Lab 5: LTI Filters: Pole-Zero Locations and Frequency Response

PURPOSE: To observe the relationship between the locations of the poles and zeros of a filter and the frequency response of a discrete-time filter. The filter's impulse response will also be observed. The meaning of a filter's frequency response will be emphasized by observing the output of a discrete-time filter to a sinusoidal input. Simple pole-zero placements resulting in frequency responses of *lowpass, bandpass, highpass, notch, comb*, and *all-pass filters* will be investigated.

BACKGROUND: An LTI filter whose output sequence, $y[n]$, is related to the input sequence, $x[n]$, according to the difference equation below:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (1)$$

has transfer function:

$$H_z(z) = \frac{b_0 + b_1 z^{-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_N z^{-N}}\,. \qquad (2)$$

The frequency response of the filter refers to the DTFT of the filter's impulse response and is denoted by $H_{\mathrm{DTFT}}(F)$:

$$H_{\mathrm{DTFT}}(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi Fn}\,. \qquad (3)$$

$H_{\mathrm{DTFT}}(F)$ is easily obtained from the system's transfer function, $H_z(z)$, using the relation:

$$H_{\mathrm{DTFT}}(F) = H_z\left(e^{j2\pi F}\right) \qquad (4)$$

A plot of a filter's frequency response function (magnitude and phase) is a convenient way of displaying how the filter will respond to sinusoids of various frequencies. In general, the ***steady-state response*** of a LTI filter to a sinusoidal input of frequency F_0 will be a sinusoid of the same frequency (F_0); however the amplitude of the output sinusoid will be equal to the amplitude of the input sinusoid multiplied by the magnitude of the filter's frequency response at $F = F_0$ and the phase of the output sinusoid will be equal to the phase of the input sinusoid plus the phase of the filter's frequency response at $F = F_0$. Thus, if $x[n] = A\cos(2\pi F_0 n)$, the steady-state output will be:

$$y_{\mathrm{ss}}[n] = A\left|H_{\mathrm{DTFT}}(F_0)\right| \cos\left(2\pi F_0 n + \measuredangle H_{\mathrm{DTFT}}(F_0)\right) \qquad (5)$$

Filters are broadly classified according to the magnitude of the filter's frequency response function. For example, *low pass filters* will attenuate high-frequency components while passing low-frequency components; *high pass filters* will attenuate low-frequency components while passing high-frequency components; *bandpass filters* attenuate both low and high frequency components while passing mid-range frequency components that fall within the filter's pass band. A *notch filter* will block (notch out) a particular frequency, F_0 (*i.e.*, its frequency response function will have a value of zero at frequency $F = F_0$). A *comb filter* typically refers to a filter which notches out several equally spaced frequencies: *e.g.,* $F_0,\ 2F_0,\ 3F_0,\ \ldots$.
An *allpass filter* passes all frequency components with equal gain/attenuation; allpass filters can be used to modify the phase of a signal.

Knowledge of how pole-zero placement influences the magnitude of a filter's frequency response function can be used as the basis for a rudimentary approach to designing a filter with desired resonances and nulls. In particular, introducing a zero at $z = \exp(j2\pi F_0)$ will create a notch (a null), at $F = F_0$, in the magnitude of the filter's frequency response function. Similarly, placing a pole at $z = r\exp(j2\pi F_0)$, with r close to but strictly less than 1, will create a peak, close to F_0, in the magnitude of the filter's frequency response function. Recall that, to ensure system stability, the poles of the filter must be strictly inside the unit circle (*i.e.*, $r < 1$).

EXERCISES:

1. Consider the LTI filter described by the difference equation:

$$y[n] = x[n] + x[n-2]$$

The transfer function of this filter is given by:

$$H_z(z) = 1 + z^{-2} = \frac{z^2 + 1}{z^2} \tag{6}$$

and hence the frequency response of the filter is given by:

$$H_{\mathrm{DTFT}}(F) = 1 + e^{-j2\pi 2F} = e^{-j2\pi F}\left(e^{j2\pi F} + e^{-j2\pi F}\right) = e^{-j2\pi F} 2\cos(2\pi F) \tag{7}$$

