4. ***Linear Time-invariant (LTI)***: A system is said to be linear time-invariant if and only if it is both linear and time-invariant. Most of the systems we will analyze in this class are LTI systems. It can be shown that *LTI systems are completely characterized by their impulse response*; that is, given the impulse response of an LTI system, one can find the output to any given input without the need for an input-output relationship.

5. ***Causal vs. Noncausal:*** A system is *causal* if and only if the output of the system at time n depends only on present and past inputs at time n, *i.e.,* if:
$$y[n] = g(x[n], x[n-1], x[n-2], \ldots)$$
Although a real-time noncausal system is unrealizable, noncausal systems are often used in non real-time applications where the signal is recorded and processed off-line (such as with image processing) or in applications where a small amount of delay is tolerable.

   It can be shown that an LTI system is causal if and only if its impulse response, $h[n]$, is causal, *i.e.*, if and only if:
$$h[n] = 0\,, \quad n < 0$$

6. ***Stable vs. Unstable:*** A relaxed system is said to be bounded-input bounded-output (BIBO) stable if and only if every bounded input signal produces a bounded output signal. (Note: a sequence, $x[n]$, is bounded if there exist some finite number, M_x, such that $|x[n]| \le M_x < \infty$ for all n.)

   It can be shown that an LTI system is BIBO stable if and only if its impulse response, $h[n]$, is absolutely summable, *i.e.,* if and only if:
$$\left[\sum_{n=-\infty}^{\infty} |h[n]|\right] < \infty$$

7. ***FIR vs. IIR:*** A system is termed as a ***finite impulse response (FIR) system*** if the impulse response is of finite length; otherwise, it is termed an ***infinite impulse response (IIR) system***.

An LTI system is causal if and only if its impulse response, $h[n]$, is causal. That is, if and only if

$$h[n] = 0, \quad n < 0$$

Proof: Let $y[n]$ denote the response of the LTI system, with impulse response $h[n]$, to the input $x[n]$.

Since the system is LTI, we may write:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \overbrace{\sum_{k=0}^{\infty} h[k]\, x[n-k]}^{\text{sum 1}} + \overbrace{\sum_{k=-\infty}^{-1} h[k]\, x[n-k]}^{\text{sum 2}}$$

$$= \overbrace{h[0]\, x[n-0] + h[1]\, x[n-1] + h[2]\, x[n-2] + \cdots}^{\text{sum 1}}$$

$$+ \overbrace{h[-1]\, x[n+1] + h[-2]\, x[n+2] + \cdots}^{\text{sum 2}}$$

proof of the 'if' part

If $h[n] = 0$ for $n < 0$, then sum 2 = 0 and thus $y[n] =$ sum 1; furthermore, sum 1 depends only on the current and past input values: $x[n], x[n-1], \ldots$

Hence, we have shown that if $h[n]$ is causal the LTI system will be causal

proof of the 'only if' part

Now suppose $h[-m] \neq 0$ for some $m > 0$. In this case, $y[n]$ will depend on the future input value $x[n+m]$.

Hence, showing that if $h[n]$ is not causal, the LTI system will not be causal which, in turn, implies that the LTI system will be causal only if $h[n]$ is causal.

Thus, we have shown that an LTI system is causal iff $h[n]$ is causal.

Stevenson

An LTI system is causal if and only if its impulse response, $h[n]$, is causal – that is, if and only if $h[n] = 0,\ n < 0$.

Example: Given the input-output relationships below, determine the impulse response, $h[n]$, of each associated system.

