The Bilinear Transform maps the s-plane into the z-plane.
It is often used for transforming a c.t. filter into a d.t. filter.

Background:

- The building blocks of c.t. filters are
  integrators, summers, and multipliers
- the building blocks of d.t. filters are
  delay elements, summers, and multipliers

In order to transform a c.t. filter into a d.t. filter, need to find the d.t. equivalent of the c.t. integrator.

The transfer function the c.t. integrator is $\boxed{H_s(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}}$

$$\Rightarrow sY(s) = X(s)$$

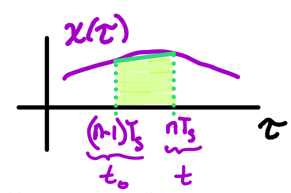
$$\Rightarrow y'(t) = x(t)$$

$$\Rightarrow y(t) = y(t_0) + \int_{t_0}^{t} x(\tau)\,d\tau$$

← input-output relationship for c.t. integrator

Letting $t = nT_s$ and $t_0 = (n-1)T_s$, we may rewrite the preceding equation as:

$$y(nT_s) = y((n-1)T_s) + \int_{(n-1)T_s}^{nT_s} x(\tau)\,d\tau$$



Using the trapezoidal rule to approximate the integral yields:

$$y(nT_s) = y((n-1)T_s) + T_s\left(\frac{x((n-1)T_s) + x(nT_s)}{2}\right)$$

Finally, we let $y[n] = y(nT_s)$ and $x[n] = x(nT_s)$

$$y[n] = y[n-1] + \frac{T_s}{2}\left(x[n-1] + x[n]\right)$$

difference equation of d.t. system approximating c.t. integrator

The integrator is, thus, approximated by the following difference equation:

$$y[n] = y[n-1] + \frac{T_s}{2}\left(x[n] + x[n-1]\right)$$

Taking the z-transform of the difference equation yields:

$$Y(z) = z^{-1} Y(z) + \frac{T_s}{2}\left(X(z) + z^{-1} X(z)\right)$$

$$\Rightarrow \quad Y(z)\left[1 - z^{-1}\right] = X(z)\,\frac{T_s}{2}\left(1 + z^{-1}\right)$$

$$\Rightarrow \quad \boxed{H_z(z) \equiv \frac{Y(z)}{X(z)} = \frac{\frac{T_s}{2}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)}} =$$

transfer function of a d.t. system that approximates an integrator

By equating the transfer function of a c.t. integrator to the transfer function of its discrete-time approximation, we find a relation between s and z that can be used to map the s-plane onto the z-plane.

$$H_s(s) = \frac{1}{s} \iff H_z(z) = \frac{T_s}{2}\,\frac{(1+z^{-1})}{(1-z^{-1})} = \frac{T_s}{2}\,\frac{(z+1)}{(z-1)}$$

Taking the reciprocal of both transfer functions yields:

$$s = \frac{2}{T_s}\,\frac{(z-1)}{(z+1)} = \frac{2f_s\,(z-1)}{(z+1)} \qquad \text{OR} \qquad \boxed{s = \frac{2f_s\,(1-z^{-1})}{(1+z^{-1})}}$$

OR solving for z in terms of s yields:

$$(z+1)s = 2f_s\,(z-1) \Rightarrow zs + s = 2f_s z - 2f_s$$

$$\Rightarrow z(s - 2f_s) = -(s + 2f_s)$$

$$\Rightarrow \boxed{z = \frac{2f_s + s}{2f_s - s}}$$

Visualizing the mapping of the Bilinear Transform

Bilinear Transform: $z = \dfrac{2f_s + s}{2f_s - s} = \dfrac{2f_s + \sigma + j\omega}{2f_s - \sigma - j\omega}$

$$|z| = \frac{\sqrt{(2f_s + \sigma)^2 + \omega^2}}{\sqrt{(2f_s - \sigma)^2 + \omega^2}}$$

