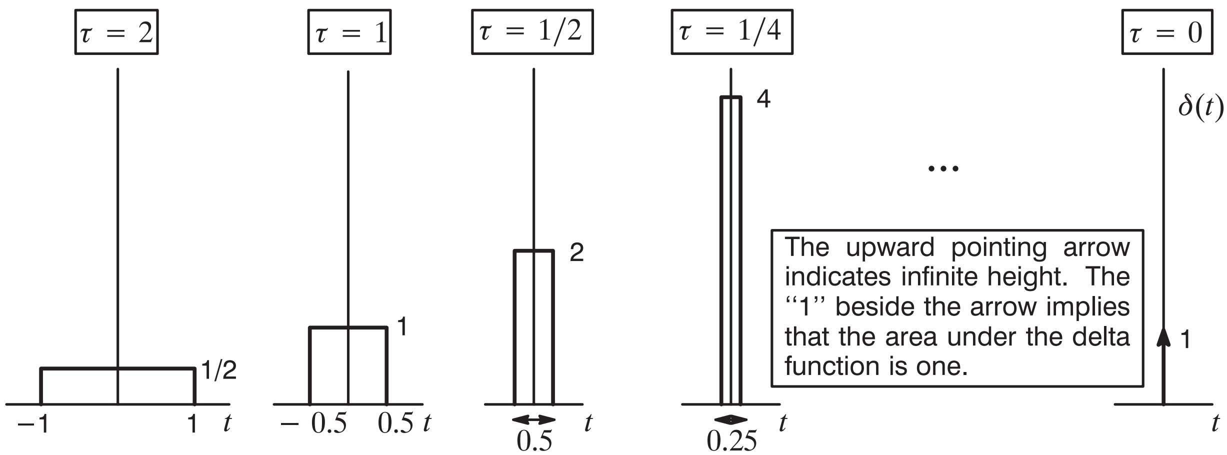
The Dirac Delta Function

The Dirac delta function is an extremely useful function in signal analysis. In particular, the Dirac delta functions allows for a continuous-time representation of an otherwise discrete-time signal. This in turn allows us to understand the connection between Fourier transforms of continuous-time signals and those of discrete-time signals. Trains of Dirac delta functions are useful in understanding sampling theory as well as providing a convenient mathematical description of periodic signals.

Definition: The Dirac delta function, $\delta(t)$, can be defined as the limiting rectangle of width τ and height $1/\tau$ as τ approaches zero :

$$\delta(t) \equiv \lim_{\tau \to 0} \frac{1}{\tau} \mathrm{rect}\left(\frac{t}{\tau}\right)$$



Note that the area under each rectangle in the sequence is 1, and thus the area under the limiting rectangle is also one; it has infinite height and zero width:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

<u>Properties of the Dirac delta function:</u>

Multiplication of a function by a Dirac Delta function: $g(t)\delta(t - t_0) = g(t_0)\delta(t - t_0)$

Sifting Property of the Dirac delta function: $\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0)$.

Time-Scaling Property of the Dirac delta function: $\delta(t/a) = |a|\delta(t)$.

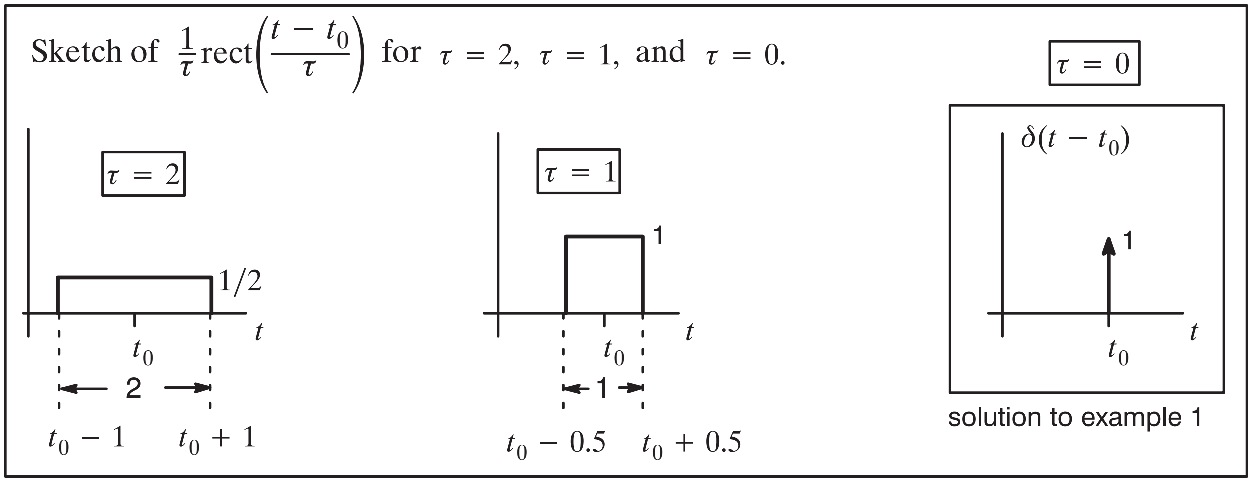
Convolution of a function $g(t)$ **with a delta function** at $t = t_0$ has the effect of shifting the function $g(t)$ by an amount t_0:

$$g(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} g(t - \tau)\delta(\tau - t_0)d\tau = g(t - t_0).$$

Time-scaled and time-shifted versions of the delta function can be understood using the definition of the Dirac Delta Function, δ, on page 1.