LTI, causal a) $y[n] = x[n]$ $\Rightarrow$ $h[n] = \delta[n]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

LTI, causal b) $y[n] = x[n-1]$ $\Rightarrow$ $h[n] = \delta[n-1]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{0}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

LTI, not causal c) $y[n] = x[n+1]$ $\Rightarrow$ $h[n] = \delta[n+1]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{0}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

LTI, causal d) $y[n] = x[n] - x[n-1]$ $\Rightarrow$ $h[n] = \delta[n] - \delta[n-1]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{-1}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

not LTI, not causal e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\Rightarrow$ $h[n] = \max(\delta[n+1], \delta[n], \delta[n-1])$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

not LTI, causal f) $y[n] = x^2[n]$ $\Rightarrow$ $h[n] = \delta^2[n]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

not LTI, not causal g) $y[n] = x[n^2]$ $\Rightarrow$ $h[n] = \delta[n^2]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$

LTI, causal h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\Rightarrow$ $h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$

Note: We cannot use the impulse response to determine the causality of systems (e), (f), or (g) because these systems are not LTI.

4. ***Linear Time-invariant (LTI)***: A system is said to be linear time-invariant if and only if it is both linear and time-invariant. Most of the systems we will analyze in this class are LTI systems. It can be shown that *LTI systems are completely characterized by their impulse response*; that is, given the impulse response of an LTI system, one can find the output to any given input without the need for an input-output relationship.

5. ***Causal vs. Noncausal:*** A system is *causal* if and only if the output of the system at time n depends only on present and past inputs at time n, *i.e.,* if:
$$y[n] = g(x[n], x[n-1], x[n-2], \ldots)$$
Although a real-time noncausal system is unrealizable, noncausal systems are often used in non real-time applications where the signal is recorded and processed off-line (such as with image processing) or in applications where a small amount of delay is tolerable.

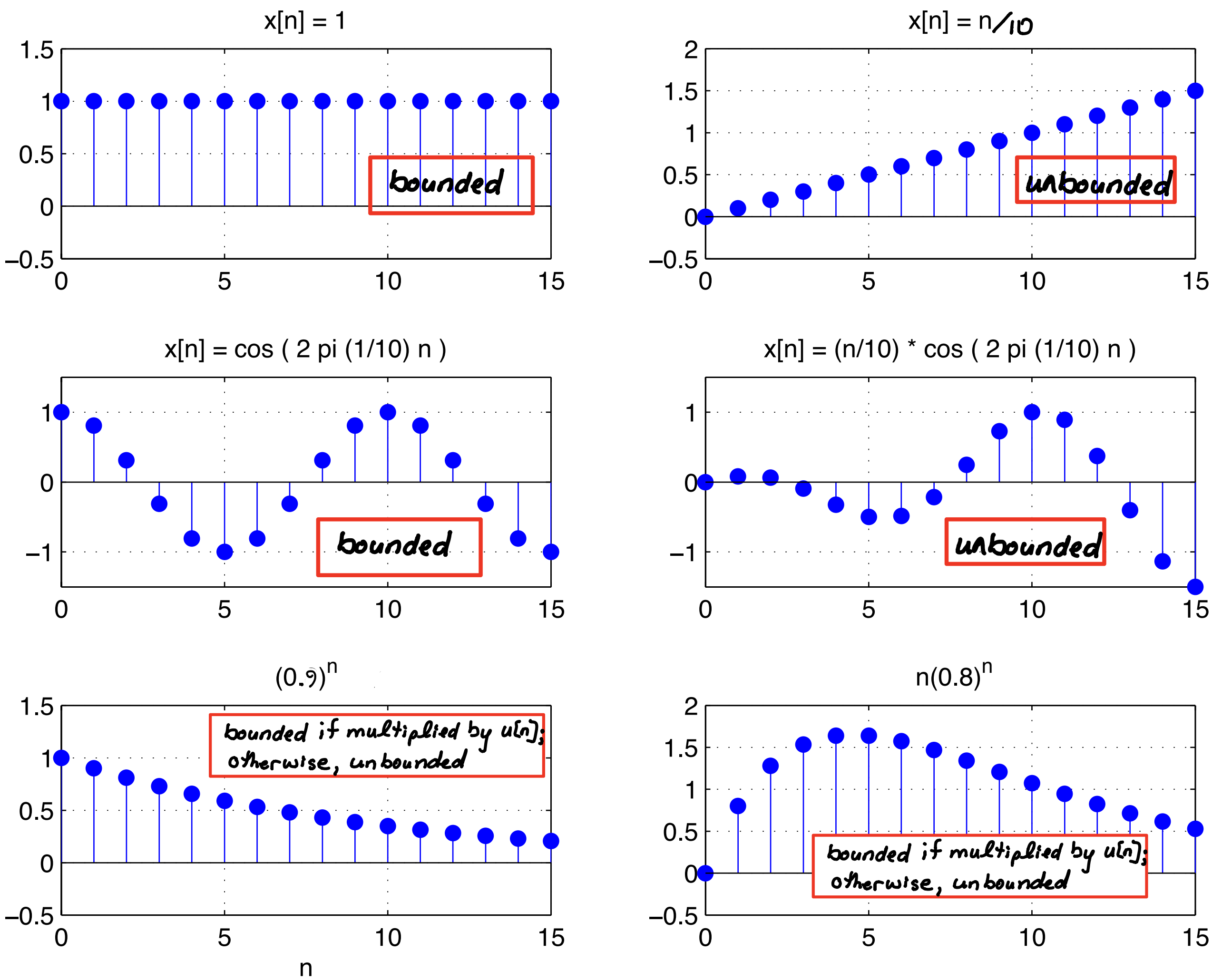
   It can be shown that an LTI system is causal if and only if its impulse response, $h[n]$, is causal, *i.e.*, if and only if:
$$h[n] = 0\,, \quad n < 0$$

