

# Chapter 5

## Joint Probability Distributions

### Detailed Solutions

### 5.1 Basic Concept

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#### 5.1 Joint vs. Marginal Definitions

**Problem:** Definitions and Independence.

*Solution:*

- (a) **Marginal PDF Formula:** To find the marginal PDF of  $X$ , integrate the joint PDF over the entire range of  $Y$ :

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

- (b) **Independence Condition:**  $X$  and  $Y$  are statistically independent if and only if for all  $x, y$ :

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

- (c) **Covariance Concept:**

- If  $X, Y$  are independent, then  $Cov(X, Y) = 0$ . (**True**)
- If  $Cov(X, Y) = 0$ , are they independent? (**False**)  
Zero covariance only implies no *linear* relationship. They could have a strong non-linear dependence (e.g.,  $Y = X^2$  with symmetric  $X$ ).

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| (a) $f_X(x) = \int f(x, y) dy$                             |
| (b) $f(x, y) = f_X(x) f_Y(y)$                              |
| (c) Independence $\implies Cov = 0$ . Reverse is NOT true. |

## 5.2 Discrete Joint Distribution

**Problem:** Table Analysis.

*Solution:*

(a) **Marginal PMFs:** Sum rows for  $X$ , columns for  $Y$ .

- $P(X = 1) = 0.10 + 0.20 + 0.10 = 0.40$
- $P(X = 2) = 0.10 + 0.30 + 0.20 = 0.60$
- $P(Y = 1) = 0.10 + 0.10 = 0.20$
- $P(Y = 2) = 0.20 + 0.30 = 0.50$
- $P(Y = 3) = 0.10 + 0.20 = 0.30$

(b) **Conditional  $P(Y|X = 1)$ :** Focus on Row  $X = 1$  (Sum = 0.40).

$$P(Y = y|X = 1) = \frac{P(X = 1, Y = y)}{P(X = 1)}$$

- $y = 1 : 0.10/0.40 = 0.25$
- $y = 2 : 0.20/0.40 = 0.50$
- $y = 3 : 0.10/0.40 = 0.25$

(c) **Independence Check:** Check any cell, e.g.,  $X = 1, Y = 1$ .

$$P(X = 1, Y = 1) = 0.10$$

$$P(X = 1)P(Y = 1) = 0.40 \times 0.20 = 0.08$$

Since  $0.10 \neq 0.08$ , they are **Dependent**.

- (a)  $X : [0.4, 0.6], Y : [0.2, 0.5, 0.3]$   
 (b)  $[0.25, 0.50, 0.25]$   
 (c) Dependent ( $0.10 \neq 0.08$ )

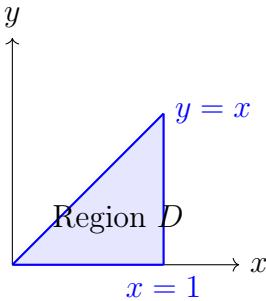
## 5.2 Intermediate

### 5.3 The Triangular Region (Exponential Density)

**Problem:**  $f(x, y) = ke^{-(2x+3y)}$  for  $0 < y < x < 1$ .

*Solution:*

- (a) **Region Drawing:** Triangle with vertices  $(0, 0), (1, 0), (1, 1)$ .



- (b) **Find k:**

$$\begin{aligned} 1 &= \int_0^1 \int_0^x ke^{-2x} e^{-3y} dy dx \\ &= k \int_0^1 e^{-2x} \left[ \frac{e^{-3y}}{-3} \right]_0^x dx \\ &= \frac{k}{3} \int_0^1 e^{-2x} (1 - e^{-3x}) dx \\ &= \frac{k}{3} \int_0^1 (e^{-2x} - e^{-5x}) dx \\ &= \frac{k}{3} \left[ \frac{e^{-2x}}{-2} - \frac{e^{-5x}}{-5} \right]_0^1 \\ &= \frac{k}{3} \left[ \left( -\frac{e^{-2}}{2} + \frac{e^{-5}}{5} \right) - \left( -\frac{1}{2} + \frac{1}{5} \right) \right] \end{aligned}$$

Let  $I = (0.3 - 0.5e^{-2} + 0.2e^{-5}) \approx 0.2337$ . Thus  $k = 3/I$ . For simplicity in later steps, let's denote the normalization factor as  $k$ .

