From pole-zero plots to frequency response

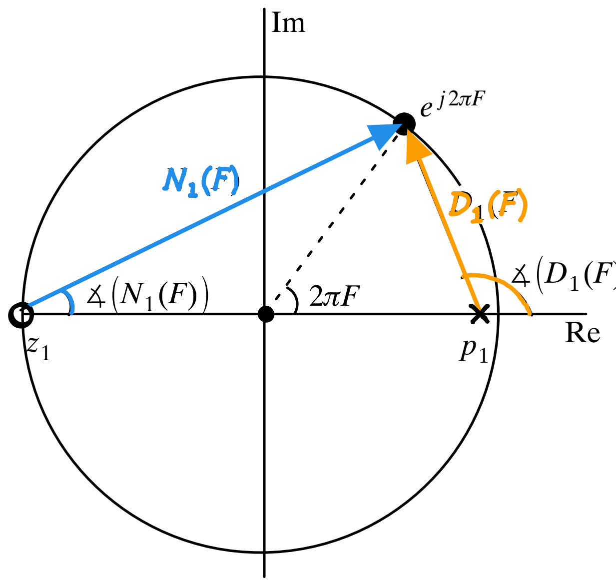
Example: Sketch the magnitude and phase of the Frequency Response Function, $H_{\mathrm{DTFT}}(F)$, for a system with transfer function:

$$H_z(z) = \frac{\frac{1}{20}(z+1)}{z-0.9} = \frac{\frac{1}{20}\big(z-(-1)\big)}{z-0.9} = \frac{\frac{1}{20}\big(z-z_1\big)}{z-p_1}$$

Note there is one zero at $z = z_1 = -1$ and one pole at $z = p_1 = 0.9$.

Solution: The frequency response function is given by:

$$H_{\mathrm{DTFT}}(F) = H_z(z)\big|_{z=e^{j2\pi F}} = \frac{1}{20}\left[\frac{e^{j2\pi F} - z_1}{e^{j2\pi F} - p_1}\right] = \frac{1}{20}\left[\frac{N_1(F)}{D_1(F)}\right]$$



$N_1(F) = e^{j2\pi F} - z_1$

$\Rightarrow N_1(F)$ is the vector from z_1 to $e^{j2\pi F}$

$D_1(F) = e^{j2\pi F} - p_1$

$\Rightarrow D_1(F)$ is the vector from p_1 to $e^{j2\pi F}$

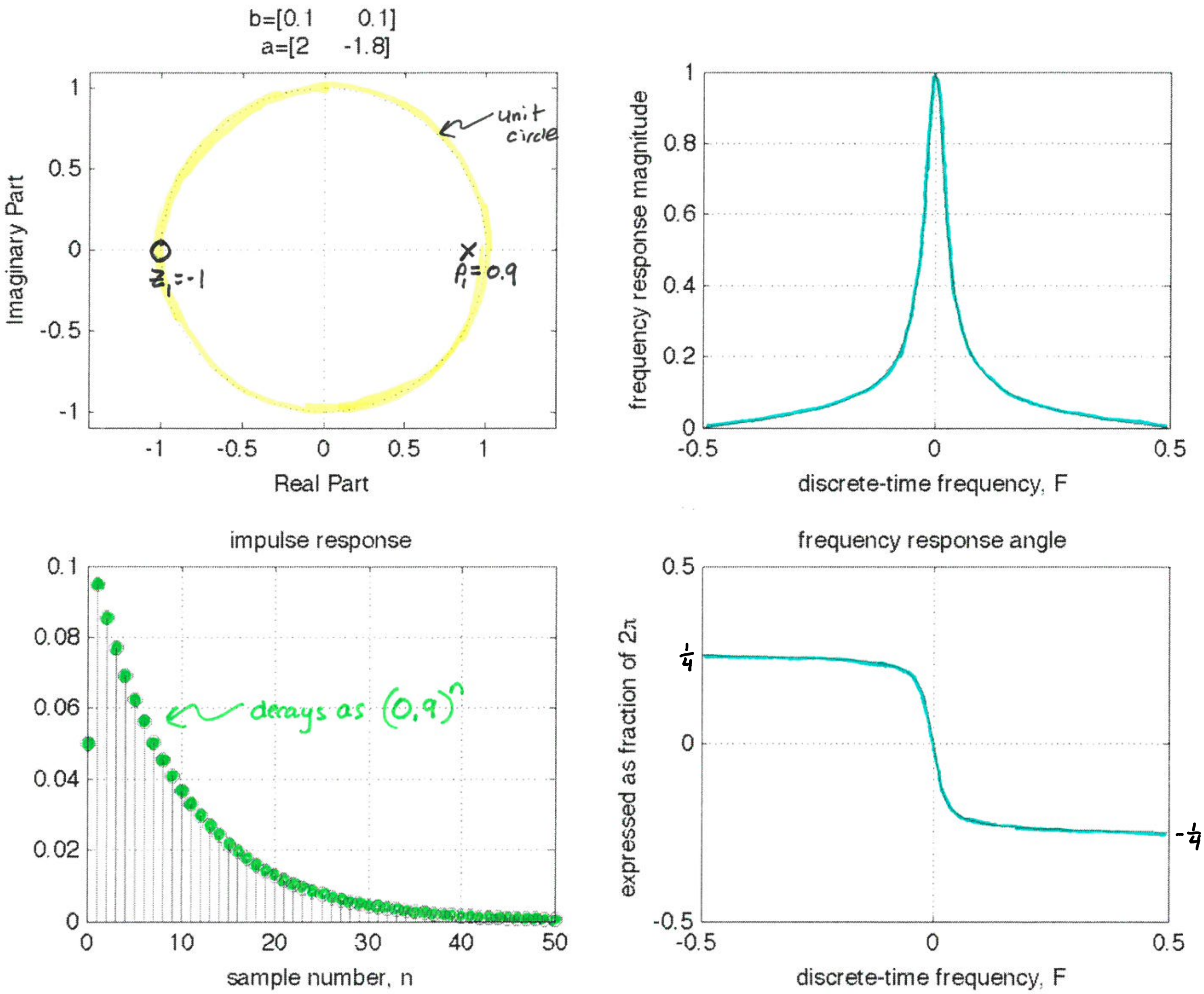
$$H_{\mathrm{DTFT}}(F) = \frac{1}{20}\left(\frac{N_1(F)}{D_1(F)}\right) \Rightarrow \begin{cases} \left|H_{\mathrm{DTFT}}(F)\right| = \frac{1}{20}\left(\frac{|N_1(F)|}{|D_1(F)|}\right) \\ \measuredangle\big(H_{\mathrm{DTFT}}(F)\big) = \measuredangle\big(N_1(F)\big) - \measuredangle\big(D_1(F)\big) \end{cases}$$

