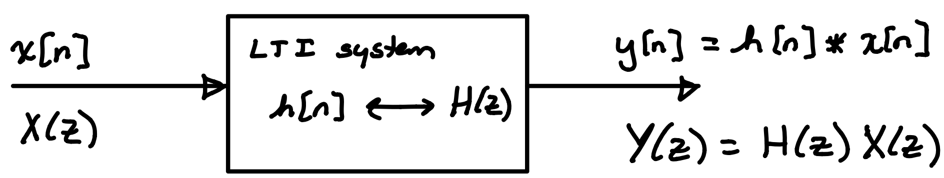
In a previous lecture we have seen that it is easy to go back and forth between the LCCDE of an LTI system and the transfer function of the system.

$$\text{LCCDE:} \quad \sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$\Rightarrow \quad \sum_{k=0}^{N} a_k\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{\sum_{k=0}^{N} a_k\, z^{-k}}$$

We also saw that the Z-transform of the system's response can be found as the product of the system's transfer function and the Z-transform of the system's input.



Now that we know how to find the inverse Z-transform, we can find $y[n] = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{H(z)\,X(z)\}$

Example 1

Use the Z-transform approach to find the step response, $y_{step}[n]$, of the causal LTI system whose transfer function is: $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$.

Solution Recall that the step response is the zero-state response of the system when $x[n] = u[n]$.

$$Y_{step}(z) = H(z)\,U(z) =$$

Example 2 Find the response of the causal LTI system whose transfer function is $H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$ when $x[n] = u[n]$ and $y[-1] = y_{-1}$.

1ST Solution (uses previous methods to find z.i. response)

In Example 1, we found the step response of this same system. The step response is, by definition, the zero-state response of the system when the input is $x[n] = u[n]$, thus we need only find the zero-input response.

$$y[n] = y_{zs}[n] + y_{zi}[n] \quad \text{where} \quad y_{zs}[n] = y_{step}[n]$$

$$\boxed{\Rightarrow y[n] = \overbrace{2 - \left(\tfrac{1}{2}\right)^n}^{y_{zs}[n]} + \overbrace{\tfrac{1}{2} y_{-1} \left(\tfrac{1}{2}\right)^n}^{y_{zi}[n]}, \quad n \geq 0}$$

see example 1   see work below

The zero-input response is a solution to the homogeneous difference eqn.

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} \Rightarrow Y(z) - \tfrac{1}{2}z^{-1}Y(z) = X(z)$$

LCCDE: $y[n] - \frac{1}{2}y[n-1] = x[n]$

char. eqn: $\lambda - \frac{1}{2} = 0 \Rightarrow \lambda = \frac{1}{2}$

homog. soln: $y_h[n] = C\left(\frac{1}{2}\right)^n \Rightarrow$ closed-form soln for $y_{zi}[n]$ $\quad y_{zi}[n] = C\left(\frac{1}{2}\right)^n$

z.i. diff. eqn: $y_{zi}[n] = \frac{1}{2}y_{zi}[n-1] \Rightarrow y_{zi}[0] = \frac{1}{2}y_{zi}[-1] = \frac{1}{2}y_{-1}$

From closed-form: $y_{zi}[0] = C$
From z.i. diff. eqn: $y_{zi}[0] = \frac{1}{2}y_{-1}$

$$\Rightarrow C = \tfrac{1}{2}y_{-1} \Rightarrow \boxed{y_{zi}[n] = \tfrac{1}{2}y_{-1}\left(\tfrac{1}{2}\right)^n}$$

can find z.i. response using time-domain method

Alternatively, we can use the one-sided Z-transform to find both the zero-state and zero-input solutions.

Assuming $H(z)$ to be the Z-transform of a causal LTI system and assuming $X(z)$ to be the Z-transform of the system's input for $n \geq 0$, we know that $Y(z) = H(z)X(z)$ will be the Z-transform of the zero-state response; in particular, $Y(z)$ will not include the Z-transform of the zero-input response ($H(z)$ does not include information about initial conditions).

To find the complete response using a Z-transform approach, we must use the one-sided Z-transform.

The one-sided Z-transform of $x[n]$ is denoted by $X^+(z)$ and is defined as:

$$X^+(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Examples

1. Let $x[n] = \{1 \quad 2 \quad \underset{\uparrow}{3} \quad 4 \quad 5\}$. Find $X^+(z)$

2. Let $g[n] = x[n-1]$. Find $G^+(z)$. Express $G^+(z)$ in terms of $X^+(z)$.

3. Let $h[n] = x[n-2]$. Find $H^+(z)$. Express $H^+(z)$ in terms of $X^+(z)$.

Time-delay property of the one-sided Z-transform

If $q[n] = x[n-k]$ where $k > 0$

then $Q^+(z) = \underbrace{x[-k]}_{m=-k} + \underbrace{z^{-1}x[-(k-1)]}_{m=-k+1} + \cdots + \underbrace{z^{-(k-1)}x[-1]}_{m=-1} + z^{-k}X^+(z)$

$$= \sum_{m=-k}^{-1} x[m]\, z^{-(k+m)} + z^{-k} X^+(z)$$

$$= z^{-k}\left(\sum_{m=-k}^{-1} x[m]\, z^{-m} + X^+(z) \right)$$

Procedure for using the one-sided Z-transform to find the output of a causal system for a given input, $x[n]$, $n>0$ and given ICs.

1) If necessary use $H(z)$ to determine LCCDE

2) If necessary, use initial state values of realization to determine the IC's for the LCCDE.

3) Take one-sided Z-transform of LCCDE and solve for $Y^+(z)$.

   [if desired, you should be able to recognize and separate the z.i. portion of $Y^+(z)$ (those terms that depend on ICs) from the zero-state portion of $Y^+(z)$ (those terms that include $X^+(z)$).]

4) Find $y[n]$, $n \ge 0$, as the inverse Z-transform of $Y^+(z)$.

Example 2

Find the response, $y[n]$, of the causal LTI system whose transfer function is $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ when $x[n] = u[n]$ and $y[-1] = y_{-1}$.

Solution - Approach #2

Find the system's LCCDE:

$$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

$$\Rightarrow y[n] - \frac{1}{2}y[n-1] = x[n]$$

Take one-sided Z-transform of LCCDE: