Q1ab

1 a)

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z)$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{8}{8 - 6z^{-1} + z^{-2}} \qquad |z| > \frac{1}{2}$$

b)

$$\frac{H(z)}{z} = \frac{8z}{8z^2 - 6z + 1} = \frac{8z}{(4\cancel{z}-1)(2\cancel{z}-1)} = \frac{A_1}{4\cancel{z}-1} + \frac{A_2}{2\cancel{z}-1}$$

$$A_1 = \left.\frac{8z}{2\cancel{z}-1}\right|_{z=\frac{1}{4}} = \frac{2}{\frac{1}{2}-1} = -4$$

$$A_2 = \left.\frac{8z}{4z-1}\right|_{z=\frac{1}{2}} = \frac{4}{2-1} = 4$$

$$\frac{H(z)}{z} = \frac{-1}{z-\frac{1}{4}} + \frac{2}{z-1/2} \Rightarrow H(z) = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n] =$$

Q1c

c) $x[n] = u[n] \implies X(z) = \dfrac{1}{1-z^{-1}} \qquad |z| > 1$

$y_{step}[n] = h[n] * u[n] \implies Y_s(z) = X(z)\,H(z)$

$$Y_s(z) = \frac{1}{(1-z^{-1})(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-1})} = \frac{z^3}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})} \qquad |z|>1$$

$$\frac{Y_s(z)}{z} = \frac{z^2}{(z-1)(z-1/2)(z-1/4)} = \frac{A_1}{z-1} + \frac{A_2}{z-1/2} + \frac{A_3}{z-1/4}$$

$$A_1 = \frac{z^2}{(z-1/2)(z-1/4)}\Big|_{z=1} = \frac{1}{1/2 \cdot \frac{3}{4}} = \frac{8}{3}$$

$$A_2 = \frac{z^2}{(z-1)(z-1/4)}\Big|_{z=1/2} = \frac{1/4}{(-1/2)(1/4)} = -2$$

$$A_3 = \frac{z^2}{(z-1)(z-1/2)}\Big|_{z=1/4} = \frac{1/16}{(\frac{-3}{4})(\frac{-1}{4})} = \frac{1}{3}$$

$$\implies Y_s(z) = \frac{8}{3}\left(\frac{1}{1-z^{-1}}\right) - 2\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) + \frac{1}{3}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad |z|>1$$

$$y_s[n] = \frac{8}{3}u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n]$$

$$= \left[\frac{8}{3} - 2\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u[n]$$

Q1d

d) $x[n] = (\frac{1}{2})^n u[n] \longrightarrow X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$

$$Y_{zs}(z) = H(z)X(z) = \frac{z^3}{(z-\frac{1}{2})(z-\frac{1}{2})(z-\frac{1}{4})} = \qquad |z| > \frac{1}{2}$$

$$\Rightarrow \frac{Y_{zs}(z)}{z} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A_1}{(z-\frac{1}{2})} + \frac{A_2}{(z-\frac{1}{2})^2} + \frac{A_3}{z-\frac{1}{4}}$$

$$A_2 = \frac{z^2}{z-\frac{1}{4}}\bigg|_{z=\frac{1}{2}} = \frac{1/4}{1/4} = 1 \qquad A_3 = \frac{z^2}{(z-\frac{1}{2})^2}\bigg|_{z=\frac{1}{4}} = 1$$

$$A_1 = 0$$

$$\lim_{z\to\infty} \left\{ \frac{z^3}{(z-\frac{1}{2})^2(z-\frac{1}{4})} = \frac{A_1 z}{z-\frac{1}{2}} + \frac{A_2 z}{(z-\frac{1}{2})^2} + \frac{A_3 z}{z-\frac{1}{4}} \right\} \Rightarrow 1 = A_1 + 0 + A_3$$

$$A_1 = 1 - A_3 = 0$$

$$Y_{zs}(z) = \frac{2(\frac{1}{2})z^{-1}}{(1-\frac{1}{2}z^{-1})^2} + \frac{1}{1-\frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$y_{zs}[n] = 2n(\tfrac{1}{2})^n u[n] + (\tfrac{1}{4})^n u[n]$$

