From Difference Equation to Transfer Function and Pole-Zero Plot

Consider a stable LTI system described by the following LCCDE:

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N] = b_0 x[n] + \dots + b_M x[n-M]$$

Taking the Z−transform of both sides of the equation yields:

$$Y(z) + a_1 z^{-1} Y(z) + \dots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \dots + b_M z^{-M} X(z)$$

or:

$$Y(z)\left[1 + a_1 z^{-1} + \dots + a_N z^{-N}\right] = X(z)\left[b_0 + b_1 z^{-1} + \dots + b_M z^{-M}\right]$$

from which we find the system function, $H_Z(z)$:

$$H_Z(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

Muliplying numerator and denominator by z^N yields:

$$H_Z(z) = \frac{z^{N-M}\left(b_0 z^M + b_1 z^{M-1} + \dots + b_M\right)}{z^N + a_1 z^{N-1} + \dots + a_N}$$

Letting $z_1, z_2, \dots,$ and z_N denote the roots of the numerator polynomial (including those at $z = 0$) and similarly letting $p_1, p_2, \dots,$ and p_N denote the roots of the denominator polynomial, we can express the system function in terms of its zeros and poles as:

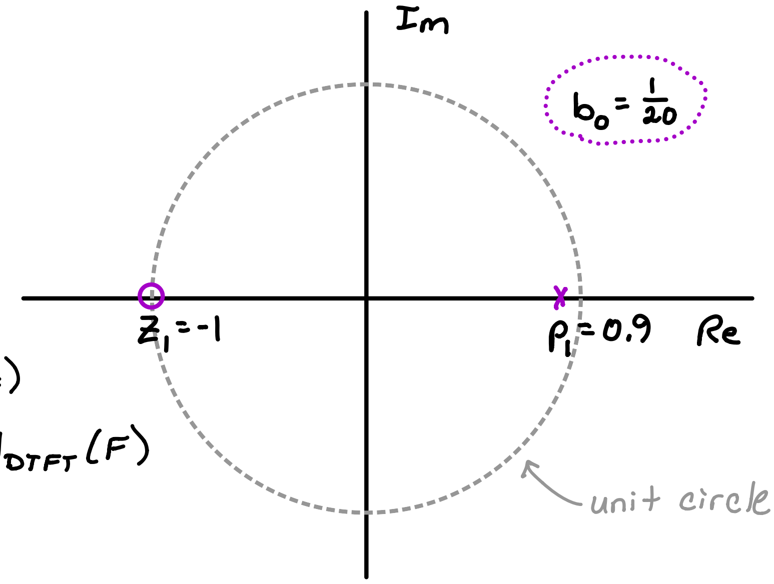
$$H_Z(z) = b_0 \frac{(z - z_1)(z - z_2)\dots(z - z_N)}{(z - p_1)(z - p_2)\dots(z - p_N)}$$

Once the transfer function is expressed in factored form, as shown above, it is straightforward to draw the pole-zero plot.

Similarly, given the value of b_0 and a pole-zero plot of the system, it is straightforward to write down the factored form of the system's transfer function.

Example

Given: the pole-zero plot of a filter as shown to the right



Find: the filter's transfer function, $H_z(z)$ and its freq. response function, $H_{DTFT}(F)$

$H_z(z) =$

$H_{DTFT}(F) =$

Would like to think about how we might use the pole-zero plot to visualize what this frequency response function looks like.

For which values of F will $|H_{DTFT}(F)|$ be largest? smallest? equal to zero?

What happens to the phase as F increases from 0 to $\frac{1}{2}$?

From pole-zero plots to frequency response

Given the pole zero plot of a system and the value of b_0, the system's transfer function can be determined as follows:

$$H_Z(z) = b_0 \frac{(z - z_1)(z - z_2)\dots(z - z_N)}{(z - p_1)(z - p_2)\dots(z - p_N)} = \frac{b_0 \prod_{i=1}^{M} (z - z_i)}{\prod_{k=1}^{N} (z - p_k)}$$

