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DeepLearning.AI

Probability and Statistics for Machine Learning and Data Science

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Week 3: Sampling and Point Estimates

W3 Lesson 1

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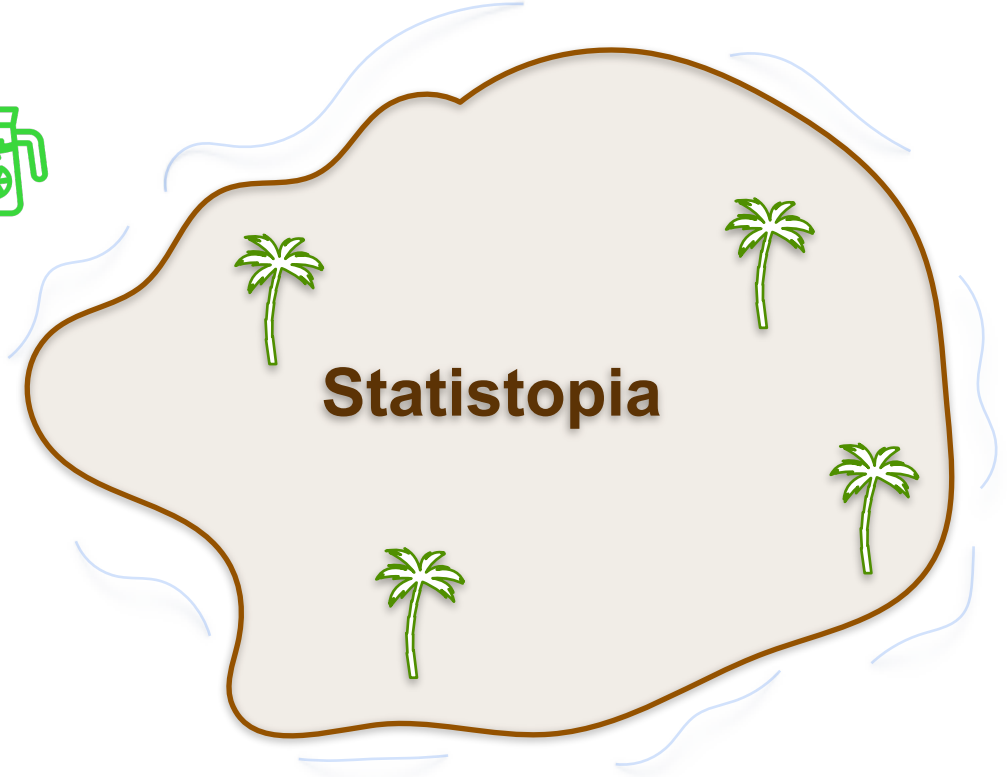
Sample and Population

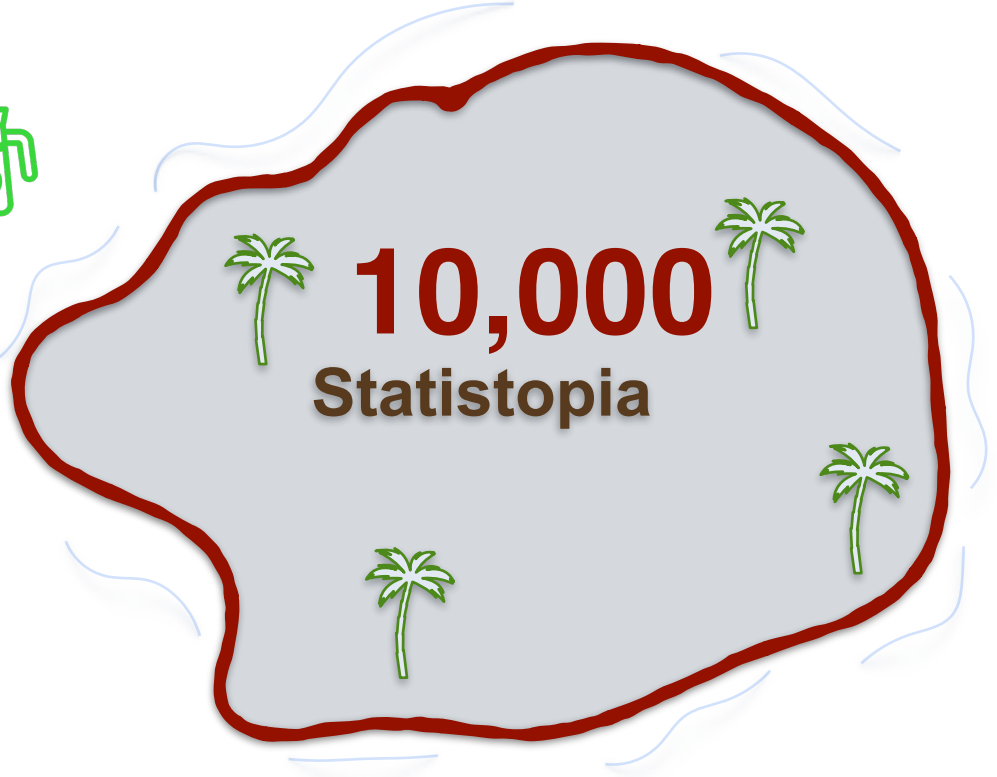
Population and Sample

Population and Sample



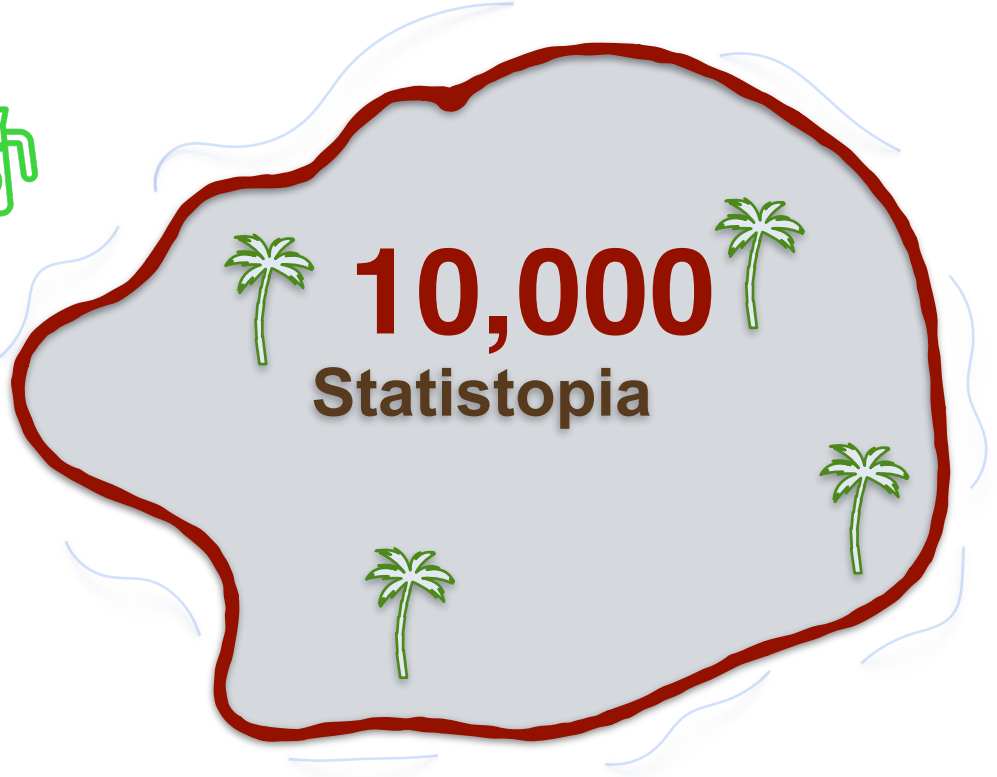
Find the **average height** of the people living on Statistopia

Population and Sample



- Ask everyone on the island for their height.
- Divide by the total number

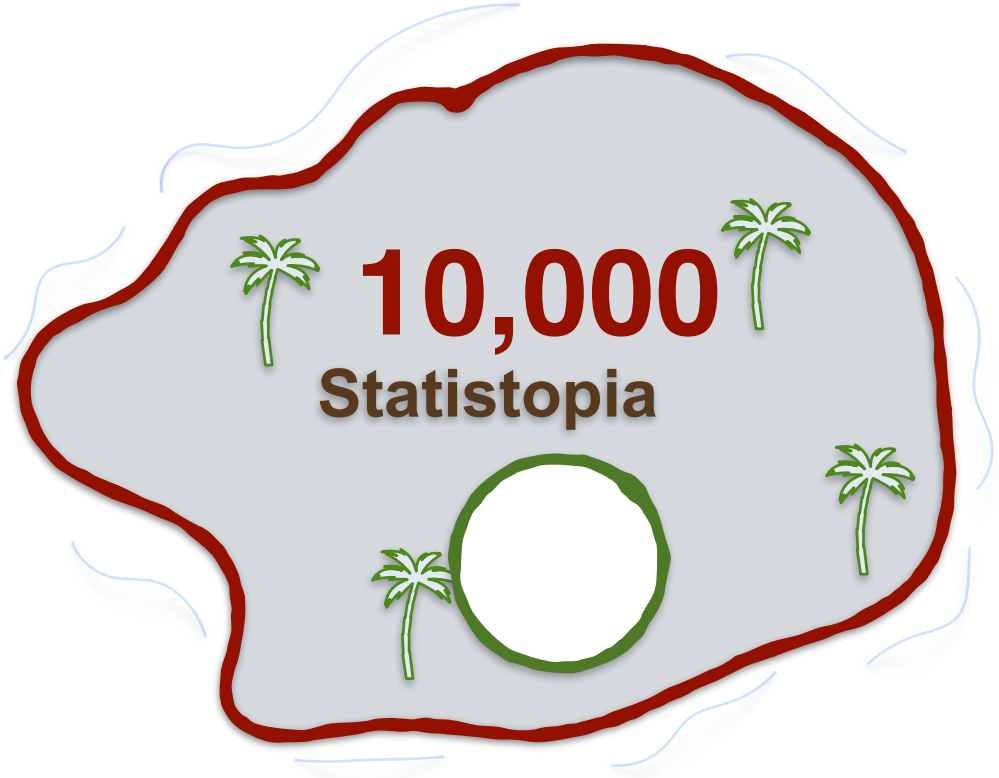
Population and Sample



- Ask everyone on the island for their height.
- Divide by the total number

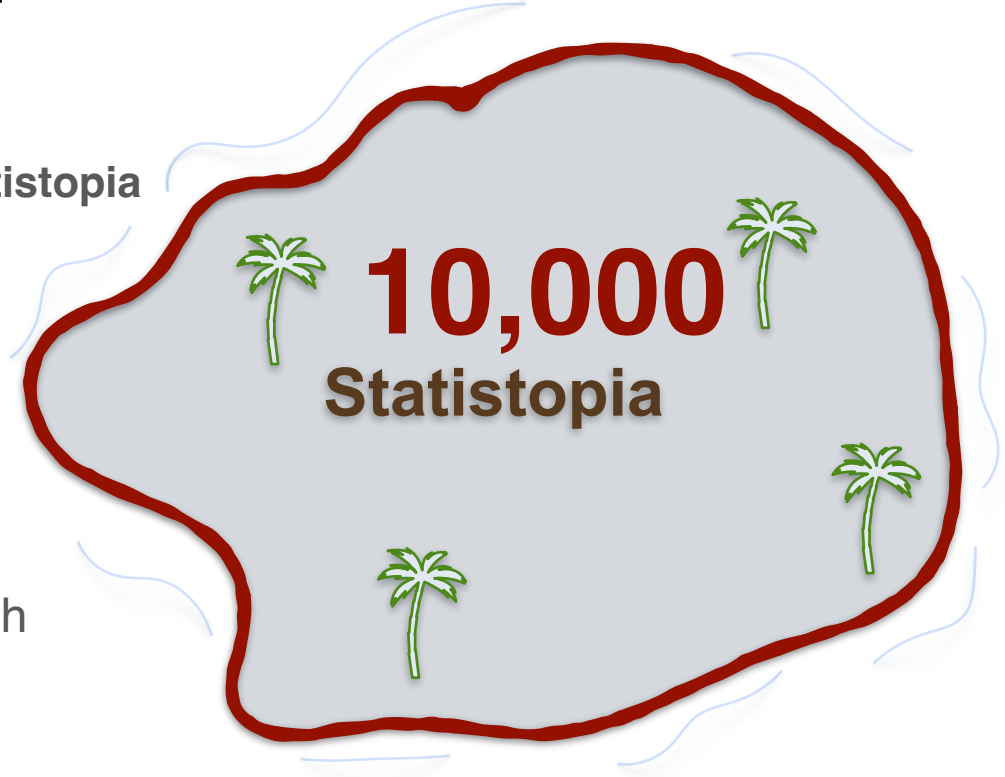
Population and Sample

- Only ask a subset of the group to estimate the average height



Population and Sample

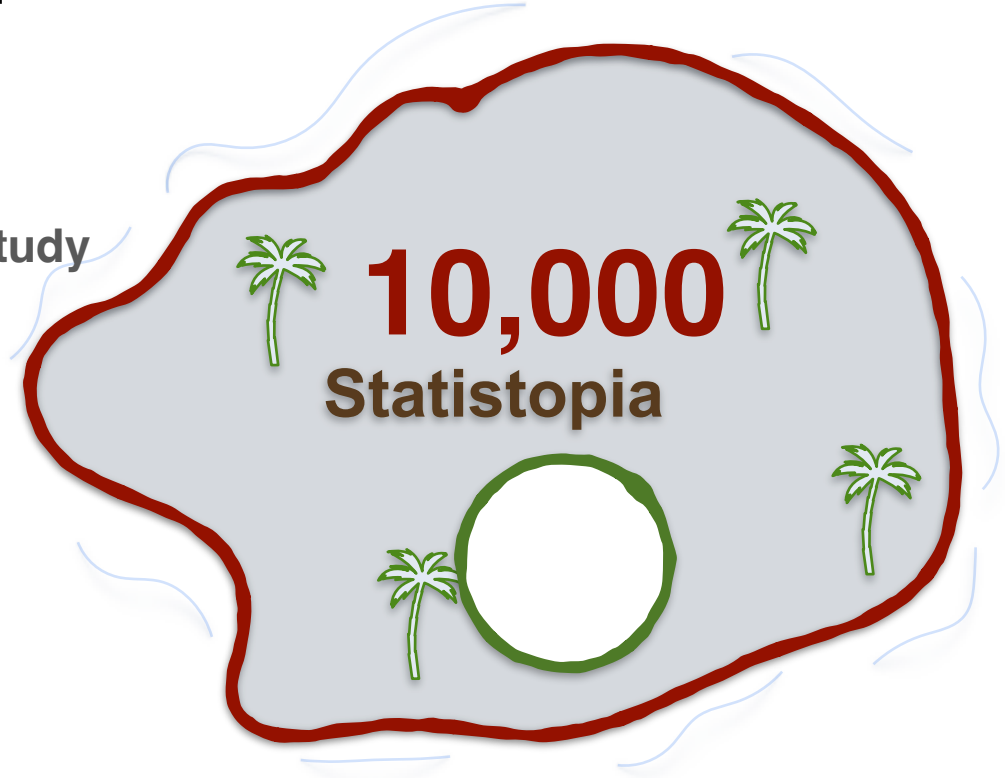
The people of statistopia



Population:
the entire group of individuals or elements you want to study which share a common behaviour

Population and Sample

The people you select for your study



Sample:
subset of the population you use to draw conclusions about the population as a whole

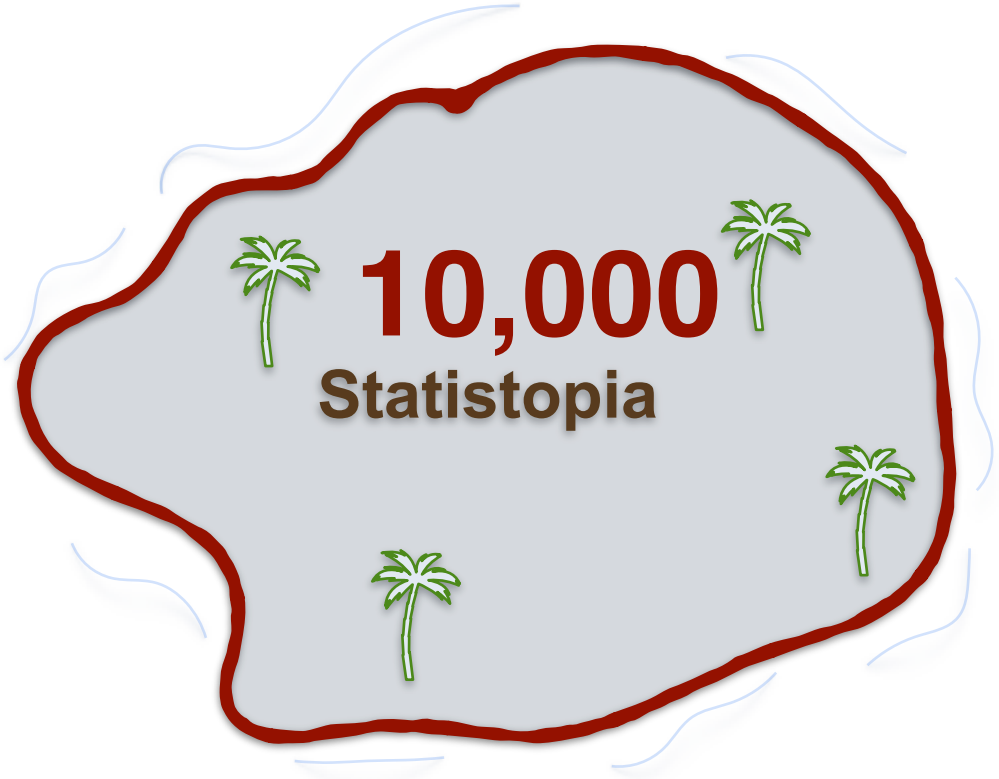
Population and Sample

Population Size (N)

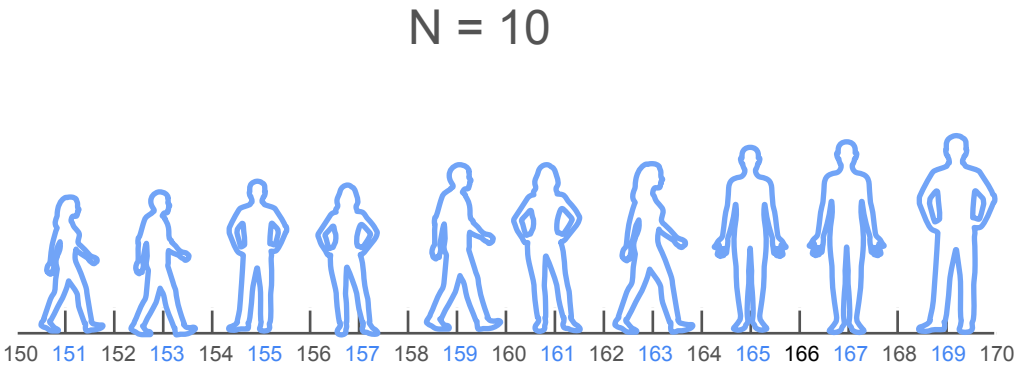
10,000

Sample Size (n)

1 - 9,999

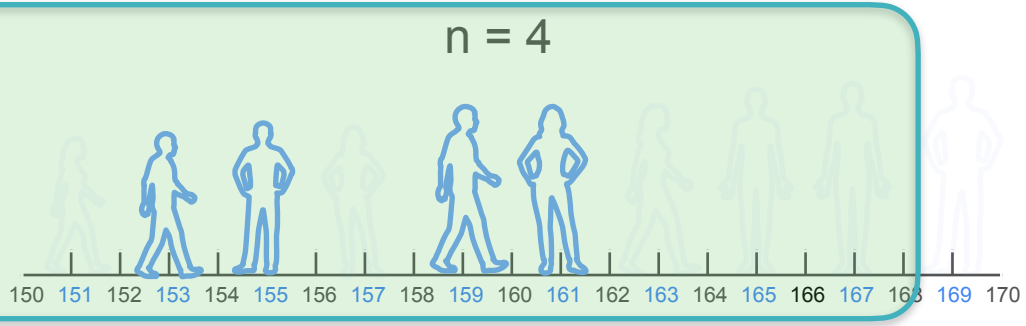


Population and Sample



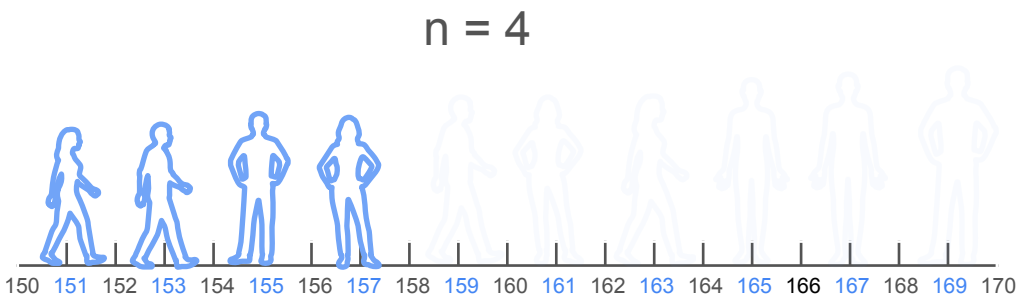
Random Sampling

A



Which is the better sample to estimate the population mean height?

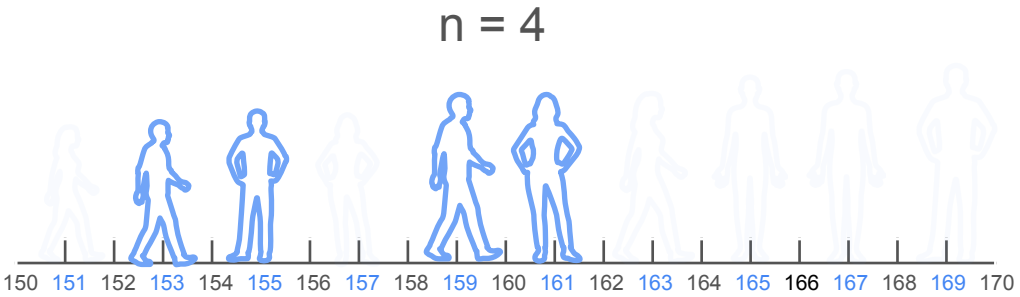
B



Independent Sample

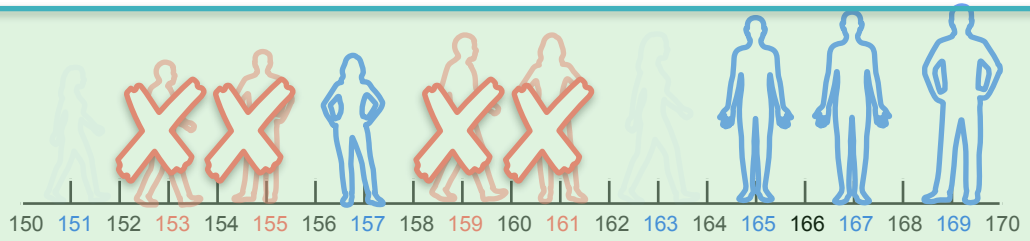
Example 1

1st sample set



Why is sample set two not a good sample?

2nd sample set

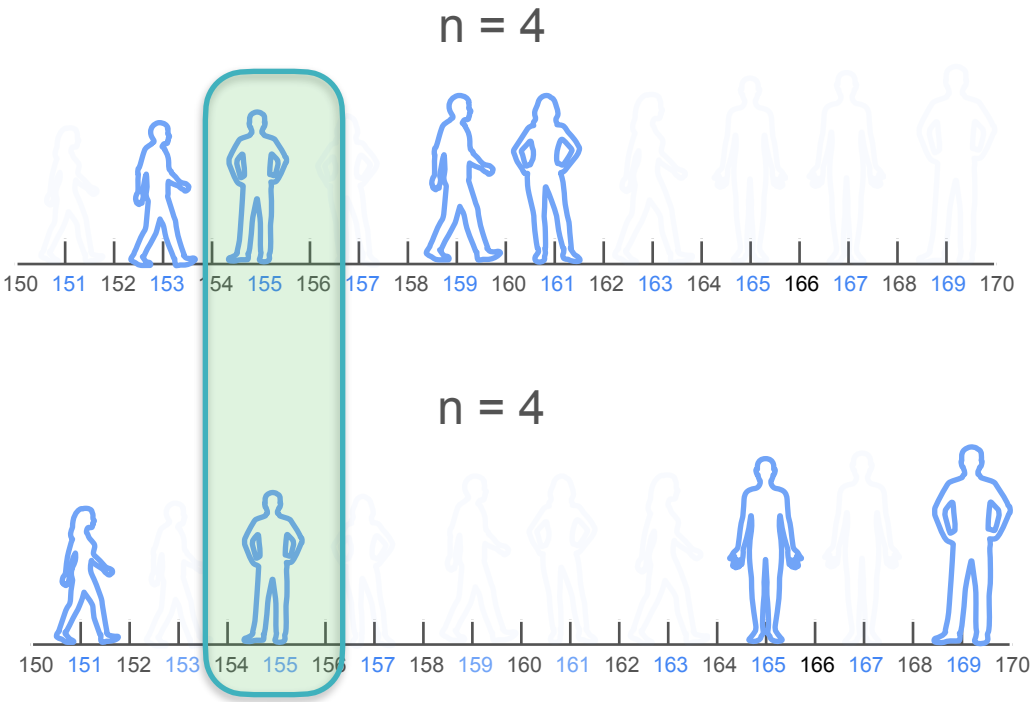


Independent Sample

Example 2

1st sample set

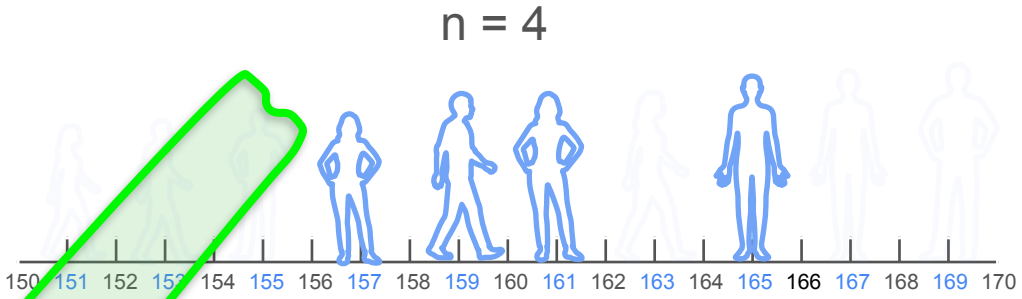
2nd sample set



Identically Distributed Samples

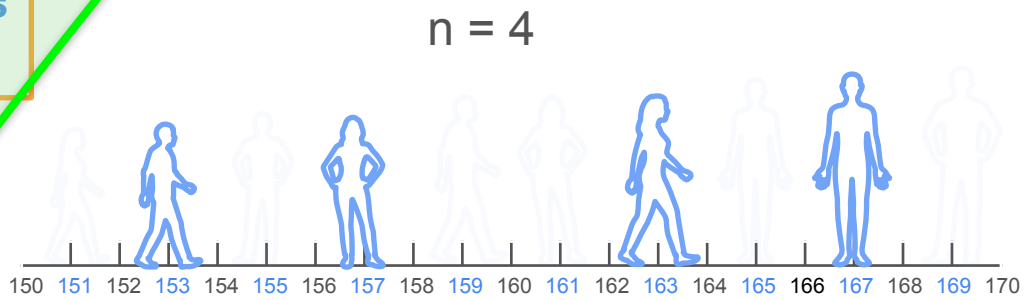
Example 1

A



Which of the following samples are identically distributed?

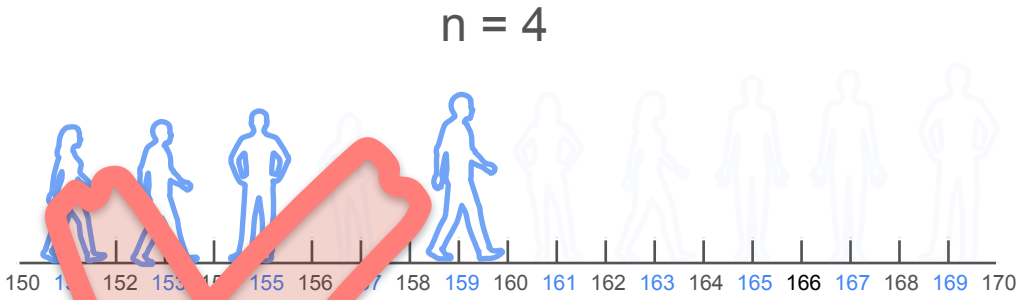
B



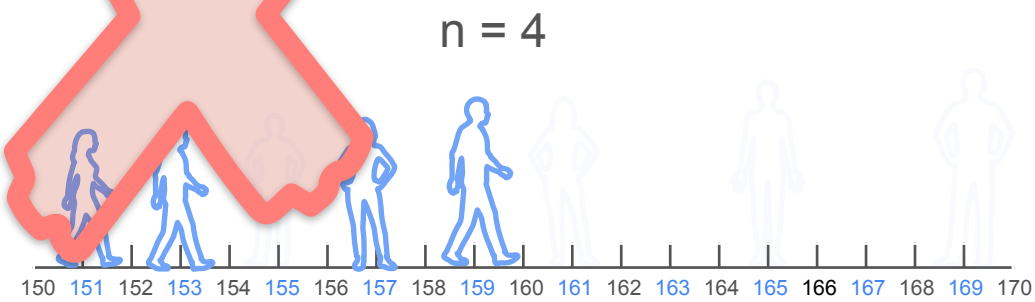
Identically Distributed Samples

Example 2

A

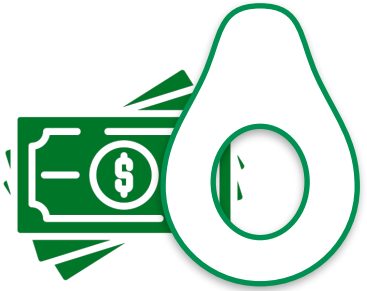


B



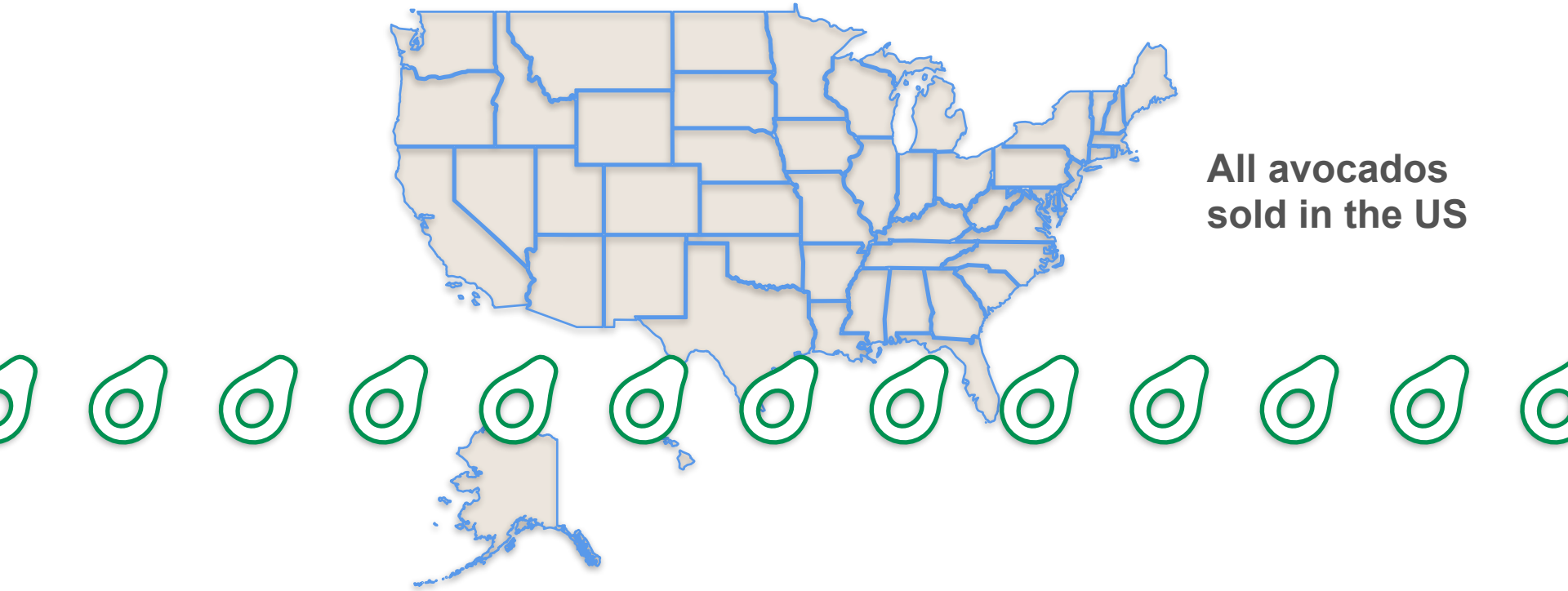
The Avocado Toast Trend

Study the price of avocados in the United States



What is the population of your study?

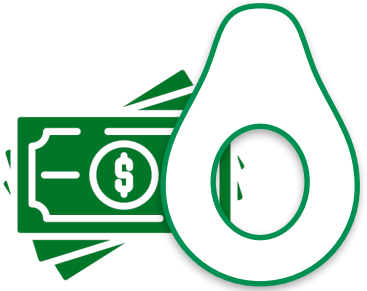
The Avocado Toast Trend



All avocados sold in the US

The Avocado Toast Trend

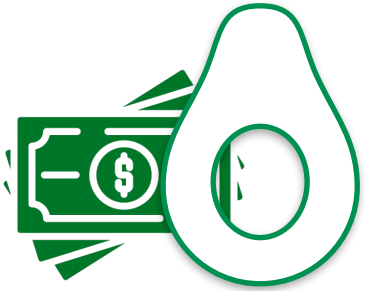
Study the price of avocados in the United States



What is the sample of your study?

The Avocado Toast Trend

Study the price of avocados in the United States



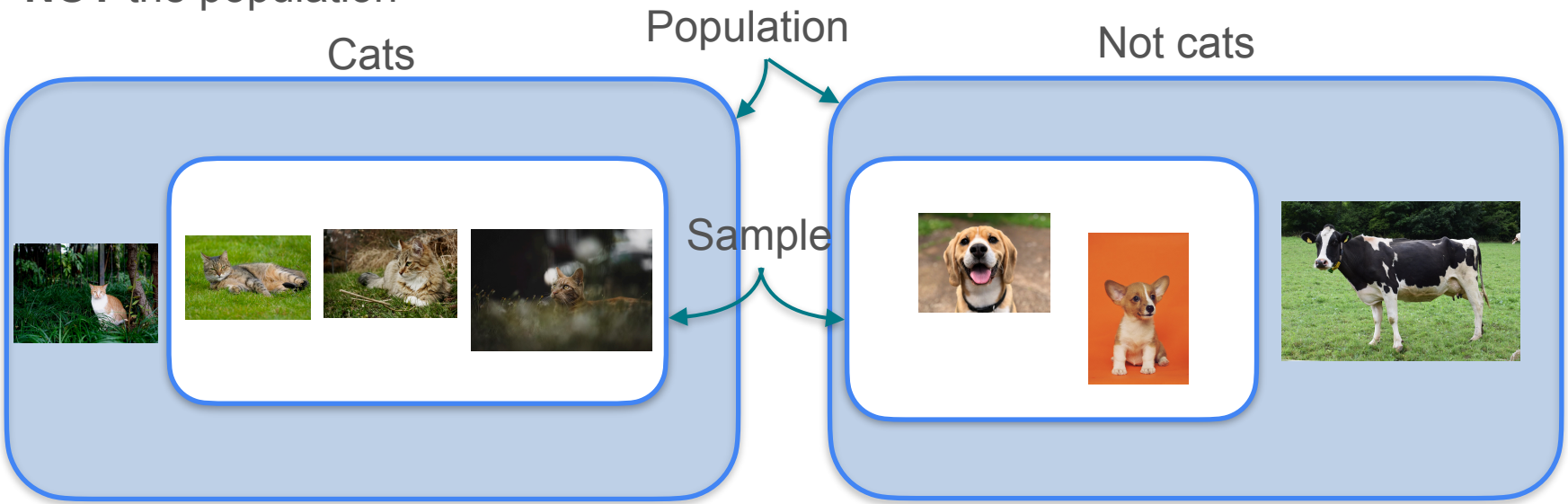
What is the sample of your study?

