Q1a) $x[n]=(1/2)^n u[n] \longrightarrow y_p[n]=k(\frac{1}{2})^n$ based on handout

but because $\lambda_1=\lambda_2=\frac{1}{2}$ then $y_p[n]=kn^2(\frac{1}{2})^n$

$y_p[n]-y_p[n-1]+\frac{1}{4}y_p[n-2]=x[n]-x[n-1]$ $\quad n\geq 2$

$n=2 \Rightarrow y_p[2]-y_p[1]+\frac{1}{4}y_p[0]=x[2]-x[1]$

$y_p[2]=k(2)^2(\frac{1}{2})^2=k$, $\quad y_p[1]=k(\frac{1}{2})^1=\frac{k}{2}$, $\quad y_p[0]=0$

$x[2]=(\frac{1}{2})^2=\frac{1}{4}$, $\quad x[1]=(\frac{1}{2})^1=\frac{1}{2}$

$\Rightarrow k-\frac{k}{2}=\frac{1}{4}-\frac{1}{2} \Rightarrow k(\frac{1}{2})=\frac{-1}{4} \Rightarrow k=\frac{-1}{2} \Rightarrow \boxed{y_p[n]=\frac{-n^2}{2}(\frac{1}{2})^n \quad n\geq 0}$

Q1-b) $y_{zs}[n] = \frac{-n^2}{2}\left(\frac{1}{2}\right)^n + C_1^{(zs)}\left(\frac{1}{2}\right)^n + C_2^{(zs)} n\left(\frac{1}{2}\right)^n$

$\Rightarrow n=0 \Rightarrow y_{zs}[0] = C_1^{(zs)} *$, $n=1 \Rightarrow y_{zs}[1] = \frac{-1}{4} + \frac{C_1^{(zs)}}{2} + \frac{C_2^{(zs)}}{2}$ $**$

$y_{zs}[n] = y_{zs}[n-1] - \frac{1}{4} y_{zs}[n-2] + x[n] - \cancel{2}\, x[n-1]$

$x[n] = \left(\frac{1}{2}\right)^n u[n]$ $\qquad y_{zs}[-1] = y_{zs}[-2] = 0$

$y_{zs}[0] = \cancel{y_{zs}[-1]}^{0} - \frac{1}{4}\cancel{y_{zs}[-2]}^{0} + x[0] - \cancel{2}\, x[-1] = 1$ $***$

$y_{zs}[1] = y_{zs}[0] - \frac{1}{4} y_{zs}[-1] + x[1] - \cancel{2}\, x[0] = 1 + \frac{1}{2} - \cancel{2}1 = \frac{1}{2}$ $****$

$*, *** \Rightarrow C_1^{(zs)} = 1$ $\qquad **, **** \Rightarrow \frac{C_1^{(zs)}}{2} + \frac{C_2^{(zs)}}{2} - \frac{1}{4} = +\frac{1}{2}$

$\Rightarrow C_2^{(zs)} + 1 - \frac{1}{2} = +1 \Rightarrow C_2^{(zs)} = \cancel{\frac{3}{2}}\ \frac{1}{2}$

$$\boxed{\Rightarrow y_{zs}[n] = \frac{-n^2}{2}\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + \frac{\cancel{3}n}{2}\left(\frac{1}{2}\right)^n}$$

C) $y_{zi}[n] = C_1^{(zi)}\left(\frac{1}{2}\right)^n + C_2^{(zi)}\, n\left(\frac{1}{2}\right)^n \Rightarrow \begin{cases} n=0 \Rightarrow y_{zi}[0] = C_1^{(zi)} & (1) \\ n=1 \Rightarrow y_{zi}[1] = \frac{C_1^{(zi)}}{2} + \frac{C_2^{(zi)}}{2} & (2) \end{cases}$

$y_{zi}[-1] = 1 \qquad y_{zi}[-2] = 0$

$y_{zi}[n] = y_{zi}[n-1] - \frac{1}{4} y_{zi}[n-2] \Rightarrow \begin{cases} n=0 \Rightarrow y_{zi}[0] = y[-1] - \frac{1}{4} y[-2] = 1 & (3) \\ n=1 \Rightarrow y_{zi}[1] = y[0] - \frac{1}{4} y[-1] = \frac{3}{4} & (4) \end{cases}$

$\xrightarrow{(1),(3)} C_1^{(zi)} = 1 \qquad \xrightarrow{(2),(4)} \frac{C_1^{(zi)} + C_2^{(zi)}}{2} = \frac{3}{4} \Rightarrow C_2^{(zi)} = \frac{1}{2}$

