Intro to the Z-transform

The Z transform plays the same role in the analysis of discrete-time systems as the Laplace transform plays for continuous-time systems.

The Z transform of a discrete-time signal, $x[n]$, is denoted by $X(z)$ or $Z\{x[n]\}$ and is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Note that the Z transform of a signal may not exists for certain values of the complex-variable z. The region of the z-plane for which $X(z)$ exists is called the region of convergence (ROC) for $X(z)$.

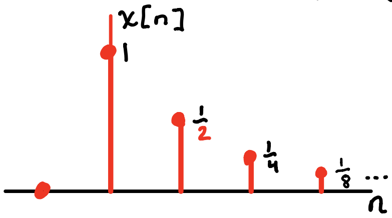
It's important to understand that two different sequences can have the same Z-transform with different ROC's.

In order to uniquely recover $x[n]$ from its Z transform, we must also know the ROC for $X(z)$.

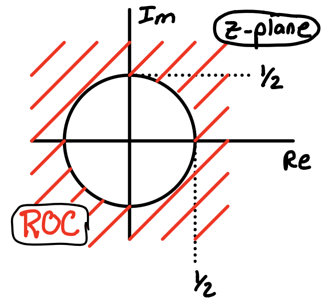
Example 1 Given $x[n] = \left(\frac{1}{2}\right)^n u[n]$, find $X(z)$ and its ROC

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=} \left(\tfrac{1}{2}\right)^n z^{-n} = \sum_{n=} \left(\tfrac{1}{2} z^{-1}\right)^n =$$

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{Z.T.} X(z) =$$



In general, the ROC for the Z-transform of a causal signal can be described by $|z| > r$.

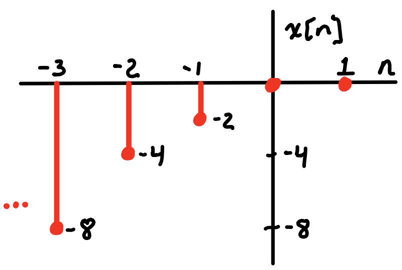


Example 2 Given $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$ find $X(z)$ and its ROC

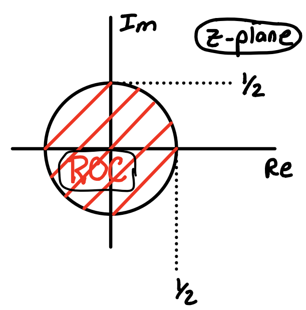
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=} -\left(\tfrac{1}{2}\right)^n z^{-n} = -\sum_{n=} \left(\tfrac{1}{2} z^{-1}\right)^n = -$$

$$=$$

$$x[n] = -\left(\tfrac{1}{2}\right)^n u[-n-1] \xleftrightarrow{Z.T.} X(z) =$$



In general, the ROC for the Z-transform of a anticausal signal can be described as $|z| > r$.



The two examples above illustrate two signals with the same Z-transform but different ROCs. The ROC must be specified in order to uniquely recover a signal from its Z-transform.

Example 3 Given that $x[n] = \left(\frac{1}{2}\right)^{|n|}$

Find $X(z)$ and ROC_x.

Solution

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^{-n}, & n < 0 \\ \left(\frac{1}{2}\right)^{n}, & n \geq 0 \end{cases}$$

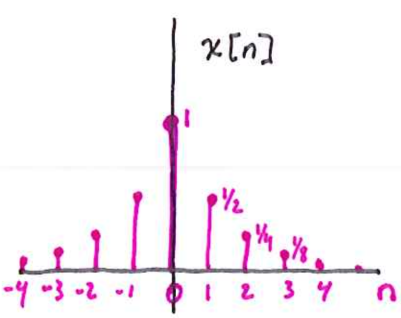
$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(2z^{-1}\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^{n}$$

$$= \frac{\left(2z^{-1}\right)^{-\infty} - 1}{1 - 2z^{-1}} + \frac{1 - \left(\frac{1}{2}z^{-1}\right)^{\infty}}{1 - \frac{1}{2}z^{-1}}$$

