Lab 3: Cross Correlation

PURPOSE: To become familiar with the Matlab function **`xcorr`** as well as to investigate some properties and applications of cross correlations.

BACKGROUND: The cross-correlation, $r_{xy}[\ell]$, of two energy sequences, $x[n]$ and $y[n]$, is defined as:

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y^*[n-\ell]$$

If $x[n]$ and $y[n]$ are finite-length seqeuences, such that:

$$x[n] = \begin{cases} 0,\ n < n_1^{(x)} \\ \text{possibly nonzero},\ n_1^{(x)} \le n \le n_2^{(x)} \\ 0,\ n > n_2^{(x)} \end{cases} \quad \text{and} \quad y[n] = \begin{cases} 0,\ n < n_1^{(y)} \\ \text{possibly nonzero},\ n_1^{(y)} \le n \le n_2^{(y)} \\ 0,\ n > n_2^{(y)} \end{cases}$$

and if vectors **x** and **y** are created in Matlab so that the vector **x** contains the $L_x = n_2^{(x)} - n_1^{(x)} + 1$ values of the sequence $x[n]$ corresponding to indices $n_1^{(x)} \le n \le n_2^{(x)}$ and the vector **y** contains the $L_y = n_2^{(y)} - n_1^{(y)} + 1$ values of the sequence $y[n]$ corresponding to indices $n_1^{(y)} \le n \le n_2^{(y)}$, then the cross-correlation sequence, $r_{xy}[\ell]$, will also be finite-length and can be computed in matlab using the command:

```
rxy = xcorr(x,y)
```

When created as described above, the vector **rxy** will contain values of $r_{xy}[\ell]$ corresponding to lags of:

$$n_1^{(x)} - n_1^{(y)} - M \le \ell \le n_1^{(x)} - n_1^{(y)} + M \tag{1}$$

$$\text{where: } M = \max(L_x, L_y) - 1$$

1. Consider the signals $x[n]$ and $y[n]$ shown below:

$$x[n] = \{ \underset{\uparrow}{1}, \ j, \ -1, \ 0, \ 0, \ ...\}$$

$$y[n] = \{ \underset{\uparrow}{j}, \ 1+j, \ 1, \ 0, \ ...\}$$

To represent these sequences in matlab, we define the vectors, **x** and **y**, as follows:

```
x = [1 , j,  -1]
y = [j, 1+j , 1]
```

a) With the vectors **x** and **y** defined as above, execute the following command in Matlab to find a vector representation of the cross correlation function $r_{xy}[\ell]$.

```
rxy = xcorr(x,y)
```

Make use of equation (1) to interpret the contents of the vector **rxy** as elements of the sequence $r_{xy}[\ell]$. That is write out the contents of the array, adding an arrow to mark the $\ell = 0$ location.

b) In class, we showed that $r_{xy}[\ell]$ can be computed as the convolution of $x[\ell]$ with $y_f^*[\ell]$ where $y_f^*[\ell] \equiv y^*[-\ell]$. Below, we will use Matlab to verify this for the sequences $x[n]$ and $y[n]$ defined above. With the vectors **x** and **y** defined as above, execute the following commands in Matlab.

```
yf = conj(fliplr(y))
xconvyf = conv(x,yf)
```

i. Let the sequence $z[\ell]$ be defined as: $z[\ell] \equiv x[\ell] * y_f^*[\ell] = x[\ell] * y^*[-\ell]$. Then we know that the vector **xconvyf** as created by the matlab commands above will contain the values of the sequence $z[\ell]$ corresponding to $n_1^{(z)} \le \ell \le n_2^{(z)}$. What are the values of $n_1^{(z)}$ and $n_2^{(z)}$? Use the contents of the vector **xconvyf** to assist you in writing out the sequence representation of $z[\ell]$. You will need to add an arrow to indicate the $\ell = 0$ location.
Hint:
For which indices ℓ are the values of the sequence $x[\ell]$ included in the vector **x**?
For which indices ℓ are the values of the sequence $y_f^*[\ell]$ included in the vector **yf**?
Recall that $y_f^*[\ell] \equiv y^*[-\ell]$.

ii. How does the vector **xconvyf** compare to the vector **rxy**?

c) In class, we showed that $r_{xy}[\ell] = r_{yx}^*[-\ell]$. With the vectors **x** and **y** as previously defined, execute the following commands in Matlab.

```
ryx = xcorr(y,x)
```

Similar to part (a), make use of equation (1) to interpret the contents of the vector **ryx** as the sequence $r_{yx}[\ell]$. That is use the contents of the vector to assist you in writing out a sequence representation of $r_{yx}[\ell]$. Compare $r_{yx}[\ell]$ to $r_{xy}[\ell]$ and verify that $r_{xy}[\ell] = r_{yx}^*[-\ell]$.

