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This is a closed-book exam. A formula sheet is provided. The time allowed is 3 hours; an additional 20 minutes are provided for printing the exam as well as scanning and uploading your solutions. **It's important to pace yourself and to not spend too much time on any one question.** The value of each question is indicated in the margin. The points total to 100. It is important to show your work and clearly indicate your answers. Points may be deducted if your writing is difficult to read or your work is difficult to follow.

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1. Consider the discrete-time sinusoids, $x_1[n] = \cos(2\pi F_1 n)$ and $x_2[n] = \cos(2\pi F_2 n)$, where $F_1 = \frac{32}{31}$ and $F_2 = \frac{59}{62}$.

( 2 pts.) a) Let $N_1$ and $N_2$ denote the fundamental periods of $x_1[n]$ and $x_2[n]$, respectively. State their values.

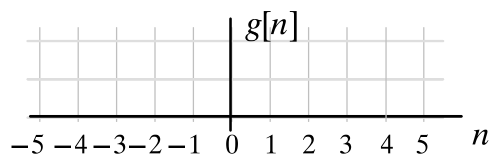
| $N_1$ = | $N_2$ = |
|---|---|

( 2 pts.) b) Of the two sinusoids, $x_1[n]$ and $x_2[n]$, which one oscillates the fastest? **Explain.** You will not receive credit unless your procedure is clearly indicated.

( 2 pts.) 2. The ***discrete-time*** unit step function is defined as follows:

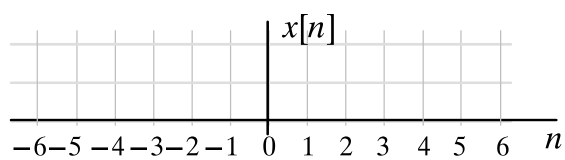
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Given that $g[n] = u[-n+2]$, sketch $g[n]$.




( 4 pts.) 3. Let $x[n] = \sum_{k=0}^{4} e^{j2\pi \frac{(n+2)}{5} k}$.

Sketch $x[n]$, $-6 \leq n \leq 6$.




***Be sure to provide a clear explanation/justification of the values illustrated in your sketch.***

4. Consider a causal LTI system with transfer function:

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{3}{4}z^{-1}}$$

( 1 pts.) a) Is this a stable system? How did you decide?

( 6 pts.) b) Find a closed-form expression for $y_{\text{step}}[n]$, the step response of the system.

( 2 pts.) c) Determine the dc gain of the system ***directly from*** $H(z)$. ***Clearly indicate*** your procedure.

( 2 pts.) d) Determine the dc gain ***directly from the system's step response***, $y_{\text{step}}[n]$, that you found in part (b). To receive credit, you must clearly explain how to determine the dc gain directly from the step response.

***continuation of question 4***. For your convenience, the system's transfer function is repeated to the right.

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{3}{4}z^{-1}}$$

( 1 pt.) e) Write the difference equation that relates the system's output, $y[n]$, to the system's input, $x[n]$.

( 3 pts.) f) Iterate the system's difference equation, with $x[n]$ appropriately replaced, to find the values of the system's impulse response, $h[n]$, for $n = 0, 1,$ and $2$. ***Show your work***.

( 1 pt.) g) Write an equation that shows how the impulse response, $h[n]$, of an LTI system can be determined from the system's step response, $y_{\text{step}}[n]$.

