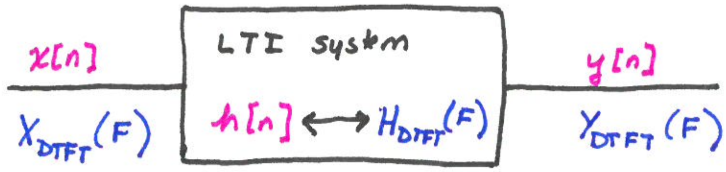
Review of discrete-time filtering:



$$\boxed{y[n] = x[n] * h[n]}$$

$$\boxed{Y_{DTFT}(F) = X_{DTFT}(F)\,H_{DTFT}(F)}$$

Given $x[n]$ and $h[n]$, we may find the output, $y[n]$, of the system above using either of the approaches below.

Approach 1:

Evaluate the convolution sum: $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

Approach 2:

a) find $X_{DTFT}(F)$

b) find $H_{DTFT}(F)$

c) find $Y_{DTFT}(F) = X_{DTFT}(F)\,H_{DTFT}(F)$

d) find $y[n]$ as the inverse DTFT of $Y_{DTFT}(F)$

In practice, approach 2 is difficult to use (since the DTFT has a continuous domain); however it is very common, and in most cases more computationally efficient, to use a modified version of approach 2 in which the DTFT's are replaced by N-point DFTs. As will be shown, it is important to choose N appropriately when using this approach.

We know that the iDTFT of a product of two DTFT's is the convolution of the two individual iDTFTs

For example : if $Y_{DTFT}(F) = X_{DTFT}(F)\, H_{DTFT}(F)$

where: $X_{DTFT}(F)$ is the DTFT of $x[n]$

$H_{DTFT}(F)$ is the DTFT of $h[n]$

and if $y[n]$ denotes the iDTFT of $Y_{DTFT}(F)$

then we know that $y[n] = x[n] * h[n]$

What can we say about the N-point iDFT of a product of two N-point DFT's?

In particular : if $Z[k] = X_{DFT,N}[k]\, H_{DFT,N}[k], \quad k=0,\dots,N-1$

where: $x[n] \xleftrightarrow{DFT,N} X_{DFT,N}[k]$

$h[n] \xleftrightarrow{DFT,N} H_{DFT,N}[k]$

and if $z[n]$ denotes the N-pt. iDFT of $Z[k]$

then what can we say about the time-domain relationship between $z[n]$, $x[n]$, and $h[n]$?

Under what circumstances may we claim that:

$$z[n] \overset{??}{=} x[n] * h[n]$$

Example to illustrate the iDFT of the product of two DFT's

$$x[n] = \{\underset{\uparrow}{1}\ 1\ 1\ 0\} \xrightarrow{\text{4-pt. DFT}} X_{DFT,4}[k] = \{3, -j, 1, j\}$$

$$h[n] = \{\underset{\uparrow}{1}\ 1\ 1\ 0\} \xrightarrow{\text{4-pt. DFT}} H_{DFT,4}[k] = \{3, -j, 1, j\}$$

$$\Downarrow$$

$$Z[k] = X_{DFT,4}[k]\, H_{DFT,4}[k]$$

$$z[n] = \{?\ ?\ ?\ ?\ \} \xleftarrow{\text{4-pt. IDFT}} Z[k] = \{9, -1, 1, -1\}$$

How do we find $z[n]$ in the time-domain?

Our first approach to answering this question will be to use the relation between the N-pt. DFT of a signal and its DTFT in conjunction with what we know about the iDTFT of a product of two DTFTs.

We know that
$$Z[k] = X_{DFT,4}[k]\, H_{DFT,4}[k]$$
$$= X_{DTFT}\left(\tfrac{k}{4}\right) H_{DTFT}\left(\tfrac{k}{4}\right)$$
$$= Y_{DTFT}\left(\tfrac{k}{4}\right) \quad \text{where } Y_{DTFT}(F) = X_{DTFT}(F)\, H_{DTFT}(F)$$

hence $Y_{DTFT}(F)$ is the DTFT of
$$y[n] = x[n] * h[n]$$

This does not allow us to conclude that $z[n]$ is equal to $y[n]$; it only says that $z[n]$ is a sequence whose DTFT has the same values as the DTFT of $y[n]$ at $F = \frac{k}{4}$.
$$Z_{DTFT}\left(\tfrac{k}{4}\right) = Y_{DTFT}\left(\tfrac{k}{4}\right)$$

