Chapter 13

Chi-square Test

Detailed Solutions

13.1 Basic Concept

13.1 Distinguishing the Tests

Problem: Match scenario to test type.
Solution:

(a) **(A) Goodness of Fit:** We are checking if one sample fits a specific theoretical distribution (Poisson).

(b) **(B) Test of Independence:** We have one sample and want to see if two variables (Job Satisfaction, Work Shift) are related.

(c) **(C) Test of Homogeneity:** We have multiple distinct samples (Machine A, B, C) and want to see if they share the same distribution (Defect Rate).

(a) Goodness of Fit
(b) Independence
(c) Homogeneity

13.2 Intermediate

13.2 Goodness of Fit (Fair Die?)

Problem: 60 rolls. Observed: 5, 8, 12, 15, 12, 8.
Solution:

(a) **Hypothesis:** H_0: The die is fair ($P_1 = P_2 = \cdots = P_6 = 1/6$). H_1: The die is not fair.

(b) **Expected Frequency (E_i):** $N = 60$. If fair, $E_i = 60 \times (1/6) = 10$ for all faces.

(c) **Chi-square Statistic:**

$$\begin{aligned}\chi^2 &= \sum \frac{(O_i - E_i)^2}{E_i} \\ &= \frac{(5-10)^2}{10} + \frac{(8-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(15-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(8-10)^2}{10} \\ &= \frac{25}{10} + \frac{4}{10} + \frac{4}{10} + \frac{25}{10} + \frac{4}{10} + \frac{4}{10} \\ &= 2.5 + 0.4 + 0.4 + 2.5 + 0.4 + 0.4 = 6.6\end{aligned}$$

(d) **Decision:** $df = k - 1 = 6 - 1 = 5$. $\chi^2_{crit} = 11.07$. Since $6.6 < 11.07$, we **Fail to Reject** H_0. There is not enough evidence to say the die is biased.

(b) $E_i = 10$
(c) $\chi^2 = 6.6$
(d) Fail to Reject.

13.3 Goodness of Fit: Binomial Distribution

Problem: $n = 5, N = 100$. Compare Observed vs Binomial($p = 0.2$).
Solution:

(a) **Visual Check:** The observed data (30, 45, 20...) follows the general decay shape of the Binomial distribution, but looking closely, the peak (45) is higher than expected (41).

(b) **Parameter Estimation:** Total items $= 100 \times 5 = 500$. Total defects $= (0 \times 30) + (1 \times 45) + (2 \times 20) + (3 \times 4) + (4 \times 1) + (5 \times 0) = 45 + 40 + 12 + 4 = 101$. $\hat{p} = 101/500 = 0.202$. (Very close to assumed 0.2).

(c) **Rule of Thumb ($E_i < 5$):** For $x = 3(5.1)$, $x = 4(0.6)$, $x = 5(0.0)$, expected values are small. We must **combine (bin)** the categories $x \geq 3$ into a single category to ensure valid Chi-square approximation.

(d) **Degrees of Freedom:** Original categories: $0, 1, 2, 3, 4, 5$ (6 categories). After grouping: $0, 1, 2, \geq 3$ (4 categories $\rightarrow k = 4$). Estimated parameters: $m = 1$ ($\hat{p}$). $df = k - 1 - m = 4 - 1 - 1 = 2$.

(b) $\hat{p} = 0.202$ (c) Combine categories $x \geq 3$. (d) $df = 2$.

13.4 Goodness of Fit: Poisson Distribution

Problem: Queueing ($\lambda = 2.5$).
Solution:

(a) **Skewness:** Poisson with low λ is right-skewed (tail extends to the right). The observed data should show fewer high values.

(b) **Calculation ($x = 0$):** $P(X = 0) = e^{-2.5} \approx 0.082$. Expected Frequency $E_0 = 100 \times 0.082 = 8.2$. Matches.