if $\sigma > 0$, then numerator > denominator $\Rightarrow |z| > 1$

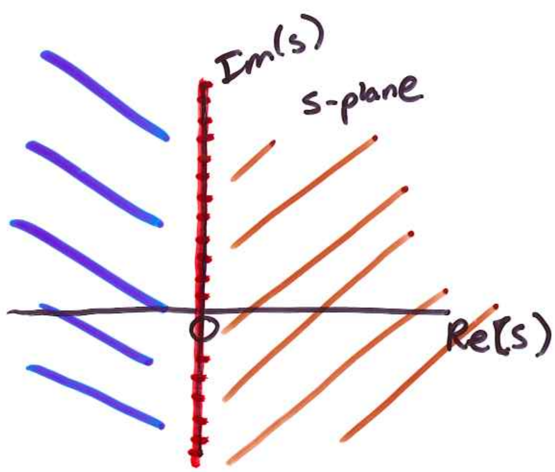
this implies that points in the right half of the s-plane get mapped to points outside the unit circle

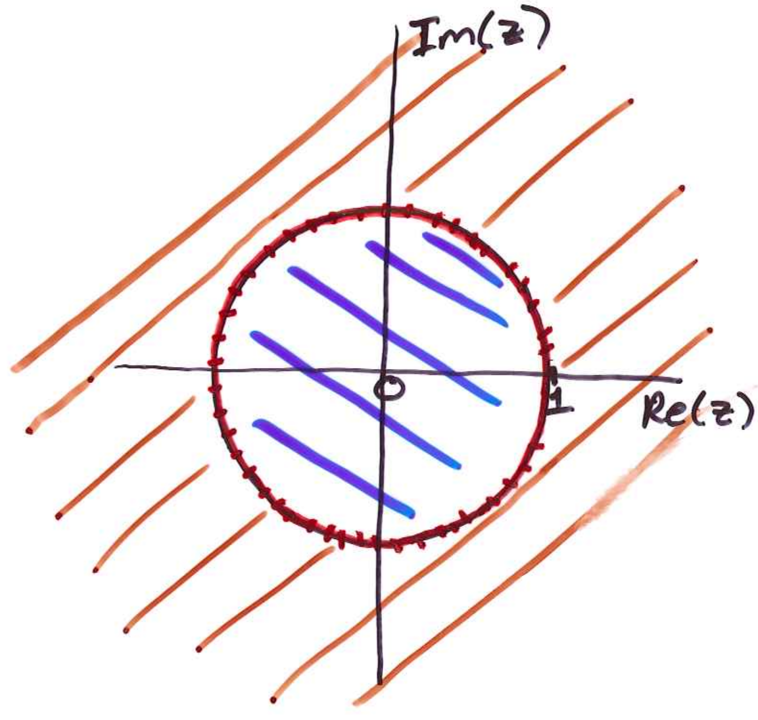
if $\sigma < 0$, then the numerator < denominator $\Rightarrow |z| < 1$

this implies that points in the left half of the s-plane get mapped to points inside the unit circle

if $\sigma = 0$, then numerator = denominator $\Rightarrow |z| = 1$

this implies that points on the $j\omega$ axis get mapped to points on the unit circle





$\boxed{\text{How does the Bilinear Transform map the c.t. frequency variable, } f\text{, to the discrete-time frequency variable } F\text{?}}$

Previously, we saw that the bilinear transform maps the $j\omega$-axis in the s-plane to the unit circle in the z-plane.

- any point s on the $j\omega$ axis can be expressed as: $s = j2\pi f$
- any point z on the unit circle can be expressed as: $z = e^{j2\pi F}$

Replacing s by $j2\pi f$ and z by $e^{j2\pi F}$ in the bilinear transform yields the following relationship between f and F.

