Chapter 14

Introduction to Non-parametric Tests

Detailed Solutions

14.1 Basic Concept

14.1 Parametric vs. Non-parametric

Problem: Advantages and Disadvantages.
Solution:

(a) **Assumption Check:** Non-parametric tests are "distribution-free". They do **not** assume the data follows a Normal distribution (or any specific shape), making them safer for small samples or unknown populations.

(b) **Data Type:** For Ordinal data (ranks, ratings) or skewed/outlier-heavy data, **Non-parametric tests** are preferred.

(c) **Efficiency:** If the data is Normal, a T-test is more powerful because it uses all the information (actual magnitudes, mean, variance). Non-parametric tests convert data to ranks, discarding the magnitude information, thus having lower power (less likely to detect a difference if one exists).

(a) No Normality assumption. (b) Non-parametric. (c) T-test is more powerful for Normal data.

14.2 The Art of Ranking (Handling Ties)

Problem: Data: 12, 5, 8, 5, 20, 8, 8, 15.
Solution:

(a) **Sorted Data:** 5, 5, 8, 8, 8, 12, 15, 20.

(b) **Assign Ranks:**

- Value 5 is at positions 1 and 2. Average Rank $= (1 + 2)/2 = 1.5$.
- Value 8 is at positions 3, 4, 5. Average Rank $= (3 + 4 + 5)/3 = 4$.
- Value 12 is at pos 6 $\rightarrow$ Rank 6.
- Value 15 is at pos 7 $\rightarrow$ Rank 7.
- Value 20 is at pos 8 $\rightarrow$ Rank 8.

(c) **Final Ranks (Original Order):** Data: 12, 5, 8, 5, 20, 8, 8, 15 Ranks: 6, 1.5, 4, 1.5, 8, 4, 4, 7

$\boxed{\text{Ranks: 6, 1.5, 4, 1.5, 8, 4, 4, 7}}$

14.2 Intermediate

14.3 The Sign Test

Problem: $H_0 : \tilde{\mu} = 15$. Data: 14, 12, 16, 13, 15, 11, 14, 18, 10, 13.
Solution:

(a) **Assign Signs ($M_0 = 15$):**

- 14 (-), 12 (-), 16 (+), 13 (-), 15 (0), 11 (-), 14 (-), 18 (+), 10 (-), 13 (-)
- The value 15 is tied with the median. It is typically **discarded**.

(b) **Counts:** Valid $n = 9$ (discarded one). Minus signs (x) = 7. Plus signs = 2.

(c) **Test:** If H_0 is true ($p = 0.5$), expected minus signs ≈ 4.5. $P(X \geq 7)$ for $B(9, 0.5)$:

$$P(X = 7) + P(X = 8) + P(X = 9)$$
$$= \binom{9}{7}(0.5)^9 + \binom{9}{8}(0.5)^9 + \binom{9}{9}(0.5)^9$$
$$= (36 + 9 + 1)/512 = 46/512 \approx 0.09$$

Two-tailed P-value ≈ 0.18. Since $0.18 > 0.05$, **Fail to Reject**. Not enough evidence.

(b) $x = 7, n = 9$.
(c) Fail to Reject.

14.4 Wilcoxon Signed-Rank Test

Problem: Paired Data.
Solution:

(a) **Diffs (d_i):** $5-8 = -3$, $6-7 = -1$, $7-6 = +1$, $4-9 = -5$, $5-5 = 0$, $2-8 = -6$. Discard 0. Valid $n = 5$. $d = \{-3, -1, +1, -5, -6\}$.

(b) **Abs Ranks:** $|d| = \{3, 1, 1, 5, 6\}$. Sorted: 1, 1, 3, 5, 6. Ranks: 1 and 1 are tied (pos 1,2) $\rightarrow$ Rank 1.5. 3 is pos 3 $\rightarrow$ Rank 3. 5 is pos 4 $\rightarrow$ Rank 4. 6 is pos 5 $\rightarrow$ Rank 5.

(c) **Signed Ranks:** Original: -3 (Rank 3), -1 (Rank 1.5), +1 (Rank 1.5), -5 (Rank 4), -6 (Rank 5). Signed: -3, -1.5, +1.5, -4, -5.

(d) **Sums:** $W_+ = 1.5$. $W_- = 3 + 1.5 + 4 + 5 = 13.5$.

