It is particularly instructive to compare the DTFT of an infinitely long sinusoid to the DTFT of a windowed version of the sinusoid.

Example

Let $x_p[n] = \cos\left(2\pi \frac{1}{8} n\right) \longleftrightarrow X_{p,\,DTFT}(F) = \left(\frac{1}{2}\delta(F+\frac{1}{8}) + \frac{1}{2}\delta(F-\frac{1}{8})\right) * \text{comb}(F)$

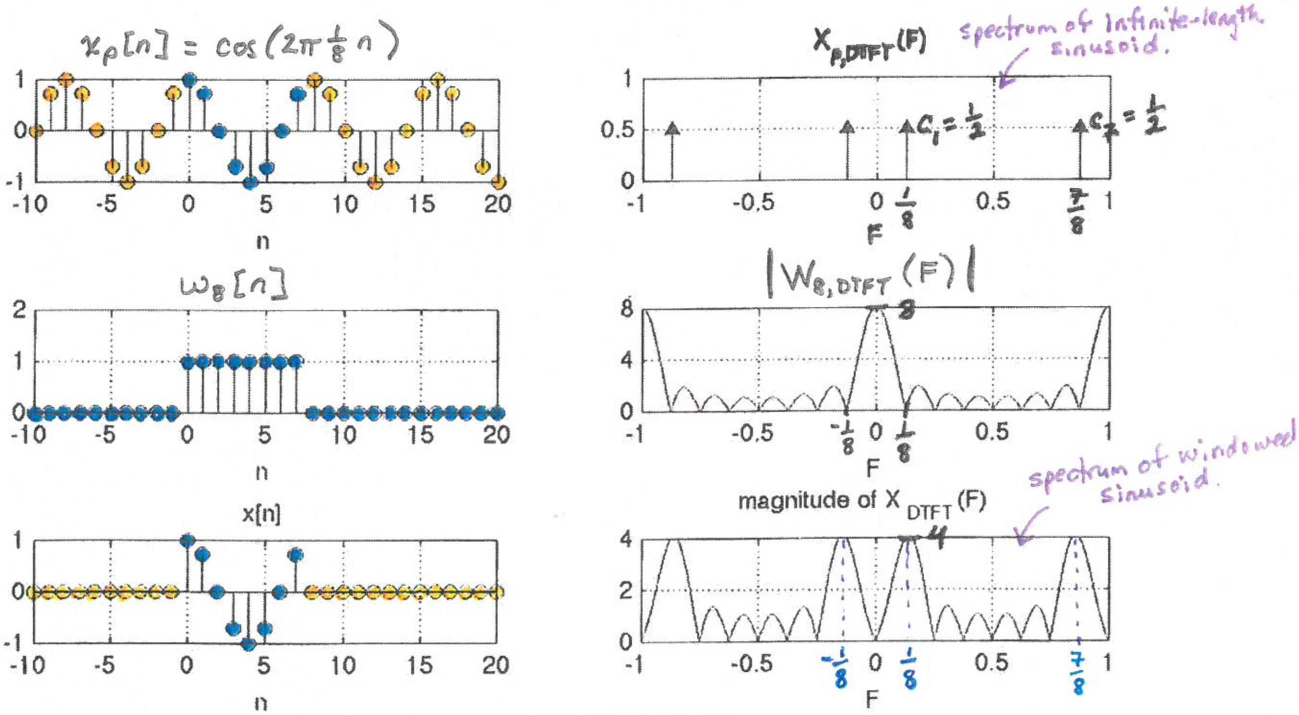
let $x[n] = x_p[n]\, w_8[n]$ where $w_8[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$

Then

$$X_{DTFT}(F) = \left[X_{p,\,DTFT}(F)\, \text{rect}(F)\right] * W_{8,\,DTFT}(F)$$

$$= \left(\frac{1}{2}\delta(F+\tfrac{1}{8}) + \frac{1}{2}\delta(F-\tfrac{1}{8})\right) * W_{8,\,DTFT}(F)$$

$$= \frac{1}{2} W_{8,DTFT}(F+\tfrac{1}{8}) + \frac{1}{2} W_{8,DTFT}(F-\tfrac{1}{8})$$



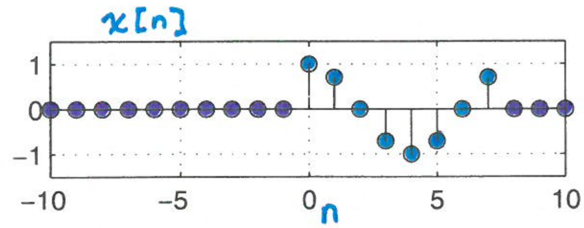
Note the spectral smearing that took place as a result of truncating/windowing the time-domain signal. The result is a loss of frequency resolution. The longer the time window, the better will be the frequency resolution.

