Correlation of Power Signals

If $x[n]$ and $y[n]$ are power signals, their cross-correlation function is defined as:

$$r_{xy}[\ell] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]\, y^*[n-\ell] \overset{\text{OR}}{=} \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n+\ell]\, y^*[n] \qquad (\star)$$

As shown below: if either $x[n]$ or $y[n]$ is periodic with period N, then $r_{xy}[\ell]$ will also be periodic with period N.

$$r_{xy}[\ell+N] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]\, y^*[n-(\ell+N)] \overset{\text{OR}}{=} \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n+(\ell+N)]\, y^*[n]$$

if $y[n]$ is periodic with period N, then $y^*[n-\ell-N] = y^*[n-\ell]$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]\, y^*[n-\ell]$$

$$= r_{xy}[\ell]$$

if $x[n]$ is periodic with period N, then $x[n+\ell+N] = x[n+\ell]$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n+\ell]\, y^*[n]$$

$$= r_{xy}[\ell]$$

If $x[n]$ and $y[n]$ are both periodic with period N, then $(\star)$ can be simplified to:

$$r_{xy}[\ell] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, y^*[n-\ell]$$

Example

Given: $x[n] = A\cos\left(2\pi \frac{1}{10} n\right)$ where A and B are real-valued

$y[n] = B\cos\left(2\pi \frac{1}{10}(n+3)\right)$

Find $r_{xy}[\ell]$.

Solution. Since $x[n]$ and $y[n]$ are both periodic with period $N =$ ____,
we may find $r_{xy}[\ell]$ as:

$x[n] = A\cos\left(2\pi \frac{1}{10} n\right)$

$y[n] = B\cos\left(2\pi \frac{1}{10}(n+3)\right)$

where A and B are real-valued

$$r_{xy}[\ell] = \frac{1}{\quad} \sum_{n=} x[n]\, y^*[n-\ell]$$

$$= \frac{1}{\quad} \sum_{n=} A\cos\left(2\pi \tfrac{1}{10} n\right) B\cos\left(2\pi \tfrac{1}{10}\left((\qquad)+3\right)\right)$$

$\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$

$$= \left(AB\right)\left(\frac{1}{\quad}\right)\left\{\sum_{n=} \cos\left(2\pi \tfrac{1}{10}\left(\qquad\right)\right) + \sum_{n=} \cos\left(2\pi \tfrac{1}{10}\left(\qquad\right)\right)\right\}$$

$$= \left(AB\right)\left(\frac{1}{\quad}\right)\left\{\sum_{n=} \cos\left(2\pi \tfrac{1}{10} \qquad\right) + \sum_{n=} \cos\left(2\pi \tfrac{1}{10} \qquad\right)\right\}$$

$$=$$

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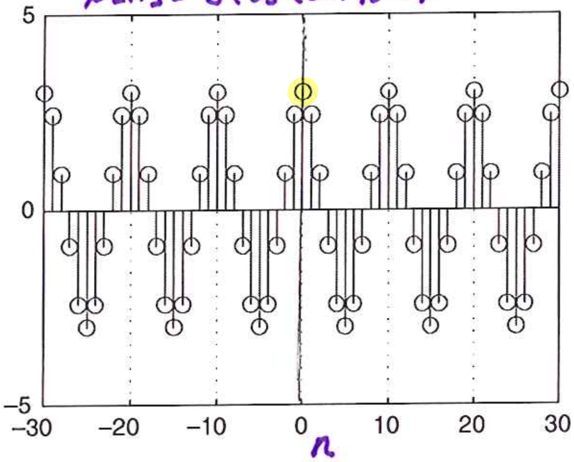
Plot of theoretically computed cross-correlation

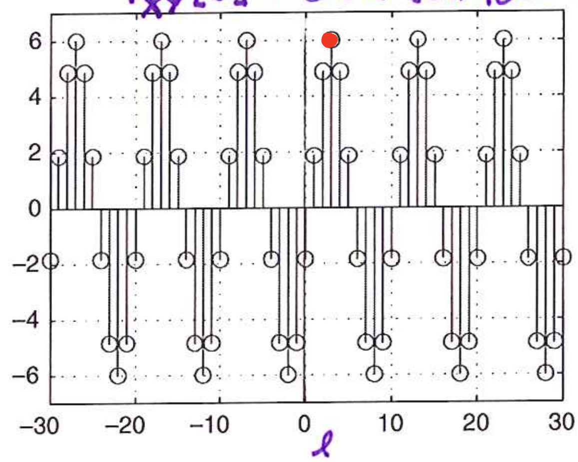
$$\left.\begin{aligned} x[n] &= 3\cos\left(2\pi\tfrac{1}{10}n\right), \quad -\infty < n < \infty \\ y[n] &= 4\cos\left(2\pi\tfrac{1}{10}(n+3)\right), \quad -\infty < n < \infty \end{aligned}\right\} \Rightarrow r_{xy}[\ell] = \frac{1}{10}\sum_{n=0}^{9} x[n]\,y^*[n-\ell] = 6\cos\left(2\pi\tfrac{1}{10}(\ell-3)\right)$$

