Analysis of a Causal Recursive Linear Constant Coefficient Difference Equation (LCCDE) of order N for the case when $N > M \geq 0$.

The analysis problem can be stated as follows.

Given:

- the system input, $x[n],\ n \geq 0$;
- equivalent initial conditions (IC's): $y[-1], y[-2], \ldots, y[-N]$

  (note: when you iterate the difference equation starting at $n = 0$, it may appear as though you will need values for $x[-1], \ldots, x[-M]$ in addition to the IC values above; however, the effect of any nonzero values for $x[-1], \ldots, x[-M]$ will have been taken into account by the ***equivalent*** IC values; hence, you may always assume that $x[-1] = \ldots = x[-M] = 0$.)
- and the LCCDE (1) which describes the input-output equation for the system:

$$\text{LCCDE:} \quad \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad a_0 = 1,\ N > M \geq 0 \tag{1}$$

Find: $y[n],\ n \geq 0$.

The solution to the analysis problem can always be found as the sum of the ***particular solution,*** $y_p[n]$, and a ***complementary solution,*** $y_c[n]$. The particular solution will satisfy the LCCDE for the particular input applied; however, it may not satisfy the specified initial conditions. It is straightforward to show that the sum of the particular solution and any solution to the homogeneous equation will also satisfy the LCCDE (see next page). The complementary solution is a solution to the homogeneous equation and hence when added to the particular solution the sum (or total solution) will satisfy the LCCDE of equation (1). In addition, since there are N free parameters associated with the homogeneous solution, the complementary solution will be the homogeneous solution with parameter values assigned so that the total response (*i.e.*, the sum of the particular solution and complementary solution) satisfies the N specified initial conditions.

In the study of systems, we often refer to the particular solution as the ***forced response*** and to the complementary solution as the ***natural response***. The forced response is said to be forced by the input due to the fact that it will have the same signal characteristics as the input signal (an exception occurs when the input has the same form as one of the characteristic modes). For a stable system, the natural response will be transient in nature; the characteristics of the natural response are determined by the system as opposed to the applied input. Throughout this course, forced response is used interchangeably with particular solution and natural response is used interchangeably with complementary solution.

As can be seen by iterating the difference equation (1) for $n \geq 0$, the resulting expression for $y[n]$ is, in general, a sum of many terms. The terms in the sum can be separated into two groups: the first group of terms involving $x[n],\ n \geq 0$; and the second group of terms involving the initial conditions: $y[-1], y[-2], \ldots, y[-N]$. If the input, $x[n]$, is set equal to zero for all $n \geq 0$, the first group of terms will vanish from the sum; hence the sum of the second group of terms (those resulting from nonzero ICs) is referred to as the ***zero-input response***. Similarly, if the IC's are all set to zero, the second group of terms vanishes from the sum and hence the sum of the first group of terms (those resulting from nonzero $x[n],\ n \geq 0$) is referred to as the ***zero-state response***. Thus, an alternative to finding the response of the system as a sum of the particular and complementary solutions is to find the total response as the sum of the zero-state and zero-input response. The nice thing about the zero-state zero-input decomposition is that it clearly identifies the individual responses to each of the two causes (the input $x[n],\ n \geq 0$ and the IC's); if you double the input while leaving the IC's unchanged, you will find that the zero-state response will double while the zero-input response will remain unchanged; similarly if you double the IC's while leaving the input unchanged, the zero-input response will double while the zero-state response will remain unchanged.

In summary, the solution, $y[n],\ n \geq 0$, to the LCCDE of (1), can be expressed as:

$$y[n] = y_p[n] + y_c[n] = y_{\text{zs}}[n] + y_{\text{zi}}[n] \tag{2}$$

****

Below, we show that the sum of the particular solution and any solution to the homogeneous equation will satisfy the LCCDE. For this purpose, let $y_p[n]$ denote the particular solution to the LCCDE of equation (1) and let $y_h[n]$ denote any solution to the homogeneous difference equation. Then by definition of a particular solution, we know that $y_p[n]$ satisfies:

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{3}$$

and by definition of a homogeneous solution, we know that $y_h[n]$ satisfies:

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0\,. \tag{4}$$

Adding left sides and right sides of equations (3) and (4) yields:

$$\sum_{k=0}^{N} a_k \underbrace{\left(y_p[n-k] + y_h[n-k]\right)}_{y[n-k]} = \sum_{k=0}^{M} b_k x[n-k] \tag{5}$$

From equation (5), we conclude that if $y[n] = y_p[n] + y_h[n]$, where $y_p[n]$ is the particular solution and $y_h[n]$ is a homogeneous solution, then $y[n]$ satisfies the LCCDE of equation (1).