equation expressing h[n] in terms of $y_{\text{step}}[]$: $h[n] =$

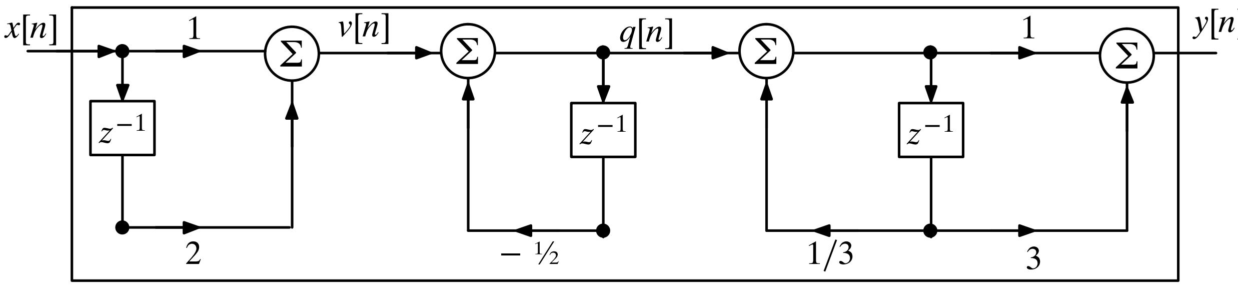
( 2 pts.) h) Knowing that $y_{\text{step}}[-1] = 0$, ***explain how*** to use the values for $h[0]$, $h[1]$, and $h[2]$ that you found in part (f) to find the value of $y_{\text{step}}[2]$. ***Then*** apply your procedure to find $y_{\text{step}}[2]$.

( 1 pt.) i) Evaluate the closed-form expression, for $y_{\text{step}}[n]$, found in part (b), at $n = 2$, to find the value of $y_{\text{step}}[2]$. ***You must show your work***. Compare with your response to part (h).

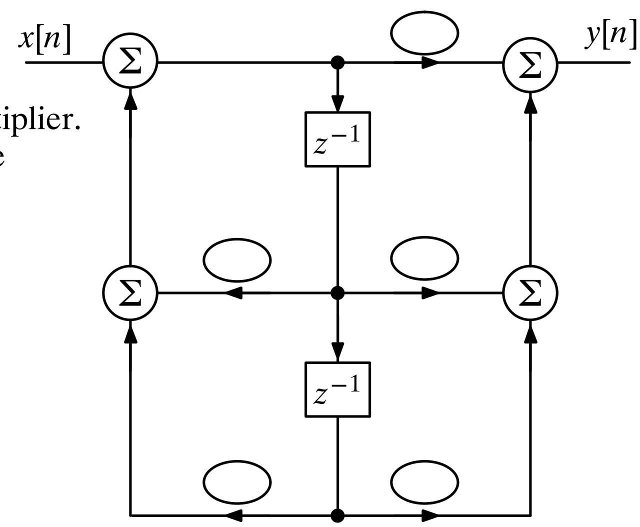


/ 8 pts

5. Consider the system shown below with input $x[n]$ and output $y[n]$.




( 5 pts.) a) Note the five ovals positioned above the arrows in the structure shown to the right. As usual, each arrow has interpretation as a multiplier. Write a number in each oval to indicate the value of the associated multiplier so that the system implemented by the structure to the right is equivalent to the system implemented by the structure shown above. ***Indicate your procedure***.




( 1 pts.) b) State the difference equation that relates $y[n]$ to $x[n]$ in the structure at the top of the page. Your equation should not refer to any signals other than $y[n]$ and $x[n]$.



/ 6 pts

( 3 pts.) 6. Determine the seven roots of $z^7 - 1 = 0$.

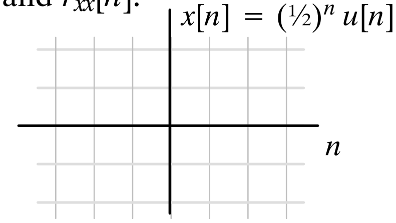
7. Let $x[n]$ and $y[n]$ denote respectively the input and output of a certain **causal LTI system**. Given that the system's response to the input $x[n] = \delta[n]$ is: $y[n] = \sin\left(2\pi \frac{1}{4} n\right)$, $n \geq 0$:

( 2 pts.) a) Determine the system's transfer function, $H(z)$.

( 3 pts.) b) Draw a structure (consisting of delay elements, multipliers, and summers) that will implement the system. ***For full credit***, you structure should include no more than 2 delay elements. Indicate your procedure.

8. Let $x[n] = (½)^n\, u[n]$ and let $r_{xx}[n]$ denote the autocorrelation function of $x[n]$. Let $X(z)$ and $R_{xx}(z)$ denote the respective $Z$ transforms of $x[n]$ and $r_{xx}[n]$.

( 1 pts.) a) Make a plot to illustrate $x[n]$, $-3 \le n \le 3$.