Theoretical computation of $r_{xy}[\ell]$

$x[n] = 3\cos(2\pi \frac{1}{10} n)$

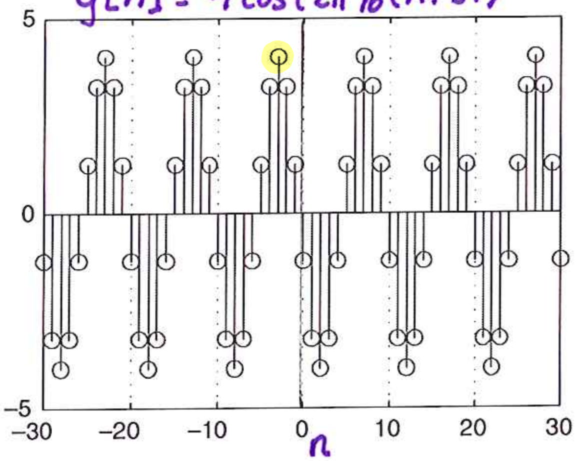


$r_{xy}[\ell] = 6\cos(2\pi\frac{1}{10}(\ell-3))$



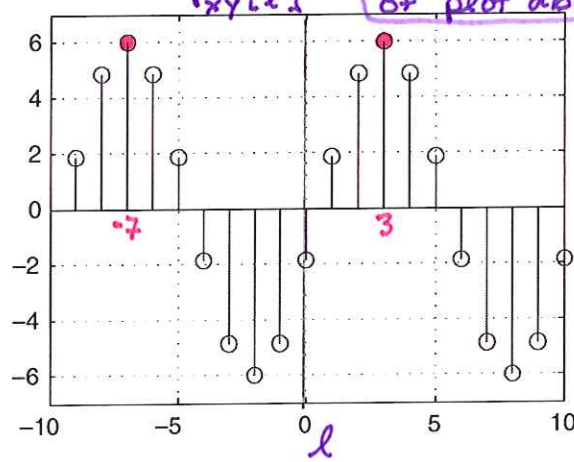
$y[n-\ell]$ is most similar to $x[n]$ when $\ell = 3, 10, 13 \ldots$ $-7, -17, \ldots$

$y[n] = 4\cos(2\pi\frac{1}{10}(n+3))$



$r_{xy}[\ell]$

Zoomed-in version of plot above



Note:

$x[n]$ is periodic with period 10 $\Rightarrow r_{xy}[\ell]$ is periodic with period 10

$y[n]$ is periodic with period 10 $\Rightarrow r_{xy}[\ell]$ is periodic with period 10

$r_{xy}[\ell]$ achieves its maximum value at $\ell = 3$ (and also at $\ell = 3 \pm 10k,\ k = 1, 2, , \ldots$); these are the values of ℓ for which $y[n-\ell]$ most resembles (or best aligns with) $x[n]$. For example, we see below that when $\ell = 3$, $y[n-\ell] = y[n-3]$ is in phase with $x[n]$.

$$y[n-3] = 4\cos\left(2\pi\frac{1}{10}((n-3)+3)\right) = 4\cos\left(2\pi\frac{1}{10}n\right)$$

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Practical computation of correlation sequences

When using matlab to cross-correlate two sequences, we use a finite number of signal samples (i.e., truncated signals). The following matlab commands were used to generate the plots below (the cross correlation was calculated using 61 samples of the signals $x[n]$ and $y[n]$).

```
> n=0:60;
> x = 3*cos(2*pi*(1/10)*n);
> y = 4*cos(2*pi*(1/10)*(n+3));
> rxy = xcorr(x,y,'biased')
```

$$x_t[n] = \begin{cases} 3\cos\left(2\pi\frac{1}{10}n\right), & 0 \le n \le 60 \\ 0, & \text{otherwise} \end{cases}$$

$$y_t[n] = \begin{cases} 4\cos\left(2\pi\frac{1}{10}(n+3)\right), & 0 \le n \le 60 \\ 0, & \text{otherwise} \end{cases}$$

