0.4 \times 1 = 0.4$$

- (c) **Mean and Variance:**

$$E[X] = \frac{a+b}{2} = \frac{-0.5+0.5}{2} = 0$$

$$Var(X) = \frac{(b-a)^2}{12} = \frac{1^2}{12} \approx 0.0833$$

- (a) $f(x) = 1$
 (b) 0.4
 (c) Mean = 0, Var = 1/12

4.4 Exponential: Component Life

Problem: Mean $\mu = 10,000$ hours.

Solution:

- (a) **Parameter λ :**

$$\lambda = \frac{1}{\mu} = \frac{1}{10000} = 0.0001$$

- (b) **More than 12,000 hours:** Using CDF $P(X > x) = e^{-\lambda x}$.

$$P(X > 12000) = e^{-0.0001 \times 12000} = e^{-1.2} \approx 0.3012$$

- (c) **Fail within 5,000 hours:**

$$P(X < 5000) = 1 - e^{-\lambda x} = 1 - e^{-0.5} \approx 1 - 0.6065 = 0.3935$$

- (d) **Median Lifetime:** Solve $F(m) = 0.5 \implies 1 - e^{-\lambda m} = 0.5 \implies e^{-\lambda m} = 0.5$.

$$-\lambda m = \ln(0.5) = -\ln(2) \implies m = \frac{\ln(2)}{\lambda} = 10000 \times 0.6931 = 6931 \text{ hours}$$

- (a) $\lambda = 10^{-4}$
 (b) 0.301
 (c) 0.394
 (d) 6,931 hours

4.5 Normal Distribution: Grades

Problem: $X \sim N(75, 100) \implies \sigma = 10$.

Solution:

- (a) **Between 60 and 80:** Convert to Z-scores: $Z = \frac{X-\mu}{\sigma}$.

$$Z_1 = \frac{60 - 75}{10} = -1.5$$

$$Z_2 = \frac{80 - 75}{10} = 0.5$$

$$\begin{aligned} P &= P(-1.5 < Z < 0.5) = \Phi(0.5) - \Phi(-1.5) \\ &= 0.6915 - 0.0668 = 0.6247 \end{aligned}$$

- (b) **Top 10% Cutoff:** We need $P(Z > k) = 0.10$, so $P(Z < k) = 0.90$. From Z-table, $k \approx 1.28$.

$$X = \mu + k\sigma = 75 + 1.28(10) = 87.8$$

- (c) **Exactly 75:** For any continuous distribution, the probability of taking on a specific exact value is 0.

$$P(X = 75) = 0$$

(Note: If the question implies discrete rounding, the answer would differ, but mathematically for Normal dist, it is 0).

- (a) 0.6247
 (b) 87.8
 (c) 0 (Continuous variable)

4.6 Standard Normal Z

Problem: Using Z-table.

Solution:

- (a) $P(Z > 1.645) = 1 - P(Z < 1.645) = 1 - 0.95 = 0.05$.
 (b) $P(-1.96 < Z < 1.96) = 0.975 - 0.025 = 0.95$.
 (c) $P(Z < k) = 0.95 \implies k = 1.645$.

- (a) 0.05
 (b) 0.95
 (c) $k = 1.645$

4.3 Challenge

4.7 Mean of Exponential (Proof)

Problem: Prove $E[X] = 1/\lambda$ using Integration by Parts.

Solution:

$$f(x) = \lambda e^{-\lambda x}.$$

$$E[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx$$

Integration by parts: Let $u = x \implies du = dx$. Let $dv = \lambda e^{-\lambda x} dx \implies v = -e^{-\lambda x}$.

$$\begin{aligned} E[X] &= [-xe^{-\lambda x}]_0^\infty - \int_0^\infty (-e^{-\lambda x}) dx \\ &= (0 - 0) + \int_0^\infty e^{-\lambda x} dx \quad (\text{Since } \lim_{x \rightarrow \infty} \frac{x}{e^{\lambda x}} = 0 \text{ by L'Hopital}) \\ &= \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \\ &= 0 - \left(-\frac{1}{\lambda} \right) = \frac{1}{\lambda} \end{aligned}$$

For Variance, perform integration by parts twice on $E[X^2]$.

Proven.

4.8 Normal Inflection Points

Problem: Prove inflection at $\mu \pm \sigma$.