part 2a: $r = 0.9$

$$H_z(z) = \frac{0.1}{2}\frac{(1+z^{-1})}{(1-0.9z^{-1})} = \frac{1}{20}\left(\frac{z-(-1)}{z-0.9}\right)$$

dc gain: $H_{DTFT}(0) = H_z(1) = \frac{1}{20}\left(\frac{1+1}{1-0.9}\right) = \frac{1}{20}\left(\frac{2}{1/10}\right) = 1$

high-freq. gain: $H_{DTFT}(\frac{1}{2}) = H_z(-1) = 0$



$$\boxed{\begin{aligned} H_z(z) &= \frac{1}{20}\left(\frac{1+z^{-1}}{1-0.9z^{-1}}\right) \\ &= \frac{1}{20}\left(\frac{1}{1-0.9z^{-1}} + \frac{z^{-1}}{1-0.9z^{-1}}\right) \end{aligned}}$$

$\xRightarrow{\text{IVT}}$ $h[0] = \lim_{z\to\infty} H_z(z) = \frac{1}{20}(1) = \frac{1}{20}$

$\Downarrow$ IZT

$$h[n] = \frac{1}{20}\left((.9)^n u[n] + (0.9)^{n-1} u[n-1]\right) = (0.9)^n \left(\frac{1}{20}\right)\left[u[n] + (.9)^{-1} u[n-1]\right]$$

