Of special interest in this course are systems whose input-output relationship can be described by a ***causal linear constant coefficient difference equation*** (LCCDE) of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad a_0 = 1,\ N \geq 0,\ M \geq 0 \tag{1}$$

where $\{y[n]\}$ denotes the system's output sequence when the input sequence is $\{x[n]\}$.

Notes:
• It can be shown that ***any system described by an LCCDE is both linear and time-invariant*** and hence any such system can be characterized by its impulse response.
• In equation (1), the coefficient a_0 is specified to equal 1. It should be mentionned that this is done for convenience and does not result in a loss of generality. Note that if $a_0 \neq 1$, we can always divide both sides of equation (1) by a_0 to find an equivalent LCCDE for which $a_0 = 1$.

The LCCDE of (1) is said to be ***recursive*** if $N \geq 1$ and ***nonrecursive*** if $N = 0$. It is sometimes useful to distinguish the nonrecursive case from the recursive case. For this purpose the LCCDE of equation (1) has been rewritten for the two cases as shown below. Equation (2a) illustrates the nonrecursive case while equation (2b) illustrates the recursive case. In both cases, a_0 is assumed equal to 1.

$$y[n] = \begin{cases} \displaystyle\sum_{k=0}^{M} b_k x[n-k], & N = 0,\ M \geq 0 \quad \text{(2a)} \\ \displaystyle -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k], & N \geq 1,\ M \geq 0 \quad \text{(2b)} \end{cases}$$

Equation (2a) is said to be nonrecursive since the current output value, $y[n]$, is determined as a linear combination of the current and past M input values; in particular, computation of the current output value does not require any past output values. In contrast, equation (2b) is said to be a ***recursive system of order N*** since determination of the current output value requires N past output values (in addition to the current and M past input values). Whereas the output of a nonrecursive system ($N = 0$ and $M < \infty$) will be of finite length whenever the input is of finite length, the output of a recursive system ($N \geq 1$) is usually[1] infinite in length. We illustrate this fact by finding the impulse response for several systems (both recursive and nonrecursive) in the section below.

Finite Impulse Response (FIR) *vs.* Infinite Impulse Response (IIR) Systems

The ***impulse response*** of a system is defined as the response of the system when the input to the system is the Kronecker Delta function or unit impulse sequence. [Recall that the unit impulse sequence is finite in length; in particular, we say it has length 1 since it is nonzero for only one value of n (*i.e.,* it has a value of 1 when $n = 0$ and has a value of 0 for all other n).] A system whose impulse response is finite in length is called an **FIR system** whereas a system whose impulse response is infinite in length is called an **IIR system**.

1. The output of a recursive system is occasionally finite in length due to pole-zero cancellations.

Since systems described by the LCCDEs of equations (2a) and (2b) are causal, the impulse response, $h[n]$, of such a system will be zero for $n < 0$. For sufficiently simple systems, the impulse response can be found by iterating the difference equation for $n \geq 0$ with $y[n]$ replaced by $h[n]$, $x[n]$ replaced by $\delta[n]$ and initial conditions (ICs): $h[-1] = ... = h[-N] = 0$. This approach is demonstrated in the three examples below. Other approaches to finding a closed form expression for the impulse response will be necessary for more complex systems and will be discussed in later handouts.

In general, the impulse response of a nonrecursive LCCDE (with $M < \infty$) will be of finite length and hence any system described by a nonrecursive LCCDE will be an FIR system. Although the impulse response of a recursive LCCDE will usually be infinite in length, there are some exceptions (see example 3 below). For now we note that whenever a recursive LCCDE is found to be FIR, the LCCDE which describes the system can always be rewritten as a nonrecursive LCCDE (as illustrated by the exercise following example 3).