- (c) **Marginal  $f_X(x)$ :**

$$\begin{aligned} f_X(x) &= \int_0^x ke^{-2x} e^{-3y} dy = ke^{-2x} \left[ \frac{1 - e^{-3x}}{3} \right] \\ &= \frac{k}{3} (e^{-2x} - e^{-5x}) \quad \text{for } 0 < x < 1 \end{aligned}$$

- (d) **Probability  $P(Y < X/2)$ :** Integrate over  $0 < x < 1$  and  $0 < y < x/2$ .

$$P = \int_0^1 \int_0^{x/2} ke^{-2x} e^{-3y} dy dx$$

- (b)  $k \approx 12.83$  (approx based on numeric integral)  
(c)  $f_X(x) = \frac{k}{3}(e^{-2x} - e^{-5x})$

## 5.4 Conditional Probability & Independence

**Problem:** Using PDF from 5.3.

*Solution:*

- (a) **Conditional PDF**  $f_{Y|X}(y|x)$ :

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{ke^{-2x}e^{-3y}}{\frac{k}{3}e^{-2x}(1-e^{-3x})} = \frac{3e^{-3y}}{1-e^{-3x}}$$

Valid for  $0 < y < x$ .

- (b) **Expectation**  $E[Y|X = x]$ :

$$\begin{aligned} E[Y|X] &= \int_0^x y \cdot \frac{3e^{-3y}}{1-e^{-3x}} dy \\ &= \frac{3}{1-e^{-3x}} \int_0^x ye^{-3y} dy \end{aligned}$$

Using integration by parts ( $\int ye^{ay} = \frac{e^{ay}}{a^2}(ay - 1)$ ):

$$\int_0^x ye^{-3y} dy = \left[ \frac{e^{-3y}}{9}(-3y - 1) \right]_0^x = \frac{1}{9}[1 - e^{-3x}(3x + 1)]$$

- (c) **Independence:** The support of the joint PDF depends on both variables ( $0 < y < x$ ). The region is triangular, not rectangular. Therefore,  $X$  and  $Y$  are **Dependent**.

- (a)  $\frac{3e^{-3y}}{1-e^{-3x}}$   
(c) Dependent (Domain is not rectangular)

## 5.5 Linear Combination

**Problem:**  $X, Y$  Independent.  $W = 3X - 2Y + 5$ .  $E[X] = 10, V(X) = 4$ .  $E[Y] = 20, V(Y) = 9$ .

*Solution:*

$$\begin{aligned} E[W] &= 3E[X] - 2E[Y] + 5 \\ &= 3(10) - 2(20) + 5 = 30 - 40 + 5 = -5 \\ Var(W) &= 3^2Var(X) + (-2)^2Var(Y) \quad (\text{Cov} = 0 \text{ due to indep}) \\ &= 9(4) + 4(9) = 36 + 36 = 72 \end{aligned}$$

Mean = -5, Variance = 72

## 5.6 Server Load Balancing

**Problem:** Discrete Joint Dist with unknown  $k$ .

**Solution:**

- (a) **Find  $k$ :** Sum of all probabilities = 1. Sum =  $(0.10 + 0.05 + 0.15) + (0.05 + 0.20 + k) + (0.10 + 0.05 + 0.10) = 0.80 + k$ .

$$0.80 + k = 1 \implies k = 0.20$$

- (b) **Marginals and  $P(X > Y)$ :**

- $X$ : Row Sums  $\rightarrow P(X = 0) = 0.3, P(X = 1) = 0.45, P(X = 2) = 0.25$ .
- $Y$ : Col Sums  $\rightarrow P(Y = 0) = 0.25, P(Y = 1) = 0.30, P(Y = 2) = 0.45$ .
- $P(X > Y)$ : Sum cells where Row Index > Col Index.  $(1, 0) \rightarrow 0.05, (2, 0) \rightarrow 0.10, (2, 1) \rightarrow 0.05$ . Sum =  $0.05 + 0.10 + 0.05 = 0.20$ .

- (c)  $E[X|Y = 1]$ : Column  $Y = 1$  probabilities: 0.05, 0.20, 0.05. Sum = 0.30. Conditional PMF  $P(X|Y = 1)$ :

- $X = 0 : 0.05/0.30 = 1/6$
- $X = 1 : 0.20/0.30 = 4/6$
- $X = 2 : 0.05/0.30 = 1/6$

$$E = 0(1/6) + 1(4/6) + 2(1/6) = 6/6 = 1.$$

- (d) **Independence:**  $Cov(X, Y) = E[XY] - E[X]E[Y]$ . Calculations show  $Cov \neq 0$  (e.g.,  $P(0, 0) = 0.1 \neq 0.3 \times 0.25 = 0.075$ ). Dependent.