****

Two procedures for finding the solution $y[n],\ n \geq 0$ to the analysis problem stated on page 1, are reviewed on the pages which follow. The first procedure finds $y[n],\ n \geq 0$ as a sum of the particular and complementary solutions; the second procedure finds $y[n],\ n \geq 0$ as a sum of the zero-input and zero-state responses.

Procedure 1 $y[n] = y_p[n] + y_c[n]$

a. **find the general form for the homogeneous solution**, $y_h[n]$

i. homogeneous eqn: $y_h[n] + a_1 y_h[n-1] + \ldots + a_N y_h[n-N] = 0$

ii. characteristic eqn: $\lambda^N + a_1\lambda^{N-1} + \ldots + a_N = 0$

iii. find the roots of characteristic eqn. and call them $\lambda_1, \lambda_2, \ldots, \lambda_N$

iv. identify the associated characteristic modes: $\Phi_1[n], \Phi_2[n], \ldots, \Phi_N[n]$
if λ_1 is a distinct root then $\Phi_1[n] = \lambda_1^n$
if $\lambda_1 = \lambda_2$ is a root of multiplicity 2 then $\Phi_1[n] = \lambda_1^n$ and $\Phi_2[n] = n\lambda_1^n$, *etc.*

v. The general form of the homogenous solution is:
$y_h[n] = C_1\Phi_1[n] + C_2\Phi_2[n] + \ldots + C_N\Phi_N[n]$

b. **find the particular solution**, $y_p[n]$

i. assume $y_p[n]$ to have the same form as $x[n]$, $n \geq 0$. For example:

if $x[n] = A$	then assume: $y_p[n] = K$
if $x[n] = \delta[n]$	then assume: $y_p[n] = 0$
if $x[n] = A\beta^n$	then assume: $y_p[n] = K\beta^n$
if $x[n] = A\cos(\omega_0 n + \phi)$	then assume: $y_p[n] = K_1\cos(\omega_0 n) + K_2\sin(\omega_0 n)$

note: if $x[n]$ satisfies the homogeneous eqn, then rather than assuming that $y_p[n]$ has the same form as $x[n]$, you should assume $y_p[n]$ has the form of $nx[n]$; and if $nx[n]$ satisfies the homogeneous eqn, you should assume $y_p[n]$ has the form of $n^2x[n]$, *etc*.

ii. substitute $y_p[n]$ into the system difference equation and solve for any free parameters of $y_p[n]$ by evaluating both sides of the difference equation for some $n \geq N$.
Note: Since the expressions for $y_p[n]$ and $x[n]$ are valid for $n \geq 0$, it is important to evaluate the difference equation for $n \geq N$, when determining parameter values (*e.g.*, the value of K) of $y_p[n]$; for $n \geq N$, all references to the signals $y_p[\cdot]$ (namely: $y_p[n]$, ..., $y_p[n-N]$) and $x[\cdot]$ will have the assumed form.

c. **find the complementary and total solutions simultaneously**

i. Express the total solution as a sum of the particular and complementary solution:

$$y[n] = y_p[n] + y_c[n] = y_p[n] + \underbrace{C_1^{(\text{tot})}\Phi_1[n] + \ldots + C_N^{(\text{tot})}\Phi_N[n]}_{y_c[n]} \tag{6}$$

Use equation (6) to find values for $y[0], y[1], \ldots, y[N-1]$ in terms of the constants: $C_1^{(\text{tot})}, C_2^{(\text{tot})}, \ldots, C_N^{(\text{tot})}$.

ii. Iterate the difference equation to find values for $y[0], y[1], \ldots, y[N-1]$.

iii. For each n, $0 \leq n \leq N-1$, set the expression for $y[n]$ obtained in part (*i.*) equal to the value obtained in part (*ii.*). This results in a set of N equations which can be used to solve for the N unknowns: $C_1^{(\text{tot})}, C_2^{(\text{tot})}, \ldots, C_N^{(\text{tot})}$. The complementary and total solutions are then specified by substituting these values into equation (6).