Avocados sold in the 4 stores you selected

Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample

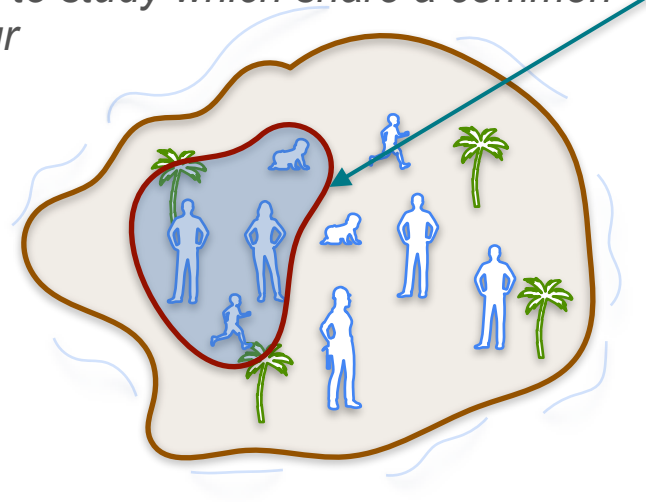
NOT the population



Recap

Population

the entire group of individuals or elements you want to study which share a common behaviour



Sample

subset of the population you use to draw conclusions about the population as a whole

Population Size: N

Sample Size: n

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Sample and Population

Sample Mean

Population and Sample Mean

What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10}$$

$$= \frac{1600}{10} = 160cm$$

Population mean μ

Population and Sample Mean

$n = 6$

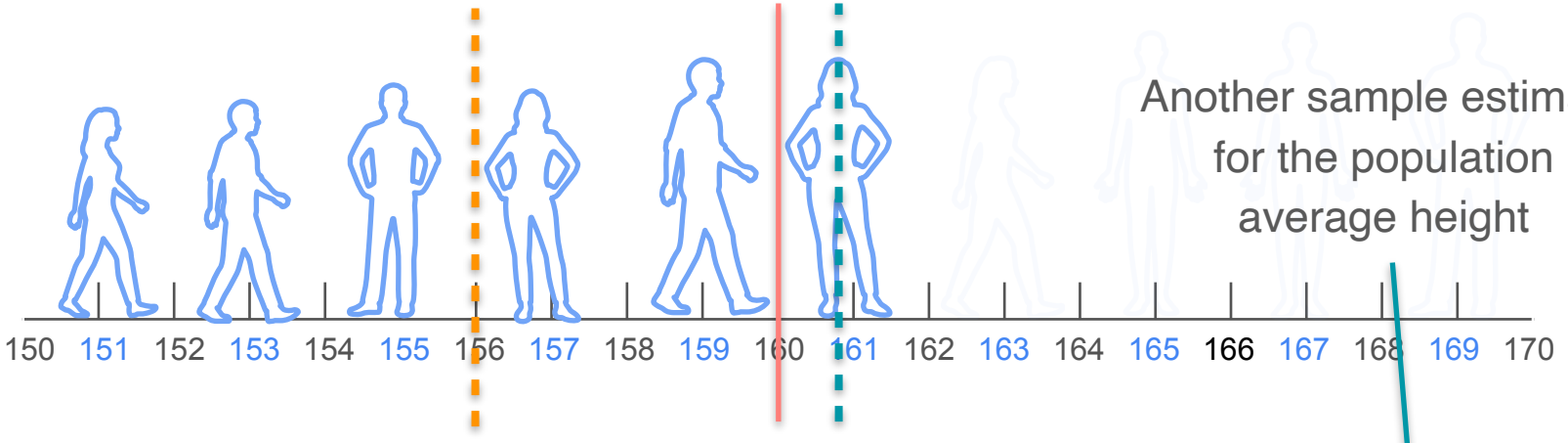
μ $\bar{x}_1$

What is the average height in statistopia?

$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6} = \frac{964}{6} = 160.97$$

$\bar{x}$

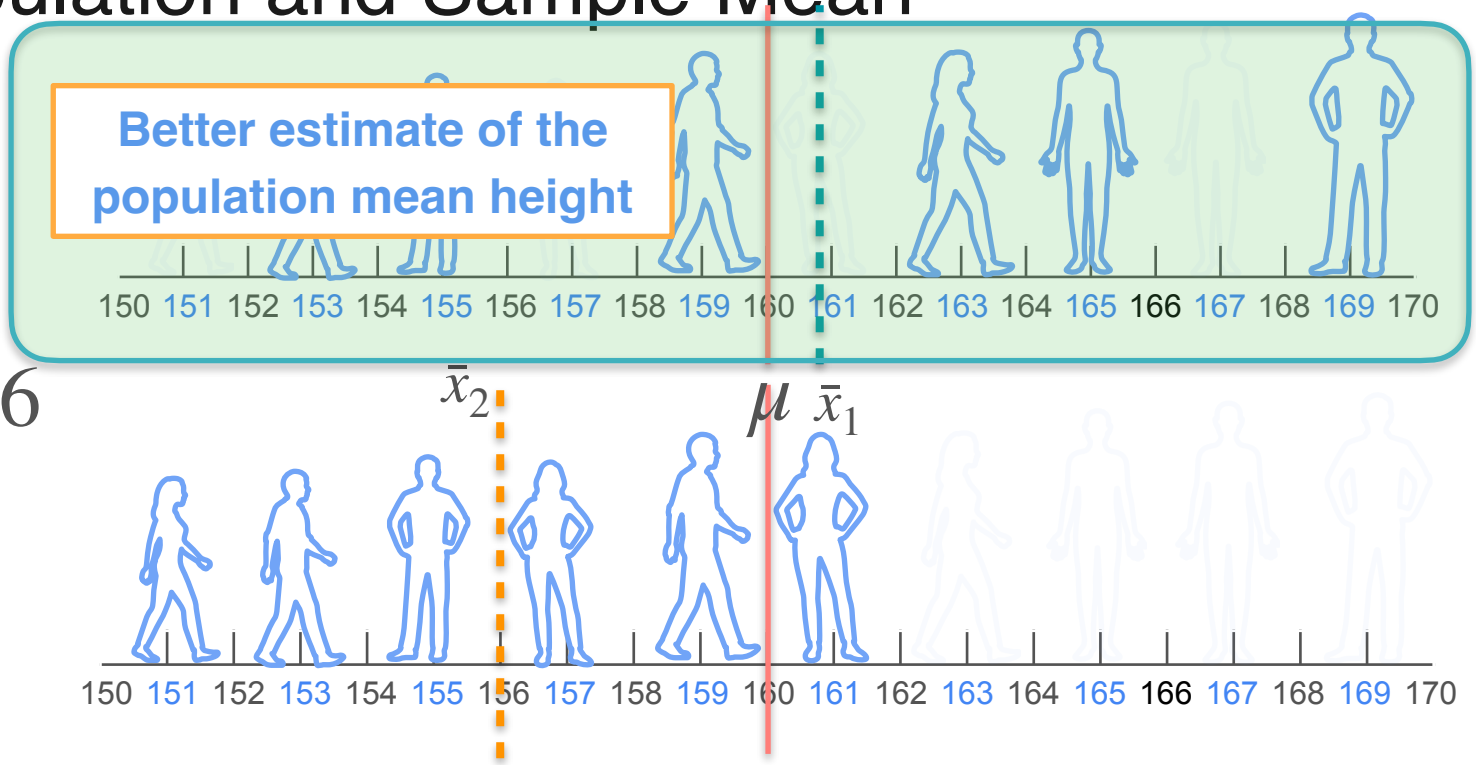
Population and Sample Mean



What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156cm$$

Population and Sample Mean

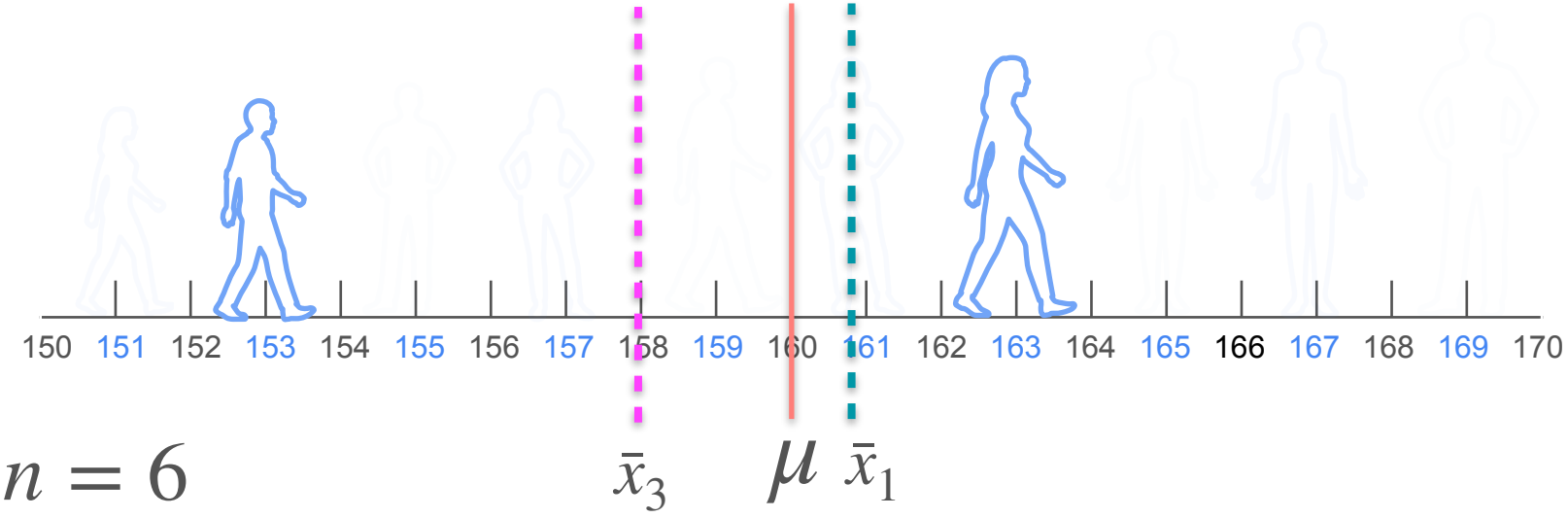


Population and Sample Mean

$n = 6$

$n = 2$

Population and Sample Mean



$n = 6$

$n = 2$

What is the average height in statistopia?

$$\frac{153 + 163}{2} = \frac{316}{2} = 158cm$$

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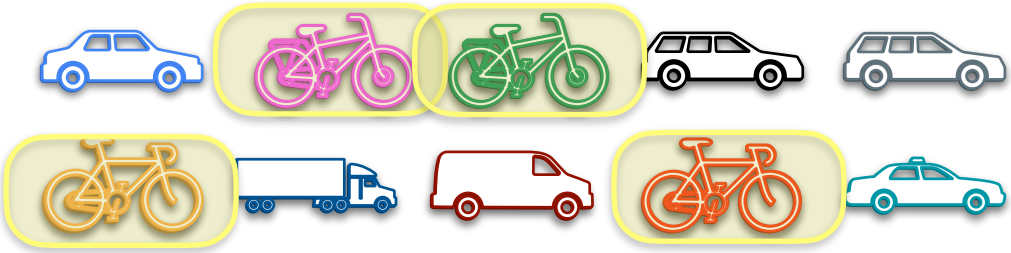
Sample and Population

Sample Proportion

Proportion

Population size: 10



What proportion of people own a bicycle?

p population proportion $= \frac{4}{10} = 0.4 = 40\,\%$

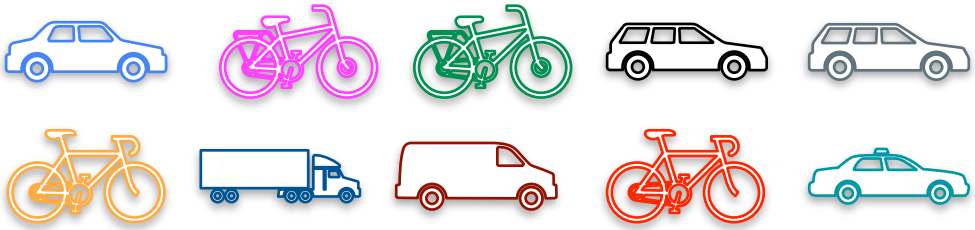
Proportion

population proportion

$$p = \frac{\text{number of items with a given characteristic } (x)}{\text{population } (N)}$$

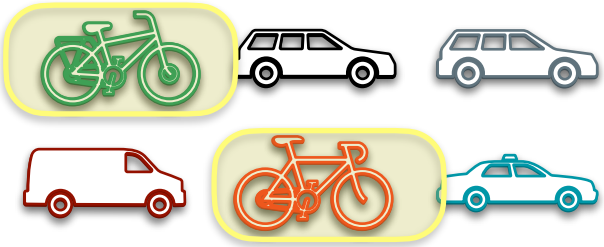
Proportion

Population size: 10



Sample Proportion

Sample size: 6



What proportion of people own a bicycle?

estimate of the population proportion

$$\hat{p} \text{ sample proportion } = \frac{2}{6} = 0.333 = 33.3\,\%$$

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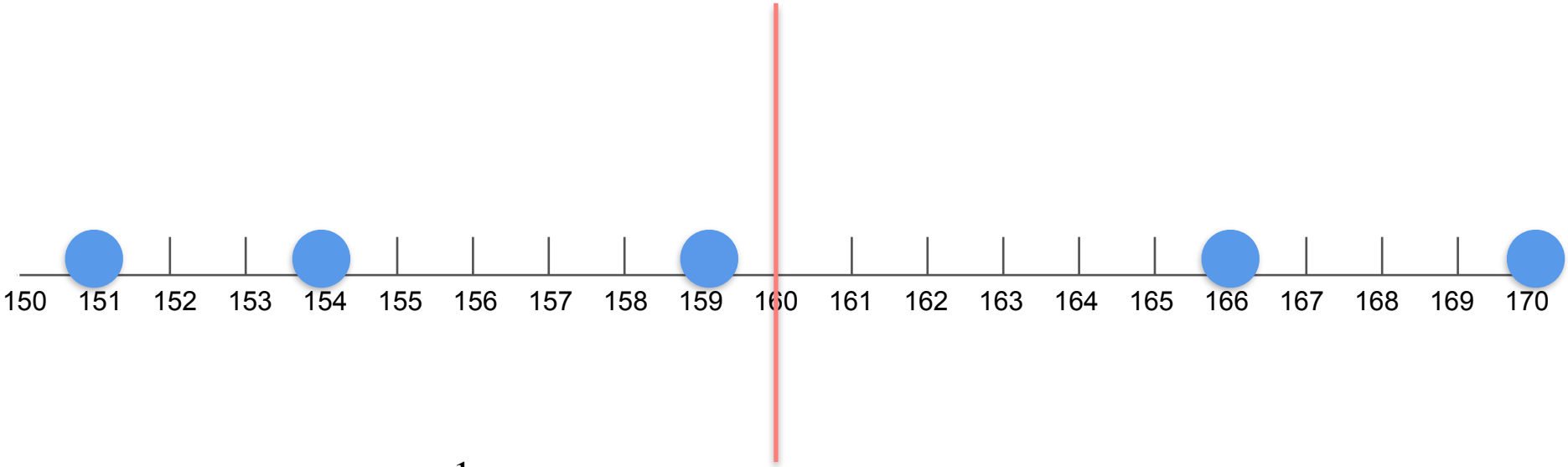
Sample and Population

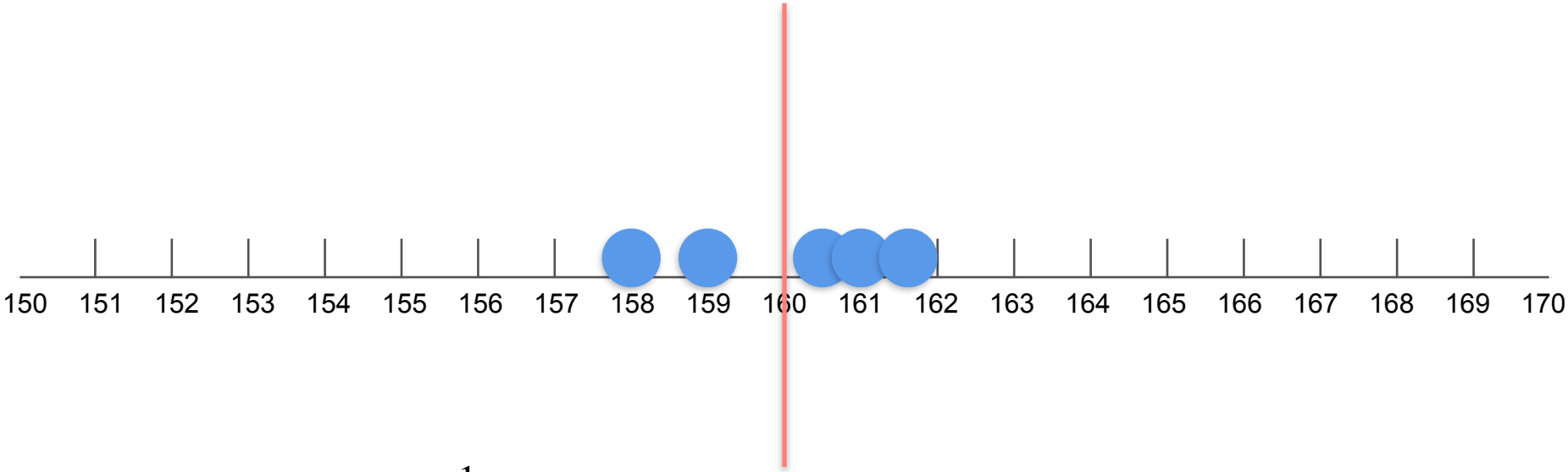
Sample Variance

Sample Variance



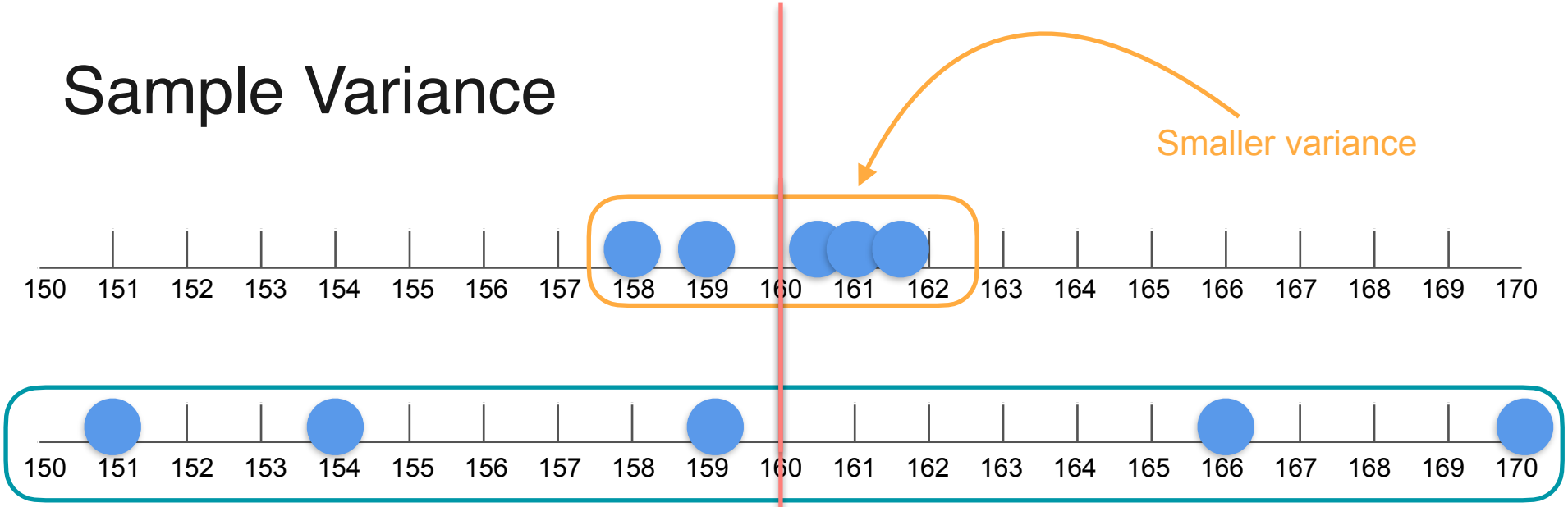
$$\mu = \frac{1}{5}\left(151 + 154 + 159 + 166 + 170\right) = 160$$

Sample Variance



$$\mu = \frac{1}{5}\left(158 + 159 + 160.5 + 161 + 161.5\right) = 160$$

Sample Variance



$$Var(X) = \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

population mean

population size

How to estimate population variance with a sample?

Sample Variance

Let's cheat and use the sample mean

$$Var(X) = \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 \longrightarrow \widehat{Var(X)} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$Y = (X - \mu)^2$$

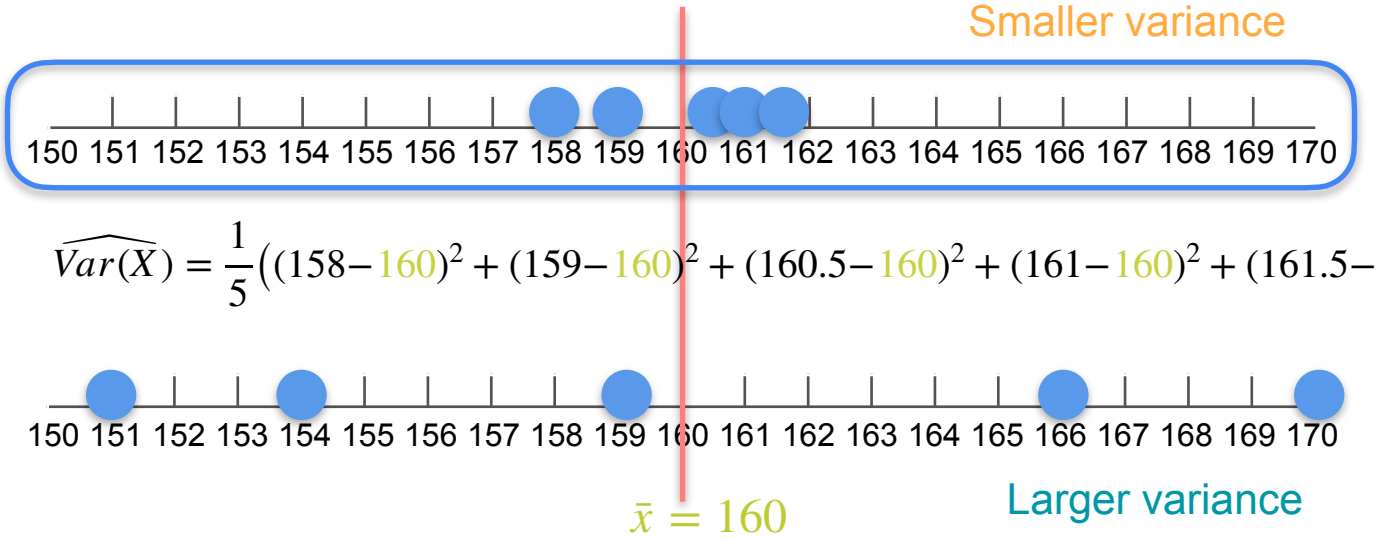
$$\mathbb{E}[Y] = \mu_Y = \frac{1}{N}\sum_{i=1}^{N} y_i \longrightarrow \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

The population mean of Y

The sample mean of Y

Sample Variance

$$Var(X) = \sigma^2 = \frac{1}{N}\sum (x-\mu)^2 \longrightarrow \widehat{Var(X)} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

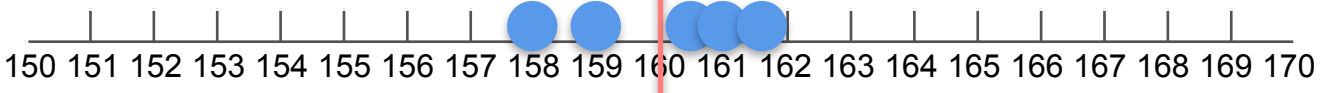


Sample Variance

This equation is “biased”
It underestimates the population variance

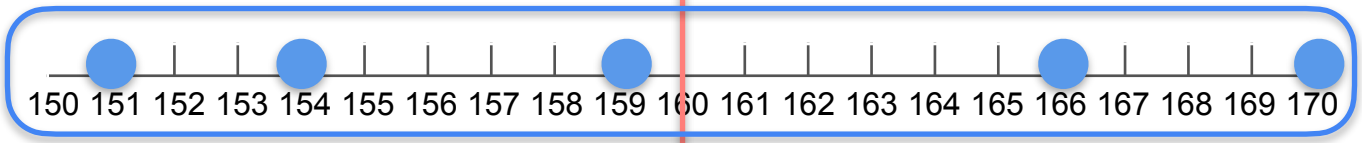
$$Var(X) = \sigma^2 = \frac{1}{N}\sum (x-\mu)^2 \longrightarrow \widehat{Var(X)} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

Smaller variance



$\widehat{Var(X)} = 1.7$

$$\widehat{Var(X)} = \frac{1}{5}\left((151-160)^2 + (154-160)^2 + (159-160)^2 + (166-160)^2 + (170-160)^2\right) = 50.8$$

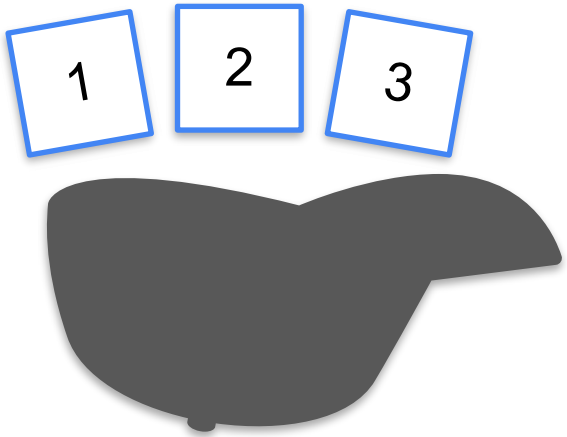


$\widehat{Var(X)} = 50.8$

$\bar{x} = 160$

Larger variance

Variance Estimation



$$\mu = \frac{1+2+3}{3} = \frac{6}{3} = 2$$

Variance Estimation

$\mu = 2$

$$\sigma^2 = \frac{1}{N}\sum (x-\mu)^2$$

$x - 2$

x	$x-\mu$	$(x-\mu)^2$
1	-1	1
2	0	0
3	1	1

$$\frac{\sum (x-\mu)^2}{N} = \frac{2}{3}$$

σ^2

Population variance

Variance Estimation

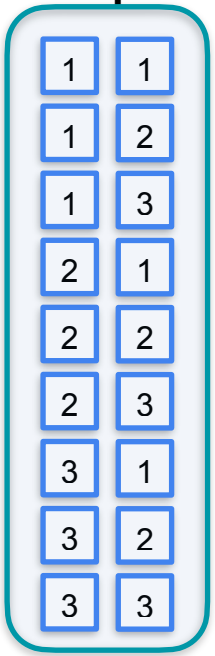
$\mu = 2$

$$\sigma^2 = \frac{1}{N}\sum (x-\mu)^2$$

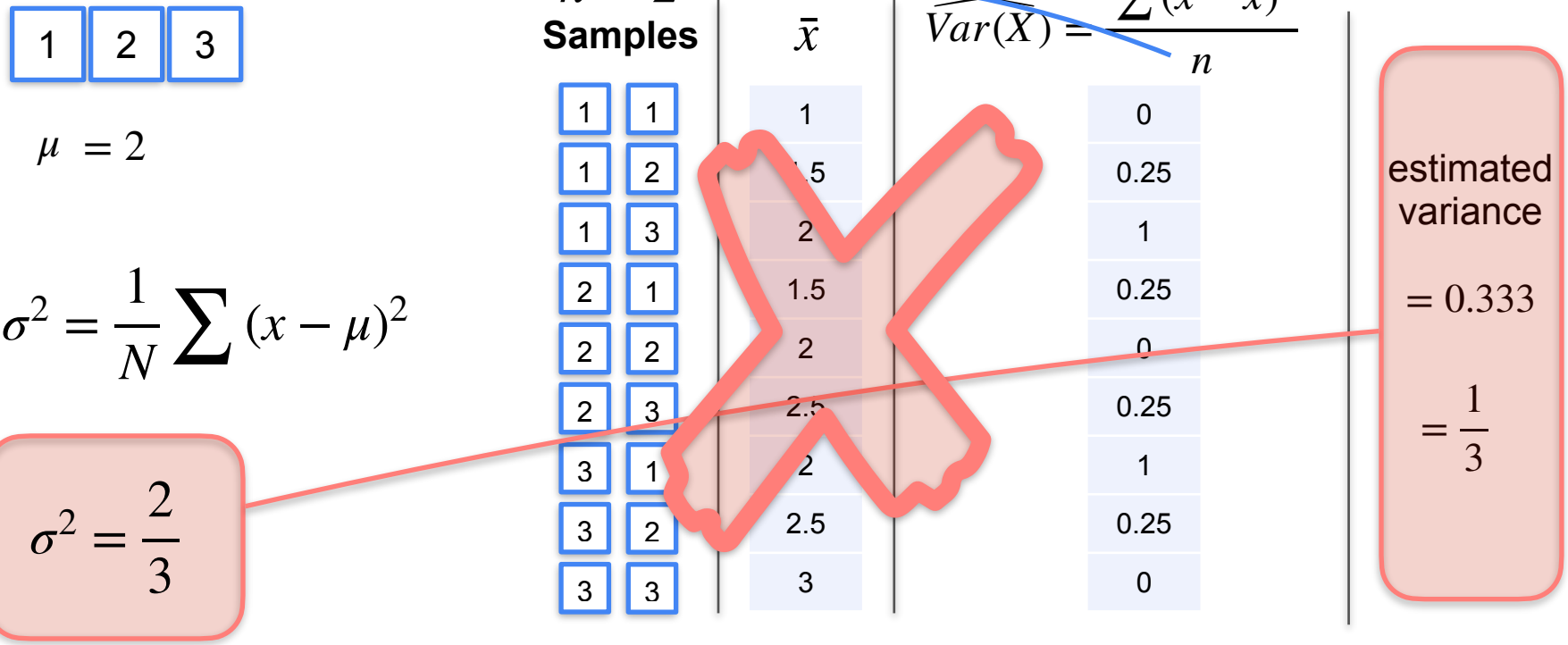
$$\sigma^2 = \frac{2}{3}$$

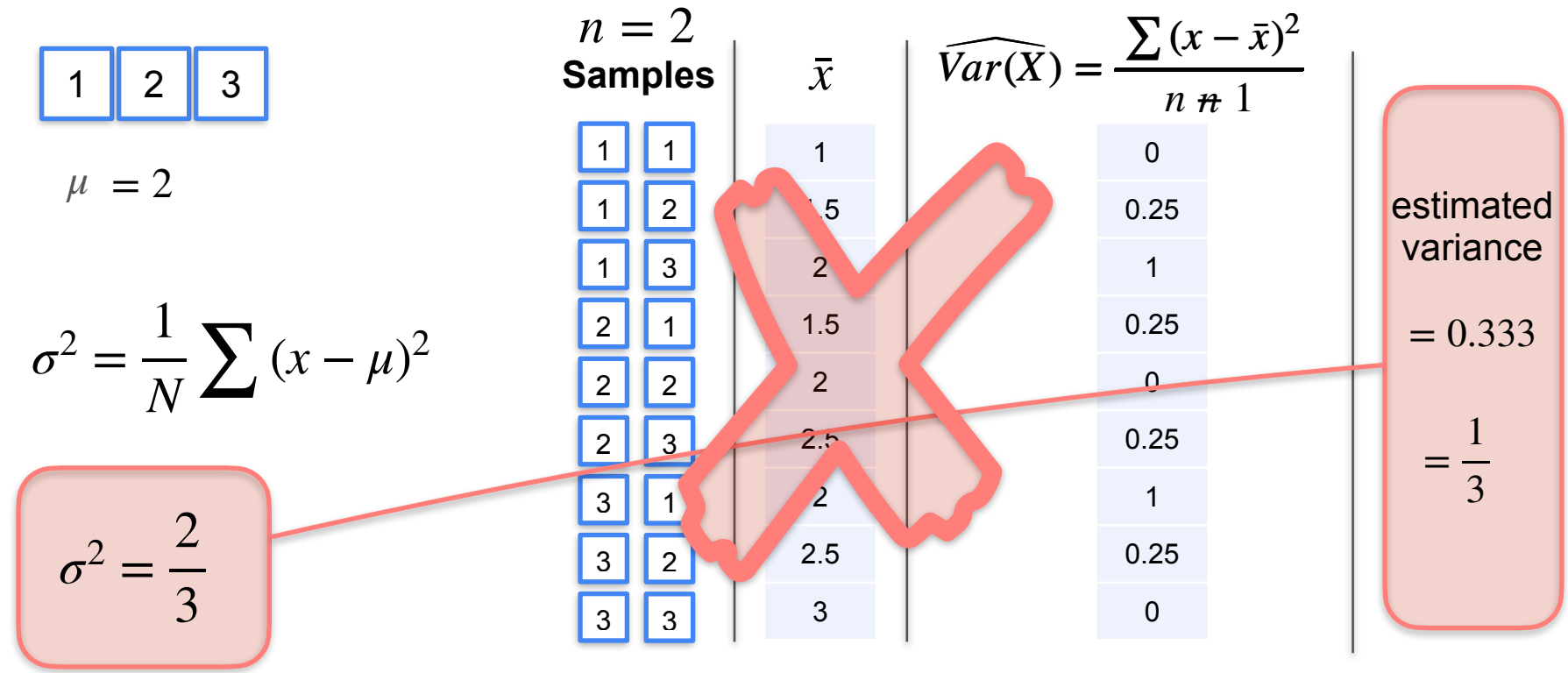
$n = 2$

Samples



Variance Estimation




1 2 3
$\mu = 2$
$\sigma^2 = \frac{1}{N}\sum(x-\mu)^2$
$\sigma^2 = \frac{2}{3}$
$n = 2$
Samples
$\bar{x}$
$\widehat{Var(X)} = \frac{\sum(x-\bar{x})^2}{n-1}$
1 1 | 1 | 0
1 2 | 1.5 | 0.25
1 3 | 2 | 1
2 1 | 1.5 | 0.25
2 2 | 2 | 0
2 3 | 2.5 | 0.25
3 1 | 2 | 1
3 2 | 2.5 | 0.25
3 3 | 3 | 0
estimated variance
$= 0.333$
$= \frac{1}{3}$

Variance Estimation

$\mu = 2$

$$\sigma^2 = \frac{1}{N}\sum (x-\mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$

Samples		$\bar{x}$	$s^2 = \frac{\sum (x-\bar{x})^2}{n-1}$
1	1	1	0
1	2	1.5	0.5
1	3	2	2
2	1	1.5	0.5
2	2	2	0
2	3	2.5	0.5
3	1	2	2
3	2	2.5	0.5
3	3	3	0

estimated variance

$= 0.667$

$= \frac{2}{3}$

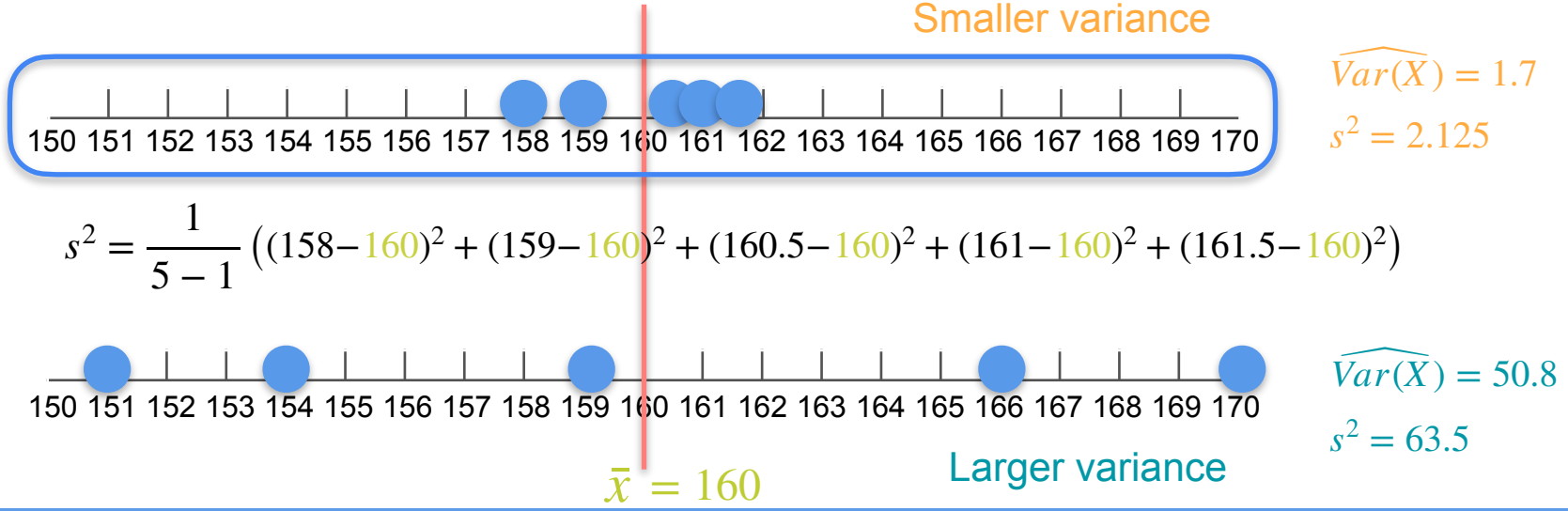
Variance Estimation

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

- n - 1 fixes bias when all you have is a sample
- As n gets big, the difference matters less
- If it matters, you may have too little data
- Some accepted statistical techniques use n

