Lab 1: Understanding Discrete-Time Frequency

PURPOSE: To become more comfortable with the concept of discrete-time frequency as well as the relationship between the discrete-time frequency, F, the continuous-time frequency, f, and the sampling rate, f_s. The student will also observe that aliasing occurs whenever a continuous-time frequency is sampled at a rate less than the Nyquist frequency and that two discrete-time frequencies are equivalent if the difference between them is an integer.

BACKGROUND INFORMATION REGARDING CT_DT SCRIPT:

For the purpose of this lab, I have written a matlab function called **ct_dt**. It is contained in the file **ct_dt.m** which you may download from the Labs>Lab1 folder on the Desire2Learn home page for this course. Once copied to your account, you will be free to make any changes you like to the function. The file ct_dt.m is included as an appendix to this document.

When executed, the function **ct_dt** will generate a plot containing **nc** cycles of the continuous-time cosine, $x_a(t)$, with amplitude **A**, frequency **`f0`** Hz, and phase **`PHI`** radians:

$$x_a(t) = A\cos(2\pi f_0 t + \mathrm{PHI}) \quad \text{for} \quad 0 \le t \le \mathrm{nc}/f_0 \text{ sec.}$$

Superimposed on the plot of $x_a(t)$, will be the discrete-time signal, $x[n] \equiv x_a(n/f_s)$, which is obtained by sampling $x_a(t)$ at a rate of f_s samples per second:

$$x[n] \equiv x_a\left(\frac{n}{f_s}\right) = A\cos(2\pi F_0 n + \mathrm{PHI}) \quad \text{for} \quad 0 \le n \le \frac{nc}{F_0} \quad \text{where} \quad F_0 = \frac{f_0}{f_s}$$

The function will also return the time vector, **t**, which was used to plot the c.t. cosine $x_a(t)$. Access to the vector **t** allows you to conveniently superimpose other continuous-time signals on the plot generated by **ct_dt**, which will be useful in parts 2 through 4 of the procedure.

The function is executed by typing the following at the MATLAB prompt:

```
t = ct_dt(A,f0,PHI,fs,nc,ifig);
```

where desired values are substituted for (or have been assigned to): **`A,f0,PHI,fs,`** and **`nc`**. **ifig** is an optional parameter used to include a figure number in the title of the figure.

<u>Example to illustrate the usage of ctdt:</u>

To plot 5 cycles of $x_a(t) = 2\cos(2\pi(0.5)t - \pi/4)$ together with a stem plot of the values of the signal: $x[n] \equiv x_a(n/10)$, type the following at the command line prompt in Matlab.

```
t = ct_dt(2,0.5,-pi/4,10,5);
```

Alternatively, you may type:

```
A = 2; f0 = 0.5; PHI = -pi/4; fs = 10; nc = 5;
t = ct_dt(A,f0,PHI,fs,nc);
```

The latter alternative is convenient in that, once executed, if you decide to change the sampling rate to **fs**=20 while keeping all other parameter values the same as above, you can simply type:

```
fs = 20;
```

and then select the following from your list of recently executed matlab commands:

```
t = ct_dt(A,f0,PHI,fs,nc);
```

Procedure/Exercises

1. Consider the continuous-time sinusoid: $x_a(t) = 3\sin(2\pi 50t) = 3\cos(2\pi 50t - \pi/2)$. Let $x[n]$ denote the discrete-time sinusoid which results from sampling $x_a(t)$ at a rate of f_s samples per second.

   a) Assuming $f_s = 200$ samples per second, use the function **ct_dt** to plot 6 cycles of $x_a(t)$ and the resulting samples, $x[n]$. ***Print your plot***. On your printed plot: add (by hand) a second scale for the horizontal axis to indicate values of n; write out the expressions for $x_a(t)$ and $x[n]$; and provide complete-sentence answers to the following questions.

      i. What fraction of a cycle of $x_a(t)$ lies between consecutive sampling instances? ***Express your answer as a ratio of two integers***.

      ii. What is the period of $x[n]$? Recall the period of a discrete-time signal must be an integer number of samples.

      iii. How many cycles of $x_a(t)$ must you sample to observe one period of $x[n]$? In answering this question, please note that associated with N samples is a time interval of length N/f_s seconds. Hence, if $x[n]$ is periodic with period N samples, the question is how many cycles of $x_a(t)$ are observed in N/f_s seconds.

      iv. What is the discrete-time frequency of $x[n]$?