(c) **Outliers (10 arrivals):** $P(X = 10)$ for $\lambda = 2.5$ is extremely small (≈ 0.000038). $E_{10} \approx 0$. If we observe $O_{10} = 1$, the term $\frac{(1-0)^2}{0}$ explodes (infinity). Even if grouped, a value this far out contributes a massive amount to χ^2, likely leading to **Rejection of** H_0. It suggests the data is not pure Poisson (maybe "bursty" traffic).

(b) $E_0 = 8.2$ (c) Massive increase in $\chi^2 \to$ Reject.

13.5 Test of Independence (Partial Table)

Problem: Age vs Coffee. $N = 200$.
Solution:

(a) **Calculate Expected Values:** Formula: $E_{ij} = \frac{\text{(Row Total)}\times\text{(Col Total)}}{N}$.

- (Young, Latte): $A = \frac{100\times90}{200} = 45$.
- (Young, Espresso): $B = \frac{100\times110}{200} = 55$.

(b) **Degrees of Freedom:** $df = (r-1)(c-1) = (2-1)(2-1) = 1$.

(c) **Interpretation:** $\chi^2_{calc} = 18.18$. $\chi^2_{crit} = 3.84$. $18.18 > 3.84 \implies$ **Reject** H_0. Age and Coffee Preference are **Dependent** (Associated). Young people prefer Latte significantly more than expected.

(a) A=45, B=55 (b) $df = 1$ (c) Dependent (Reject H_0).

13.6 Test of Homogeneity (Defect Rates)

Problem: Morning (10/200), Afternoon (15/200), Night (35/200).
Solution:

(a) **Contingency Table:**

	Defect	Good	Total
Morning	10	190	200
Afternoon	15	185	200
Night	35	165	200
Total	60	540	600

(b) **Expected (Night, Defect):** Overall Defect Rate = 60/600 = 0.10 (10%). Under H_0 (Homogeneity), Night shift should also have 10% defects. $E = 200 \times 0.10 = 20$. (Or formula: $\frac{200\times60}{600} = 20$).

(c) **Visual Check:** Observed Night Defect = 35. Expected = 20. The deviation $(35-20)^2/20 = 225/20 = 11.25$ is huge. Without full calc, Night shift clearly has a **significantly higher** defect rate.

(b) $E_{Night,Defect} = 20$.
(c) Yes, Night shift has a problem.

13.3 Applications

13.7 Chi-square Test with Python

Problem: Python Code Analysis.
Solution:

(a) **P-value Check:** The code prints 'p'. If $p < 0.05$, the result is significant. Given the data (Males prefer C, Females prefer A/B), we expect a very small p-value.

(b) **Expected Frequency Check:** It is important to check 'expected' array because the Chi-square approximation becomes inaccurate if any $E_{ij} < 5$. If this happens, we might need Fisher's Exact Test (for 2x2) or combine categories.

(c) **Implication:** Small P-value $\implies$ Reject Independence. Gender and Product Preference are **related**. You cannot market to them the same way.

13.8 Interpreting Excel Output

Problem: Defect Type vs Machine.
Solution:

(a) **Manual Expected (Machine A, Crack):** Row Total (A) = 40. Col Total (Crack) = 40. Grand Total = 100. $E = \frac{40 \times 40}{100} = 16$. Observed = 15. $15 \approx 16$. Machine A produces cracks roughly as expected (average rate).

(b) **Hypothesis Conclusion:** P-value = $0.000216 < 0.05$. **Reject** H_0. Defect Type **depends** on the Machine.

(c) **Business Insight (Dents):** Machine B: Observed Dents = 25. Expected Dents for B = $\frac{60 \times 30}{100} = 18$. Observed (25) > Expected (18). Machine B is disproportionately causing **Dents**. The engineer should inspect Machine B's clamps or impact points.

(a) $E = 16$. Close to observed.
(b) Reject H_0. Dependent.
(c) Fix Machine B (causing Dents).