1. Consider a real causal and stable LTI system with transfer function: $H_z(z) = \dfrac{N(z)}{D(z)}$

   where: $D(z) = z^2 + a_1 z + a_2$ and $N(z) = a_2 z^2 + a_1 z + 1$.

   Note that the coefficients of $N(z)$ are the same (but in reverse order) as those of $D(z)$. This is characteristic of the transfer function of an allpass filter.

(1 pts.) a) What piece(s) of information (provided in the problem statement) allow(s) you to assume that the coefficients, a_1 and a_2, are real-valued.

(1 pts.) b) What piece(s) of information allow(s) you to assume that the poles, p_1 and p_2, of the system have magnitude less than 1.

(2 pts.) c) Show that $N(z) = z^2 D\left(z^{-1}\right)$.

(3 pts.) d) Show that if p_1 and p_2 are the roots of $D(z)$, (*i.e.,* if $D(p_1) = D(p_2) = 0$), then $1/p_1$ and $1/p_2$ are roots of $N(z)$. Hence the zeros of the system transfer function are the reciprocals of the poles. ***Hint:*** Use the result of part (c).

(3 pts.) e) Show that the frequency response of the system will have magnitude 1 for all F. ***Hint***: Use the result of part (c). It may also be helpful for you to recall that in problem 6 (parts c and d) of Assignment 9 you showed that:

$$\left|H_{\mathrm{DTFT}}(F)\right|^2 = \left(H_z(z) H_z\left(z^{-1}\right)\right)\Big|_{z = e^{j2\pi F}}$$

(4 pts.) f) Let $\theta_H(F) = \measuredangle H_{DTFT}(F) = \measuredangle H_z\left(e^{j2\pi F}\right)$ and let $\theta_D(F) = \measuredangle D\left(e^{j2\pi F}\right)$. Express $\theta_H(F)$ in terms of $\theta_D(F)$.

(6 pts.) 2. Let $Z_{\mathrm{DTFT}}(F) = \sum_{k=-\infty}^{\infty} \delta\left(F - \frac{k}{N}\right)$ where $\delta(\cdot)$ denotes the Dirac Delta function. Let $z[n]$ denote the IDTFT of $Z_{\mathrm{DTFT}}(F)$. Evaluate the IDTFT to find $z[n]$; ***show your work***. Then sketch the DTFT pair, $z[n]$ and $Z_{\mathrm{DTFT}}(F)$, for the case when $N = 10$.
Hint: You can find the IDTFT either by evaluating the IDTFT integral or by using the connection to the CTFT. In the latter case, we know that if $Z_{\mathrm{DTFT}}(F) = Z_a(F) * \mathrm{comb}(F)$, then $z[n] = z_a(n)$ where $z_a(t)$ is the inverse continuous-time FT of $Z_a(f)$. Letting $Z_a(f)$ denote one period of $Z_{\mathrm{DTFT}}(F)$, we note that $Z_a(f)$ is easily expressed as a sum of N Dirac Delta functions, and hence you may use the linearity property of the inverse CTFT to find $z_a(t)$ as a sum of N c.t. complex exponentials; hence $z[n]$ will be a sum of N d.t. complex exponentials. It should be a very simple task to evaluate the sum of the N d.t. complex exponentials for any integer n (no need for math just think carefully about what you are summing and then justify your answer). You've seen the sum lots of times before!

3. Let $w[n]$ denote a rectangular window of length $L = 10$: $w[n] = \begin{cases} 1, & 0 \le n \le (L-1) \\ 0, & \text{otherwise} \end{cases}$

   Furthermore, let:

$$x[n] = \cos\left(2\pi \tfrac{1}{5} n\right) \quad \text{and} \quad y[n] = \cos\left(2\pi \tfrac{1}{5} n\right) + \cos\left(2\pi \tfrac{1}{4} n\right)$$
$$g[n] = x[n] w[n] \quad \text{and} \quad h[n] = y[n] w[n]$$

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(6 pts.) a) Sketch $X_{\text{DTFT}}(F)$ and $Y_{\text{DTFT}}(F)$, $-1 \le F \le 1$.
Hint: In class we derived the DTFT pair:

$$\cos(2\pi F_0 n) \leftrightarrows \frac{1}{2}\left[\delta(F + F_0) + \delta(F - F_0)\right] * \text{comb}(F)$$

b) Let $W_{\text{DTFT}}(F)$ denote the DTFT of $w[n]$.