Thus we know that $z[n]$ and $y[n]$ have the same 4-periodic extensions. Thus
$$z[n] = \begin{cases} y[n] * \mathrm{comb}_4[n], & n = 0, 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Solution to example on previous page:

$$\underbrace{\{\underset{\uparrow}{1}, 1, 1\}}_{x[n]} * \underbrace{\{\underset{\uparrow}{1}, 1, 1\}}_{h[n]} = \underbrace{\{\underset{\uparrow}{1} \quad 2 \quad 3 \quad 2 \quad 1\}}_{y[n]}$$

$$\begin{array}{rrrrr}
 & & 1. & 1 & 1 \\
 & & 1. & 1 & 1 \\
\hline
 & & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
\hline
1. & 2 & 3 & 2 & 1
\end{array}$$

Compute the 4-periodic extension of y[n]:

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
$y[n]$	0	0	0	0	1	2	3	2	1	0	0	0	0	0
$y[n-4]$	0	0	0	0	0	0	0	0	1	2	3	2	1	0
$y[n+4]$	1	2	3	2	1	0	0	0	0	0	0	0	0	0
			3	2	2	2	3	2	2	2	3	2		

Thus will find $z[n] = \{\underset{\uparrow}{2} \quad 2 \quad 3 \quad 2\}$

Since $z[n]$ has the same 4-periodic extension as $y[n]$, their DTFT's will agree in value at $F = \frac{k}{4}$

$$Z_{DTFT}\left(\frac{k}{4}\right) = Y_{DTFT}\left(\frac{k}{4}\right)$$

Can also derive the mathematical relationship between $z[n]$, $x[n]$, and $h[n]$.

Let $x[n]$ and $h[n]$ be finite-length sequences whose nonzero values are confined to the interval $0 \le n \le N-1$. Furthermore, let $X[k]$ and $H[k]$ denote the N-pt. DFT's of these sequences, and let $Z[k]$ denote the product of $X[k]$ and $H[k]$:

$$x[n] \xleftrightarrow{\text{N-pt. DFT}} X[k]$$

$$h[n] \xleftrightarrow{\text{N-pt. DFT}} H[k]$$

$$z[n] \xleftrightarrow{\text{N-pt. DFT}} Z[k] = X[k]\,H[k]$$

Find $z[n]$, the N-pt. IDFT of $Z[k]$

$$z[n] = \frac{1}{N}\sum_{k=0}^{N-1} Z[k]\, e^{j2\pi\frac{k}{N}n}$$ by defn. of N-pt. IDFT

$$= \frac{1}{N}\sum_{k=0}^{N-1} \overbrace{X[k]\,H[k]}^{Z[k]}\, e^{j2\pi\frac{k}{N}n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \overbrace{\sum_{m=0}^{N-1} x[m]\, e^{-j2\pi\frac{k}{N}m}}^{X[k]}\, H[k]\, e^{j2\pi\frac{k}{N}n}$$

$$= \sum_{m=0}^{N-1} x[m]\, \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} H[k]\, e^{j2\pi\frac{k}{N}(n-m)}}_{h_N[n-m]}$$

$$h[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, e^{j2\pi\frac{k}{N}n}, & n = 0, \ldots, N-1 \\ 0, & n<0 \text{ or } n>N-1 \end{cases}$$

$$= \sum_{m=0}^{N-1} x[m]\, h_N[n-m] \quad (*)$$

where $h_N[n]$ denotes the N-periodic extension of $h[n]$.

$$= x[n] \; \textcircled{N} \; h[n]$$

$\textcircled{N}$ denotes the N-pt. circular convolution

Reversing the roles of $x[n]$ and $h[n]$ in derivation above, it is easily shown that:

$$\boxed{z[n] = x[n] \; \textcircled{N} \; h[n] = \sum_{m=0}^{N-1} x[m]\, h_N[n-m] = \sum_{m=0}^{N-1} h[m]\, x_N[n-m]}$$

In general, if you multiply two N-pt. DFT's, the IDFT of the product will be given by the N-point circular convolution of the individual IDFT's.

if $Z[k] = X[k]\, H[k]$, $k = 0, .., N-1$

where $x[n] \xleftrightarrow{\text{N-pt. DFT}} X[k]$

$h[n] \xleftrightarrow{\text{N-pt. DFT}} H[k]$

N-pt. circular convolution

then $z[n] = x[n] \, Ⓝ \, h[n]$

$= \sum_{k=0}^{N-1} x_N[k]\, h_N[n-k]$

regular convolution

$= x[n] * h_N[n]$

$= x_N[n] * h[n]$

where:

