Application of Z-transforms to Difference Equations

LCCDE: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

Let $Z\{y[n]\} = Y(z)$. Then $Z\{y[n-k]\} = z^{-k} Y(z)$

let $Z\{x[n]\} = X(z)$. Then $Z\{x[n-k]\} = z^{-k} X(z)$

Taking Z-transform of LCCDE:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = H(z),$$ the system's transfer function

Note: The z transform of the system's output is the product of the z transform of the system's input and the system's transfer function.

$$Y(z) = H(z) X(z)$$

we also know:

$$y[n] = h[n] * x[n]$$

The system's transfer function, $H(z)$, is the z transform of the system's impulse response.

$\boxed{\text{The impulse response of an LTI system characterized by an LCCDE has a rational z-transform.}}$

Example

Let $h[n] = a^n u[n]$.

Find a system which models $h[n]$. That is, find a system which, when excited by a unit impulse, will generate the response $h[n]$.

Solution We know that $H(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$

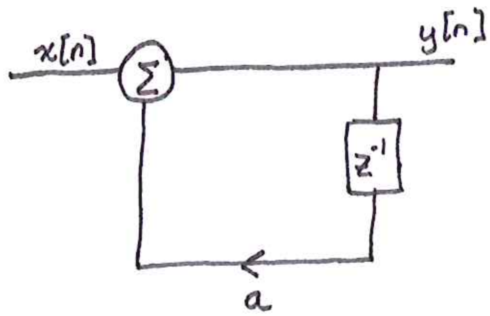
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$$

$$\Rightarrow \quad Y(z)\left[1-az^{-1}\right] = X(z)$$

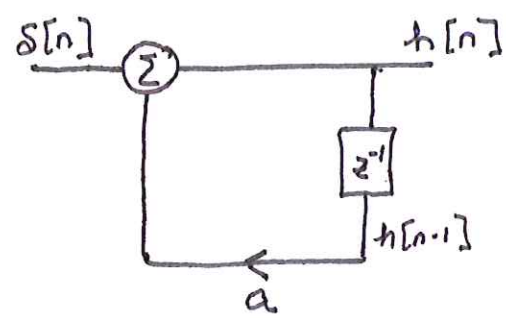
$$\Rightarrow \quad Y(z) - az^{-1}Y(z) = X(z)$$

$$\Rightarrow \quad y[n] - a\,y[n-1] = x[n]$$

$$\Rightarrow \quad y[n] = a\,y[n-1] + x[n]$$



$\Rightarrow$



$h[-1] = 0$

In general, the Z-transform of a causal signal (with rational z-transform) can be viewed as a model for the signal; it is the transfer function of an LTI system whose impulse response is the signal.

A rational z-transform has the form of a ratio of polynomials in z or z^{-1}:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$= \frac{b_0 z^{-M}\left(z^M + \frac{b_1}{b_0} z^{M-1} + \dots + \frac{b_M}{b_0}\right)}{a_0 z^{-N}\left(z^N + \frac{a_1}{a_0} z^{N-1} + \dots + \frac{a_N}{a_0}\right)}$$

$$= \frac{b_0}{a_0} z^{N-M} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

Assuming $b_0 \neq 0$:

$H(z)$ has M zeroes at $z = z_1, \dots, z_M$

$H(z)$ has N poles at $z = p_1, \dots, p_N$

In addition to the poles and zeros mentionned above:

if $N > M$, $X(z)$ has $N-M$ zeroes at $z=0$

if $M > N$, $X(z)$ has $M-N$ poles at $z=0$

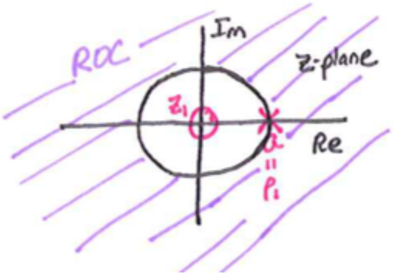
Example

Let $h[n] = a^n u[n]$.

Sketch the pole-zero plot and ROC for $H(z)$.

Solution

$$H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z|>|a|$$



$z_1 = 0$
$p_1 = a$

For causal signals, the growing/decaying nature of the signal's envelope will depend on whether the poles are contained within the unit circle, on the unit circle, or outside the unit circle.

Referring to the example above

If $|a|>1$, the pole is outside the unit circle and $h[n]$ grows without bound as $n \to \infty$

Ex. $h[n] = 2^n u[n] \Rightarrow h[n] = \{\underset{\uparrow}{1} \;\; 2 \;\; 4 \;\; 8 \;\; 16 \ldots\}$

If $|a|=1$, the pole is on the unit circle and the envelope of $h[n]$ remains constant as $n \to \infty$

Ex: $h[n] = u[n] \Rightarrow h[n] = \{\underset{\uparrow}{1} \;\; 1 \;\; 1 \;\; 1 \;\; 1 \ldots\}$

If $|a|<1$, the pole is inside the unit circle and $h[n]$ decays to zero as $n \to \infty$

Ex. $h[n] = (\frac{1}{2})^n u[n] \Rightarrow h[n] = \{\underset{\uparrow}{1} \;\; \frac{1}{2} \;\; \frac{1}{4} \;\; \frac{1}{8}, \ldots\}$

Recall also that a causal LTI system is stable iff all of its characteristic roots are inside the unit circle.

We now know that the poles of a system are the same as the system's characteristics roots, and thus a causal system is BIBO stable if and only if all of its poles are inside the unit circle.

> The ROC for the Transfer Function of a stable causal system will always contain the unit circle.