Examples

Let $x[n] = e^{j2\pi\frac{1}{8}n}$. find $X_{DTFT}(F)$

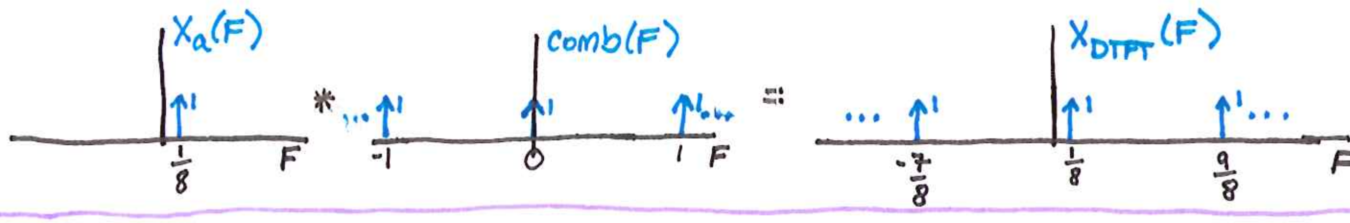
Solution

Recall: $x_a(t) = e^{j2\pi\frac{1}{8}t} \overset{CTFT}{\longleftrightarrow} X_a(f) = \delta(f - \frac{1}{8})$

Since $x[n] = x_a(t)\big|_{t=n}$,

we know that $X_{DTFT}(F) = X_a(F) * \text{comb}(F)$

$$\Rightarrow X_{DTFT}(F) = \delta(F - \tfrac{1}{8}) * \text{comb}(F)$$



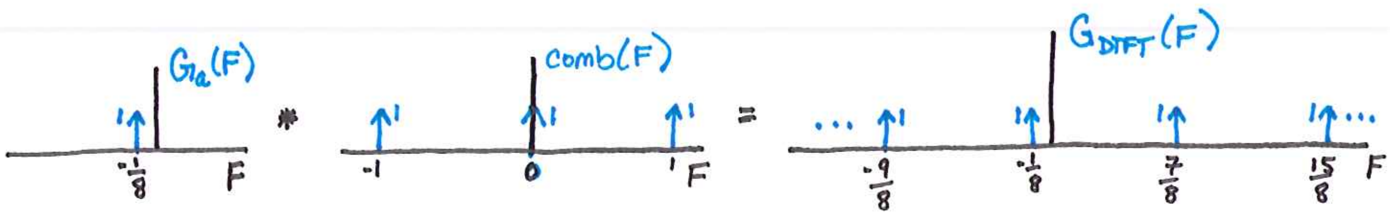
Let $g[n] = e^{-j2\pi\frac{1}{8}n}$. Find $G_{DTFT}(F)$

Solution

$$g_a(t) = e^{-j2\pi\frac{1}{8}t} \longleftrightarrow G_a(f) = \delta(f + \tfrac{1}{8})$$

Furthermore $g[n] = g_a(n) \Longrightarrow G_{DTFT}(F) = G_a(F) * \text{comb}(F)$

$$\Rightarrow G_{DTFT}(F) = \delta(F + \tfrac{1}{8}) * \text{comb}(F)$$



Problem: Given a discrete-time LTI system with impulse response $h[n]$, find the system response, $y[n]$, when the input is $x[n] = e^{j2\pi F_0 n}$.