Since $x_p[n] = x[n] * \text{comb}_8[n]$ ⟵ see illustrations in left-side column below.

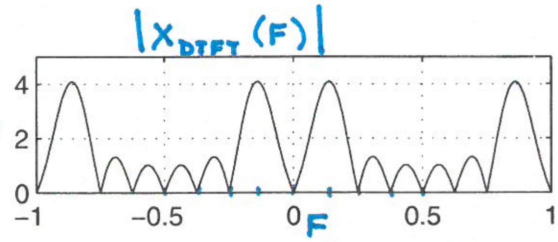
see illustrations in right-side column below.

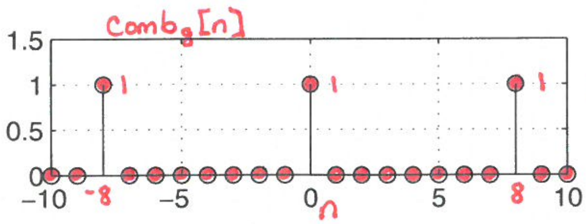
We should find: $X_{p,DTFT}(F) = X_{DTFT}(F)\,\text{comb}(8F) = X_{DTFT}(F)\sum_{k=-\infty}^{\infty} \frac{1}{8}\delta(F-\frac{k}{8})$

$$\Rightarrow X_{p,DTFT}(F) = \sum_{k=-\infty}^{\infty} \frac{1}{8} X_{DTFT}\left(\frac{k}{8}\right)\delta\left(F-\frac{k}{8}\right)$$

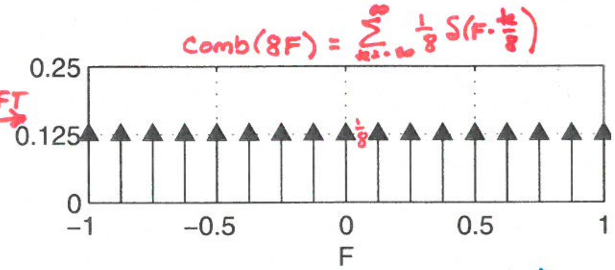


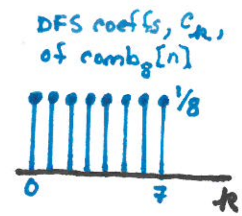
DTFT

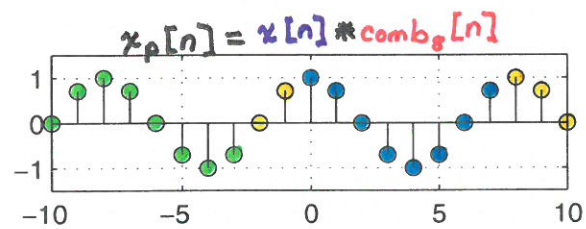




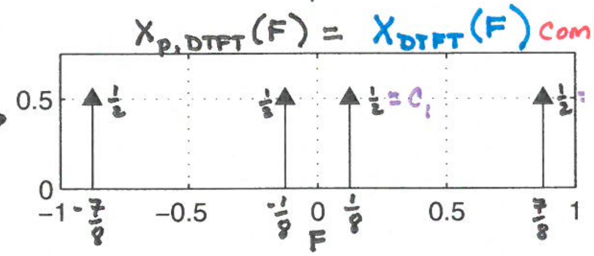
DTFT

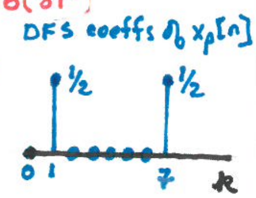




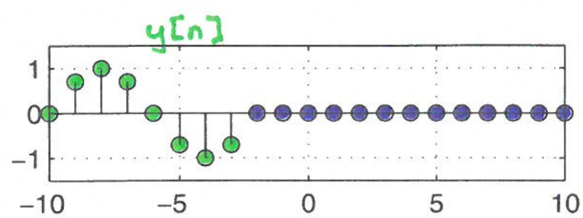


DTFT





Note that $y[n]$, shown below, has the same 8-periodic extension as $x[n]$.