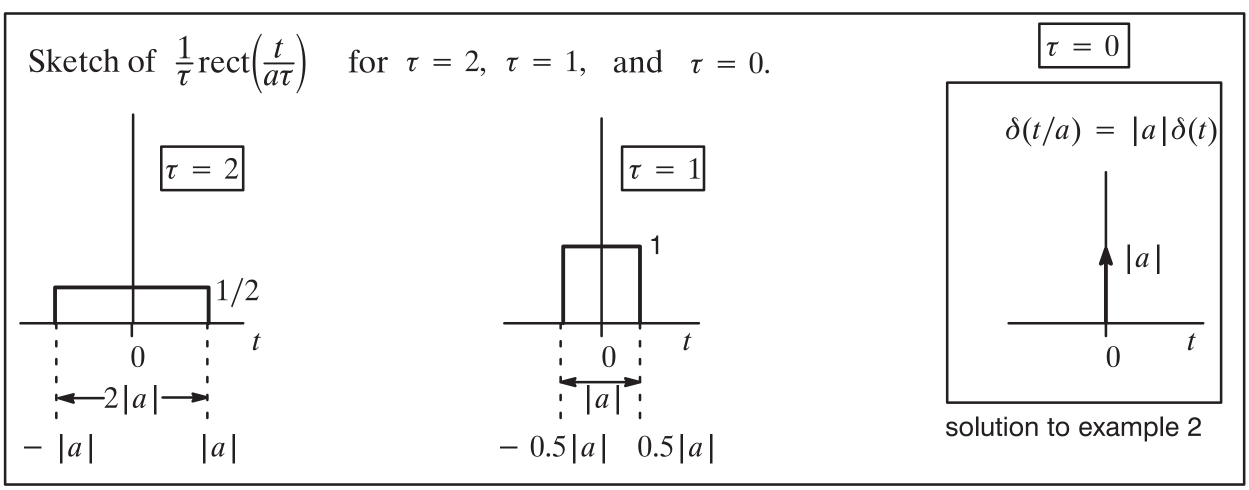
Example 1: (Time Shifting) Sketch $\delta(t - t_0)$ as a function of t.

Procedure: Using the definition on the page 1, we find: $\delta(t - t_0) = \lim_{\tau \to 0} \frac{1}{\tau} \mathrm{rect}\left(\frac{t - t_0}{\tau}\right)$.
Sketching several rectangles using various values of τ, we find that each rectangle is centered at $t = t_0$ and has area one (width $= \tau$ and height $1/\tau$). Thus, $\delta(t - t_0)$ can be sketched as shown below.



Example 2: (Time Scaling) Sketch $\delta(t/a)$ as a function of t.

Procedure: Using the definition above, we find: $\delta(t/a) = \lim_{\tau \to 0} \frac{1}{\tau} \mathrm{rect}\left(\frac{t}{a\tau}\right)$.
Sketching several rectangles using various values of τ, we find that each rectangle is centered at $t = 0$ and has area equal to $|a|$ (width $= |a|\tau$ and height $= 1/\tau$). Thus, $\delta(t/a)$ can be sketched as shown below.



The sifting and time-scaling properties of the Delta Function make it extremely easy to evaluate integrals with delta functions in the integrand such as the one shown to the right.

$$\int_a^b g(x)\delta\left(\frac{x - x_0}{c}\right)dx$$

The following steps detail a procedure for evaluation of integrals involving delta functions.

1. Identify the dummy variable of integration (I will refer to this variable as x.) and find the value of the variable of integration that causes the argument of the delta function to have a value of zero. [I will refer to this value as x_0.]
2. Check to see that x_0 lies within the limits of integration (is $a < x_0 < b$ true?). If so, proceed to step 3. Otherwise, the integral is zero since the integrand is 0 over the interval of integration.
3. Identify $g(x)$ (the rest of the integrand which multiplies the delta function). Note that $g(x)$ may include other delta functions as long as they are not located at $x = x_0$.
4. Use the time-scaling property of the delta function to rewrite the integrand as $g(x)|c|\delta(x - x_0)$
5. Apply the sifting property of the delta function to find that the result of the integration is $|c|g(x_0)$.

A mathematical summary of the above steps is shown below:

$$\int_a^b g(x)\delta\left(\frac{x - x_0}{c}\right)dx = \int_a^b |c|g(x)\delta(x - x_0)dx = \begin{cases} 0, & x_0 < a \\ |c|g(x_0), & a < x_0 < b \\ 0, & x_0 > b \end{cases}$$

Examples:

1. $$\int_{-1}^{1} (t^2 + 2)\delta(t/2)dt = \int_{-1}^{1} 2(t^2 + 2)\delta(t)dt = 2(t^2 + 2)\big|_{t=0} = 4$$

2. $$\int_1^4 \cos(t)\delta(2t - 6)dt = \int_1^4 \cos(t)\delta\left(\frac{t - 3}{½}\right)dt = ½\cos(t)\big|_{t=3} = ½\cos(3)$$

3. $$\int_1^4 (t^2 + 2)\delta(t + 3)dt = 0,$$ since t=−3 lies outside the interval of integration

4. $$\int_{-\infty}^{\infty} (t^2 + \tau)\delta(t - \tau + 3)d\tau = (t^2 + \tau)\big|_{\tau=t+3} = t^2 + t + 3$$

5. $$\int_{-\infty}^{t} \delta(x)dx = u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$