From the first expression for $H_{\mathrm{DTFT}}(F)$ in equation (7), we observe that $H_{\mathrm{DTFT}}(F)$ is periodic in F with period 0.5 (it is also periodic with period 1). From the last expression above, it is easily observed that the magnitude of the frequency response function is given by:

$$|H_{\mathrm{DTFT}}(F)| = 2|\cos(2\pi F)| \tag{8}$$

and the phase of the frequency response function is given by:

$$\measuredangle H_{\mathrm{DTFT}}(F) = \begin{cases} -2\pi F, & \text{when } \cos(2\pi F) > 0 \\ \pi - 2\pi F, & \text{when } \cos(2\pi F) < 0 \end{cases} \tag{9}$$

a) From equation (6), we note the filter's transfer function has two zeros at $z = \pm j$ and two poles at $z = 0$. Since the zeros have magnitude one, they are located on the unit circle and can thus be expressed as $z = e^{\pm j2\pi F_0}$. For which value of F_0 are the zeros of the system transfer function, $H_z(z)$, described by $z = e^{\pm j2\pi F_0}$?

b) Use equations (8) and (9) to find the values of the magnitude and phase of $H_{\mathrm{DTFT}}(F)$ at $F = \frac{1}{8}$ and at $F = \frac{1}{4}$. Note that if $|H_{\mathrm{DTFT}}(F_o)| = 0$, the phase of $H_{\mathrm{DTFT}}(F_o)$ can be anything and is thus not clearly defined. In the case that the value of the phase is not clearly defined at one of the requested values of F, you should instead find the two limiting values of $\measuredangle H_{\mathrm{DTFT}}(F_o)$:

$$\lim_{\epsilon \to 0} \measuredangle H_{\mathrm{DTFT}}(F_o + \epsilon) \quad \text{and} \quad \lim_{\epsilon \to 0} \measuredangle H_{\mathrm{DTFT}}(F_o - \epsilon)$$

c) The Matlab function **freqz** can be used to evaluate the frequency response function at specified frequencies. Type in the following matlab commands to illustrate the use of the **freqz** function as well as to check some of your answers to part (b). Please note that if $|H_{\mathrm{DTFT}}(F_{\mathrm{o}})| = 0$, the matlab commands (as presented below) will not be useful (but could be modified so as to be useful) in confirming the answers that you found for $\lim_{\epsilon \to 0} \measuredangle H_{\mathrm{DTFT}}(F_{\mathrm{o}} + \epsilon)$ and $\lim_{\epsilon \to 0} \measuredangle H_{\mathrm{DTFT}}(F_{\mathrm{o}} - \epsilon)$.

```
b = [1, 0, 1];
a = [1];
F = [0.125 0.25];
H = freqz(b,a,2*pi*F);
magH = abs(H); %magnitude of H_DTFT(F)
angleH = angle(H)/(2*pi); %angle of H_DTFT(F) as a fraction of 2π radians
```

d) Let $y_{1_{\mathrm{ss}}}[n]$ and $y_{2_{\mathrm{ss}}}[n]$ denote the respective steady-state responses of the filter to the input signals $x_1[n] = \cos\left(2\pi\frac{1}{8}n\right)$ and $x_2[n] = \cos\left(2\pi\frac{1}{4}n\right)$. Use your results to parts (b) and/or (c) to determine closed-form expressions for $y_{1_{\mathrm{ss}}}[n]$ and $y_{2_{\mathrm{ss}}}[n]$. (For help with this, read the BACKGROUND section.) So as to simplify the task of comparing the network response to the network input, rewrite your expressions in the form of $y_{1_{\mathrm{ss}}}[n] = A_1 x_1[n - n_1]$ and $y_{2_{\mathrm{ss}}}[n] = A_2 x_2[n - n_2]$. What are the values of A_1, n_1, A_2, and n_2?

e) With the vectors **b** and **a** as defined in part (c), use the Matlab function **filter** to find the zero-state response of the filter to the input signals $x_1[n]$ and $x_2[n]$ as defined in part (d). In each case, make a plot of the filter's response superimposed on the filter's input.

```
n = 0:16;
x1 = cos(2 * pi * (1/8) * n);
x2 = cos(2 * pi *(1/4) * n);
y1 = filter(b,a,x1);
y2 = filter(b,a,x2);
figure, clf
subplot(2,1,1), plot(n,x1,'-o',n,y1,'-s'),hold on, grid on
subplot(2,1,2), plot(n,x2,'-o',n,y2,'-s'),hold on, grid on
```

Note that the plot function has been used instead of the stem function so as to better distinguish the input and output signals. In general, the response of a stable filter to a sinusoidal input applied at n=0 can be expressed as a sum of a transient, which eventually dies out, and a steady-state sinusoidal signal, which remains as long as the input is applied. In most cases, the transient response will have a duration comparable to that of the filter's impulse response. Referring to the filter used above: what is the length of the filter's impulse response? how many samples should we wait before assuming that the transient response has died out and hence that the filter's response has reached steady-state values? For n sufficiently large, do the responses shown in your plots agree with the relations you proposed in part (d)?

