The DTFT is a spectral analysis tool intended for application to an energy signal, $x[n]$.

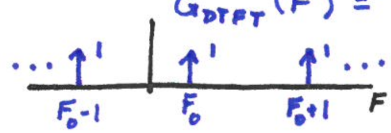
$$x[n] = \int_0^1 X_{DTFT}(F)\, e^{j2\pi F n}\, dF \;\xleftrightarrow{\text{DTFT}}\; X_{DTFT}(F) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi F n}$$

The DFS coefficients convey the spectral content of an N-periodic signal, $x_p[n]$:

$$x_p[n] = \sum_{k=0}^{N-1} c_k\, e^{j2\pi \frac{k}{N} n} \qquad \text{where} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n]\, e^{-j2\pi \frac{k}{N} n}$$

The Dirac Delta Function allows us to talk about the DTFT of certain power signals. For example, consider the following DTFT pair:

$$g[n] = e^{j2\pi F_0 n} \;\xleftrightarrow{\text{DTFT}}\; G_{DTFT}(F) = \delta(F - F_0) * \text{comb}(F) = \sum_{k=-\infty}^{\infty} \delta(F - F_0 - k)$$



The DTFT pair above can be easily verified using the IDTFT integral:

$$g[n] = \int_0^1 G_{DTFT}(F)\, e^{j2\pi F n}\, dF = \int_0^1 \delta(F - F_0)\, e^{j2\pi F n}\, dF = e^{j2\pi F_0 n}$$

In general, if $x_p[n]$ is an N-periodic discrete-time signal, then $x_p[n]$ can be expressed as a linear combination of N complex exponentials (see the DFS expansion of $x_p[n]$ shown above). Hence, we may use the linearity property of the DTFT together with the DTFT of a complex exponential to find the DTFT of any N-periodic signal.

used linearity property of DTFT

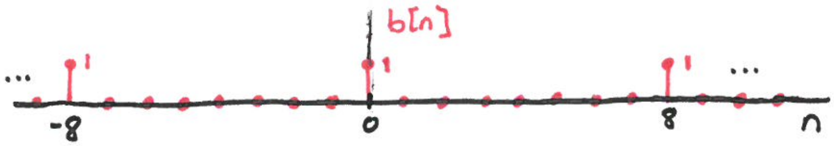
$$\text{DTFT}\{x_p[n]\} = \text{DTFT}\left\{\sum_{k=0}^{N-1} c_k\, e^{j2\pi \frac{k}{N} n}\right\} = \sum_{k=0}^{N-1} c_k\, \text{DTFT}\left\{e^{j2\pi \frac{k}{N} n}\right\}$$

$$= \sum_{k=0}^{N-1} c_k \left(\delta\left(F - \tfrac{k}{N}\right) * \text{comb}(F)\right) = \left(\sum_{k=0}^{N-1} c_k\, \delta\left(F - \tfrac{k}{N}\right)\right) * \text{comb}(F)$$

$$= \sum_{k=-\infty}^{\infty} c_k\, \delta\left(F - \tfrac{k}{N}\right)$$

used the fact that the coefficients $\{c_k\}$ are periodic with period N. $c_{k+N} = c_k$

Example Let $b[n] = comb_8[n] = \sum_{k=-\infty}^{\infty} \delta[n-8k]$ as illustrated below.



a) Find the DFS representation of $b[n]$. ← We did this in L25.

b) Plot the DFS coefficients

c) Find and plot the DTFT of $b[n]$.

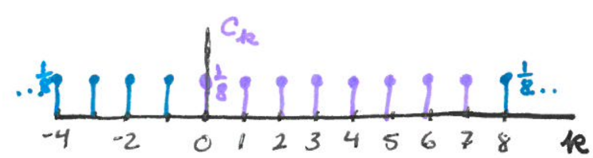
Solution

a) Since $b[n]$ is periodic with period 8, its DFS representation may be written as:

$$b[n] = \sum_{k=0}^{7} c_k e^{j2\pi \frac{k}{8} n}$$

where: $c_k = \frac{1}{8} \sum_{n=0}^{7} b[n] e^{-j2\pi \frac{k}{8} n} = \frac{1}{8} b[0] e^{-j0} = \frac{1}{8}$