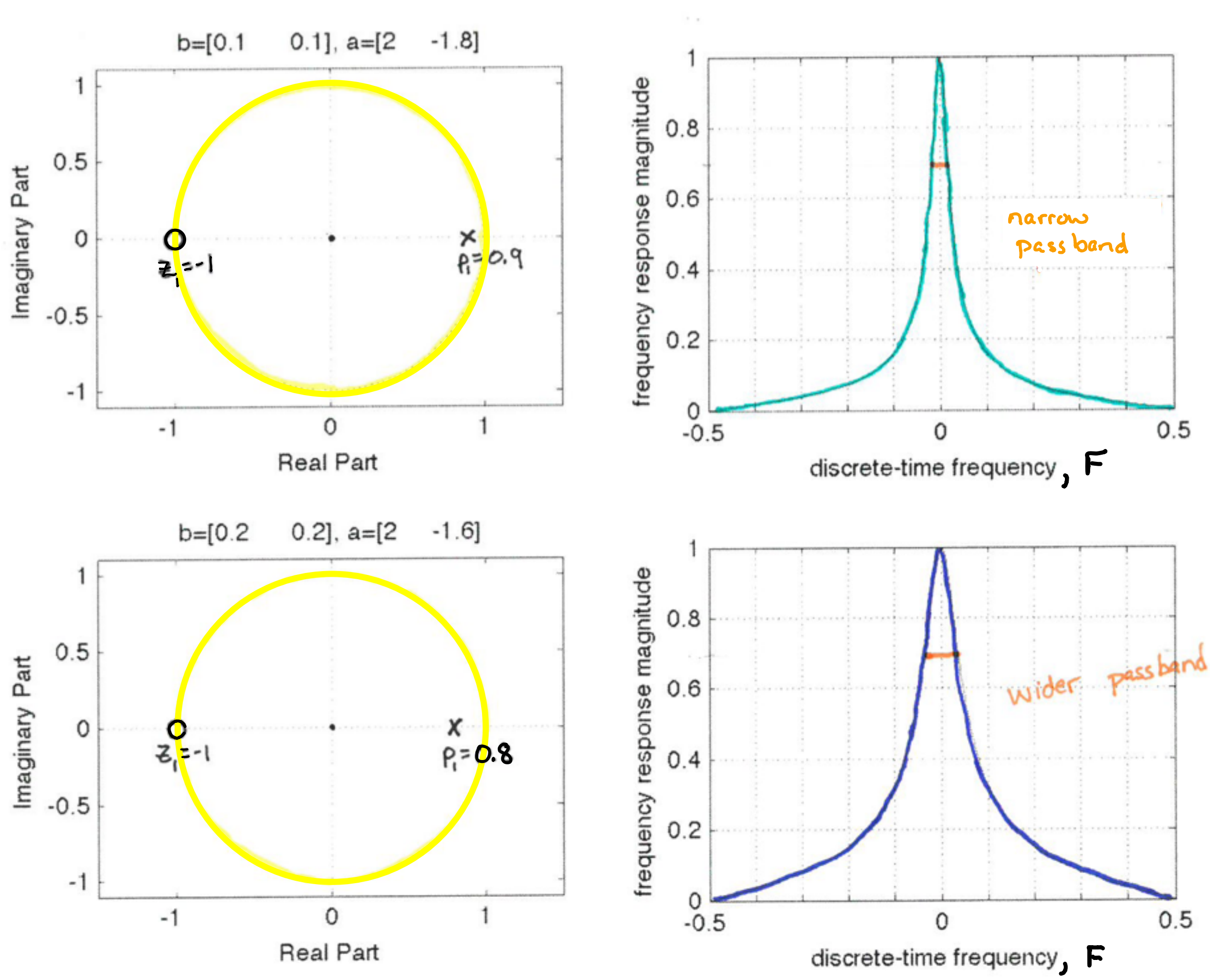
$$= \begin{cases} (0.9)^n \left(\frac{1}{20}\right), & n = 0 \\ (0.9)^n \left(\frac{1}{20}\right)\left[1 + \frac{10}{9}\right] = (0.9)^n \left(\frac{1}{20}\right)\left(\frac{19}{9}\right), & n \geq 1 \end{cases}$$

Question

What changes will you observe in the frequency response and impulse response of the LPF in part 2(a) of Lab 5 when the filter's pole is moved from $z=0.9$ to $z=0.8$?

- will the 3 dB cutoff frequency increase or decrease?
- will the effective duration of the impulse response increase or decrease?

Comparison of the frequency response magnitude of two lowpass filters: one with pole at $z=0.9$, the other with pole at $z=0.8$



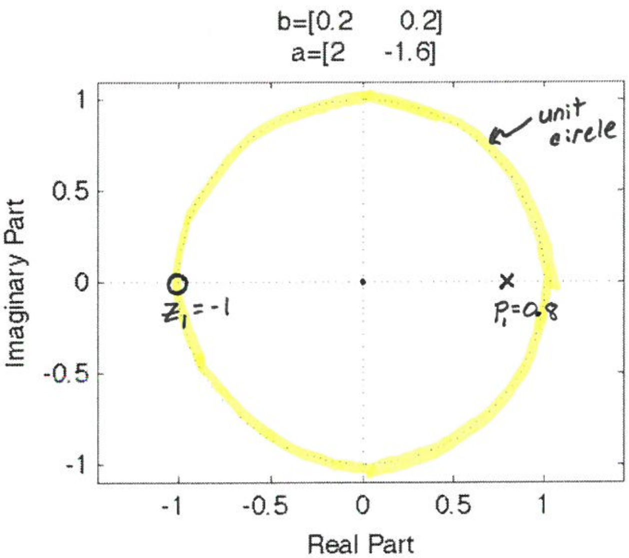
part 2a: $r = 0.8$

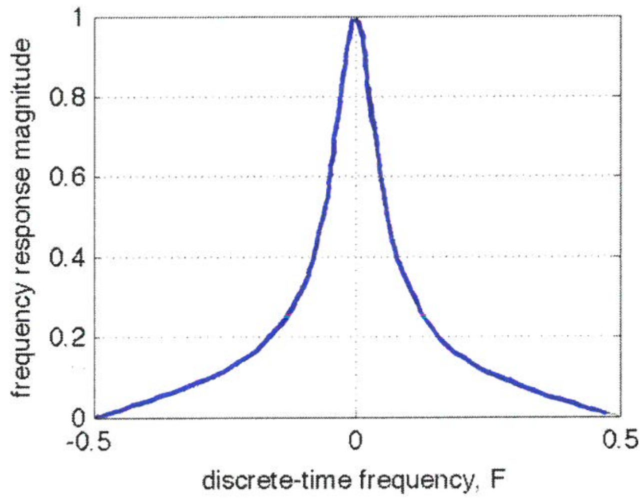
$$H_z(z) = \frac{0.2 + 0.2z^{-1}}{2 - 1.6z^{-1}} = \frac{0.2}{2}\left(\frac{1+z^{-1}}{1-0.8z^{-1}}\right) = \frac{1}{10}\left(\frac{z+1}{z-0.8}\right)$$

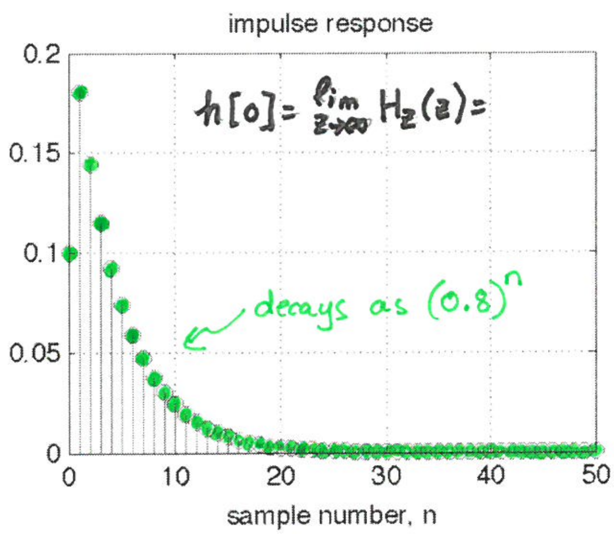
$$H_{DTFT}(0) = H_z(1) = \frac{1}{10}\left(\frac{2}{2/10}\right) = 1$$

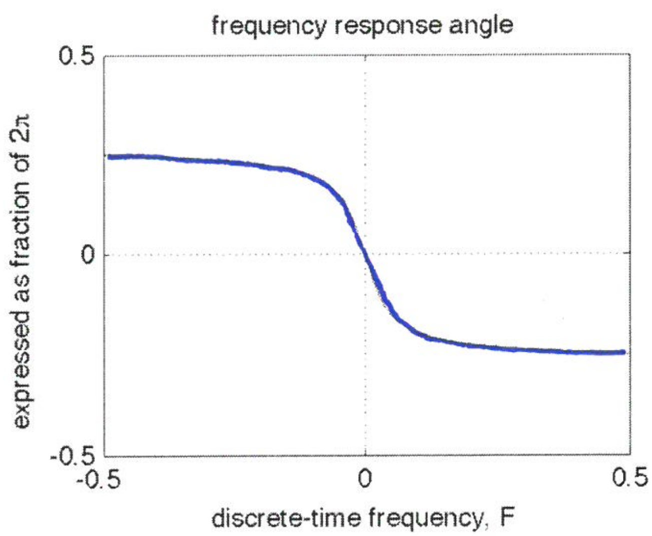
$$H_{DTFT}\left(\tfrac{1}{2}\right) = H_z(-1) = 0$$

note: in addition to moving the pole from 0.9 to 0.8, I also changed the value of b_0 from $1/20$ to $1/10$ so as to maintain a dc gain of 1.









Question

What kind of filter results if the locations of the poles and zeros of the LPF in exercise 2a are rotated by 180° with respect to their current locations?

$$H_z^{(a)}(z) = \frac{1}{20}\left(\frac{z+1}{z-0.9}\right)$$

Solution

Rotating the poles and zeros of the LPF above, by 180°, results in a new filter with transfer function $H_z^{(b)}(z)$ below.