B.L.T.

$$z = \frac{2f_s + s}{2f_s - s} \xrightarrow[z = e^{j2\pi F},\ s = j2\pi f]{} \boxed{e^{j2\pi F} = \frac{2f_s + j2\pi f}{2f_s - j2\pi f}}$$

Equating the angles of both sides of previous expression yields:

$$\underbrace{2\pi F}_{\text{angle of LHS}} = \overbrace{\text{atan}\left(\frac{2\pi f}{2f_s}\right)}^{\text{angle of numerator on RHS}} - \overbrace{\left(-\,\text{atan}\left(\frac{2\pi f}{2f_s}\right)\right)}^{\text{angle of denominator on RHS}} = \underbrace{2\,\text{atan}\left(\frac{\pi f}{f_s}\right)}_{\text{angle of RHS}}$$

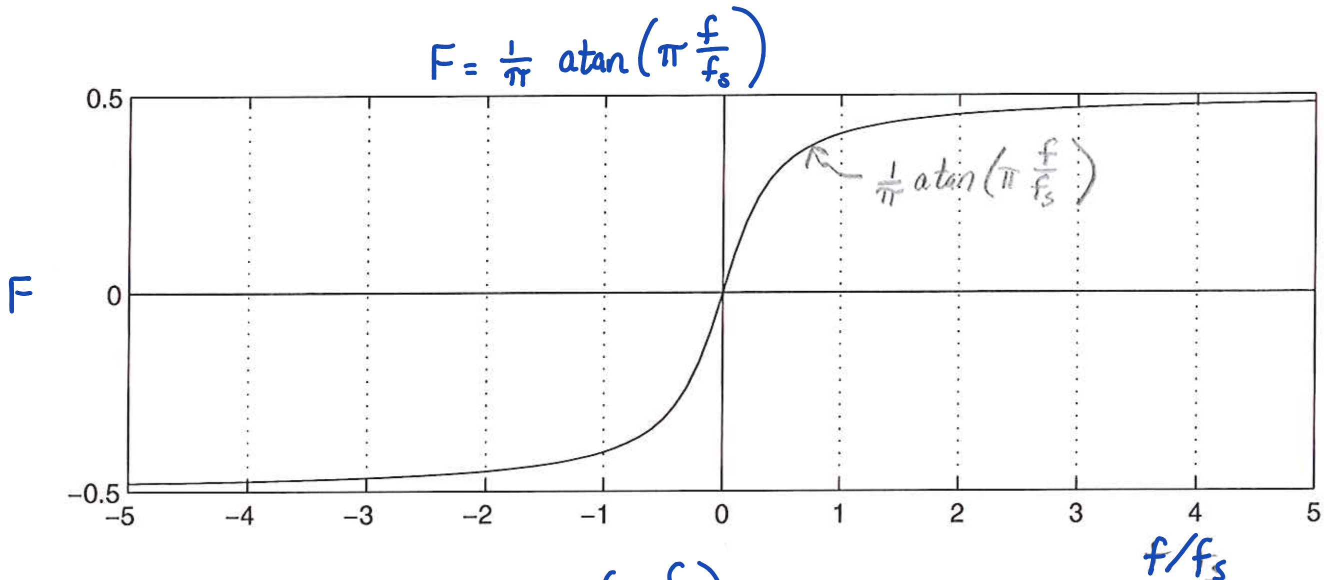
$$\Rightarrow \pi F = \text{atan}\left(\pi \frac{f}{f_s}\right) \Rightarrow \boxed{F = \frac{1}{\pi}\,\text{atan}\left(\pi \frac{f}{f_s}\right)}$$