Furthermore, the system's Frequency Response Function, $H_{\text{DTFT}}(F)$, can be found by evaluating $H_Z(z)$ at $z = e^{j2\pi F}$:

$$H_{\text{DTFT}}(F) = H_Z\left(e^{j2\pi F}\right) = \frac{b_0 \prod_{i=1}^{M} \overbrace{\left(e^{j2\pi F} - z_i\right)}^{N_i(F)}}{\prod_{k=1}^{N} \underbrace{\left(e^{j2\pi F} - p_k\right)}_{D_k(F)}} = \frac{b_0 \prod_{i=1}^{M} N_i(F)}{\prod_{k=1}^{N} D_k(F)}$$

where:

$$N_i(F) = e^{j2\pi F} - z_i \text{ and } D_k(F) = e^{j2\pi F} - p_k$$

Expressing $N_i(F)$ and $D_k(F)$ in polar form, we may write:

$$H_{\text{DTFT}}(F) = \frac{b_0 \prod_{i=1}^{M} |N_i(F)| e^{j\measuredangle(N_i(F))}}{\prod_{k=1}^{N} |D_k(F)| e^{j\measuredangle(D_k(F))}}$$

from which it follows that:

$$|H_{\text{DTFT}}(F)| = \frac{|b_0| \prod_{i=1}^{M} |N_i(F)|}{\prod_{k=1}^{N} |D_k(F)|}$$

and

$$\measuredangle\left(H_{\text{DTFT}}(F)\right) = \measuredangle\left(b_0\right) + \sum_{i=1}^{M} \measuredangle\left(N_i(F)\right) - \sum_{k=1}^{N} \measuredangle\left(D_k(F)\right)$$

From pole-zero plots to frequency response

Example: Sketch the magnitude and phase of the Frequency Response Function, $H_{\text{DTFT}}(F)$, for a system with transfer function:

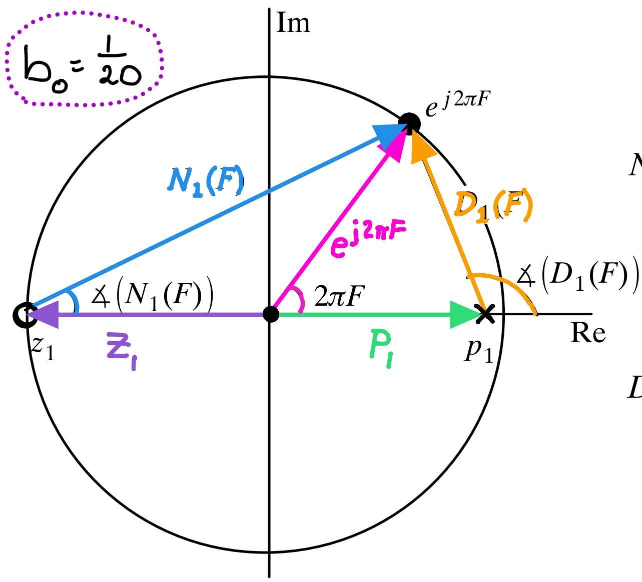
$$H_z(z) = \frac{\frac{1}{20}(z+1)}{z-0.9} = \frac{\frac{1}{20}\big(z-(-1)\big)}{z-0.9} = \frac{\frac{1}{20}(z-z_1)}{z-p_1}$$

$\boxed{\text{This is the filter in part 2a of Lab 5.}}$

Note there is one zero at $z = z_1 = -1$ and one pole at $z = p_1 = 0.9$.

Solution: The frequency response function is given by:

$$H_{\text{DTFT}}(F) = H_z(z)\big|_{z=e^{j2\pi F}} = \frac{1}{20}\left[\frac{e^{j2\pi F}-z_1}{e^{j2\pi F}-p_1}\right] = \frac{1}{20}\left[\frac{N_1(F)}{D_1(F)}\right]$$



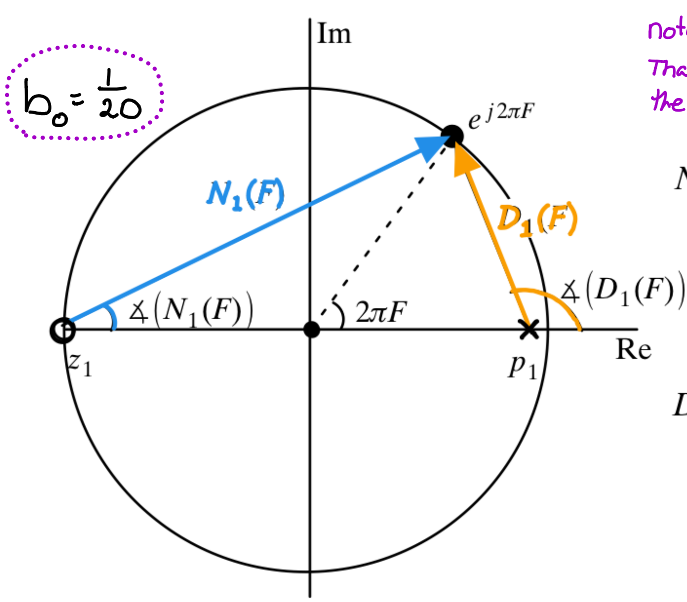
$N_1(F) = e^{j2\pi F} - z_1 \Rightarrow \boxed{z_1 + N_1(F) = e^{j2\pi F}}$