$$H_z^{(b)}(z) = \frac{1}{20}\left(\frac{z+1e^{j\pi}}{z-0.9e^{j\pi}}\right)$$

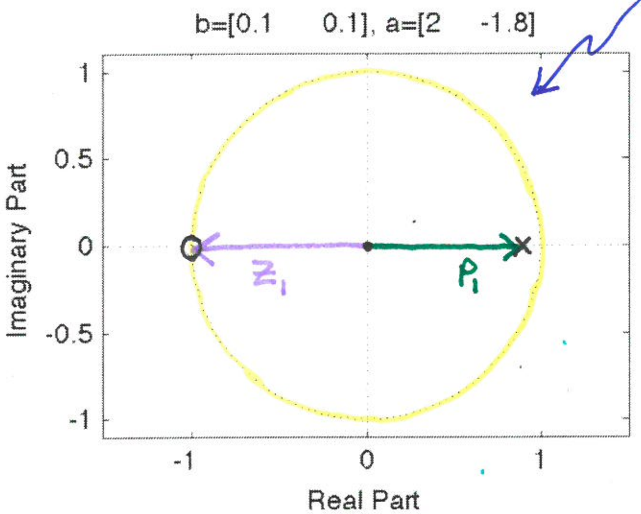
$$\Rightarrow H_{DTFT}^{(b)}(F) =$$

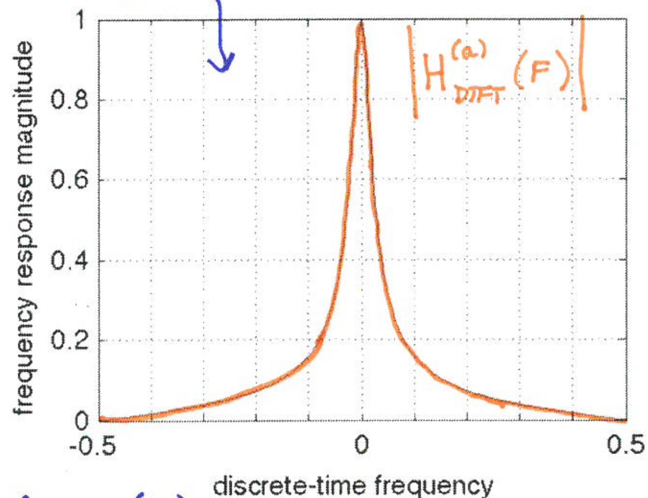
$\therefore$ Rotating the poles and zeroes by 180° (i.e., half way around the circle on which they are located) serves to shift the Frequency Response of the filter by $\Delta F = \boxed{}$ cycle/sample. What used to happen at $F=0$ now happens at $\boxed{}$.

part a vs. part b

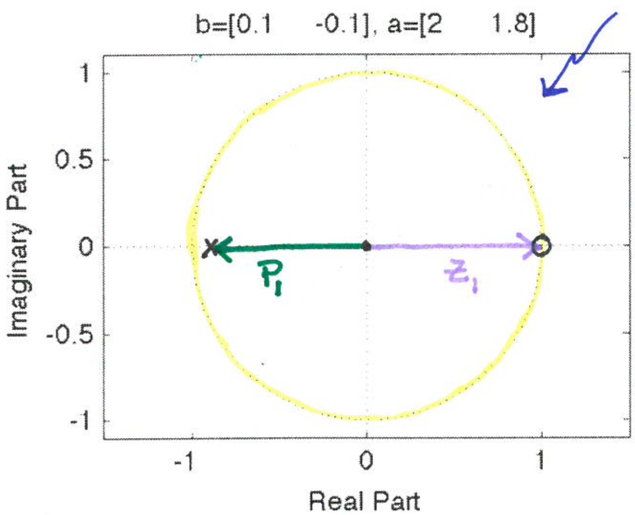
When the locations of the poles and zeroes are rotated by 180° (i.e., $\frac{1}{2}$ of the way around the unit circle), the frequency response function is shifted by $\frac{1}{2}\frac{\text{cycle}}{\text{sample}}$

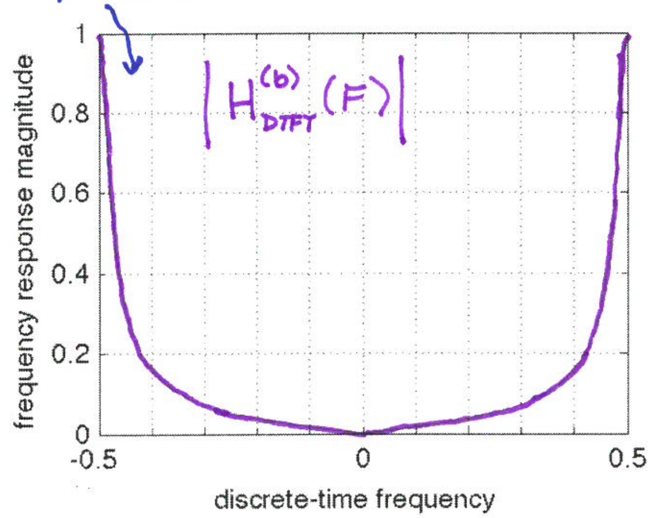
Filter of part (a)





Filter of part (b)



