Special Case of Distinct poles: A pair of complex-conjugate poles

For a real system (i.e., a system described by an LCCDE with real-valued coefficients $\{a_1, a_2, \ldots, a_N\}$ and $\{b_0, b_1, \ldots, b_M\}$), it is easy to show that:

- complex-valued poles will occur in complex-conjugate pairs

  note: $$(z-p_1)(z-p_1^*) = z^2 \underbrace{- (p_1 + p_1^*)}_{a_1} z + \underbrace{p_1 p_1^*}_{a_2}$$

  note: a_1 is real and a_2 is real

- if $p_2 = p_1^*$, then the coefficients, A_1 and A_2, of the partial fraction expansion will also satisfy $A_2 = A_1^*$.

  This is easy to show using the Heaviside cover up method:

$$\frac{X(z)}{z} = \frac{b_0 z + b_1}{(z-p_1)(z-p_1^*)} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_1^*}$$

where:
$$A_1 = \left. \frac{b_0 z + b_1}{z - p_1^*} \right|_{z=p_1} = \frac{b_0 p_1 + b_1}{p_1 - p_1^*}$$

$$A_2 = \left. \frac{b_0 z + b_1}{z - p_1} \right|_{z=p_1^*} = \frac{b_0 p_1^* + b_1}{p_1^* - p_1}$$

Since b_0 + b_1 are real, it is easily seen from the expressions above that $A_2 = A_1^*$

Example

Suppose a PFE yields:

$$\frac{X(z)}{z} = \frac{\overbrace{2e^{j\pi/3}}^{A}}{z - \underbrace{0.8e^{j2\pi\frac{1}{8}}}_{P}} + \frac{\overbrace{2e^{-j\pi/3}}^{A^*}}{z - \underbrace{0.8e^{-j2\pi\frac{1}{8}}}_{P^*}} \quad , \; |z| > 0.8$$

$$\Rightarrow X(z) = \frac{2e^{j\pi/3}\, z}{z - 0.8e^{j2\pi\frac{1}{8}}} + \frac{2e^{-j\pi/3}\, z}{z - 0.8e^{-j2\pi\frac{1}{8}}} \quad , \; |z| > 0.8$$

$$\Rightarrow X(z) = \frac{2e^{j\pi/3}}{1 - 0.8e^{j2\pi\frac{1}{8}} z^{-1}} + \frac{2e^{-j\pi/3}}{1 - 0.8e^{-j2\pi\frac{1}{8}} z^{-1}} \quad , \; |z| > 0.8$$

Selected Z Transform Pairs

$x[n]$		$X(z)$	ROC
$\delta[n]$	$\leftrightarrows$	1	All z
$u[n]$	$\leftrightarrows$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
$\boxed{a^n u[n]}$	$\leftrightarrows$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$

$$\Rightarrow x[n] =$$

$a^n \cos(2\pi F_0 n + \theta) u[n]$	$\leftrightarrows$	$\frac{\cos(\theta) - a\cos(2\pi F_0 - \theta) z^{-1}}{1 - 2a\cos(2\pi F_0) z^{-1} + a^2 z^{-2}}$	$\|z\| > \|a\|$

Procedure for finding the inverse Z-transform of a proper rational $X(z)$ via PFE.

(illustrated below for the case of a repeated pole of multiplicity ℓ at $z=p_1$ and distinct poles at $z=p_{\ell+1}, \dots, z=p_N$)

Let $$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$ where: $M<N$ and $a_N \neq 0$

STEP 1: multiply numerator and denominator by z^N to get rid of negative powers of z

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \dots + a_N}$$

STEP 2: divide by z to find an expression for $\frac{X(z)}{z}$

STEP 3: Re-write expression for $\frac{X(z)}{z}$ with factored denominator

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_M z^{N-M-1}}{\boxed{(z-p_1)^\ell}\ \boxed{(z-p_{\ell+1})\cdots(z-p_N)}}$$

STEP 4: Find PFE of $\frac{X(z)}{z}$

Assuming a repeated pole of multiplicity ℓ at $z=p_1$ and distinct poles at $z=p_{\ell+1}, \dots, z=p_N$, the PFE is:

$$\frac{X(z)}{z} = \left(\frac{A_1}{z-p_1} + \frac{A_2}{(z-p_1)^2} + \dots + \frac{A_\ell}{(z-p_1)^\ell}\right) + \left(\frac{A_{\ell+1}}{z-p_{\ell+1}} + \dots + \frac{A_N}{z-p_N}\right)$$

where the coefficients $A_1, \dots, A_\ell$ may be found as:

$$A_\ell = \left((z-p_1)^\ell \frac{X(z)}{z}\right)\Bigg|_{z=p_1}$$

$$A_{\ell-m} = \frac{1}{m!}\left(\frac{d^m}{dz^m}\left((z-p_1)^\ell \frac{X(z)}{z}\right)\right)\Bigg|_{z=p_1} \qquad m=0,\dots,\ell-1$$

note: $m=0$ corresponds to the coefficient A_ℓ

and the coefficients $A_{\ell+1}, \dots, A_N$ can be found as:

$$A_k = (z-p_k)\frac{X(z)}{z}\Bigg|_{z=p_k} \qquad k=\ell+1,\dots,N$$

STEP 5: Convert PFE for $\frac{X(z)}{z}$ into PFE for $X(z)$ and then rewrite using negative powers of z.