$z[n]$ = N-pt. IDFT of $Z[k]$

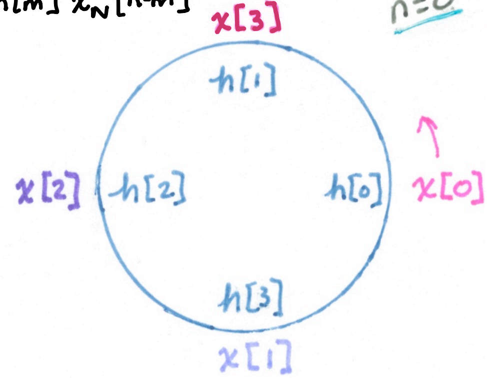
$h_N[n]$ = N-periodic extensi of $h[n]$

$x_N[n]$ = N-periodic extensi of $x[n]$

As shown previously, $z[n]$ can also be found as the N-periodic extension of $y[n]$ where $y[n] = x[n] * h[n]$.

Illustration of the 4-point Circular Convolution of $x[n]$ and $h[n]$

$$z[n] = \sum_{m=0}^{N-1} h[m]\, x_N[n-m]$$



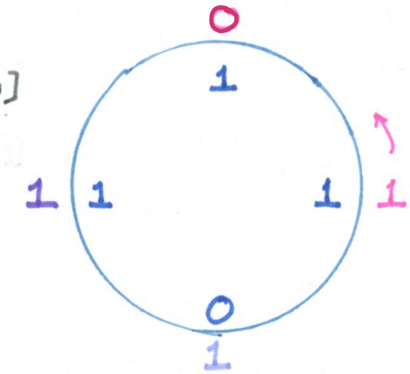
$n=0$

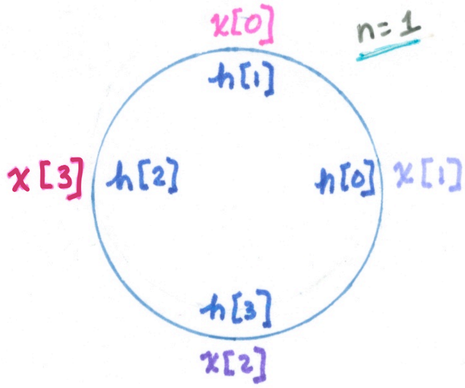
$z[0] = h[0]\,x_N[0] + h[1]\,x_N[-1] + h[2]\,x_N[-2] + h[3]\,x_N[-3]$

$z[0] = h[0]\,x[0] + h[1]\,x[3] + h[2]\,x[2] + h[3]\,x[1]$

$\Rightarrow z[0] = 1+0+1+0$

$\boxed{z[0] = 2}$





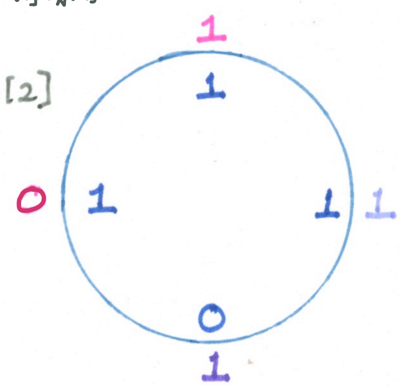
$n=1$

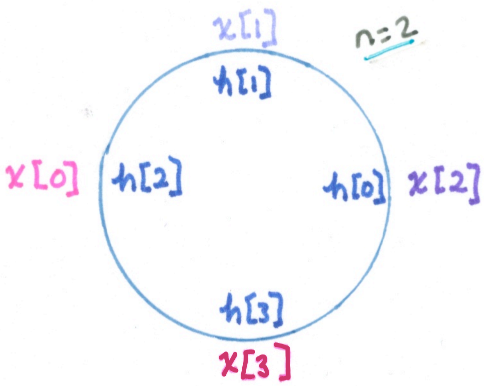
$z[1] = h[0]\,x_N[1] + h[1]\,x_N[0] + h[2]\,x_N[-1] + h[3]\,x_N[-2]$