Does $Y_{DTFT}(F) = X_{DTFT}(F)$?

Does $Y_{DTFT}\left(\frac{k}{8}\right) = X_{DTFT}\left(\frac{k}{8}\right)$?

Since: $y[n] * \text{comb}_8[n] = x[n] * \text{comb}_8[n]$

we know that: $Y_{DTFT}(F)\,\text{comb}(8F) = X_{DTFT}(F)\,\text{comb}(8F)$

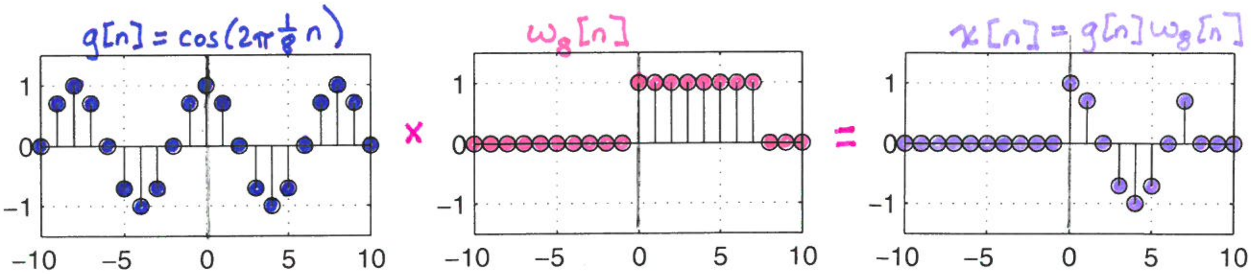
$$\Rightarrow \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{8} Y_{DTFT}\left(\frac{k}{8}\right)}_{c_k^{(y)}} \delta\left(F-\frac{k}{8}\right) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{8} X_{DTFT}\left(\frac{k}{8}\right)}_{c_k^{(x)}} \delta\left(F-\frac{k}{8}\right)$$

$$\Rightarrow Y_{DTFT}(k/8) = X_{DTFT}(k/8) \implies$$

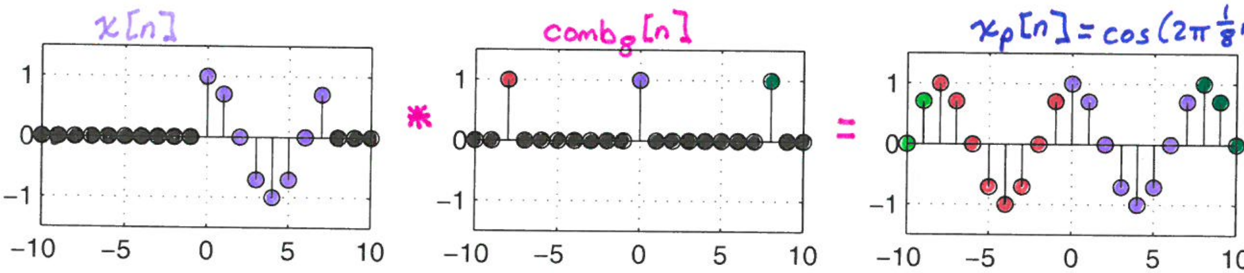
Important finding

> If two signals, $x[n]$ and $y[n]$, have the same N-periodic extension, their DTFTs will agree in value at $F = k/N$, $k = 0, \pm 1, \pm 2, \ldots$

Let $x[n]$ denote an energy signal whose nonzero values are contained within the interval $0 \le n \le N-1$. For example consider $x[n] = g[n]\, w_8[n]$ as shown below.



Let $x_p[n]$ denote the N-periodic extension of $x[n]$: $x_p[n] = x[n] * \mathrm{comb}_N[n]$



Then the DTFT of $x[n]$ is:

$$X_{DTFT}(F) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi F n} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi F n}$$

since $x[n] = 0$ for $n<0$ and for $n \ge N$

And the DFS coefficients of $x_p[n]$ are given by:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n]\, e^{-j2\pi \frac{k}{N} n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{k}{N} n} = \frac{1}{N} X_{DTFT}\left(\frac{k}{N}\right)$$

since $x_p[n] = x[n]$ for $0 \le n \le N-1$

The N-point DFT of a signal $x[n]$ is defined as:

$$X_{DFT,N}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{k}{N} n} = N c_k \quad , \qquad k = 0, .., N-1$$