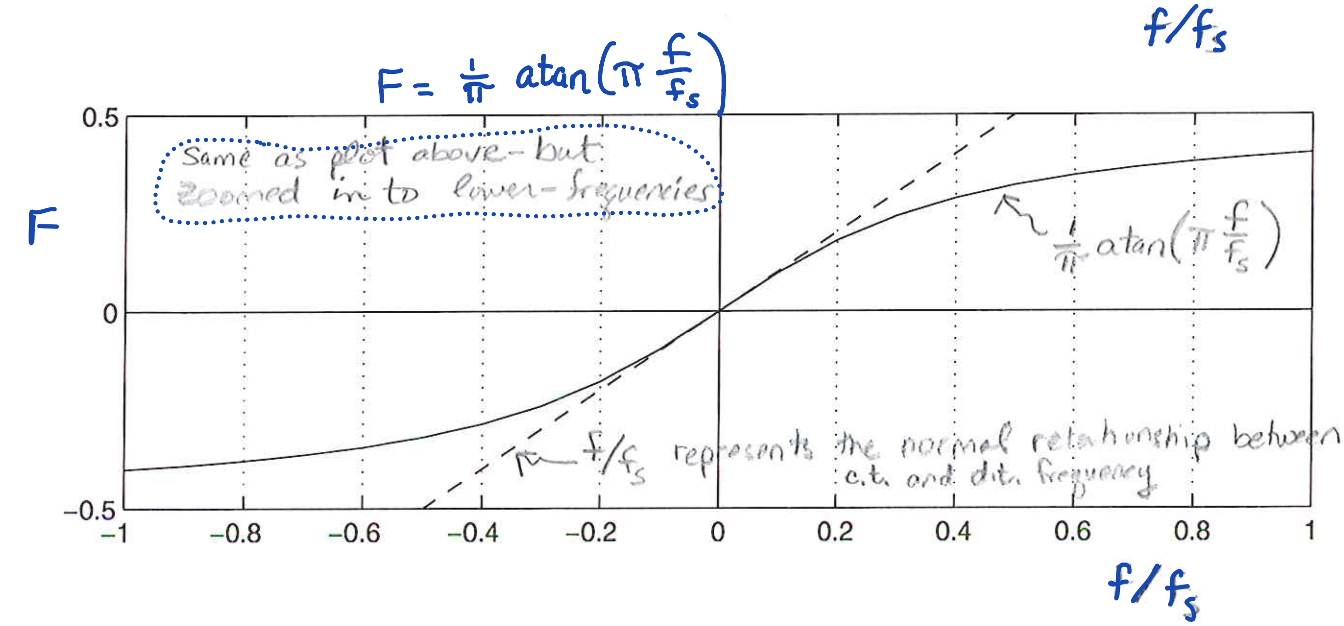
OR

$$\boxed{f = \frac{f_s}{\pi}\tan(\pi F)}$$

The relationships between F and f/f_s resulting from the bilinear transform are approximately equivalent to the normal relationships ($F = \frac{f}{f_s}$ or $f = f_s F$) provided that F or f/f_s is small (less than $\frac{1}{10}$). For larger values of F and/or f/f_s, the frequency response of the transformed system will be distorted due to the nonlinear mapping between f and F.

Note: the bilinear transform maps the continuous-time frequency, f, to the discrete-time frequency, F, according to $F = \frac{1}{\pi} \operatorname{atan}\left(\pi \frac{f}{f_s}\right)$.





Note that for $|f| < 0.1 f_s$, the mapping from continuous-time frequency, f, to discrete-time frequency, F, is as would be expected.

$$F = \frac{1}{\pi} \operatorname{atan}\left(\pi \frac{f}{f_s}\right) \simeq f/f_s$$

Example:

The transfer function of a continuous-time 1^{st} order Low Pass Filter (LPF) with 3dB cutoff frequency f_c Hz. is Known to be:

$$H_s(s) = \frac{2\pi f_c}{s + 2\pi f_c}$$

Use the bilinear transform together with the analog prototype above to design a discrete-time LPF with 3dB cutoff $F_c = 0.1 \ \frac{\text{cycle}}{\text{sample}}$.

Solution

Step 1: Determine the c.t. frequency f_c which will be mapped to a d.t. frequency $F_c = 0.1$.

$F = \frac{1}{\pi} \operatorname{atan}\left(\pi \frac{f}{f_s}\right)$

$f = \frac{f_s}{\pi} \tan(\pi F)$

$$f_c = \frac{f_s}{\pi} \tan(\pi F_c) = \frac{f_s}{\pi} \tan\left(\frac{\pi}{10}\right) = 0.1034\, f_s$$