Sample Variance

$$Var(X) = \sigma^2 = \frac{1}{N}\sum (x-\mu)^2 \longrightarrow s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$



Variance Estimation

Population Variance Formula

$$Var(X) = \sigma^2 = \frac{1}{N}\sum (x - \mu)^2$$

Sample Variance Formula

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}$$

$$\widehat{\sigma^2} = \widehat{Var(X)} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

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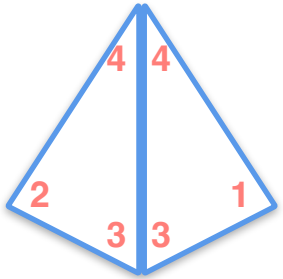
Sample and Population

Law of Large Numbers

Law of Large Numbers



1 2 3 4

$\mu = 2.5$

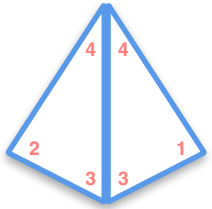
Experiment:

Toss the 4-sided dice twice and record the average of your outcomes

	1	2	3	4
1	1	1.5	2	2.5
2	1.5	2	2.5	3
3	2	2.5	3	3.5
4	2.5	3	3.5	4

	1	2	3	4
1	μ			
2	**2.5**			
3				
4				

Law of Large Numbers



1 2 3 4

$\mu = 2.5$

1 trial

4,3

$\bar{x}_1$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
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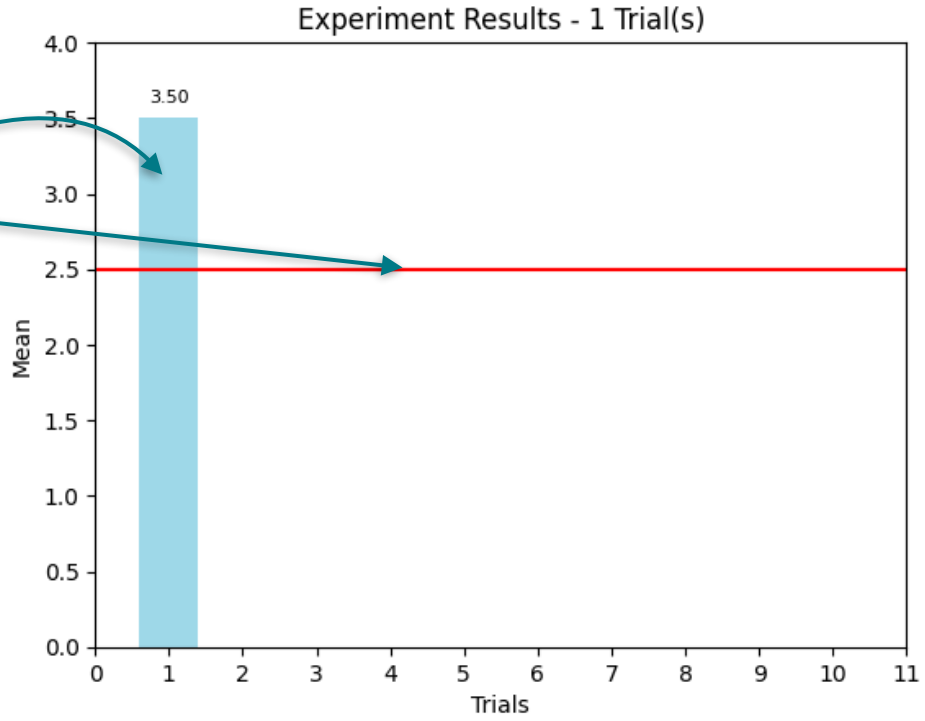
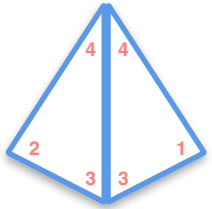


Law of Large Numbers



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4	4,1	4,2	4,3	4,4

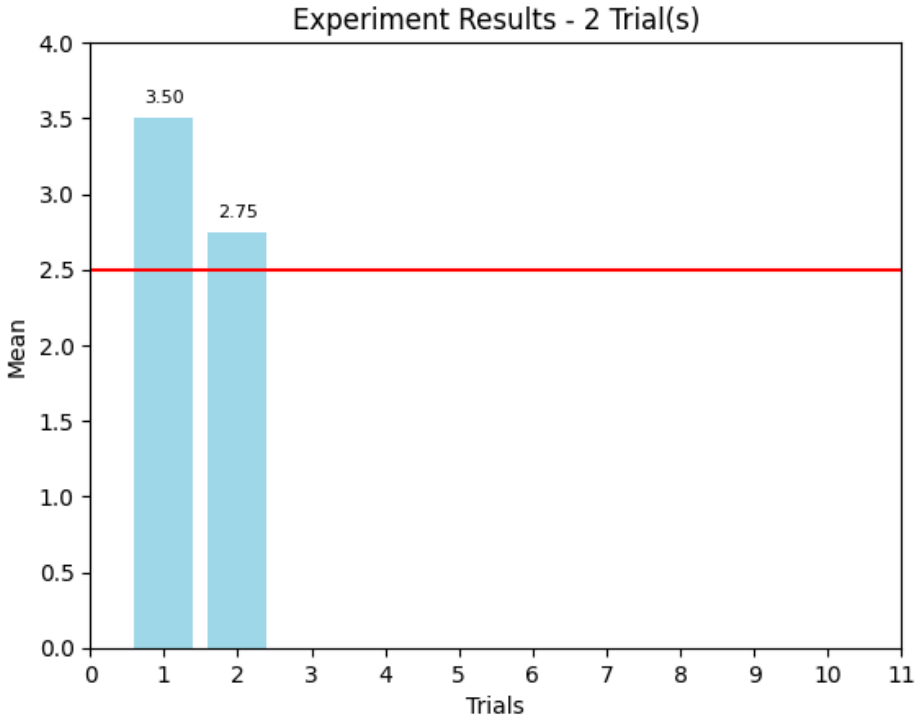
1 trial

4,3

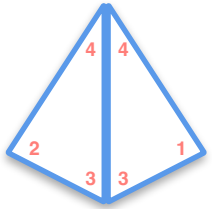
2 trials

3,4

1,3



Law of Large Numbers



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1 trial

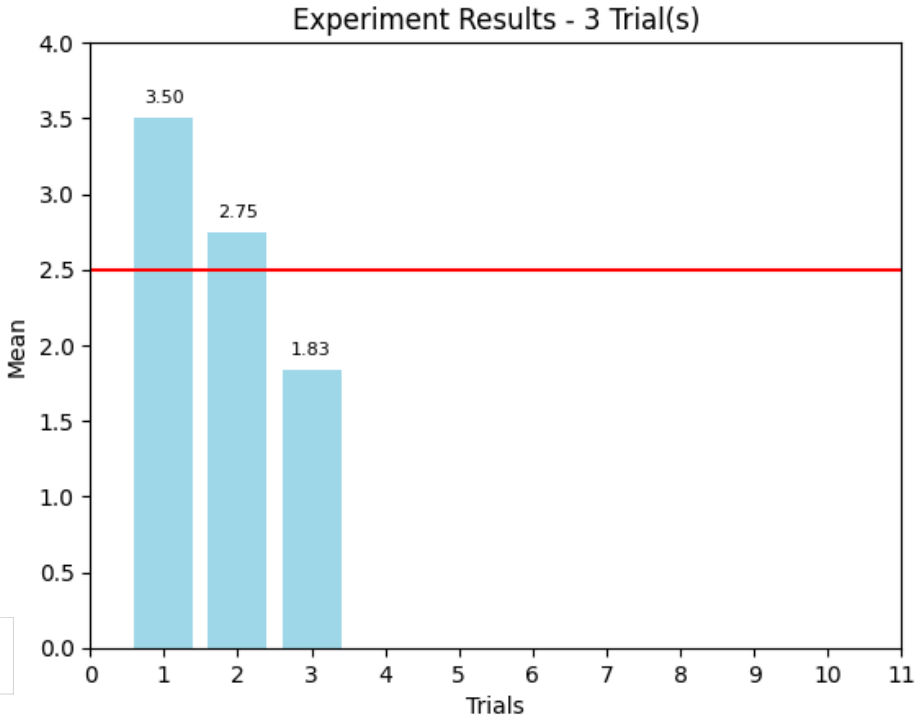
4,3

2 trials

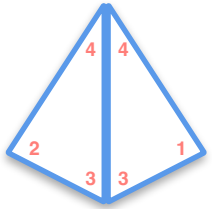
3,4
1,3

3 trials

3,1	1,4	1,1



Law of Large Numbers



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3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

2 trials

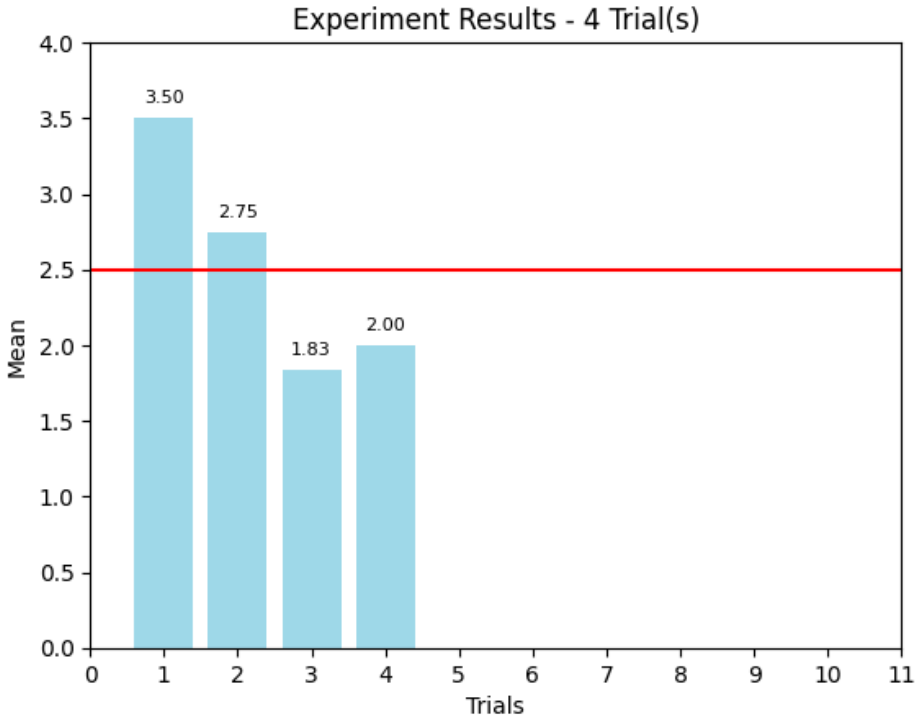
3,4	
1,3	

3 trials

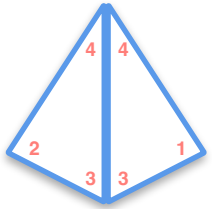
3,1	1,4	1,1

4 trials

3,1	3,1
1,2	3,2



Law of Large Numbers



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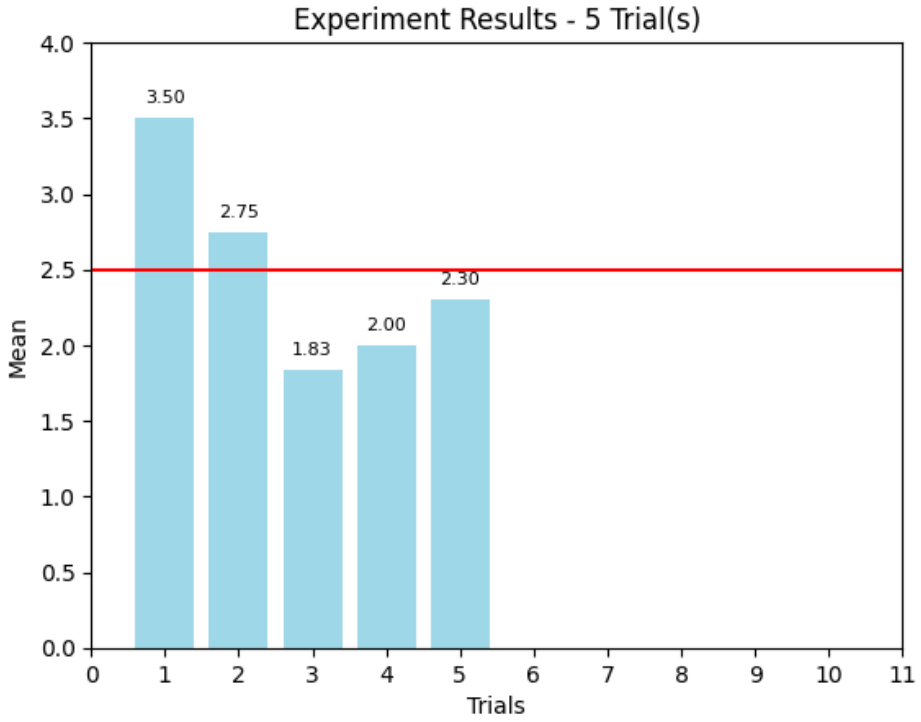
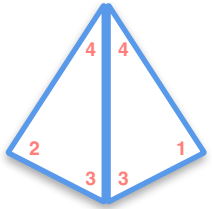


Law of Large Numbers



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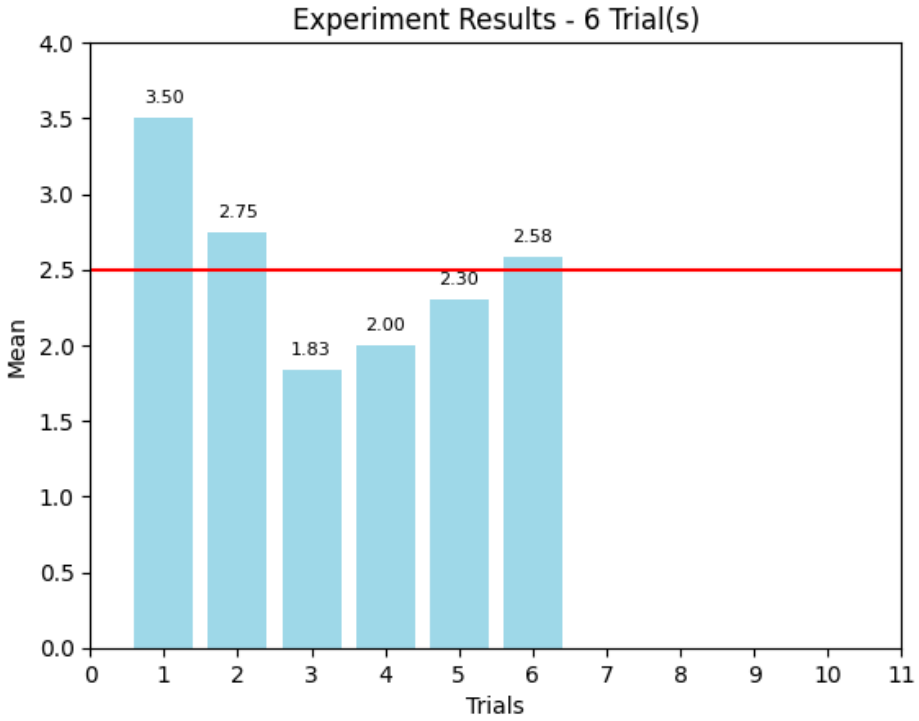
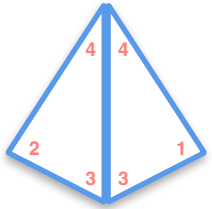


Law of Large Numbers



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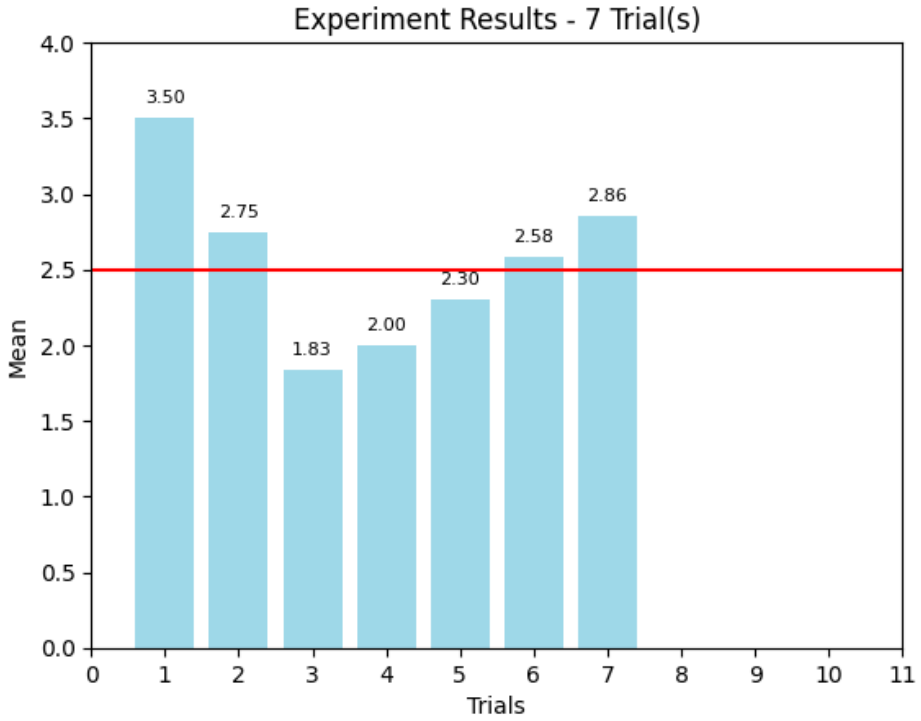

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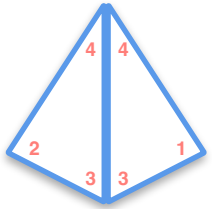

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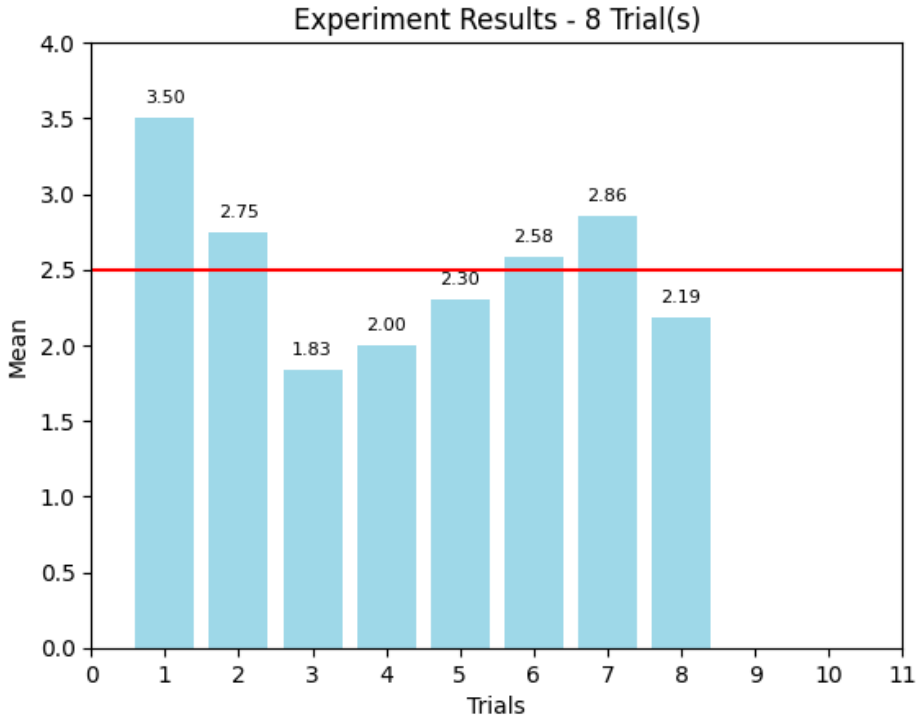
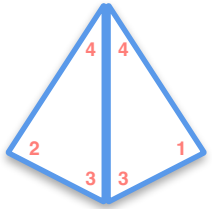


Law of Large Numbers



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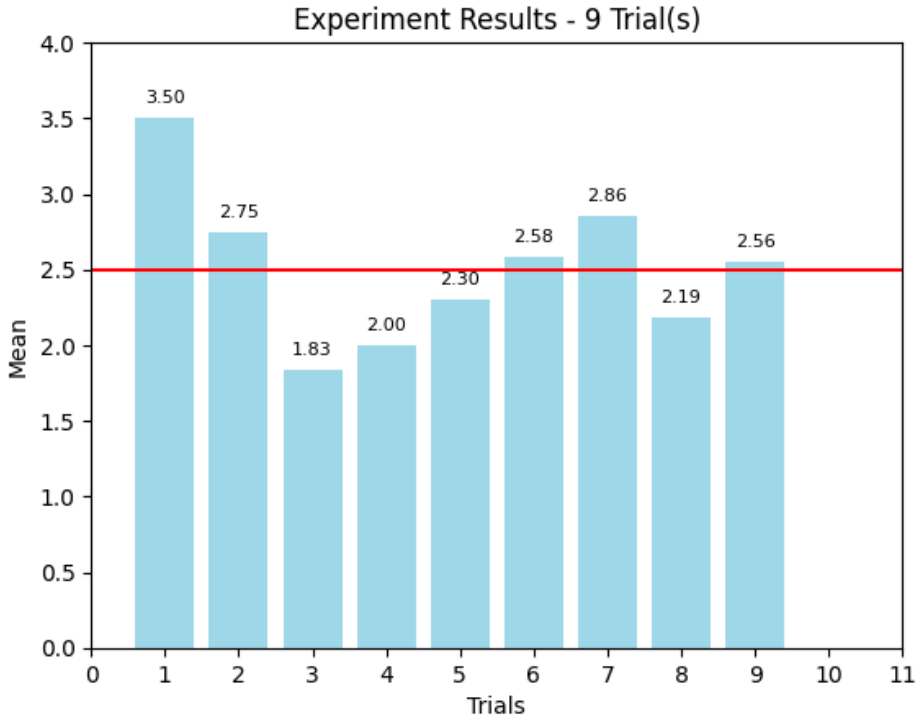
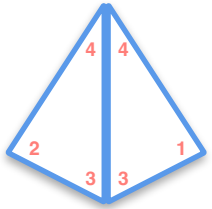


Law of Large Numbers



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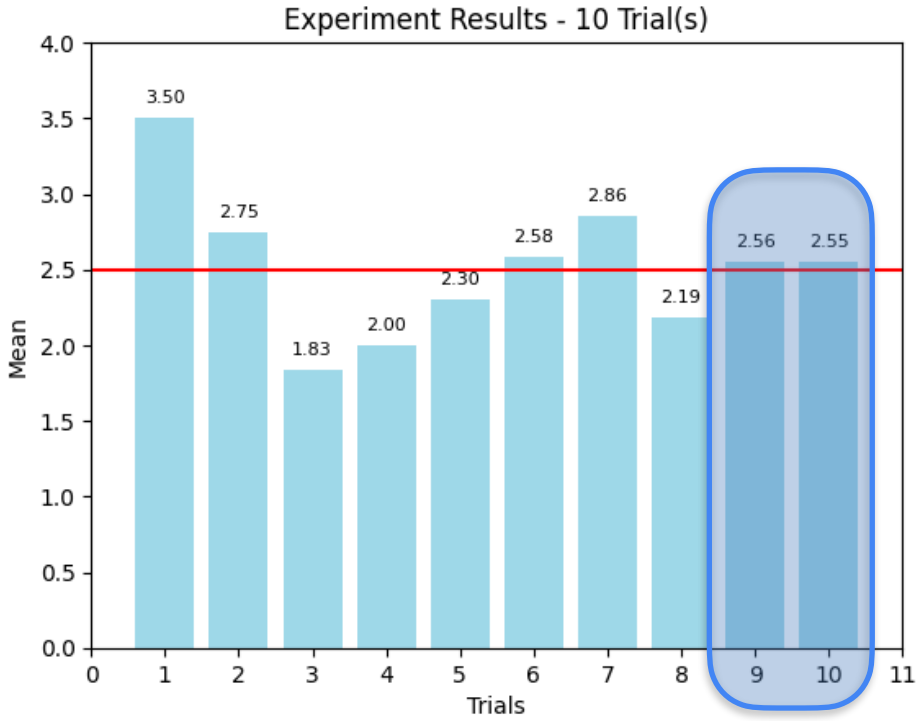
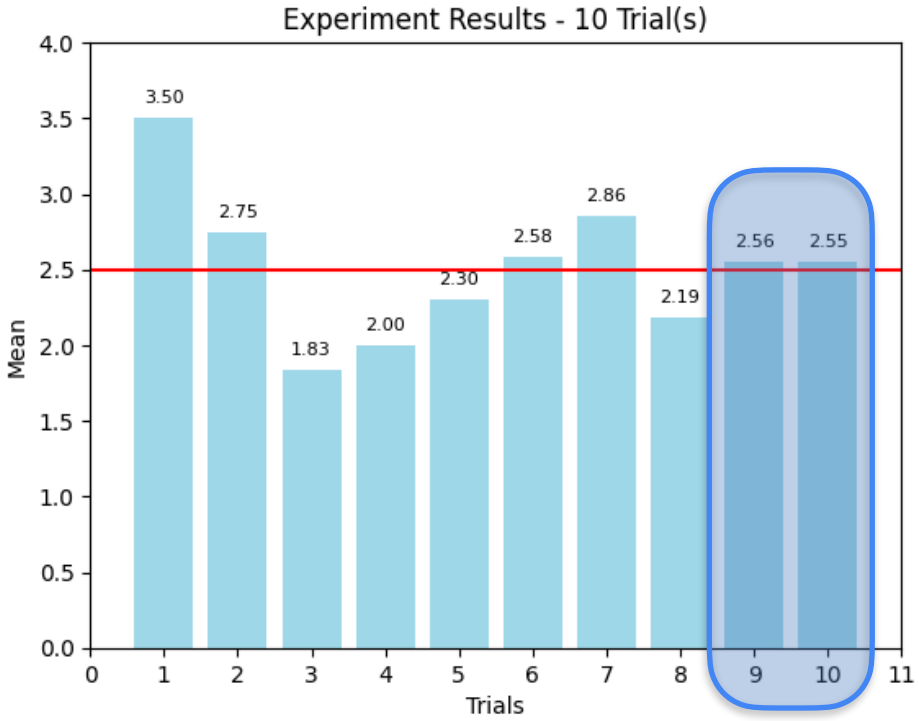


Law of Large Numbers

As the sample size increases, the average of the sample will tend to get closer to the average of the entire population.

Law of Large Numbers



Law of Large Numbers

Law of Large Numbers

n : number of samples

X_i : is the i-th random sample from the population.

Each X_i are independent and identically distributed (i.i.d.)

as $n \to \infty$

$$\frac{X_1 + X_2 + X_3 + \ \ldots + \ X_n}{n} \longrightarrow \mathbb{E}[X] = \mu_X$$

UNDER CERTAIN CONDITIONS

as $n \to \infty$

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}[X] = \mu_X$$

Law of Large Numbers

UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.
- Sample size must be sufficiently large.
- Independent observations.

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Sample and Population

Central Limit Theorem
Discrete Random Variable

Central Limit Theorem (CLT) - Example 1

Random variable → X **number of heads when a coin is flipped n times**

If $n = 1$ $X = 1$ $X = 0$

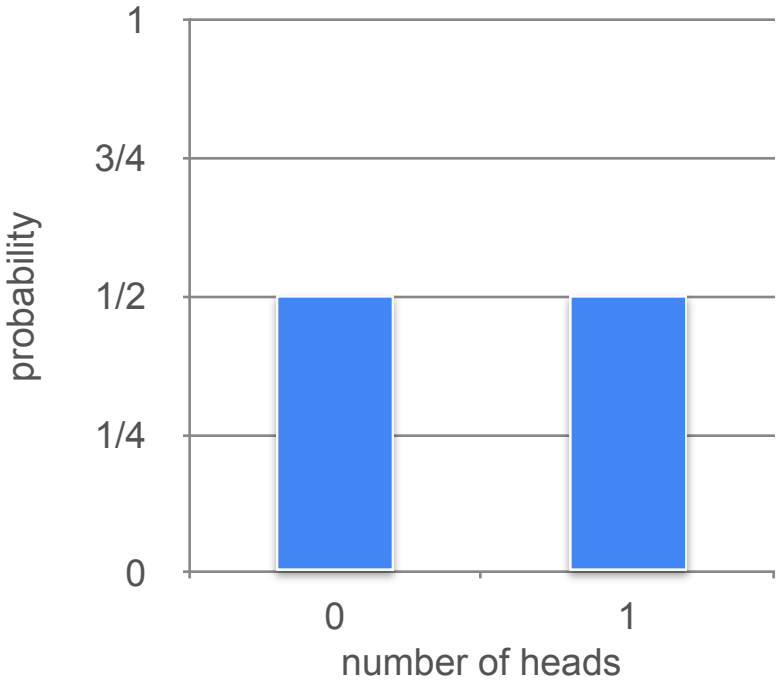
Discrete Random Variable

Central Limit Theorem (CLT) - Example 1

$\mathbf{P}(H) = 0.5$ $\mathbf{P}(T) = 0.5$

$X = 1$ $X = 0$

What can we say about the probability distribution when the number of coin flips increases?



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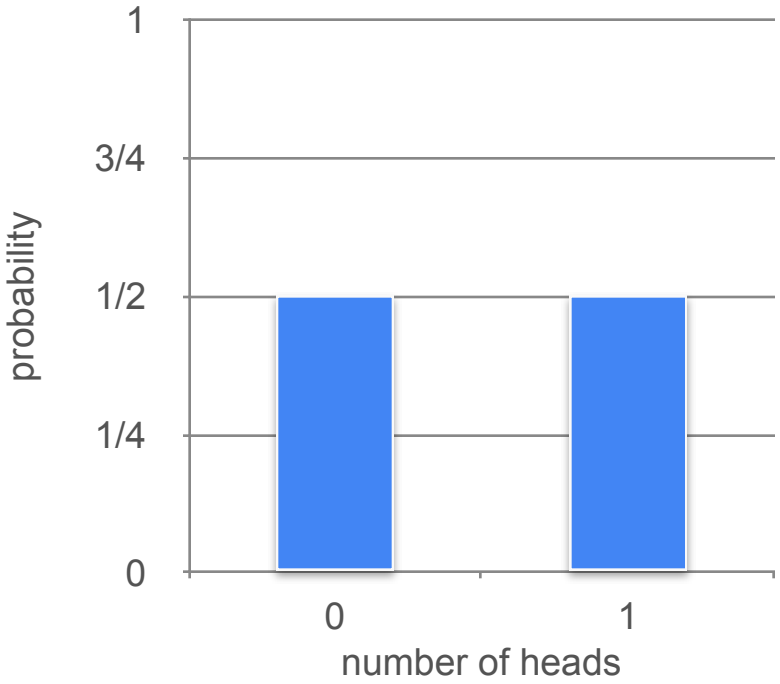

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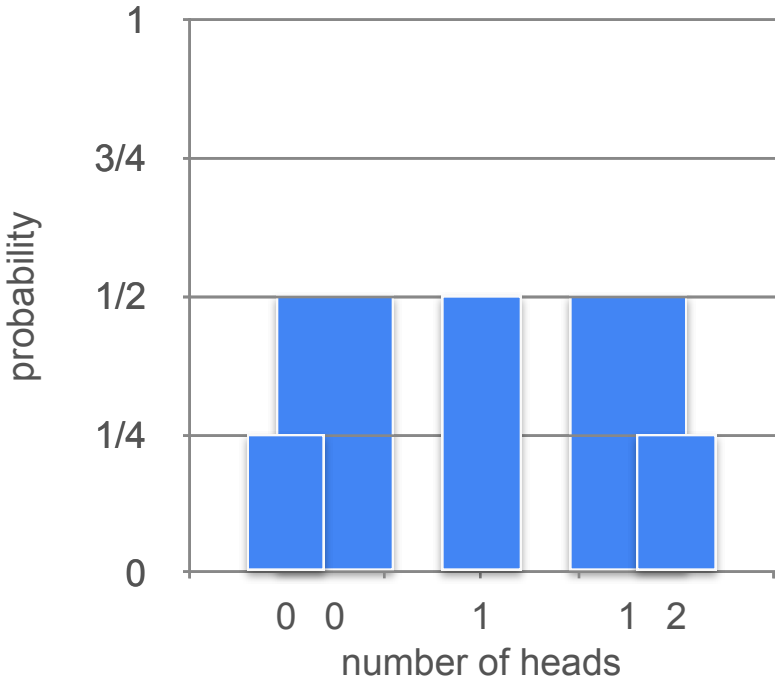

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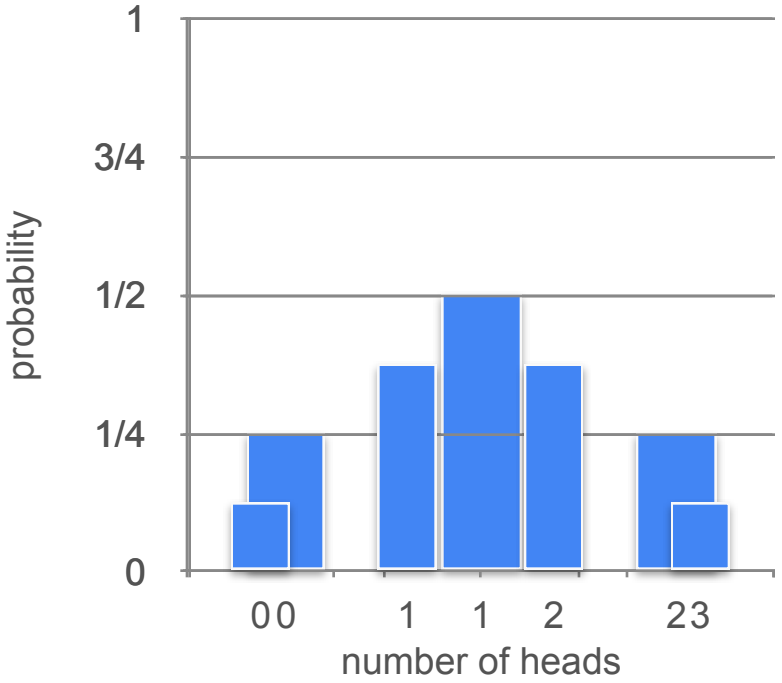

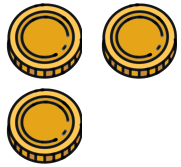

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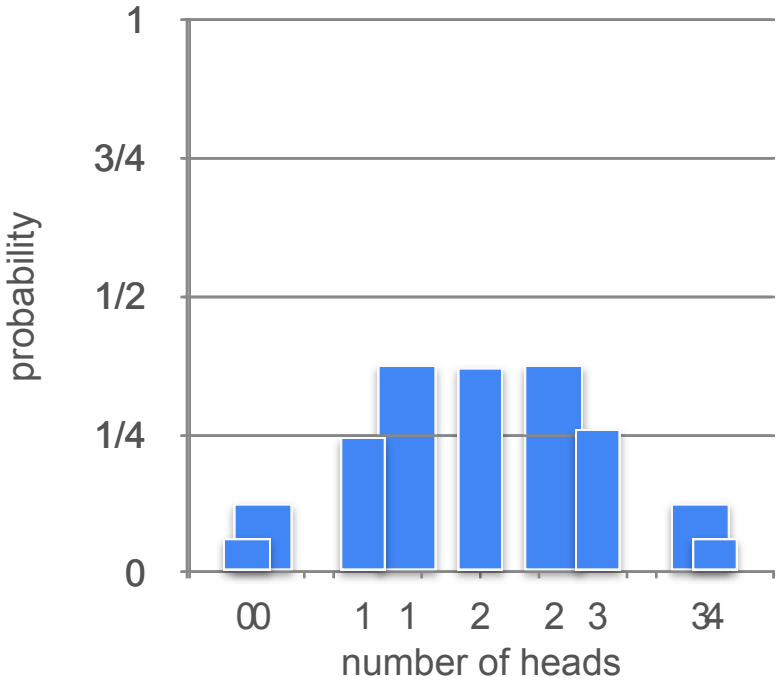
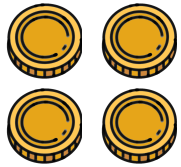


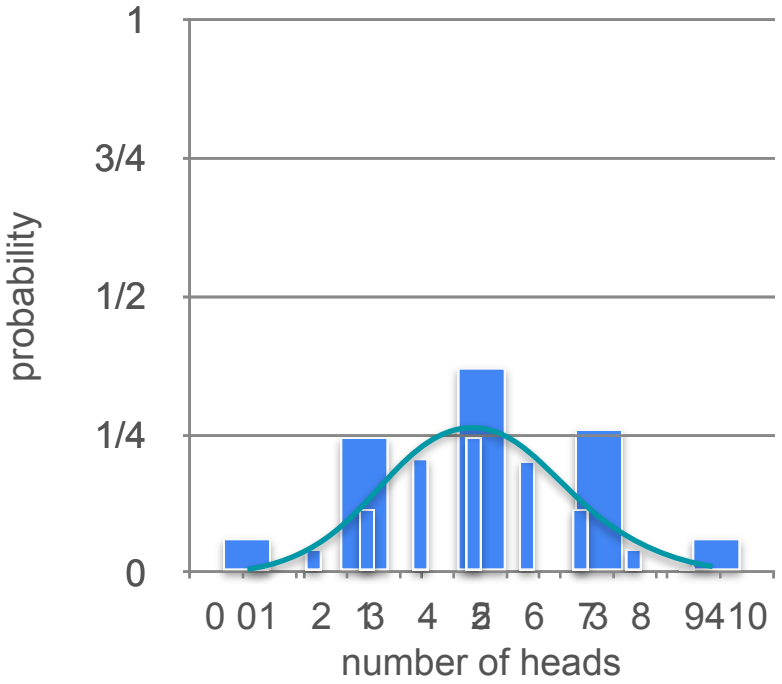


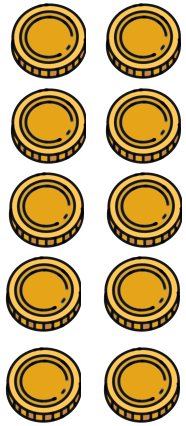
Central Limit Theorem (CLT) - Example 1

$\mathbf{P}(H) = 0.5$ $\mathbf{P}(T) = 0.5$

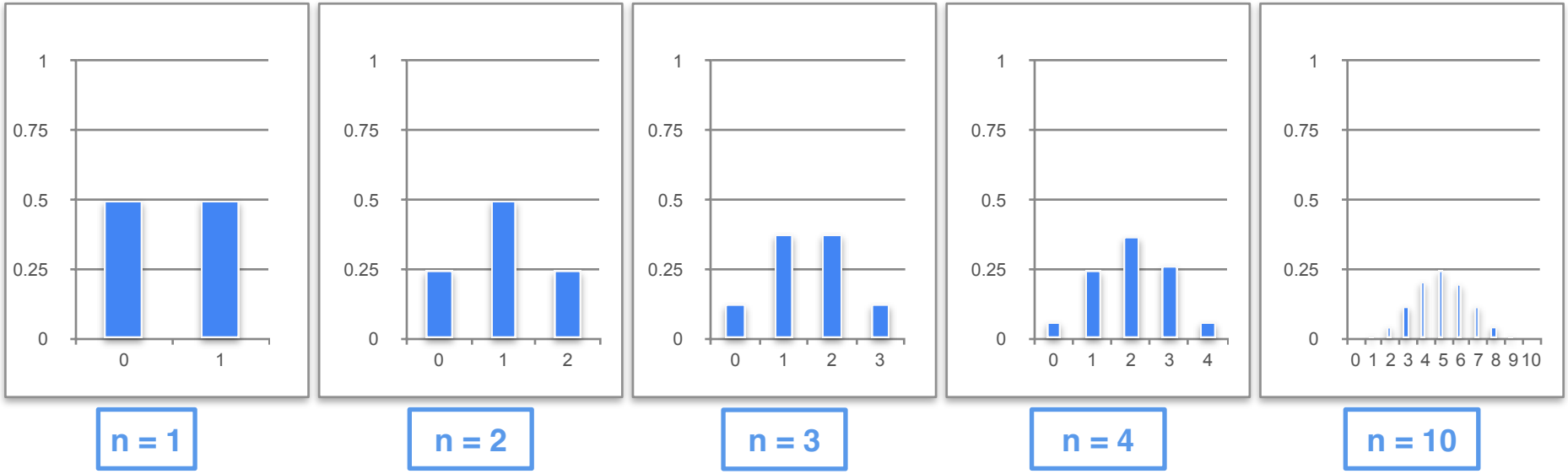
$X = 1$ $X = 0$

What can we say about the distribution when the number of coins we flip increases?



