

# Chapter 4

## Continuous Random Variables & Distributions

### Detailed Solutions

#### 4.1 Basic Concept

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##### 4.1 PDF Properties

**Problem:** Validating PDF properties (True/False).

**Solution:**

- (a) **True.** The probability density  $f(x)$  represents height, not probability. For example, a Uniform distribution on  $[0, 0.1]$  has height  $f(x) = \frac{1}{0.1} = 10$ .
- (b) **True.** For continuous variables, the probability of any single exact point is zero because the area under a single point is zero ( $\int_c^c f(x)dx = 0$ ).
- (c) **True.** The total area under the PDF curve must equal 1 (Normalization axiom).
- (d) **True.** Since  $f(x) = F'(x)$ , if  $f(x)$  is increasing ( $f'(x) > 0$ ), then  $F''(x) > 0$ . A function with a positive second derivative is convex.

- (a) True ( $f(x)$  is density, not probability)
- (b) True ( $P(X = c) = 0$ )
- (c) True (Total Prob = 1)
- (d) True ( $F''(x) > 0$ )

#### 4.2 Fundamental of Continuous RV

**Problem:**  $f(x) = k(1 - x^2)$  for  $-1 < x < 1$ .

**Solution:**

(a) **Find k:** Integrate over the support and set to 1.

$$\begin{aligned}\int_{-1}^1 k(1-x^2)dx &= 1 \\ k \left[ x - \frac{x^3}{3} \right]_{-1}^1 &= 1 \\ k \left[ \left( 1 - \frac{1}{3} \right) - \left( -1 - \frac{-1}{3} \right) \right] &= 1 \\ k \left[ \frac{2}{3} - \left( -\frac{2}{3} \right) \right] &= 1 \implies k \left( \frac{4}{3} \right) = 1 \implies k = \frac{3}{4}\end{aligned}$$

(b) **Find CDF  $F(x)$ :** For  $-1 < x < 1$ :

$$\begin{aligned}F(x) &= \int_{-1}^x \frac{3}{4}(1-t^2)dt = \frac{3}{4} \left[ t - \frac{t^3}{3} \right]_{-1}^x \\ &= \frac{3}{4} \left[ \left( x - \frac{x^3}{3} \right) - \left( -1 + \frac{1}{3} \right) \right] \\ &= \frac{3}{4} \left( x - \frac{x^3}{3} + \frac{2}{3} \right) = \frac{3x - x^3 + 2}{4}\end{aligned}$$

(c) **Calculate Probability:**

$$\begin{aligned}P(-0.5 < X < 0.5) &= \int_{-0.5}^{0.5} \frac{3}{4}(1-x^2)dx \\ &= 2 \times \int_0^{0.5} \frac{3}{4}(1-x^2)dx \quad (\text{Symmetry}) \\ &= \frac{3}{2} \left[ x - \frac{x^3}{3} \right]_0^{0.5} = \frac{3}{2} \left[ 0.5 - \frac{0.125}{3} \right] \\ &= \frac{3}{2} \left[ \frac{1.5 - 0.125}{3} \right] = \frac{1.375}{2} = 0.6875\end{aligned}$$

(d) **Expectation and Variance:**  $E[X] = \int_{-1}^1 x \cdot k(1-x^2)dx = 0$  (Odd function over symmetric interval).

$$Var(3X) = 3^2 Var(X) = 9E[X^2] \quad (\text{since } E[X] = 0).$$

$$\begin{aligned}E[X^2] &= \int_{-1}^1 x^2 \cdot \frac{3}{4}(1-x^2)dx = \frac{3}{4} \int_{-1}^1 (x^2 - x^4)dx \\ &= \frac{3}{4} \cdot 2 \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{3}{2} \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{3}{2} \left( \frac{2}{15} \right) = \frac{1}{5} = 0.2\end{aligned}$$

$$Var(3X) = 9(0.2) = 1.8.$$

(a)  $k = 3/4$   
 (b)  $F(x) = \frac{-x^3+3x+2}{4}$   
 (c) 0.6875  
 (d)  $E[X] = 0, Var(3X) = 1.8$

## 4.2 Intermediate

### 4.3 Uniform Distribution: Rounding Error

**Problem:**  $X \sim U(-0.5, 0.5)$ .

**Solution:**

(a) **PDF:** Since length  $b - a = 0.5 - (-0.5) = 1$ . Height  $= 1/(b - a) = 1$ .

$$f(x) = \begin{cases} 1 & -0.5 < x < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

(b) **Probability:** Area of rectangle from -0.2 to 0.2. Width = 0.4, Height = 1.