Solution:

Let $z = \frac{x-\mu}{\sigma}$. PDF is proportional to $e^{-z^2/2}$.

$$\begin{aligned} f'(x) &= f(x) \cdot \frac{d}{dx} \left(-\frac{(x-\mu)^2}{2\sigma^2} \right) = f(x) \cdot \left(-\frac{x-\mu}{\sigma^2} \right) \\ f''(x) &= f'(x) \left(-\frac{x-\mu}{\sigma^2} \right) + f(x) \left(-\frac{1}{\sigma^2} \right) \quad (\text{Product Rule}) \\ &= f(x) \left(-\frac{x-\mu}{\sigma^2} \right)^2 - \frac{f(x)}{\sigma^2} \\ &= \frac{f(x)}{\sigma^2} \left[\left(\frac{x-\mu}{\sigma} \right)^2 - 1 \right] \end{aligned}$$

Set $f''(x) = 0$. Since $f(x) \neq 0$:

$$\left(\frac{x-\mu}{\sigma} \right)^2 = 1 \implies \frac{x-\mu}{\sigma} = \pm 1 \implies x = \mu \pm \sigma$$

Proven.

4.9 Constructing a Triangular PDF

Problem: Triangle from a to b with peak at c .

Solution:

- (a) **Value of k:** Area of triangle = $\frac{1}{2} \times \text{Base} \times \text{Height} = 1$.

$$\frac{1}{2}(b-a)k = 1 \implies k = \frac{2}{b-a}$$

- (b) **Piecewise Function:** L_1 goes from $(a, 0)$ to (c, k) . Slope $m_1 = \frac{k}{c-a}$. Equation: $f(x) = \frac{2(x-a)}{(b-a)(c-a)}$. L_2 goes from (c, k) to $(b, 0)$. Slope $m_2 = \frac{-k}{b-c}$. Equation: $f(x) = \frac{2(b-x)}{(b-a)(b-c)}$.

- (c) **Expectation:**

$$E[X] = \int_a^c xL_1 dx + \int_c^b xL_2 dx$$

After evaluating these integrals (algebraically intensive), the result simplifies to the centroid of the triangle's x-projection:

$$E[X] = \frac{a+b+c}{3}$$

- (d) **Geometry Calculation ($a = 0, c = 2, b = 4$):** $k = 2/4 = 0.5$. We want $P(X > 3)$. This region is a small triangle from $x = 3$ to $x = 4$. Base = $4 - 3 = 1$. Height at $x = 3$: Equation of line L_2 is $y = -0.25(x - 4)$. At $x = 3, y = 0.25$.

$$\text{Area} = \frac{1}{2} \times 1 \times 0.25 = 0.125$$

- (a) $k = \frac{2}{b-a}$
 (c) $E[X] = (a+b+c)/3$
 (d) 0.125

4.4 Application

4.10 Numerical Integration (Python)

Problem: Code for Trapezoidal rule.

Solution:

```

1 import numpy as np
2 from scipy.stats import norm
3
4 def standard_normal(x):
5     return (1/np.sqrt(2*np.pi)) * np.exp(-0.5 * x**2)
6
7 # Parameters
8 x_min = -10 # Approximating negative infinity
9 x_max = 2
10 n = 1000    # Number of trapezoids
11 dx = (x_max - x_min) / n
12
13 # Trapezoidal Rule Implementation
14 area = 0
15 for i in range(n):
16     x1 = x_min + i*dx
17     x2 = x_min + (i+1)*dx
18     # Area = dx * average height
19     area += dx * (standard_normal(x1) + standard_normal(x2)) / 2
20
21 exact_val = norm.cdf(2)
22
23 print(f"Numerical Integration P(Z<2): {area:.5f}")
24 print(f"Scipy Exact Value: {exact_val:.5f}")
25 print(f"Error: {abs(area - exact_val):.5e}")

```

4.11 Fitting a Distribution

Problem: Fitting Exponential to data.

Solution:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3 from scipy.stats import expon
4
5 # 1. Generate Dummy Data (for demonstration)
6 np.random.seed(42)
7 data = expon.rvs(scale=100, size=1000) # True lambda = 1/100 = 0.01
8
9 # 2. Fit Distribution
10 # expon.fit returns (loc, scale). For exponential, scale = 1/lambda
11 loc, scale = expon.fit(data, floc=0)
12 lambda_est = 1/scale
13
14 print(f"Estimated Lambda: {lambda_est:.5f}")
15
16 # 3. Plotting

```

```
17 plt.figure(figsize=(10, 6))
18
19 # Histogram of data
20 plt.hist(data, bins=30, density=True, alpha=0.6, color='g', label='Data
    Hist')
21
22 # Fitted PDF
23 x = np.linspace(0, max(data), 1000)
24 pdf = expon.pdf(x, scale=scale)
25 plt.plot(x, pdf, 'r-', linewidth=2, label=f'Fitted Exp($\lambda$={lambda_
    est:.4f}))')
26
27 plt.title('Fitting Exponential Distribution to Machine Failure Times')
28 plt.xlabel('Time')
29 plt.ylabel('Density')
30 plt.legend()
31 plt.show()
```

The `scipy.stats.expon.fit` function uses Maximum Likelihood Estimation (MLE) to find the parameter that maximizes the probability of observing the given data.