$$\Rightarrow 2\pi f_c = 2 f_s \tan(\pi/10) = 0.65 f_s$$

$\Rightarrow$ desired c.t. prototype is $H_s(s) = \frac{0.65 f_s}{s + 0.65 f_s}$

Step 2: Apply Bilinear transformation to $H_s(s)$ to obtain $H_z(z)$

BLT: $s = 2f_s \frac{1-z^{-1}}{1+z^{-1}}$

$$\Rightarrow H_z(z) = \frac{0.65 \cancel{f_s}}{2\cancel{f_s} \frac{1-z^{-1}}{1+z^{-1}} + 0.65 \cancel{f_s}} = \frac{0.65(1+z^{-1})}{2(1-z^{-1}) + 0.65(1+z^{-1})}$$

$$\Rightarrow H_z(z) = \frac{0.65(1+z^{-1})}{2.65 + (0.65-2)z^{-1}}$$

$$= \frac{0.65}{2.65} \frac{(1+z^{-1})}{1 - \frac{1.35}{2.65} z^{-1}}$$

$$= 0.2453 \frac{1+z^{-1}}{1-0.509 z^{-1}}$$

$$= 0.2453 \left(\frac{z+1}{z-0.509} \right) \Rightarrow \begin{cases} \text{one pole at } z=0.509 \\ \text{one zero at } z=-1 \end{cases}$$

How do the pole and zero locations of the d.t. filter, $H_z(z)$, compare to those of the c.t. filter, $H_s(s)$?

Recall: $H_s(s) = \dfrac{0.65 f_s}{s+0.65 f_s} \Rightarrow \begin{cases} \text{one pole at } s=-0.65 f_s \\ \text{one zero at } s=\infty \end{cases}$

Note that the bilinear transform maps the c.t. pole at $s=-0.65 f_s$ to the d.t. pole at $z=0.509$; similarly it maps the c.t. zero at $s=\infty$ to the d.t. zero at $z=-1$.

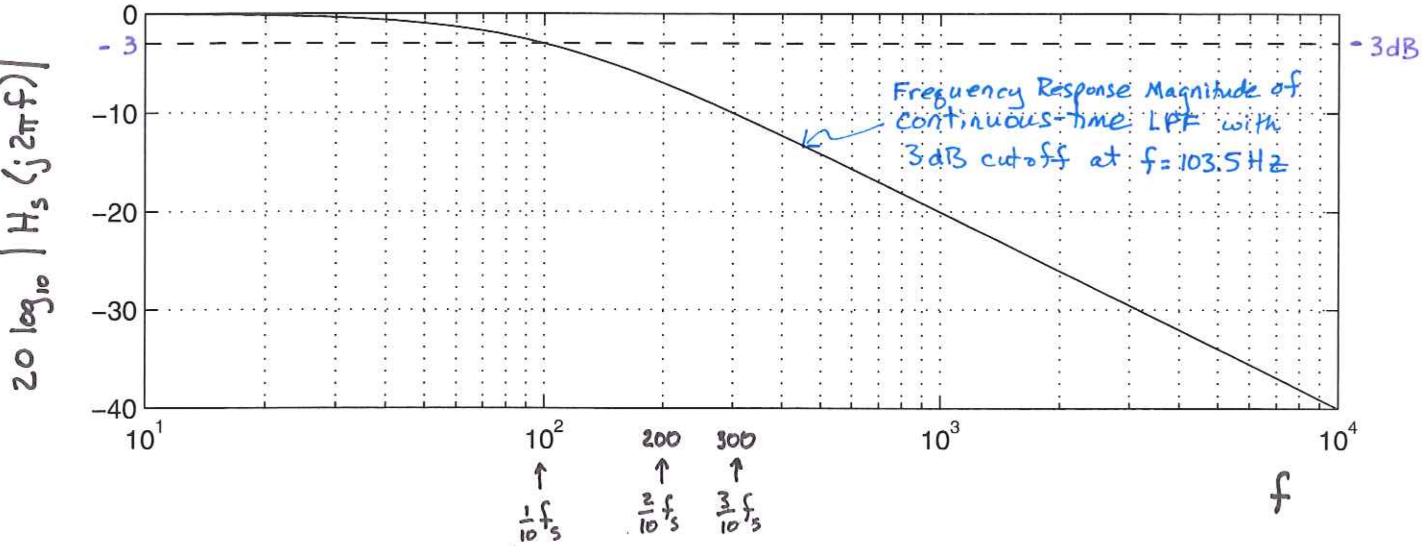
$$s=-0.65 f_s \Longrightarrow z = \frac{2f_s+s}{2f_s-s} = \frac{2f_s-0.65f_s}{2f_s+0.65f_s} = \frac{1.35}{2.65} = 0.509$$

$$s=\infty \Longrightarrow z = \lim_{s\to\infty} \frac{2f_s+s}{2f_s-s} = \lim_{s\to\infty} \frac{s}{-s} = -1$$