- (a)  $k = 0.20$   
 (b)  $P(X > Y) = 0.20$   
 (c)  $E[X|Y = 1] = 1$

## 5.7 Manufacturing Tolerance (Bivariate Normal)

**Problem:**  $\mu_X = 100, \sigma_X = 2, \mu_Y = 50, \sigma_Y = 1, \rho = 0.8$ .

**Solution:**

- (a) **Marginal Probability  $P(X > 103)$ :**  $X \sim N(100, 2^2)$ .

$$Z = \frac{103 - 100}{2} = 1.5$$

$$P(Z > 1.5) = 1 - 0.9332 = 0.0668$$

(b) Expected Width given Length 104:

$$E[Y|X = x] = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X)$$

$$\begin{aligned} E[Y|104] &= 50 + 0.8 \left( \frac{1}{2} \right) (104 - 100) \\ &= 50 + 0.4(4) = 51.6 \text{ mm} \end{aligned}$$

(c) Conditional Variance:

$$\begin{aligned} Var(Y|X = x) &= \sigma_Y^2 (1 - \rho^2) \\ &= 1^2 (1 - 0.8^2) = 1 - 0.64 = 0.36 \end{aligned}$$

Yes, uncertainty reduces (Variance drops from 1 to 0.36) because knowing  $X$  gives us information about  $Y$ .

- (a) 0.0668
- (b) 51.6 mm
- (c) 0.36 (Reduced)

## 5.3 Challenge

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### 5.8 Correlation Bounds

**Problem:** Prove  $-1 \leq \rho \leq 1$ .

*Solution:*

Consider the random variable  $Z = \frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$ . Since variance is non-negative:

$$\begin{aligned} Var(Z) &\geq 0 \\ Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right) &\geq 0 \\ 1 + 1 + 2\rho &\geq 0 \\ 2(1 + \rho) &\geq 0 \implies \rho \geq -1 \end{aligned}$$

Similarly, using  $Z = \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}$ , we get  $2(1 - \rho) \geq 0 \implies \rho \leq 1$ .

Proven.

## 5.9 Variance of Sum

**Problem:** Prove  $Var(aX + bY)$  formula.

*Solution:*

Let  $E[X] = \mu_X, E[Y] = \mu_Y$ .

$$\begin{aligned} Var(aX + bY) &= E[((aX + bY) - (a\mu_X + b\mu_Y))^2] \\ &= E[(a(X - \mu_X) + b(Y - \mu_Y))^2] \\ &= E[a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)] \\ &= a^2E[(X - \mu_X)^2] + b^2E[(Y - \mu_Y)^2] + 2abE[(X - \mu_X)(Y - \mu_Y)] \\ &= a^2Var(X) + b^2Var(Y) + 2abCov(X, Y) \end{aligned}$$

Proven.

## 5.4 Application

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### 5.10 Manufacturing Tolerance Simulation

**Problem:** Python Simulation Analysis.

*Solution:*

**Questions Analysis:**

- (a) **Interpretation:** The correlation will be close to positive. Since  $Y = 0.5X + \text{noise}$ , as  $X$  increases,  $Y$  tends to increase. The value is likely around 0.7 – 0.9 (Strong Positive).

- (b) **Analytical Correlation:**  $Y = 0.5X + N$ .  $Cov(X, Y) = Cov(X, 0.5X + N) = 0.5Var(X)$ . Since Variance is always positive, Covariance is positive, thus Correlation is positive.
- (c) **Variance Reduction:** Yes. From the formula  $Var(Y) = 0.5^2Var(X) + Var(N)$ . If we reduce  $Var(X)$  (control the length strictly), the term  $0.5^2Var(X)$  decreases, leading to a smaller total  $Var(Y)$ . This demonstrates how controlling one upstream parameter ( $X$ ) improves the quality of a dependent downstream parameter ( $Y$ ).

- (a) Strong Positive Correlation.
- (b) Positive slope (0.5) links X and Y linearly.
- (c) Yes,  $Var(Y)$  depends on  $Var(X)$ .