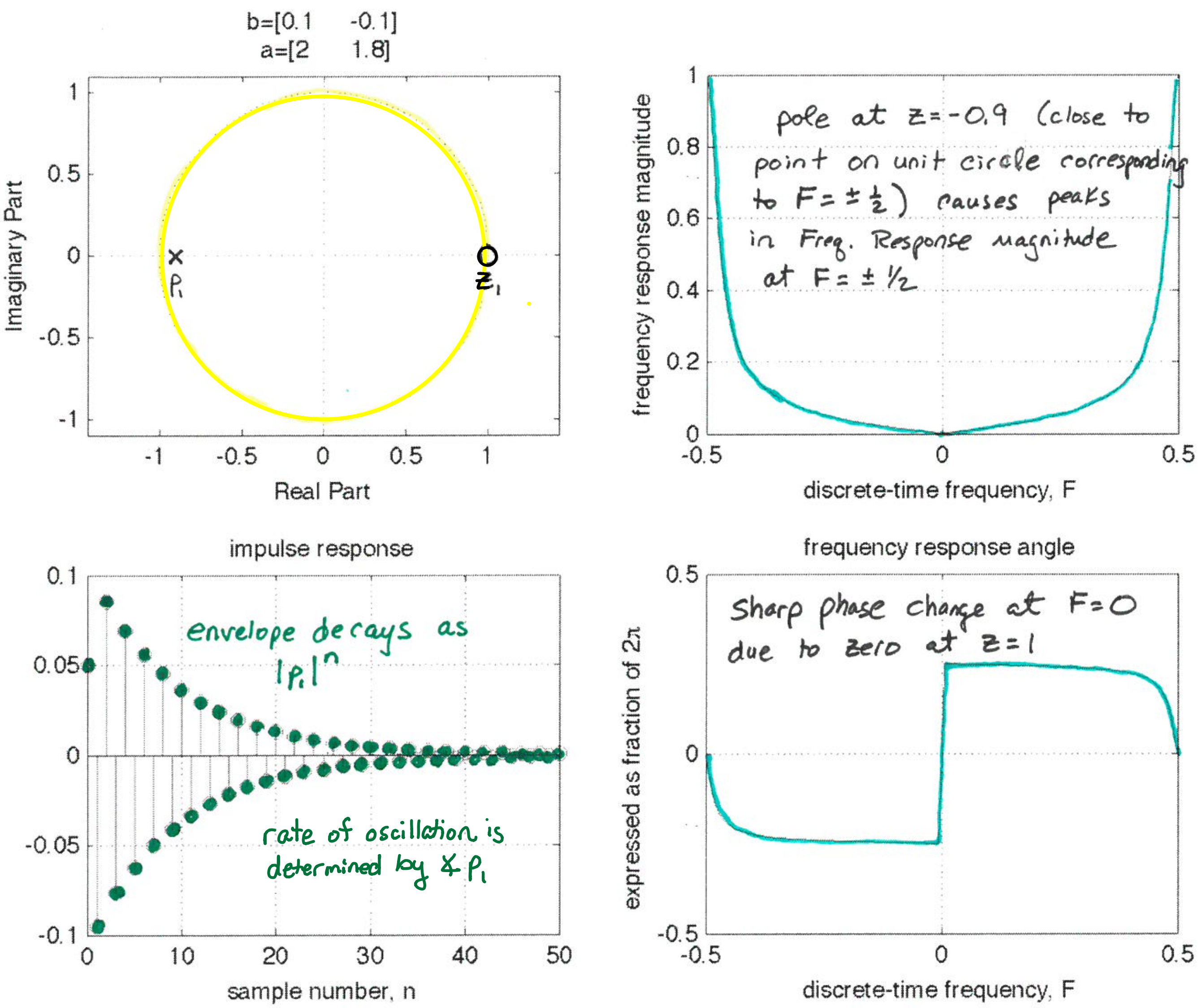


part b

$$H_z(z) = \frac{0.1\,(1 - z^{-1})}{2\,(1 + 0.9 z^{-1})} = \frac{1}{20}\left(\frac{z-1}{z+0.9}\right)$$

$$H_{DTFT}(0) = H_z(1) = 0$$

$$H_{DTFT}\left(\tfrac{1}{2}\right) = H_z(-1) = \frac{1}{20}\left(\frac{-2}{-0.1}\right) = \frac{1}{20}(20) = 1$$



$$h[n] = \frac{1}{20}\left[(-0.9)^n u[n] - (-0.9)^{n-1} u[n-1]\right] = \frac{1}{20}(-0.9)^n \left[u[n] + \frac{10}{9} u[n-1]\right]$$

$$= \begin{cases} \frac{1}{20}(-0.9)^n = \frac{1}{20}, & n = 0 \\ \frac{1}{20}\left(\frac{19}{9}\right)(-0.9)^n, & n \geq 1 \end{cases}$$

part (c)

