1. Use the **convolution-in-time** property of the Z transform to:

(5 pts.) a) express $Y(z)$ and ROC_Y in terms of $X(z)$ and ROC_X given that:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Hint: in order to use the convolution-in-time property, you must first determine how to express $y[n]$ as a convolution of some other signal with $x[n]$. Can you show that: $y[n] = x[n] * u[n]$? Provide detailed justification and then proceed with the question.

(7 pts.) b) determine $X(z)$ and ROC_X given that $x[n] = (n+1)u[n]$.
(***hint:*** first show (providing detailed justification) that $x[n] = u[n] * u[n]$.)
How many poles does $X(z)$ have? Specify their values.
How many zeros? specify their values.

2. Let $X(z)$ be defined as follows: $X(z) = \dfrac{1 + \frac{1}{2}z^{-1} + z^{-2}}{1 + \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$

(5 pts.) a) Use long division to find $x[n]$ for $n = 0, 1, ..., 5$, when ROC for $X(z)$ is: $|z| > 1$.

(5 pts.) b) Use long division to find $x[n]$ for $n = 0, -1, ..., -5$, when ROC for $X(z)$ is: $|z| < 1/2$.

(3 pts.) c) In its current form, $X(z)$ is not a proper rational function. Use long division to express $X(z)$ in terms of a polynomial plus a proper rational function ($M < N$). Your answer should be in the form of:

$$X(z) = P(z) + Y(z)$$

where: $P(z)$ is a simple polynomial whose inverse Z transform is a finite-length sequence that can be written down by inspection and $Y(z)$ is a strictly proper rational function of z^{-1} (so that the highest power of z^{-1} appearing in the numerator is less than the highest power of z^{-1} in the denominator.)

(2 pts.) d) Find the inverse Z transform, $p[n]$, of the polynomial $P(z)$, found in part (c).

(5 pts.) e) Do a partial fraction expansion of $Y(z)$ with $Y(z)$ as defined in part (c).

f) Use your partial fraction expansion of $Y(z)$ along with a table of Z transform pairs to find a closed-form expression for $y[n]$ when:

(3 pts.) i. the ROC for $Y(z)$ is: $|z| > 1$.

(3 pts.) ii. the ROC for $Y(z)$ is: $|z| < \frac{1}{2}$.

(3 pts.) iii. the ROC for $Y(z)$ is: $\frac{1}{2} < |z| < 1$.

(3 pts.) g) Verify that the samples of $x[n]$ you found in part (a) are given by the sum of $p[n]$ from part (d) and $y[n]$ from part (f)-(i.). Show details of your verification calculations.

(3 pts.) h) Verify that the samples of $x[n]$ you found in part (b) are given by the sum of $p[n]$ from part (d) and $y[n]$ from part (f)-(ii.). Show details of your verification calculations.

3. Consider the difference equation descriptions of two LTI systems shown below:

   System 1: $y[n] = 0.2y[n-1] + x[n] - 0.3x[n-1] + 0.02x[n-2]$
   System 2: $y[n] = x[n] - 0.1x[n-1]$

(4 pts.) a) Let $H_1(z)$ and $H_2(z)$ denote the transfer functions of System 1 and System 2 respectively. Find $H_1(z)$ and $H_2(z)$.

(3 pts.) b) Two LTI systems are equivalent if they have the same transfer function. Show that System 1 is equivalent to System 2.

(5 pts.) 4. Let z_1, z_2, and z_3 denote the three roots of $z^3 - 1 = 0$. Without the use of any extra computing power (*i.e.,* a calculator, matlab, *etc.*) find z_1, z_2, and z_3. Hint: $1 = e^{j2\pi k}$
Verify your answer by expanding the product $(z - z_1)(z - z_2)(z - z_3)$ using the values you found for z_1, z_2, and z_3 to show that $(z - z_1)(z - z_2)(z - z_3) = z^3 - 1$. ***Show details of your verification.***

5. Consider the causal LTI system with impulse response: $h[n] = (1/3)^n u[n]$

(2 pts.) a) Find the system transfer function, $H(z)$. (Note this is a first-order system with one pole, in part (b), I refer to this pole as p_1)

b) Follow the procedure detailed in the steps below to find a closed-form expression for the system's z.s. (zero-state) response when: $x[n] = (1/2)^n(\cos(\pi n/3))u[n]$

(2 pts.) i. Find $X(z)$. You are welcome to use a Z-transform table; however, you should always keep in mind that the derivation of $X(z)$ provides you with an easy method for identifying the poles of $X(z)$ and hence the factors of its denominator.

(2 pts.) ii. Find $Y_{zs}(z)$ as the product of $H(z)$ and $X(z)$.

(5 pts.) iii. Note that $X(z)$ has complex-valued poles (p_2 and p_2^*); a simple inspection of $x[n]$ should allow you to identify the magnitude and phase of p_2. Since the Z-transform of the z.s. response is the product of $H(z)$ and $X(z)$, $Y_{zs}(z)$ will have three poles: p_1 (as identifed in part (a)), and the pair p_2 and p_2^*. Do a partial fraction expansion of $Y_{zs}(z)/z$ to find the constants: A_1, A_2 such that:

$$\frac{Y_{zs}(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_2^*}{z - p_2^*} \Rightarrow Y_{zs}(z) = \frac{A_1 z}{z - p_1} + \frac{A_2 z}{z - p_2} + \frac{A_2^* z}{z - p_2^*}$$

A_1 and A_2 are both easily found using the Heaviside cover-up method. If you would like to use matlab (not necessary) to help evaluate the expression you find for A_2, you may do this provided you specify the expression you found for A_2, state the matlab command you used to evaluate A_2 and state the value that matlab returned.

(4 pts.) iv. Express both p_2 and A_2 in polar form. (This will be useful for part *v.*)

(4 pts.) v. Find $y_{zs}[n]$ as the inverse Z-transform of $Y_{zs}(z)$. note: since this is a real system with a real-valued input, $y_{zs}[n]$ should also be real-valued. Please combine the iZTs of the last two terms of the PFE, of part (*iii.*) into a single cosine. (see notes on D2L regarding inversion of a ZT with a pair of complex-conjugate poles)