STEP 6: Find the inverse Z-transform of $X(z)$ using table look-up and linearity property

$$X(z) = A_1 X_1(z) + \dots + A_N X_N(z) \implies x[n] = A_1 x_1[n] + \dots + A_N x_N[n]$$

To understand the formulae for finding the coefficients of the partial fraction expansion for the case of repeated poles... consider the case shown below for which p_1 is a pole of multiplicity 4:

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{(z-p_1)^2} + \frac{A_3}{(z-p_1)^3} + \frac{A_4}{(z-p_1)^4}$$

Then

$$(z-p_1)^4 \frac{X(z)}{z} = \frac{A_1}{z-p_1}(z-p_1)^4 + \frac{A_2}{(z-p_1)^2}(z-p_1)^4 + \frac{A_3}{(z-p_1)^3}(z-p_1)^4 + \frac{A_4}{(z-p_1)^4}(z-p_1)^4$$

$$\Rightarrow \left[(z-p_1)^4 \frac{X(z)}{z}\right] = A_1(z-p_1)^3 + A_2(z-p_1)^2 + A_3(z-p_1) + A_4 \qquad (1)$$

$$\frac{d}{dz}\left[(z-p_1)^4 \frac{X(z)}{z}\right] = 3A_1(z-p_1)^2 + 2A_2(z-p_1) + A_3 \qquad (2)$$

$$\frac{d^2}{dz^2}\left[(z-p_1)^4 \frac{X(z)}{z}\right] = 3\cdot 2\cdot A_1(z-p_1) + \underbrace{2}_{2!}A_2 \qquad (3)$$

$$\frac{d^3}{dz^3}\left[(z-p_1)^4 \frac{X(z)}{z}\right] = \underbrace{3\cdot 2}_{3!}\cdot A_1 \qquad (4)$$

Evaluating (1) at $z=p_1$ $\Rightarrow$ $A_4 = \left[(z-p_1)^4 \frac{X(z)}{z}\right]\Big|_{z=p_1}$

Evaluating (2) at $z=p_1$ $\Rightarrow$ $A_3 = \left(\frac{d}{dz}\left[(z-p_1)^4 \frac{X(z)}{z}\right]\right)\Big|_{z=p_1}$

Evaluating (3) at $z=p_1$ $\Rightarrow$ $A_2 = \frac{1}{2}\left(\frac{d^2}{dz^2}\left[(z-p_1)^4 \frac{X(z)}{z}\right]\right)\Big|_{z=p_1}$

Evaluating (4) at $z=p_1$ $\Rightarrow$ $A_1 = \frac{1}{3\cdot 2}\left(\frac{d^3}{dz^3}\left[(z-p_1)^4 \frac{X(z)}{z}\right]\right)\Big|_{z=p_1}$

Example to illustrate Z-transform inversion for the case of a proper rational Z-transform with a repeated pole.

Given $X(z) = \dfrac{1}{(1-z^{-1})^2 \;\; (1+z^{-1})}$, $|z| > 1$

Find $x[n]$

Solution $X(z) = \dfrac{\qquad}{(\quad)^2\,(\quad)}$

$$\frac{X(z)}{z} = \frac{\qquad}{(\quad)^2\,(\quad)} = \frac{\qquad}{\qquad} + \frac{\qquad}{\qquad} + \frac{\qquad}{\qquad} \qquad (*)$$

Rather than evaluating the derivative expression for A_1, a nicer approach is to multiply both sides of $(*)$ by z and take the limit as $z \to \infty$.

On previous page, we found:

$$\frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-1)^2} + \frac{A_3}{(z+1)} \quad , \quad |z| > 1$$

$A_1 = 3/4$
$A_2 = 1/2$
$A_3 = 1/4$

$$\Rightarrow \quad X(z) = \frac{A_1 z}{(z-1)} + \frac{A_2 z}{(z-1)^2} + \frac{A_3 z}{(z+1)} \quad , \quad |z| > 1$$

$$\Rightarrow \quad X(z) = \underline{\qquad} + \underline{\qquad} + \underline{\qquad} \quad , \quad |z| > 1$$

$$\Rightarrow \quad x[n] =$$

Selected Z Transform Pairs

$x[n]$		$X(z)$	ROC
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$u[n]$	$\leftrightarrows$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
$a^n u[n]$	$\leftrightarrows$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
$na^n u[n]$	$\leftrightarrows$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\leftrightarrows$	$\frac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$
$-na^n u[-n-1]$	$\leftrightarrows$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| < \|a\|$

$A_1 = 3/4$
$A_2 = 1/2$
$A_3 = 1/4$