      General discussion: How do your answers to parts (i), (ii), and (iii) relate to the discrete-time frequency identified in part (iv)?

   b) Repeat part (a) for $f_s = 120$ samples per second.

   c) Repeat part (a) for $f_s = 40$ samples per second.

   d) Of the f_s values considered above, which two resulted in the same $\{x[n]\}$? Explain.

2. Define the functions $x_a(t)$ and $y_a(t)$ as follows:

$$x_a(t) = 6\cos(2\pi 50t - \pi/3) \qquad \text{and} \qquad y_a(t) = 3\cos(2\pi 50t)$$

   The Nyquist rate for both these signals is $f_N = 100$ (samples/sec). The Sampling Theorem states that a signal must be sampled at a rate greater than the Nyquist rate in order that the signal can be uniquely recovered from its samples.

   a) What will be the discrete-time frequency of the discrete-time signal obtained by sampling either of these signals at a rate of $f_s = f_N = 100$ samples/sec.

   b) Use the function **ct_dt** to generate and plot two cycles of $x_a(t)$ along with its samples when using a sampling rate of f_N samples/sec. Superimpose on your figure, a plot of $y_a(t)$. Assuming you assigned the time vector returned by **ct_dt** to the variable **`t`**, you can superimpose a plot of $y_a(t)$ by typing the following two lines after executing the function ct_dt:

```
y = 3*cos(2*pi*50*t);
plot(t,y,':')
```

   Print a copy of your plot. On your hard copy, add labels to clearly identify the waveforms: $x_a(t)$ and $y_a(t)$. Add a scale for n along the horizontal axis. Write out expressions for

$x_a(t)$, $y_a(t)$, $x[n] = x_a(n/f_N)$, and $y[n] = y_a(n/f_N)$. Using complete sentences, answer the following question on the printed copy of your plot. Will you be able to uniquely recover $x_a(t)$ from its samples when using a sampling rate equal to the Nyquist rate of 100 samples/sec.? Explain. Your explanation should make reference to your plot.

3. Define the functions $x_a(t)$ and $y_a(t)$ as follows:

$$x_a(t) = \cos(2\pi 50t - \pi/2) \quad \text{and} \quad y_a(t) = \cos(2\pi(-30)t - \pi/2) = \cos(2\pi 30t + \pi/2)$$

Use the function **ct_dt** to plot 5 cycles of $x_a(t)$ and its samples, $x[n]$, when sampled at a rate of 80 samples per second. Be sure to assign the time vector returned by **ct_dt** to the variable **`t`**. Then execute the following matlab commands so as to superimpose a plot of $y_a(t)$.

```
y = cos(2*pi*30*t + pi/2);
plot(t,y,':')
```

Print the resulting plot. On your printed copy, add a scale for n along the horizontal axis and clearly label the signals $x_a(t)$ and $y_a(t)$. Write expressions for $x_a(t)$ and $y_a(t)$ on the printed copy of the plot and use complete sentences to answer the following questions.

a) Note that $x[n] \equiv x_a\left(\frac{n}{80}\right) = \cos\left(2\pi F_x n - \frac{\pi}{2}\right)$ where $F_x = \frac{5}{8}$.

   i. What fraction of a cycle of $x_a(t)$ lies between consecutive samples?

   ii. What is the period of $x[n]$?

   iii. How many cycles of $x_a(t)$ must you sample to observe one period of $x[n]$?

b) Note that $y[n] \equiv y_a\left(\frac{n}{80}\right) = \cos\left(2\pi F_y n - \frac{\pi}{2}\right)$ where $F_y = \frac{-3}{8}$.