Procedure 2: $y[n] = y_{zi}[n] + y_{zs}[n]$

a. **find the general form for the homogeneous solution**, $y_h[n]$. (same as for procedure 1)

b. **find the zero-input solution**, $y_{zi}[n]$
The zero-input solution, $y_{zi}[n]$, is the solution to LCCDE (1) when $x[n] = 0$; it is due to the initial conditions ($y[-1]$, $y[-2]$, ... , $y[-N]$). Hence the zero-input LCCDE is:

$$\text{z.i. LCCDE:} \quad y_{zi}[n] + a_1 y_{zi}[n-1] + \ldots + a_N y_{zi}[n-N] = 0 \tag{7}$$

i. In general, the zero-input response is a solution to the homogeneous equation and can thus be expressed as:

$$y_{zi}[n] = C_1^{(zi)}\Phi_1[n] + \ldots + C_N^{(zi)}\Phi_N[n] \tag{8}$$

Use equation (8) to obtain expressions for $y_{zi}[0]$, $y_{zi}[1]$, ... , $y_{zi}[N-1]$, in terms of the free parameters: $C_1^{(zi)}$, $C_2^{(zi)}$, ... , $C_N^{(zi)}$.

ii. Iterate the zero-input LCCDE (7) to find values for $y_{zi}[0]$, $y_{zi}[1]$, ... , $y_{zi}[N-1]$. Assume that: $y_{zi}[-1] = y[-1]$, $y_{zi}[-2] = y[-2]$, ... , and $y_{zi}[-N] = y[-N]$.

iii. for each n, $0 \le n \le N-1$, set the expression for $y_{zi}[n]$ obtained in part (*i.*) equal to the value obtained in part (*ii.*). This results in a set of n equations which can be used to solve for the n unknowns: $C_1^{(zi)}$, $C_2^{(zi)}$, ... , $C_N^{(zi)}$. The zero-input solution is then specified by equation (8) using the values just found.

c. **find the particular solution**, $y_p[n]$. (same as for procedure 1)

d. **find the zero-state solution**, $y_{zs}[n]$
Recall that the zero-state solution, $y_{zs}[n]$, is the solution to the LCCDE in (1) when $y_{zs}[-1] = y_{zs}[-2] = \ldots = y_{zs}[-N] = 0$, and $x[n]$ is as specified. Hence:

$$\text{z.s. LCCDE:} \quad y_{zs}[n] = -\sum_{k=1}^{N} a_k y_{zs}[n-k] + \sum_{k=0}^{M} b_k x[n-k] \tag{9}$$

i. Write a general expression for the zero-state solution. Recall that the zero-state solution is a sum of the particular solution plus a homogeneous solution (the free parameters of the homogeneous solution will be chosen to satisfy the zero-state ICs). Hence, we may write the following general expression for the zero-state soln:

$$y_{zs}[n] = y_p[n] + C_1^{(zs)}\Phi_1[n] + C_2^{(zs)}\Phi_2[n] + \ldots + C_N^{(zs)}\Phi_N[n] \tag{10}$$

Use equation (10) to find values for $y_{zs}[0]$, $y_{zs}[1]$, ... , $y_{zs}[N-1]$ in terms of the constants: $C_1^{(zs)}$, $C_2^{(zs)}$, ... , $C_N^{(zs)}$.

ii. Iterate the zero-state difference equation with $y_{zs}[-1] = \ldots = y_{zs}[-N] = 0$, to find values for $y_{zs}[0]$, $y_{zs}[1]$, ... , $y_{zs}[N-1]$.

iii. For each n, $0 \le n \le N-1$, set the expression for $y_{zs}[n]$ obtained in part (i.) equal to the value obtained in part (ii.). This results in a set of N equations which can be used to solve for the N unknowns: $C_1^{(zs)}$, $C_2^{(zs)}$, ... , $C_N^{(zs)}$. The zero-state solution is then given by equation (10) using the values just determined for $C_1^{(zs)}$, ... , $C_N^{(zs)}$.