$$= \underbrace{\frac{-1}{1 - 2z^{-1}}}_{\substack{\text{provided } |2z^{-1}| > 1 \\ \Rightarrow |z| < 2}} + \underbrace{\frac{1}{1 - \frac{1}{2}z^{-1}}}_{\substack{\text{provided } |\frac{1}{2}z^{-1}| < 1 \\ \Rightarrow |z| > \frac{1}{2}}} \quad \boxed{\text{provided } \tfrac{1}{2} < |z| < 2}$$

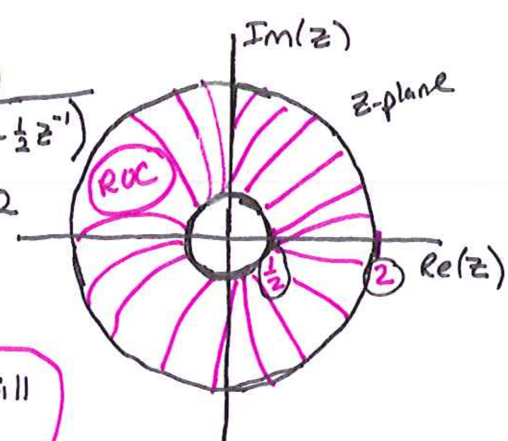
$$= \frac{-\left(1 - \frac{1}{2}z^{-1}\right) + \left(1 - 2z^{-1}\right)}{\left(1 - 2z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = \frac{-\frac{3}{2}z^{-1}}{\left(1 - 2z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \quad \frac{1}{2} < |z| < 2$$



$\longleftrightarrow$

$$X(z) = \frac{-\frac{3}{2}(z^{-1})}{\left(1 - 2z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

ROC_x: $\frac{1}{2} < |z| < 2$



$\boxed{\text{The ROC for infinite-length two sided sequences will be annular (ring-shaped).}}$

The Z-transform and ROC of Finite Duration Signals

Exercise:

Let $x_1[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3\}$, $x_2[n] = \{1 \quad 2 \quad \underset{\uparrow}{3}\}$, and $x_3[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$

a) Find the Z-transform and ROC for each of the signals above.

(i) $x_1[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3\} \Rightarrow X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n}$

$$\Rightarrow X_1(z) =$$

$$ROC_1 = \{ z \in \mathbb{C} : \qquad \qquad \}$$

(ii) $x_2[n] = \{1 \quad 2 \quad \underset{\uparrow}{3}\} \Rightarrow X_2(z) =$

$$ROC_2 = \{ z \in \mathbb{C} : \qquad \qquad \}$$

(iii) $x_3[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\} \Rightarrow X_3(z) =$

$$ROC_3 = \{ z \in \mathbb{C} : \qquad \qquad \}$$

b) For each of the signals above, determine whether or not the signal is causal, anticausal, or neither (i.e., two-sided)?
Does the ROC of each signal have the characteristics we would expect given its causal, anticausal, or two-sided classification?

c) Find the relationship between $x_3[n]$ and $x_2[\cdot]$, between $X_3(z)$ and $X_2(z)$?

$x_3[n] = x_2[\qquad]$ $\qquad\qquad$ $X_3(z) =$

d) Draw a block diagram to illustrate the relationship between $x_3[n]$ and $x_2[\cdot]$

Time Shift Property of the Z-transform

Let $g[n] = x[n-k]$

If $x[n] \longleftrightarrow X(z)$ then $G(z) \longleftrightarrow z^{-k} X(z)$

Furthermore ROC_G will be the same as ROC_x except possibly at $z=0$ and $z=\infty$.