$$P(-0.2 < X < 0.2) = 0.4 \times 1 = 0.4$$

(c) **Mean and Variance:**

$$E[X] = \frac{a + b}{2} = \frac{-0.5 + 0.5}{2} = 0$$

$$Var(X) = \frac{(b - a)^2}{12} = \frac{1^2}{12} \approx 0.0833$$

(a)  $f(x) = 1$   
 (b) 0.4  
 (c) Mean = 0, Var = 1/12

### 4.4 Exponential: Component Life

**Problem:** Mean  $\mu = 10,000$  hours.

**Solution:**

(a) **Parameter  $\lambda$ :**

$$\lambda = \frac{1}{\mu} = \frac{1}{10000} = 0.0001$$

(b) **More than 12,000 hours:** Using CDF  $P(X > x) = e^{-\lambda x}$ .

$$P(X > 12000) = e^{-0.0001 \times 12000} = e^{-1.2} \approx 0.3012$$

(c) **Fail within 5,000 hours:**

$$P(X < 5000) = 1 - e^{-\lambda x} = 1 - e^{-0.5} \approx 1 - 0.6065 = 0.3935$$

(d) **Median Lifetime:** Solve  $F(m) = 0.5 \implies 1 - e^{-\lambda m} = 0.5 \implies e^{-\lambda m} = 0.5$ .

$$-\lambda m = \ln(0.5) = -\ln(2) \implies m = \frac{\ln(2)}{\lambda} = 10000 \times 0.6931 = 6931 \text{ hours}$$

- (a)  $\lambda = 10^{-4}$   
 (b) 0.301  
 (c) 0.394  
 (d) 6,931 hours

## 4.5 Normal Distribution: Grades

**Problem:**  $X \sim N(75, 100) \implies \sigma = 10$ .

**Solution:**

- (a) **Between 60 and 80:** Convert to Z-scores:  $Z = \frac{X - \mu}{\sigma}$ .

$$Z_1 = \frac{60 - 75}{10} = -1.5$$

$$Z_2 = \frac{80 - 75}{10} = 0.5$$

$$\begin{aligned} P &= P(-1.5 < Z < 0.5) = \Phi(0.5) - \Phi(-1.5) \\ &= 0.6915 - 0.0668 = 0.6247 \end{aligned}$$

- (b) **Top 10% Cutoff:** We need  $P(Z > k) = 0.10$ , so  $P(Z < k) = 0.90$ . From Z-table,  $k \approx 1.28$ .

$$X = \mu + k\sigma = 75 + 1.28(10) = 87.8$$

- (c) **Exactly 75:** For any continuous distribution, the probability of taking on a specific exact value is 0.

$$P(X = 75) = 0$$

(Note: If the question implies discrete rounding, the answer would differ, but mathematically for Normal dist, it is 0).

- (a) 0.6247  
 (b) 87.8  
 (c) 0 (Continuous variable)

## 4.6 Standard Normal Z

**Problem:** Using Z-table.

**Solution:**

(a)  $P(Z > 1.645) = 1 - P(Z < 1.645) = 1 - 0.95 = 0.05$ .

(b)  $P(-1.96 < Z < 1.96) = 0.975 - 0.025 = 0.95$ .

(c)  $P(Z < k) = 0.95 \implies k = 1.645$ .

- (a) 0.05  
 (b) 0.95  
 (c)  $k = 1.645$

## 4.3 Challenge

### 4.7 Mean of Exponential (Proof)

**Problem:** Prove  $E[X] = 1/\lambda$  using Integration by Parts.

**Solution:**

$$f(x) = \lambda e^{-\lambda x}.$$

$$E[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

Integration by parts: Let  $u = x \implies du = dx$ . Let  $dv = \lambda e^{-\lambda x} dx \implies v = -e^{-\lambda x}$ .

$$\begin{aligned} E[X] &= [-xe^{-\lambda x}]_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x}) dx \\ &= (0 - 0) + \int_0^{\infty} e^{-\lambda x} dx \quad (\text{Since } \lim_{x \rightarrow \infty} \frac{x}{e^{\lambda x}} = 0 \text{ by L'Hopital}) \\ &= \left[ -\frac{1}{\lambda} e^{-\lambda x} \right]_0^{\infty} \\ &= 0 - \left( -\frac{1}{\lambda} \right) = \frac{1}{\lambda} \end{aligned}$$

For Variance, perform integration by parts twice on  $E[X^2]$ .

Proven.

### 4.8 Normal Inflection Points

**Problem:** Prove inflection at  $\mu \pm \sigma$ .

**Solution:**

Let  $z = \frac{x-\mu}{\sigma}$ . PDF is proportional to  $e^{-z^2/2}$ .

$$\begin{aligned} f'(x) &= f(x) \cdot \frac{d}{dx} \left( -\frac{(x-\mu)^2}{2\sigma^2} \right) = f(x) \cdot \left( -\frac{x-\mu}{\sigma^2} \right) \\ f''(x) &= f'(x) \left( -\frac{x-\mu}{\sigma^2} \right) + f(x) \left( -\frac{1}{\sigma^2} \right) \quad (\text{Product Rule}) \\ &= f(x) \left( -\frac{x-\mu}{\sigma^2} \right)^2 - \frac{f(x)}{\sigma^2} \\ &= \frac{f(x)}{\sigma^2} \left[ \left( \frac{x-\mu}{\sigma} \right)^2 - 1 \right] \end{aligned}$$

Set  $f''(x) = 0$ . Since  $f(x) \neq 0$ :

$$\left( \frac{x-\mu}{\sigma} \right)^2 = 1 \implies \frac{x-\mu}{\sigma} = \pm 1 \implies x = \mu \pm \sigma$$

Proven.