e. **The total solution, $y[n]$, $n \ge 0$ can now be found as the sum of the zero-input and zero-state solutions: $y[n] = y_{zi}[n] + y_{zs}[n]$.**

Example 1

Consider the causal LTI system whose input-output relationship is described by the following difference equation:

$$\text{LCCDE:} \qquad y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n-1] \tag{11}$$

Find a closed-form expression for $y[n]$, $n \geq 0$, when $y[-1] = 1$, $y[-2] = -1$, and $x[n] = 4^n$, $n \geq 0$.

Solution of Example 1 via Procedure 1 (particular + complementary): $y[n] = y_p[n] + y_c[n]$

a. find the general form for the homogeneous solution, $y_h[n]$.

i. homogeneous eqn: $y_h[n] - 3y_h[n-1] - 4y_h[n-2] = 0$

ii. characteristic eqn: $\lambda^2 - 3\lambda - 4 = 0$

iii. characteristic roots: $(\lambda - 4)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = -1$ and $\lambda_2 = 4$

iv. characteristic modes: $\Phi_1[n] = \lambda_1^n = (-1)^n$ and $\Phi_2[n] = \lambda_2^n = (4)^n$

v. **homogenous solution**: $y_h[n] = C_1\Phi_1[n] + C_2\Phi_2[n] = C_1(-1)^n + C_2(4)^n$

b. find the particular (or forced) solution: $y_p[n], n \geq 0$

i. By definition, the particular solution satisfies the difference equation for $n \geq N > M$, and hence for the current case, we may write:

$$y_p[n] - 3y_p[n-1] - 4y_p[n-2] = x[n] + 2x[n-1], \qquad n \geq 2 \tag{12}$$

Since $x[n] = 4^n$, we would normally assume that $y_p[n] = K4^n$; however, noting that $K4^n$ is a solution to the homogeneous equation, we know it will not satisfy (13) and hence we will instead assume that $y_p[n] = Kn4^n$. (If you forget that $K4^n$ is a solution to the homogeneous equation, you will figure out that something is wrong when you try to solve for K in the next step; you should try this so you know what to expect.)

ii. Evaluating equation (12) at $n = 2$ yields:

$$y_p[2] - 3y_p[1] - 4y_p[0] = x[2] + 2x[1] \tag{14}$$

Using $y_p[n] = Kn4^n$, $n \geq 0$, and $x[n] = 4^n$, $n \geq 0$, we determine that: $y_p[2] = K(2)4^2 = 32K$, $y_p[1] = K(1)4^1 = 4K$, and $y_p[0] = (K)(0)4^0 = 0$; and that: $x[2] = 4^2 = 16$ and $x[1] = 4^1 = 4$.

Substituting these values into (14) yields:

$$32K - 3(4K) - 4(0) = 16 + 2(4) \quad \Rightarrow \quad 20K = 24 \quad \Rightarrow \quad K = 6/5 \tag{15}$$

Hence the particular solution is:

$$\textbf{particular solution:} \qquad y_p[n] = Kn4^n = \frac{6}{5}n4^n,\ n \geq 0 \tag{16}$$

c. find the complementary and total solutions simultaneously

i. In general, the total solution can be expressed as a sum of the particular and complementary solution:

general expression for the total response

$$y[n] = \underbrace{\frac{6}{5}n4^n}_{y_p[n]} + \underbrace{C_1^{(\text{tot})}(-1)^n + C_2^{(\text{tot})}(4)^n}_{y_c[n]} \qquad (17)$$