$\Rightarrow N_1(F)$ is the vector from z_1 to $e^{j2\pi F}$

$D_1(F) = e^{j2\pi F} - p_1 \Rightarrow \boxed{p_1 + D_1(F) = e^{j2\pi F}}$

$\Rightarrow D_1(F)$ is the vector from p_1 to $e^{j2\pi F}$

$$H_{\text{DTFT}}(F) = \underbrace{\frac{1}{20}}_{b_0}\left(\frac{N_1(F)}{D_1(F)}\right) \Rightarrow \begin{cases} \left|H_{\text{DTFT}}(F)\right| = \frac{1}{20}\left(\frac{|N_1(F)|}{|D_1(F)|}\right) \\ \measuredangle\left(H_{\text{DTFT}}(F)\right) = \measuredangle\left(N_1(F)\right) - \measuredangle\left(D_1(F)\right) \end{cases}$$



note that $|H_{\text{DTFT}}(F)|$ will be biggest when $|D_1(F)|$ is smallest. That will happen for values of F associated with points on the unit circle close to p_1.

$N_1(F) = e^{j2\pi F} - z_1$

$\Rightarrow N_1(F)$ is the vector from z_1 to $e^{j2\pi F}$

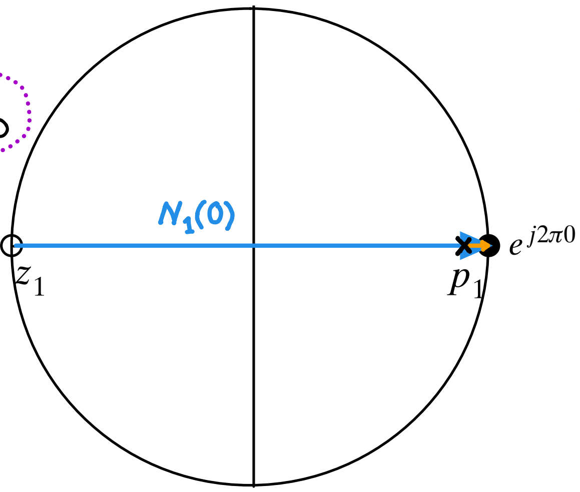
$D_1(F) = e^{j2\pi F} - p_1$

$\Rightarrow D_1(F)$ is the vector from p_1 to $e^{j2\pi F}$

From pole-zero plots to frequency response

Case 1: $F = 0$:

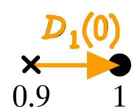
$b_0 = \frac{1}{20}$



$N_1(0)$ is the vector from z_1 to $e^{j2\pi 0}$

$$\left.\begin{array}{l}|N_1(0)| = 2 \\ \measuredangle\left(N_1(0)\right) = 0^\circ\end{array}\right\} N_1(0) = 2\angle 0^\circ$$

$D_1(0)$ is the vector from p_1 to $e^{j2\pi 0}$



$$\left.\begin{array}{l}|D_1(0)| = 0.1 \\ \measuredangle\left(D_1(0)\right) = 0^\circ\end{array}\right\} D_1(0) = 0.1\angle 0^\circ$$

