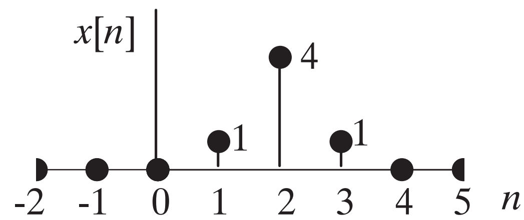
Representation of Discrete-Time Signals

1. Graphical representation:



2. Functional representation:

$$x[n] = \begin{cases} 1, & n = 1, 3 \\ 4, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

or

$$y[n] = \begin{cases} (0.8)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

3. Tabular representation:

n	⋯	-2	-1	0	1	2	3	4	5	⋯
$x[n]$	⋯	0	0	0	1	4	1	0	0	⋯

4. Sequence representation:
   Note: unless otherwise specified the arrow points to the $n = 0$ location.

$$x[n] = \{..., 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, ...\}$$

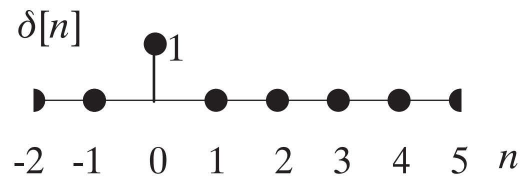
or

$$x[n] = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, ...\}$$

Commonly Used Discrete-Time Signals

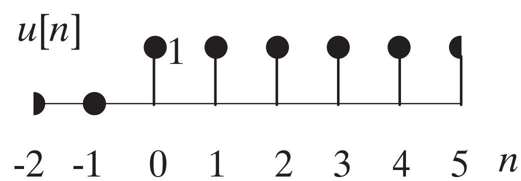
1. Unit Sample Sequence or Unit Impulse or Kronecker Delta, $\delta[n]$:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$



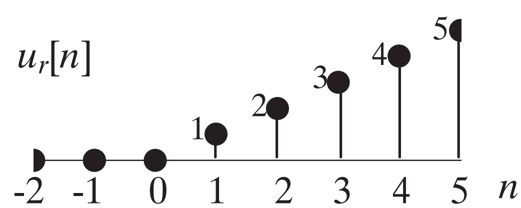
2. Unit Step Sequence, $u[n]$:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



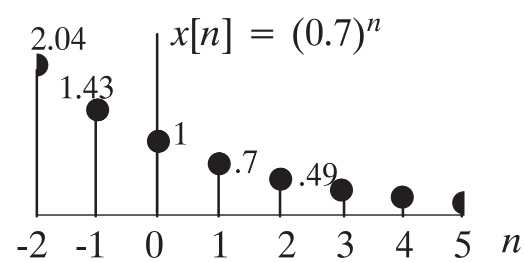
3. Unit Ramp Sequence, $u_r[n]$:

$$u_r[n] = \begin{cases} n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



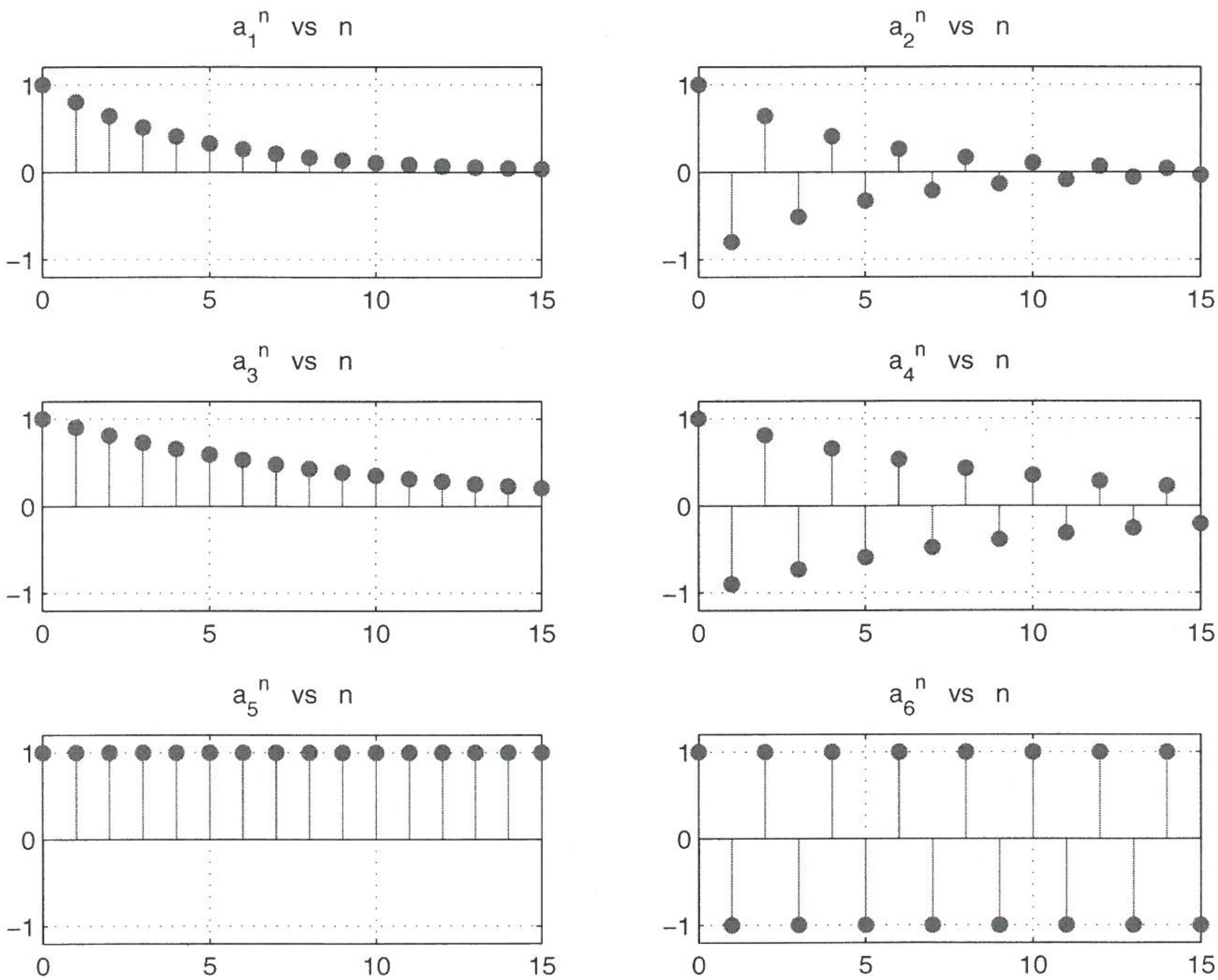
4. Exponential Sequence:

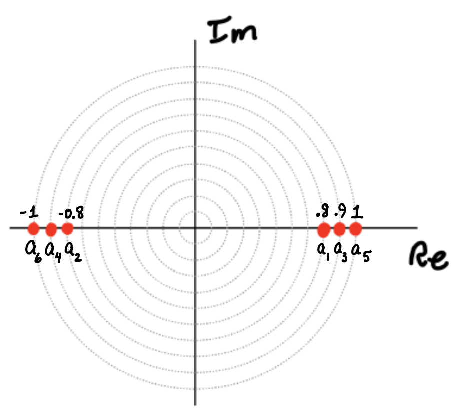
$$x[n] = a^n$$



note: in general, a can be real-valued or complex-valued. For the signal shown to the right, a is real (a=0.7), and there is no oscillation. In the case that a is complex-valued, the signal will be complex-valued; the real and imaginary parts will oscillate in accordance with the phase of a and will decay or grow according to the magnitude of a as shown by the equations below.

if: $a = re^{j2\pi F_0}$, then: $x[n] = r^n e^{j2\pi F_0 n} = r^n\left(\cos(2\pi F_0 n) + j\sin(2\pi F_0 n)\right)$





$a_1 = 0.8 = 0.8e^{j2\pi 0}$

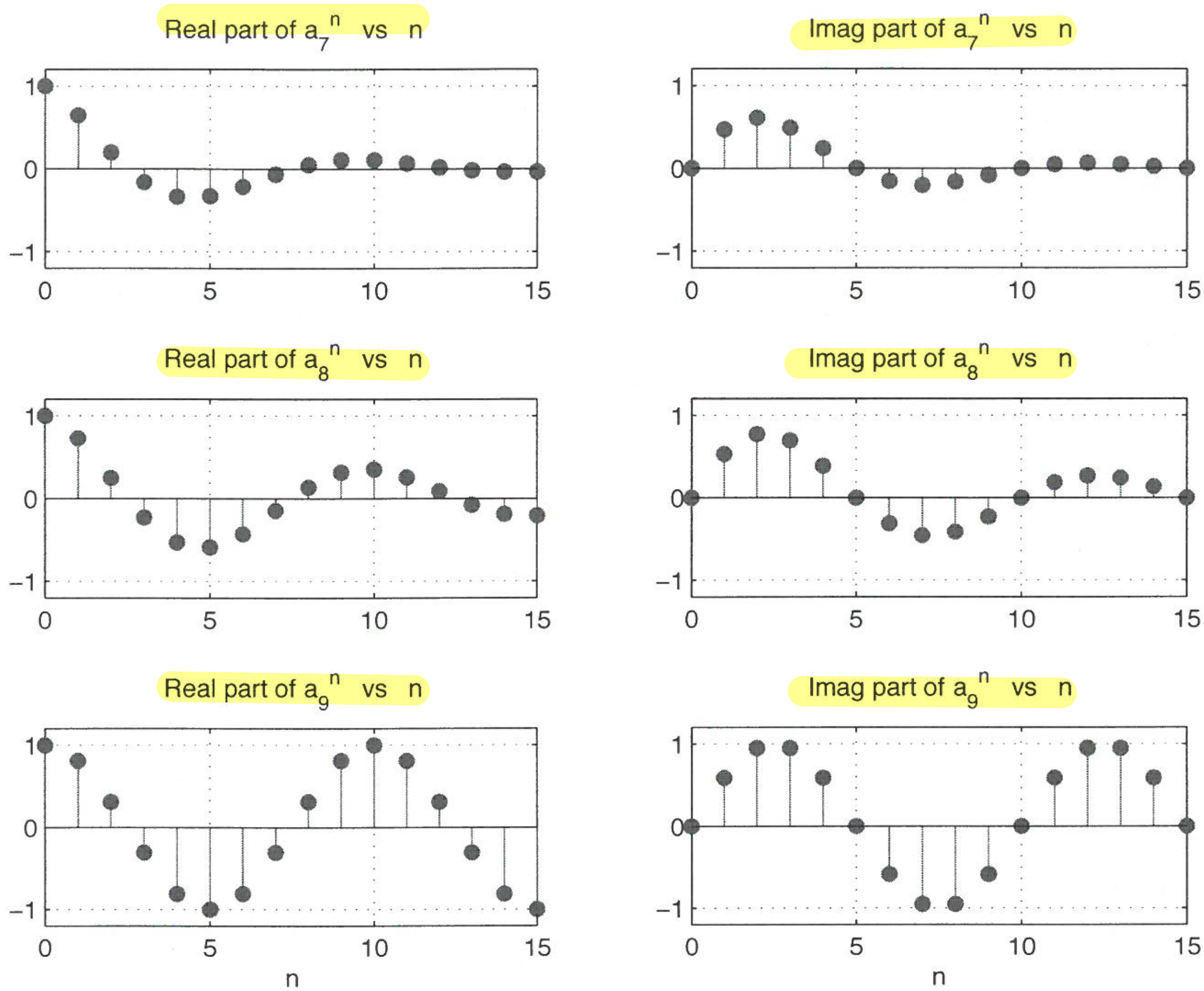
$a_3 = 0.9 = 0.9e^{j2\pi 0}$

$a_5 = 1 = 1\,e^{j2\pi 0}$

$a_2 = -0.8 = 0.8e^{j2\pi \frac{1}{2}}$

$a_4 = -0.9 = 0.9e^{j2\pi \frac{1}{2}}$

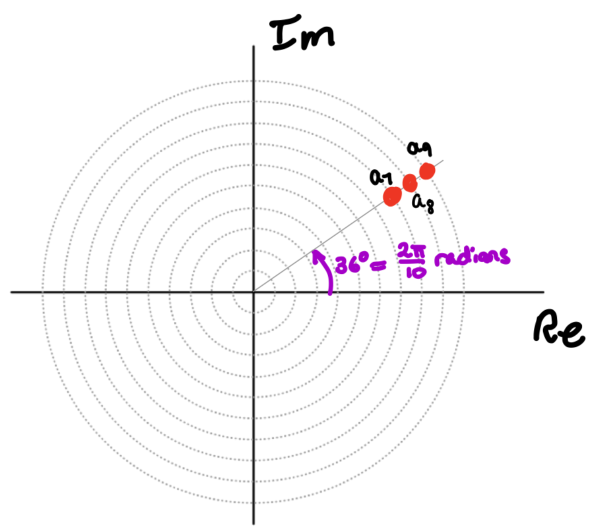
$a_6 = -1 = 1\,e^{j2\pi \frac{1}{2}}$



$a_7 = 0.8\, e^{j 2\pi \frac{1}{10}}$

$a_8 = 0.9\, e^{j 2\pi \frac{1}{10}}$

$a_9 = 1\, e^{j 2\pi \frac{1}{10}}$



Energy Signals vs Power Signals

The **energy**, E_x, of a discrete-time signal, $x[n]$, is defined as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

If $0 \le E_x < \infty$, then $x[n]$ is an ***energy signal***.

The energy of a discrete-time signal, $x[n]$, with respect to the interval, $n_1 \le n \le n_2$, is denoted by $E_{x,[n_1,n_2]}$ and defined as:

$$E_{x,\,[n_1,n_2]} = \sum_{n=n_1}^{n_2} |x[n]|^2$$

The **average power**, P_x, of a discrete-time signal, $x[n]$, is defined as:

$$P_x = \lim_{N\to\infty}\left[\frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2\right]$$

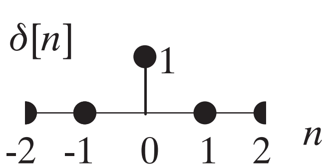
If $0 < P_x < \infty$, then $x[n]$ is a ***power signal***.