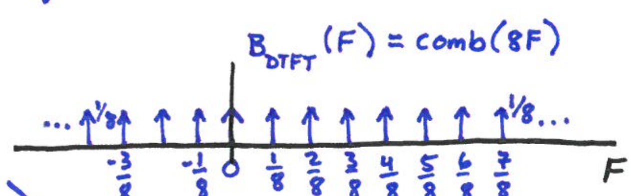
b) The DFS coefficients are shown plotted to the right.



c) $\text{DTFT}\{b[n]\} = \text{DTFT}\left\{\sum_{k=0}^{7} \frac{1}{8} e^{j2\pi \frac{k}{8} n}\right\}$

$$= \sum_{k=0}^{7} \frac{1}{8} \text{DTFT}\left\{e^{j2\pi \frac{k}{8} n}\right\}$$

$$= \sum_{k=0}^{7} \frac{1}{8} \left(\delta(F - \tfrac{k}{8}) * comb(F)\right)$$



$$B_{DTFT}(F) = \sum_{k=-\infty}^{\infty} \frac{1}{8} \delta(F - \tfrac{k}{8})$$

Example Let $x_p[n] = \cos(2\pi \frac{1}{8} n)$

a) Find the DFS representation of $x_p[n]$

b) Plot the DFS coefficients $\{c_k\}$

c) Find and plot the DTFT of $x_p[n]$

Solution

a) Since $x_p[n]$ is periodic with period $N = 8$, its DFS representation is:

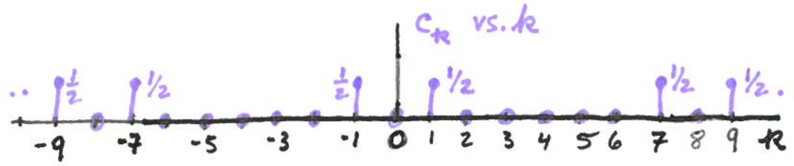
$$x_p[n] = \sum_{k=0}^{7} c_k e^{j2\pi \frac{k}{8} n}$$

By Euler's formula: $x_p[n] = \frac{1}{2} e^{j2\pi \frac{1}{8} n} + \frac{1}{2} e^{-j2\pi\frac{1}{8} n}$

$e^{-j2\pi\frac{1}{8}n} = e^{j2\pi\frac{7}{8}n}$ for all integer n

$$\Rightarrow x_p[n] = \underbrace{\tfrac{1}{2}}_{c_1} e^{j2\pi \frac{1}{8} n} + \underbrace{\tfrac{1}{2}}_{c_7} e^{j2\pi\frac{7}{8} n}$$

b) The DFS coefficients are shown plotted to the right.

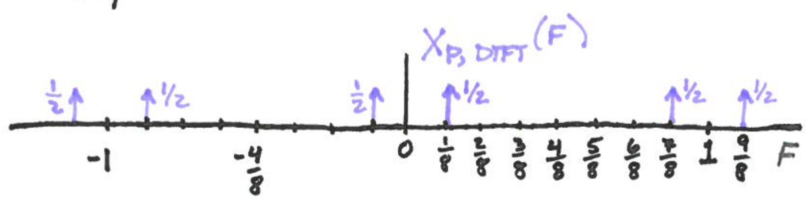


The coefficients are periodic with period 8.

c) $X_{P,DTFT}(F) = \frac{1}{2}\,\text{DTFT}\{e^{j2\pi\frac{1}{8}n}\} + \frac{1}{2}\,\text{DTFT}\{e^{j2\pi\frac{7}{8}n}\}$

$$= \frac{1}{2}\left(\delta(F-\tfrac{1}{8}) * \text{comb}(F)\right) + \frac{1}{2}\left(\delta(F-\tfrac{7}{8}) * \text{comb}(F)\right)$$

$$= \left(\frac{1}{2}\delta(F-\tfrac{1}{8}) + \frac{1}{2}\delta(F-\tfrac{7}{8})\right) * \text{comb}(F)$$