6. ***Stable vs. Unstable:*** A relaxed system is said to be bounded-input bounded-output (BIBO) stable if and only if every bounded input signal produces a bounded output signal. (Note: a sequence, $x[n]$, is bounded if there exist some finite number, M_x, such that $|x[n]| \le M_x < \infty$ for all n.)

   It can be shown that an LTI system is BIBO stable if and only if its impulse response, $h[n]$, is absolutely summable, *i.e.,* if and only if:
$$\left[\sum_{n=-\infty}^{\infty} |h[n]|\right] < \infty$$

7. ***FIR vs. IIR:*** A system is termed as a ***finite impulse response (FIR) system*** if the impulse response is of finite length; otherwise, it is termed an ***infinite impulse response (IIR) system***.

A signal $x[n]$ is bounded if there exists a constant B such that $|x[n]| \leq B$ for all n.



Stevenson

A system is said to be bounded-input bounded-output (BIBO) stable if and only if every bounded input signal produces a bounded output signal.

Label each of the systems below as BIBO stable or not BIBO stable.

a) $y[n] = x[n]$

note: if $|x[n]| \le M_x$ for all n, then $|y[n]| \le M_x$ for all n.

$\Rightarrow$ $\boxed{\text{system is BIBO stable.}}$

b) $y[n] = x[n-1]$

if $|x[n]| \le M_x \; \forall n$, then $|y[n]| = |x[n-1]| \le M_x \; \forall n$

$\Rightarrow$ $\boxed{\text{system is BIBO stable.}}$

c) $y[n] = x[n+1]$

similar to part b

d) $y[n] = x[n] - x[n-1]$

if $|x[n]| \le M_x \; \forall n$,

then $|y[n]| = |x[n] - x[n-1]| \le |x[n]| + |x[n-1]| \le$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$

if $|x[n]| \le M_x$ for all n

then $|y[n]| = |\max(x[n+1], x[n], x[n-1])| \le M_x \; \forall n$

$\Rightarrow$ $\boxed{\text{BIBO stable}}$

f) $y[n] = x^2[n]$

if $|x[n]| \le M_x \; \forall n$, then $|y[n]| = |x^2[n]| \le$

g) $y[n] = x[n^2]$

if $|x[n]| \le M_x \; \forall n$, then $|y[n]| = |x[n^2]| \le$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$

if $|x[n]| \le M_x \; \forall n$ then $|y[n]| = \left| \sum_{k=-\infty}^{n} x[k] \right|$

