1. Classify each of the following systems as: static or dynamic (2 pts.); linear or nonlinear (4 pts.); time-invariant or time-varying (4 pts.); causal or noncausal (2 pts.); and stable or unstable (2 pts.). **In each case, you will lose points if your response is not clearly and logically justified.**

(14 pts.) a) $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$ according to the input-output relationship: $y[n] = x\left[n^2\right]$

(14 pts.) b) $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$ according to the input-output relationship: $y[n] = \cos(x[n])$

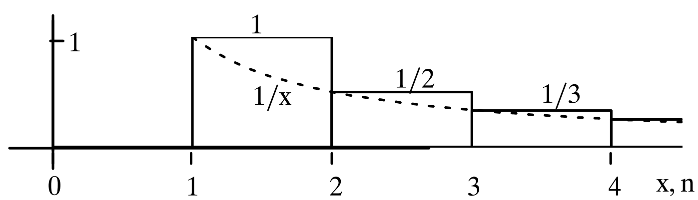
(6 pts.) 2. The following input-output pairs have been observed during the operation of a system that is known to be ***time-invariant***. Provide a clear explanation as to how this information allows you to conclude that the system is **not** linear.

$$x_1[n] = \{\ldots,\ \underset{\uparrow}{0},\ 0,\ 3,\ 0,\ 0,\ \ldots\} \quad \Rightarrow \quad y_1[n] = \{\ldots,\ 0,\ 0,\ \underset{\uparrow}{0},\ 1,\ 0,\ 2,\ 0,\ \ldots\}$$

$$x_2[n] = \{\ldots,\ \underset{\uparrow}{0},\ 0,\ 0,\ 1,\ 0,\ \ldots\} \quad \Rightarrow \quad y_2[n] = \{\ldots,\ 0,\ 1,\ \underset{\uparrow}{2},\ 1,\ 0,\ 0,\ 0,\ \ldots\}$$

3. Let $h[n]$ denote the impulse response of an ***LTI system***. Comment on the stability and causality of the system, when:

(3 pts.) a) $h[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1}{n+1}, & n \geq 0 \end{cases}$



Hint: The figure above illustrates that the area under the infinitely long staircase function (composed of the sequence of rectangles of width 1 and height $1/n$, $1 \leq n < \infty$) is greater than the area under the curve $1/x$ on the interval $1 \leq x < \infty$.

$$\text{area under staircase function} = \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x} dx = \text{area under } 1/x \text{ curve}$$

Note also that: $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$

(3 pts.) b) $h[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 1/n^2, & n > 0 \end{cases}$ ***Hint***: it can be shown that: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(2 pts.) c) $h[n] = \{-2,\ -1,\ \underset{\uparrow}{0},\ 1,\ 2\}$

(3 pts.) d) $h[n] = \begin{cases} 2^n, & n < 0 \\ 1, & n = 0 \\ (\tfrac{1}{2})^n, & n > 0 \end{cases}$

(3 pts.) e) $h[n] = n(\frac{1}{2})^n u[n]$

note: you should be able to determine the value of $\sum_{n=0}^{\infty} n(\frac{1}{2})^n$ using the hint below.

Being able to derive a closed-form expression for this type of sum will be important later in the course; although, the hint below provides a derivation, I would like you to ***include the derivation in your solutions*** (as opposed to simply saying "according to the hint, ...").

Hint: if $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ then: $\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

furthermore: $\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{x}\sum_{n=0}^{\infty} nx^n$

and $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

therefore: $\sum_{n=0}^{\infty} nx^n = ??$

4. Consider an initially relaxed $(y[-1] = 0)$ causal recursive system whose input-output relationship is as specified below for $n \geq 0$; we assume zero input is applied for $n < 0$.

$$y[n] = \frac{n}{n+1}y[n-1] + \frac{1}{n+1}x[n], \qquad n \geq 0$$

(4 pts.) a) Iterate the input-output equation for $n \geq 0$ to show that the impulse response, $h[n]$, of this system is the same as the impulse response of the system in question 3(a), *i.e.,* $h[n] = \frac{1}{n+1}u[n]$. Recall that the impulse response is the response of the system when $x[n] = \delta[n]$ and the system is initially relaxed (*i.e.*, all internal storage registers are set to zero which for this system implies that: $h[-1] = y[-1] = 0$).

(4 pts.) b) Assuming that the input sequence, $x[n]$, is bounded for all $n \geq 0$ (*i.e.,* assuming there exists some finite number B so that $|x[n]| \leq B, \forall n \geq 0$), show that $y[n]$ will also be bounded for $n \geq 0$ (*i.e.*, show that the system is stable). To do this, you need only find a finite number C for which $|y[n]| \leq C, \forall n > 0$; be sure to clearly explain the relation between the value of C and the value of B. Thus, conclude that the system is stable. ***Hint***: Given that q is a weighted average of a and b (*i.e.,* $q = wa + (1-w)b$ for some $0 \leq w \leq 1$), show that $\min(a,b) \leq q \leq \max(a,b)$ and hence conclude that $|q| \leq \max(|a|,|b|)$. Note that $y[n]$ is a weighted average of $y[n-1]$ and $x[n]$.

(6 pts.) c) Let $x_2[n] = \delta[n-1]$ and let $y_2[n]$ denote the response of the initially relaxed system when the input is $x_2[n]$. Find $y_2[n], n \geq 0$. Recall that the fact that the system is initially relaxed implies that $y_2[-1] = 0$. If the system were time-invariant what would be the relation between $y_2[n]$ and $h[n]$? Is the system time-invariant? Explain.

(2 pts.) d) As demonstrated in question 3(a), the impulse response of this system is not absolutely summable. The fact that the impulse response is not absolutely summable often allows us to conclude that the system is not stable. Why can we not make that conclusion for this system?