c)
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r = 0.9;
F0 = 1/5;
p1 = r*exp(j*2*pi*F0);
G = (1-r)*sqrt(1+ r^2 - 2*r*cos(2*pi*2*F0))/(2*sin(2*pi*F0))
b = G*conv([1, -1],[1, 1]);
a = conv([1, -p1],[1, -conj(p1)]);
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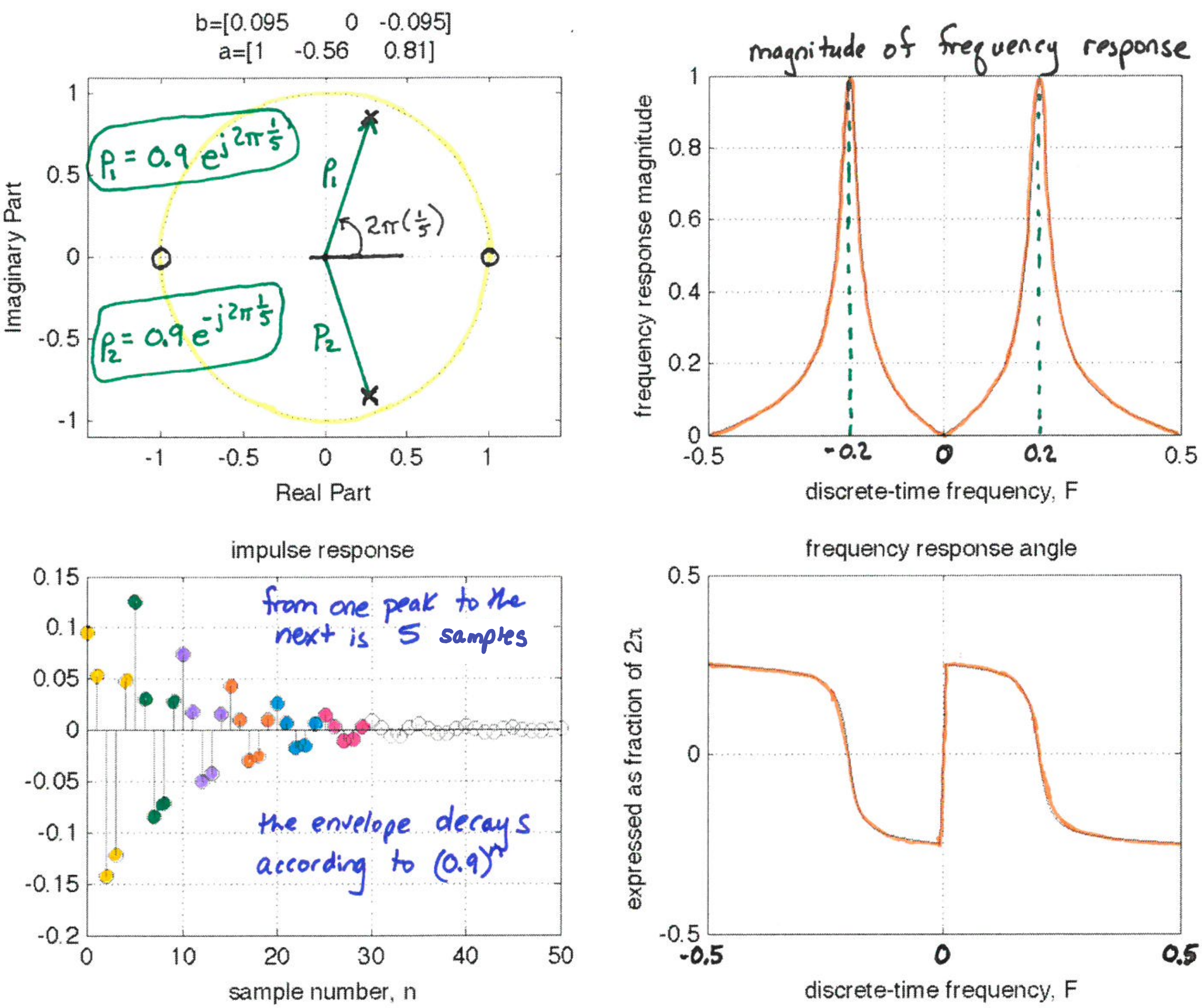
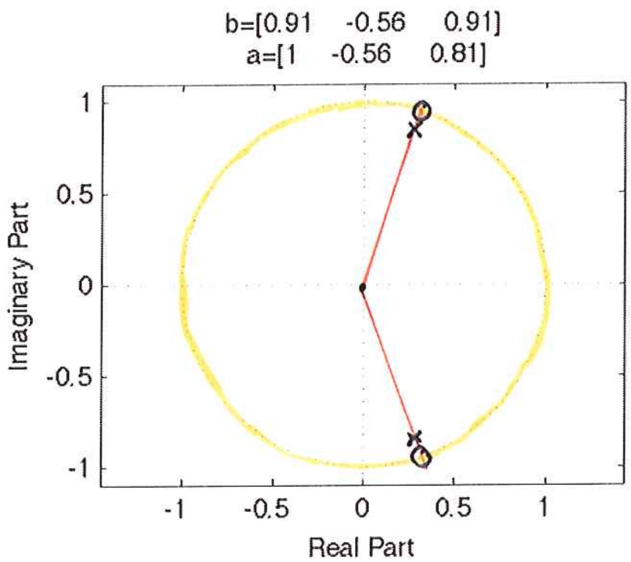
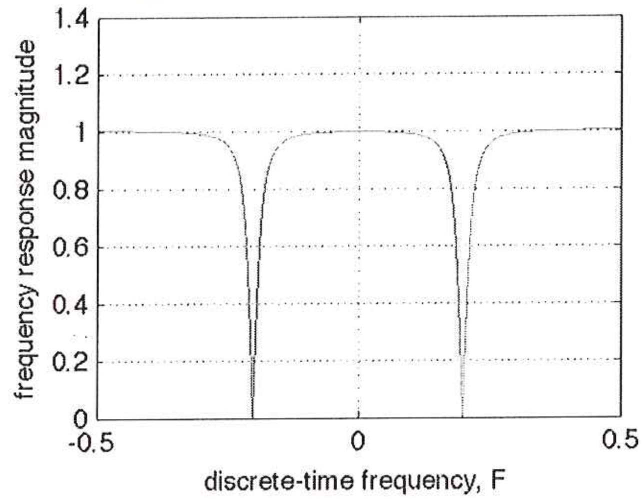