If $x[n]$ is periodic with period N, the average power of $x[n]$ can be computed as:

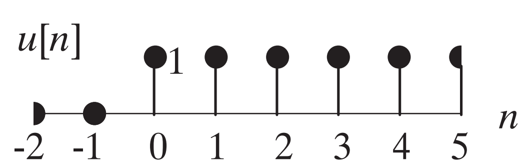
$$P_x = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} |x[n]|^2$$

Exercise: Determine whether each of the following is a power signal, energy signal, or neither.

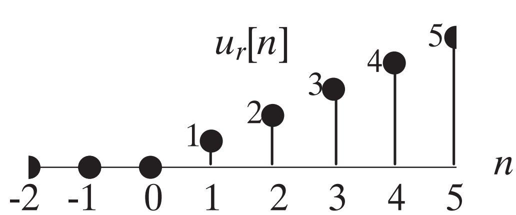
a)



b)

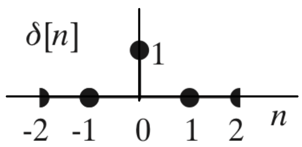


c)



Exercise: Determine whether each of the following is a power signal, energy signal, or neither.

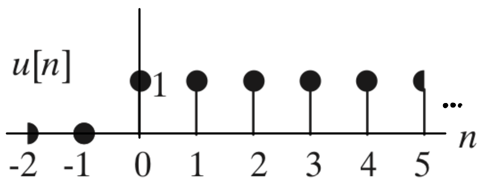
a)



$$E_\delta = \sum_{n=-\infty}^{\infty} |\delta[n]|^2 =$$

$$P_u = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\delta[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \left(\qquad \right) =$$

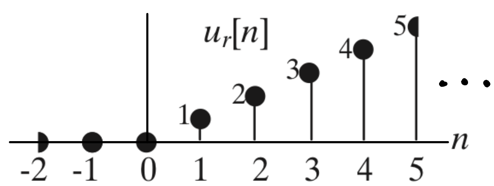
b)



$$E_u = \sum_{n=-\infty}^{\infty} |u[n]|^2 =$$

$$P_u = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \left(\qquad \right)$$

c)



$$E_{u_r} = \sum_{n=-\infty}^{\infty} |u_r[n]|^2 =$$

$$P_{u_r} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u_r[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \left(\qquad \right)$$

Average Power Examples

Let $y[n] = A\cos\left(2\pi \frac{1}{5} n\right)$

a) Find the average power, P_y, of $y[n]$.

Soln: Since $y[n]$ is periodic with period 5, we may find P_y as follows:

$$P_y = \frac{1}{5}\sum_{n=0}^{4} |y[n]|^2 = \frac{1}{5}\sum_{n=0}^{4} A^2\cos^2\left(2\pi\frac{1}{5}n\right)$$

$$= \frac{1}{5}\sum_{n=0}^{4}\frac{A^2}{2}\left[1+\cos\left(2\pi\frac{2}{5}n\right)\right]$$

$$= \frac{1}{5}\sum_{n=0}^{4}\frac{A^2}{2} + \underbrace{\frac{1}{5}\sum_{n=0}^{4}\frac{A^2}{2}\underbrace{\cos\left(2\pi\frac{2}{5}n\right)}_{\text{periodic with period 5}}}_{\text{the sum over one period of a sinusoid is 0}}$$

a proof of this is provided on the next page.

$$= \frac{1}{5}\left(\frac{A^2}{2}+\frac{A^2}{2}+\frac{A^2}{2}+\frac{A^2}{2}+\frac{A^2}{2}\right) = \frac{A^2}{2}$$

b) Find the RMS value of $y[n]$.

RMS stands for Root Mean Square.
It is the square root of the mean square value.
P_y is the mean of the squared magnitude of $y[n]$.
RMS value of $y[n] = \sqrt{P_y} = \frac{A}{\sqrt{2}}$

Stevenson

Proof that $\sum_{n=0}^{4} \cos\left(2\pi \frac{2}{5} n\right) = 0$

$x = x_R + j x_I$	
$y = y_R + j y_I$	
$x+y = x_R + y_R + j(x_I + y_I)$	
$Re\{x+y\} = x_R + y_R = Re\{x\} + Re\{y\}$	
$Im\{x+y\} = x_I + y_I = Im\{x\} + Im\{y\}$	

$$\sum_{n=0}^{4} \cos\left(2\pi \tfrac{2}{5} n\right) = \sum_{n=0}^{4} Re\left\{e^{j2\pi \frac{2}{5} n}\right\}$$

using Euler's

$$= Re\left\{\sum_{n=0}^{4} e^{j2\pi \frac{2}{5} n}\right\}$$

the sum of the real parts is equal to the real part of the sum

$$= Re\{0 + j0\}$$

using the geometric sum (see below)

$$= 0$$

Use of the Geometric Sum in the problem above:

$$\boxed{\sum_{n=n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a}, \quad a \neq 1}$$