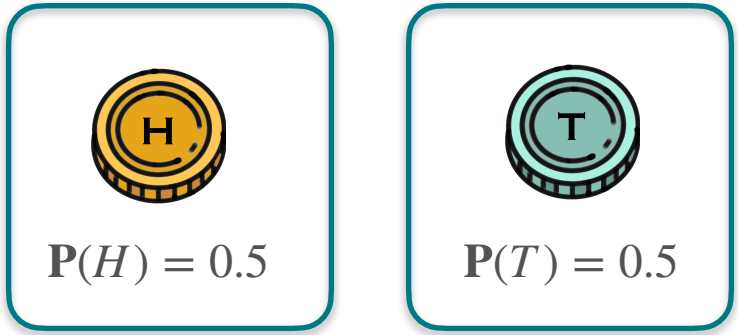


Central Limit Theorem (CLT) - Example 1



As n increases, the probability distribution becomes closer to a Gaussian distribution

Central Limit Theorem (CLT) - Example 1



Random variable → X **number of heads when a coin is flipped n times**

$$\mu = np = n\mathbf{P}(H)$$

$$\sigma^2 = np(1-p) = n\mathbf{P}(H)\mathbf{P}(T)$$

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$\mu = np = n\mathbf{P}(H)$

$\sigma^2 = np(1-p) = n\mathbf{P}(H)\mathbf{P}(T)$

$n = 1$

$\mu = np$

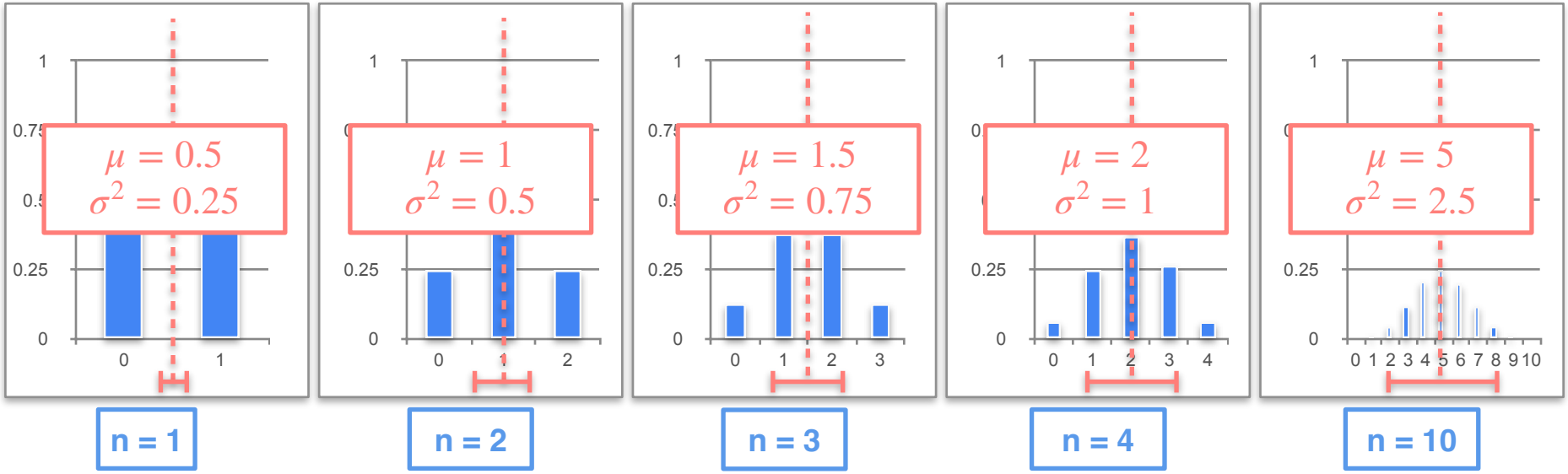
$\mu = 1 \times 0.5 \quad = 0.5$

$\sigma^2 = np(1-p)$

$\sigma^2 = (1 \times 0.5)(0.5) = 0.25$

Central Limit Theorem (CLT) - Example 1

$$\mu = np \qquad \sigma^2 = np(1-p)$$



As n increases, the probability distribution becomes closer to a gaussian distribution

Central Limit Theorem (CLT) - Example 1

$$\mu = np \qquad \sigma^2 = np(1-p)$$

$\mu = 0.5$ $\sigma^2 = 0.25$	$\mu = 1$ $\sigma^2 = 0.5$	$\mu = 1.5$ $\sigma^2 = 0.75$	$\mu = 2$ $\sigma^2 = 1$	$\mu = 5$ $\sigma^2 = 2.5$

as n become sufficiently large we will get a normal distribution with parameters

$$\mu = np$$

$$\sigma^2 = np(1-p)$$

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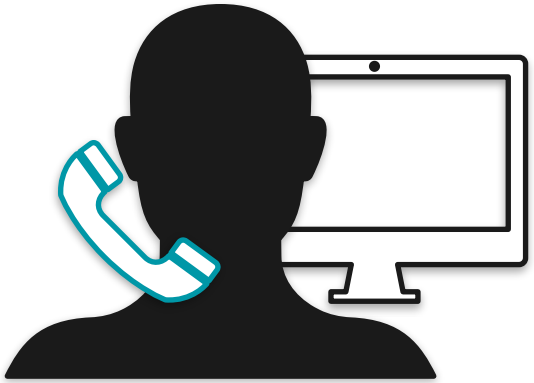
Sample and Population

Central Limit Theorem
Contiuous Random Variable

Uniform Distribution: Motivation



You're calling a tech support line.
They can answer any time between zero and 15 minutes and if they don't answer in this time, the line is disconnected.

$X =$ "Wait time for a called to be answered "

$$X \sim \mathcal{U}(0,15)$$

Central Limit Theorem (CLT) - Example 2

$$n = 1 \qquad Y_1 = \frac{X_1}{1}$$

$$n = 2 \qquad Y_2 = \frac{X_1 + X_2}{2}$$

$$n = 3 \qquad Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

$$\vdots \qquad \vdots$$

Record the average of all n experiments

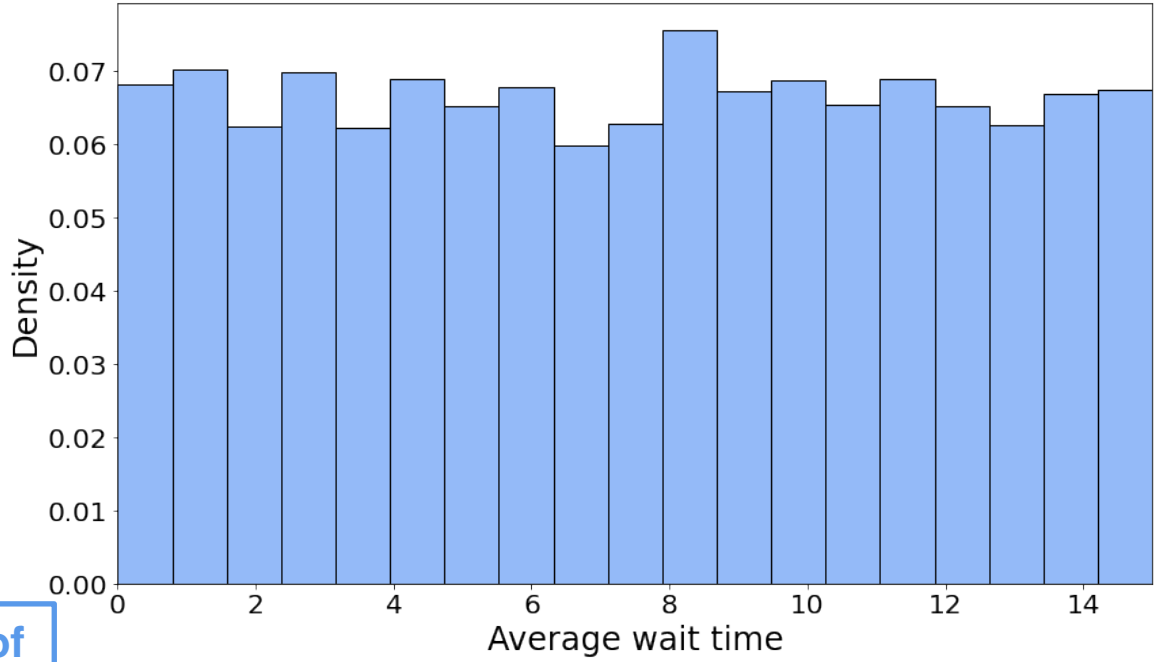
$$Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Can we say anything about the distribution of this average?

Central Limit Theorem (CLT) - Example 2

$$n = 1 \qquad Y_1 = \frac{X_1}{1}$$

Create many samples of Y_1 so you can get a pretty histogram

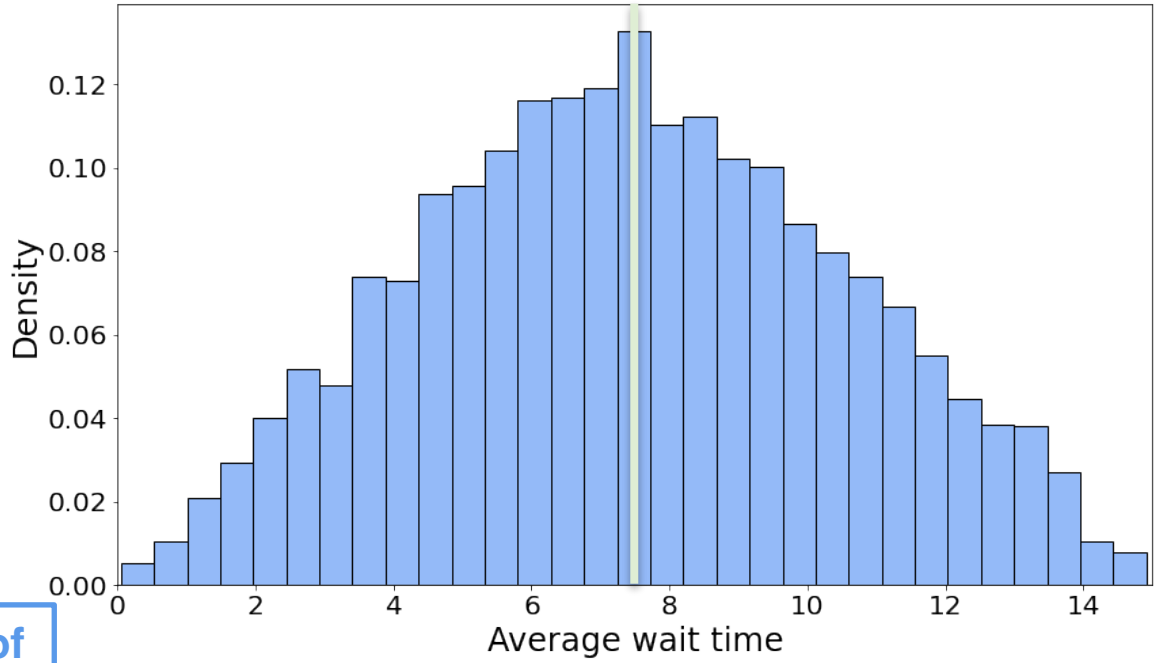


What happens to the distribution of these averages as n increases?

Central Limit Theorem (CLT) - Example 2

$$n = 2 \qquad Y_2 = \frac{X_1 + X_2}{2}$$

Create many samples of Y_2 so you can get a pretty histogram

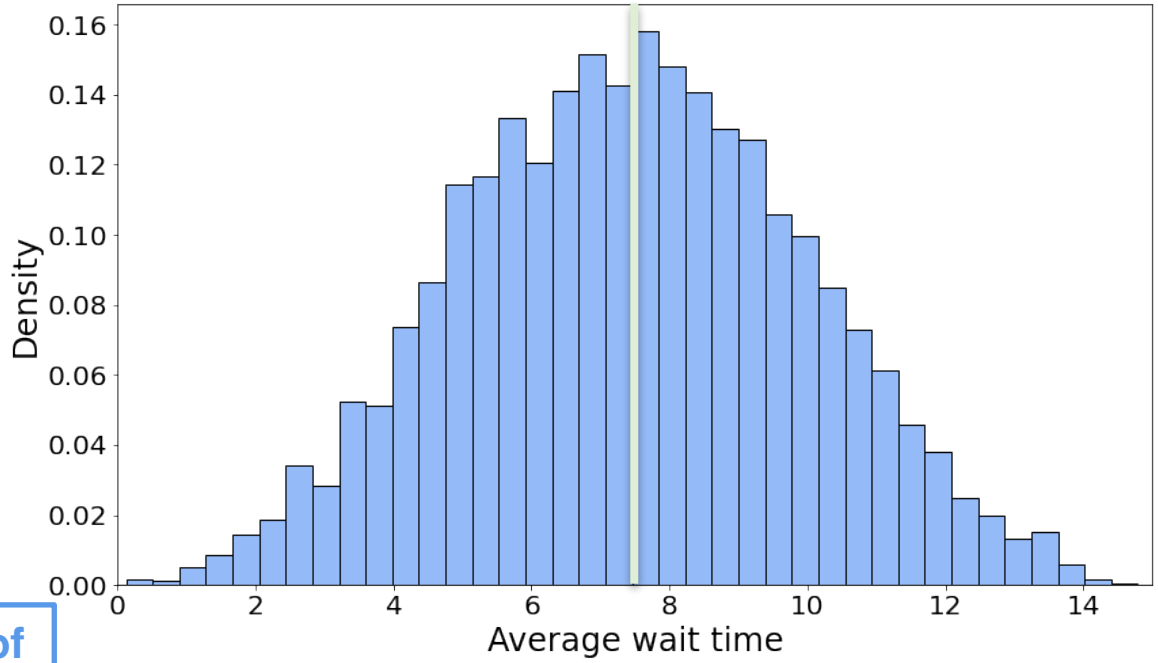


What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

$$n = 3 \quad Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

Create many samples of Y_3 so you can get a pretty histogram

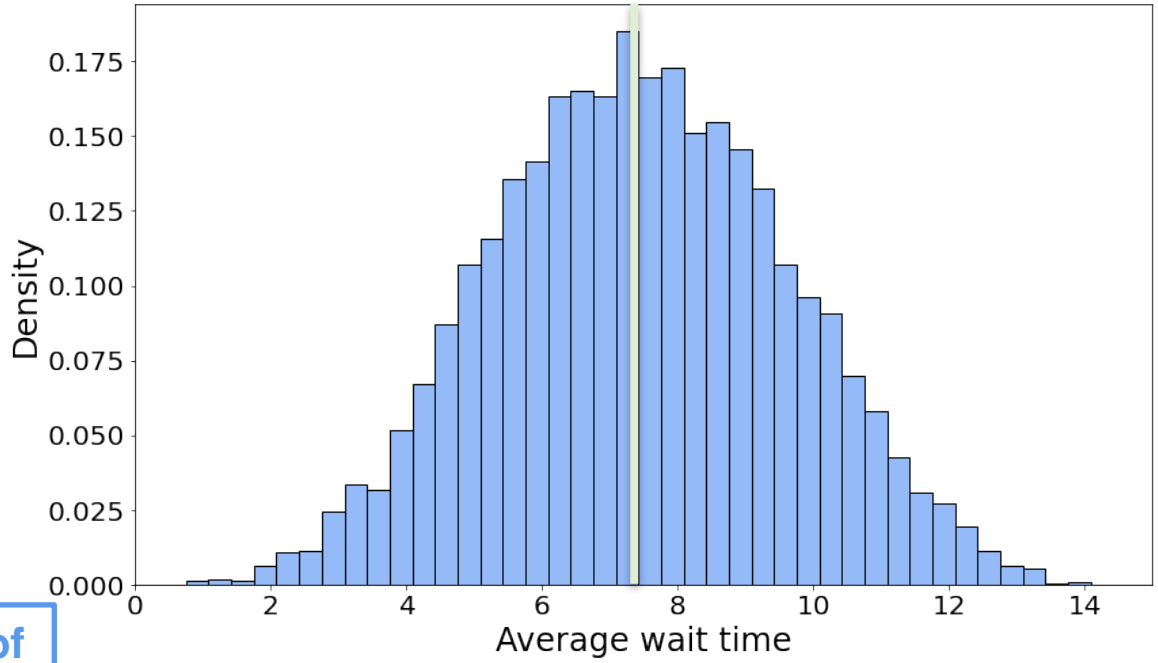


What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

$$n = 4 \quad Y_4 = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

Create many samples of Y_4 so you can get a pretty histogram

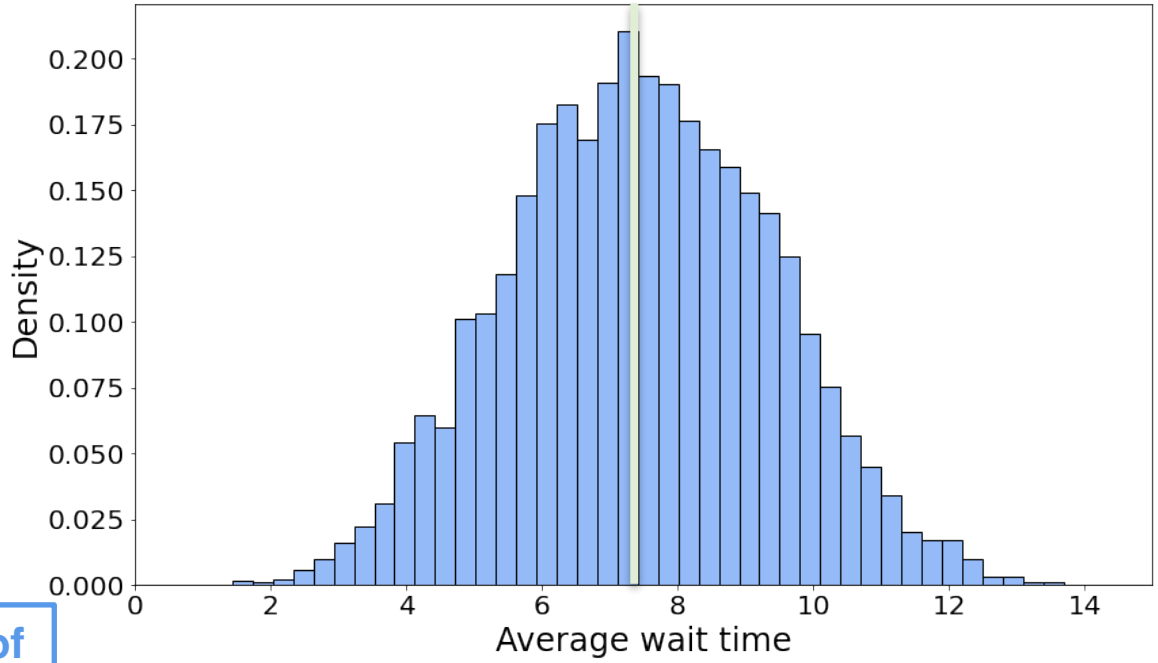


What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

$$n = 5 \quad Y_5 = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}$$

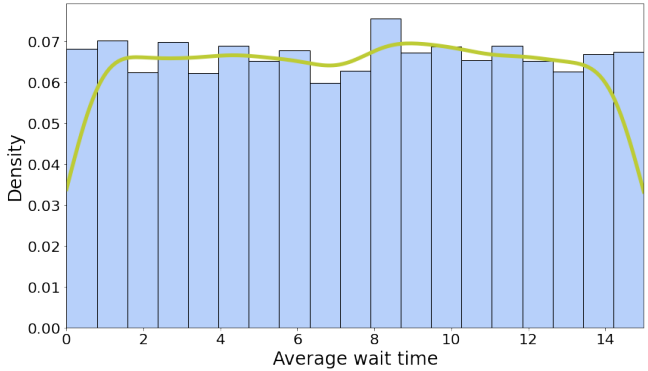
Create many samples of Y_5 so you can get a pretty histogram

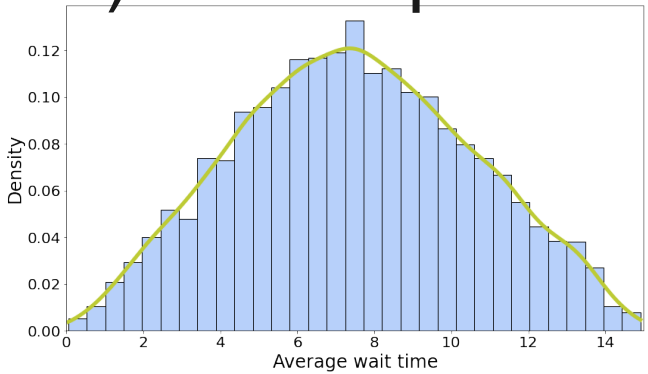


What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

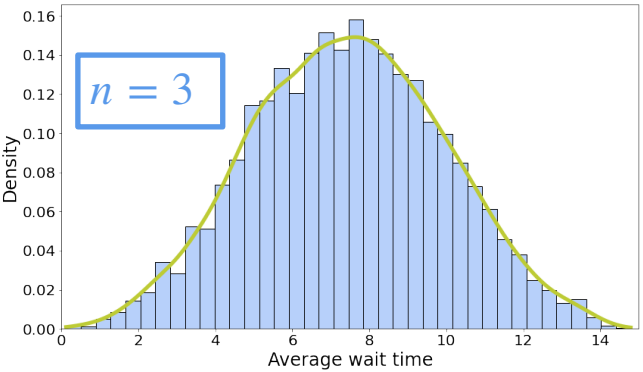
$n = 1$



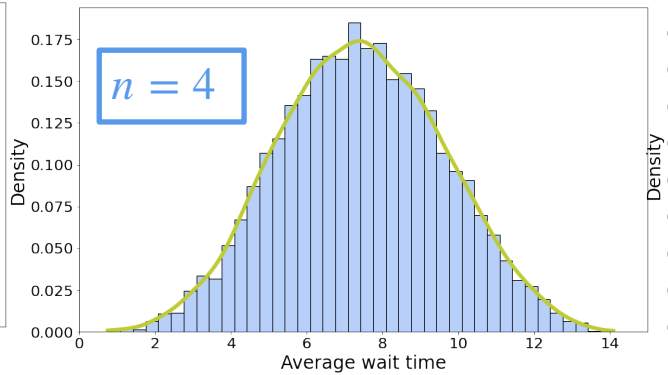


$n = 2$

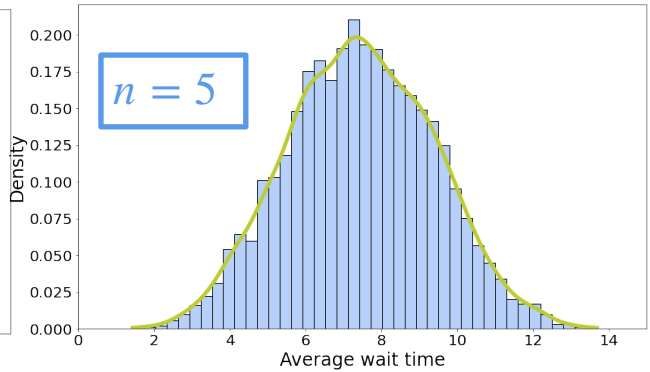
$n = 3$



$n = 4$



$n = 5$



Central Limit Theorem (CLT) - Example 2

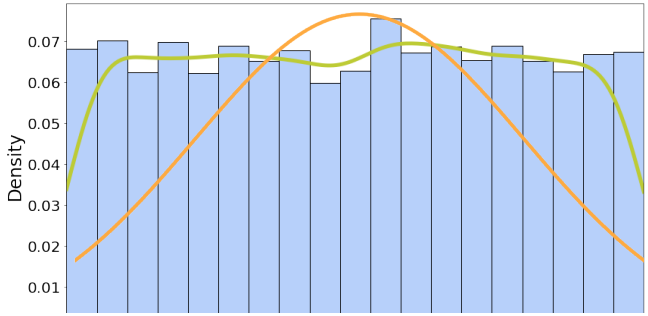
$$\mathbb{E}[Y_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}n\mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

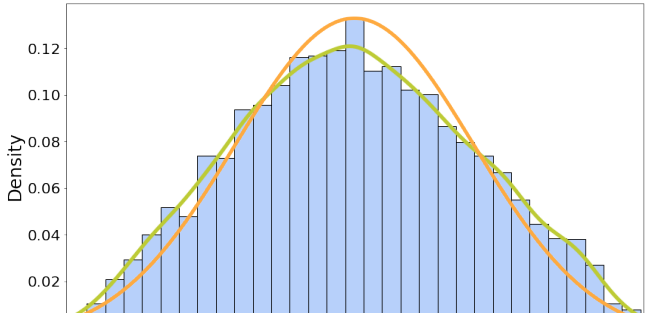
$$Var[Y_n] = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2}nVar(X) = \frac{Var(X)}{n} = \frac{18.75}{n}$$

Central Limit Theorem (CLT) - Example 2

$n = 1$

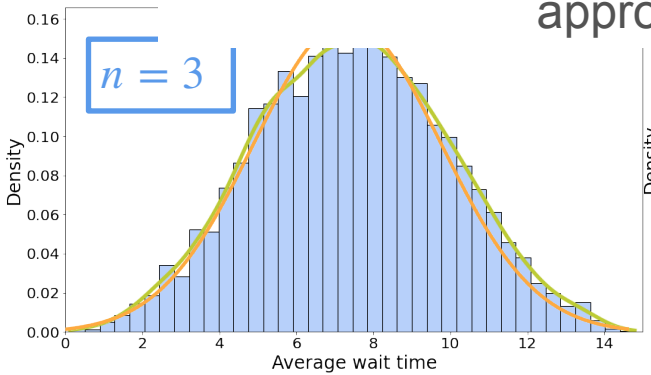




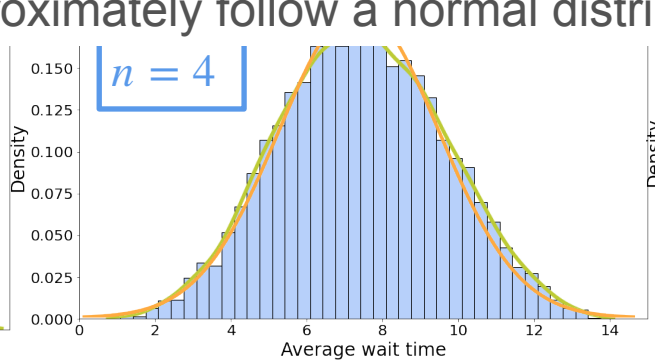
$n = 2$

When you average a large enough number of variables, the distribution will approximately follow a normal distribution

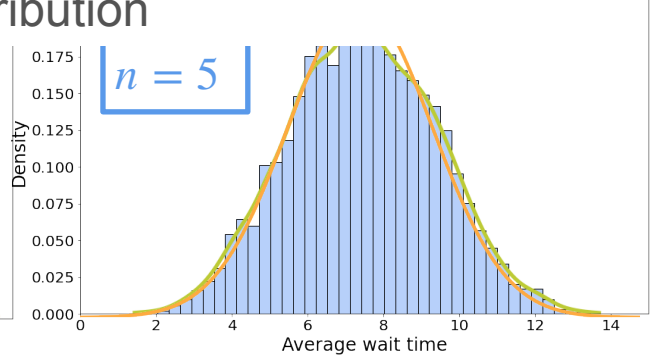
$n = 3$



$n = 4$

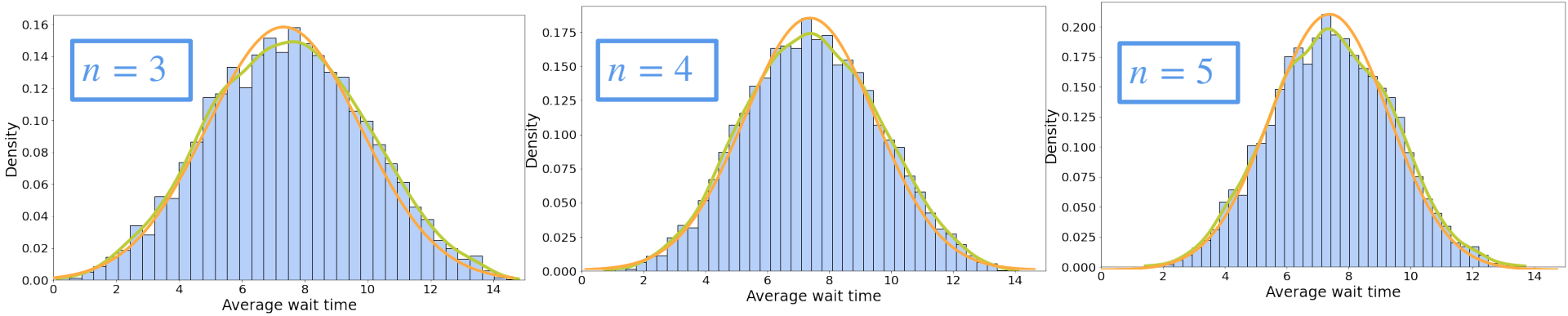


$n = 5$



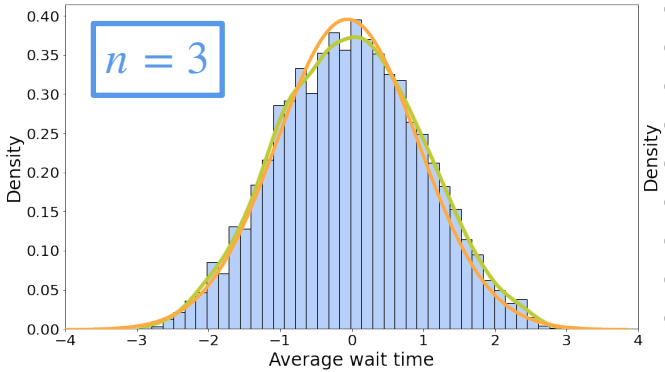
Central Limit Theorem (CLT) - Example 2

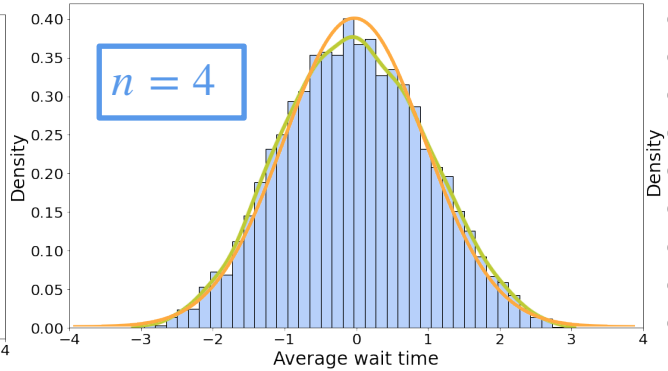
$$\frac{Y_n - 7.5}{\sqrt{18.75/n}}$$

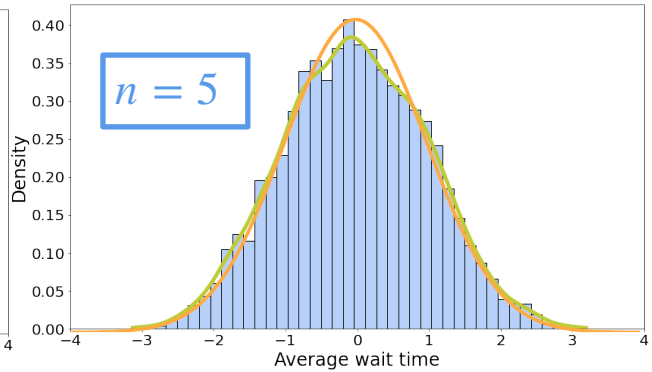


Central Limit Theorem (CLT) - Example 2

$$\frac{Y_n - 7.5}{\sqrt{18.75/n}} \xrightarrow{n\uparrow} \mathcal{N}(0,1)$$







Central Limit Theorem (CLT) - Example 2

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

mean of Y_n → $\mu_{Y_n} = \mu$ ← population mean

$$\sigma^2_{Y_n} = \frac{\sigma^2}{n}$$

variance of Y_n → $\sigma^2_{Y_n}$ ← population variance

Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \to \infty \quad \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X]}{\sigma_X}\sqrt{n} \sim \mathcal{N}\left(0, 1^2\right)$$

$$\text{As } n \to \infty \quad \frac{1}{\sqrt{n}}\left(\frac{\sum_{i=1}^{n} X_i - \frac{1}{n}\mathbb{E}[X]}{\sigma_X}\right)\sqrt{n} \sim \mathcal{N}\left(0, 1^2\right)$$

Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \to \infty \quad \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X]}{\sigma_X}\sqrt{n} \sim \mathcal{N}\left(0, 1^2\right)$$

$$\text{As } n \to \infty \quad \frac{\sum_{i=1}^{n} X_i - n\mathbb{E}[X]}{\sqrt{n}\sigma_X} \sim \mathcal{N}\left(0, 1^2\right)$$

W3 Lesson 2

DeepLearning.AI

Point Estimation

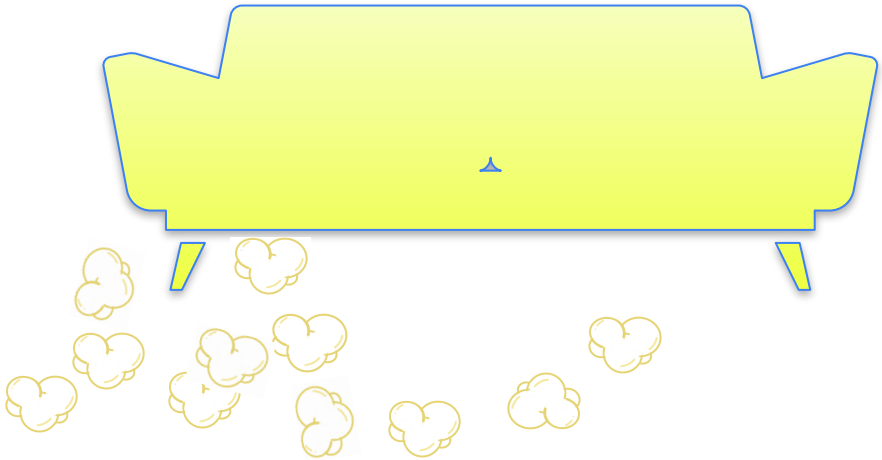
Point Estimation

DeepLearning.AI

Point Estimation

Maximum Likelihood Estimation: Motivation

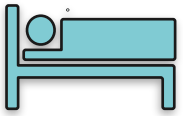
There’s Popcorn on the Floor. What Happened?



