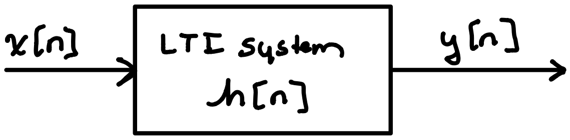
We've seen that an LTI system is characterized by its impulse response:



$$\Rightarrow y[n] = x[n] * h[n]$$

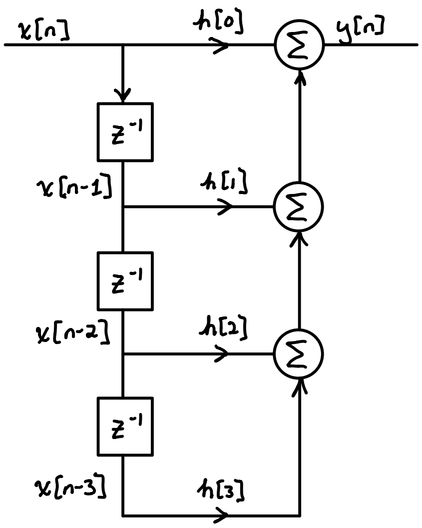
$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

If the LTI system is causal then $h[k]=0$ for $k<0$.
If the nonzero portion of the impulse response is of finite length (for example, if $h[k]=0$ for $k<0$ and for $k>M$), then the system is said to be a finite impulse response (FIR) system.

$$y[n] = \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}_{\text{if LTI}} = \underbrace{\sum_{k=0}^{\infty} h[k]\, x[n-k]}_{\text{if causal and LTI}} = \underbrace{\sum_{k=0}^{M} h[k]\, x[n-k]}_{\text{if causal, FIR and LTI}}$$

Realization / Implementation of a causal, FIR, LTI system

A causal, FIR, LTI system can be implemented using the structure shown below. (illustrated for the case $M=3$)



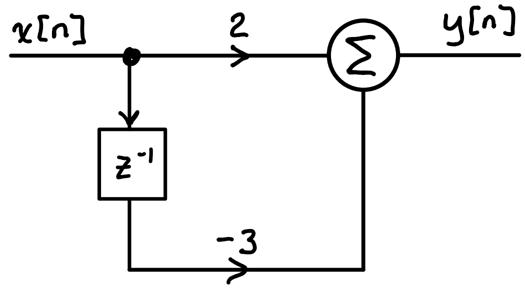
$$y[n] = h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] + h[3]\, x[n-3]$$

From the Handout on D.T. systems:

> 7. ***FIR vs. IIR:*** A system is termed as a ***finite impulse response (FIR) system*** if the impulse response is of finite length; otherwise, it is termed an ***infinite impulse response (IIR) system***.

Example

Find the impulse response and the input-output relationship of the system shown to the right.



Solution

The input-output equation can be found by inspection of the block diagram.

The impulse response, $h[n]$, is the response of the system when $x[n] = \delta[n]$. Hence:

How to implement an IIR system?

If the impulse response, $h[n]$, is nonzero over an infinite-length interval of n, the system is IIR.

Example

Consider an LTI causal system with impulse response

$$h[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then $$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=0}^{\infty} x[n-k]$$

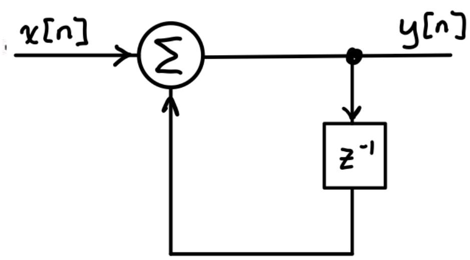
We cannot use the FIR structure to implement this system because it would require an infinite number of storage units.

The solution is to use a recursive implementation.

Note:

$$y[n] = \sum_{k=0}^{\infty} x[n-k] = x[n] + x[n-1] + x[n-2] + \cdots = \sum_{k=-\infty}^{n} x[k]$$

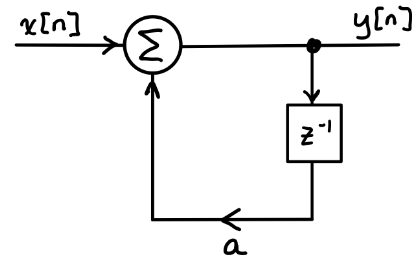
$$= x[n] + \underbrace{\sum_{k=-\infty}^{n-1} x[k]}_{y[n-1]}$$



A recursive implementation implies that the current output value depends on past output values.

Example

Consider the system to the right.



a) Find the system's input-output equation:

b) Find the system's impulse response, $h[n]$.

Recall that $h[n]$ is the response of the system when $x[n] = \delta[n]$.

To find the values of $h[n]$, $n \geq 0$, we will iterate the LCCDE. Since the system is causal, we know that $h[n] = 0$, $n < 0$.

$n=0$: $h[\;\;] = \delta[\;\;] + a\,h[\;\;]$

$n=1$: $h[\;\;] = \delta[\;\;] + a\,h[\;\;]$

$n=2$: $h[\;\;] = \delta[\;\;] + a\,h[\;\;]$

$n=3$: $h[\;\;] = \delta[\;\;] + a\,h[\;\;]$

$\vdots$

c) For which values of a will the system be stable?

