ECE 4531
Stevenson

Autocorrelation is an effective way to observe periodicities in a signal corrupted by noise.

Let $s[n]$ be a periodic signal of interest.

Let $\nu[n]$ denote a zero-mean noise waveform that is uncorrelated with $s[n]$.

Let $z[n] = s[n] + \nu[n]$

$$r_{zz}[\ell] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \underbrace{\left(s[n]+\nu[n]\right)}_{z[n]} \underbrace{\left(s^*[n-\ell]+\nu^*[n-\ell]\right)}_{z^*[n-\ell]}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} s[n]s^*[n-\ell] + s[n]\nu^*[n-\ell] + \nu[n]s^*[n-\ell] + \nu[n]\nu^*[n-\ell]$$

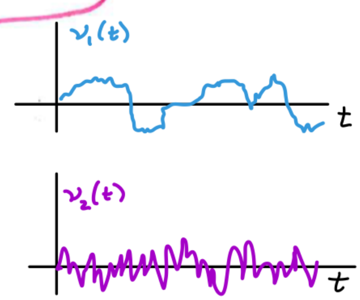
$$= r_{ss}[\ell] + r_{s\nu}[\ell] + r_{\nu s}[\ell] + r_{\nu\nu}[\ell]$$

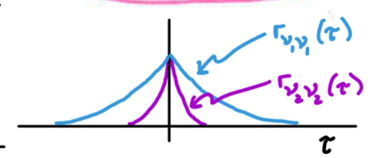
if $s[n]$ is periodic with period N, then so will be $r_{ss}[\ell]$. The max value of $r_{ss}[\ell]$ occurs at $\ell=0$, $\ell=\pm N$, $\ell=\pm 2N$, ...

both of these terms will be equal to zero because $\nu[n]$ is zero-mean and uncorrelated with $s[n]$

the elements of a random noise process are only correlated over short time intervals (depending on the frequency content of the process).

$\Rightarrow \lim_{\ell\to\infty} r_{\nu\nu}[\ell] = 0$





$$r_{zz}[0] = r_{ss}[0] + r_{\nu\nu}[0]$$

$= $ signal power + noise power

assumes $s[n]$ + $\nu[n]$ are uncorrelated

Autocorrelation of a noisy sinusoid

Autocorrelation is an effective way of revealing periodicities in noisy signals.

Let $z_t[n] = \begin{cases} s[n] + v[n], & 0 \le n \le 500 \\ 0, & \text{otherwise} \end{cases}$

where $s[n] = \cos\left(2\pi \frac{1}{10}(n-5)\right)$ is a sinusoid of amplitude 1 and period 10 samples

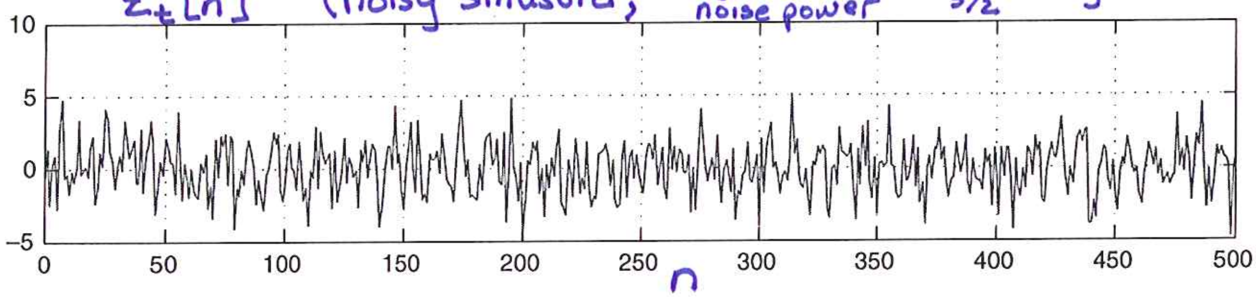
and $v[n]$ is a zero-mean white gaussian noise with variance 5/2 $\Rightarrow$ SNR $= \frac{1}{5}$