Movies

Board Games

Nap



Quiz

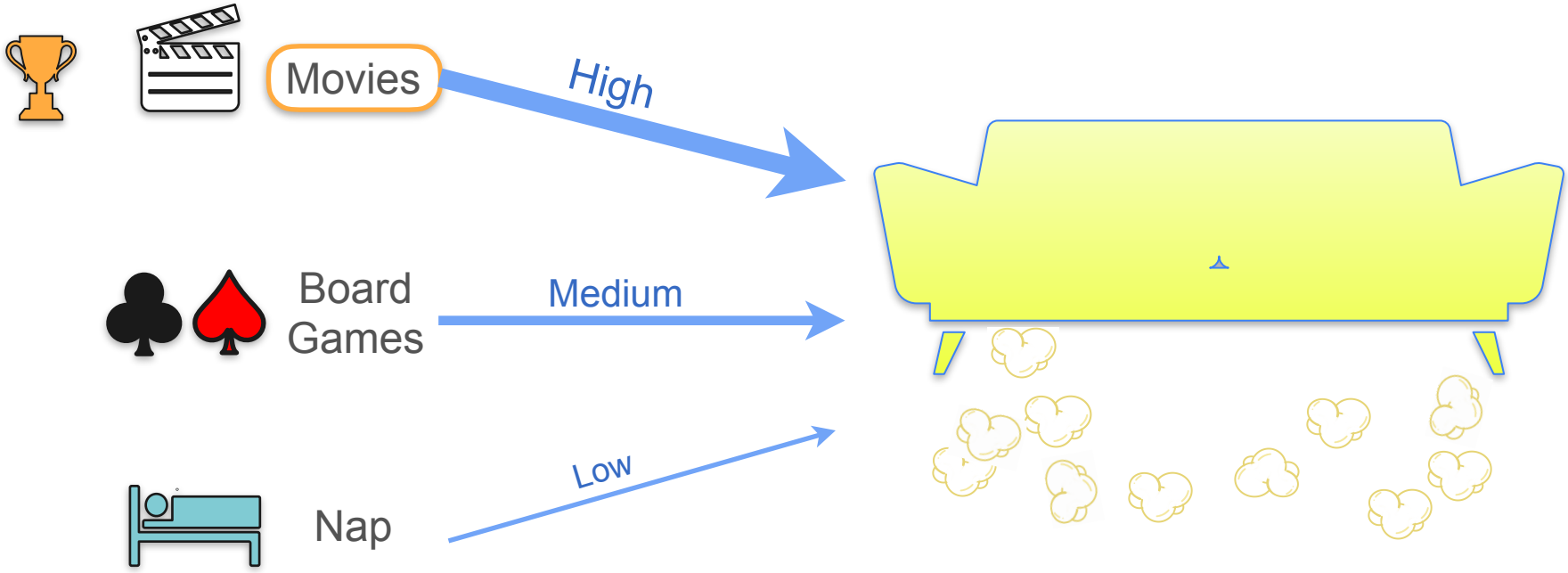
What do you think happened?

A. People were watching a movie

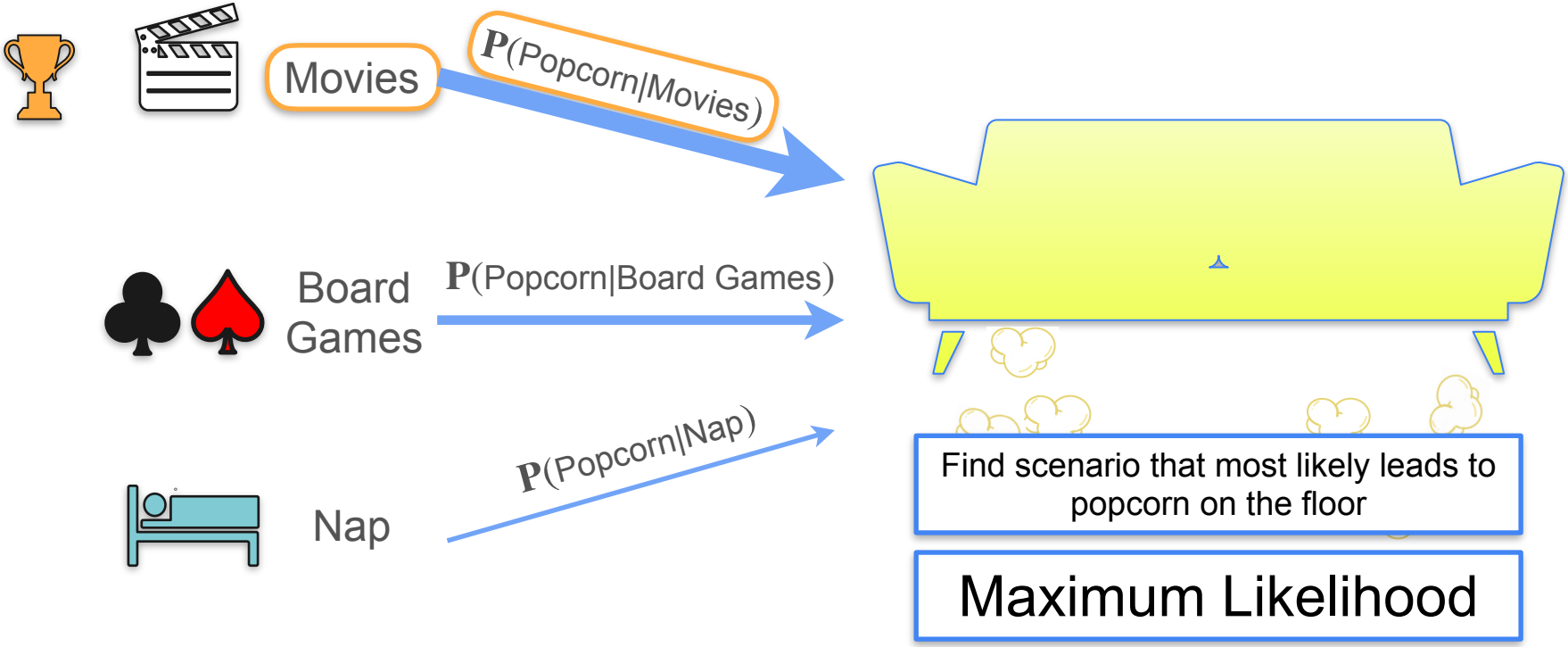
B. People were playing boardgames

C. People were taking a nap

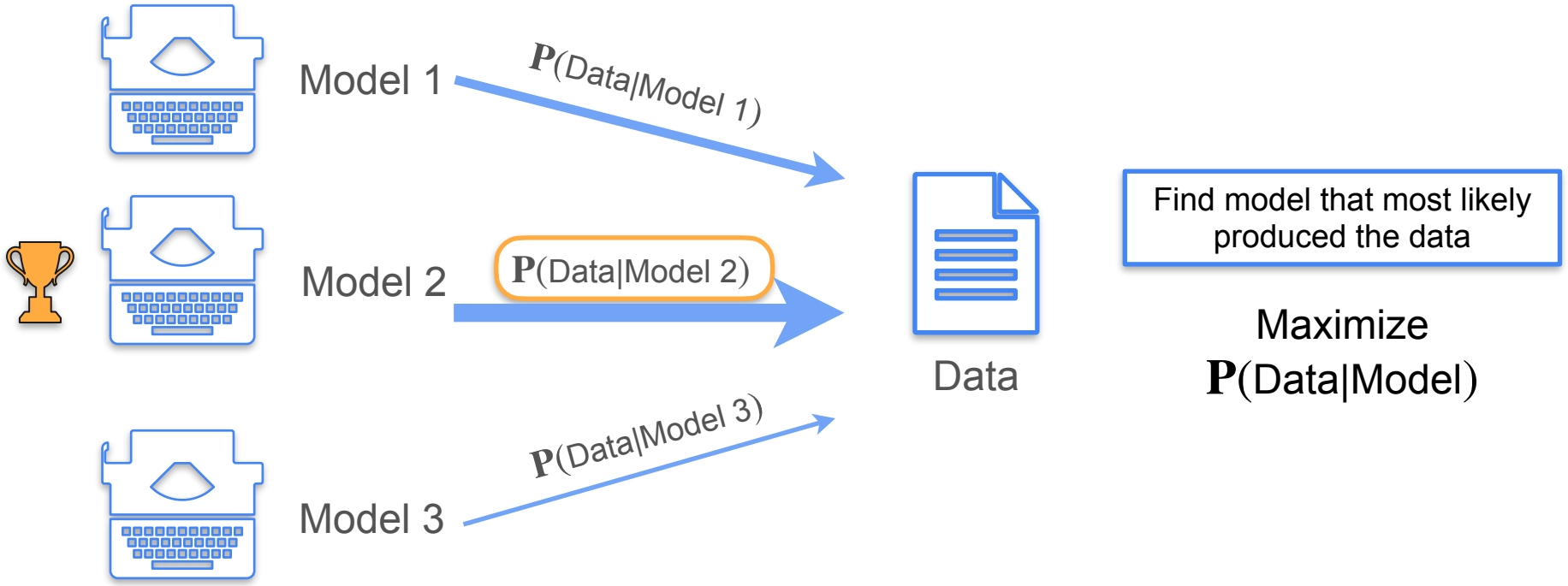
There's Popcorn on the Floor. What Happened?



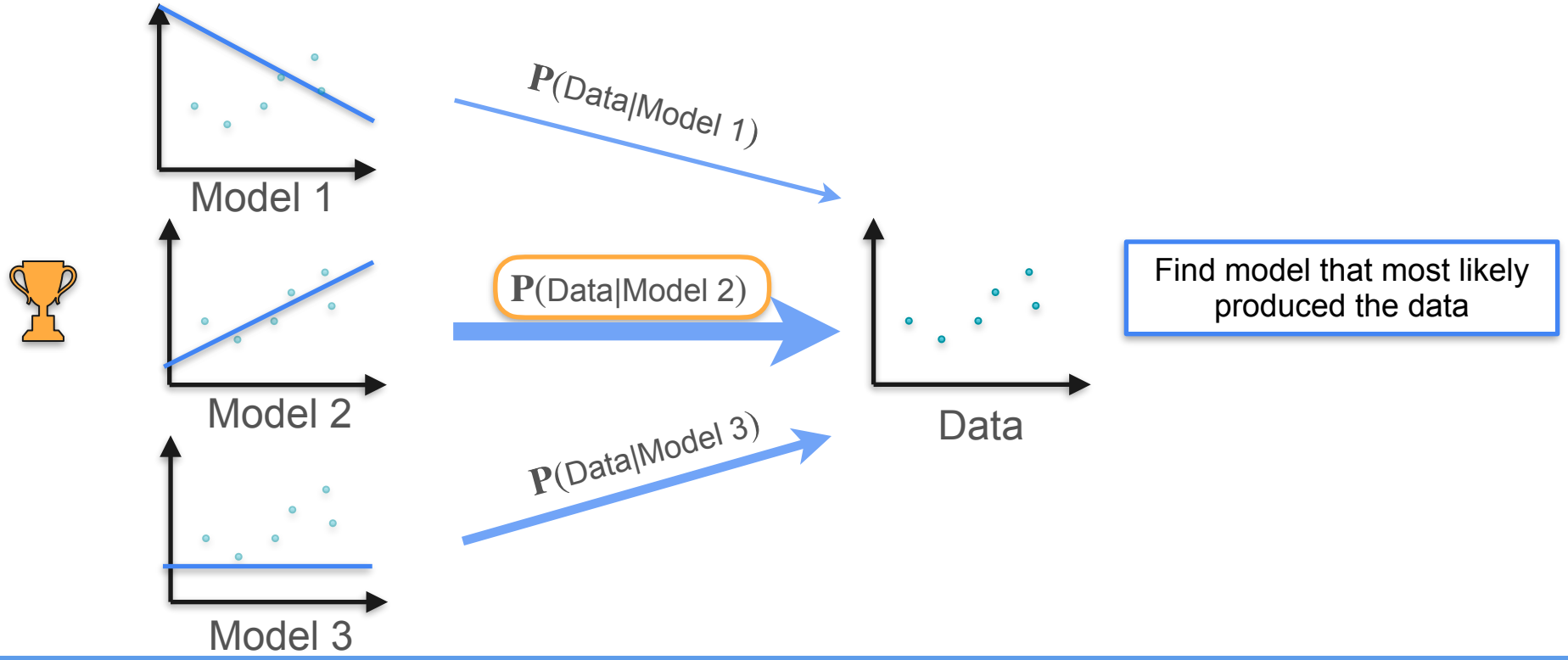
There's Popcorn on the Floor. What Happened?



Maximum Likelihood



Example: Linear Regression



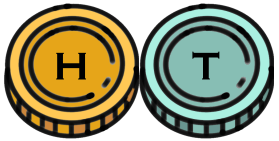
DeepLearning.AI

Point Estimation

MLE: Bernoulli Example

Maximum Likelihood: Bernoulli Example

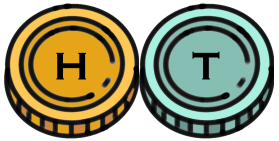
Coin 1



$\mathbf{P}(H) = 0.7$

$\mathbf{P}(T) = 0.3$

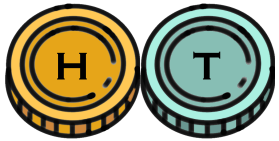
Coin 2



$\mathbf{P}(H) = 0.5$

$\mathbf{P}(T) = 0.5$

Coin 3



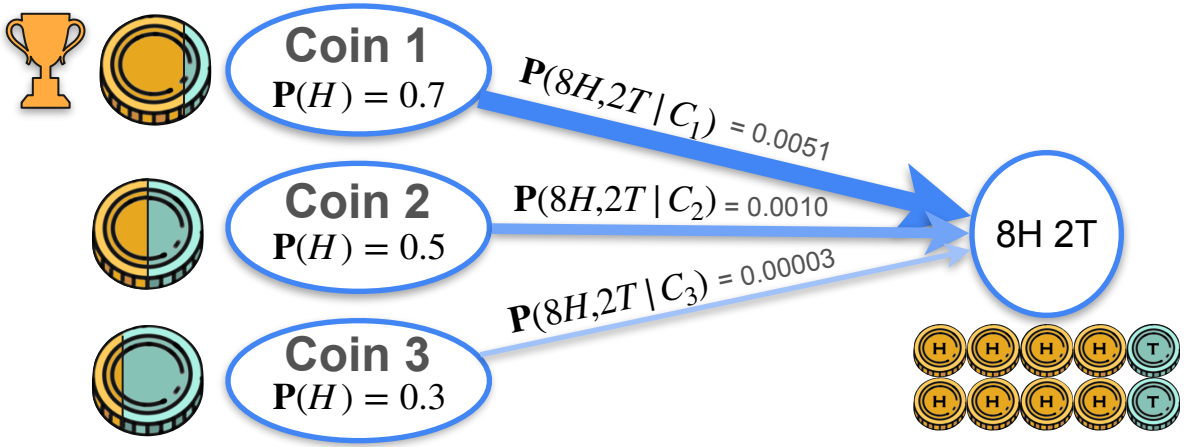
$\mathbf{P}(H) = 0.3$

$\mathbf{P}(T) = 0.7$

Maximum Likelihood: Bernoulli Example

Coin 1	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.3	0.3 = 0.0051
Coin 2	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5 = 0.0010
Coin 3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.30	0.7	0.7 = 0.00003

Maximum Likelihood: Bernoulli Example

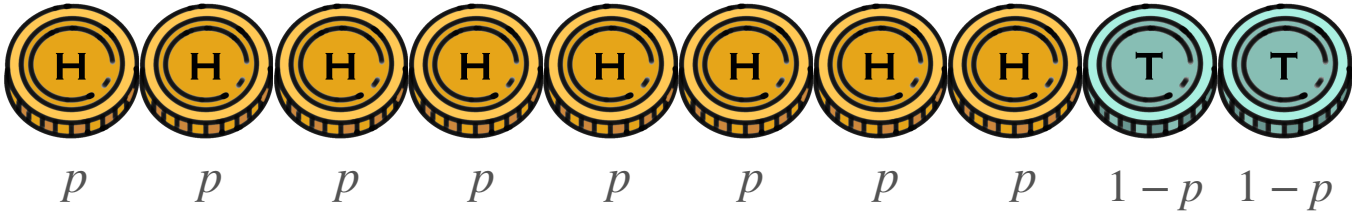


Find coin that most likely produced the 8 heads and 2 tails

Maximize

$\mathbf{P}(8H, 2T \mid \text{Coin})$

Maximum Likelihood: Bernoulli Example



Can you do any better?

$p = \mathbf{P}(H)$ $\quad\quad$ $p^8(1-p)^2$

You want p that maximizes the chances of seeing 8H

Maximum Likelihood: Bernoulli Example

$p = \mathbf{P}(H)$ Likelihood $L(p; 8H) = p^8(1-p)^2$

Function of p

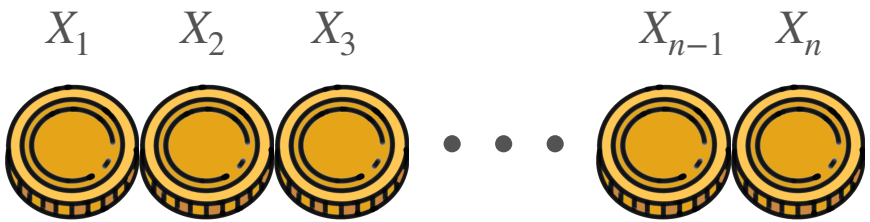
You want p that maximizes the chances of seeing 8H

Log-likelihood $\ell(p; 8H) = \log\left((p^8(1-p)^2\right) = 8\log(p) + 2\log(1-p)$

$$\frac{d}{dp}\left(8\log(p) + 2\log(1-p)\right) = \frac{8}{p} + \frac{2}{1-p}(-1) = 0 \rightarrow \hat{p} = \frac{8}{10}$$

Maximum Likelihood: Bernoulli Example

n coins
k heads



$$\mathbf{X} = (X_1, \ldots, X_n)$$

$$X_i \overset{i.i.d}{\sim} Bernoulli(p)$$

Likelihood

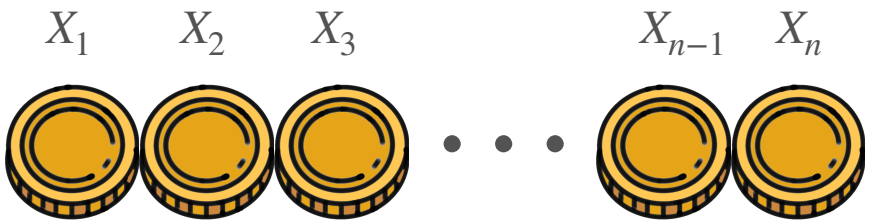
$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^{n} p_{X_i}(x_i) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

If $x_i = 1, p^{[x_i]}(1-p)^{[1-x_i]} = p$
If $x_i = 0, p^{[x_i]}(1-p)^{[1-x_i]} = (1-p)$

$\sum_{i=1}^{n} x_i$ = # heads

$n - \sum_{i=1}^{n} x_i$ = # tails

Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \ldots, X_n)$$

$$X_i \overset{i.i.d}{\sim} Bernoulli(p)$$

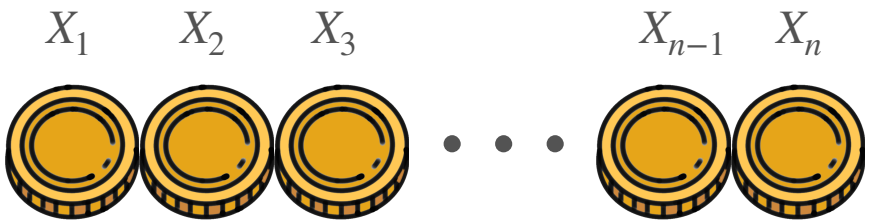
Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^{n} p_{X_i}(x_i) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\left(\sum_{i=1}^{n} x_i\right)}(1-p)^{\left(n-\sum_{i=1}^{n} x_i\right)}$$

Log-likelihood

$$\ell(p; \mathbf{x}) = \log\left((p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}\right) = \left(\sum_{i=1}^{n} x_i\right)\log(p) + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-p)$$

Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \ldots, X_n)$$

$$X_i \overset{i.i.d}{\sim} Bernoulli(p)$$

$$\ell(p; \mathbf{x}) = \log\left((p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}\right) = \left(\sum_{i=1}^n x_i\right)\log(p) + \left(n - \sum_{i=1}^n x_i\right)\log(1-p)$$

Find the maximum!

$$\frac{d}{dp}\ell(p; \mathbf{x}) = \frac{d}{dp}\left(\left(\sum_{i=1}^n x_i\right)\log(p) + \left(n - \sum_{i=1}^n x_i\right)\log(1-p)\right)$$

$$= \frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p}(-1) = 0 \quad \rightarrow \quad \hat{p} = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$$

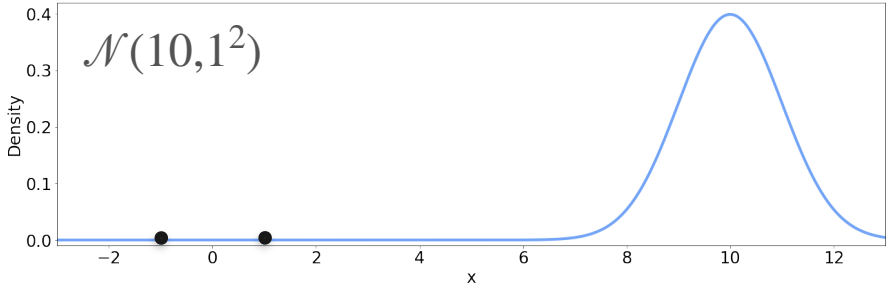
DeepLearning.AI

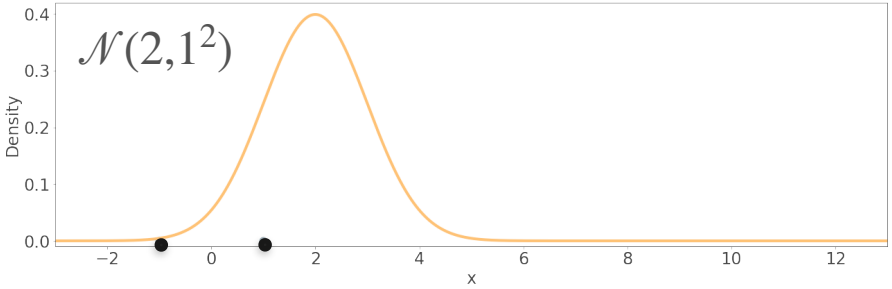
Point Estimation

MLE: Gaussian Example

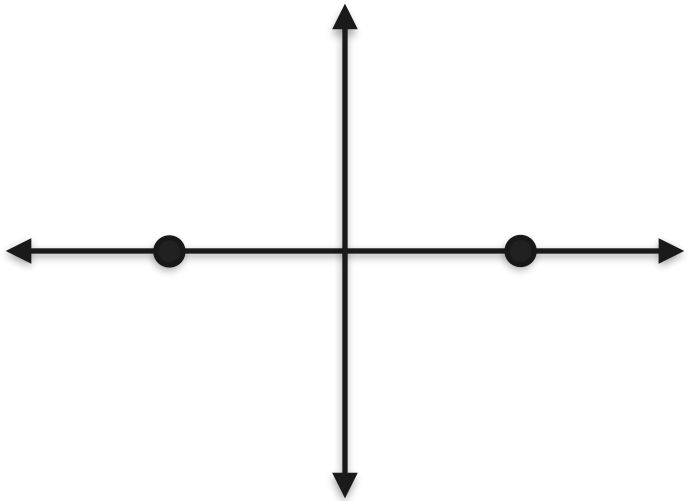
Maximum Likelihood: Gaussian Example

Candidates



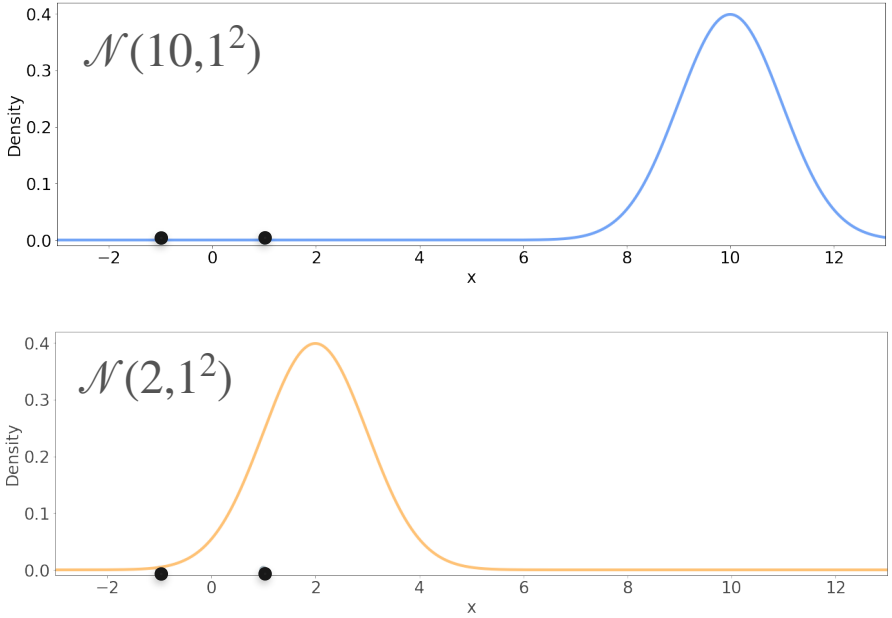


Observations

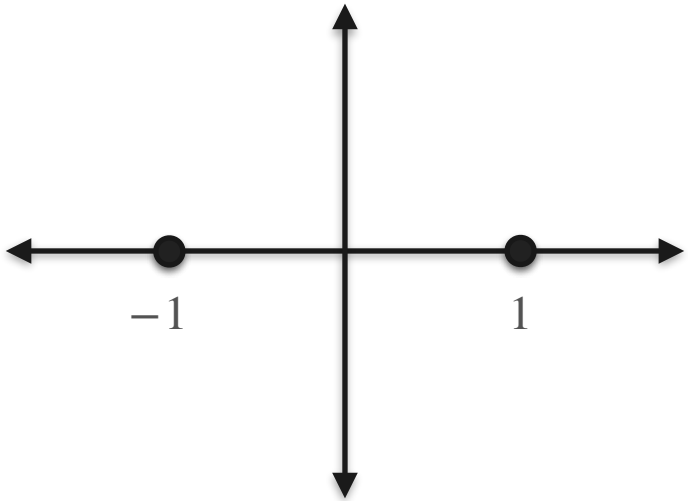


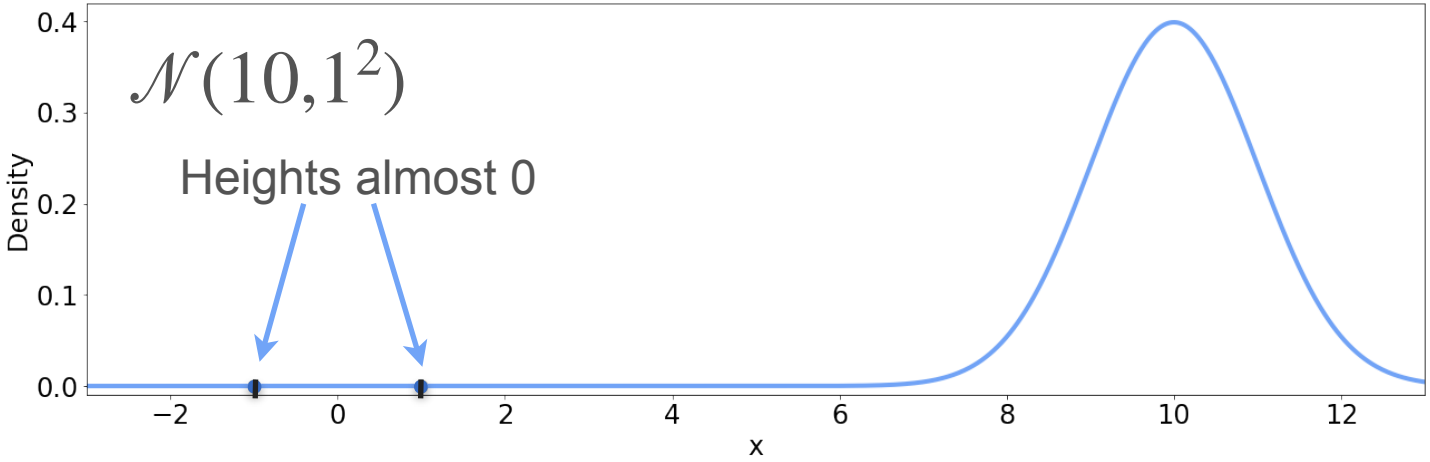
Maximum Likelihood: Gaussian Example

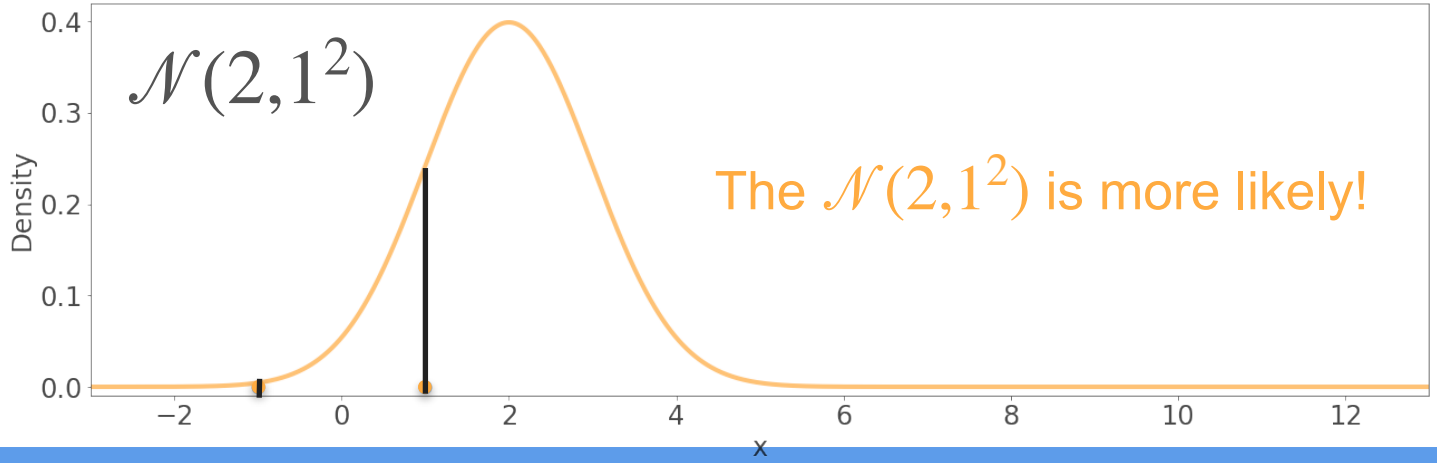
Candidates



Observations

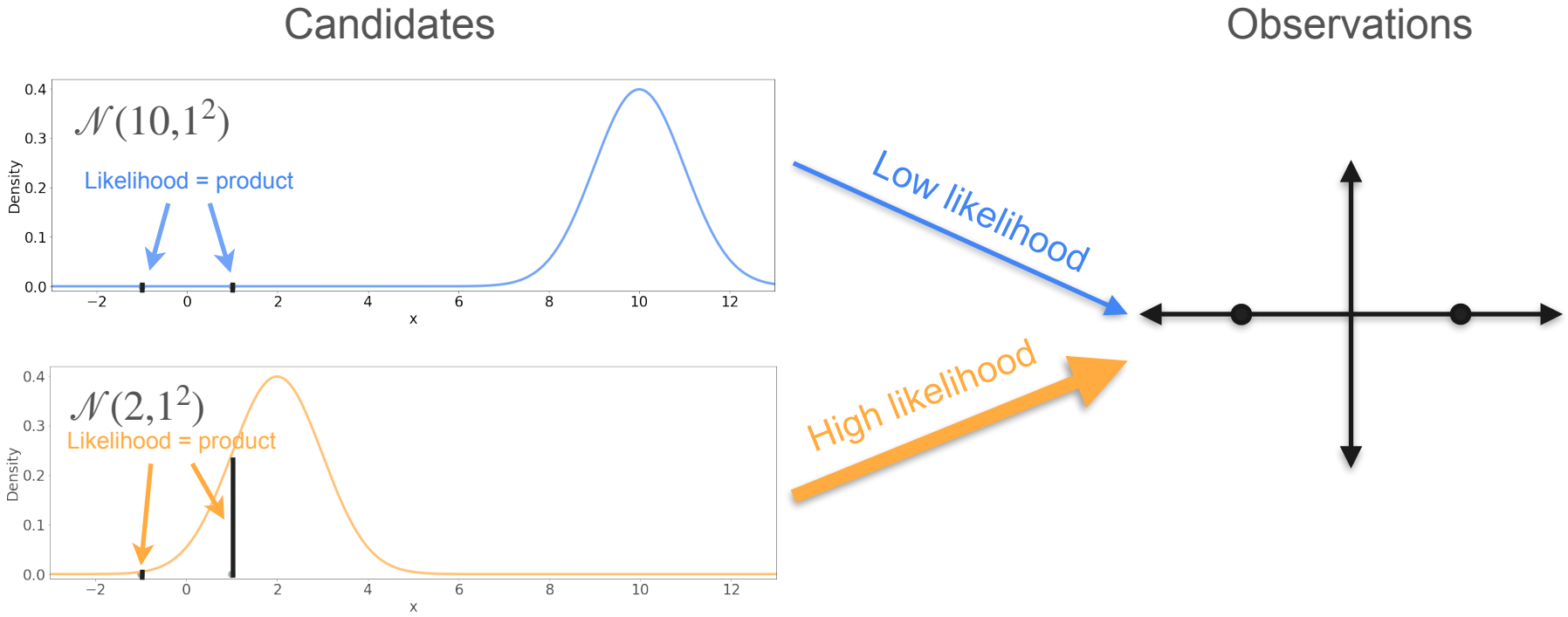






The $\mathcal{N}(2,1^2)$ is more likely!