part (e) is continued on the next page

Superimpose the steady-state responses on the plots you made above. (This can be done with the code below assuming that **F, magH,** and **angleH** are defined as in part (c), and that **n** is defined as in part (e).)

```
y1ss = magH(1)*cos(2*pi*F(1)*n + 2*pi*angleH(1));
y2ss = magH(2)*cos(2*pi*F(2)*n + 2*pi*angleH(2));
subplot(2,1,1), plot(n,y1ss,'r-x')
subplot(2,1,2), plot(n,y2ss,'r-x')
```

At which sample values do the zero-state responses converge to the steady-state responses?

f) With the vectors **b** and **a** as defined in part (c), implement the commands below to make a plot of the magnitude and phase of the filter's frequency response function for $-0.5 \leq F < 0.5$. Confirm that the magnitude and phase functions that you plot are in agreement with expressions (8) and (9).

```
F = [-0.5:0.005:0.5];
H = freqz(b,a,2*pi*F);
magH = abs(H);
angleH = angle(H)/(2*pi); %angle of H as a fraction of 2π
figure, clf,
subplot(2,1,1), plot(F,magH)
title('magnitude of frequency response')
xlabel('discrete-time frequency, F')
subplot(2,1,2), plot(F,angleH)
title('phase of frequency response')
xlabel('discrete-time frequency, F')
ylabel('phase as a fraction of 2\pi')
```

2. The purpose of this exercise is to develop a deeper understanding of the connection between pole-zero placement and the resulting frequency response function. For your convenience in completing this exercise, I have written a script named **lab5.m** which can be downloaded from D2L. The script assumes that filter coefficient vectors **b** and **a** have been defined. Once the vectors **b** and **a** are defined in your current workspace, execution of the script will result in plots of: the filter's poles and zeros, the filter's frequency response function (magnitude and phase), and the filter's impulse response. Run the script for each filter (as defined by the filter's coefficient vectors, **b** and **a**) in parts (a)-(h) below. Feel free to modify the script.

In each case, classify the resulting filter as one of the following: low-pass, band-pass, high-pass, notch, comb, or all-pass and discuss the connection between the frequency response plots and the locations of the filter's poles and zeros. In addition, your discussion for each filter should address all questions that follow the filter's definition.

a)
```
r = 0.9;
b = (1-r)*[1, 1];
a = 2*[1, -r];
```

Write out the transfer function of the filter, $H_z(z)$. Verify that the dc gain of the filter (*i.e.*, the value of $H_{\mathrm{DTFT}}(F)$ at $F = 0$), can be found by evaluating the transfer function at $z = 1$. Verify that the value of the filter's impulse response at $n = 0$ is in accordance with the initial value theorem as applied to the the filter's transfer function.

Determine the approximate 3dB cutoff frequency of this filter (*i.e.*, the frequency at which the filter's frequency response magnitude has fallen to approximately 0.707 of what it was at the center of the filter's passband) as well as the approximate effective length of the filter's impulse response, $h[n]$. (Note: for purposes of this lab, we will define, L_{eff}, the effective length of a filter's impulse response as the smallest value of k for which we may claim that $|h[n]| \le 0.02(\mathrm{hmax})$ for all $n \ge k$, where $\mathrm{hmax} = \max_n |h[n]|$.) Feel free to determine the 3dB cutoff frequency and the effective length of the filter's impulse response by zooming into your plots of the frequency response magnitude and the filter's impulse response or by making changes to the script. What changes do you observe in the filter's 3dB cutoff frequency and the effective length of the filter's impulse response when you change **r** from a value of 0.9 to a value of 0.8? (Throughout this lab, you should think about how a pole's proximity to the unit circle influences the length of the filter's impulse response and the width of associated peaks or notches in the frequency response.)

b)
```
r = 0.9;
b = (1-r)*[1, -1];
a = 2*[1,  r];
```

Write out the transfer function of the filter, $H_z(z)$. Verify that the high-frequency gain of the filter (*i.e.*, the value of $H_{\mathrm{DTFT}}(F)$ at $F = 1/2$), can be found by evaluating the transfer function at $z = -1$. Verify that the value of the filter's impulse response at $n = 0$ is in accordance with the initial value theorem as applied to the the filter's transfer function.