Q1e

e) $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \Rightarrow$

$$Y^+(z) - \frac{3}{4}\left(y[-1] + z^{-1}Y^+(z)\right) + \frac{1}{8}\left(y[-2] + y[-1]z^{-1} + z^{-2}Y^+(z)\right) = X^+(z) \quad (*)$$

$$X^+(z) = 0 \qquad y[-1] = 2 \qquad y[-2] = 28$$

$$Y^+(z) - \frac{3}{2} - \frac{3}{4}z^{-1}Y^+(z) + \frac{28}{8} + \frac{2}{8}z^{-1} + \frac{z^{-2}}{8}Y^+(z) = 0$$

$$Y^+(z)\left(1 - \frac{3}{4}z^{-1} + \frac{z^{-2}}{8}\right) = \frac{3}{2} - \frac{28}{8} - \frac{2}{8}z^{-1} \Rightarrow Y^+(z) = \frac{-1/4\, z^{-1} - 2}{1 - \frac{3}{4}z^{-1} + \frac{z^{-2}}{8}}$$

$$Y^+(z) = \frac{-2z^2 - \frac{1}{4}z}{z^2 - \frac{3}{4}z + \frac{1}{8}} \Rightarrow \frac{Y^+(z)}{z} = \frac{-2z - 1/4}{(z - \frac{1}{2})(z - \frac{1}{4})} = \frac{A_1}{z - 1/2} + \frac{A_2}{z - 1/4}$$

$$A_1 = \left.\frac{-2z - 1/4}{z - 1/4}\right|_{z=1/2} = \frac{-1 - 1/4}{1/4} = -5$$

$$A_2 = \left.\frac{-2z - 1/4}{z - 1/2}\right|_{z=1/4} = \frac{-1/2 - 1/4}{-1/4} = 3$$

$$Y^+(z) = \frac{3}{1 - \frac{1}{4}z^{-1}} - \frac{5}{1 - \frac{1}{2}z^{-1}} \qquad |z| > 1/2$$

$$y_{zi}[n] = \left(3\left(\tfrac{1}{4}\right)^n - 5\left(\tfrac{1}{2}\right)^n\right)u[n]$$

Q1f

f) from (*) $\quad Y^{+}(z)\left(1-\frac{3}{4}z^{-1}+\frac{z^{-2}}{8}\right) = X^{+}(z) - 2 - \frac{1}{4}z^{-1}$

$$\Rightarrow Y^{+}(z) = \frac{X^{+}(z)}{1-\frac{3}{4}z^{-1}+\frac{z^{-2}}{8}} + \frac{-2-\frac{1}{4}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{z^{-2}}{8}}, \quad X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z|>\frac{1}{2}$$

$$y[n] = y_{zs}[n] + y_{zi}[n] \Rightarrow Y(z) = Y_{zs}(z) + Y_{zi}(z)$$

$$\Rightarrow y[n] = 2n\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] + \left(3\left(\frac{1}{4}\right)^n - 5\left(\frac{1}{2}\right)^n\right)u[n]$$

from the close form

$n=0 \quad y[0] = 0 + 1 + 3 - 5 = -1$

$n=1 \quad y[1] = 2\left(\frac{1}{2}\right) + \frac{1}{4} + \frac{3}{4} - \frac{5}{2} = 1 + 1 - \frac{5}{2} = \frac{4-5}{2} = \frac{-1}{2}$

$n=2 \quad y[2] = 4\left(\frac{1}{4}\right) + \frac{1}{16} + 3\left(\frac{1}{16}\right) - 5\left(\frac{1}{4}\right) = 1 + \frac{1}{4} - \frac{5}{4} = 0$

$n=3 \quad y[3] = 6\left(\frac{1}{8}\right) + \frac{1}{4\times16} + \frac{3}{4\times16} - 5\left(\frac{1}{8}\right) = \frac{6}{8} - \frac{5}{8} + \frac{1}{16} = \frac{3}{16}$

from the diffrence eqn:

$n=0 \quad y[0] = \frac{3}{4}(2) - \frac{1}{8}(28) + 1 = \frac{5}{2} - \frac{28}{8} = -1$ ✓

$n=1 \quad y[1] = \frac{3}{4}(-1) - \frac{1}{8}(2) + \frac{1}{2} = \frac{-3}{4} - \frac{1}{4} + \frac{1}{2} = -1 + \frac{1}{2} = \frac{-1}{2}$ ✓

$n=2 \quad y[2] = \frac{3}{4}\left(\frac{-1}{2}\right) - \frac{1}{8}(-1) + \frac{1}{4} = \frac{-3}{8} + \frac{1}{8} + \frac{1}{4} = 0$ ✓

$n=3 \quad y[3] = \frac{3}{4}(0) - \frac{1}{8}\left(\frac{-1}{2}\right) + \frac{1}{8} = \frac{1}{16} + \frac{2}{16} = \frac{3}{16}$ ✓

Q2abcd

2) a) $x[n] = 2 + 2\cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right)$