$$N_1(0) = e^{j2\pi 0} - z_1 = 1 - (-1) = 2$$

$$D_1(0) = e^{j2\pi 0} - p_1 = 1 - 0.9 = 0.1$$

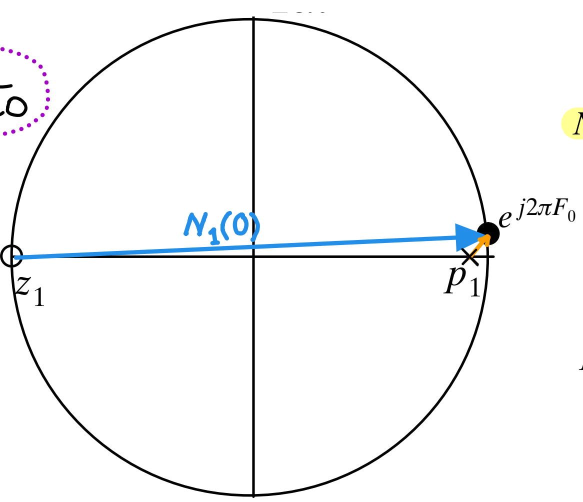
$$H_{DTFT}(0) = \underbrace{\frac{1}{20}}_{b_0} \frac{N_1(0)}{D_1(0)} = \frac{1}{20} \frac{2\angle 0^\circ}{0.1\angle 0^\circ} = \frac{20}{20} = 1$$

Case 2: Determine the approximate value of the 3-dB cutoff frequency, F_0.

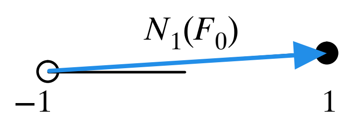
By Defn: $|H_{DTFT}(F_0)| = \frac{1}{\sqrt{2}} |H_{DTFT}(0)| \Rightarrow \frac{N_1(F_0)}{D_1(F_0)} = \frac{1}{\sqrt{2}} \frac{N_1(0)}{D_1(0)} \Rightarrow$ will find F_0 such that: $N_1(F_0) \simeq N_1(0)$ and $D_1(F_0) \simeq \sqrt{2}\, D_1(0)$

For F close to 0: $\frac{|N_1(F)|}{|N_1(0)|} \simeq 1$ whereas $\frac{|D_1(F)|}{|D_1(0)|}$ changes quickly.

$b_0 = \frac{1}{20}$

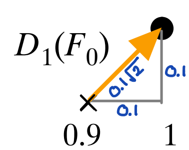


$N_1(F_0)$ is the vector from z_1 to $e^{j2\pi F_0}$



$$\left.\begin{array}{l}|N_1(F_0)| \approx 2 \\ \measuredangle\left(N_1(F_0)\right) \approx 0^\circ\end{array}\right\} N_1(F_0) \simeq 2\angle 0^\circ$$

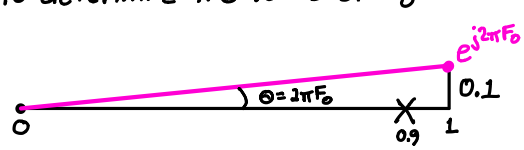
$D_1(F_0)$ is the vector from p_1 to $e^{j2\pi F_0}$



$$\left.\begin{array}{l}|D_1(F_0)| \approx \frac{\sqrt{2}}{10} \\ \measuredangle\left(D_1(F_0)\right) = 45^\circ\end{array}\right\} D_1(F_0) \simeq \frac{\sqrt{2}}{10}\angle 45^\circ$$

$$H_{DTFT}(F_0) = \underbrace{\frac{1}{20}}_{b_0} \frac{N_1(F_0)}{D_1(F_0)} = \frac{1}{20} \frac{2\angle 0^\circ}{0.1\sqrt{2}\angle 45^\circ} = \frac{1}{\sqrt{2}} \angle -45^\circ$$

To determine the value of F_0:

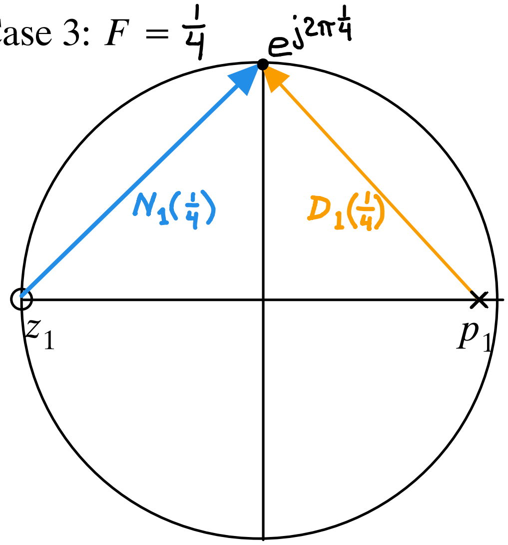


$$\Theta = 2\pi F_0 \simeq \operatorname{atan}\left(\frac{0.1}{1}\right) \simeq 0.1 \Rightarrow 2\pi F_0 \simeq \frac{1}{10} \Rightarrow \boxed{F_0 \simeq \frac{1}{20\pi}}$$