Examples

$$x_3[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\} \longleftrightarrow X_3(z) = z + 2 + 3z^{-1}, \overbrace{0<|z|<\infty}^{ROC_3}$$

$$x_1[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3\} \longleftrightarrow X_1(z) = 1 + 2z^{-1} + 3z^{-2}, \overbrace{|z|>0}^{ROC_1}$$

$$x_1[n] = x_3[n-1] \longleftrightarrow X_1(z) = z^{-1} X_3(z), \quad ROC_1 = ROC_3 + \{\infty\}$$

$$x_2[n] = \{1 \quad 2 \quad \underset{\uparrow}{3}\} \longleftrightarrow X_2(z) = z^2 + 2z + 3, \overbrace{|z|<\infty}^{ROC_2}$$

$$x_2[n] = x_3[n+1] \longleftrightarrow X_2(z) = z X_3(z), \quad ROC_2 = ROC_3 + \{0\}$$

Proof of time-shift property:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-k] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+k)}$$

let $m = n-k \Rightarrow n = m+k$

$n=-\infty \Rightarrow m=-\infty$

$n=+\infty \Rightarrow m=+\infty$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-k} X(z)$$

The following z-transform pair can be used in conjunction with z-transform properties to find many many other z-transform pairs.

$$\boxed{x[n] = a^n u[n] \xleftrightarrow{Z.T.} X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}}, \quad \boxed{|z| > |a|}$$

Linearity Property of the Z-transform

If $x_1[n] \longleftrightarrow X_1(z)$ with ROC_1

and $x_2[n] \longleftrightarrow X_2(z)$ with ROC_2

then $x_3[n] = a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow X_3(z) = a_1 X_1(z) + a_2 X_2(z)$

with $ROC_3 \supseteq ROC_1 \cap ROC_2$

Note: in most cases, you will find that $ROC_3 = ROC_1 \cap ROC_2$. However, in the case that a linear combination of $X_1(z)$ and $X_2(z)$ results in the cancellation of a pole, then it is possible that ROC_3 will be larger than the intersection of ROC_1 and ROC_2. This will be illustrated when we do example 1 below.

CAUTION: It is not always obvious that a pole cancellation has occured. Any time that a linear combination of two infinite length sequences results in a finite-length sequence, you should anticipate the occurence of a pole cancellation. As demonstrated by previous examples, the ROC of a finite-duration sequence is easily determined by its causality.

Example 1: Use the Z-transform of $a^n u[n]$ together with the linearity and time-shift properties of the Z.T. to find the Z-transform of $\delta[n] = u[n] - u[n-1]$. Be sure to specify the ROC.

Example 2: Use the Z-transform of $a^n u[n]$ together with the linearity property of the Z.T. to find the Z-transform of:

$$x[n] = \cos(2\pi F_0 n)\, u[n].$$

Be sure to specify the ROC.

Examples illustrating the use of the linearity property

Example 1: Use the Z-transform of $a^n u[n]$ together with the linearity and time-shift properties of the Z.T. to find the Z-transform of $\delta[n] = u[n] - u[n-1]$. Be sure to specify the ROC.

As found above: $a^n u[n] \overset{Z.T.}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}$, $|z| > |a|$

Since $u[n] = a^n u[n]$ when $a =$ _____, we know that:

$$u[n] \overset{Z.T.}{\longleftrightarrow} \text{\_\_\_\_\_}, \quad |z| >$$

By the time-shift property, we know:

$$u[n-1] \overset{Z.T.}{\longleftrightarrow} \text{\_\_\_\_\_}, \quad |z| >$$

By the linearity property of the Z.T., we find:

$$\delta[n] = u[n] - u[n-1] \overset{Z.T.}{\longleftrightarrow} \text{\_\_\_\_\_} - \text{\_\_\_\_\_} =$$

and $ROC_\delta \supseteq \{z \in \mathbb{C} : |z| > \quad \} \cap \{z \in \mathbb{C} : |z| > \quad \} = \{z \in \mathbb{C} : |z| > \quad \}$

Check: can also find $Z\{\delta[n]\}$ directly from the Z.T. sum.

$$Z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \qquad ROC_\delta = \{ z \in \mathbb{C} : \qquad \}$$