Geometric Sum
(will be on your formula sheet)

$$\sum_{n=0}^{4} e^{j2\pi \frac{2}{5} n} = \sum_{n=0}^{4} \Big(\underbrace{e^{j2\pi \frac{2}{5}}}_{a}\Big)^n = \frac{\left(e^{j2\pi\frac{2}{5}}\right)^0 - \left(e^{j2\pi\frac{2}{5}}\right)^{(4+1)}}{1 - e^{j2\pi\frac{2}{5}}}$$

$$= \frac{1 - e^{j2\pi \frac{2}{\cancel{5}}\cancel{5}}}{1 - e^{j2\pi\frac{2}{5}}} = \underbrace{\frac{1-1}{1 - e^{j2\pi\frac{2}{5}}}}$$

note the numerator is equal to zero & the denominator is not equal to 0. Therefore the quotient is 0.

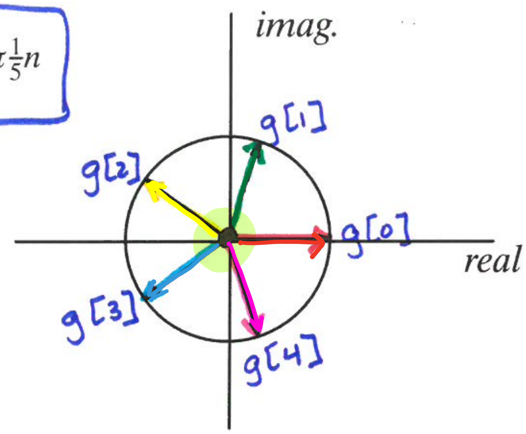
$$= 0$$

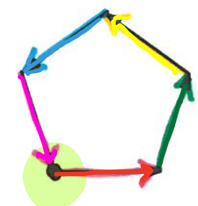
By the way:
if the denominator is zero, then $a=1$ which means the sum is easily determined without the geometric formula.

Included this handout from ECE 3511 in case you are interested.

Illustration that the sum of a d.t. complex exponential over one period is equal to zero. (This statement assumes that the period, N, is greater than 1 sample.)

$$g[n] = e^{j2\pi\frac{1}{5}n}$$



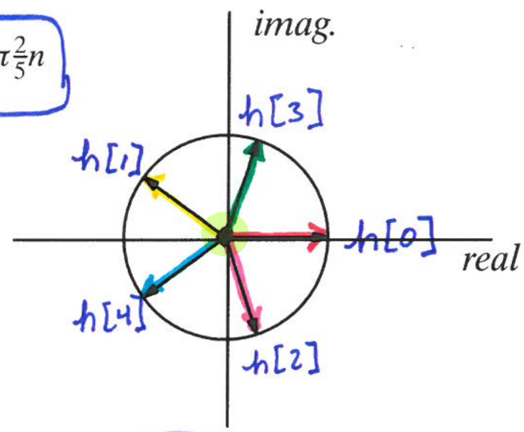


note: the vector sum is zero

$$\sum_{n=0}^{4} g[n] = 0$$

The sum over one period of a complex exponential with period N, amounts to the sum of N points equally spaced around the unit circle.

$$h[n] = e^{j2\pi\frac{2}{5}n}$$



$h[0] = g[0]$
$h[1] = g[2]$
$h[2] = g[4]$
$h[3] = g[1]$
$h[4] = g[3]$

$$\sum_{n=0}^{4} h[n] = \sum_{n=0}^{4} g[n] = 0$$

$$\sum_{n=0}^{4} h[n] = 0 \implies \begin{cases} \mathrm{Re}\left\{\sum_{n=0}^{4} h[n]\right\} = 0 \Rightarrow \sum_{n=0}^{4} \mathrm{Re}\{h[n]\} = 0 \\ \qquad \Rightarrow \sum_{n=0}^{4} \cos(2\pi\frac{2}{5}n) = 0 \\ \mathrm{Im}\left\{\sum_{n=0}^{4} h[n]\right\} = 0 \Rightarrow \sum_{n=0}^{4} \mathrm{Im}\{h[n]\} = 0 \\ \qquad \Rightarrow \sum_{n=0}^{4} \sin(2\pi\frac{2}{5}n) = 0 \end{cases}$$

sum_complex

Note: The average power of a sum of signals is not usually equal to the sum of the individual average powers.

if $z[n] = x[n] + y[n]$, we may <u>not</u> conclude that $P_z = P_x + P_y$.