$\simeq 0.016 \frac{\text{cycles}}{\text{sample}}$

From pole-zero plots to frequency response

Case 3: $F = \frac{1}{4}$



$N_1(\frac{1}{4}) =$

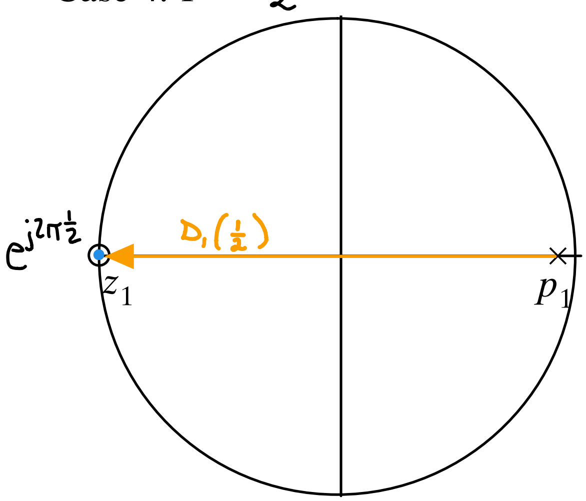
$D_1(\frac{1}{4}) \simeq$

$$\Rightarrow H_{DTFT}(\tfrac{1}{4}) = b_0 \frac{N_1(\frac{1}{4})}{D_1(\frac{1}{4})}$$

$$\simeq \frac{1}{20} \underline{\qquad\qquad}$$

$$=$$

Case 4: $F = \frac{1}{2}$



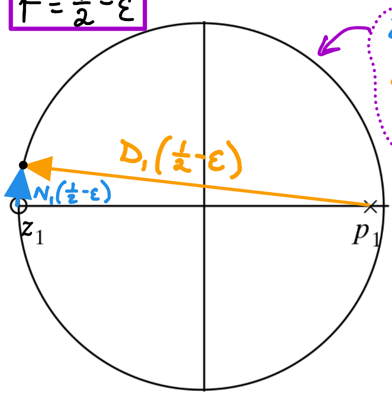
$N_1(\frac{1}{2}) =$

$D_1(\frac{1}{2}) =$

$$H_{DTFT}(\tfrac{1}{2}) = b_0 \frac{N_1(\frac{1}{2})}{D_1(\frac{1}{2})} = \frac{1}{20} \underline{\qquad\quad} =$$