## 4.9 Constructing a Triangular PDF

**Problem:** Triangle from  $a$  to  $b$  with peak at  $c$ .

**Solution:**

- (a) **Value of  $k$ :** Area of triangle  $= \frac{1}{2} \times \text{Base} \times \text{Height} = 1$ .

$$\frac{1}{2}(b-a)k = 1 \implies k = \frac{2}{b-a}$$

- (b) **Piecewise Function:**  $L_1$  goes from  $(a, 0)$  to  $(c, k)$ . Slope  $m_1 = \frac{k}{c-a}$ . Equation:  $f(x) = \frac{2(x-a)}{(b-a)(c-a)}$ .  $L_2$  goes from  $(c, k)$  to  $(b, 0)$ . Slope  $m_2 = \frac{-k}{b-c}$ . Equation:  $f(x) = \frac{2(b-x)}{(b-a)(b-c)}$ .

- (c) **Expectation:**

$$E[X] = \int_a^c x L_1 dx + \int_c^b x L_2 dx$$

After evaluating these integrals (algebraically intensive), the result simplifies to the centroid of the triangle's x-projection:

$$E[X] = \frac{a+b+c}{3}$$

- (d) **Geometry Calculation** ( $a = 0, c = 2, b = 4$ ):  $k = 2/4 = 0.5$ . We want  $P(X > 3)$ . This region is a small triangle from  $x = 3$  to  $x = 4$ . Base  $= 4 - 3 = 1$ . Height at  $x = 3$ : Equation of line  $L_2$  is  $y = -0.25(x - 4)$ . At  $x = 3, y = 0.25$ .

$$\text{Area} = \frac{1}{2} \times 1 \times 0.25 = 0.125$$

- (a)  $k = \frac{2}{b-a}$   
 (c)  $E[X] = (a+b+c)/3$   
 (d) 0.125

## 4.4 Application

### 4.10 Numerical Integration (Python)

**Problem:** Code for Trapezoidal rule.

**Solution:**

```
1 import numpy as np
2 from scipy.stats import norm
3
4 def standard_normal(x):
5     return (1/np.sqrt(2*np.pi)) * np.exp(-0.5 * x**2)
6
7 # Parameters
8 x_min = -10 # Approximating negative infinity
9 x_max = 2
10 n = 1000 # Number of trapezoids
11 dx = (x_max - x_min) / n
12
13 # Trapezoidal Rule Implementation
14 area = 0
15 for i in range(n):
16     x1 = x_min + i*dx
17     x2 = x_min + (i+1)*dx
18     # Area = dx * average height
19     area += dx * (standard_normal(x1) + standard_normal(x2)) / 2
20
21 exact_val = norm.cdf(2)
22
23 print(f"Numerical Integration P(Z<2): {area:.5f}")
24 print(f"Scipy Exact Value: {exact_val:.5f}")
25 print(f"Error: {abs(area - exact_val):.5e}")
```

### 4.11 Fitting a Distribution

**Problem:** Fitting Exponential to data.

**Solution:**

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3 from scipy.stats import expon
4
5 # 1. Generate Dummy Data (for demonstration)
6 np.random.seed(42)
7 data = expon.rvs(scale=100, size=1000) # True lambda = 1/100 = 0.01
8
9 # 2. Fit Distribution
10 # expon.fit returns (loc, scale). For exponential, scale = 1/lambda
11 loc, scale = expon.fit(data, floc=0)
12 lambda_est = 1/scale
13
14 print(f"Estimated Lambda: {lambda_est:.5f}")
15
16 # 3. Plotting
```

```
17 plt.figure(figsize=(10, 6))
18
19 # Histogram of data
20 plt.hist(data, bins=30, density=True, alpha=0.6, color='g', label='Data
    Hist')
21
22 # Fitted PDF
23 x = np.linspace(0, max(data), 1000)
24 pdf = expon.pdf(x, scale=scale)
25 plt.plot(x, pdf, 'r-', linewidth=2, label=f'Fitted Exp($\lambda$={
    lambda_est:.4f})')
26
27 plt.title('Fitting Exponential Distribution to Machine Failure Times')
28 plt.xlabel('Time')
29 plt.ylabel('Density')
30 plt.legend()
31 plt.show()
```

The `scipy.stats.expon.fit` function uses Maximum Likelihood Estimation (MLE) to find the parameter that maximizes the probability of observing the given data.