In the case that $x[n]$ & $y[n]$ are orthogonal, we will find that $P_z = P_x + P_y$

<u>Ex. 1</u> Show that:

if $z[n] = A\cos(2\pi \frac{1}{5} n) + B\cos(2\pi \frac{1}{5} n + 45°)$

then $P_z = \frac{A^2}{2} + \frac{B^2}{2} + AB\cos(45°)$

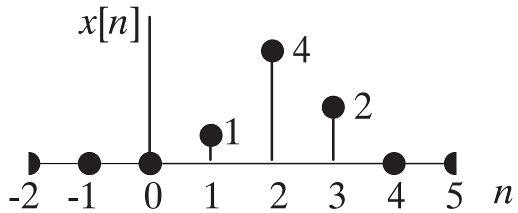
<u>Ex. 2</u> Show that:

if $z[n] = A\cos(2\pi \frac{1}{5} n) + B\sin(2\pi \frac{1}{5} n)$

then $P_z = \frac{A^2}{2} + \frac{B^2}{2}$

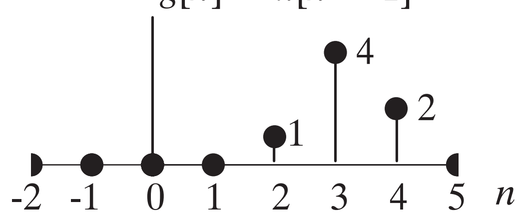
Simple Manipulations of Discrete-Time Signals

1. time shift: The signal $x[n-k]$, where k is an integer, denotes a shifted version of the signal $x[n]$. If $k > 0$, the signal $x[n-k]$ is delayed by k samples relative to the signal $x[n]$; and if $k < 0$, the signal $x[n-k]$ is advanced by $|k|$ samples relative to the signal $x[n]$.

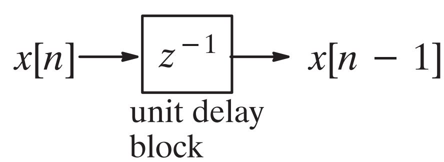


$$x[n] = \{..., 0, \underset{\uparrow}{0}, 1, 4, 2, 0, 0, ...\}$$

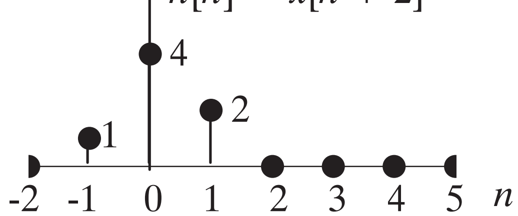
$g[n] \equiv x[n-1]$



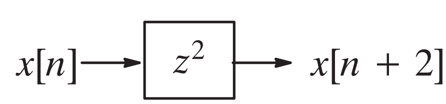
$$g[n] = \{..., \underset{\uparrow}{0}, 0, 1, 4, 2, 0, 0, ...\}$$



$h[n] \equiv x[n+2]$



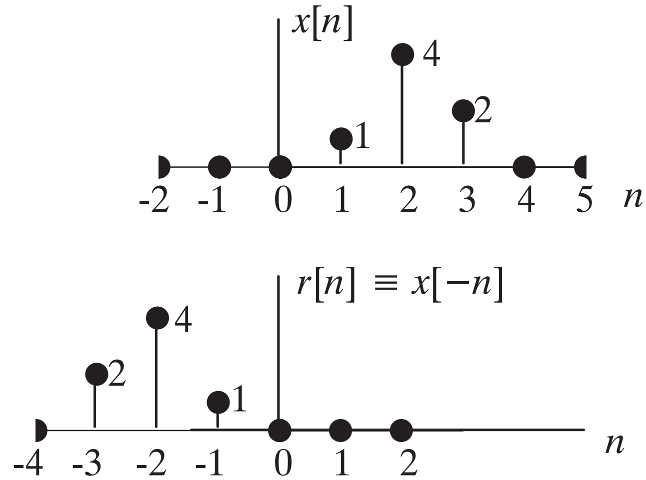
$$h[n] = \{..., 0, 0, 1, \underset{\uparrow}{4}, 2, 0, 0, ...\}$$