After implementing the commands above:

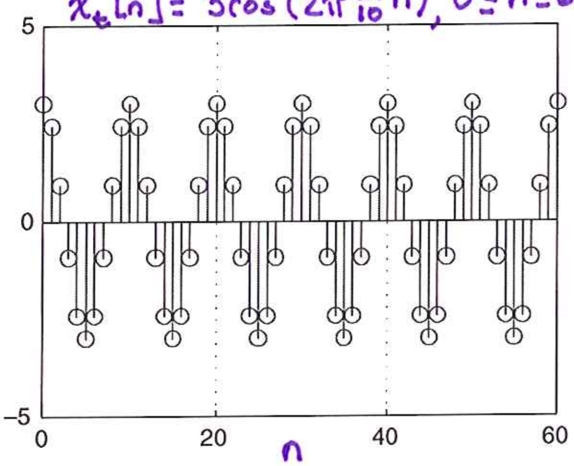
the vector x will contain values of $x[n]$, $n = 0, ..., 60$

the vector y will contain values of $y[n]$, $n = 0, ..., 60$

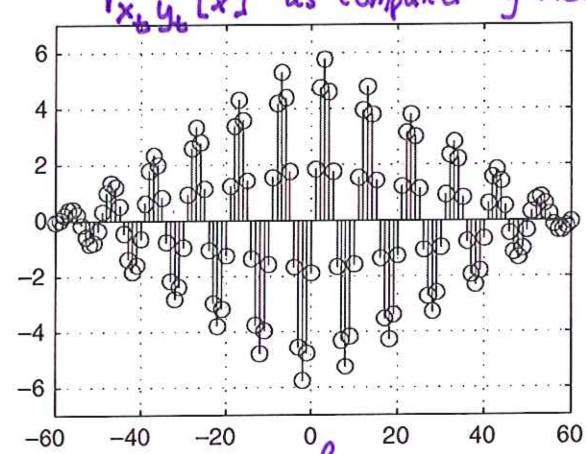
the vector rxy will contain values of $r_{xy_t}[\ell]$, $\ell = -60, ..., 60$

$$\boxed{r_{xy_t}[\ell] = \frac{1}{61}\sum_{n=0}^{60} x_t[n]y_t[n-\ell]}$$

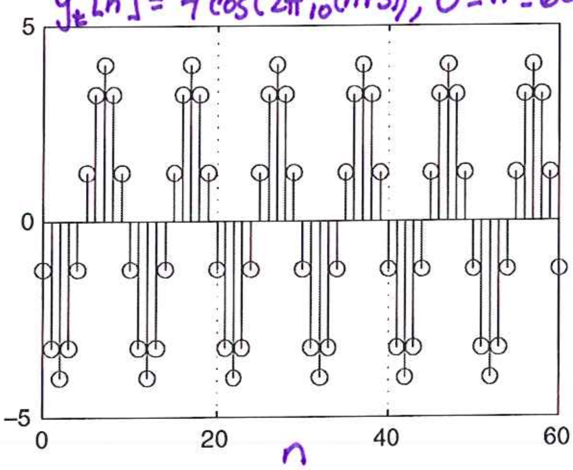
$x_t[n] = 3\cos(2\pi\frac{1}{10}n), 0 \le n \le 60$



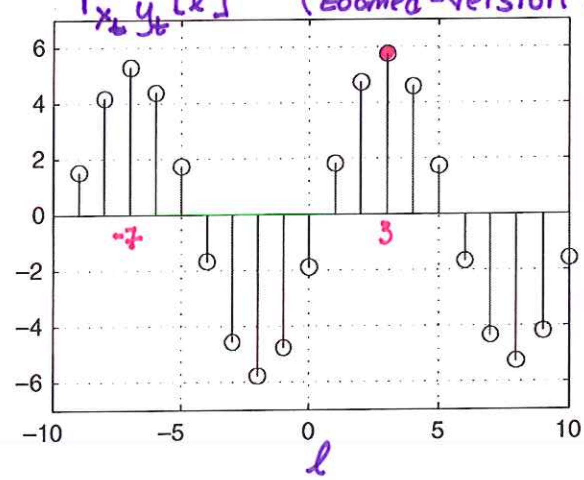
$r_{x_t y_t}[\ell]$ as computed by xcorr(·)



$y_t[n] = 4\cos(2\pi\frac{1}{10}(n+3)), 0 \le n \le 60$



$r_{x_t y_t}[\ell]$ (Zoomed-version)



Note:

for $|\ell| > 60$, there is no overlap between the nonzero portions of $x_t[n]$ and $y_t[n-\ell]$

for $|\ell| \ll 60$, $r_{xy_t}[\ell] \approx r_{xy}[\ell]$ where $r_{xy}[\ell]$ is the cross correlation of the infinitely long sequences, $x[n]$ and $y[n]$.