2. *Time Delay Estimation in Radar.*
(adapted from problem 2.62 in Proakis and Manolakis, third edition, 1996)
In brief, a radar system transmits a pulse-type signal which is reflected back to the radar system after hitting an object. The time between when the pulse is transmitted and when it is received back at the system can then be used to determine the distance to the object.

Let $x_a(t)$ denote the transmitted signal and let $y_a(t)$ denote the received signal for a radar system. In general the received signal will be a noisy delayed and attenuated version of the transmitted signal. Thus, we can write:
$$y_a(t) = Ax_a(t - t_d) + v_a(t) \tag{2}$$
where $v_a(t)$ is an additive random noise. The signals $x_a(t)$ and $y_a(t)$ are sampled in the receiver at a rate of $1/T$ samples per second (where the sampling rate has been chosen in accordance with the sampling theorem), and are processed digitally to determine the time delay and hence the distance to the object. The resulting discrete-time signals are:
$$x[n] = x_a(nT)$$
$$y[n] = y_a(nT) = Ax_a(nT - n_dT) + v_a(nT) \;\; = Ax[n - n_d] + v[n]$$
where $n_dT = t_d$.

a) Let the sampled transmitted signal, $x[n]$, be the 13-point Barker sequence followed by an all zero sequence as shown below:
$$x[n] = \{+\underset{\uparrow}{1}, +1, +1, +1, +1, -1, -1, +1, +1, -1, +1, -1, +1, 0, 0, \ldots\}$$
Generate a vector B which contains the 13-point Barker sequence and a vector **x** of length 200 which includes the values of $x[n]$, as described above, for $0 \le n \le 199$.

suggested commands:
```
B = ones(1,13);
B(1,[6,7,10,12]) = -1;
x = zeros(1,200);
x(1,[1:13]) = B;
```

b) Use Matlab to find and stem the autocorrelation function of the Barker sequence. Note: the autocorrelation of B can be computed as the cross-correlation of B with itself. Hence:
```
rBB = xcorr(B,B)
```
To which values of ℓ will the elements of the vector rBB correspond? Be sure to label the horizontal axis with appropriate lag values. To which value of ℓ does the maximum value of rBB correspond? What is the maximum value of rBB? What is the second highest value of rBB?

c) Generate a zero-mean gaussian random noise sequence, **v**, of length 200 and variance 0.1. Note: a gaussian random sequence of length **L**, zero-mean, and variance **var** can be generated with the following matlab command.

```
v = sqrt(var) * randn(1,L)
```

d) Assuming an attenuation factor of $A = .9$ and a delay of $n_d = 30$, generate a vector **y** containing the sampled received signal, $y[n]$ for $0 \leq n \leq 199$, where:

$$y[n] = A(x[n - n_d] + v[n])$$

Note: the attenuation factor was applied to the noise waveform so that the signal-to-noise ratio is easily calculated as the reciprocal of the noise variance.
Note also that the equation above is easily implemented in Matlab as:

```
A = 0.9
nd = 30
xd = zeros(1,200);
xd(1,[1+nd:13+nd])=B;%xd contains values of x[n − nd] for 0 ≤ n ≤ 199
y = A*(xd + v);
```

e) Compute the cross-correlation $r_{yx}[\ell]$. Note: if **ryx = xcorr(y,x)**, then **ryx** will be a vector of length 399 corresponding to lags: $-199 \leq \ell \leq 199$. If the first element of the vector **ryx** corresponds to lag $\ell = -199$, which element will correspond to lag $\ell = 0$? which element will correspond to lag $\ell = 60$? Use the subplot command to make plots of the sampled received signal, $y[n]$, $0 \leq n \leq 60$, and the cross-correlation sequence $r_{yx}[\ell]$ for $0 \leq \ell \leq 60$. ***Label your axes and provide a descriptive title for your plot***. Feel free to use the plot command as opposed to the stem command (it is sometimes easier to see where the peak value of a function occurs when plot is used). In general, the peak value of $r_{yx}[\ell]$ occurs at the value of ℓ for which the sequence $\{x[n - \ell]\}$ most resembles the sequence $\{y[n]\}$. From your plot, determine the value of ℓ associated with the max value of $r_{yx}[\ell]$. Explain how we can use the cross-correlation of the received and transmitted signals in a radar system to determine n_d and hence the distance to the object.

f) Repeat parts (c), (d), and (e) for lower Signal to Noise Ratios (SNRs). Use noise variances of 0.5, 1, and 2. What happens to the effectiveness of this method as the SNR of the received signal decreases? In general longer sequences with properties similar to the 13 point Barker sequences can be used to operate with lower SNRs.