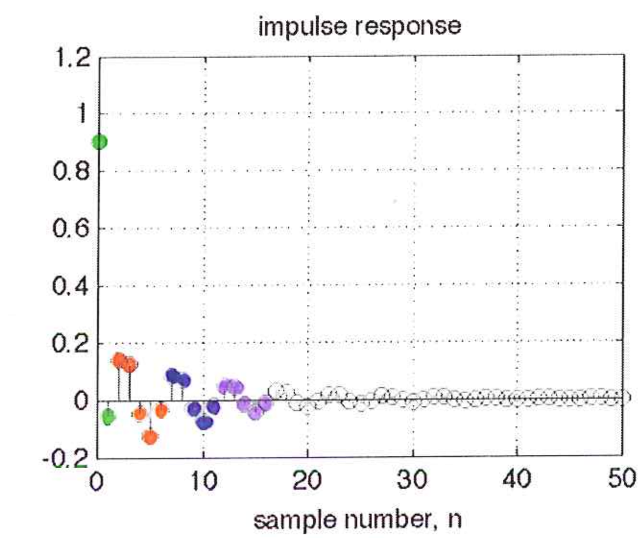
- Note how the frequency associated with the peak value of the magnitude of the frequency response function compares to the location of the poles.
- How will you measure the 3 dB passband width of the filter?
- If the radius of the poles is decreased from r=0.9 to r=0.8, what will happen to the 3dB passband width of the filter? Will it increase or decrease?
- Explain how the nature of the impulse response is determined by the pole locations? How does the value of r influence the impulse response? What about the value of Fo?

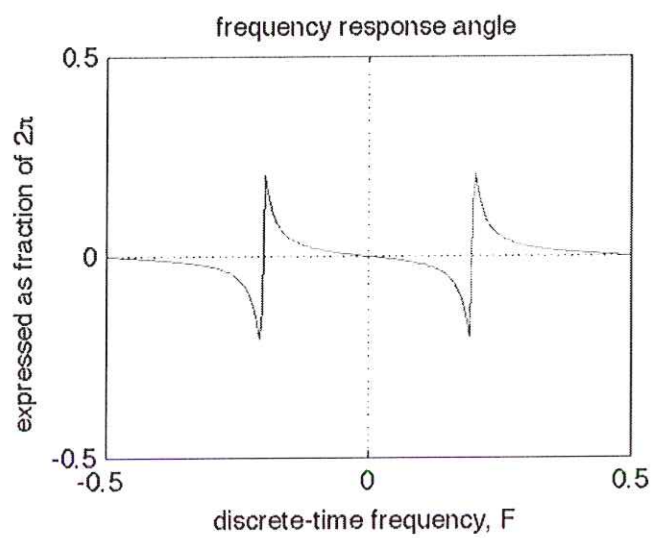
part (d)

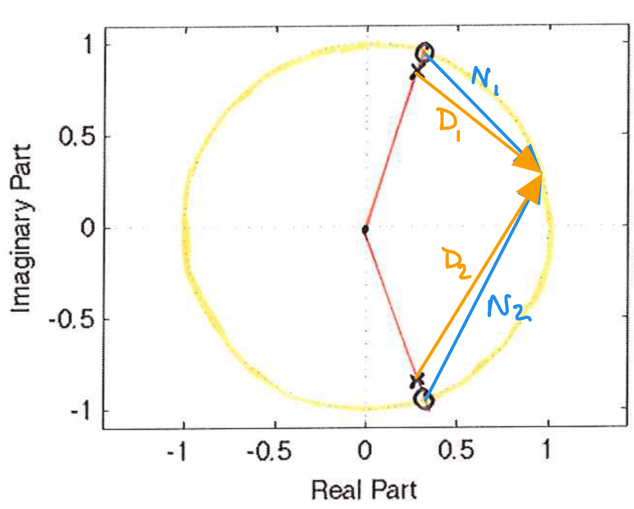
The poles of the filter in part (d), shown below, are at the same location as those of the filter in part (c). However, the zeros have been moved from 1 and -1 to their new locations on the unit circle very close to the poles. Whereas the filter of part (c) has peaks in its frequency response at F=+-1/5, the filter of part (d) now has notches at these same frequencies.
Explain how each pole-zero pair works to achieve a gain of 1 for all frequencies outside of the notch area.
How will you measure the width of the notch?











note how the pole-zero pairs work to achieve unit magnitude at frequencies outside of the notch area.