(e) **Evidence Against:** The drug is effective if pain *decreases* (After < Before $\rightarrow$ Negative Diff). So large W_- supports the drug. Small W_+ suggests few cases where pain increased. The test statistic is usually $\min(W_+, W_-) = 1.5$.

(d) $W_+ = 1.5, W_- = 13.5$.

14.5 Mann-Whitney U Test

Problem: A: 40, 45, 50, 200. B: 35, 38, 42, 44, 48.
Solution:

(a) **Ranking Combined:** Sorted: 35(B), 38(B), 40(A), 42(B), 44(B), 45(A), 48(B), 50(A), 200(A). Ranks: B: 1, 2, 4, 5, 7. A: 3, 6, 8, 9.

(b) **Sum Ranks A (R_1):** $R_1 = 3 + 6 + 8 + 9 = 26$.

(c) **Calculate U:**

$$\begin{aligned} U_1 &= n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1 \\ &= (4)(5) + \frac{4(5)}{2} - 26 \\ &= 20 + 10 - 26 = 4 \end{aligned}$$

(d) **Outlier Effect:** In a T-test, the value 200 would massively pull the mean of Group A, increasing variance. In Mann-Whitney, 200 is simply "the largest value" (Rank 9). Whether it is 200 or 2,000,000, its rank remains 9. Thus, rank tests are robust to outliers.

(b) $R_A = 26$.
(c) $U = 4$.

14.6 Kruskal-Wallis Test

Problem: Diet A (1,3,5), B (2,4,6), C (7,8,9).
Solution:

Analysis: Diet C has exclusively the highest ranks (7, 8, 9). Diet A and B are mixed in the lower range. The sum of ranks for C will be much higher than expected under the null hypothesis (random mixing). This separation suggests a **significant difference**. Specifically, Diet C yields much higher values (weight loss?) than A and B.

14.3 Applications

14.7 T-Test vs. Mann-Whitney on Skewed Data

Problem: Group B has outlier 1000.
Solution:

(a) **Means:** Mean A ≈ 32.5. Mean B will be huge (≈ 188) due to the 1000. Mean B is not representative.

(b) **T-Test:** Because of the huge variance in B (caused by outlier), the standard error is large. The T-statistic will be small (despite the mean difference). High P-value $\rightarrow$ Fail to Reject.

(c) **Mann-Whitney:** It ranks the data. 1000 becomes just "Rank 12". Most values in A (30s) are larger than most values in B (20s). The test sees that A consistently ranks higher than B (except for the one outlier). Low P-value $\rightarrow$ Reject H_0. It correctly identifies the shift in the median/distribution.

14.8 Spearman's Rank Correlation

Problem: Study vs Rank.
Solution:

- Study Hours: [10, 2, 8, 20, 5]
- Sorted: 2, 5, 8, 10, 20
- Ranks (R_X): 10 $\rightarrow$ 4 2 $\rightarrow$ 1 8 $\rightarrow$ 3 20 $\rightarrow$ 5 5 $\rightarrow$ 2 $R_X = [4, 1, 3, 5, 2]$.
- Exam Rank (R_Y): Given as [2, 5, 3, 1, 4]. Wait, let's assume "Rank 1" is best (highest hours?). Usually Rank 1 = Smallest. Let's check correlation. R_X: 4, 1, 3, 5, 2. R_Y: 2, 5, 3, 1, 4. This looks like a **perfect negative correlation**. Low hours (Rank 1) $\rightarrow$ High Exam Rank number (Rank 5 = Worst?). Or High hours (Rank 5) $\rightarrow$ Low Exam Rank number (Rank 1 = Best).

Calculation: $d = R_X - R_Y$: $4 - 2 = 2$ $1 - 5 = -4$ $3 - 3 = 0$ $5 - 1 = 4$ $2 - 4 = -2$ $\sum d^2 = 4 + 16 + 0 + 16 + 4 = 40$.

$$\rho = 1 - \frac{6 \sum d^2}{n(n^2 - 1)} = 1 - \frac{6(40)}{5(24)} = 1 - \frac{240}{120} = 1 - 2 = -1$$

$\boxed{\rho = -1. \text{ Perfect negative correlation. (More study hours} \rightarrow \text{Better rank (lower number)).}}$