How does the frequency response of $H_z(z)$ compare to that of $H_s(s)$?

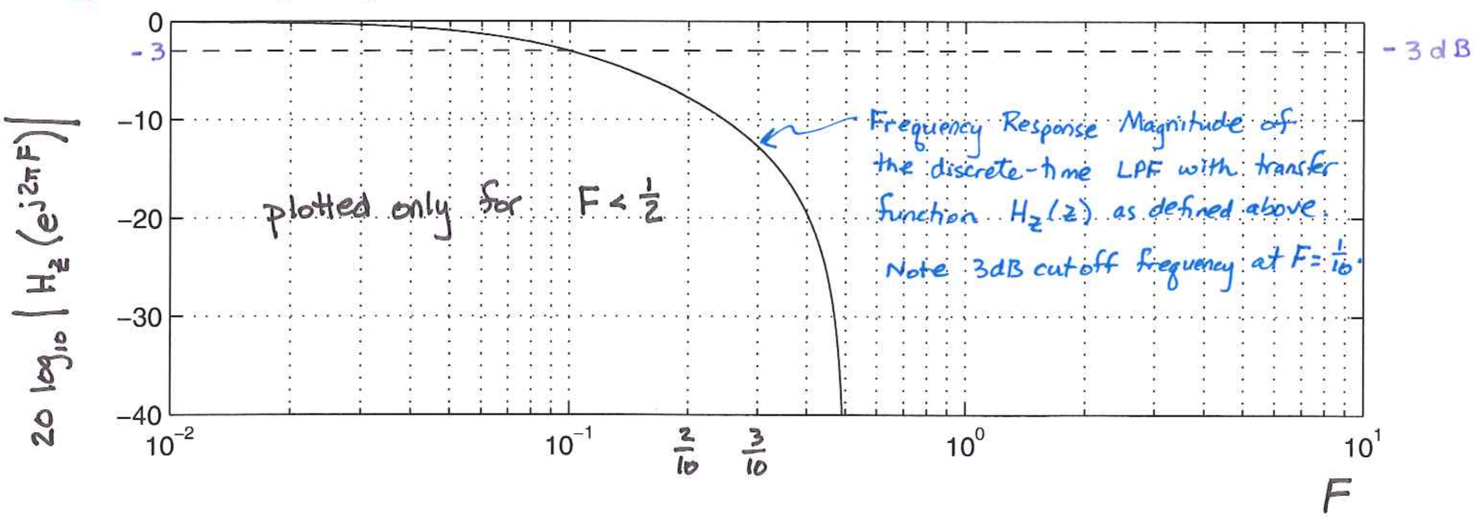
Application of the Bilinear Transform (with $f_s = 1000 \frac{\text{samples}}{\text{sec}}$) to the continuous-time filter shown below

$$H_s(s) = \left.\frac{0.65\, f_s}{s + 0.65\, f_s}\right|_{f_s = 1000} = \frac{650}{s+650} = \frac{2\pi(103.5)}{s + 2\pi(103.5)} \approx \frac{2\pi(100)}{s + 2\pi(100)}$$



yields the discrete-time LPF: $H_z(z) = \dfrac{0.245\,(1+z^{-1})}{(1 - 0.509\, z^{-1})}$

whose frequency response is shown below.



Note: For $F < \frac{1}{10}$, $20 \log_{10} |H_z(e^{j2\pi F})| \approx 20 \log_{10} |H_s(j2\pi f)|_{f = f_s F}$

As F increases above $\frac{1}{10}$, this relationship is no longer valid.