or $10\log_{10}\left(\frac{1}{5}\right) = -7\text{dB}$

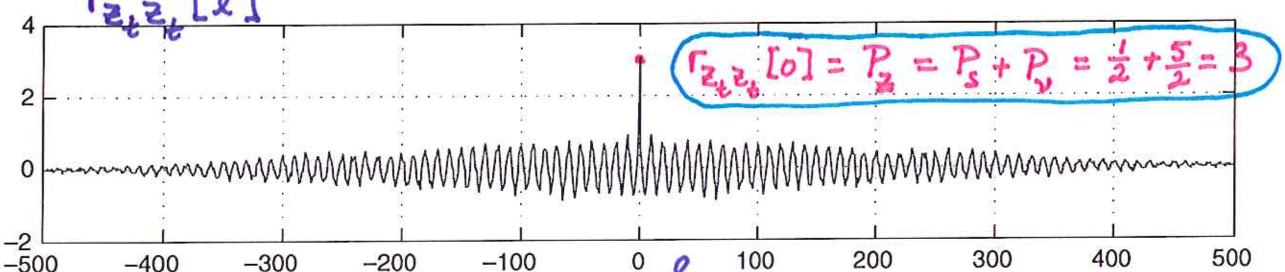
Then can use matlab to compute: $r_{z_t z_t}[\ell] = \frac{1}{501}\sum_{n=0}^{500} x_t[n]y_t[n-\ell]$

Plots of $z_t[n]$, $r_{z_t z_t}[\ell]$, and a zoomed version of $r_{z_t z_t}[\ell]$ are shown below.

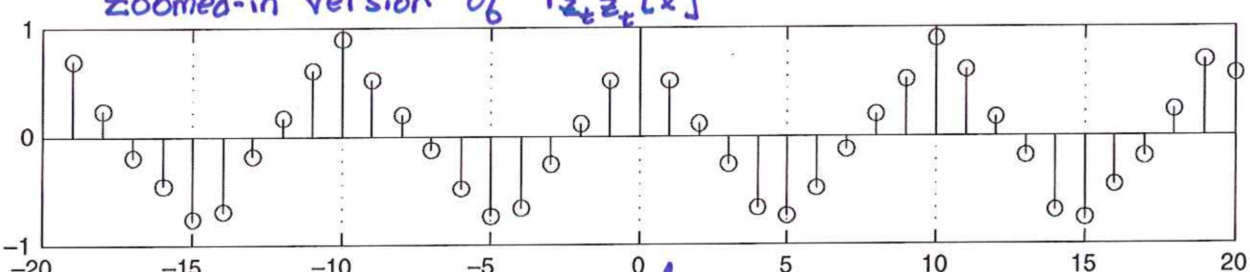
$z_t[n]$ (noisy sinusoid, $\frac{\text{Signal power}}{\text{noise power}} = \frac{1/2}{5/2} = \frac{1}{5}$ OR -7dB)



$r_{z_t z_t}[\ell]$



Zoomed-in version of $r_{z_t z_t}[\ell]$



Note

- autocorrelation function has the same period as $s[n]$
- the autocorrelation function does not retain phase information

Correlations involving the input and output of an LTI system

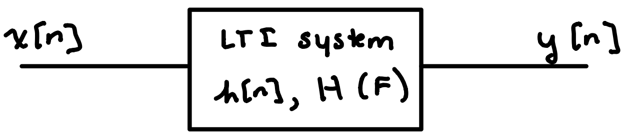
Let $x[n]$ denote an energy signal with autocorrelation function:

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-\ell] \implies r_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

The energy spectral density of $x[n]$ is denoted by $S_{xx}(f)$ and may be found as the DTFT of $r_{xx}[\ell]$. $\Rightarrow S_{xx}(F) = X(F)\, X^*(F)$

Exercise

Given that $y[n]$ is the response of an LTI system with impulse response $h[n]$ when the system's input is $x[n]$.
Find the relationship between $r_{yx}[\ell]$ and $r_{xx}[\ell]$.



Solution

(Hint: start by expressing $r_{yx}[\ell]$ as a convolution of two signals)

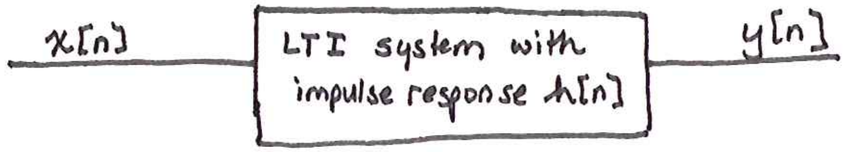
$r_{yx}[\ell] =$

The cross spectral density $S_{yx}(F)$ is the DTFT of $r_{yx}[\ell]$.