Recall that the frequency response of a filter is the DTFT of the filter's impulse response. Note that the frequency response of this filter can be viewed as a frequency shifted version of the frequency response of the filter in part (a). In particular, if we let $H_{\mathrm{b}}(F)$ denote the frequency response of this filter and $H_{\mathrm{a}}(F)$ denote the frequency response of the filter of part (a), we can write $H_{\mathrm{b}}(F) = H_{\mathrm{a}}(F - F_0)$. What is the appropriate value of F_0 in the previous relation? Recall frequency responses are periodic functions of F with period 1. Furthermore, the frequency shifting property of the DTFT says that if $H_{\mathrm{b}}(F) = H_{\mathrm{a}}(F - F_0)$, then $h_b[n] = e^{j2\pi F_0 n} h_a[n]$. Verify that this is indeed the relationship between the impulse responses of these two filters. Be sure to comment on the characteristics of the sequence $e^{j2\pi F_0 n}$ for the specified value of F_0.

Letting $H_z^{(b)}(z)$ denote the transfer function of this filter and $H_z^{(a)}(z)$ denote the transfer function of the filter in part (a), determine the relationship between the two transfer functions $\left(H_z^{(b)}(z) = H_z^{(a)}(??)\right)$.

c)
```
r = 0.9;
F0 = 1/5;
p1 = r*exp(j*2*pi*F0);
G = (1-r)*sqrt(1+ r^2 - 2*r*cos(2*pi*2*F0))/(2*sin(2*pi*F0))
b = G*conv([1, -1],[1, 1]);
a = conv([1, -p1],[1, -conj(p1)]);
```

At approximately which frequency does the magnitude of the filter's frequency response achieve its biggest value? Find the approximate 3dB passband width (*i.e.,* find $F_u - F_\ell$ where F_u is the upper 3dB cutoff frequency and F_ℓ is the lower 3dB cutoff frequency) and the effective length of the filter's impulse response.

If **r** is decreased to a value of 0.8: does the width of the filter's passband increase or decrease? does the filter's impulse response get shorter or longer?

For which filter (r=0.9 or r=0.8) will the transient response have the longest duration?

d)
```
r = 0.9;
F0 = 1/5;
p1 = r*exp(j*2*pi*F0);
z1 = exp(j*2*pi*F0);
b = conv([1, -z1],[1, -conj(z1)]);
a = conv([1, -p1],[1, -conj(p1)])
b = b*sum(a)/sum(b)
```

Note that in comparison to the filter of part (c), the pole locations are the same, but the zero locations have changed (forcing a notch in the frequency response where there used to be a peak). Quantify the width of the notch.

If the magnitude of the poles is changed to a value of 0.8 instead of 0.9, does the notch width increase or decrease?

e)
```
b = 0.5*[1, 0, 0, 0, 0, -1];
a = [1];
```

For which frequencies does the frequency response magnitude have a value of zero? How do these frequencies compare to the zero locations of the filter's transfer function? Write out the difference equation for this filter. Based on the filter's difference equation, provide a very simple time-domain explanation as to how this filter notches out the frequencies that it does. Hint: what is the common period associated with the various notch frequencies?

f)
```
b = [1, 1, 1, 1, 1];
a = [1]
b = b*sum(a)/sum(b)
```

How do the frequencies notched out by this filter differ from those notched out by the filter of part (e)? Write the difference equation for this filter and then use the difference equation to provide a simple time-domain explanation as to how this filter notches out the frequencies that it does.

g)
```
N = 4;
k = 0:N;
b = ones(size(k));
r = 0.9;
a = r.^k;
b = b*sum(a)/sum(b)
```

Compare the pole-zero plot of this filter to the pole-zero plot of the filter in part (f). Note that the locations of the zeros are the same for both filters. But that the locations of the poles have been changed. Discuss how the change in the pole locations affected the frequency response. Note that this type of filter can be used to notch out 60Hz. interference from electrical outlets along with its harmonics. What pole-zero movement might you suggest if you needed to sharpen up the notches (*i.e.*, make them more narrow)? How would this affect the length of the filter's transient response?

h)
```
r = 0.9;
F0 = 1/5;
p1 = r*exp(j*2*pi*F0);
a = conv([1, -p1],[1, -conj(p1)]);
b = fliplr(a)
```

Note the reciprocal relation between the poles and zeros of this filter. Describe the magnitude of the filter's frequency response function. Try a few other filters from this family. For example, let **`a`** be defined as in part (g) and let **`b=fliplr(a)`**. What does the magnitude of the filter's frequency response function look like?