Summary

$\boxed{\text{Relationship between the DFS coefficients and the DTFT of an } N\text{-periodic sequence, } x_p[n].}$

If $$x_p[n] = \sum_{k=0}^{N-1} c_k \, e^{j2\pi \frac{k}{N} n}$$

Then $$X_{p,\,DTFT}(F) = \left(\sum_{k=0}^{N-1} c_k \, \delta\!\left(F - \frac{k}{N}\right) \right) * \text{comb}(F)$$

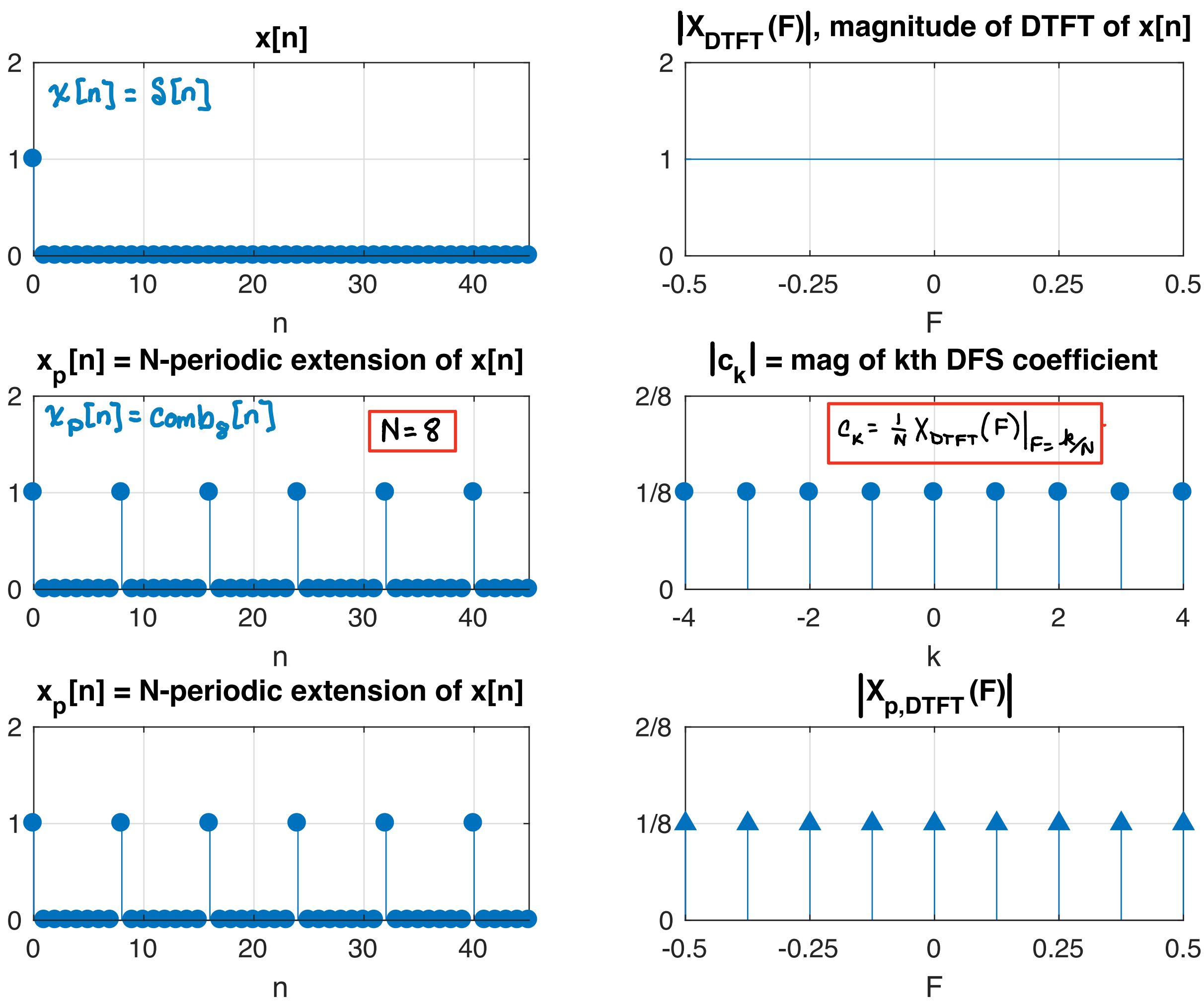
In words: The DTFT of an N-periodic sequence, $x_p[n]$, is the 1-periodic extension of a sum of Dirac Delta Functions at $F = \frac{k}{N}$, $k = 0, 1, .., N-1$. The area under the Delta Function at $F = \frac{k}{N}$ is c_k, the k^{th} DFS coefficient.