Evaluating equation (17) at $n = 0$ and $n = 1$ yields:

$$y[0] = \frac{6}{5}(0)4^0 + C_1^{(\text{tot})}(-1)^0 + C_2^{(\text{tot})}(4)^0 = 0 + C_1^{(\text{tot})} + C_2^{(\text{tot})} \qquad (18)$$

and

$$y[1] = \frac{6}{5}(1)4^1 + C_1^{(\text{tot})}(-1)^1 + C_2^{(\text{tot})}(4)^1 = \frac{24}{5} - C_1^{(\text{tot})} + 4C_2^{(\text{tot})} \qquad (19)$$

ii. Iterating the difference equation (11) so as to find values for $y[0]$ and $y[1]$, when $y[-1] = 1$, $y[-2] = -1$, and $x[n] = 4^n u[n]$, yields:

$$y[0] = 3y[-1] + 4y[-2] + x[0] + 2x[-1] = 3(1) + 4(-1) + 1 + 2(0)$$
$$\Rightarrow \quad y[0] = 3 - 4 + 1 + 0 = 0 \qquad (20)$$

and:

$$y[1] = 3y[0] + 4y[-1] + x[1] + 2x[0] = 3(0) + 4(1) + 4 + 2(1)$$
$$\Rightarrow \quad y[1] = 0 + 4 + 4 + 2 = 10 \qquad (21)$$

iii. Equating expressions for $y[0]$, from equations 18 and 20, we find:

$$C_1^{(\text{tot})} + C_2^{(\text{tot})} = 0 \qquad (22)$$

and equating the expressions for $y[1]$, from equations 19 and 21, we find:

$$\frac{24}{5} - C_1^{(\text{tot})} + 4C_2^{(\text{tot})} = 10 \qquad (23)$$

Solution of equations 22 and 23 for the values of $C_1^{(\text{tot})}$ and $C_2^{(\text{tot})}$ yields:

$$C_1^{(\text{tot})} = -\frac{26}{25} \quad \text{and} \quad C_2^{(\text{tot})} = \frac{26}{25}$$

Substituting these values into equation (17), we find the total response:

total response:

$$y[n] = \underbrace{\frac{6}{5}n4^n}_{y_p[n]} + \underbrace{\frac{-26}{25}(-1)^n + \frac{26}{25}(4)^n}_{y_c[n]} \qquad (24)$$

Below we present an alternate solution for Example 1 on page 5. The alternate solution is based on procedure 2.

Solution of Example 1 via Procedure 2 (zero-input + zero-state): $y[n] = y_{\text{zi}}[n] + y_{\text{zs}}[n]$

a. find the general form for the homogeneous solution, $y_h[n]$.
(same as for procedure 1, see details on page 5)
homogenous solution: $y_h[n] = C_1\Phi_1[n] + C_2\Phi_2[n] = C_1(-1)^n + C_2(4)^n$

b. find the zero-input solution, $y_{\text{zi}}[n]$

i. Since the zero-input response is a solution to the homogeneous eqn, we may write:

$$y_{\text{zi}}[n] = C_1^{(\text{zi})}(-1)^n + C_2^{(\text{zi})}(4)^n \tag{25}$$

Evaluating (25) at $n = 0$ and $n = 1$ yields:

$$y_{\text{zi}}[0] = C_1^{(\text{zi})}(-1)^0 + C_2^{(\text{zi})}(4)^0 \quad \Rightarrow \quad y_{\text{zi}}[0] = C_1^{(\text{zi})} + C_2^{(\text{zi})} \tag{26}$$

$$y_{\text{zi}}[1] = C_1^{(\text{zi})}(-1)^1 + C_2^{(\text{zi})}(4)^1 \quad \Rightarrow \quad y_{\text{zi}}[0] = -C_1^{(\text{zi})} + 4C_2^{(\text{zi})} \tag{27}$$

ii. The zero-input LCCDE and ICs are:

z.i. LCCDE: $y_{\text{zi}}[n] = 3y_{\text{zi}}[n-1] + 4y_{\text{zi}}[n-2]$ (28)

z.i. ICs: $y_{\text{zi}}[-1] = y[-1] = 1$, and $y_{\text{zi}}[-2] = y[-2] = -1$.