$\Rightarrow \boxed{y_{zi}[n] = \left(\frac{1}{2}\right)^n + \frac{n}{2}\left(\frac{1}{2}\right)^n}$

d) $\left.\begin{array}{l} y_{zi}[0] = y[-1] - \frac{1}{4} y[-2] \\ y_{zi}[1] = y[0] - \frac{1}{4} y[-1] \end{array}\right\} \Rightarrow \begin{cases} y_{zi}[0] = -1/4 = C_1^{(zi)}, \quad C_2^{(zi)} = -\frac{1}{4} \\ y_{zi}[1] = -1/4 = \frac{C_1^{(zi)} + C_2^{(zi)}}{2} \end{cases}$

$y[-1] = 0 \qquad y[-2] = 1 \Rightarrow \boxed{y_{zi}[n] = \frac{-1}{4}\left(\frac{1}{2}\right)^n - \frac{n}{4}\left(\frac{1}{2}\right)^n}$

e) based on part (c) and (d), it can conclude that

$y_{zi}^{(S)}[n] = y_{zi}^{\text{summary}}[n] = \underbrace{2}_{y^S_{zi}[-1]=2}\, y_{zi}^{(c)}[n] + \underbrace{(-4)}_{y^S_{zi}[-2]=(-4)}\, y_{zi}^{(d)}[n]$

S mean summary

or in other words, we can find zero input response with linear combiniation of zero-input response in part c and d

for example, if $y[-1] = 2 \quad y[-2] = -4$ then

$y_{zi}[n] = y[-1]\, y_{zi}^{(c)}[n] + y[-2]\, y_{zi}^{(d)}[n] = 2\left[\left(\frac{1}{2}\right)^n + \left(\frac{n}{2}\right)\left(\frac{1}{2}\right)^n\right] + (-4)\left[\frac{-1}{4}\left(\frac{1}{2}\right)^n - \frac{n}{4}\left(\frac{1}{2}\right)^n\right] = 2\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n =$

$= 3\left(\frac{1}{2}\right)^n + 2n\left(\frac{1}{2}\right)^n$

f) $y[-1]=1 \quad y[-2]=1 \quad x[n]=(\frac{1}{2})^n u[n]$

the total solution $y[n]$, $n>0$ can be find as the sum of zi and zs response.

$$\Rightarrow y_{zi}[n] = y[-1]\, y_{zi}^{(c)}[n] + y[-2]\, y_{zi}^{(d)}[n]$$

$$y_{zi}[n] = \frac{3}{4}(\frac{1}{2})^n + \frac{n}{4}(\frac{1}{2})^n \qquad y_{zs} = \text{Part } \underline{b}$$

$$y[n] = \frac{3}{4}(\frac{1}{2})^n + \frac{n}{4}(\frac{1}{2})^n - \frac{n^2}{2}(\frac{1}{2})^n + (\frac{1}{2})^n + \frac{n}{2}(\frac{1}{2})^n$$

$$y[n] = \frac{7}{4}(\frac{1}{2})^n + \boxed{\frac{3}{4} n (\frac{1}{2})^n - \frac{n^2}{2}(\frac{1}{2})^n}$$

$y[0] = \frac{7}{4}$ ✓ $\quad y[1] = \frac{7}{8} + \frac{3}{8} - \frac{1}{4} = \underline{1}$ ✓ $\quad y[2] = \frac{7}{16} + \frac{6}{16} - \frac{4}{8} = \frac{5}{16}$ ✓

$y[3] = \frac{7}{32} + \frac{9}{32} - \frac{9}{16} = \frac{-2}{32} = \frac{-1}{16}$ ✓

g) $y[n] = y[n-1] - \frac{1}{4}y[n-2] + x[n] - x[n-1]$

$n=0 \Rightarrow y[0] = y[-1] - \frac{1}{4}y[-2] + x[0] - x[-1] = 1 - \frac{1}{4} + 1 = \frac{7}{4}$ ✓

$n=1 \Rightarrow y[1] = y[0] - \frac{1}{4}y[-1] + x[1] - x[0] = \frac{7}{4} + \frac{1}{2} - 1 = \frac{1}{4} = \underline{1}$ ✓

$n=2 \Rightarrow y[2] = y[1] - \frac{1}{4}y[0] + x[2] - x[1] = 1 - \frac{7}{16} + \frac{1}{4} - \frac{1}{2} = \frac{5}{16}$ ✓

$n=3 \Rightarrow y[3] = y[2] - \frac{1}{4}y[1] + x[3] - x[2] = \frac{5}{16} - \frac{1}{2} + \frac{1}{8} = \frac{-1}{16}$ ✓

both methods have the same result.