$\Rightarrow S_{yx}(F) =$

$\Rightarrow H(F) =$

Correlations involving the input, $x[n]$, and the output, $y[n]$, of an LTI system with impulse response $h[n]$.



__Problem__ Given that $x[n]$ has autocorrelation $r_{xx}[\ell]$

a) Find the relation between $r_{yx}[\ell]$ and $r_{xx}[\ell]$

b) Find the relation between $r_{yy}[\ell]$ and $r_{xx}[\ell]$

c) Find the relation between $r_{xy}[\ell]$ and $r_{xx}[\ell]$

> The relationships determined in this problem are very useful for system identification.

Solution:

a) $r_{yx}[\ell] = y[\ell] * x^*[-\ell]$ Using relation of cross correlation and convolution

$= \overbrace{h[\ell] * x[\ell]}^{y[\ell]} * x^*[-\ell]$ output of LTI system is the convolution of system's impulse response and the input

$= h[\ell] * \underbrace{x[\ell] * x^*[-\ell]}_{r_{xx}[\ell]}$

$= h[\ell] * r_{xx}[\ell]$ using relation between cross correlation and convolution

b) $r_{yy}[\ell] = y[\ell] * y^*[-\ell]$ expressing cross-correlation in terms of convolution

$= \overbrace{h[\ell] * x[\ell]}^{y[\ell]} * \overbrace{x^*[-\ell] * h^*[-\ell]}^{y^*[-\ell]}$ input-output relation of LTI system + property of convolution shown below.

$= h[\ell] * h^*[-\ell] * \overbrace{x[\ell] * x^*[-\ell]}^{r_{xx}[\ell]}$ convolution is commutative and associative

$= h[\ell] * h^*[-\ell] * r_{xx}[\ell]$ using relation between cross correlation and convolution

$= r_{hh}[\ell] * r_{xx}[\ell]$

c) $r_{xy}[\ell] = r_{yx}^*[-\ell]$ using property of cross-correlations

$= h^*[-\ell] * r_{xx}^*[-\ell]$ using result of part (a) together with the convolution property below

$= h^*[-\ell] * r_{xx}[\ell]$ autocorrelation functions are Hermitian

Useful property of convolution:

$$\boxed{\text{If } z[\ell] = w[\ell] * v[\ell] \quad \text{then} \quad z[-\ell] = w[-\ell] * v[-\ell]}$$

From which it follows that

$$\boxed{\text{if } z[\ell] = w[\ell] * v[\ell] \quad \text{then} \quad z^*[-\ell] = w^*[-\ell] * v^*[-\ell]}$$

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$$\boxed{\text{If } z[\ell] = w[\ell] * v[\ell] \text{ then } z[-\ell] = w[-\ell] * v[-\ell]}$$

Proof

$$z[\ell] = \sum_{k=-\infty}^{\infty} w[k]\, v[\ell-k]$$

using definition of convolution

$$\Rightarrow z[-\ell] = \sum_{k=-\infty}^{\infty} w[k]\, v[-\ell-k]$$

replaced ℓ by $-\ell$ on both sides of equation

$$= \sum_{m=-\infty}^{\infty} w[-m]\, v[-\ell+m]$$

change of variables: $k=-m$

let $w_f[m] = w[-m]$

let $v_f[m] = v[-m] \Rightarrow v_f[\ell-m] = v[m-\ell]$

$$= \sum_{m=-\infty}^{\infty} w_f[m]\, v_f[\ell-m]$$

using previously defined relationships

$$= w_f[\ell] * v_f[\ell]$$

definition of convolution

$$= w[-\ell] * v[-\ell]$$

relations between $w_f[\cdot]$ & $w[\cdot]$, $v_f[\cdot]$ & $v[\cdot]$

Q.E.D.

$$\boxed{\text{If } z[\ell] = w[\ell] * v[\ell] \text{ then } z^*[-\ell] = w^*[-\ell] * v^*[-\ell]}$$

Proof

From 3rd last equality in the proof above, we have that:

$$z[-\ell] = \sum_{m=-\infty}^{\infty} w_f[m]\, v_f[\ell-m]$$

$$\Rightarrow z^*[-\ell] = \sum_{m=-\infty}^{\infty} w_f^*[m]\, v_f^*[\ell-m]$$

$$= w_f^*[\ell] * v_f^*[\ell]$$

$$= w^*[-\ell] * v^*[-\ell]$$

Q.E.D.