The cross correlation of truncated (or windowed) signals will have a triangular envelope.

$x[n]$: ...0 0 0 1 1 1 1 1 0 0 0...

window where $x[n] \neq 0$

$y[n-0]$: 0 0 0 1 1 1 1 1 0 0 0

$x[n]\,y[n-0]$: 0 0 0 1 1 1 1 1 0 0 0 $\qquad r_{xy}[0] = 5$

$x[n]$: ...0 1 1 1 1 1 0 0 0...

$y[n-1]$: 0 0 1 1 1 1 1 0 0

$x[n]\,y[n-1]$: 0 0 1 1 1 1 0 0 0 $\qquad r_{xy}[1] = 4$

$x[n]$: ...0 1 1 1 1 1 0 0 0...

$y[n-2]$: 0 0 0 1 1 1 1 1 0

$x[n]\,y[n-2]$: 0 0 1 1 1 0 0 0 $\qquad r_{xy}[2] = 3$

$y[n-3]$: 0 0 0 0 1 1 1 1 1 0 $\qquad r_{xy}[3] = 2$

$y[n-4]$: 0 0 0 0 0 1 1 1 1 $\qquad r_{xy}[4] = 1$

$y[n-5]$: 0 0 0 0 0 0 1 1 1 $\qquad r_{xy}[5] = 0$

$y[n+1]$: 1 1 1 1 1 0 0 0 0 $\qquad r_{xy}[-1] = 4$

$y[n+2]$: 1 1 1 1 0 0 0 0 0 $\qquad r_{xy}[-2] = 3$

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Cross Correlation of noisy sinusoids

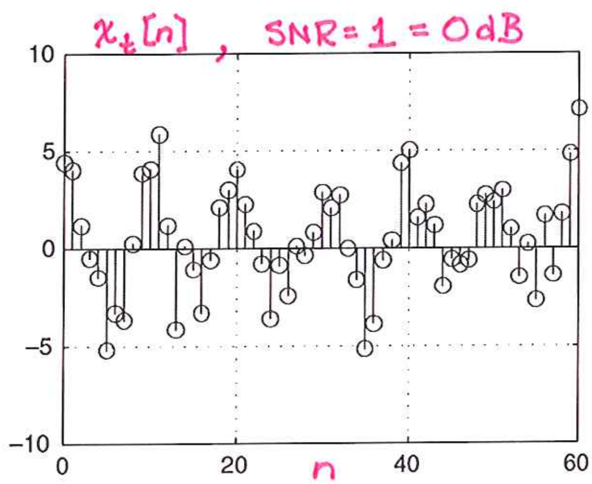
Cross correlation is an effective way to find similarities between noisy signals.

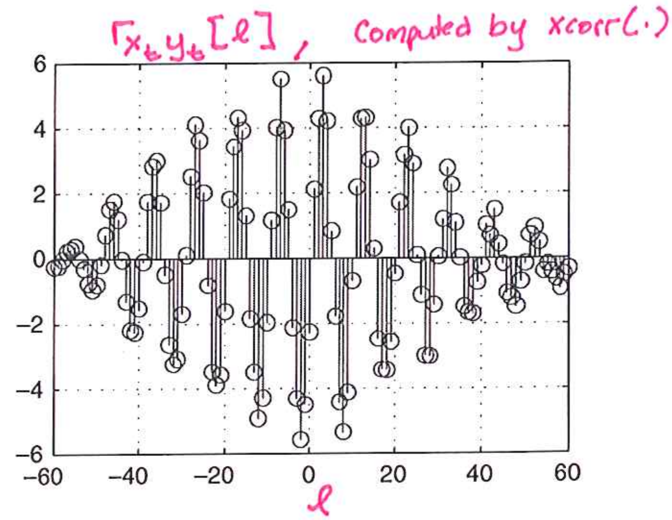
$$\text{Let } x_t[n] = \begin{cases} 3\cos\left(2\pi \frac{1}{10} n\right) + \nu_1[n],\ 0 \le n \le 60 \\ 0,\ \text{otherwise} \end{cases}$$

