Linearity Property of the Z-transform

If $x_1[n] \longleftrightarrow X_1(z)$ with ROC_1

and $x_2[n] \longleftrightarrow X_2(z)$ with ROC_2

then $x_3[n] = a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow X_3(z) = a_1 X_1(z) + a_2 X_2(z)$

with $ROC_3 \supseteq ROC_1 \cap ROC_2$

Note: in most cases, you will find that $ROC_3 = ROC_1 \cap ROC_2$

However, in the case that a linear combination of $X_1(z)$ and $X_2(z)$ results in the cancellation of a pole, then it is possible that ROC_3 will be larger than the intersection of ROC_1 and ROC_2. This will be illustrated when we do example 1 below.

CAUTION: It is not always obvious that a pole cancellation has occured. Any time that a linear combination of two infinite length sequences results in a finite-length sequence, you should anticipate the occurence of a pole cancellation. As demonstrated by previous examples, the ROC of a finite-duration sequence is easily determined by its causality.

Example 1: Use the Z-transform of $a^n u[n]$ together with the linearity and time-shift properties of the Z.T. to find the Z-transform of $\delta[n] = u[n] - u[n-1]$.

Be sure to specify the ROC.

Example 2: Use the Z-transform of $a^n u[n]$ together with the linearity property of the Z.T. to find the Z-transform of:

$$x[n] = \cos(2\pi F_0 n)\, u[n].$$

Be sure to specify the ROC.

Example 2 Use the Z-transform of $a^n u[n]$ together with the linearity property of the Z.T. to find the Z-transform of

$$x[n] = \cos(2\pi F_0 n)\, u[n]$$

Be sure to specify the R.O.C.

Solution - will first use Euler's formula to rewrite $x[n]$ as:

$$x[n] = \cos(2\pi F_0 n)\, u[n] =$$

Thus, allowing us to use linearity together with:

$$a^n u[n] \overset{Z.T.}{\longleftrightarrow} \frac{1}{1 - a z^{-1}} \,, \quad |z| > |a|$$

to find:

$$X(z) =$$

Multiplication by a^n in time domain

$\Rightarrow$ scaling by a^{-1} in z-domain

if $x[n] \longleftrightarrow X(z)$, $ROC_x: r_1 < |z| < r_2$

and if $g[n] = a^n x[n]$

then $g[n] \longleftrightarrow G(z) = X\left(\frac{z}{a}\right)$, $ROC_G: |a| r_1 < |z| < |a| r_2$

Proof: $$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] (a^{-1} z)^{-n}$$

Recall $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$\therefore X(a^{-1}z) = \sum_{n=-\infty}^{\infty} x[n] (a^{-1}z)^{-n}$

$$= X(a^{-1} z)$$

if ROC of $X(z)$ is: $r_1 < |z| < r_2$

then ROC of $X(a^{-1}z)$ is: $r_1 < |a^{-1}z| < r_2$

$\Rightarrow |a| r_1 < |z| < |a| r_2$

Ex. Determine the Z-transform of $a^n \cos(2\pi F_0 n) u[n]$

Previously, we found that if $x[n] = \cos(2\pi F_0 n) u[n]$

then $$X(z) = \frac{1 - \cos(2\pi F_0) z^{-1}}{1 - 2\cos(2\pi F_0) z^{-1} + z^{-2}}$$ $ROC_x: |z| > 1$

Time Reversal Property of Z-transform

If $x[n] \longleftrightarrow X(z)$, $ROC_x : r_1 < |z| < r_2$

and if $g[n] = x[-n]$

then $g[n] \longleftrightarrow G(z) = X(z^{-1})$, $ROC_G = \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Examples

Illustration of Property

Ex 1

$x[n] = \{ \underset{\uparrow}{1} \quad 2 \quad 3 \} \Rightarrow X(z) = 1 + 2z^{-1} + 3z^{-2}, \quad |z| > 0$

$g[n] = x[-n] = \{ 3 \quad 2 \quad \underset{\uparrow}{1} \} \Rightarrow G(z) = 3z^2 + 2z + 1, \quad |z| < \infty$

$= X(z^{-1})$

Ex 2

Given $u[n] \longleftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$

- Find $Z\{u[-n]\}$.
- Find $Z\{u[-n-1]\}$ (may require additional properties)

Will do Ex 2 on the next page.

Ex 2 - illustrating the time-reversal or time-negation property.

Given: $u[n] \xleftrightarrow{Z.T.} U(z) = \dfrac{1}{1-z^{-1}}, \; |z|>1$

a) Let $g[n] = u[-n]$. Find $G(z)$.

Soln: By the time-reversal property we know:

$$g[n] = u[-n] \xleftrightarrow{Z.T.} G(z) = U(z^{-1}) = \frac{1}{1-(\quad)^{-1}}, \; | \quad | > 1$$

b) Let $h[n] = u[-n-1]$. Find $H(z)$.

Soln: - will first express $h[n]$ as a time-shifted version of $g[n]$, and then apply the time-shift property of the Z.T. to find $H(z)$.

$$g[n] = u[-n] \Rightarrow g[\qquad] = u[-n-1]$$

Thus, by the time-shift property:

$$h[n] = g[\qquad] \xleftrightarrow{Z.T.} H(z) =$$

Multiplication-by-n property

If $g[n] = n\,x[n]$ and $x[n] \xleftrightarrow{\text{Z.T.}} X(z),\ r_1 < |z| < r_2$

then $g[n] \xleftrightarrow{\text{Z.T.}} G(z) = -z \frac{d}{dz} X(z),\quad ROC_G = ROC_X$

Exercises

a) Show that if $g[n] = n\,x[n]$, then $G(z) = -z \frac{d}{dz} X(z)$

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} n\,x[n]\, z^{-n}$$

(note: the solution to a similar problem was illustrated by the hint for question 3e of Ass 2)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty}$$

b) Use previously-found ZT pairs in conjunction with the multiplication-by-n property of the Z.T. to find the z-transform of $g[n] = n\,a^n u[n]$.

__Soln:__ Let $g[n] = n\,x[n]$ where $x[n] = a^n u[n]$. Then use the known Z.T. pair:

$$x[n] = a^n u[n] \xleftrightarrow{\text{Z.T.}} X(z) = \frac{1}{1-az^{-1}},\quad |z| > |a|$$

together with the multiplication-by-n property to find $G(z)$ as shown below:

$$\therefore\ G(z) = -z \frac{d}{dz} X(z) = -z \left(\frac{\qquad}{(1-az^{-1})^2} \right) =$$

$$\frac{d}{dz}\left(\frac{Hi}{Ho}\right) = \frac{Ho\, d(Hi) - Hi\, d(Ho)}{Ho\, Ho}$$

$ROC_G =$

c) Given that: $g[n] = n a^n u[n]$ and $x[n] = a^n u[n]$,

how do the poles of $G(z)$ compare to the poles of $X(z)$?

In particular: are any new pole locations introduced in $G(z)$ that were not present in $X(z)$?

What about the multiplicity of the poles?

The multiplicity of the poles in $G(z)$ is ______ the multiplicity of the poles in $X(z)$.

d) Recall that a^n is the characteristic mode of a discrete-time system with characteristic root, $\lambda = a$. What is the additional characteristic mode of a system with a repeated root of multiplicity 2 at $\lambda = a$ (i.e., with $\lambda_1 = \lambda_2 = a$)?

$\Phi_1[n] = a^n$, $\Phi_2[n] =$

We will soon see that $X(z)$ is the transfer function of a system with impulse response, $x[n]$. The poles of $X(z)$ are the characteristic roots of the system.