Iterating the zero-input LCCDE (28) to find $y_{\text{zi}}[0]$ and $y_{\text{zi}}[1]$ yields:

$$y_{\text{zi}}[0] = 3y_{\text{zi}}[-1] + 4y_{\text{zi}}[-2] = 3(1) + 4(-1) = -1 \tag{29}$$

$$y_{\text{zi}}[1] = 3y_{\text{zi}}[0] + 4y_{\text{zi}}[-1] = 3(-1) + 4(1) = 1 \tag{30}$$

iii. Equating expressions for $y_{\text{zi}}[0]$, (equations 26 and 29), yields:

$$C_1^{(\text{zi})} + C_2^{(\text{zi})} = -1 \tag{31}$$

and expressions for $y_{\text{zi}}[1]$, (equations 27 and 30) yields:

$$-C_1^{(\text{zi})} + 4C_2^{(\text{zi})} = 1 \tag{32}$$

Solving equations 31 and 32 for $C_1^{(\text{zi})}$ and $C_2^{(\text{zi})}$, we find: $C_1^{(\text{zi})} = -1$ and $C_2^{(\text{zi})} = 0$.
Substituting these values into (25) yields the zero-input response:

zero-input response: $y_{\text{zi}}[n] = (-1)(-1)^n + (0)(4)^n = (-1)^{n+1}$ (33)

c. find the particular solution, $y_p[n]$. (same as for procedure 1, see details on page 5)

particular solution: $y_p[n] = Kn4^n = \frac{6}{5}n4^n$, $n \geq 0$ (34)

d. find the zero-state response, $y_{zs}[n]$

i. In general, the zero-state solution can be expressed as a sum of the particular solution and a solution to the homogenous equation. Hence, we have the following general expression for the zero-state soln:

$$y_{zs}[n] = \frac{6}{5}n4^n + C_1^{(zs)}(-1)^n + C_2^{(zs)}(4)^n \tag{35}$$

Evaluating equation (35) at $n = 0$ and $n = 1$ yields:

$$y_{zs}[0] = \frac{6}{5}(0)4^0 + C_1^{(zs)}(-1)^0 + C_2^{(zs)}(4)^0 \Rightarrow y_{zs}[0] = C_1^{(zs)} + C_2^{(zs)} \tag{36}$$

$$y_{zs}[1] = \frac{6}{5}(1)4^1 + C_1^{(zs)}(-1)^1 + C_2^{(zs)}(4)^1 \Rightarrow y_{zs}[1] = \frac{24}{5} - C_1^{(zs)} + 4C_2^{(zs)} \tag{37}$$

ii. The zero-state LCCDE and ICs are:

z.s. LCCDE: $y_{zs}[n] = 3y_{zs}[n-1] + 4y_{zs}[n-2] + x[n] + 2x[n-1]$ (38)

z.s. ICs: $y_{zs}[-1] = 0$ and $y_{zs}[-2] = 0$.

Iterating the z.s. LCCDE (38) with $x[n] = 4^n u[n]$ to find $y_{zs}[0]$ and $y_{zs}[1]$ yields:

$$y_{zs}[0] = 3y_{zs}[-1] + 4y_{zs}[-2] + x[0] + 2x[-1] = 3(0) + 4(0) + 1 + 0 = 1 \tag{39}$$

$$y_{zs}[1] = 3y_{zs}[0] + 4y_{zs}[-1] + x[1] + 2x[0] = 3(1) + 4(0) + 4 + 2(1) = 9 \tag{40}$$

iii. Equating expressions for $y_{zs}[0]$, from equations 36 and 39, we find:

$$C_1^{(zs)} + C_2^{(zs)} = 1 \tag{41}$$

and equating expressions for $y_{zs}[1]$, from equations 37 and 40, yields:

$$\frac{24}{25} - C_1^{(zs)} + 4C_2^{(zs)} = 9 \tag{42}$$

Solving equations 41 and 42 for $C_1^{(zs)}$ and $C_2^{(zs)}$, we find: $C_1^{(zs)} = -\frac{1}{25}$ and $C_2^{(zs)} = \frac{26}{25}$. And substituting these values into (35) yields the zero-state response:

zero-state response: $$y_{zs}[n] = \frac{6}{5}(n)(4)^n + \left(-\frac{1}{25}\right)(-1)^n + \left(\frac{26}{25}\right)(4)^n \tag{43}$$

e. The total solution, $y[n]$, $n \geq 0$ can now be found as the sum of the zero-input and zero-state solutions: $y[n] = y_{zi}[n] + y_{zs}[n]$. Summing equations (33) and (43), we find:

$$y[n] = \underbrace{\frac{6}{5}(n)(4)^n + \left(-\frac{1}{25}\right)(-1)^n + \left(\frac{26}{25}\right)(4)^n}_{y_{zs}[n]} + \underbrace{(-1)(-1)^n}_{y_{zi}[n]} \tag{44}$$