Maximum Likelihood: Gaussian Example



Gaussians With Three Different Means

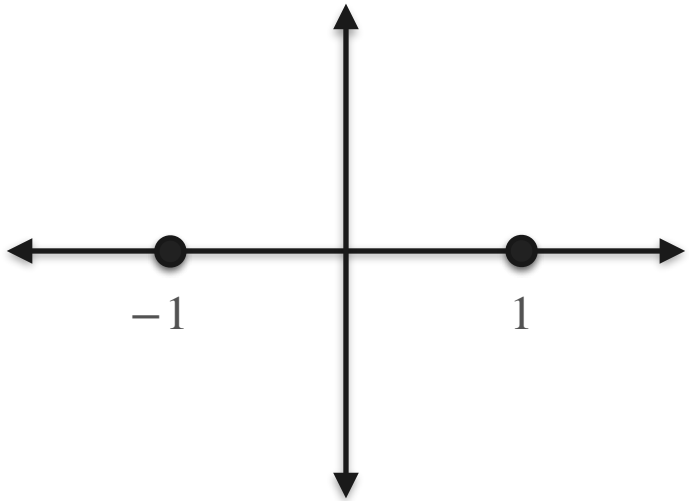
Candidates

$\mathcal{N}(-1,1^2)$

$\mathcal{N}(0,1^2)$

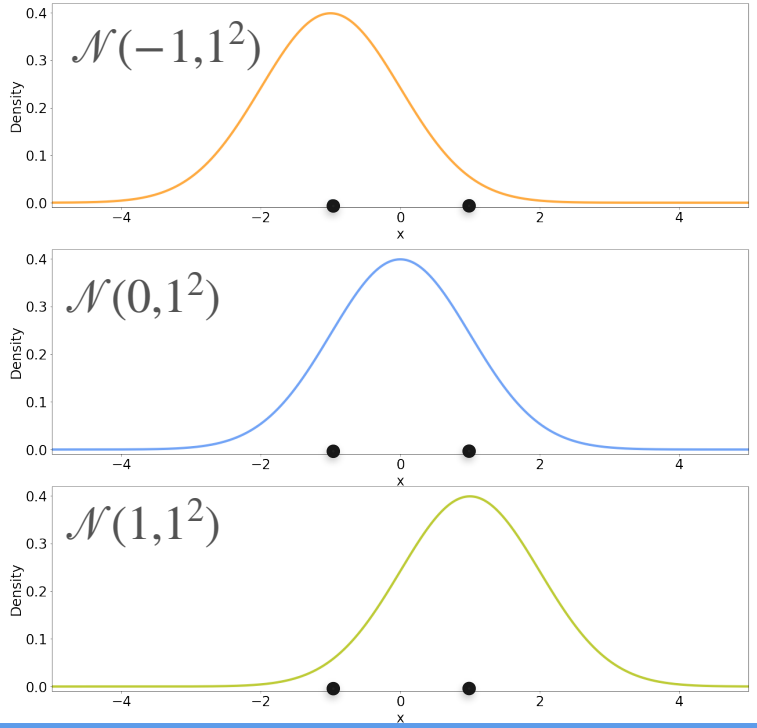
$\mathcal{N}(1,1^2)$

Observations

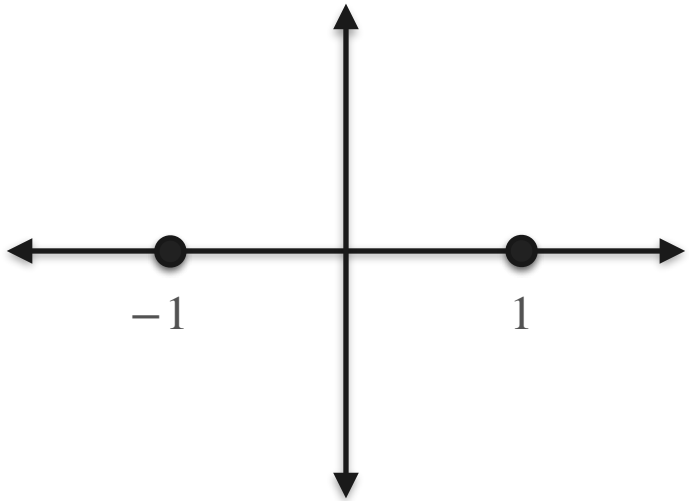


Gaussians With Three Different Means

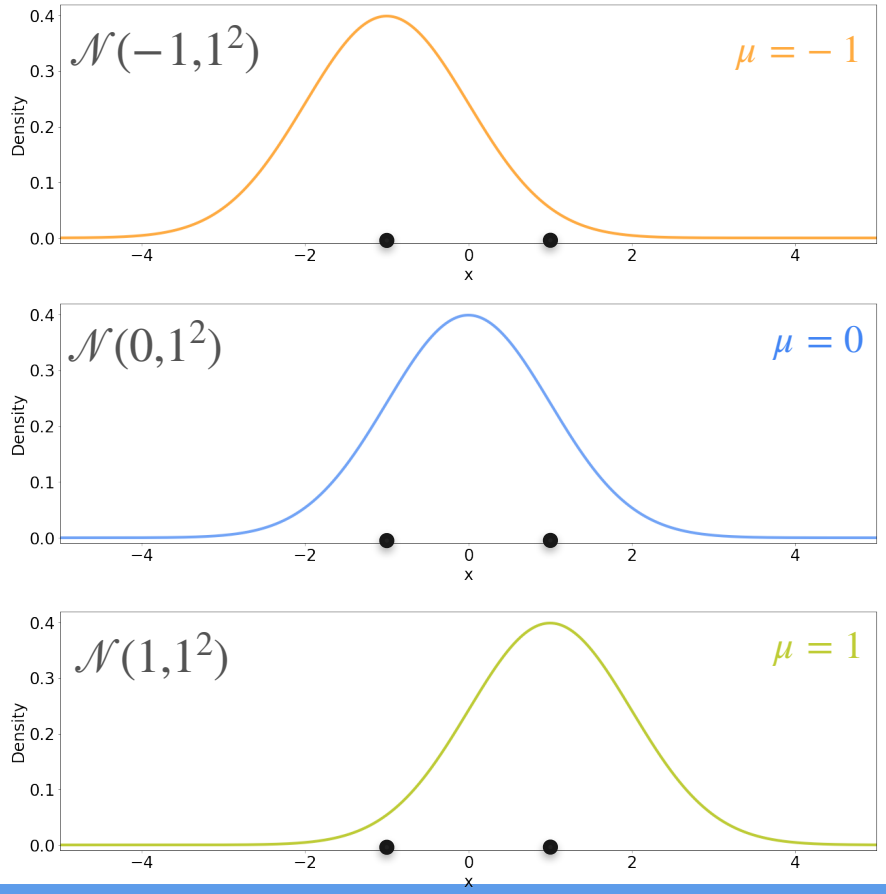
Candidates



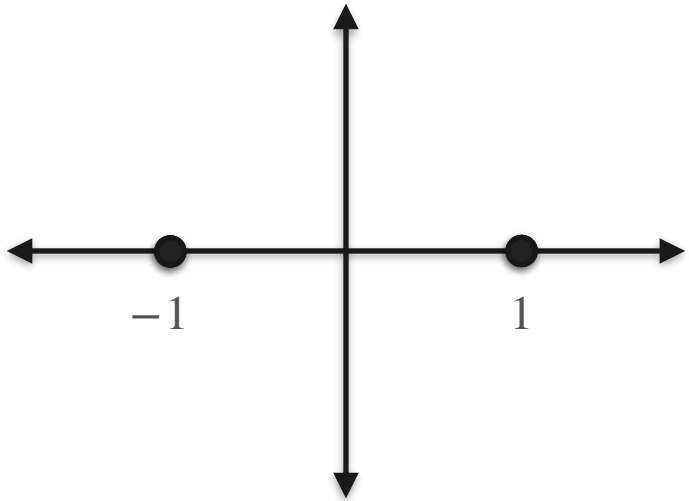
Observations



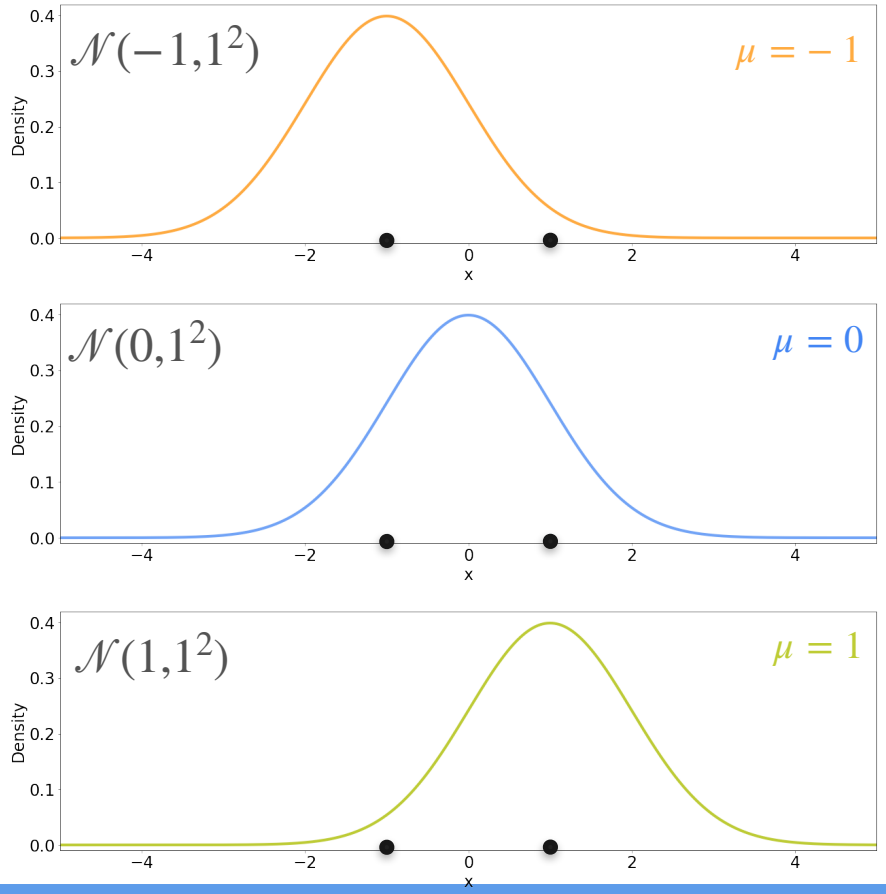
Candidates



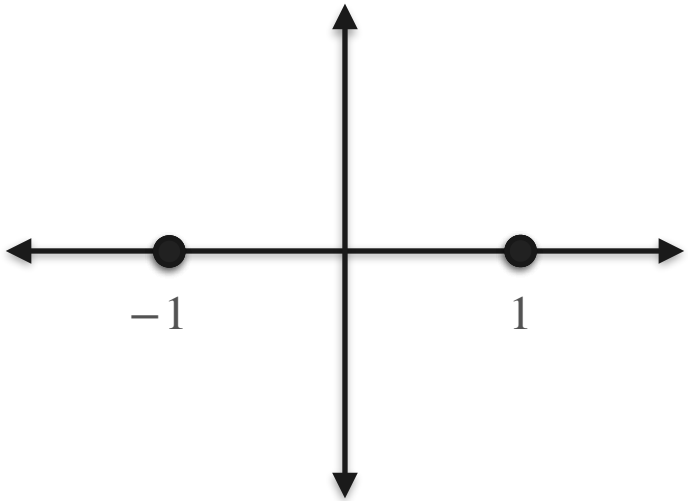
Observations



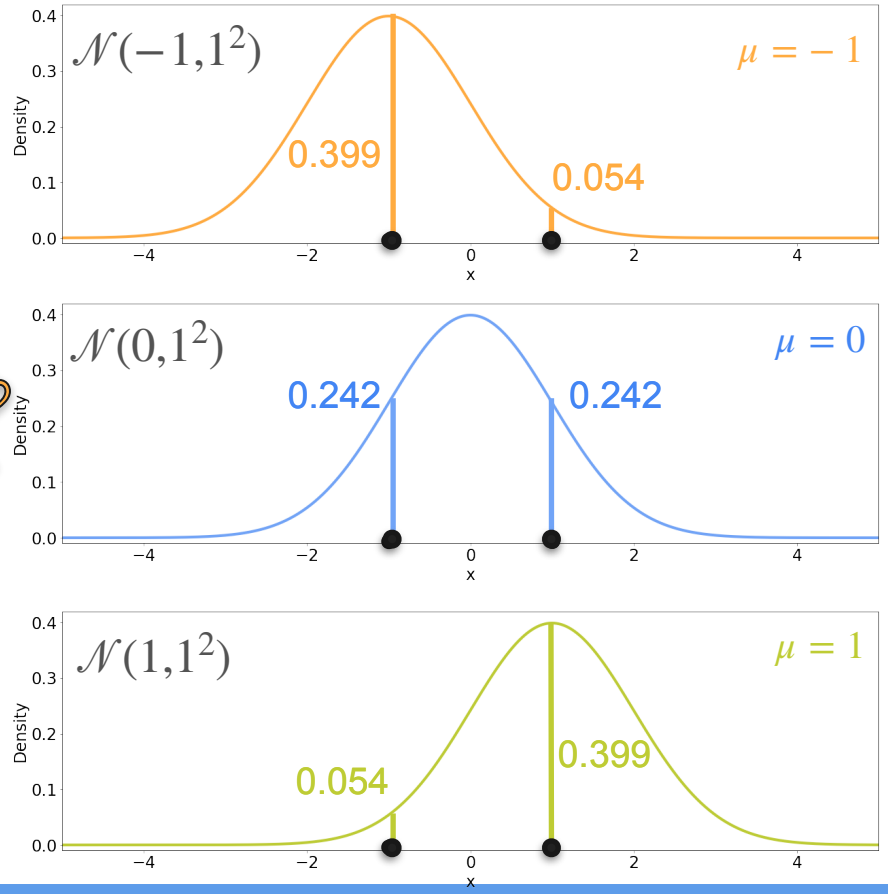
Candidates



Observations



Candidates



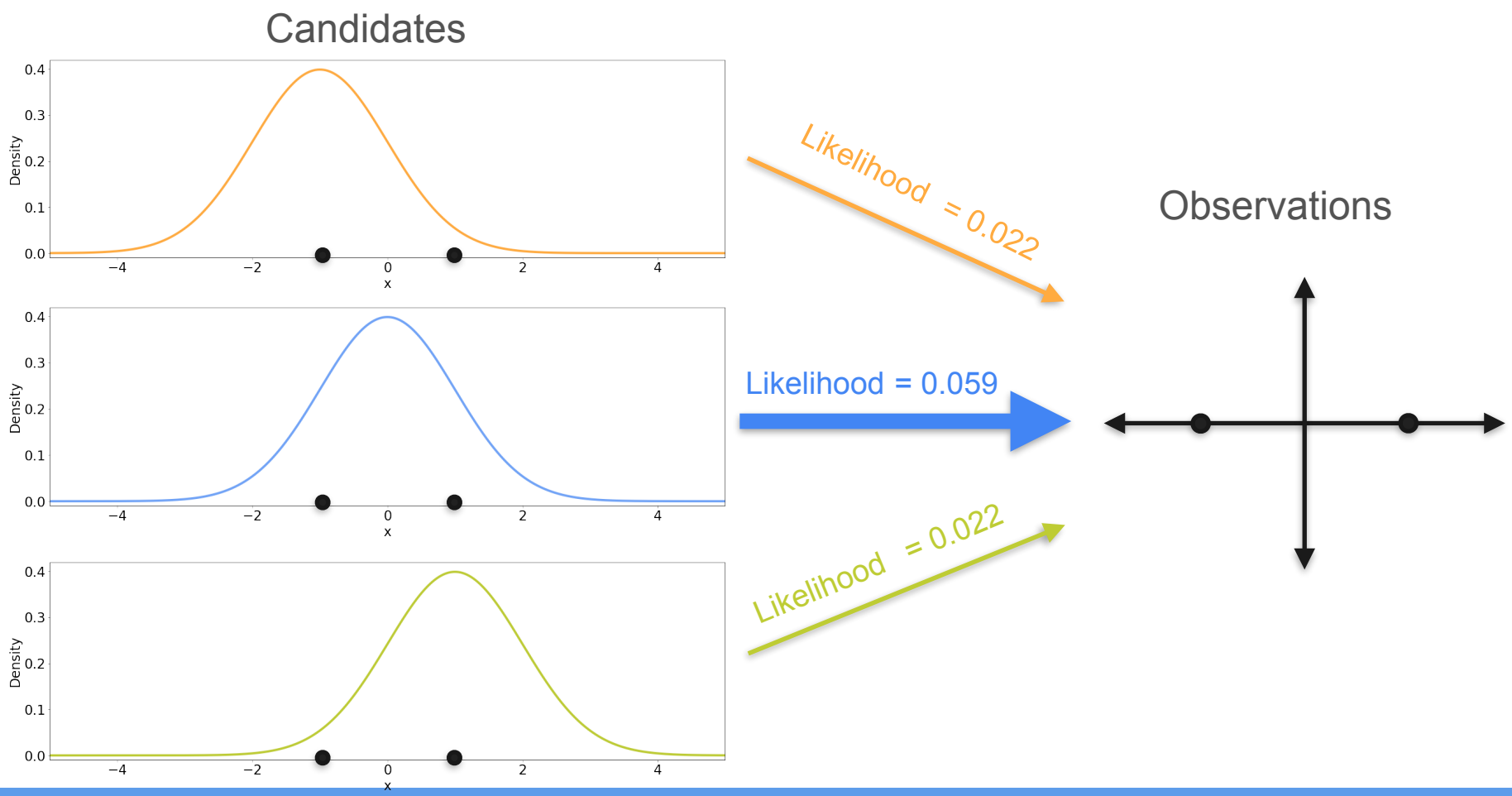
Observations

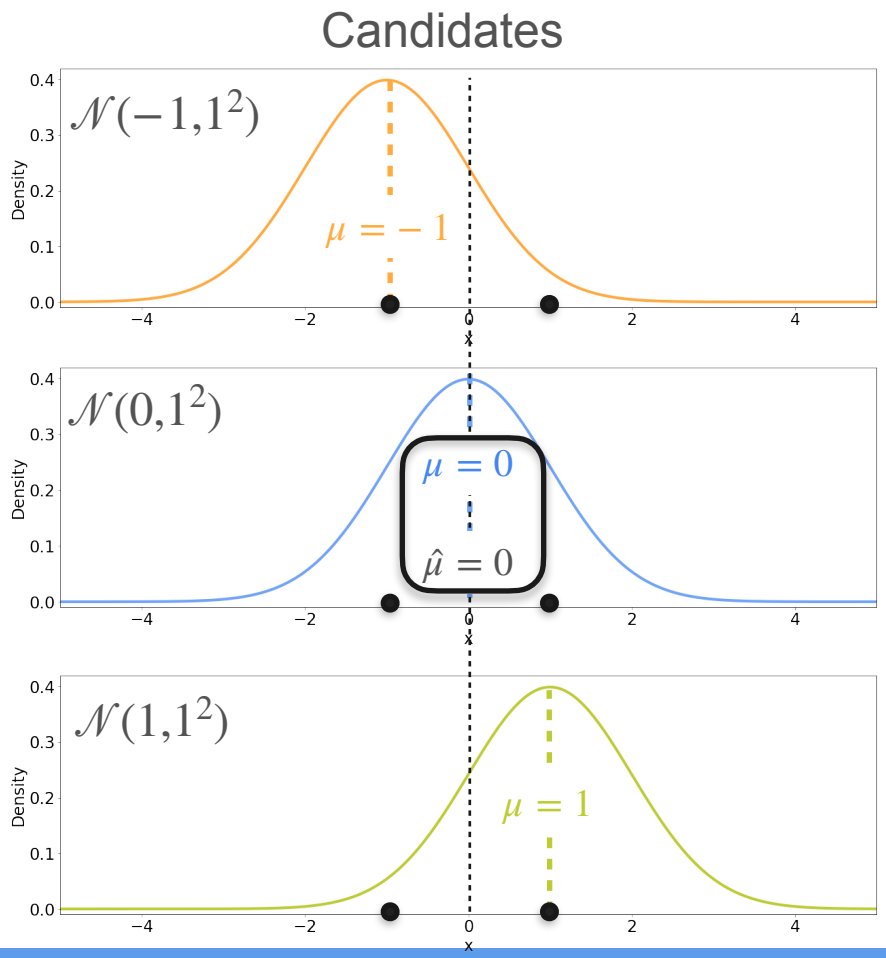
· = 0.022

The $\mathcal{N}(0,1)$ is more likely!

· = 0.059

· = 0.022


Candidates
Density
x
Likelihood = 0.022
Likelihood = 0.059
Likelihood = 0.022
Observations



The best distribution is the one where the **mean** of the distribution is the **mean** of the sample

Gaussians With Three Different Variance

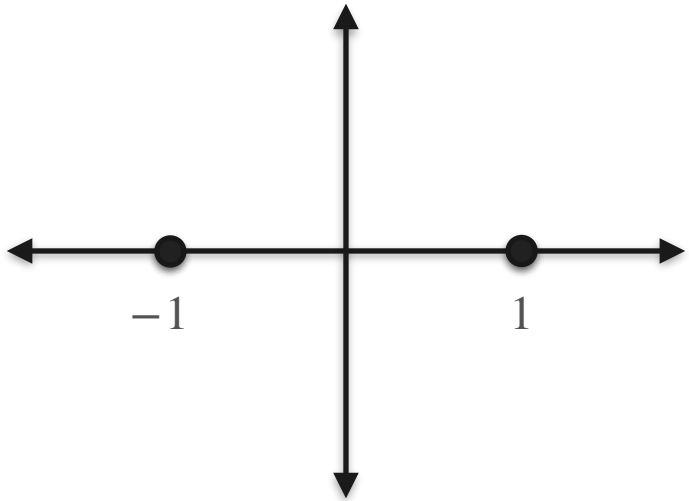
Candidates

$\mathcal{N}(0,0.5^2)$

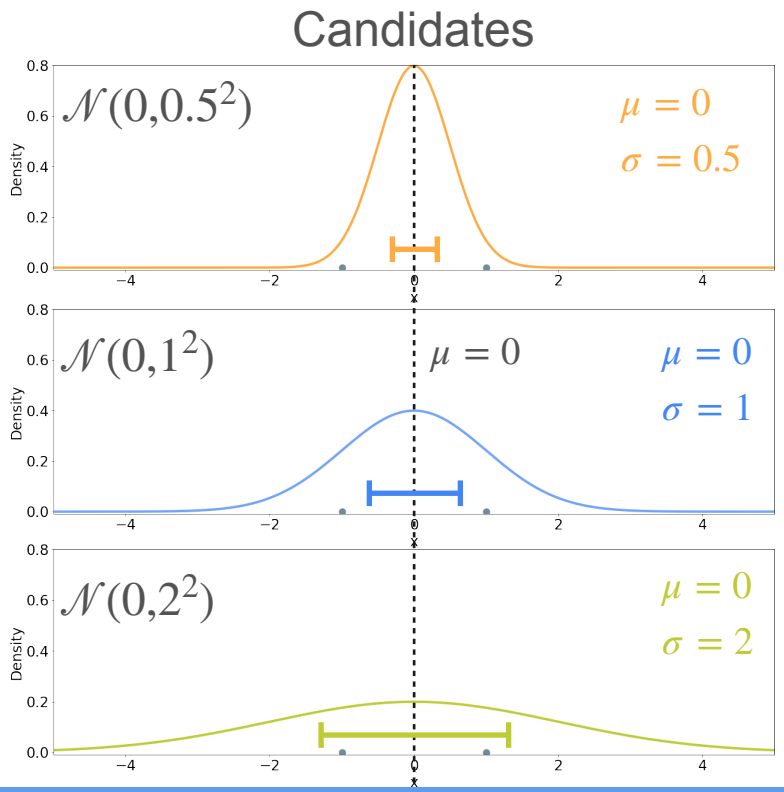
$\mathcal{N}(0,1^2)$

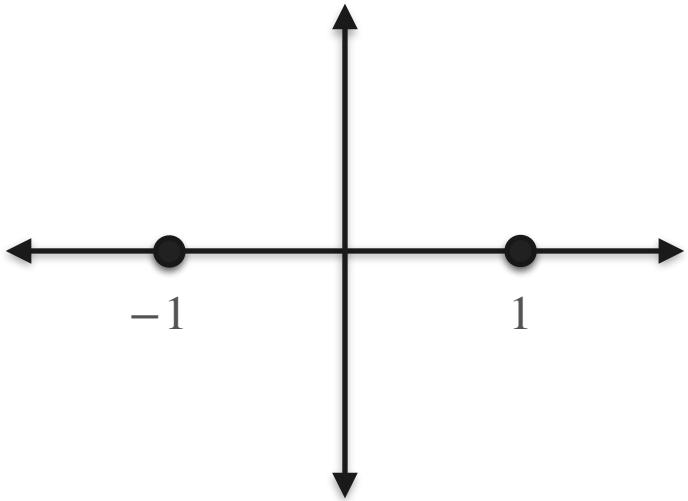
$\mathcal{N}(0,2^2)$

Observations

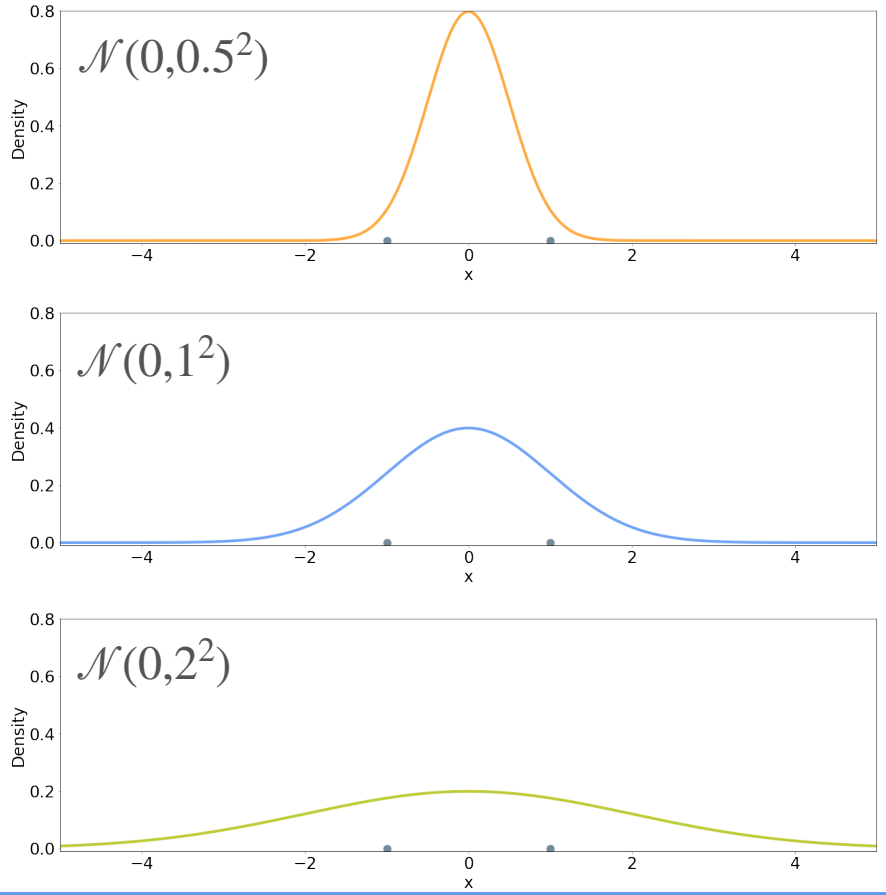


Gaussians With Three Different Variance

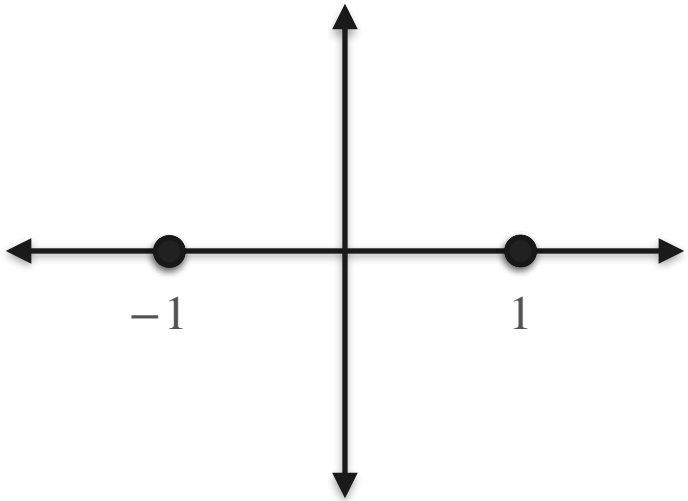




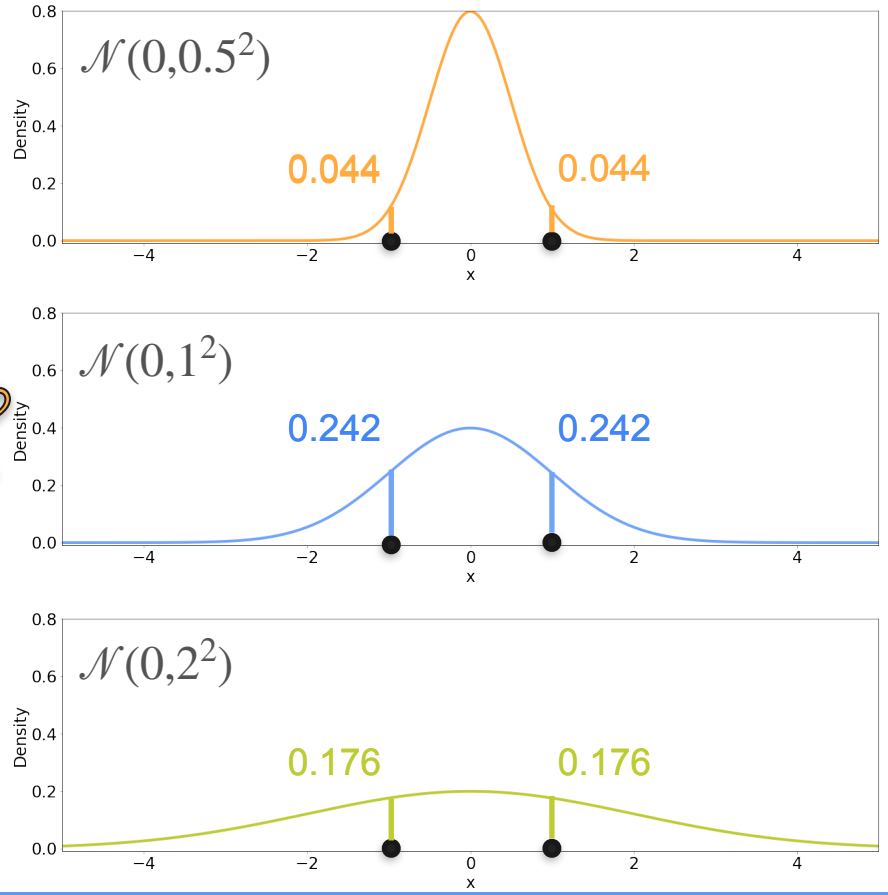
Candidates



Observations



Candidates



Observations

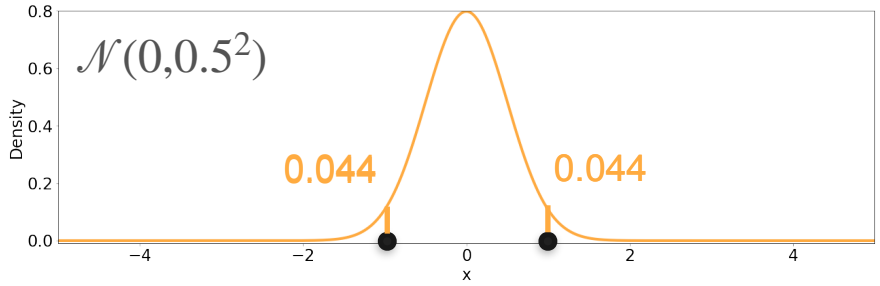
· = 0.002

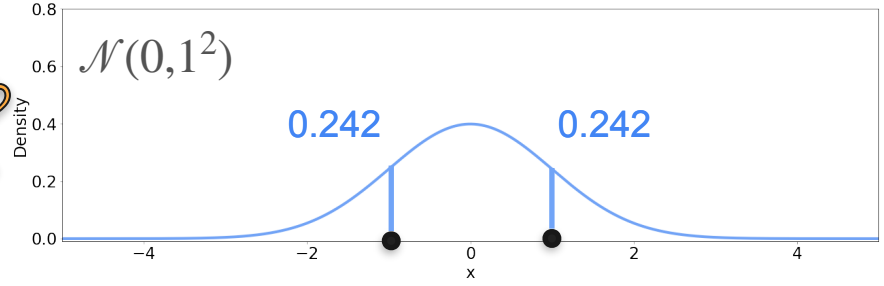
The $\mathcal{N}(0,1^2)$ is more likely!