Solution:

$$y[n] = h[n] * x[n] \implies Y_{DTFT}(F) = H_{DTFT}(F)\, X_{DTFT}(F)$$

$$\Rightarrow Y_{DTFT}(F) = H_{DTFT}(F)\Big[\underbrace{\delta(F-F_0) * \text{comb}(F)}_{X_{DTFT}(F)}\Big] = H_{DTFT}(F) \underbrace{\sum_{k=-\infty}^{\infty} \delta(F-F_0-k)}_{X_{DTFT}(F)}$$

$$= \sum_{k=-\infty}^{\infty} H_{DTFT}(F)\,\delta(F-F_0-k) = \sum_{k=-\infty}^{\infty} H_{DTFT}(F_0+k)\,\underbrace{\delta(F-F_0-k)}_{\text{has a value of } 0 \text{ for all } F \neq F_0+k}$$

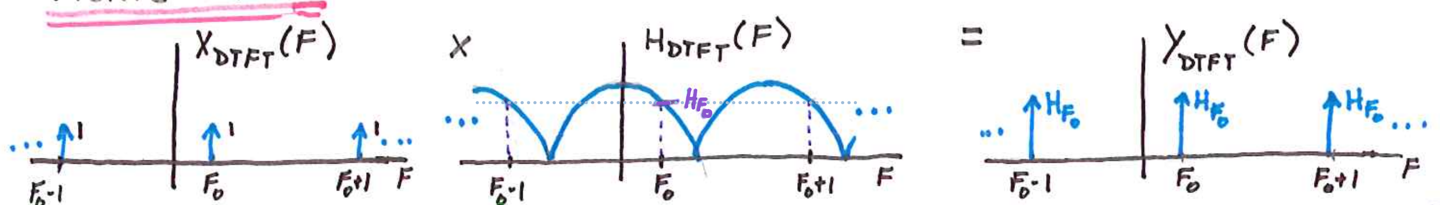
$$= \sum H_{DTFT}(F_0)\,\delta(F-F_0-k) = H_{DTFT}(F_0) \sum_{k=-\infty}^{\infty} \delta(F-F_0-k)$$

$H_{DTFT}(F) = H_{DTFT}(F+k)$, $k = 0, \pm 1, \pm 2, \ldots$

$$= H_{DTFT}(F_0)\, X_{DTFT}(F)$$

$$\Rightarrow y[n] = H_{DTFT}(F_0)\, x[n] = H_{DTFT}(F_0)\, e^{j2\pi F_0 n}$$

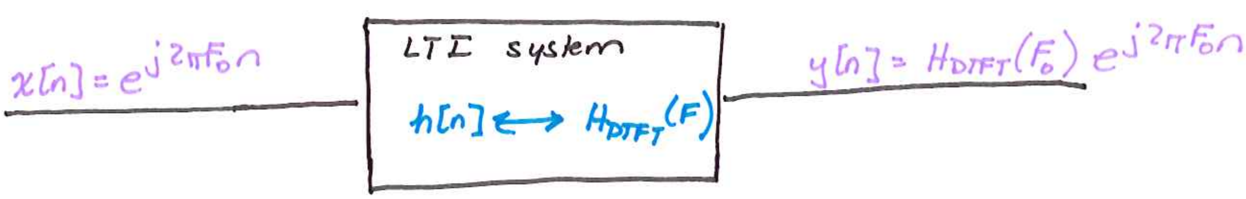
Picture Solution



where $H_{F_0} \equiv H_{DTFT}(F_0)$

From picture above, we see that $Y_{DTFT}(F) = H_{F_0}\, X_{DTFT}(F)$

$$\Rightarrow y[n] = H_{F_0}\, x[n] = H_{DTFT}(F_0)\, e^{j2\pi F_0 n}$$



A complex-exponential is an eigenfunction for an LTI system. When the input to the system is a complex exponential, the output will be a complex-valued constant times the input.

Problem: Given a real LTI discrete-time system with impulse response $h[n]$, find the system responses, $y_1[n]$ and $y_2[n]$, to the respective inputs:

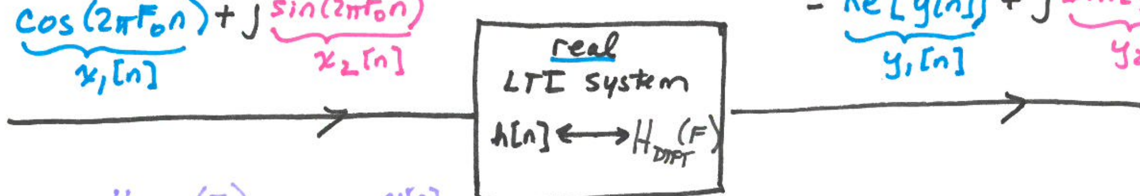
$x_1[n] = \cos(2\pi F_0 n)$ and $x_2[n] = \sin(2\pi F_0 n)$.