An LTI system is stable if and only if its impulse response is absolutely summable, i.e., if and only if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

<u>Proof</u> Let $y[n]$ denote the response of the LTI system with impulse response $h[n]$ when the input is $x[n]$.

a) To prove the "if" part, we need to show that

$$\text{if } \sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{and} \quad |x[k]| < M_x \ \forall k$$

then we will be able to find a constant M_y such that $|y[k]| < M_y \ \forall k$

b) To prove the "only if" part, we need to show that if $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, then there exists a bounded sequence $x[n]$ for which the system's response, $y[n]$, is not bounded.

<u>Proof of "if" part:</u>

must show that if $h[n]$ is absolutely summable and $x[n]$ is bounded then $y[n]$ will be bounded.

Since the system is LTI we know $y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

$$\Rightarrow |y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\leq M_x \underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}_{S_h} = M_x S_h = M_y$$

Thus we have shown that if the impulse response of an LTI system is absolutely summable, the response of the system will be bounded provided the input is bounded. (proof of "only if" part is on next page.)

(continued from previous page)

must show that if $h[k]$ is not absolutely summable, then there will be a bounded sequence $x[n]$ for which the system's output is not bounded.

Proof of the "only if" part:

Consider the input sequence $x[n] = \dfrac{h^*[-n]}{|h[-n]|}$

Note that $|x[n]|^2 = x[n]\, x^*[n] = \dfrac{h^*[-n]\, h[-n]}{|h[-n]|^2} = 1$

Thus $|x[n]| \le 1 \quad \forall n$, i.e. $x[n]$ is bounded.

Since the system is LTI, we may write that:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k] \frac{h^*[-(n-k)]}{|h[-(n-k)]|}$$

$$= \sum_{k=-\infty}^{\infty} h[k] \frac{h^*[k-n]}{|h[k-n]|}$$

Evaluating the expression above at $n=0$ yields:

$$y[0] = \sum_{k=-\infty}^{\infty} h[k] \frac{h^*[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

if $h[k]$ not absolutely summable

Thus we have shown that if $h[n]$ is not absolutely summable, there exists a bounded sequence $x[n]$ for which the response of the system at $n=0$ will be infinitely big.

Stevenson

An LTI system is BIBO stable if and only if its impulse response is absolutely summable, i.e., if and only if:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Examples

a) $y[n] = x[n]$ LTI $\Rightarrow$ $h[n] = \delta[n]$ $\sum_{k=-\infty}^{\infty} |h[k]| = 1 < \infty$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$

b) $y[n] = x[n-1]$ LTI $\Rightarrow$ $h[n] = \delta[n-1]$ $\sum_{k=-\infty}^{\infty} |h[k]| = 1 < \infty$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{0}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$

c) $y[n] = x[n+1]$ LTI $\Rightarrow$ $h[n] = \delta[n+1]$ $\sum_{k=-\infty}^{\infty} |h[k]| = 1 < \infty$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{0}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$

d) $y[n] = x[n] - x[n-1]$ LTI $\Rightarrow$ $h[n] = \delta[n] - \delta[n-1]$ $\sum_{k=-\infty}^{\infty} |h[k]| = 2 < \infty$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{-1}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ not LTI $\Rightarrow$ $h[n] = \max(\delta[n+1], \delta[n], \delta[n-1])$ $\sum_{k=-\infty}^{\infty} |h[k]| = 3 < \infty$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$ but system is not LTI

f) $y[n] = x^2[n]$ not LTI $\Rightarrow$ $h[n] = \delta^2[n]$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$ not LTI

g) $y[n] = x[n^2]$ not LTI $\Rightarrow$ $h[n] = \delta[n^2]$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$ not LTI

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ LTI $\Rightarrow$ $h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots \}$ $\sum_{k=-\infty}^{\infty} |h[k]| = ??$