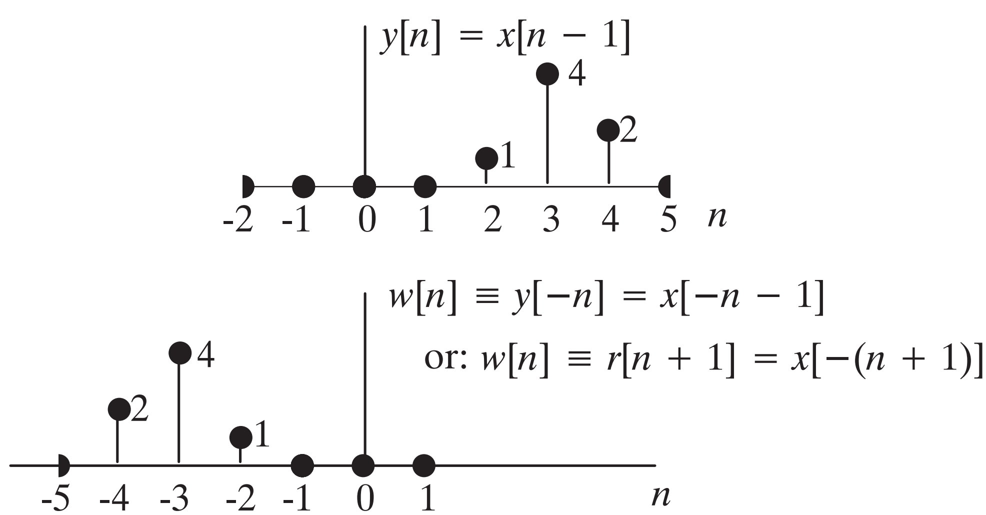
or time-negation

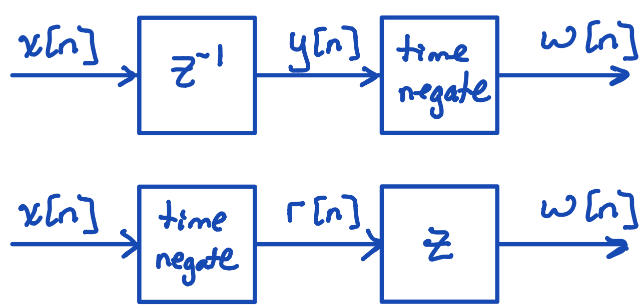
2. **Reflection** (or folding) about the time origin. A signal, $x[n]$, may be reflected about the time origin by replacing the independent variable, n, by $-n$.

Example 1:



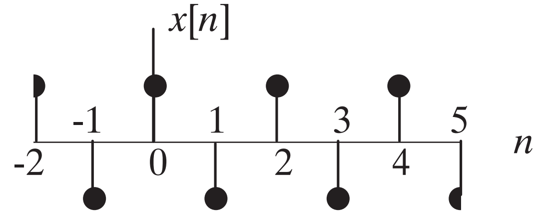
Example 2:

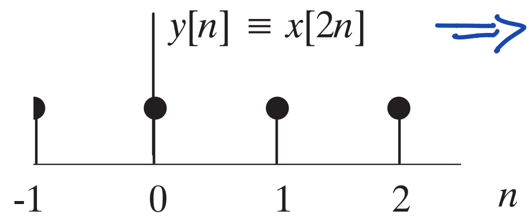




3. **Down-sampling** (or time-scaling). ***Down sampling*** a signal refers to replacing the independent variable, n, by μn, where μ is an integer.

Example 1:





$\Longrightarrow$ y[0] = x[0]
y[1] = x[2]
y[2] = x[4]
⋮

Note: if the original $x[n]$ was obtained by sampling an continuous-time signal every T_s seconds:

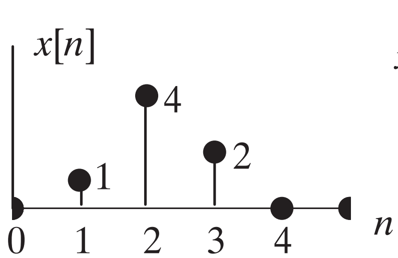
$$x[n] = x_a(nT_s)$$

then the sequence $y[n] = x[2n]$ is what you get by sampling the same continuous-time signal once every $2T_s$ seconds:

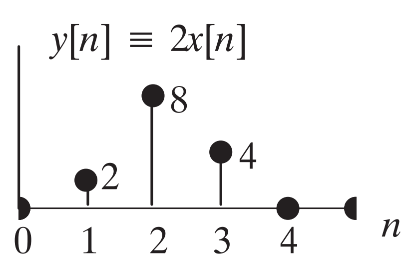
$$y[n] = x[2n] = x_a(n2T_s)$$

i.e., the sampling rate has been cut in half.

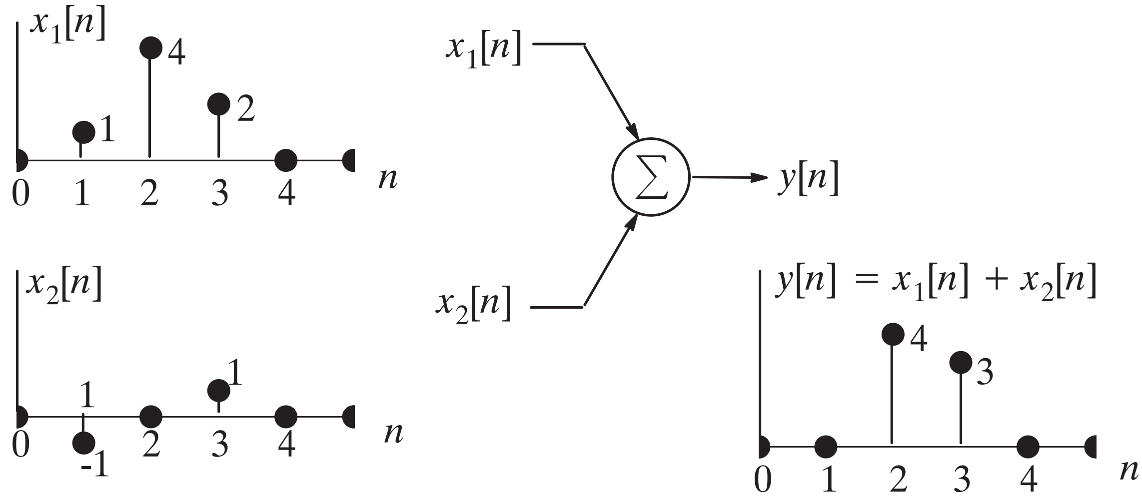
4. **Amplitude Scaling** by A refers to replacing every signal sample by the original sample multiplied by A.