Q2. a)

$$h[n] - \frac{3}{2}h[n-1] + \frac{1}{2}h[n-2] = 2\delta[n] + 3\delta[n-1]$$

it is causal

$$n=0 \Rightarrow h[0] - \frac{3}{2}\cancelto{0}{h[-1]} + \frac{1}{2}\cancelto{0}{h[-2]} = \cancelto{2}{2\delta[0]} + 3\cancelto{0}{\delta[-1]} \Rightarrow h[0] = 2$$

$$n=1 \Rightarrow h[1] = \cancelto{3}{\frac{3}{2}h[0]} - \frac{1}{2}\cancelto{0}{h[-1]} + 2\cancelto{0}{\delta[1]} + \cancelto{3}{3\delta[0]} \Rightarrow h[1] = 6$$

$$n=2 \Rightarrow h[2] - \cancelto{9}{\frac{3}{2}h[1]} - \frac{1}{2}\cancelto{1}{h[0]} + 2\cancelto{0}{\delta[2]} + 3\cancelto{0}{\delta[1]} \Rightarrow h[2] = 8$$

$n=3 \Rightarrow h[3] = 9$ $\qquad$ $n=4 \Rightarrow h[4] = 9{,}5$

$n \Rightarrow$	0	1	2	3	4
$h \Rightarrow$	2	6	8	9	9,5
$b \Rightarrow$	4	2	1	$\frac{1}{2}$	

$b_0 = h_1 - h_0 = 6 - 2 = 4$ $\qquad$ $b_2 = h_3 - h_2 = 9 - 8 = 1$

$b_1 = h_2 - h_1 = 8 - 6 = 2$ $\qquad$ $b_3 = h_4 - h_3 = 9{,}5 - 9 = 1/2$

$$\sum_{i=0}^{n-1} b_i = h[n] - h[0] \qquad b_i = 2^{2-i}$$

$$\Rightarrow \sum_{i=0}^{n-1} 2^{2-i} = h[n] - h[0] \Rightarrow h[n] = 4\sum_{i=0}^{n-1} 2^{-i} + h[0]$$

$$\Rightarrow h[n] = 4\left(\frac{1-(1/2)^n}{1-1/2}\right) + 2 = 8\left(1 - \left(\tfrac{1}{2}\right)^n\right) + 2 = \boxed{10 - 8\left(\tfrac{1}{2}\right)^n}$$

b) $y_{step}[n] = s[n]$ $\qquad x[n] = u[n]$

$$s[n] = \frac{3}{2}s[n-1] + \frac{1}{2}s[n-2] + 2u[n] + 3u[n-1]$$

$$n=0 \Rightarrow s[0] = \frac{3}{2}\cancelto{0}{s[-1]} - \frac{1}{2}\cancelto{0}{s[-2]} + 2\cancelto{2}{u[0]} + 3\cancelto{0}{u[-1]} = 2$$

$$n=1 \Rightarrow s[1] = \frac{3}{2}\cancelto{3}{s[0]} - \frac{1}{2}\cancelto{0}{s[-1]} + 2\cancelto{2}{u[1]} + 3\cancelto{3}{u[0]} = 8$$

$n=2 \Rightarrow s[2] = 16$ $\qquad n=3 \Rightarrow s[3] = 25$

$n=4 \Rightarrow s[4] = 34{,}5$

n	0	1	2	3	4	...
s	2	8	16	25	34,5	...
$b_n = s_{n-1} - s_n$	6	8	9	9,5	...	
$d_n = b_{n-1} - b_n$	2	1	$\frac{1}{2}$	$\frac{1}{4}$	...	

$$d_0 + d_1 + \dots d_{n-1} = b[n] - b[0] \Rightarrow \sum_{i=0}^{n-1} d_i = b[n] - b[0]$$

$$d[n] = 2^{1-n} \Rightarrow b[n] = \sum_{i=0}^{n-1} 2^{1-i} + b[0] = 2\sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^i + 6 = 2\left(\frac{1-(1/2)^n}{1-1/2}\right) +$$

$$\Rightarrow b[n] = 10 - 4\left(\frac{1}{2}\right)^n$$

$$s[n] - s[0] = \sum_{i=0}^{n-1} b_i = \sum_{i=0}^{n-1}\left(10 - 4\left(\frac{1}{2}\right)^i\right) = 10n - 4\left(\frac{1-(1/2)^n}{1-1/2}\right)$$

$$\Rightarrow s[n] = 2 + 10n - 8 + 8\left(\frac{1}{2}\right)^n = 8\left(\frac{1}{2}\right)^n + 10n - 6$$

c) $h[n] = s[n] - s[n-1]$

$$s[n-1] = 8\left(\tfrac{1}{2}\right)^{n-1} + 10(n-1) - 6 = 16\left(\tfrac{1}{2}\right)^{n} + 10n - 16$$

$$\Rightarrow h[n] = 8\left(\tfrac{1}{2}\right)^{n} + 10n - 6 - \left(16\left(\tfrac{1}{2}\right)^{n} + 10n - 16\right)$$

$$= 8\left(\tfrac{1}{2}\right)^{n} \cancel{+10n} - 6 - 16\left(\tfrac{1}{2}\right)^{n} \cancel{-10n} + 16 = 10 - 8\left(\tfrac{1}{2}\right)^{n}$$