( 2 pts.) b) Evaluate the sum below to find the energy, $E_x$, of $x[n]$. ***Show your work***. Recall $x[n] = (½)^n\, u[n]$.

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 =$$

( 4 pts.) c) Since $x[n]$ is real, its autocorrelation is given by the sum below. Evaluate the sum to find a closed-form expression for $r_{xx}[\ell]$ **for the case** $\ell \ge 0$. ***Show your work***. Recall $x[n] = (½)^n\, u[n]$.

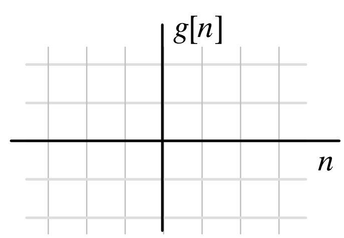
$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x[n-\ell] =$$

( 2 pts.) d) Find $X(z)$ **and its ROC**. ***It is not necessary to show any work.***

e) Let $g[n] = x[-n]$.

( 1 pts.) *i.* Make a plot to illustrate $g[n]$, $-3 \le n \le 3$.

( 3 pts.) *ii.* Find $G(z)$ and its **ROC**.
***Hint***: what should be the relation between $G(z)$ and $X(z)$?
what about the relation between the ROC of $G(z)$ and that of $X(z)$?





Question 8 is continued on the next page.

(continuation of question 8)

( 2 pts.) f) Find $R_{xx}(z)$ **and its ROC**. ***Hint***: Recall that $r_{xx}[n] = x[n] * x[-n] = x[n] * g[n]$.

( 5 pts.) g) Obtain a closed-form expression for $r_{xx}[n]$ by finding the inverse $Z$ transform of $R_{xx}(z)$. ***Caution***: Pay attention to the ROC of $R_{xx}(z)$. ***Note:*** you will only receive credit for this part if your response to part (f) is correct and you successfully invert $R_{xx}(z)$ to find $r_{xx}[n]$.

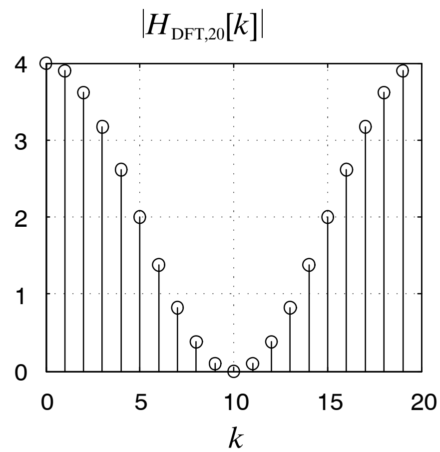
( 3 pts.) h) Evaluate the expression you found for $r_{xx}[n]$ at $n = 0, +1,$ and $-1$. What should be the relationship between $r_{xx}[-1]$ and $r_{xx}[1]$ ? How can you determine $E_x$ from $r_{xx}[n]$?

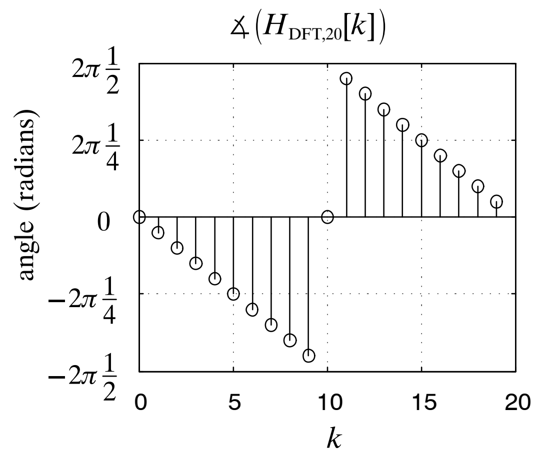


**/ 10 pts**

This exam is solely for the use of students registered in ECE 4531 during Winter of 2021; it should not be shared with others.

9. Consider a ***causal FIR*** filter with impulse response $h[n]$ and frequency response $H_{\text{DTFT}}(F)$. Given that $h[n] = 0$ for all $n > 19$ and that the magnitude and phase of the 20-point DFT of $h[n]$ are as illustrated below:




(3 pts.) a) ***Use the plots above*** to determine the value of $H_{\text{DTFT}}(F)$ at $F = 1/4$. ***Your procedure must be clearly indicated.***.