The N-point iDFT is defined as:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{X_{DFT,N}[k]}_{N c_k}\, e^{j2\pi \frac{k}{N} n} = x_p[n] \, , \quad 0 \le n \le N-1$$

Recall: $x_p[n] = \sum_{k=0}^{N-1} c_k\, e^{j2\pi \frac{k}{N} n}$

Assuming that $x[n]$ is equal to zero outside of the interval $0 \le n \le N-1$, we may claim that:

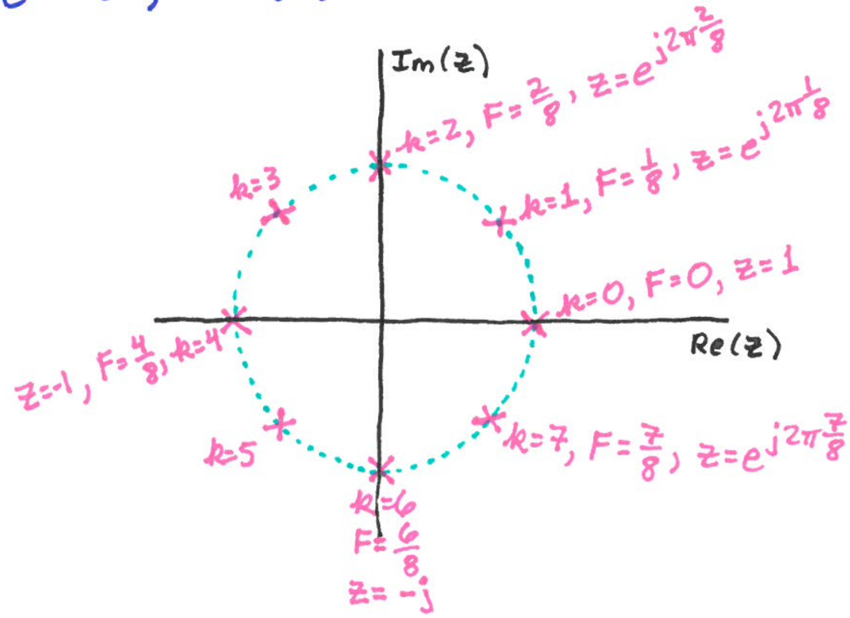
$$X_z(z) = \sum_{n=0}^{N-1} x[n]\, z^{-n}$$

$$X_{DTFT}(F) = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi F n} = \sum_{n=0}^{N-1} x[n] \left(e^{j2\pi F}\right)^{-n} = X_z\left(e^{j2\pi F}\right)$$

$$X_{DFT,N}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{k}{N} n} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{k}{N} n} = X_{DTFT}\left(\tfrac{k}{N}\right)$$

Thus: $\boxed{X_{DFT,N}[k] = X_{DTFT}\left(\tfrac{k}{N}\right) = X_z\left(e^{j2\pi \frac{k}{N} n}\right)}$

Example: Assuming $x[n] = 0$ for $n < 0$ and $n \ge 8$, the 8-point DFT of $x[n]$ consists of a set of $N = 8$ values (indexed by $k = 0, 1, \ldots, 7$) which may be viewed as samples of $X_{DTFT}(F)$ at $F = \frac{k}{8}$, $k = 0, \ldots, 7$ or as samples of $X_z(z)$ at $z = e^{j2\pi \frac{k}{8}}$, $k = 0, \ldots, 7$.



Example : Find the 4-point DFT of the sequence

$$x[n] = \{ 0 \ \underset{\uparrow}{1} \ 1 \ 1 \ 0 \}$$

Solution

$$X_z(z) = 1 + z^{-1} + z^{-2}$$

$$X_{DTFT}(F) = X_z(e^{j2\pi F}) = 1 + e^{-j2\pi F} + e^{-j2\pi 2F}$$

$$X_{DFT,4}[k] = X_{DTFT}\left(\frac{k}{4}\right) = 1 + e^{-j2\pi\frac{k}{4}} + e^{-j2\pi 2\frac{k}{4}}$$

$$= 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

k	1	$e^{-j\frac{\pi}{2}k}$	$e^{-j\pi k}$	$X_{DFT,4}[k]$
0	1	1	1	3
1	1	$-j$	-1	$-j$
2	1	-1	1	1
3	1	j	-1	j

In matlab, the 4-point DFT of $x[n]$ may be computed as:

```
> x = [1 1 1];
> X = fft(x, 4)
X = [3  -j  1  +j]
```