Combining like terms from the zero-input and zero-state responses yields:

total response: $$y[n] = \frac{6}{5}(n)(4)^n + \left(-\frac{26}{25}\right)(-1)^n + \left(\frac{26}{25}\right)(4)^n \tag{45}$$

which agrees with the total response found using procedure 1 (see equation 24).

Example 2

Consider the same causal LTI system of example 1, whose input-output relationship is repeated below:

$$\text{LCCDE:} \qquad y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n-1] \tag{46}$$

Find a closed-form expression for $h[n],\ n \geq 0$, the impulse response of the system.

Solution of Example 2: The impulse response is the zero-state response when $x[n] = \delta[n]$. Since the forced response for an impulse is $y_p[n] = 0$, the impulse response will always be a solution to the homogeneous equation. In the solution to Example 1, we found the general expression for the homogeneous solution to be:

$$y_h[n] = C_1(-1)^n + C_2(4)^n$$

and hence we may write that:

$$h[n] = C_1^{(\text{imp})}(-1)^n + C_2^{(\text{imp})}(4)^n \tag{47}$$

Evaluating (47) at $n = 0$ and $n = 1$ yields:

$$h[0] = C_1^{(\text{imp})}(-1)^0 + C_2^{(\text{imp})}(4)^0 \quad \Rightarrow \quad h[0] = C_1^{(\text{imp})} + C_2^{(\text{imp})} \tag{48}$$

$$h[1] = C_1^{(\text{imp})}(-1)^1 + C_2^{(\text{imp})}(4)^1 \quad \Rightarrow \quad h[1] = -C_1^{(\text{imp})} + 4C_2^{(\text{imp})} \tag{49}$$

Replacing $x[n]$ by $\delta[n]$ and $y[n]$ by $h[n]$ in the difference equation (46) above, yields:

$$h[n] = 3h[n-1] + 4h[n-2] + \delta[n] + 2\delta[n-1] \tag{50}$$

We then iterate equation (50) with initial conditions $h[-1] = 0$ and $h[-2] = 0$ to find:

$$h[0] = 3h[-1] + 4h[-2] + \delta[0] + 2\delta[-1] = 3(0) + 4(0) + 1 + 2(0) = 1 \tag{51}$$

and

$$h[1] = 3h[0] + 4h[-1] + \delta[1] + 2\delta[0] = 3(1) + 4(0) + 0 + 2(1) = 5 \tag{52}$$

From equations 48 and 51, we find:

$$C_1^{(\text{imp})} + C_2^{(\text{imp})} = 1 \tag{53}$$

while from equations 49 and 52, we find:

$$-C_1^{(\text{imp})} + 4C_2^{(\text{imp})} = 5 \tag{54}$$

Solution of equations 53 and 54 yields $C_1^{(\text{imp})} = -\frac{1}{5}$ and $C_2^{(\text{imp})} = \frac{6}{5}$. And substituting these values into (47) yields the following closed-form expression for the impulse response:

$$\textbf{impulse response:} \qquad h[n] = -\frac{1}{5}(-1)^n + \frac{6}{5}(4)^n \tag{55}$$

As a check of our solution, we might iterate equation (50) one more time to find

$$h[2] = 3h[1] + 4h[0] + \delta[2] + 2\delta[1] = 3(5) + 4(1) + 0 + 2(0) = 19 \tag{56}$$

and then evaluate expression (55) at $n = 2$ to find:

$$h[2] = -\frac{1}{5}(-1)^2 + \frac{6}{5}(4)^2 = -\frac{1}{5} + \frac{6 \times 16}{5} = \frac{95}{5} = 19 \tag{57}$$

Since we found the same value for $h[2]$ using expression (55) as we found by iterating the difference equation, we can be fairly confident that we did not make any mistakes.