$x_1[n] = 2 = 2\cos(2\pi n) \Rightarrow N_1 = 1$

$x_2[n] = 2\cos\left(\frac{\pi n}{4}\right) = 2\cos\left(2\pi \frac{1}{8} n\right) \Rightarrow N_2 = 8$

$x_3[n] = \cos\left(\frac{\pi n}{2}\right) = \cos\left(2\pi \frac{1}{4} n\right) \Rightarrow N_3 = 4$

$x_4[n] = \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right) = \frac{1}{2}\cos\left(2\pi \frac{3}{8} n\right) \Rightarrow N_4 = 8$

$N = 8 = \mathrm{lcm}(8, 1, 4, 8)$

b) $C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$

$C_k = \frac{1}{8} \sum_{n=0}^{7} \left(2 + e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}} + \frac{1}{2} e^{j\frac{\pi n}{2}} + \frac{1}{2} e^{-j\frac{\pi n}{2}} + \frac{1}{4} e^{j\frac{3\pi n}{4}} + \frac{1}{4} e^{-j\frac{\pi 3n}{4}}\right) e^{-j\frac{2\pi k}{8} n}$

$C_k = \frac{1}{8}\Big[2 \underbrace{\sum_{n=0}^{7} e^{j\frac{2\pi k}{8} n}}_{S_1} + \underbrace{\sum_{n=0}^{7} e^{j\pi\left(\frac{1-k}{4}\right) n}}_{S_2} + \underbrace{\sum_{n=0}^{7} e^{-j\pi\left(\frac{1+k}{4}\right) n}}_{S_3} + \frac{1}{2} \underbrace{\sum_{n=0}^{7} e^{+j\pi\left(\frac{2-k}{4}\right) n}}_{S_4}$

$+ \frac{1}{2} \underbrace{\sum_{n=0}^{7} e^{-j\left(\frac{2+k}{4}\right)\pi n}}_{S_5} + \frac{1}{4} \underbrace{\sum_{n=0}^{7} e^{j\pi\left(\frac{3-k}{4}\right) n}}_{S_6} + \frac{1}{4} \underbrace{\sum_{n=0}^{7} e^{-j\pi\left(\frac{3+k}{4}\right) n}}_{S_7} =$

$S_1 = \begin{cases} 0 & \text{other} \\ 8 & k=0 \end{cases}$ $S_2 = \begin{cases} 0 & \text{others} \\ 8 & k=1 \end{cases}$ $S_3 = \begin{cases} 0 & \text{others} \\ 8 & k=7 \end{cases}$

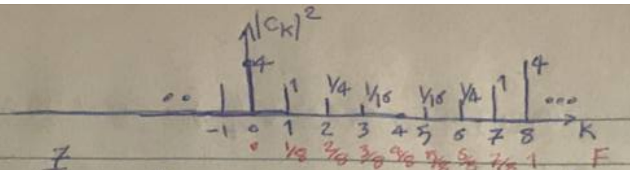
$S_4 = \begin{cases} 0 & \text{other} \\ 8 & k=2 \end{cases}$ $S_5 = \begin{cases} 0 & \text{other} \\ 8 & k=6 \end{cases}$ $S_6 = \begin{cases} 0 & \text{other} \\ 8 & k=3 \end{cases}$ $S_7 = \begin{cases} 0 & \text{others} \\ 8 & k=5 \end{cases}$

$\Rightarrow C_k = \frac{1}{8}\left(2S_1 + S_2 + S_3 + \frac{1}{2} S_4 + \frac{1}{2} S_5 + \frac{1}{4} S_6 + \frac{1}{4} S_7\right)$

$\Rightarrow C_0 = 2,\ C_1 = 1,\ C_2 = \frac{1}{2},\ C_3 = \frac{1}{4},\ C_4 = 0$

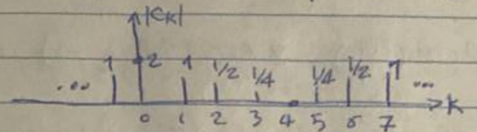
$C_5 = \frac{1}{4},\ C_6 = \frac{1}{2},\ C_7 = 1$

c)



$$P_x = \sum_{k=0}^{7} |c_k|^2 = \left(4 + \left(1 + \frac{1}{4} + \frac{1}{16}\right)2\right) = 4 + 2 + \frac{1}{2} + \frac{1}{8} = 6{,}625 = \frac{53}{8}$$

(**)



$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x_1[n] + x_2[n] + x_3[n] + x_4[n]|^2$$

