1. Consider the following periodic signal (one period of the signal is shown in bold and bracketed above):

$$x[n] = \{..., 1, \overbrace{\mathbf{0, 1, 2, \underset{\uparrow}{3}, 2, 1},} 0, 1, ...\}$$

(5 pts.) a) Note that $x[n]$ is a real and even sequence and it is periodic with period $N = 6$. As discussed in class, the DFS coefficients will be periodic with the same period as $x[n]$. It is also straightforward to prove that the DFS coefficients of a real and even periodic sequence will also be real and even. Find a closed-form expression for the Discrete Fourier Series (DFS) coefficients, $\{c_k\}$, of $x[n]$. Your closed-form expression should be expressed in a form that makes it easy to see that the DFS coefficients $\{c_k\}$ can be viewed as a real and even periodic sequence with period 6.
Hint: Since $x[n]$ is an even function of n, you will find that once you expand the DFS analysis sum, you will be able to use Euler's identity to combine pairs of complex exponentials into cosines (depending on the interval of n that you used in the analysis sum, you may need to think about equivalent discrete-time frequencies before it becomes obvious how to combine the exponentials into cosines.); this will result in an expression for c_k as a sum of cosines, all of which will be periodic with period 6.

(3 pts.) b) Evaluate the expression you found for the DFS coefficients in part (a) to find the values of c_k, $k = 0, ..., N-1$. Recalling that $\cos(\pi/3) = ½$ and $\cos(2\pi/3) = -½$, it should be easy to evaluate your expressions without the use of a calculator; ***leave your values expressed as fractions***. Making use of the periodicity of the coefficients $\{c_k\}$, provide a sequence representation (similar to that provided above for $x[n]$) of the DFS coefficients, $\{c_k\}$, on an interval symmetric about the origin. The interval should contain at least one complete period of the coefficients.

(3 pts.) c) Verify Parseval's theorem for discrete-time periodic signals by computing the average power of $x[n]$ in both the time and frequency domain. ***Show your calculations. Express your answer (i.e., the average power) as a fraction.***

2. Consider the following aperiodic signal (note the relation to the signal, $x[n]$, in question 1):

$$y[n] = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

(3 pts.) a) Evaluate the Discrete-Time Fourier-Transform (DTFT) sum to find $Y_{\mathrm{DTFT}}(F)$, the DTFT of $y[n]$.

(3 pts.) b) Compare the DFS coefficients for $x[n]$ of question 1 to $Y_{\mathrm{DTFT}}(F)$. Express c_k in terms of $Y_{\mathrm{DTFT}}(\cdot)$.

(5 pts.) c) Similar to the relationship you established in part (b), what should be the relationship between $Y_{\mathrm{DTFT}}(\cdot)$ and the DFS coefficients of the periodic signal $z[n]$ shown below? (note that one period of the signal is shown in bold and bracketed above.) Use your stated relationship together with the result of part (a) to find a closed-form expression for the DFS coefficients of $z[n]$. Your final closed-form expression should not refer to the function $Y_{\mathrm{DTFT}}(\cdot)$.

$$z[n] = \{..., 1, 0, \overbrace{\mathbf{0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0},} 0, 1, ...\}$$

3. Consider the d.t. periodic signal: $x[n] = \begin{cases} 1, & (n \bmod 9) = 0,\ 1,\ \text{or } 8 \\ 0, & \text{otherwise} \end{cases}$

(2 pts.) a) Sketch $x[n]$. Let N denote the fundamental period of $x[n]$. What is the value of N?

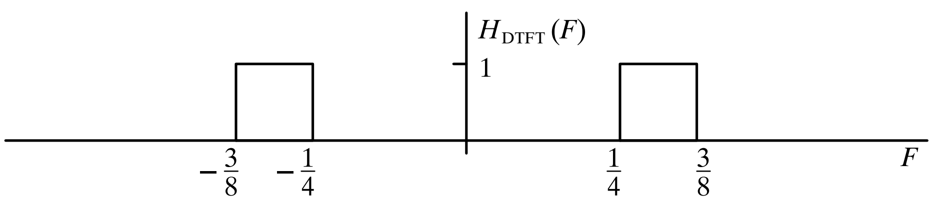
(3 pts.) b) Find a closed-form expression for the DFS coefficients $\{c_k\}$ of $x[n]$.