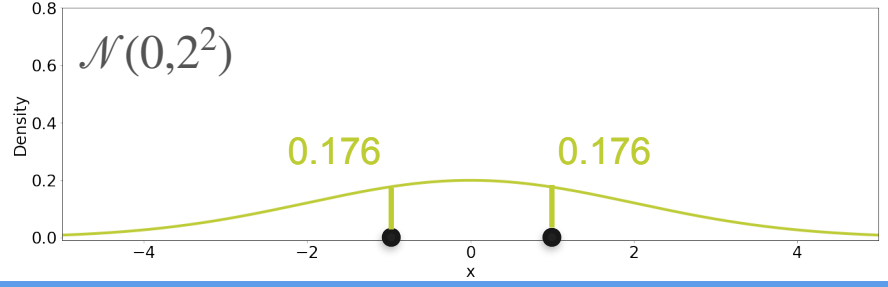
· = 0.059

· = 0.031

Candidates







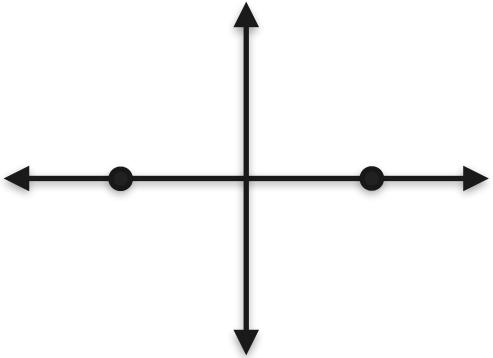
Observations

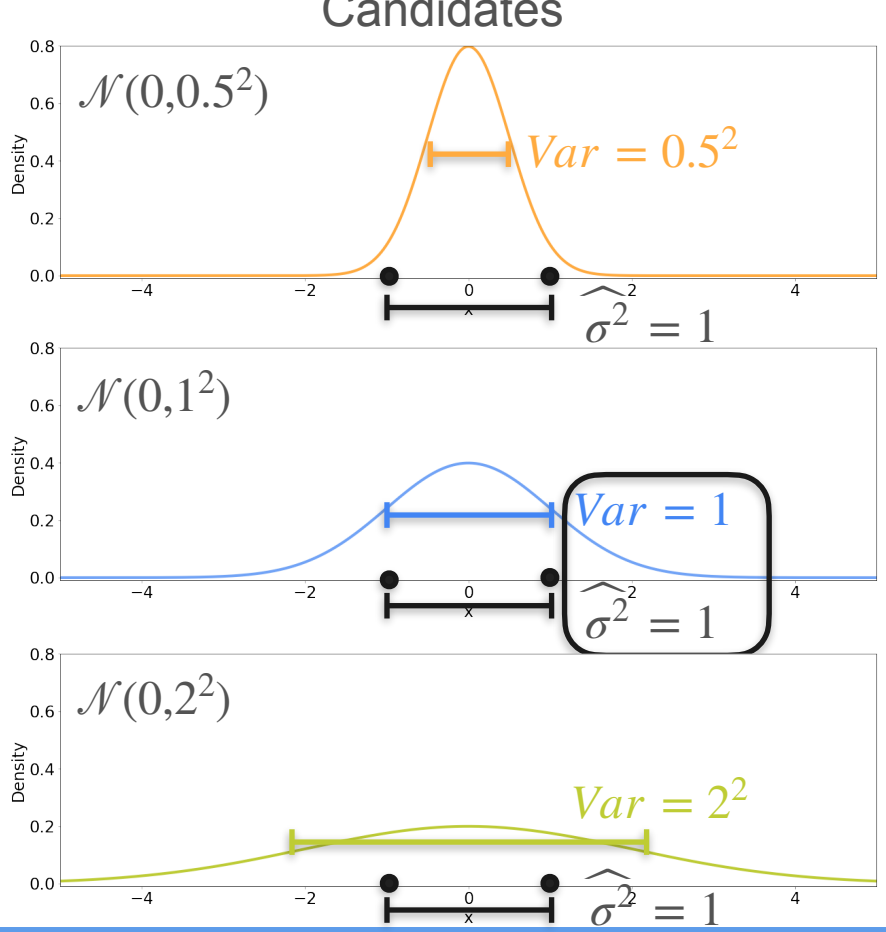
Likelihood = 0.002

Likelihood = 0.059

Likelihood = 0.031

Observations





Observations

Variance of the observations

$$\widehat{\sigma^2} = \frac{1}{2}\left((0-1)^2 + (0+1)^2\right) = 1$$

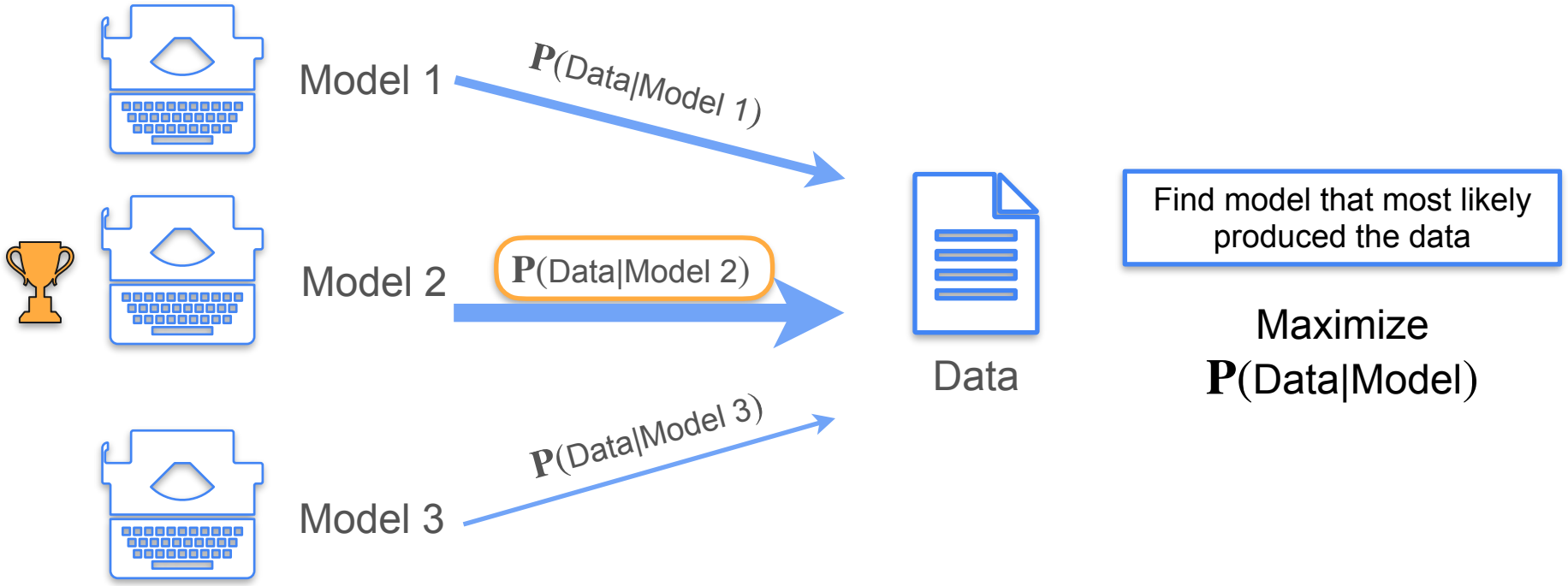
The best distribution is the one where the **variance** of the distribution is the **variance** of the sample

DeepLearning.AI

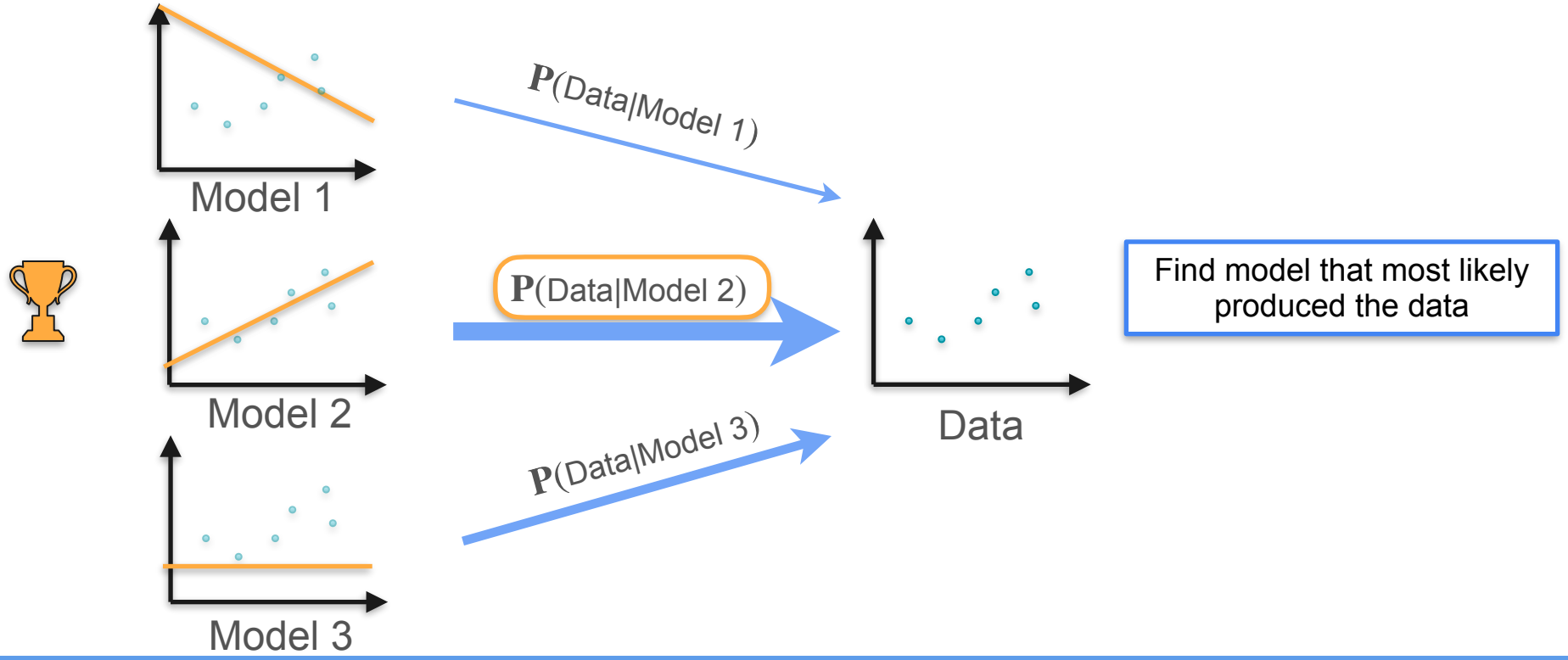
Point Estimation

MLE: Linear Regression

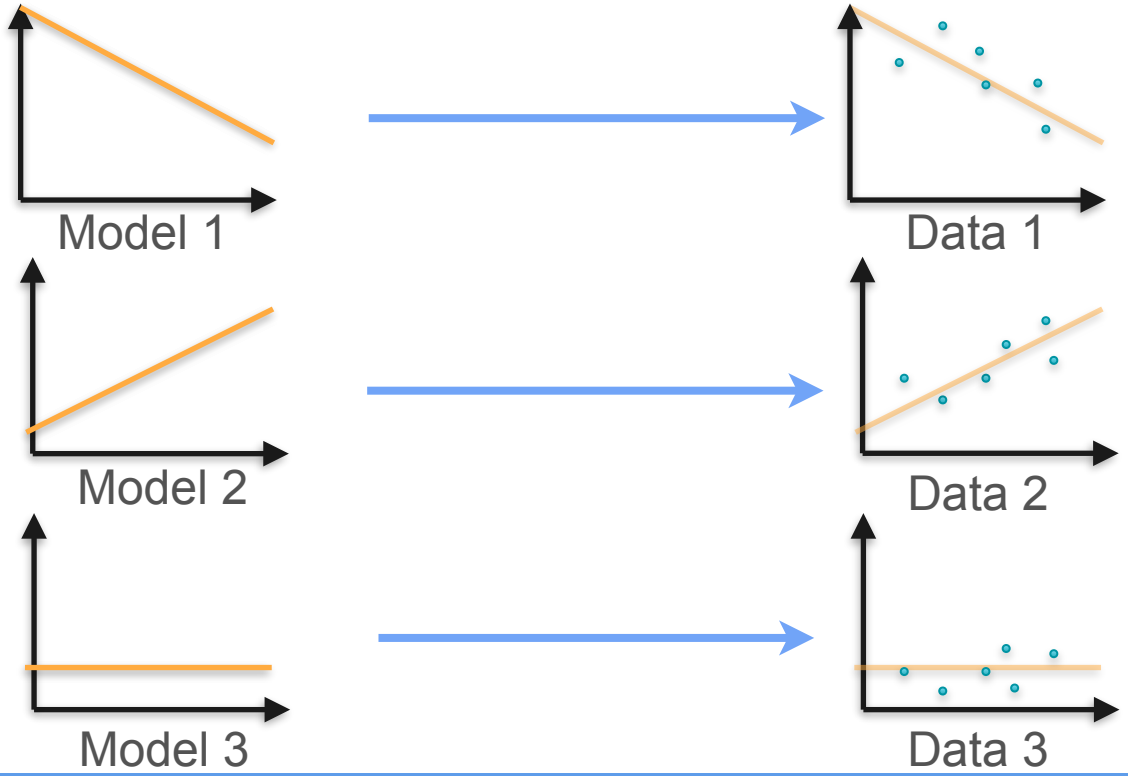
Maximum Likelihood



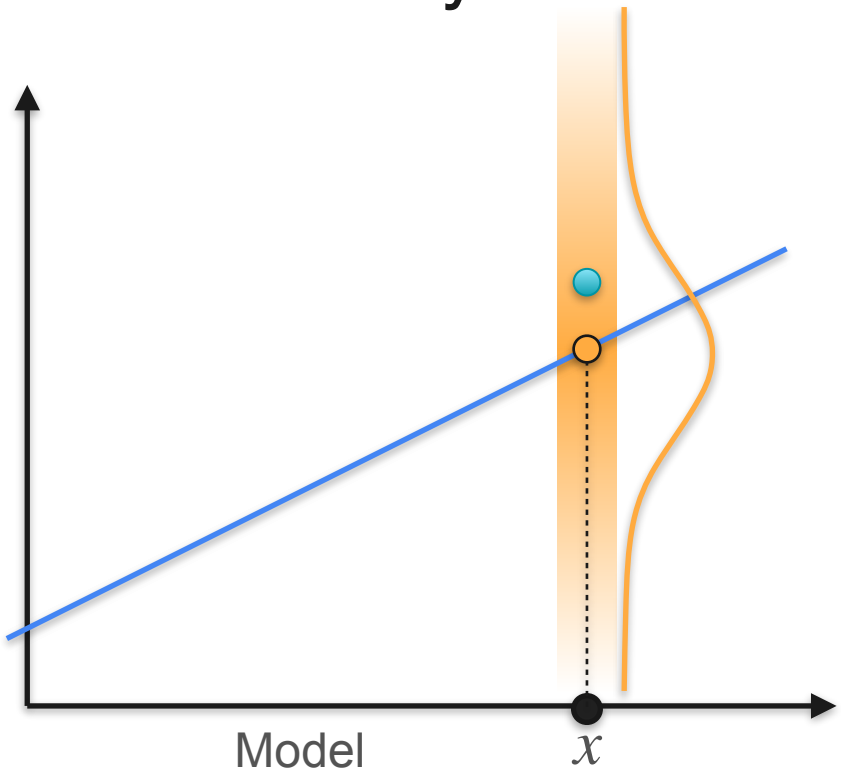
Example: Linear Regression

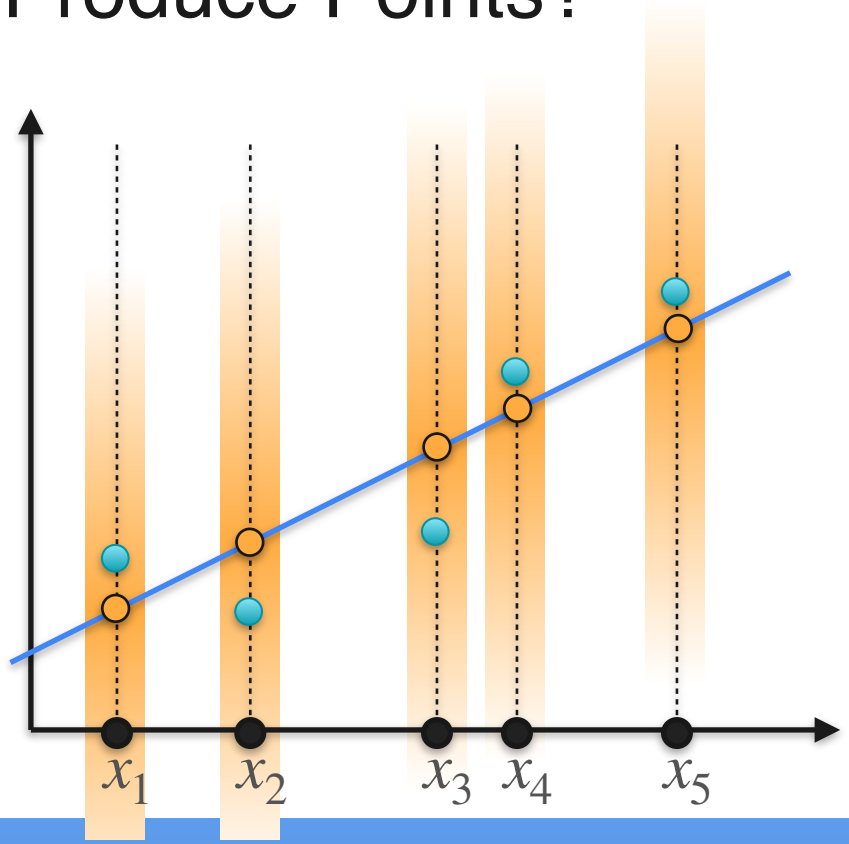


How Exactly Does a Line Produce Points?



How Exactly Does a Line Produce Points?



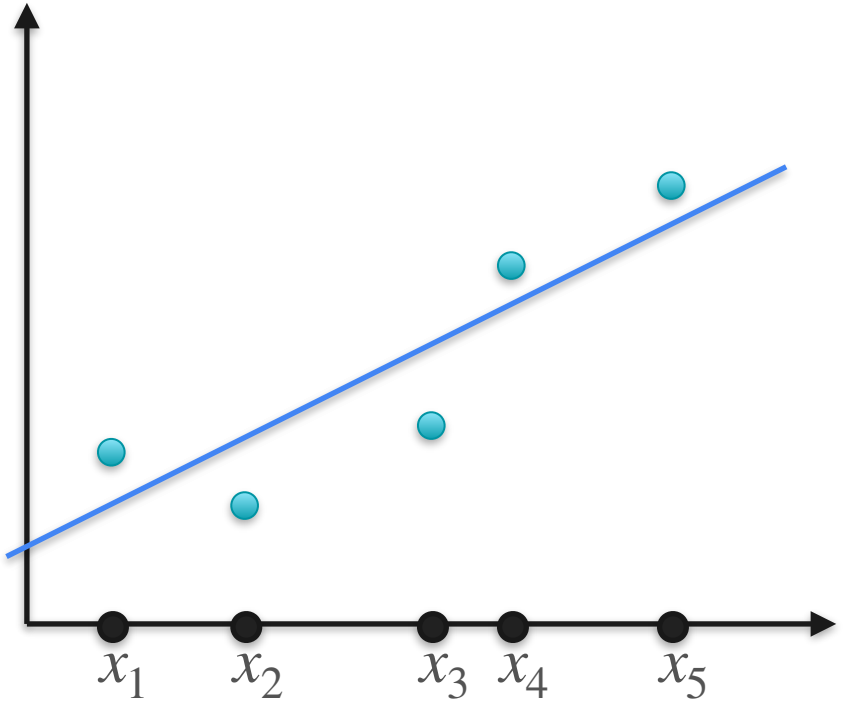


Linear Regression

Line that best produced the points

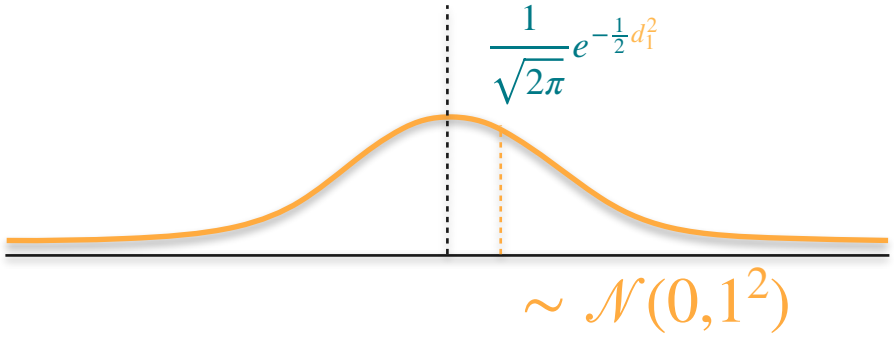
How?

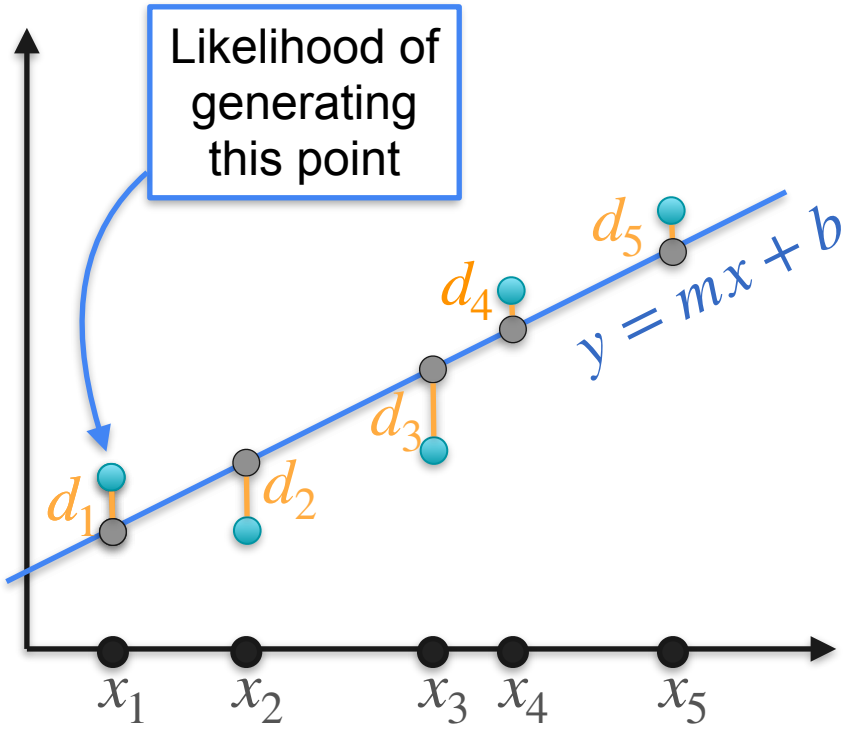
Line that best fits the data (linear regression)



Linear Regression and Likelihood

Likelihood:





Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

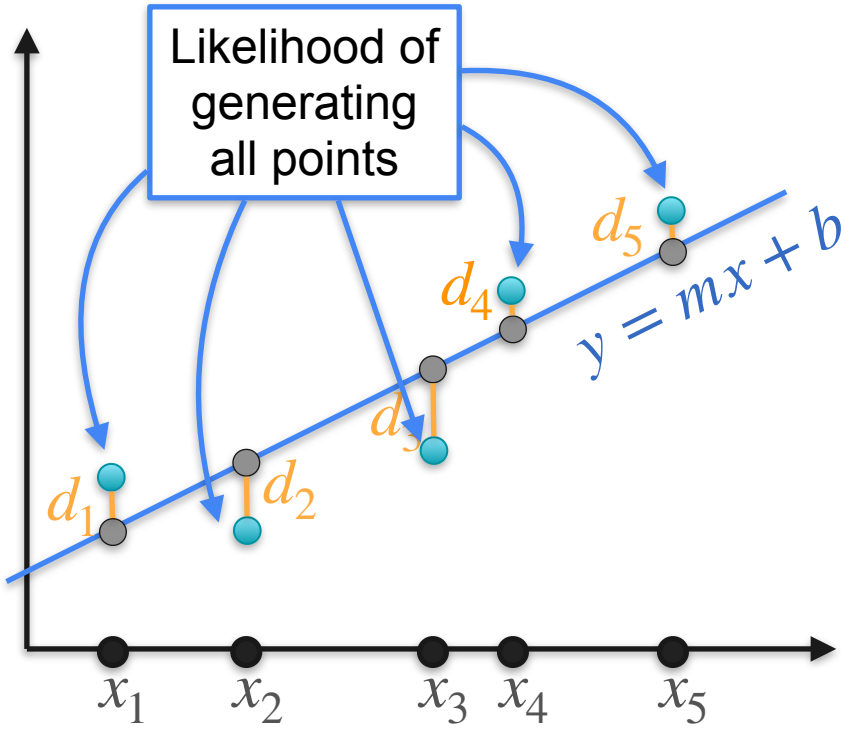
$$e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}$$

Minimize

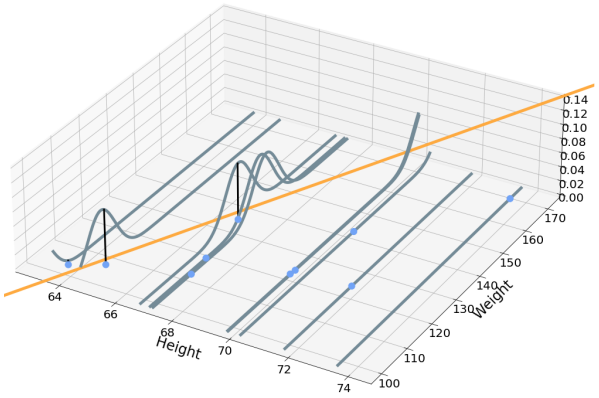
$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

Least squares error!

Linear regression!

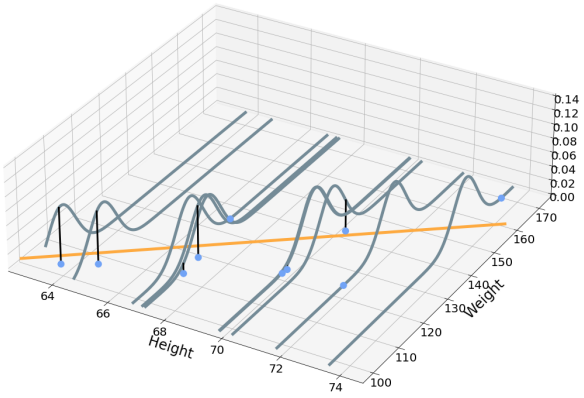


Picking the Right Model



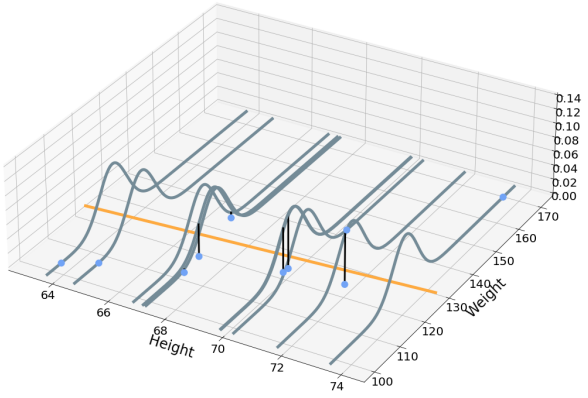
Model 1:

Likelihood $= 4.91 \cdot 10^{-260}$



Model 2:

Likelihood $= 8.16 \cdot 10^{-28}$



Model 3:

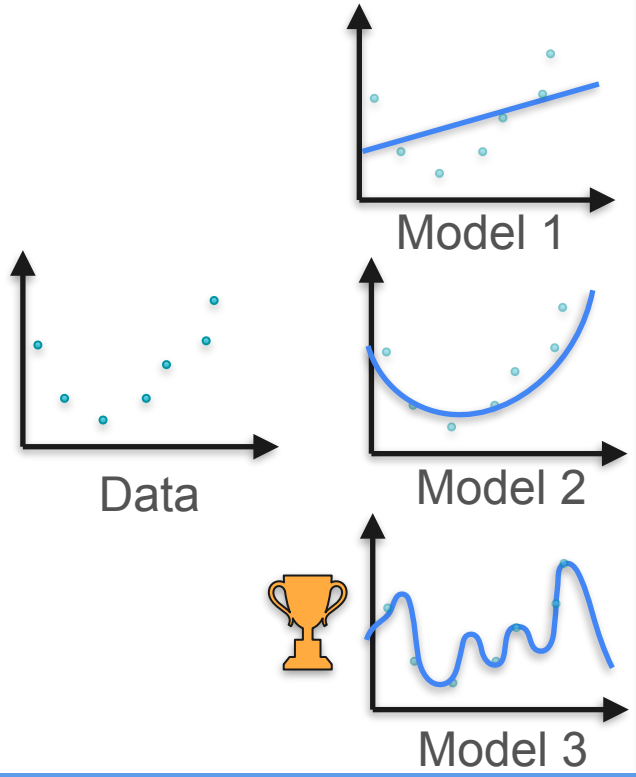
Likelihood $= 3.48 \cdot 10^{-49}$

DeepLearning.AI

Point Estimation

Regularization

Example: Polynomial Regression



Loss	Equation	Penalty	New loss
10	$y = 4x + 3$	$L_2 = 4^2 = 16$	26
2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	22
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	246.1

Regularization Term

Model: $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Log-loss: $\ell\ell$

L2 Regularization Error: $a_n^2 + a_{n-1}^2 + \cdots + a_1^2$

Regularization parameter: λ

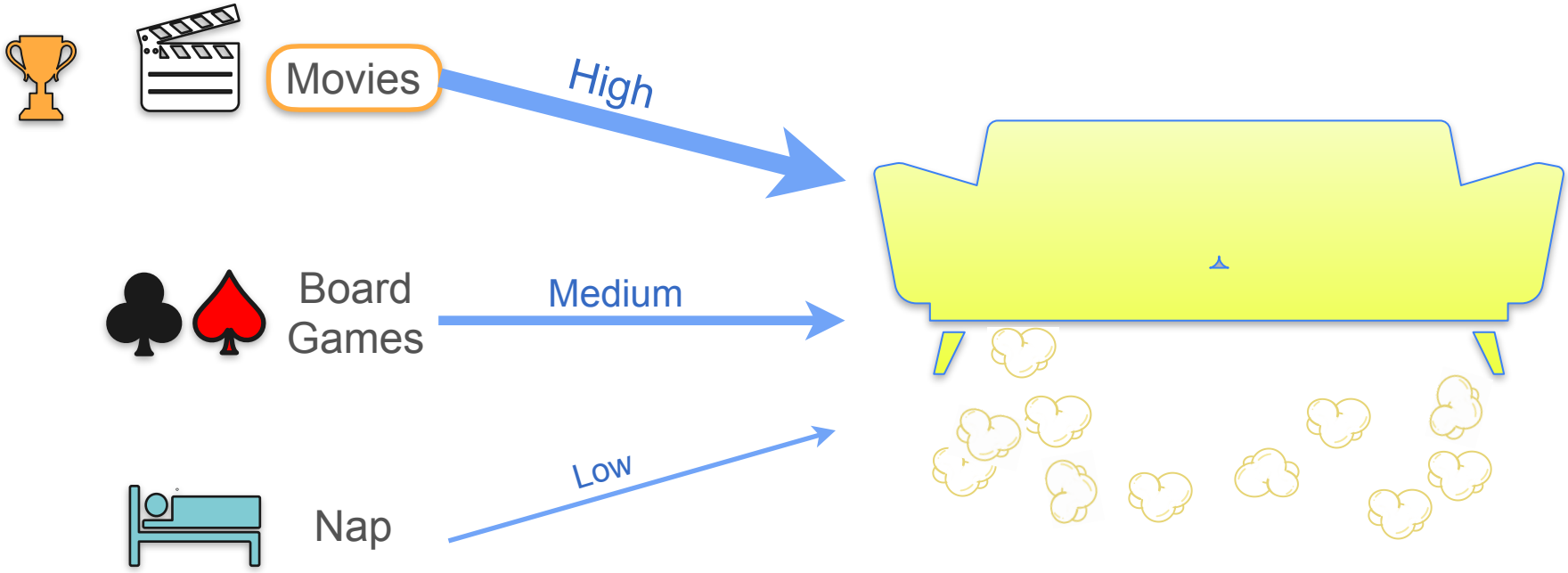
Regularized error: $+ \quad (\qquad\qquad\qquad\qquad)$

DeepLearning.AI

Point Estimation

Back to “Bayesics”

There's Popcorn on the Floor. What Happened?



There's Popcorn on the Floor. What Happened?

$\mathbf{P}(\text{Movies})$ Movies

High

$\mathbf{P}(\text{Popcorn}|\text{Movies})$

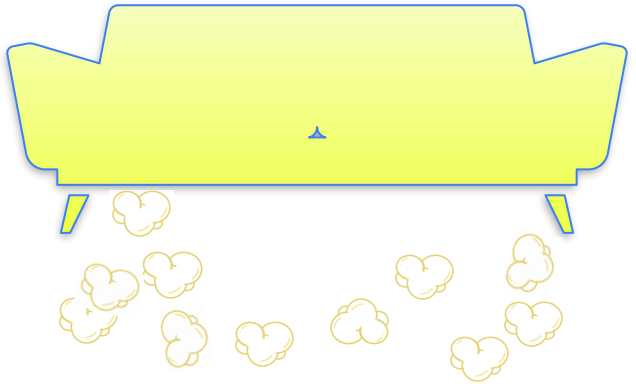
$\mathbf{P}(\text{Movies}) \gg \mathbf{P}(\text{Contest})$

$\mathbf{P}(\text{Contest})$

Popcorn throwing contest

Very high

$\mathbf{P}(\text{Popcorn}|\text{Contest})$



$$\mathbf{P}(\text{Popcorn}|\text{Movies}) < \mathbf{P}(\text{Popcorn}|\text{Contest})$$

$$\mathbf{P}(\text{Popcorn} \cap \text{Movie}) > \mathbf{P}(\text{Popcorn} \cap \text{Contest})$$

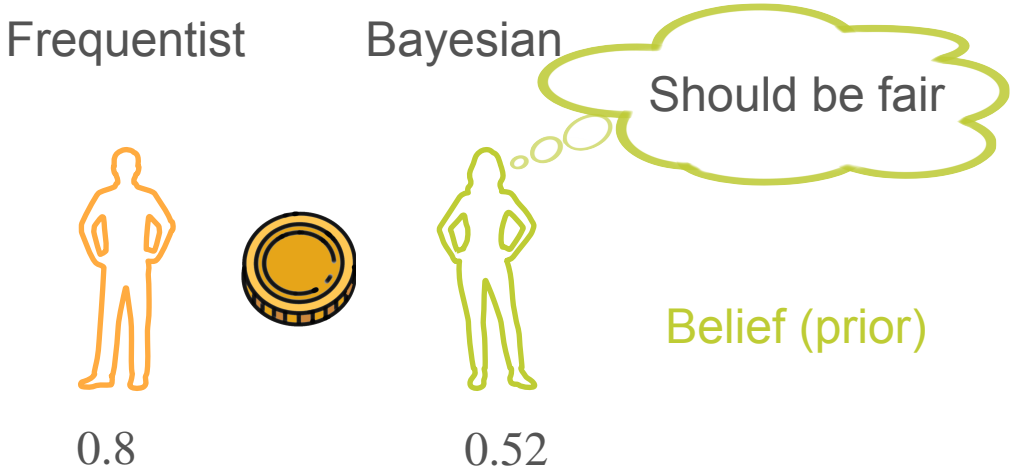
$$\mathbf{P}(A \mid B)\mathbf{P}(B) = \mathbf{P}(A \cap B)$$

DeepLearning.AI

Point Estimation

Frequentist vs Bayesian Statistics

Frequentists vs. Bayesians



Frequentist Vs. Bayesian Statistics

Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood
- Goal: Find the model that most likely generated the observed data

Bayesians

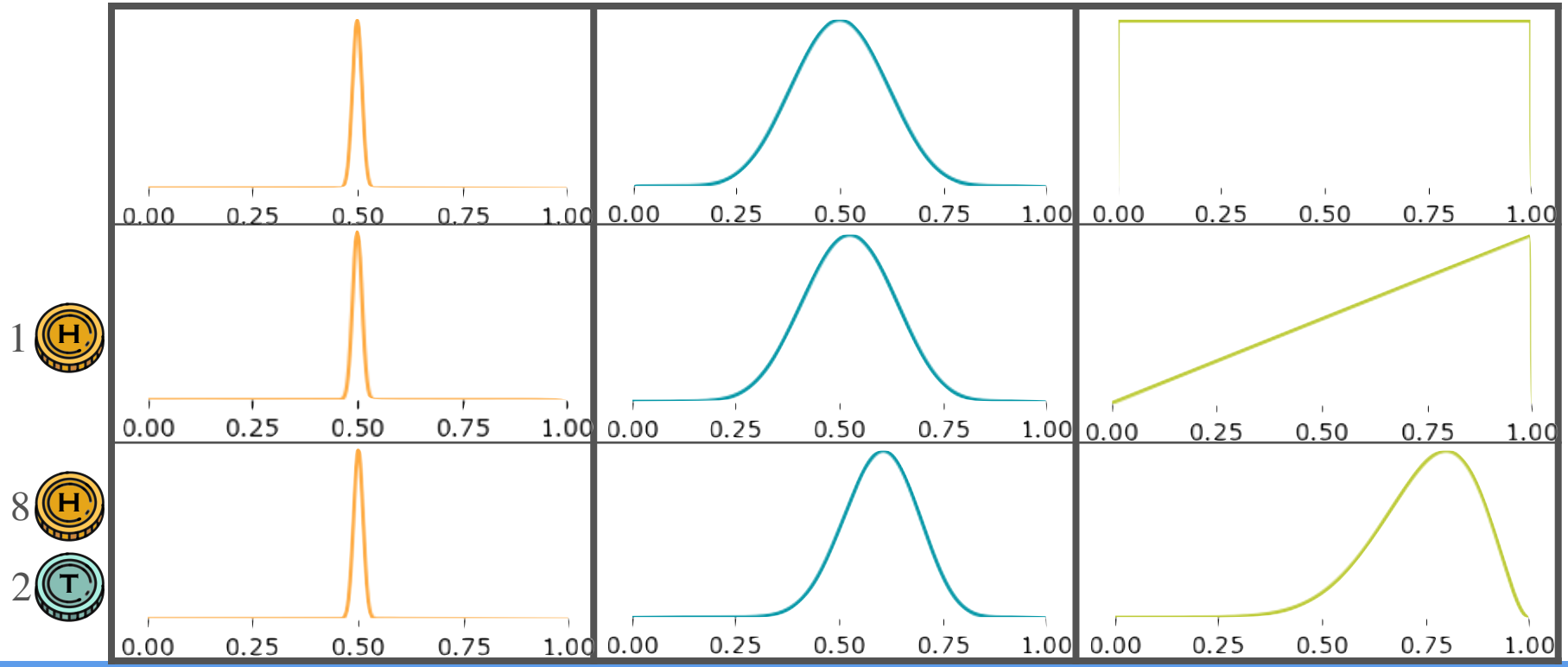
- Probabilities represent the degree of belief (or certainty)
- Concept of Prior
- Goal: update prior belief based on observations