(see previous two examples)

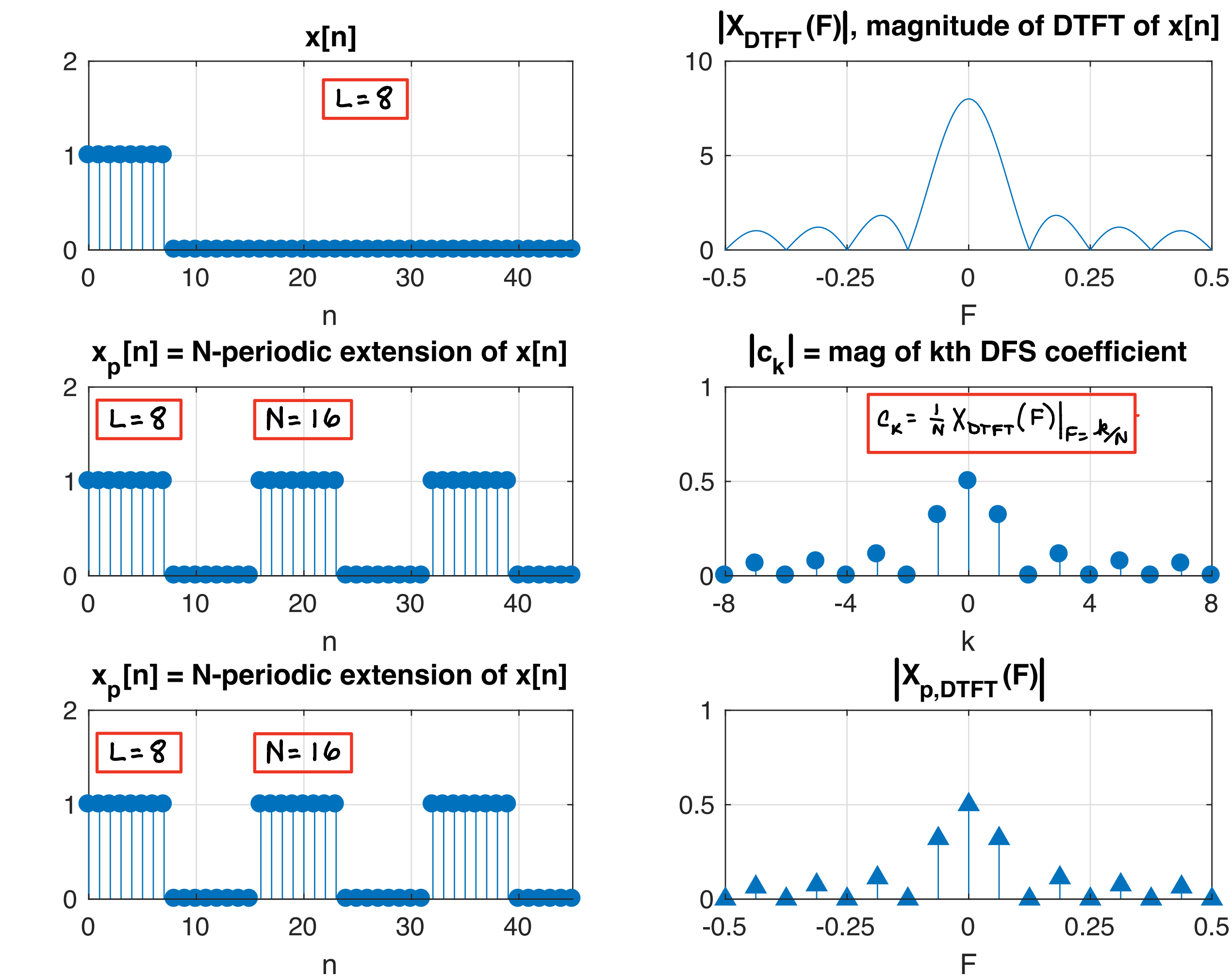
In Lecture 26, we determined the following relationship between the DFS coefficients, $\{c_k\}$, of an N-periodic signal, $x_N[n]$, and the DTFT, $X_{DTFT}(F)$, of a single period, $x[n]$, of $x_N[n]$:

$$c_k = \frac{1}{N} X_{DTFT}(F)\Big|_{F=\frac{k}{N}} = \frac{1}{N} X_{DTFT}\left(\frac{k}{N}\right)$$

This relationship is demonstrated below, along with our newly derived relationship between the DTFT of a periodic signal and its DFS coefficients.



Another illustration of the relationship between the DFS coefficients, $\{c_k\}$, of an N-periodic signal, $x_p[n] \equiv x_N[n]$, the DTFT, $X_{p,DTFT}(F)$, of $x_p[n]$, and the DTFT, $X_{DTFT}(F)$, of a single period, $x[n]$, of $x_p[n]$.



It is not possible to numerically evaluate the DTFT sum of an infinite-length signal; instead, we will approximate the DTFT of interest by the DTFT of a truncated version of the signal.

How does the DTFT of the truncated signal compare to the true DTFT of the infinite-length signal?

To answer this question:

Let $g[n]$ denote the original signal

Let $x[n]$ denote a truncated version of $g[n]$

If $$x[n] = \begin{cases} g[n], & n = 0, 1, \ldots, L-1 \\ 0, & \text{otherwise} \end{cases}$$

Then we may write:

$$x[n] = g[n]\, w[n] \quad \text{where} \quad w[n] = \begin{cases} 1, & n = 0, \ldots, L-1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, using the multiplication-in-time property of the DTFT, we know that:

$$X_{DTFT}(F) = \left[G_{DTFT}(F)\, \text{rect}(F) \right] * W_{DTFT}(F)$$

Previously, we found the DTFT of a rectangular window of length L samples.

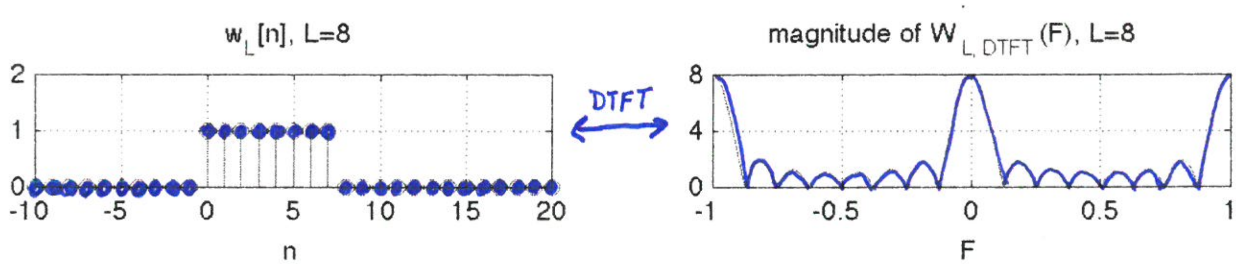
Let $w_L[n] = \begin{cases} 1, & n = 0, \dots, L-1 \\ 0, & \text{otherwise} \end{cases}$

Let $W_{L,DTFT}(F)$ denote the DTFT of $w_L[n]$.

We found $\quad |W_{L,DTFT}(F)| = \left| \dfrac{\sin(\pi F L)}{\sin(\pi F)} \right|$

$$\Rightarrow \quad |W_{DTFT}(0)| = \; ?$$

$$W_{DTFT}(F) = 0 \quad \text{when} \quad F = \; ?$$



It is particularly instructive to compare the DTFT of an infinitely long sinusoid to the DTFT of a windowed version of the sinusoid.