$z[1] = h[0]\,x[1] + h[1]\,x[0] + h[2]\,x[3] + h[3]\,x[2]$

$\Rightarrow z[1] = 1+1+0+0$

$\boxed{z[1] = 2}$





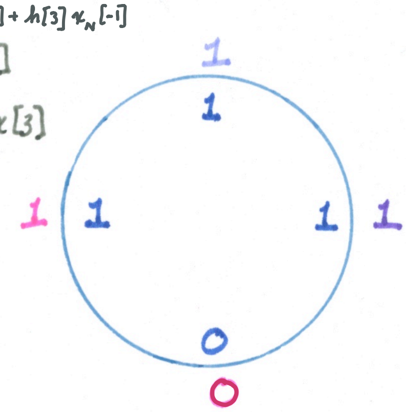
$n=2$

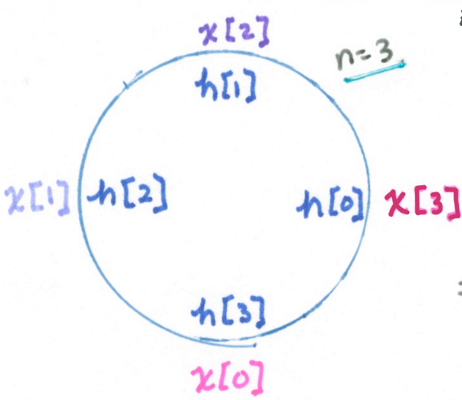
$z[2] = h[0]\,x_N[2] + h[1]\,x_N[1] + h[2]\,x_N[0] + h[3]\,x_N[-1]$

$z[2] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0] + h[3]\,x[3]$

$\Rightarrow z[2] = 1+1+1+0$

$\boxed{z[2] = 3}$





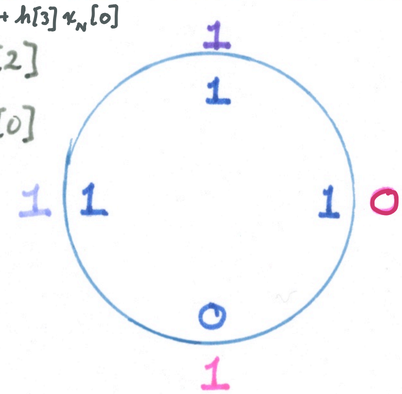
$n=3$

$z[3] = h[0]\,x_N[3] + h[1]\,x_N[2] + h[2]\,x_N[1] + h[3]\,x_N[0]$

$z[3] = h[0]\,x[3] + h[1]\,x[2] + h[2]\,x[1] + h[3]\,x[0]$

$\Rightarrow z[3] = 0+1+1+0$

$\boxed{z[3] = 2}$



$h[n] = \{\underset{\uparrow}{1}\ \ 1\ \ 1\ \ 0\}$

$x[n] = \{\underset{\uparrow}{1}\ \ 1\ \ 1\ \ 0\}$

$z[n] = x[n]\ ④\ h[n] \Rightarrow z[n] = \{\underset{\uparrow}{2}\ \ 2\ \ 3\ \ 2\}$

Given $h[n] = \{\underset{\uparrow}{1} \; 1 \; 1 \; 0\}$ and $x[n] = \{\underset{\uparrow}{1} \; 1 \; 1 \; 0\}$

we can also evaluate $x[n] \; ④ \; h[n]$ using the strip-of-paper method to find:

$$z[n] = h[n] * x_N[n] = \sum_{m=-\infty}^{\infty} h[m]\, x_N[n-m] = \sum_{m=-\infty}^{\infty} h[m]\, g_n[m]$$

$$x[m] = \{ 0 \;\; 0 \;\; 0 \;\; 0 \;\; \underset{\uparrow}{1} \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \}$$

$$x_N[m] = \{ \ldots 1 \;\; 1 \;\; 1 \;\; 0 \;\; \underset{\uparrow}{1} \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \ldots \}$$

Let $g_n[m] = x_N[n-m]$

$$g_n[m] = \{ \ldots 1 \;\; 0 \;\; 1 \;\; 1 \;\; \underset{\underset{m=n}{\uparrow}}{1} \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \ldots \}$$

$n=0$

$$g_0[m] = \{ \ldots 1 \;\; 0 \;\; 1 \;\; 1 \;\; \underset{\underset{m=0}{\uparrow}}{1} \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \ldots \}$$

$n=1$

$$g_1[m] = \{ \ldots 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; \underset{\underset{m=1}{\uparrow}}{1} \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \ldots \}$$

$n=2$

$$g_2[m] = \{ \ldots 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; \underset{\underset{m=2}{\uparrow}}{1} \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \ldots \}$$

$n=3$

$$g_3[m] = \{ \ldots 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; \underset{\underset{m=3}{\uparrow}}{1} \;\; 0 \;\; 1 \;\; 1 \;\; 1 \ldots \}$$

$$h[m] = \{ \ldots 0 \;\; 0 \;\; 0 \;\; 0 \;\; \underset{\uparrow}{1} \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \ldots \}$$

$$z[n] = \sum_{m=-\infty}^{\infty} h[m]\, g_n[m] = g_n[0] + g_n[1] + g_n[2] \Rightarrow \begin{cases} z[0] = \\ z[1] = \\ z[2] = \\ z[3] = \end{cases}$$