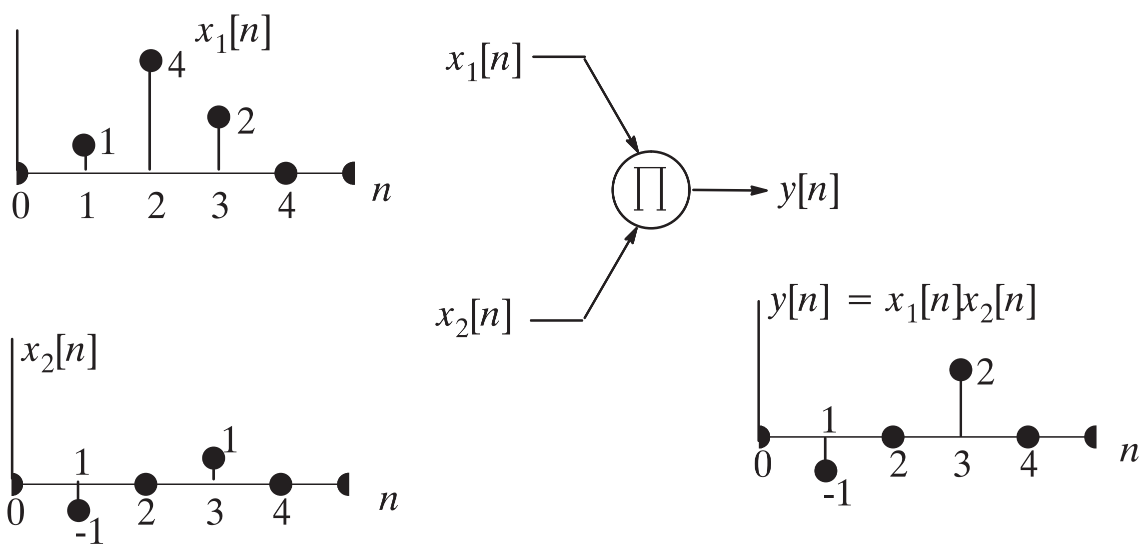
$$x[n] \xrightarrow{2} y[n]$$



5. **Sum of two signals**



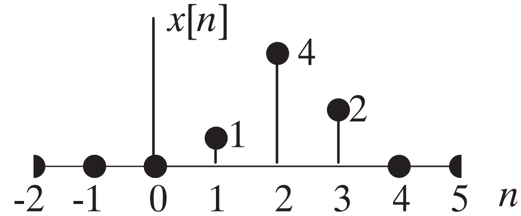
6. **Product of two signals**



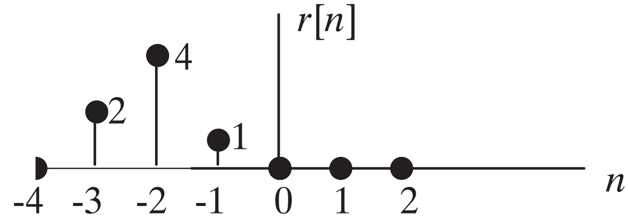
7. **Causal, Anticausal, and Noncausal signals:**

A signal is said to be ***causal*** if and only if it is equal to 0 for all $n < 0$; a signal is said to be ***anticausal*** if it is equal to 0 for all $n > 0$; a signal is said to be ***non-causal*** if it is not causal.

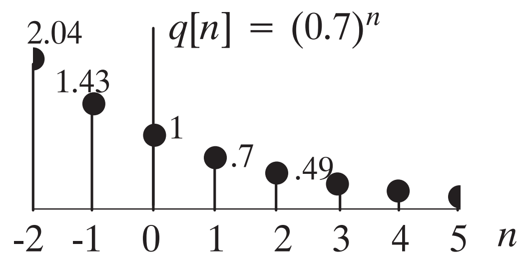
Examples:



$x[n]$ is causal



$r[n]$ is anticausal



$q[n]$ is noncausal