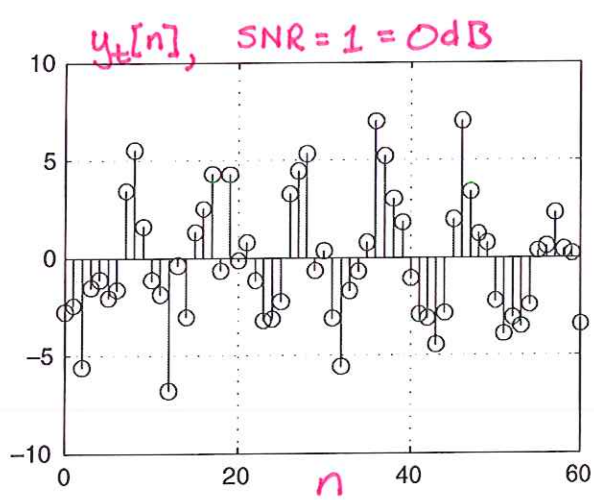
$$\text{and } y_t[n] = \begin{cases} 4\cos\left(2\pi \frac{1}{10} (n+3)\right) + \nu_2[n],\ 0 \le n \le 60 \\ 0,\ \text{otherwise} \end{cases}$$

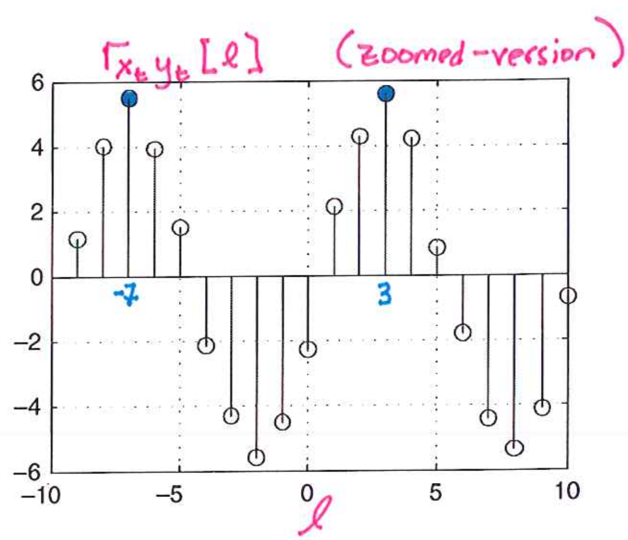
$$\text{and let } r_{xy_t}[\ell] = \frac{1}{61} \sum_{n=0}^{60} x_t[n] y_t[n-\ell]$$

The plots below show $x_t[n]$, $y_t[n]$, and $r_{xy_t}[\ell]$ for the case where $\nu_1[n]$ and $\nu_2[n]$ are both white gaussian noise sequences with with variances of 9/2 and 8 respectively, resulting in $x_t[n]$ and $y_t[n]$ both having signal-to-noise ratios of 1 or 0dB. From the cross correlation sequence, we see that $y_t[n-\ell]$ is most similar to $x_t[n]$ for $\ell = 3$ and -7.









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Cross Correlation of noisy sinusoids

This page is similar to the previous page except that we have increased the number of samples included in the noisy truncated signals. Except for the zoomed-in version of the cross correlation sequence, the discrete-time signals have been illustrated using the plot command instead of the stem command. This was done because of the increased number of samples and the fact that it is difficult to see the evolution of the signals when the stems are too close together.

$$\text{Let } x_t[n] = \begin{cases} 3\cos\left(2\pi\frac{1}{10}n\right) + \nu_1[n], \; 0 \le n \le 200 \\ 0, \;\text{otherwise} \end{cases}$$

$$\text{and } y_t[n] = \begin{cases} 4\cos\left(2\pi\frac{1}{10}(n+3)\right) + \nu_2[n], \; 0 \le n \le 200 \\ 0, \;\text{otherwise} \end{cases}$$

$$\text{and let } r_{xy_t}[\ell] = \frac{1}{201}\sum_{n=0}^{200} x_t[n]y_t[n-\ell]$$

The plots below show $x_t[n]$, $y_t[n]$, and $r_{xy_t}[\ell]$ for the case where $\nu_1[n]$ and $\nu_2[n]$ are both white gaussian noise sequences with with variances of 9/2 and 8 respectively, resulting in $x_t[n]$ and $y_t[n]$ both having signal-to-noise ratios of 1 or 0dB. From the cross correlation sequence, we see that $y_t[n-\ell]$ is most similar to $x_t[n]$ for $\ell = 3$ and -7.