Example 1: Find the impulse response, $h[n]$, of the nonrecursive system described by the LCCDE below and determine whether the associated system is FIR or IIR.

$$y[n] = 2x[n] - 3x[n-1] \tag{3}$$

Solution: Since the impulse response, $h[n]$, is the response of the system when $x[n] = \delta[n]$, we can iterate equation (3) with $y[n]$ replaced by $h[n]$ and $x[n]$ replaced by $\delta[n]$ to find:

$$\left.\begin{array}{l} h[0] = 2\delta[0] - 3\delta[0-1] = 2\times 1 - 3\times 0 = 2 \\ h[1] = 2\delta[1] - 3\delta[1-1] = 2\times 0 - 3\times 1 = -3 \\ h[2] = 2\delta[2] - 3\delta[2-1] = 2\times 0 - 3\times 0 = 0 \\ \text{for n>1:} \\ h[n] = 2\delta[n] - 3\delta[n-1] = 2\times 0 - 3\times 0 = 0 \end{array}\right\} \Rightarrow \quad h[n] = \begin{cases} 2 = b_0, & n = 0 \\ -3 = b_1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Since the impulse response is found to be of finite length (in particular, the nonzero portion of $h[n]$ is of length 2), the system described by equation (3) is an FIR system.

Note that the nonzero elements of the impulse response for the system associated with equation (3) have values which are equal to the coefficients of the difference equation. This is not a coincidence; it will always be the case. To see this, we note that the impulse response for an arbitrary nonrecursive causal system is found (by replacing $y[n]$ by $h[n]$ and $x[n]$ by $\delta[n]$ in equation 2a) to be:

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where the second equality follows from the sifting property of the Kronecker delta function.

From equation (4), it follows that the length of the impulse response of a nonrecursive system described by equation (2a) will be at most $M+1$; it will be $M+1$ provided $b_0 \neq 0$ and $b_M \neq 0$; otherwise, it will be less than $M+1$. Hence any system described by an LCCDE with $N = 0$ and $0 \leq M < \infty$ has an impulse response which is finite in length and is therefore an FIR system.

Example 2: Find the impulse response, $h[n]$, of the recursive system described by the LCCDE below and determine whether the associated system is FIR or IIR.

$$y[n] = -\frac{1}{2}y[n-1] + x[n] \tag{5}$$

Solution: Recall that the impulse response, $h[n]$, is the response of the system when $x[n] = \delta[n]$. Furthermore, since the system is causal and $\delta[n] = 0$ for $n < 0$, we can assume that $h[n] = 0$ for $n < 0$. Hence, as shown below, we can find $h[n]$ by iteratating equation (5) for $n \geq 0$ with $y[n]$ replaced by $h[n]$, $x[n]$ replaced by $\delta[n]$, and $h[-1] = 0$.

$$\begin{aligned}
h[0] &= -\frac{1}{2}h[-1] + \delta[0] = \left(-\frac{1}{2} \times 0\right) + 1 = 1 \\
h[1] &= -\frac{1}{2}h[0] + \delta[1] = \left(-\frac{1}{2} \times 1\right) + 0 = -\frac{1}{2} \\
h[2] &= -\frac{1}{2}h[1] + \delta[2] = \left(-\frac{1}{2} \times \left(-\frac{1}{2}\right)\right) + 0 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4} \\
h[3] &= -\frac{1}{2}h[2] + \delta[3] = \left(-\frac{1}{2} \times \left(-\frac{1}{2}\right)^2\right) + 0 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}
\end{aligned}$$

In general we may have to continue this procedure indefinitely. For this particular difference equation, it is straightforward to see that for $n \geq 1$, $h[n] = -½h[n-1]$, and since we found that $h[0] = 1$, we may conclude that:

$$h[n] = \left(-\frac{1}{2}\right)^n u[n] = \begin{cases} 0, & n < 0 \\ \left(-\frac{1}{2}\right)^n, & n \geq 0 \end{cases}$$

Since the impulse response is found to be of infinite length (*i.e.*, the sequence $h[n]$ includes an infinite number of nonzero elements), we conclude that the recursive system described by equation (5) is an IIR system.