~~because x_1~~ x_2, x_3, x_4 are orthonogal

$$= \frac{1}{N}\sum_{n=0}^{N-1} |x_1[n]|^2 + \frac{1}{N}\sum_{n=0}^{N-1} |x_2[n]| + \frac{1}{N}\sum_{n=0}^{N-1} |x_3[n]| + \frac{1}{N}\sum_{n=0}^{N-1} |x_4[n]|^2$$

$$= \frac{4}{2} + \frac{1}{2} + \frac{1}{8} + 4 = 2 + 4 + \frac{1}{2} + \frac{1}{8} = \frac{53}{8}$$ (*)

based on $x[n] = A\cos(2\pi F n)$ $\therefore$ $P_x = \frac{A^2}{2}$

and $P_{x_1} = \frac{1}{8}\sum_{n=0}^{N-1} |2|^2 = 4$

d)

As you can see the both results are ~~agree~~ equal

Q3abc

3 a) $C_k = \cos(k\frac{\pi}{4}) + \sin(3\frac{k\pi}{4})$

$C_1[k] = \cos(\frac{\pi}{4}k) = \cos(2\pi\frac{1}{8}k) \Rightarrow N=8$

$C_2[k] = \sin(\frac{3\pi}{4}k) = \sin(2\pi\frac{3}{8}k) \Rightarrow N=8$

C_k - is peridic with $N=8$

b) $C_k = \frac{1}{2}e^{jk\frac{\pi}{4}} + \frac{1}{2}e^{-jk\frac{\pi}{4}} + \frac{1}{2j}e^{+j\frac{3k\pi}{4}} - \frac{1}{2j}e^{-j\frac{3k\pi}{4}}$

$x[n] = \frac{1}{2}\sum_{k=0}^{7}\left(e^{jk\pi/4} + e^{-jk\frac{\pi}{4}} + \frac{1}{j}e^{j\frac{3k\pi}{4}} - \frac{1}{j}e^{-j\frac{3k\pi}{4}}\right)e^{j2\pi\frac{k}{8}n}$

$x[n] = \frac{1}{2}\sum_{k=0}^{7} \underbrace{e^{j2\pi k(\frac{1+n}{8})}}_{S_1} + \underbrace{e^{-j2\pi k(\frac{1-n}{8})}}_{S_2} - j\underbrace{e^{j2\pi k(\frac{3+n}{8})}}_{S_3} + j\underbrace{e^{-j2\pi k(\frac{3-n}{8})}}_{S_4}$

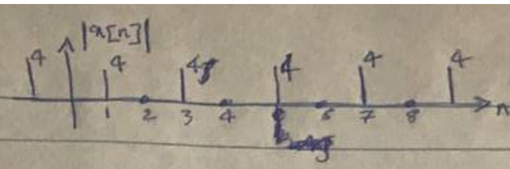
$S_1 = \begin{cases} 8 & n=7 \\ 0 & \text{others} \end{cases}$ $S_2 = \begin{cases} 8 & n=1 \\ 0 & \text{others} \end{cases}$ $S_3 = \begin{cases} 8 & n=5 \\ 0 & \text{others} \end{cases}$ $S_4 = \begin{cases} 8 & n=3 \\ 0 & \text{other} \end{cases}$

$\Rightarrow x[n] = \frac{1}{2}(S_1 + S_2 - jS_3 + jS_4)$

$\Rightarrow x[n] = 4\delta[n-7] + 4\delta[n-1] - j4\delta[n-5] + j4\delta[n-3]$

$0 \leq n \leq 7$

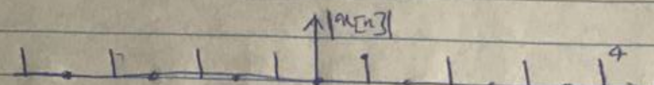
$x[n] =$



$$x[n] = \{\dots, \underset{\uparrow}{0}, \overline{4, 0, 4j, 0, -4j, 0, 4}, \dots\}$$

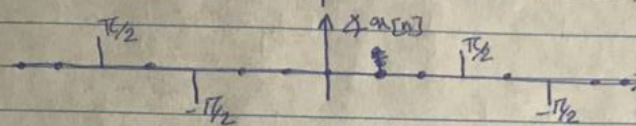
$$\{\dots, 0, 4, 0, 4j, 0, -4j, 0, 4, \underset{\uparrow}{0}, 4, 0, 4j, 0, -4j, 0, 4, \}$$