   i. Observe from your plot how the values of $y_a(t)$ compare to those of $x_a(t)$ at the sampling instances: $t = n/80$, $n = 0, \pm 1, \pm 2, \ldots$

   ii. What fraction of a cycle of $y_a(t)$ lies between consecutive samples?

   iii. What is the period of $y[n]$?

   iv. How many cycles of $y_a(t)$ must you sample to observe one period of $y[n]$?

c) If we use an ideal reconstruction filter (based on $f_s = 80$ samples per second) to reconstruct a continuous-time signal from the samples $x[n]$ of $x_a(t)$, what signal will be produced? ***Hint***: According to the sampling theorem, there will be at most one c.t. signal which is both bandlimited to some frequency strictly less than 40 Hz. and has the values $x[n]$ at $t = n/80$. This is the signal that will be reconstructed.

4. Define the functions $x_a(t)$ and $y_a(t)$ as follows:

$$x_a(t) = \cos(2\pi 60 t) \qquad \text{and} \qquad y_a(t) = \cos(2\pi f_y\, t)$$

   a) Given that $f_s = 50$ samples/second, find a value for f_y such that $|f_y| < f_s/2$ and such that when $x_a(t)$ and $y_a(t)$ are sampled at a rate of f_s samples/second, the samples of $x_a(t)$ will be identical to those of $y_a(t)$.

   b) Verify your response to part (a) as follows:

      i. Use **ct_dt** to generate a plot of $x_a(t)$ and its samples when $f_s = 50$ samples/second

      ii. Use appropriate matlab commands to superimpose a plot of $y_a(t)$ on the plot obtained in part i. Label your plot and explain how you determined the value for f_y.

The contents of the file **ct_dt.m** are shown below:

```
function t = ct_dt(A,f0,PHI,fs,nc, ifig)
%A           amplitude of cosine;
%f0          CT frequency of cosine (cycles/sec);
%PHI         phase of cosine (radians);
%fs          sampling frequency (samples/sec.)
%nc          number of CT cycles to be displayed
%ifig        optional Figure number to use in the title of the plot
%t           time vector used to plot CT cosine
if (nargin < 5 | nargin > 6)
   error('in call to ct_dt: there should be 5 arguments')
end
if (A < 0 )
   error(['in call to ct_dt: Amplitude of cosine, A, should not be negative'])
end
if (fs<0)
   error(['in call to ct_dt: the sampling frequency, fs,should be positive'])
end
if (nc<0)
   error('in call to ct_dt: nc should be positive')
end
if (exist('ifig'))
   pFig = ['Fig. ', num2str(ifig),':  '];
else
   pFig = [''];
end
   figure, clf
Ts=1/fs; %time between samples
Tp=1/abs(f0);  %period of CT cosine (sec/cycle)
F0 = f0/fs; %DT frequency (cycles/sample)
%DT plot will display samples n=0 to n=nmax
nmax = nc/abs(F0);
%CT plot will display t=0 to t=tmax
tmax = nmax * Ts;
% define t vector for CT plots to:
%  have a length greater than or equal to 200
%  with every kth element corresponding to a sampling instant
k = ceil(200/nmax);
t=0:Ts/k:tmax;
xa = A*cos(2*pi*f0*t + PHI);
plot(t,xa);
hold on
n=0:nmax;
nTs = n*Ts;
xn = A*cos(2*pi*F0*n + PHI);
stem(nTs,xn);p0=['x_a(t)=A cos(2\pi f_0 t), '];
```

```
p1=[' A='];
p2=[', f0='];
p3=[', PHI='];
p4=[', fs='];
p5=[', nc='];
p6=[', User='];
name=getenv('USER'); % gets the users login id
title( [pFig p0 p1 num2str(A) p2 num2str(f0) p3 num2str(PHI) ...
        p4 num2str(fs) p5 num2str(nc) p6 name ] )
xlabel('t (seconds)')
```