Note: the phase changes by 180° at $F = \frac{1}{2}$

$\boxed{F = \frac{1}{2} - \varepsilon}$

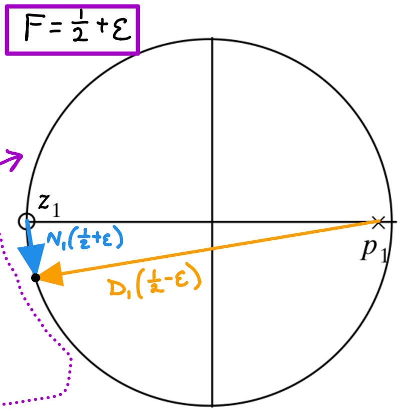


$\measuredangle N_1(\frac{1}{2}-\varepsilon) \simeq$

$\measuredangle D_1(\frac{1}{2}-\varepsilon) \simeq$

$\measuredangle H_{DTFT}(\frac{1}{2}-\varepsilon) = (\qquad) - (\qquad) =$

$\boxed{F = \frac{1}{2} + \varepsilon}$



$\measuredangle N_1(\frac{1}{2}+\varepsilon) \simeq$

$\measuredangle D_1(\frac{1}{2}+\varepsilon) \simeq$

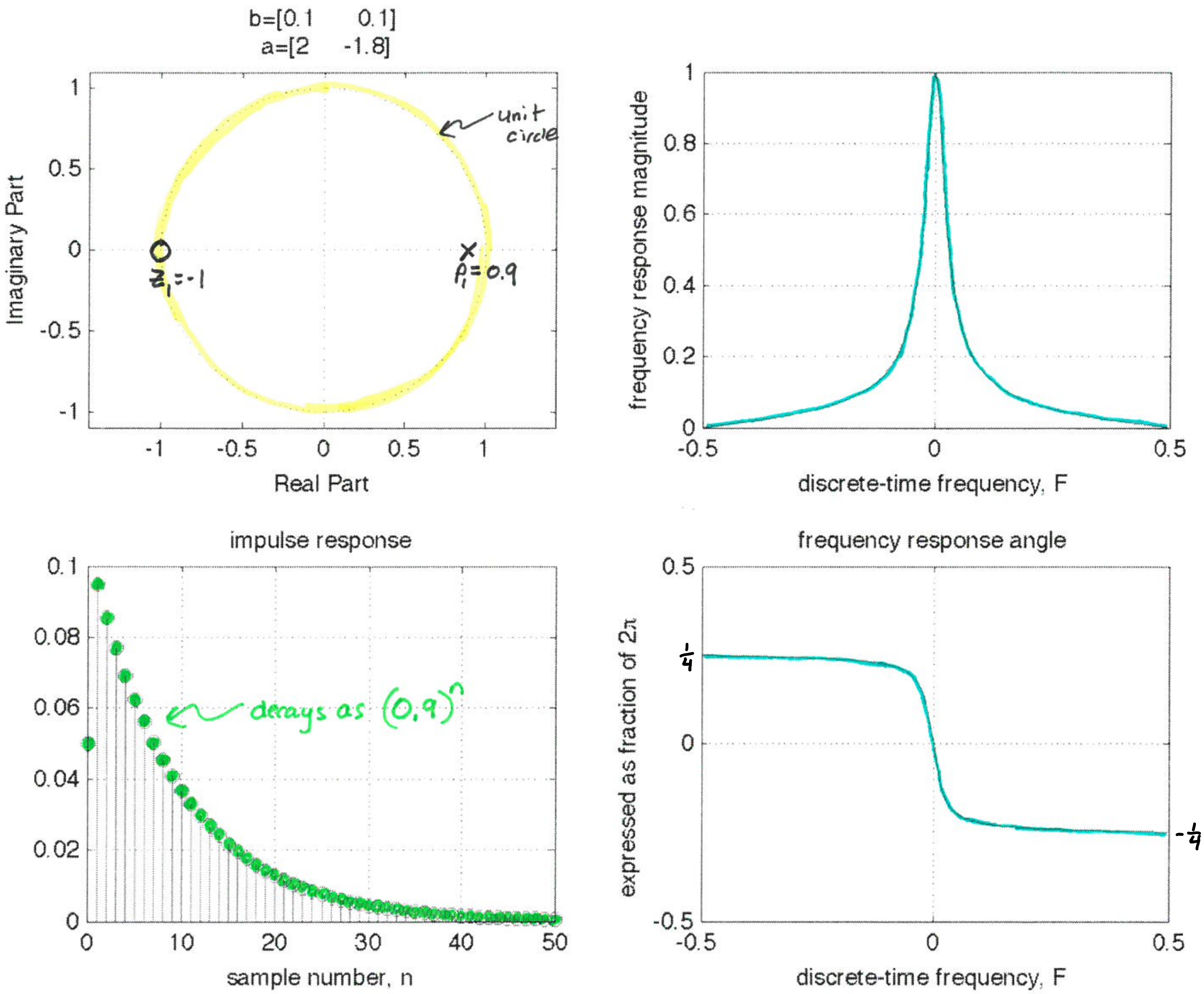
$\measuredangle H_{DTFT}(\frac{1}{2}+\varepsilon) = (\qquad) - (\qquad) =$

part 2a: $r=0.9$

$$H_z(z) = \frac{0.1}{2}\frac{(1+z^{-1})}{(1-0.9z^{-1})} = \frac{1}{20}\left(\frac{z-(-1)}{z-0.9}\right)$$

dc gain: $H_{DTFT}(0) = H_z(1) = \frac{1}{20}\left(\frac{1+1}{1-0.9}\right) = \frac{1}{20}\left(\frac{2}{1/10}\right) = 1$

high-freq. gain: $H_{DTFT}(\frac{1}{2}) = H_z(-1) = 0$



$$\boxed{\begin{aligned} H_z(z) &= \frac{1}{20}\left(\frac{1+z^{-1}}{1-0.9z^{-1}}\right) \\ &= \frac{1}{20}\left(\frac{1}{1-0.9z^{-1}} + \frac{z^{-1}}{1-0.9z^{-1}}\right)\end{aligned}}$$

$\xRightarrow{\text{IVT}}$ $h[0] =$

$\Downarrow$ IZT

$h[n] =$