Since this is an LTI system, we know it will be stable provided its impulse response, $h[n]$, is absolutely summable. Furthermore, $h[n] = a^n u[n]$ is absolutely summable provided ____________

$$\sum_{n=-\infty}^{\infty} |h[n]| =$$

Most of the systems we analyze in this course will be causal LTI systems which can be described by an input-output equation of the form:

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

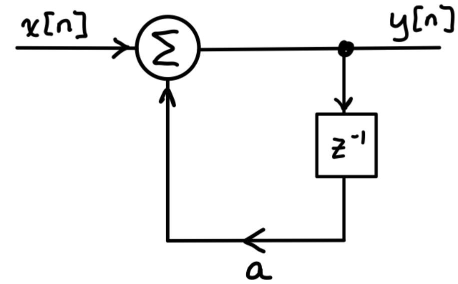
$$\Rightarrow \quad y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \qquad (*)$$

$(*)$ is called a Linear Constant Coefficient Difference Equation OR LCCDE.

<u>Analysis Task</u>

Given $x[n]$, $n \geq 0$ and ICs $y[-1], \dots, y[-N]$, we'd like to be able to find $y[n]$, $n \geq 0$.

(preferably as a closed-form expression rather than a string of numbers.)



Analysis of a simple recursive system

Given: $x[n]$, $n \geq 0$ and I.C. $y[-1] = y_{-1}$

Find: a nonrecursive expression for $y[n]$, $n \geq 0$

Solution:

From the block diagram, we note the system's LCCDE may be written as shown to the right.

LCCDE:
$$\boxed{y[n] = a\,y[n-1] + x[n]}$$

Below, we will iterate the LCCDE to find a nonrecursive expression for $y[n]$, $n \geq 0$ in terms of $x[n]$, $n \geq 0$ and the I.C. y_{-1}.

I will refer to this as a brute force approach to analysis.

$n=0$: $y[0] = a\,y[-1] + x[0] = a\,y_{-1} + x[0]$

$n=1$: $y[1] = a\,y[0] + x[1] = a^2 y_{-1} + a\,x[0] + x[1]$

$n=2$: $y[2] = a\,y[1] + x[2] = a^3 y_{-1} + a^2 x[0] + a\,x[1] + x[2]$

$\vdots$

arbitrary $n \geq 0$: $y[n] = a\,y[n-1] + x[n] = a^{n+1} y_{-1} + a^n x[0] + a^{n-1} x[1] + \ldots + a\,x[n-1] + x[n]$

$$\Rightarrow y[n] = \underbrace{a^{n+1}\, y_{-1}}_{\substack{y_{zi}[n] \\ \text{zero-input response}}} + \underbrace{\sum_{k=0}^{n} a^{n-k}\, x[k]}_{\substack{y_{zs}[n] \\ \text{zero-state response}}}, \quad n \geq 0$$

if $x[k] = 0$ for all $k \geq 0$, then $y[n] = y_{zi}[n]$ = zero-input response

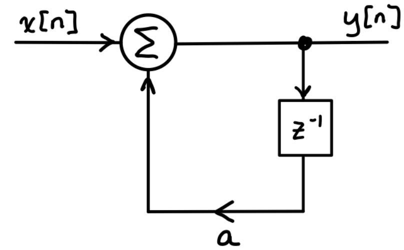
note: the zero-input response is caused by the initial conditions

if initial state of the system is zero (i.e., all registers are cleared $\Rightarrow$ IC's are all zero) then $y[n] = y_{zs}[n]$ = zero-state response

note: the zero-state response is due to $x[n]$, $n \geq 0$

Example

Use the result of the brute-force analysis on previous page to find the impulse response $h[n]$.



From previous page:

$$y[n] = \underbrace{a^{n+1} y_{-1}}_{y_{zi}[n]} + \underbrace{\sum_{k=0}^{n} a^{n-k} x[k]}_{y_{zs}[n]} \quad , \quad n \ge 0$$

By definition, the impulse response is the zero-state response of a system to $\delta[n]$.

Replacing $y[n]$ by $h[n]$ and $x[k]$ by $\delta[k]$ in (1), yields:

$$h[n] = a^{n+1} \cancel{h_{-1}}^{0} + \underbrace{\sum_{k=0}^{n} a^{n-k} \delta[k]}_{\text{only one nonzero term in this sum — the term corresponding to } k=0} \quad , \quad n \ge 0$$

$$\Rightarrow \boxed{h[n] = \begin{cases} a^n , & n \ge 0 \\ 0 , & n < 0 \end{cases}}$$

note that $y_{zs}[n] = \sum_{k=0}^{n} h[n-k]\, x[k]$

$$= h[n] * (x[n]\, u[n]) \quad \text{(see below)}$$

$$h[n] * (x[n]\, u[n]) = \sum_{k=-\infty}^{\infty} h[n-k]\, (x[k]\, u[k]) = \sum_{k=0}^{\infty} h[n-k]\, x[k] = \sum_{k=0}^{n} h[n-k]\, x[k]$$

$u[k] = 0$ for $k < 0$

if $h[n]$ is causal then $h[n-k] = 0$ for $k > n$

The effects of the input $x[n]$, $n < 0$ are accounted for by the ICs at $n = 0$.