Example 3: Find the impulse response, $h[n]$, of the system described by the following recursive LCCDE and determine whether the associated system is FIR or IIR.

$$y[n] = y[n-1] + x[n] - \frac{1}{3}x[n-1] - \frac{2}{3}x[n-2] \tag{6}$$

Solution: Similar to the previous example, we can find $h[n]$ by iteratating equation (6) for $n \geq 0$ with $y[n]$ replaced by $h[n]$, $x[n]$ replaced by $\delta[n]$, and $h[-1] = 0$. Hence:

$$\begin{aligned}
h[0] &= h[-1] + \delta[0] - \frac{1}{3}\delta[-1] - \frac{2}{3}\delta[-2] = 0 + 1 - \left(\frac{1}{3} \times 0\right) - \left(\frac{2}{3} \times 0\right) = 1 \\
h[1] &= h[0] + \delta[1] - \frac{1}{3}\delta[0] - \frac{2}{3}\delta[-1] = 1 + 0 - \left(\frac{1}{3} \times 1\right) - \left(\frac{2}{3} \times 0\right) = \frac{2}{3} \\
h[2] &= h[1] + \delta[2] - \frac{1}{3}\delta[1] - \frac{2}{3}\delta[0] = \frac{2}{3} + 0 - \left(\frac{1}{3} \times 0\right) - \left(\frac{2}{3} \times 1\right) = 0 \\
h[3] &= h[2] + \delta[3] - \frac{1}{3}\delta[2] - \frac{2}{3}\delta[1] = 0 + 0 - \left(\frac{1}{3} \times 0\right) - \left(\frac{2}{3} \times 0\right) = 0
\end{aligned}$$

For $n \geq 3$, we find that $h[n] = h[n-1]$ and since we found that $h[2] = 0$, it is clear that $h[n] = 0$ for $n \geq 2$. Thus the impulse response of the system described by LCCDE (6) is given by:

$$h[n] = \begin{cases} 1, \; n = 0 \\ \frac{2}{3}, \; n = 1 \\ 0, \; \text{otherwise} \end{cases} \tag{7}$$

Since the impulse response is finite in length (it has length 2), this is a FIR system.

As mentionned previously, the fact that the recursive system of example 3 was found to be FIR indicates that the LCCDE of equation (6) can be rewritten as a nonrecursive LCCDE. This is illustrated in the exercise below.

Exercise:
Find a nonrecursive LCCDE which is equivalent to the recursive LCCDE of Example 3.
Solution:
Having already found the impulse response of the system in example 3, it is straightforward to find an equivalent nonrecursive LCCDE. Recall that any system described by a LCCDE is LTI and hence is characterized by its impulse response. In particular, we can find the response of such a system by convolving the input with the impulse response (equation 7). Hence:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{8}$$

Since the impulse response of LCCDE (6) has only 2 nonzero values, the infinite sum in equation (8) can be reduced to a sum of two terms (namely, the terms corresponding to $k = 0$ and $k = 1$). Hence:

$$y[n] = h[0]x[n] + h[1]x[n-1] \tag{9}$$

Substituting $h[0] = 1$ and $h[1] = 2/3$ into equation (9), yields the following nonrecursive LCCDE:

$$y[n] = x[n] + \frac{2}{3}x[n-1] \tag{10}$$

To verify that the nonrecursive LCCDE of equation (10) is equivalent to the LCCDE of equation (6), we first note (see equation 10) that:

$$y[n-1] = x[n-1] + \frac{2}{3}x[n-2] \tag{11}$$

Adding and substracting $y[n-1]$ from the right hand side of equation (10) yields:

$$y[n] = y[n-1] - y[n-1] + x[n] + \frac{2}{3}x[n-1] \tag{12}$$

and substituting expression (11) for the second instance of $y[n-1]$ on the right hand side of equation (12) to yields:

$$y[n] = y[n-1] - \left(x[n-1] + \frac{2}{3}x[n-2]\right) + x[n] + \frac{2}{3}x[n-1] \tag{13}$$

which upon simplification results in the original LCCDE of equation (6).