$H_{\text{DTFT}}(1/4) =$

(2 pts.) b) Find the steady-state output, $y_{ss}[n]$, of the filter when the input to the filter is:

$$x[n] = \cos\left(2\pi\frac{1}{4}n\right)$$

***Indicate your procedure***.

$y_{ss}[n] =$

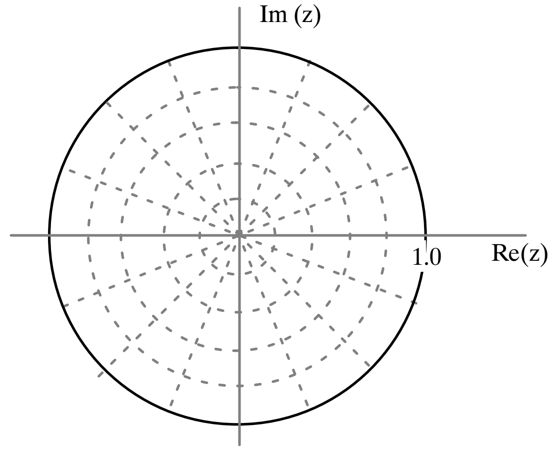


/ 5 pts

10. Consider a discrete-time causal LTI filter with transfer function:

$$H(z) = \frac{A(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

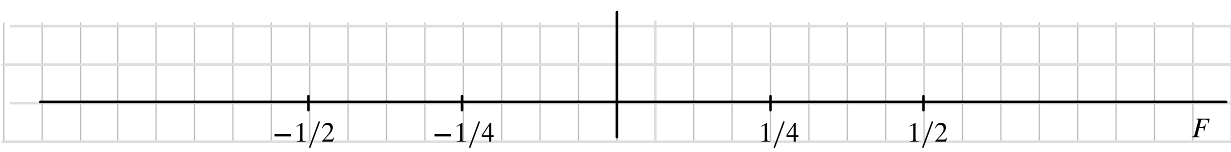
where $z_1 = e^{j2\pi\frac{1}{8}}$, $z_2 = e^{-j2\pi\frac{1}{8}}$, $p_1 = 0.8e^{j2\pi\frac{1}{8}}$, and $p_2 = 0.8e^{-j2\pi\frac{1}{8}}$.




( 2 pts.) a) Sketch the pole-zero plot of this filter on the z-plane provided to the right. As always, mark poles with an 'x' and zeros with an 'o'.

( 3 pts.) b) Assuming ***the constant $A$ is chosen so that the filter has a DC gain of 1***, sketch the magnitude of the filter's frequency response for $-0.5 < F < 0.5$. Note: your sketch does not have to be completely accurate. However, your sketch should ***convey the general nature of the frequency response magnitud***e; and it should ***accurately convey information that is easily deduced from inspection of the pole-zero plot***.

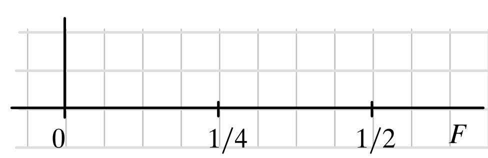
*Magnitude of Filter's frequency response*




( 2 pts.) c) Let $h[n]$ denote the filter's impulse response. Without finding $h[n]$, describe its nature. (*i.e.*, does $h[n]$ oscillate? if so, what is the avg number of samples between neighboring peaks? does the envelope of $h[n]$ grow, decay, or remain constant with time? if not constant, is the growth/decay best characterized as linear? exponential? *etc.*).

d) If the pole locations are changed to $p_1 = 0.9e^{j2\pi\frac{1}{8}}$ and $p_2 = 0.9e^{-j2\pi\frac{1}{8}}$ and the constant $A$ is updated to maintain a DC gain of 1, while leaving the zero locations unchanged:

( 2 pts.) *i.* What will be the effect on the filter's frequency response magnitude? Illustrate the effect with a sketch comparing the new frequency response curve to the old one. Be sure to label the two curves as old and new.