DeepLearning.AI

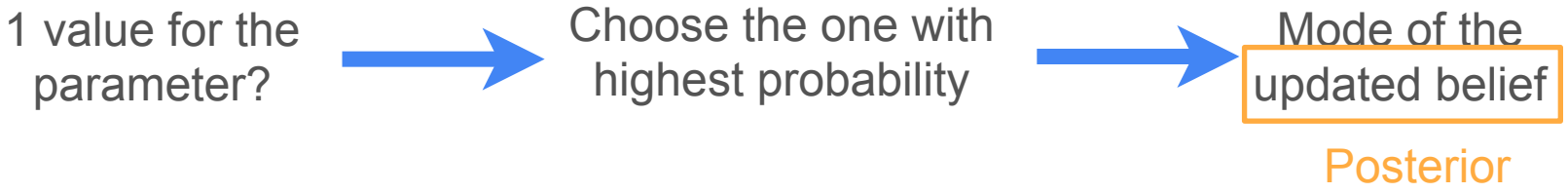
Point Estimation

Bayesian Statistics
MAP

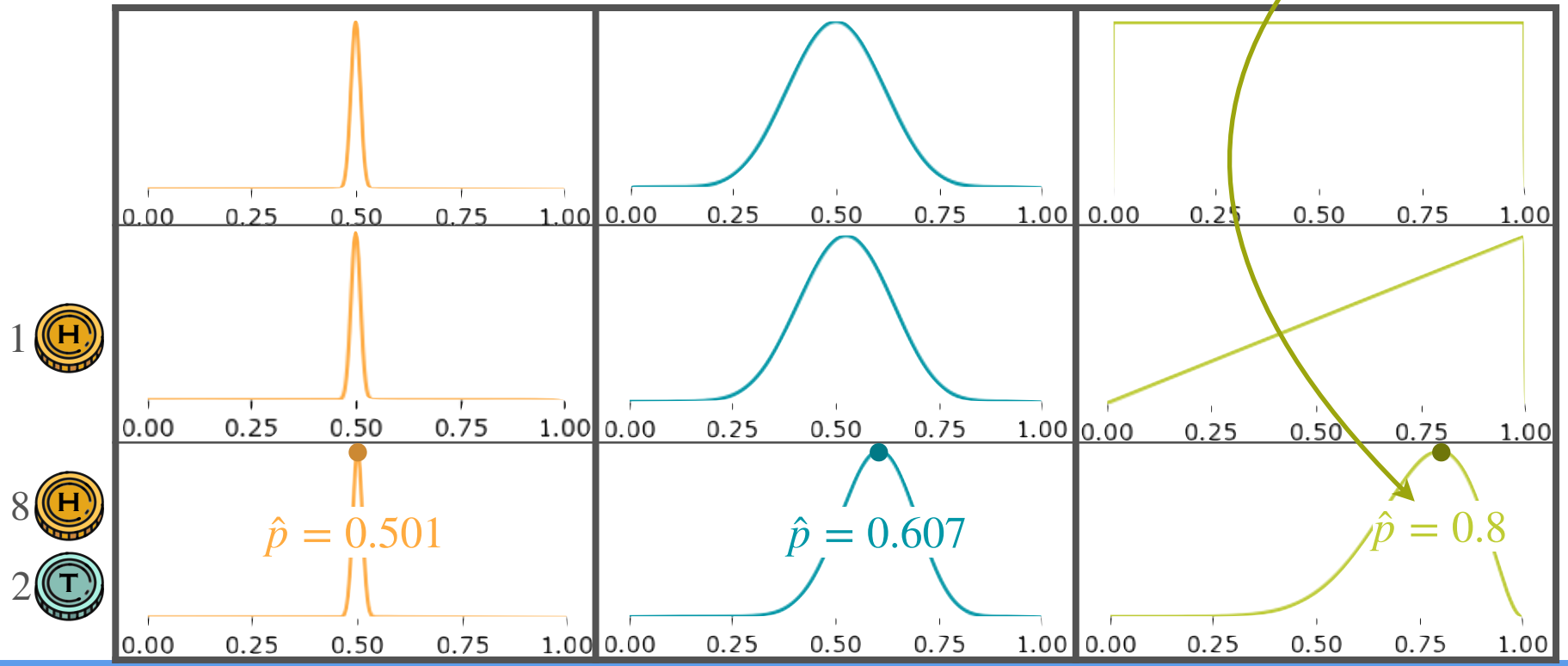
Updating Your Beliefs



Maximum A Posteriori (MAP)



Maximum A Posteriori (MAP)

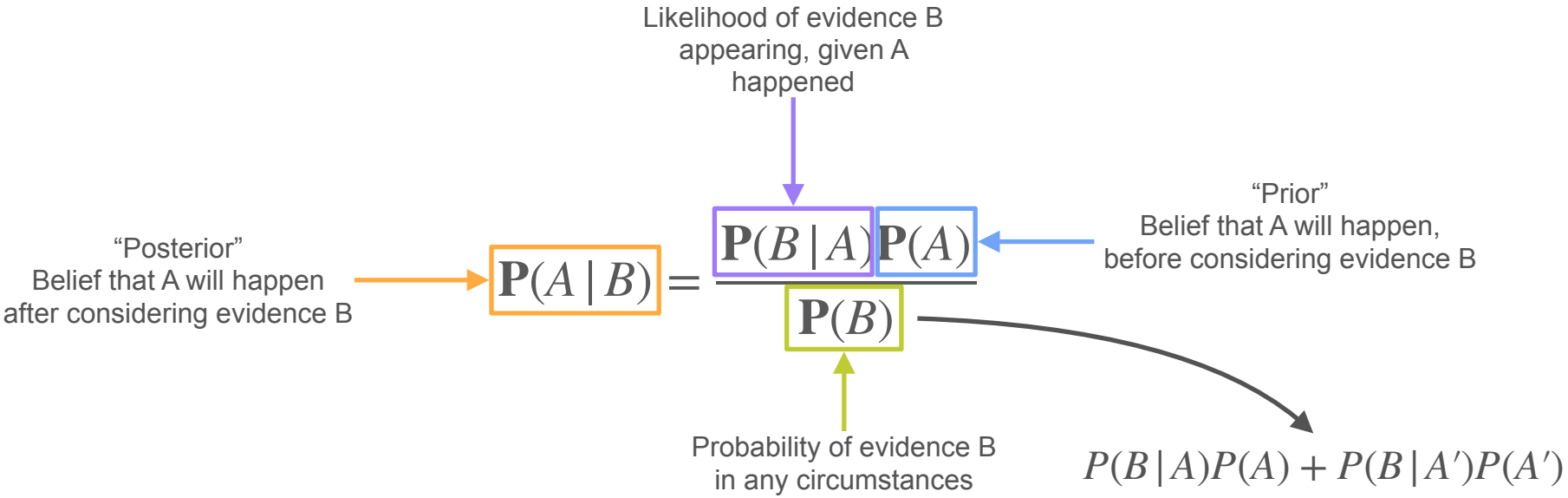


DeepLearning.AI

Point Estimation

Bayesian Statistics
Updating Priors

Bayesian Statistics

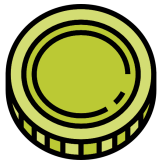


A: an event you are trying to predict (you are offered a job)

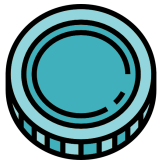
B: another event, or evidence, that helps refine your prediction (you were asked to a follow-up phone call)

Bayesian Statistics

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)}$$



"Fair"
$\mathbf{P}(H) = 0.5$



"Biased"
$\mathbf{P}(H) = 0.8$

"Mystery"
Either Fair or Biased

Event to Predict
$A \rightarrow Y$ takes some value
Y: odds of H for your coin

$$Y = \begin{cases} 0.5 & if \text{ coin is fair} \\ 0.8 & if \text{ coin is biased} \end{cases}$$

Evidence
$B \rightarrow X$ take some value
X: result of coin flip

$$X = \begin{cases} 0 & if\ T \\ 1 & if\ H \end{cases}$$

$\mathbf{P}(Y = 0.5) = 0.75$
$\mathbf{P}(Y = 0.80) = 0.25$
Priors

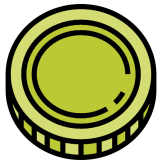
$x = 1$

$$p_{Y|X=1}(0.5) = \frac{p_{X|Y=0.5}(1)\ p_Y(0.5)}{p_X(1)} = \frac{0.5\ 0.75}{0.5\ 0.75 + 0.8\ 0.25}$$

$\mathbf{P}(Y = 0.5 \mid X = 1) = 0.652$
$\mathbf{P}(Y = 0.8 \mid X = 1) = 0.348$
Posterior

Bayesian Statistics

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)}$$



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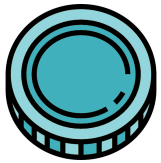

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Evidence

$B \rightarrow X$ take some value

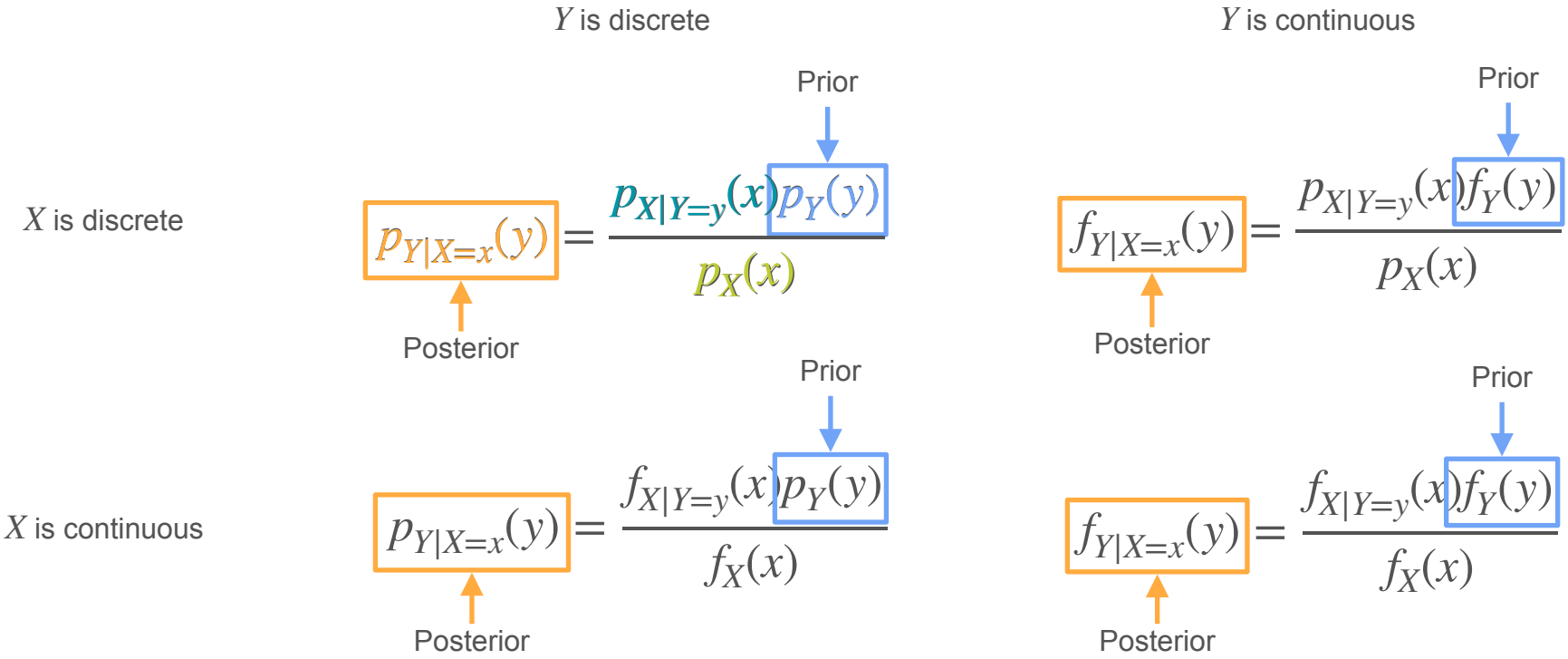
X: result of coin flip

$$X = \begin{cases} 0 & if \ T \\ 1 & if \ H \end{cases}$$

$x = 1$

$$p_{Y|X=x}(y) = \frac{p_{X|Y=y}(x) \quad p_Y(y)}{p_X(x)}$$

Bayesian Statistics



Bayesian Statistics

	Θ is discrete	Θ is continuous
X is discrete	$p_{\Theta\|X=x}(\theta) = \frac{p_{X\|\Theta=\theta}(x)p_{\Theta}(\theta)}{p_X(x)}$	$f_{\Theta\|X=x}(\theta) = \frac{p_{X\|\Theta=\theta}(x)f_{\Theta}(\theta)}{p_X(x)}$
X is continuous	$p_{\Theta\|X=x}(\theta) = \frac{f_{X\|\Theta=\theta}(x)p_{\Theta}(\theta)}{f_X(x)}$	$f_{\Theta\|X=x}(\theta) = \frac{f_{X\|\Theta=\theta}(x)f_{\Theta}(\theta)}{f_X(x)}$

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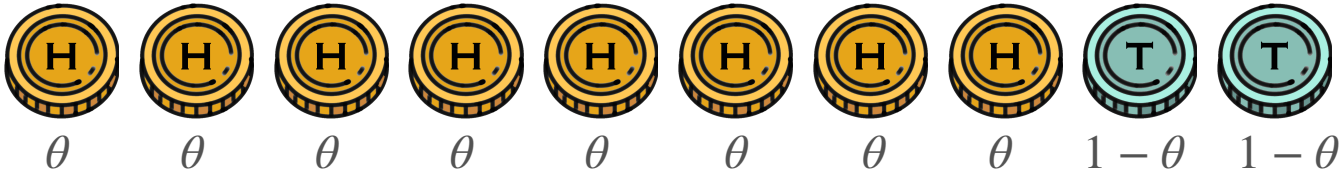
Point Estimation

Bayesian Statistics
Full Worked Example

Bayesian Statistics: Bernoulli Example



$$\Theta = \mathbf{P}(H)$$

$$\mathbf{X} = (X_1, X_2, \ldots, X_{10})$$

$X_i = 1$ if H, 0 if T

$X_i \mid \Theta = \theta \sim Bernoulli(\theta)$

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{p_{\mathbf{X}|\Theta=\theta}(\mathbf{x}) f_{\Theta}(\theta)}{p_{\mathbf{X}}(\mathbf{x})}$$

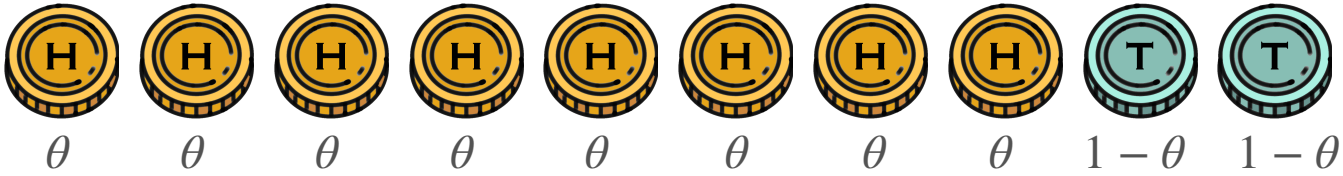
Θ is a continuous random variable

$\mathbf{X}$ is a discrete random variable

Use this version of Bayes' Theorem

$$p_{\mathbf{X}|\Theta=\theta}(1,1,\ldots,1,0,0) = \mathbf{P}(X_1 = 1, X_2 = 1, \ldots, X_8 = 1, X_9 = 0, X_{10} = 0 \mid \Theta = \theta) = \theta^8(1-\theta)^2$$

Bayesian Statistics: Bernoulli Example



$$X_i \mid \Theta = \theta \sim Bernoulli(\theta)$$

$$\Theta \sim Uniform(0,1)$$

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{p_{\mathbf{X}|\Theta=\theta}(\mathbf{x}) f_{\Theta}(\theta)}{p_{\mathbf{X}}(\mathbf{x})}$$

$$p_{\mathbf{X}|\Theta=\theta}(1,1,\ldots,1,0,0) = \theta^8(1-\theta)^2$$

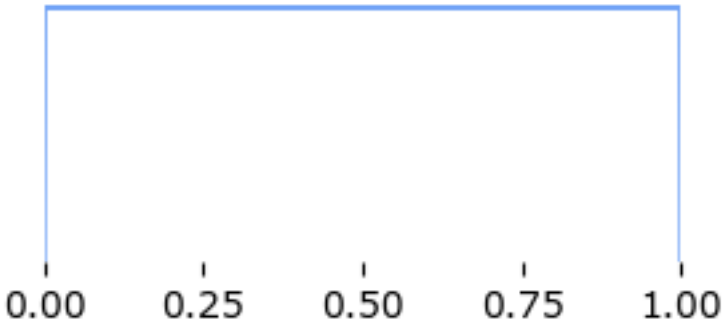
$$f_{\Theta}(\theta) = 1, \quad 0 \le \theta \le 1$$

$$f_{\Theta|X=x}(\theta) = \frac{\theta^8(1-\theta)^2}{\text{constant}} = \frac{1}{\text{constant}}\theta^8(1-\theta)^2$$

This is a Beta Distribution, it's possible to calculate this constant, but it's unnecessary

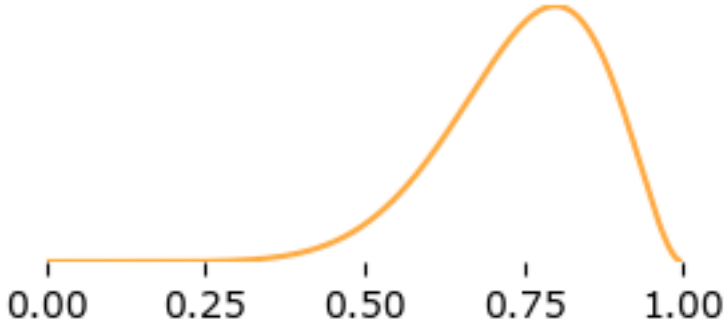
Bayesian Statistics: Bernoulli Example

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{\theta^8(1-\theta)^2\ 1}{\text{constant}}$$



Prior Beliefs

$$f_{\Theta}(\theta) = 1$$



Posterior Beliefs

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{1}{\text{constant}}\theta^8(1-\theta)^2$$

Bayesian Statistics: MAP Estimator

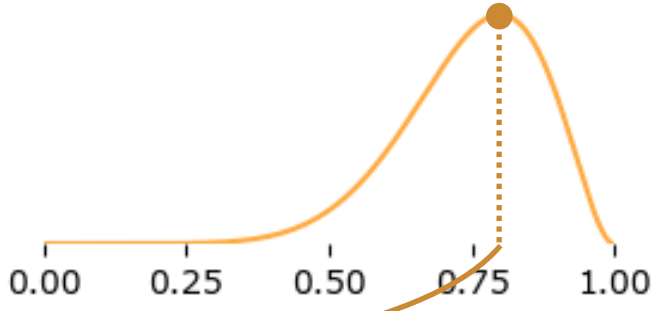
$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) \propto \theta^8(1-\theta)^2 \boxed{1}$$

Likelihood of the model

θ: a model of the coin where $P(H) = \theta$

$\mathbf{P}(\text{data} \mid \text{model})$

Maximizing this is all MLE does!

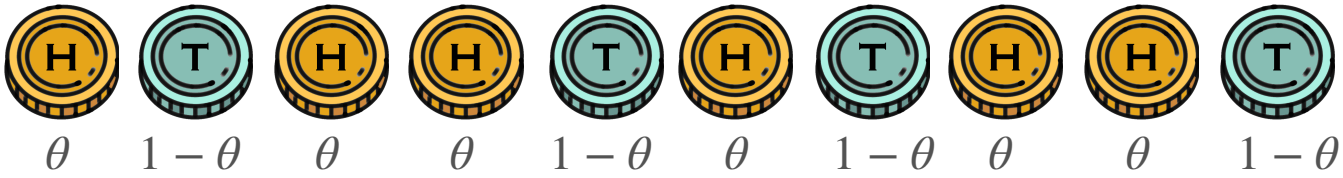


Maximum a Posteriori (MAP)

$$\hat{\theta} = \arg\max_{\theta} f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{8}{10}$$

Non informative prior: MAP = MLE

Bayesian Statistics: Bernoulli Example Continued



$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \theta^8(1-\theta)^2$

$p_{\mathbf{X}|\Theta=\theta}(1,1,0,\ldots,1,1,0) = \theta^6(1-\theta)^4$

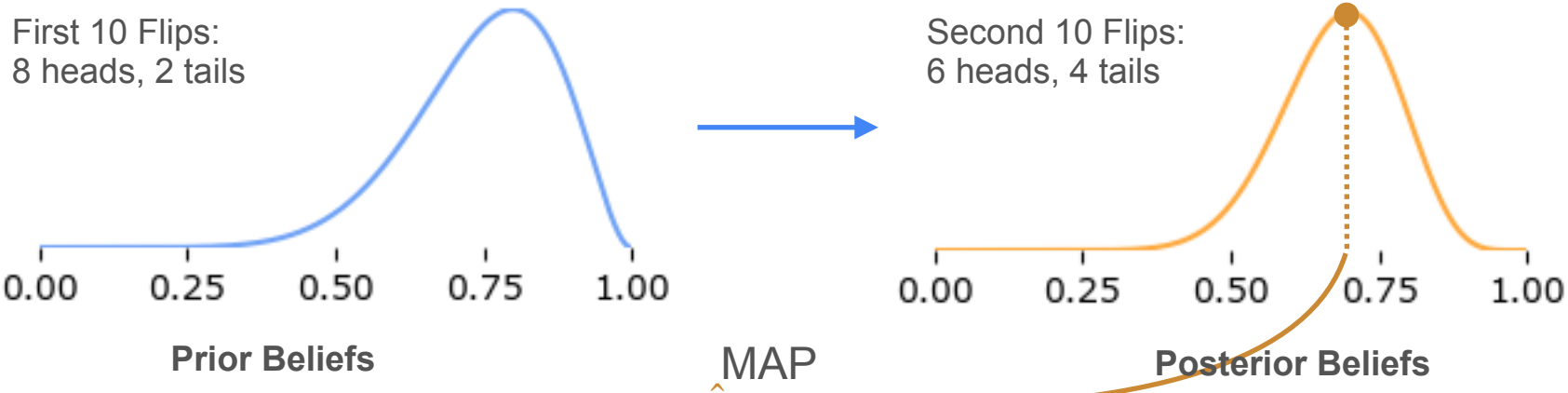
New prior!

Repeat process

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{\qquad\qquad}{\text{constant}} = \frac{1}{\text{constant}}\theta^{14}(1-\theta)^6$$

Bayesian Statistics: Bernoulli Example Continued

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{\theta^6(1-\theta)^4\,\theta^8(1-\theta)^2}{\text{constant}}$$



$$f_{\Theta}(\theta) = \frac{1}{\text{constant}}\theta^8(1-\theta)^2$$

MAP

$$\hat{\theta} = 0.7$$

14 out of 20 heads
Same result as Frequentist

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{1}{\text{constant}}\theta^{14}(1-\theta)^6$$

Bayesian Statistics: Final Summary

- Bayesians update prior beliefs
- MAP with uninformative priors is just MLE
- With enough data, MLE and MAP estimates usually converge
- Good for instances when you have limited data or strong prior beliefs
- Wrong priors, wrong conclusions

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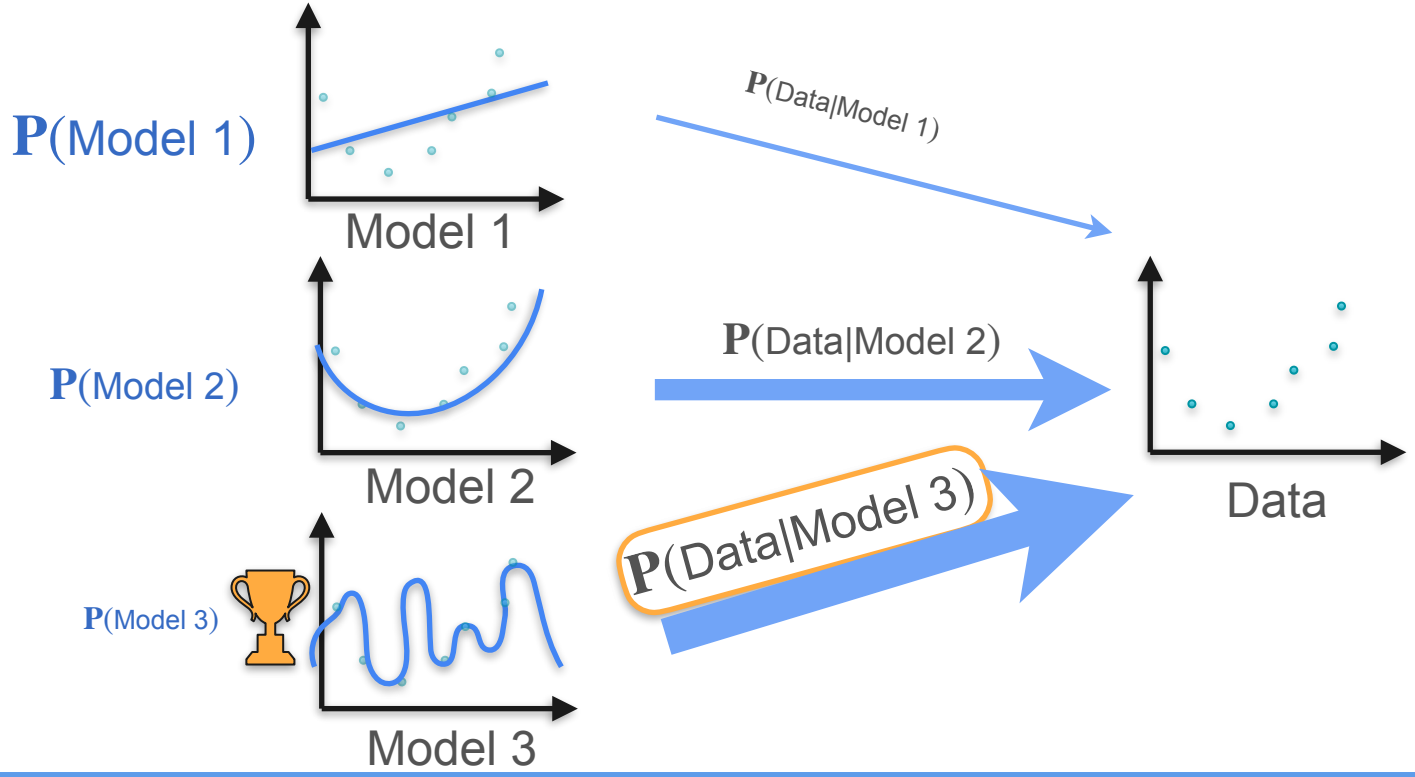
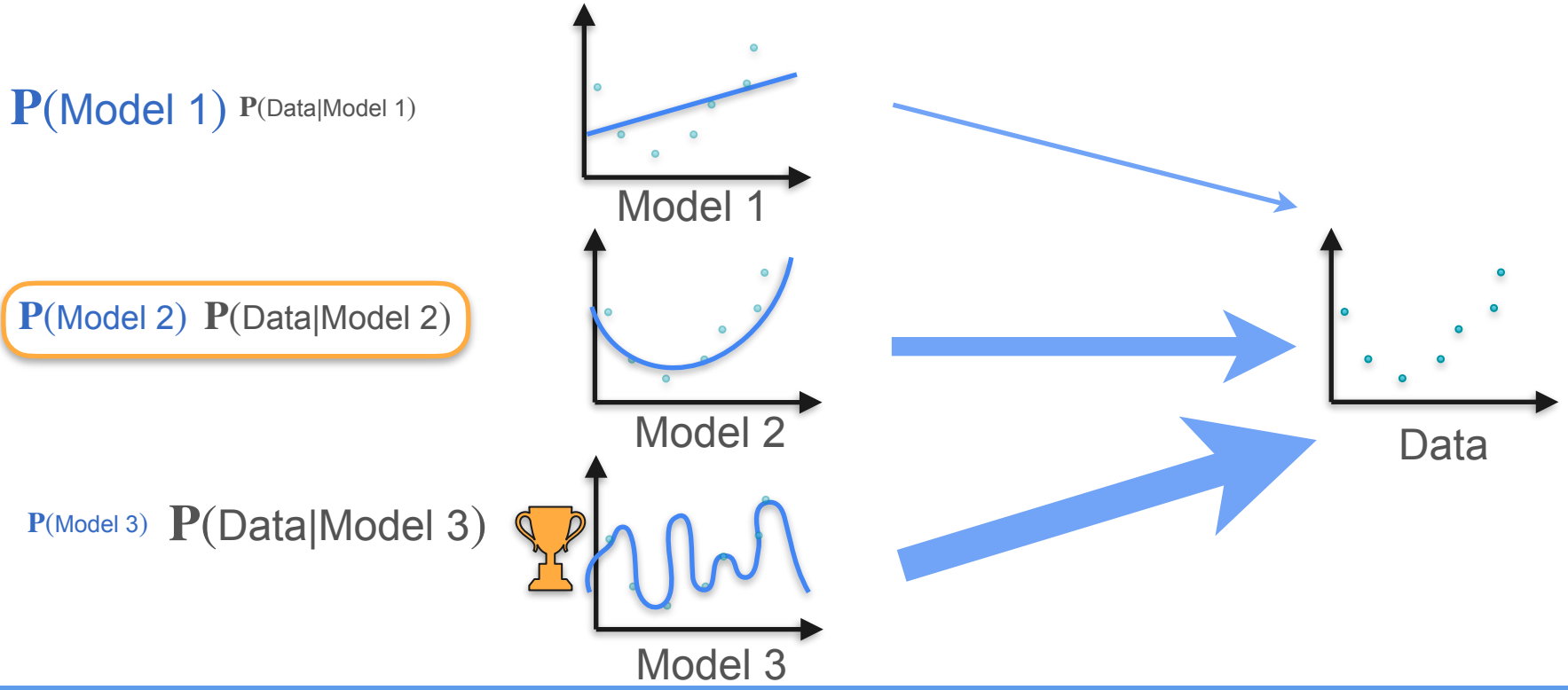
Point Estimation

Relationship between MAP, MLE, and Regularlization

Example: Polynomial Regression

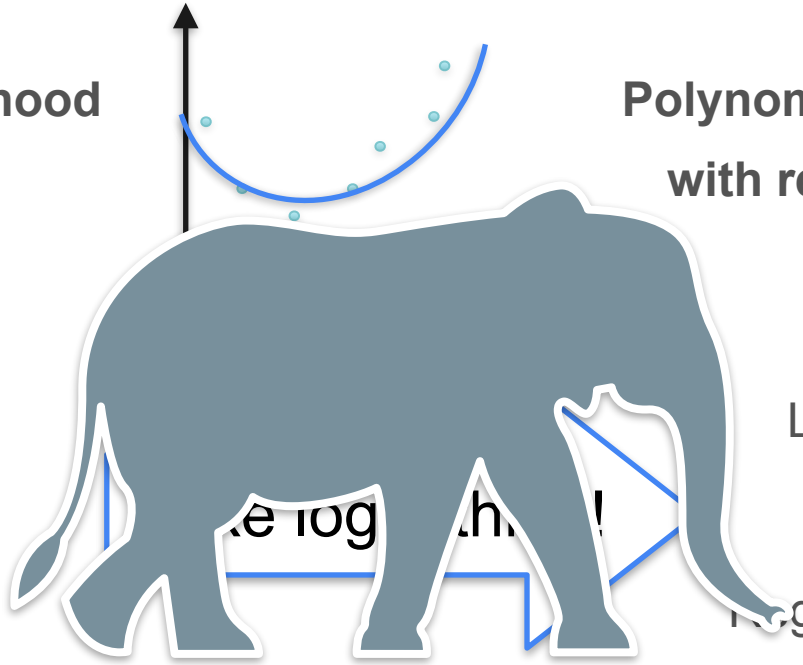


Example: Polynomial Regression



Maximum likelihood with Bayes

Polynomial regression with regularization



$\mathbf{P}(\text{Data}|\text{Model})$

$\cdot$

$\mathbf{P}(\text{Model})$

?

Log-loss

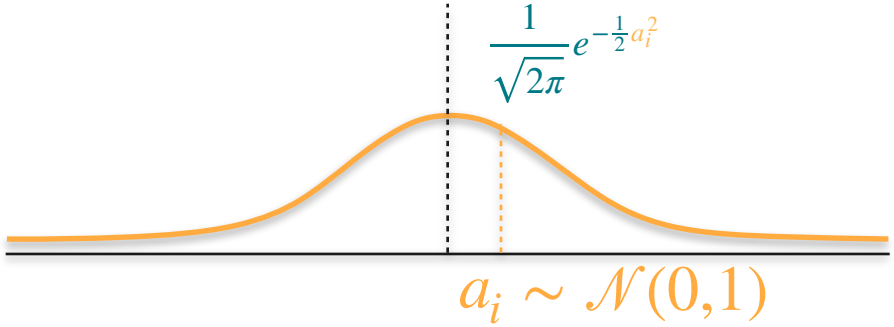
+

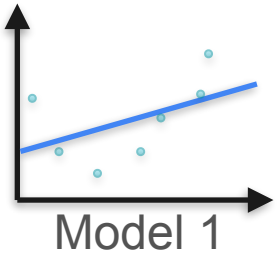
Regularization term

What Is the Probability of a Model?

$$\mathbf{P}(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

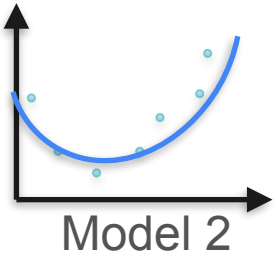
$a_1 x + b$





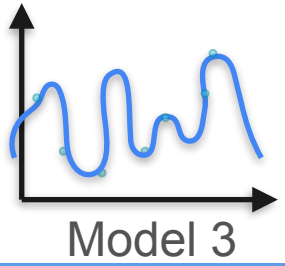
$$\mathbf{P}(\text{Model 2})$$

$$a_1 x^2 + a_2 x + b = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$



$$\mathbf{P}(\text{Model 3})$$

$$a_1 x^{10} + a_2 x^9 + \cdots + a_{10} x + b = \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_i^2}$$



Bayes and Regularization

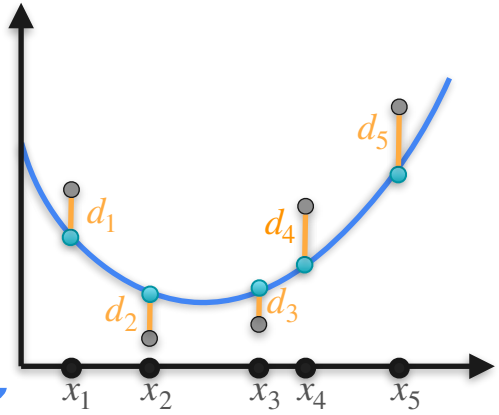
$\mathbf{P}(\text{Data}|\text{Model})$ $\mathbf{P}(\text{Model})$

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_5^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a_2^2}$$

log log log

Maximize

$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2) \quad + \quad -\frac{1}{2}(a_1^2 + a_2^2)$$



$a_1x^2 + a_2x + b$

Minimize $d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 \quad + \quad a_1^2 + a_2^2$

Regularization

New Loss

$\mathbf{P}$(Model 1)

$\mathbf{P}$(Data|Model 1)

Minimize $x_1^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$

$\mathbf{P}$(Model 2)

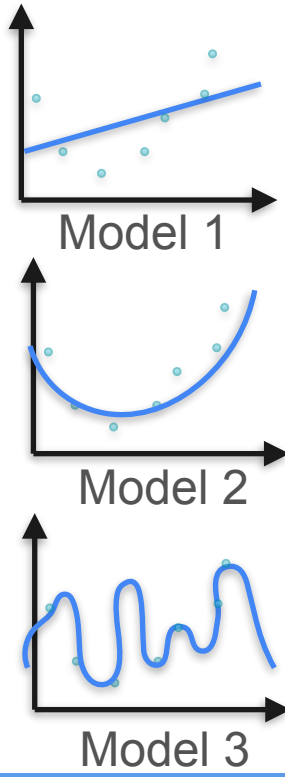
$\mathbf{P}$(Data|Model 2)

Minimize $x_1^2 + x_2^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$

$\mathbf{P}$(Model 3)

$\mathbf{P}$(Data|Model 3)

Minimize $x_1^2 + \cdots + x_{10}^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$



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Point Estimation

Conclusion