$|x[n]| = |x[-n]|$



is a even function

$\measuredangle x[n] = -\measuredangle x[-n]$



so $\measuredangle x[n]$ is a odd function

d) $$g[n] = \frac{1}{2} \sum_{k=0}^{15} \left(e^{j\frac{k\pi}{4}} + e^{-j\frac{k\pi}{4}} - je^{j\frac{k\pi 3}{4}} + je^{-j\frac{3k\pi}{4}}\right) e^{j\frac{2\pi kn}{16}}$$

$$= \frac{1}{2} \sum_{k=0}^{15} \underbrace{e^{j2\pi k\left(\frac{2+n}{16}\right)}}_{S_1} + \underbrace{e^{-j2\pi k\left(\frac{2-n}{16}\right)}}_{S_2} - j\underbrace{e^{j2\pi k\left(\frac{6+n}{16}\right)}}_{S_3} + j\underbrace{e^{-j2\pi k\left(\frac{6-n}{16}\right)}}_{S_4}$$

$$S_1 = \begin{cases} 16 & n = 14 \\ 0 & \text{others} \end{cases} \quad S_2 = \begin{cases} 16 & n = 2 \\ 0 & \text{others} \end{cases} \quad S_3 = \begin{cases} 16 & n = 10 \\ 0 & \text{others} \end{cases} \quad S_4 = \begin{cases} 16 & n = 6 \\ 0 & \text{others} \end{cases}$$

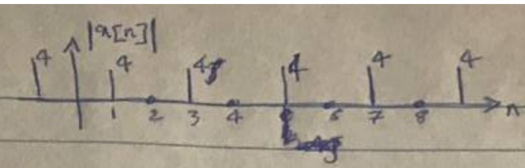
$$g[n] = \frac{1}{2}(S_1 + S_2 - jS_3 + jS_4) = \{$$

$$\{\overline{\underset{\uparrow}{0}, 0, 8, 0, 0, 0, j8, 0, 0, 0, -j8, 0, 0, 0, 8, 0}, \dots\}$$

there are four non zero values in the interval of lenght M

Q3d

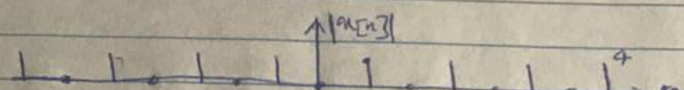
$x[n] =$



$x[n] = \{\ldots, \underset{\uparrow}{0}, \overline{4, 0, 4J, 0, -4J, 0, 4}, \ldots\}$

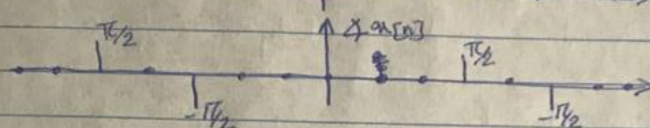
$\{\ldots, 0, 4, 0, 4J, 0, -4J, 0, 4, \underset{\uparrow}{0}, 4, 0, 4J, 0, -4J, 0, 4, \ldots\}$

$|x[n]| = |x[-n]|$



is a even function

$\measuredangle x[n] = -\measuredangle x[-n]$



so $\measuredangle x[n]$ is a odd function

d) $g[n] = \frac{1}{2} \sum_{k=0}^{15} \left(e^{\frac{Jk\pi}{4}} + e^{\frac{-Jk\pi}{4}} - Je^{\frac{Jk\pi 3}{4}} + Je^{-J\frac{3k\pi}{4}}\right) e^{\frac{J2\pi kn}{16}}$

$= \frac{1}{2} \sum_{k=0}^{15} \underbrace{e^{J2\pi k\left(\frac{2+n}{16}\right)}}_{S_1} + \underbrace{e^{-J2\pi k\left(\frac{2-n}{16}\right)}}_{S_2} - J\underbrace{e^{J2\pi k\left(\frac{6+n}{16}\right)}}_{S_3} + J\underbrace{e^{-J2\pi k\left(\frac{6-n}{16}\right)}}_{S_4}$

$S_1 = \begin{cases} 16 & n=14 \\ 0 & \text{others} \end{cases}$ $S_2 = \begin{cases} 16 & n=2 \\ 0 & \text{others} \end{cases}$ $S_3 = \begin{cases} 16 & n=10 \\ 0 & \text{others} \end{cases}$ $S_4 = \begin{cases} 16 & n=6 \\ 0 & \text{others} \end{cases}$

$g[n] = \frac{1}{2}(S_1 + S_2 - JS_3 + JS_4) = \{$

$\{\underset{\uparrow}{0}, \overline{0, 8, 0, 0, 0, J8, 0, 0, 0, -J8, 0, 0, 0, 8, 0}, \ldots\}$

there are four non zero values in the interval of lenght M