So $h[n]$ here is equal to part (a)

because the system is LTI we can apply a linear operation on input and we can do it ,this operation, on the output

(Q3 a)

$y[n] + \frac{1}{6} y[n-1] - \frac{1}{6} y[n-2] = x[n] - 2x[n-1]$ $\quad N=2$

$\Rightarrow \lambda^2 + \frac{1}{6}\lambda - \frac{1}{6} = 0$, $y_h[n] + \frac{1}{6} y_h[n-1] - \frac{1}{6} y_h[n-2] = 0$

characteristic eqn $\qquad$ homogeneous eqn

$\Rightarrow (\lambda + \frac{1}{2})(\lambda - \frac{1}{3}) = 0 \Rightarrow \lambda_1 = \frac{-1}{2}$, $\lambda_2 = \frac{1}{3}$

$|\lambda_1|, |\lambda_2|$ are less than 1 and system is a causal LTI then it is BIBO stable.

b) homogeneous eqn : $y_h[n] - \frac{1}{\sqrt{2}} y_h[n-1] + \frac{1}{4} y[n-2] = 0$ $\quad N=2$

characteristic eqn: $\lambda^2 - \frac{1}{\sqrt{2}}\lambda + \frac{1}{4} = 0$

$\Rightarrow \lambda_1 = \sqrt{2} + \sqrt{2}i \quad \lambda_2 = \sqrt{2} - \sqrt{2}i \quad |\lambda_1| = |\lambda_2|$

$|\lambda_1| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$, so it is not BIBO stable

c) homo. eqn : $y_h[n] - \frac{5}{2} y_h[n-1] + y_h[n-2] = 0$ $\quad N=2$

charac. eqn: $\lambda^2 - \frac{5}{2}\lambda + 1 = 0 \qquad (\lambda - 2)(\lambda - \frac{1}{2}) = 0$

$\lambda_1 = 2 \quad \lambda_2 = 1/2$ $\qquad$ because $|\lambda_1| > 1$ then it is not BIBO stable.

Q4-a)

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$y_{zi}[n] = 5\left(\frac{1}{3}\right)^n \cos\left(2\pi\frac{1}{8}n + \frac{\pi}{3}\right) = 5\left(\frac{1}{3}\right)^n \left(\frac{e^{\frac{2\pi n}{8}j+\frac{\pi}{3}j} + e^{-\frac{2\pi n}{8}j-\frac{\pi}{3}j}}{2}\right)$$

$$= \left(\frac{5}{2}\right)\left(\frac{1}{3}\right)^n \left[e^{j\frac{2\pi n}{8}} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi n}{8}} e^{-j\frac{\pi}{3}}\right]$$

$$= \frac{5}{2}e^{j\pi/3}\left(\frac{1}{3}e^{j\frac{2\pi}{8}}\right)^n + \frac{5}{2}e^{-j\pi/3}\left(\frac{1}{3}e^{-j\frac{2\pi}{8}}\right)^n = C_1(\lambda_1)^n + C_2(\lambda_2)^n$$

$$\Rightarrow \lambda_1 = \frac{1}{3}e^{j\pi/4} \qquad \lambda_2 = \frac{1}{3}e^{-j\pi/4} \qquad \lambda_2 = \lambda_1^*$$

$$\lambda_1 = \frac{1}{3}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{6} + i\frac{\sqrt{2}}{6} \qquad \lambda_2 = \frac{\sqrt{2}}{6} - i\frac{\sqrt{2}}{6}$$

b) $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \Rightarrow \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$

$$\Rightarrow \lambda_1 + \lambda_2 = \frac{\sqrt{2}}{6} + \cancel{\frac{\sqrt{2}}{6}i} + \frac{\sqrt{2}}{6} - \cancel{i\frac{\sqrt{2}}{6}} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

$$\lambda_1\lambda_2 = \frac{1}{3}e^{j\pi/4} \cdot \frac{1}{3}e^{-j\pi/4} = \frac{1}{9}e^0 = \frac{1}{9}$$

$$\Rightarrow \lambda^2 - \frac{\sqrt{2}}{3}\lambda + \frac{1}{9} = 0$$ ← charactristic eqn $N=2$

$$\Rightarrow y_h[n] - \frac{\sqrt{2}}{3}y_h[n-1] + \frac{1}{9}y_h[n-2] = 0$$ ← homo. eqn.

$$\Rightarrow a_1 = -\frac{\sqrt{2}}{3} \qquad a_2 = 1/9$$