( 2 pts.) *ii.* Will the duration of the impulse response become longer or shorter? Provide a brief explanation.

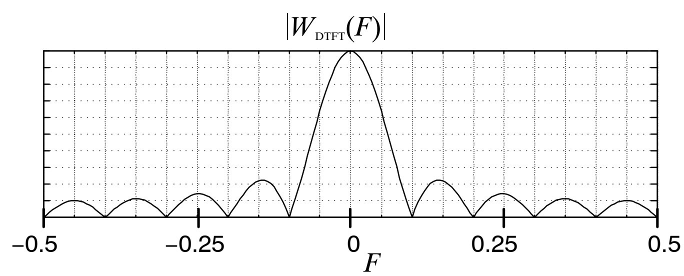
11. Let $w[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$ and let $W_{\mathrm{DTFT}}(F)$ denote the DTFT of $w[n]$.

( 4 pts.) a) Evaluate the DTFT sum to ***find a closed-form expression for*** $W_{\mathrm{DTFT}}(F)$. Then show that $|W_{\mathrm{DTFT}}(F)| = \left|\dfrac{\sin(\pi L F)}{\sin(\pi F)}\right|$. ***Show your work.***

( 2 pts.) b) Determine the value of $W_{\mathrm{DTFT}}(0)$. (This will also be the value of $W_{\mathrm{DTFT}}(F)$ for $F = \pm 1, \pm 2, \ldots$ ). ***Indicate your procedure.***

( 2 pts.) c) State all the values of $F$, $0 \le F \le 1$, for which $W_{\mathrm{DTFT}}(F)$ will be equal to zero?

( 1 pts.) d) Given that $|W_{\mathrm{DTFT}}(F)|$ is as shown to the right for $-0.5 \le F \le 0.5$. Determine the window length $L$.






/ 9 pts

continuation of question 11

( 4 pts.) e) Let $x[n] = \cos\left(2\pi \frac{2}{20} n\right) + \cos\left(2\pi \frac{3}{20} n\right)$ and let $h[n] = x[n]w[n]$ where $w[n]$ ***is the signal whose DTFT is shown in part (d)***. Note that the signal $h[n]$ is a windowed version of $x[n]$. If you examine a plot of the magnitude of $X_{\text{DTFT}}(F)$, you will see two distinct peaks at the frequencies $F = 2/20$ and $F = 3/20$. Will the same be true of $H_{\text{DTFT}}(F)$? (*i.e.*, when looking at a plot of the magnitude of $H_{\text{DTFT}}(F)$, will you see distinct peaks at $F = 2/20$ and at $F = 3/20$?) ***Provide an explanation to support your answer. The points you receive will be based on the clarity of your explanation.***

To facilitate your explanation, you may wish to: illustrate $X_{\text{DTFT}}(F)$; explain the relation between $H_{\text{DTFT}}(F)$, $X_{\text{DTFT}}(F)$ and $W_{\text{DTFT}}(F)$; use this relationship together with your plot of $X_{\text{DTFT}}(F)$ and the plot of $|W_{\text{DTFT}}(F)|$ provided in part (d) to provide an approximate sketch of $|H_{\text{DTFT}}(F)|$.



**/ 4 pts**

12. Let $x[n] = \{\underset{\uparrow}{1}, \ 2, \ 3, \ 0\}$ and let $y[n] = \{\underset{\uparrow}{1}, \ 2, \ -1, \ 1\}$.

( 2 pts.) a) Let $z[n] = x[n] * y[n]$, where $*$ denotes the convolution operator. Find $z[n]$.

( 3 pts.) b) Let $X_{\text{DFT},4}[k]$ and $Y_{\text{DFT},4}[k]$ denote the respective 4-point DFT's of $x[n]$ and $y[n]$. Let $W_{\text{DFT},4}[k] = X_{\text{DFT},4}[k] \times Y_{\text{DFT},4}[k]$ and let $w[n]$ denote the 4-point IDFT of $W_{\text{DFT},4}[k]$. Work in the time-domain to find $w[n], \ n = 0, 1, 2, 3$. ***Indicate your procedure***.



**/ 5 pts**