(4 pts.) *i.* Evaluate the DTFT sum to find a closed-form expression for $W_{\text{DTFT}}(F)$. We did a similar problem in class.

(3 pts.) *ii.* As is the case for any DTFT, $W_{\text{DTFT}}(F)$ is periodic with period 1. Confirm that the expression you found in part (a) satisfies $W_{\text{DTFT}}(F + 1) = W_{\text{DTFT}}(F)$.

(2 pts.) *iii.* Determine the value of $W_{\text{DTFT}}(0)$. (This will also be the value of $W_{\text{DTFT}}(F)$ for $F = \pm\, 1, \pm\, 2, \ldots$).

(2 pts.) *iv.* For which values of F will $W_{\text{DTFT}}(F)$ be equal to zero?

(3 pts.) *v.* Sketch $|W_{\text{DTFT}}(F)|$ vs. F for $-0.5 \le F \le 0.5$. Your sketch does not need to be too accurate but should accurately reflect the findings of parts (iii.) and (iv.) and the general nature of the function throughout the interval of interest. [If you would like to check your sketch, please note that the matlab commands: **F=−0.5:0.005:0.5; L= 10; plot(F, abs(L*diric(2*pi*F,L)))** will generate a plot of $|\sin(\pi FL)/\sin(\pi F)|$ *vs*. F .]

(5 pts.) c) Since $g[n] = x[n]w[n]$, we can use the multiplication-in-time property of the DTFT to find $G_{\text{DTFT}}(F)$ as the *circular* or *periodic convolution* of $W_{\text{DTFT}}(F)$ and $X_{\text{DTFT}}(F)$; this is equivalent to the normal convolution of $W_{\text{DTFT}}(F)\text{rect}(F)$ and $X_{\text{DTFT}}(F)$. Hence:

$$G_{\text{DTFT}}(F) = \left(W_{\text{DTFT}}(F)\text{rect}(F)\right) * X_{\text{DTFT}}(F)$$

Sketch $|G_{\text{DTFT}}(F)|$ vs. F for $0 \le F \le 1$.
Hint: Note that $X_{\text{DTFT}}(F)$ is a sum of Dirac Delta functions and recall that $Q(F) * \delta(F - F_0) = Q(F - F_0)$. Note also that $\text{rect}(F)$ has a value of 1 for $-0.5 \le F \le 0.5$ and a value of 0 otherwise.
You can always check your sketch of $G_{\text{DTFT}}(F)$ using the following matlab commands: **L=10; n=0:(L−1); g = cos(2*pi*n/5); fftsize=1000; k=0:(fftsize−1); F = k/fftsize; G = fft(g,fftsize); plot(F,abs(G)), grid on**

(5 pts.) d) Note that $g[n]$ is a windowed version of the infinitely long sinusoid, $x[n]$. In general, when we window a signal in time, we sacrifice resolution in frequency. Note that the infinitesimally narrow peaks in $X_{\text{DTFT}}(F)$ have been replaced by fatter peaks in $|G_{\text{DTFT}}(F)|$. The fatter peaks limit our ability to resolve frequencies. With $h[n]$ defined as above (*i.e.*, as a windowed version of $y[n]$) use matlab to generate a plot of $|H_{\text{DTFT}}(F)|$. When looking at $Y_{\text{DTFT}}(F)$, there is no difficulty resolving the two frequency components ($F_1 = 0.2$ and $F_2 = 0.25$) present in $y[n]$. Are the two components easily resolved in your Matlab-generated plot of $|H_{\text{DTFT}}(F)|$ when $L = 10$? Try increasing the window length, L, to 40 and make a plot of $|H_{\text{DTFT}}(F)|$, for the case when $L = 40$. Is it any easier to resolve the two frequency components when $L = 40$? In general, frequency resolution increases as the time window is made longer.