> The fact that the system is real means that the coefficients of the LCCDE describing the system are real. This, in turn, means that a real-valued input will result in a real-valued response (hence, $h[n]$ is real), and a purely imaginary input will result in an imaginary-valued response. Furthermore when the input is complex-valued, we know that the real part of the response is the response to the real part of the input and that the imaginary part of the response is the response to the imaginary part of the input.

Solution

From the previous problem, we know that the response of the system to the input signal $x[n] = e^{j2\pi F_0 n} = \underbrace{\cos(2\pi F_0 n)}_{x_1[n]} + j\underbrace{\sin(2\pi F_0 n)}_{x_2[n]}$ is $y[n] = H_{DTFT}(F_0)e^{j2\pi F_0 n}$. Since the system is real, the response of the system to $x_1[n]$ is the real part of $y[n]$ and the response of the system to $x_2[n]$ is the imaginary part of $y[n]$.

$$x[n] = e^{j2\pi F_0 n} = \underbrace{\cos(2\pi F_0 n)}_{x_1[n]} + j\underbrace{\sin(2\pi F_0 n)}_{x_2[n]}$$



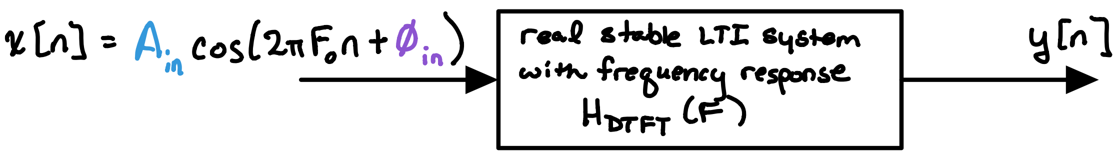
$$y[n] = H_{DTFT}(F_0)e^{j2\pi F_0 n} = \underbrace{\mathrm{Re}[y[n]]}_{y_1[n]} + j\underbrace{\mathrm{Im}[y[n]]}_{y_2[n]}$$

$$y[n] = \overbrace{|H_{DTFT}(F_0)|e^{j\measuredangle H_{DTFT}(F_0)}}^{H_{DTFT}(F_0)} \overbrace{e^{j2\pi F_0 n}}^{x[n]} = |H_{DTFT}(F_0)|e^{j(2\pi F_0 n + \measuredangle H_{DTFT}(F_0))}$$

$$\Rightarrow y[n] = \underbrace{|H_{DTFT}(F_0)|\cos(2\pi F_0 n + \measuredangle H_{DTFT}(F_0))}_{\text{This is } y_1[n]\text{, the response of the system to the input } x_1[n] = \cos(2\pi F_0 n)} + j\underbrace{|H_{DTFT}(F_0)|\sin(2\pi F_0 n + \measuredangle H_{DTFT}(F_0))}_{\text{This is } y_2[n]\text{, the response of the system to the input } x_2[n] = \sin(2\pi F_0 n)}$$

Conclusion of previous exercise:

Given the frequency response function, $H_{DTFT}(F)$, of a real stable LTI system, we know that the steady-state response of the system to a sinusoidal excitation may be written down by inspection of the system's frequency response function as illustrated below.



where $$y_{ss}[n] = \underbrace{|H_{DTFT}(F_o)|\, A_{in}}_{A_{out}} \cos\big(2\pi F_o n + \underbrace{\phi_{in} + \measuredangle H_{DTFT}(F_o)}_{\phi_{out}}\big)$$