(2 pts.) c) How would you describe the real-imag/even-odd characteristics of $x[n]$? of $\{c_k\}$?

(5 pts.) d) For which values of k, $0 \le k \le 8$, is $|c_k| = 0$? To which discrete-time frequencies, do these values of k correspond? In the case that the corresponding discrete-time frequency is outside the interval $-½ \le F < ½$, state an equivalent d.t. frequency that lies within the interval.

(2 pts.) e) What is the value of c_0 ? of c_9?

(8 pts.) 4. Find a closed-form expression that describes the impulse response, $h[n]$, of the discrete-time ideal bandpass filter whose frequency response, $H_{\text{DTFT}}(F)$, is as shown below for $-0.5 \le F \le 0.5$. Since $H_{\text{DTFT}}(F)$ is real and even, we know that $h[n]$ should also be real and even. Your final expression for $h[n]$ must be simplified, if necessary, so as to eliminate any reference to j $\left(\text{where } j = \sqrt{-1}\right)$; it should also be obvious, via a simple inspection of your expression that $h[n]$ is even.



(4 pts.) 5. Given a stable LTI system with frequency response function:

$$H_{\text{DTFT}}(F) = [1 + \cos(2\pi F)]e^{-j2\pi F},$$

find $y_{ss}[n]$, the steady-state response of the system when the input is $x[n] = \cos\left(2\pi \frac{1}{10} n\right)$.

6. Let $h[n]$ be a ***real-valued*** sequence with DTFT: $H_{\text{DTFT}}(F)$, and Z-transform: $H_z(z)$.

(5 pts.) a) Show that $H_{\text{DTFT}}(F)$ is Hermitian, *i.e.*, show that $H^*_{\text{DTFT}}(F) = H_{\text{DTFT}}(-F)$.

(3 pts.) b) Let $G_z(z) = H_z(z^{-1})$. Knowing that $H_{\text{DTFT}}(F) = H_z(z)|_{z=e^{j2\pi F}}$, show that $H_{\text{DTFT}}(-F)$ may be expressed as: $H_{\text{DTFT}}(-F) = G_z(z)|_{z=e^{j2\pi F}}$ where $G_z(z) = H_z(z^{-1})$. Letting $g[n]$ denote the inverse Z transform of $G_z(z)$, determine the relationship between $G_{\text{DTFT}}(F)$ and $H_{\text{DTFT}}(\cdot)$ as well as between $g[n]$ and $h[\cdot]$.

(4 pts.) c) Let $q[n]$ denote the inverse Z transform of $Q_z(z) = H_z(z)H_z(z^{-1}) = H_z(z)G_z(z)$. What is the relation between $q[n]$ and $h[n]$? Show that $q[0] = E_h$ where E_h denotes the energy of the *real-valued* signal $h[n]$.

question 6 is continued on the next page.

(4 pts.) d) By the "area-under" property of the DTFT, we know that $q[0] = \int_0^1 Q_{\mathrm{DTFT}}(F)dF$. Furthermore, in part (c) we showed that $q[0] = E_h$. Since the area under one period of $Q_{\mathrm{DTFT}}(F)$ is equal to the energy of the signal $h[n]$, $Q_{\mathrm{DTFT}}(F)$ is called the *energy spectral density* of $h[n]$; it shows how the energy of the signal $h[n]$ is distributed over frequency. Show that $Q_{\mathrm{DTFT}}(F) = |H_{\mathrm{DTFT}}(F)|^2$.
Hint: What is the relationship between $Q_{\mathrm{DTFT}}(F)$ and $Q_z(z)$? What is the relation between $Q_z(z)$ and $H_z(z)$? What is the relation between $Q_{\mathrm{DTFT}}(F)$ and $H_{\mathrm{DTFT}}(F)$? Recall also that, as shown in part (a), $H_{\mathrm{DTFT}}(F)$ is Hermitian.