Example

Let $x_p[n] = \cos\left(2\pi \tfrac{1}{8} n\right) \longleftrightarrow X_{p,\,DTFT}(F) = \left(\tfrac{1}{2}\delta(F+\tfrac{1}{8}) + \tfrac{1}{2}\delta(F-\tfrac{1}{8})\right) * \mathrm{comb}(F)$

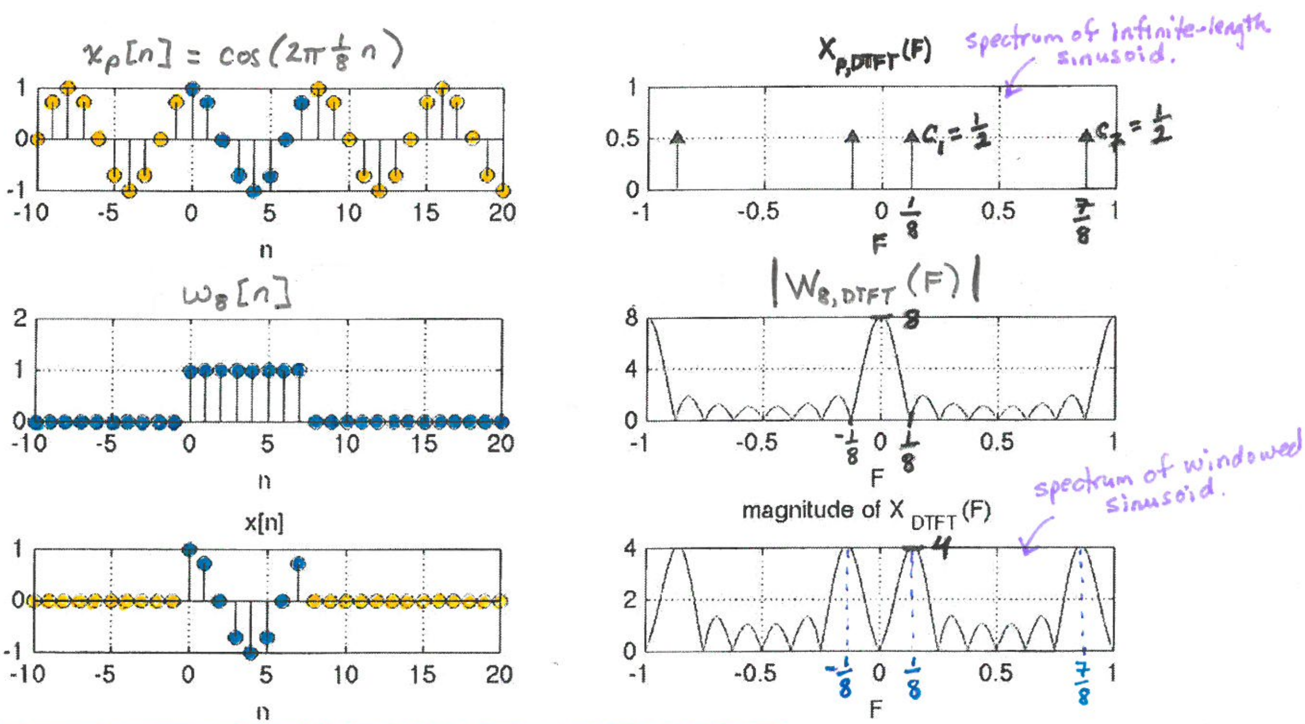
Let $x[n] = x_p[n]\, w_8[n]$ where $w_8[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$

Then

$$X_{DTFT}(F) = \left[X_{p,\,DTFT}(F)\,\mathrm{rect}(F)\right] * W_{8,\,DTFT}(F)$$

$$= \left(\tfrac{1}{2}\delta(F+\tfrac{1}{8}) + \tfrac{1}{2}\delta(F-\tfrac{1}{8})\right) * W_{8,\,DTFT}(F)$$

$$= \tfrac{1}{2} W_{8,\,DTFT}(F+\tfrac{1}{8}) + \tfrac{1}{2} W_{8,\,DTFT}(F-\tfrac{1}{8})$$



Note the spectral smearing that took place as a result of truncating/windowing the time-domain signal. The result is a loss of frequency resolution. The longer the time window, the better will be the frequency resolution.