Definition: A ***discrete-time system*** is a device that operates on one discrete-time signal (the input) to produce another d.t. signal (the output).

$$x[n] \longrightarrow \boxed{\mathcal{T}} \longrightarrow y[n] = \mathcal{T}\{x[n]\}$$

input signal (excitation) — output signal (response)

The information depicted in the diagram above can also be conveyed by the notation: $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$ which is read as: *the system $\mathcal{T}$ maps the discrete-time signal $x[n]$ to the discrete-time signal $y[n]$*. A system definition must also convey the mapping from input to output. This is often accomplished via an input-output equation which relates $y[n]$ to $x[n]$. Given the system definition, you should be able to determine the system output for any input.

Example: Given that $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$. Find the output, $y[n]$, for the input sequence $x[n]$, when the input-output relationship of $\mathcal{T}$ is as stated below and:

$$x[n] = \begin{cases} |n|, & |n| \le 2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow x[n] = \{\ldots,\ 0,\ 0,\ 2,\ 1,\ \underset{\uparrow}{0},\ 1,\ 2,\ 0,\ 0,\ \ldots\}$$

system input

input-output equation — system output

a) $y[n] = x[n]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ 2,\ 1,\ \underset{\uparrow}{0},\ 1,\ 2,\ 0,\ 0,\ \ldots\}$

b) $y[n] = x[n-1]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ 0,\ 2,\ \underset{\uparrow}{1},\ 0,\ 1,\ 2,\ 0,\ \ldots\}$

c) $y[n] = x[n+1]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 2,\ 1,\ 0,\ \underset{\uparrow}{1},\ 2,\ 0,\ 0,\ 0,\ \ldots\}$

d) $y[n] = x[n] - x[n-1]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ 2,\ -1,\ \underset{\uparrow}{\_\_},\ \_\_,\ \_\_,\ \_\_,\ \_\_,\ \ldots\}$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\Rightarrow y[n] = \{\ldots,\ 0,\ \_\_,\ \_\_,\ \_\_,\ \underset{\uparrow}{\_\_},\ \_\_,\ \_\_,\ \_\_,\ \_\_,\ \ldots\}$

f) $y[n] = x^2[n]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ 4,\ 1,\ \underset{\uparrow}{0},\ 1,\ 4,\ 0,\ 0,\ \ldots\}$

g) $y[n] = x[n^2]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ \_\_,\ \_\_,\ \underset{\uparrow}{0},\ \_\_,\ \_\_,\ 0,\ 0,\ \ldots\}$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\Rightarrow y[n] = \{\ldots,\ 0,\ 0,\ 2,\ \_\_,\ \underset{\uparrow}{\_\_},\ \_\_,\ \_\_,\ \_\_,\ \_\_,\ \ldots\}$

The ***impulse response*** of a system is the response of the system to a Kronecker Delta function (*i.e.*, to a unit impulse).

Example: Given the input-output relationships below, determine the impulse response, $h[n]$, of each associated system.

a) $y[n] = x[n]$ $\Rightarrow$ $h[n] = \delta[n]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

b) $y[n] = x[n-1]$ $\Rightarrow$ $h[n] = \delta[n-1]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{0}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

c) $y[n] = x[n+1]$ $\Rightarrow$ $h[n] = \delta[n+1]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{0}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

d) $y[n] = x[n] - x[n-1]$ $\Rightarrow$ $h[n] = \delta[n] - \delta[n-1]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{\hspace{1em}}, \underset{\uparrow}{\underline{\hspace{1em}}}, \underline{\hspace{1em}}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\Rightarrow$ $h[n] = \max(\delta[n+1], \delta[n], \delta[n-1])$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{\hspace{1em}}, \underset{\uparrow}{\underline{\hspace{1em}}}, \underline{\hspace{1em}}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

f) $y[n] = x^2[n]$ $\Rightarrow$ $h[n] = \delta^2[n]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{\hspace{1em}}, \underset{\uparrow}{\underline{\hspace{1em}}}, \underline{\hspace{1em}}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

g) $y[n] = x\left[n^2\right]$ $\Rightarrow$ $h[n] = \delta[n^2]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{\hspace{1em}}, \underset{\uparrow}{\underline{\hspace{1em}}}, \underline{\hspace{1em}}, \underline{0}, \underline{0}, \underline{0}, \ldots \}$$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\Rightarrow$ $h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$$\Rightarrow h[n] = \{ \ldots, \underline{0}, \underline{0}, \underline{0}, \underline{\hspace{1em}}, \underset{\uparrow}{\underline{\hspace{1em}}}, \underline{\hspace{1em}}, \underline{\hspace{1em}}, \underline{\hspace{1em}}, \underline{\hspace{1em}}, \ldots \}$$

The ***step response*** of a system is the response of the system to the unit step function, $u[n]$.

Example: Given the input-output relationships below, determine the step response, $y_{\text{step}}[n]$, of each associated system.

a) $y[n] = x[n] \quad \Rightarrow \quad y_{\text{step}}[n] = u[n]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{0},\ \underline{0},\ \underline{0},\ \underline{0},\ \underset{\uparrow}{\underline{1}},\underline{1},\ \underline{1},\ \underline{1},\ \underline{1},\ldots\}$$

b) $y[n] = x[n-1] \quad \Rightarrow \quad y_{\text{step}}[n] = u[n-1]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{0},\ \underline{0},\ \underline{0},\ \underline{0},\ \underset{\uparrow}{\underline{0}},\underline{1},\ \underline{1},\ \underline{1},\ \underline{1},\ldots\}$$

c) $y[n] = x[n+1] \quad \Rightarrow \quad y_{\text{step}}[n] = u[n+1]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{0},\ \underline{0},\ \underline{0},\ \underline{1},\ \underset{\uparrow}{\underline{1}},\underline{1},\ \underline{1},\ \underline{1},\ \underline{1},\ldots\}$$

d) $y[n] = x[n] - x[n-1] \quad \Rightarrow \quad y_{\text{step}}[n] = u[n] - u[n-1]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underset{\uparrow}{\underline{\quad}},\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ldots\}$$

e) $y[n] = \max(x[n+1], x[n], x[n-1]) \quad \Rightarrow \quad y_{\text{step}}[n] = \max(u[n+1], u[n], u[n-1])$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{0},\ \underline{0},\ \underline{0},\ \underline{1},\ \underset{\uparrow}{\underline{1}},\underline{1},\ \underline{1},\ \underline{1},\ \underline{1},\ldots\}$$

f) $y[n] = x^2[n] \quad \Rightarrow \quad y_{\text{step}}[n] = u^2[n]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{0},\ \underline{0},\ \underline{0},\ \underline{0},\ \underset{\uparrow}{\underline{1}},\underline{1},\ \underline{1},\ \underline{1},\ \underline{1},\ldots\}$$

g) $y[n] = x\left[n^2\right] \quad \Rightarrow \quad y_{\text{step}}[n] = u[n^2]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underset{\uparrow}{\underline{\quad}},\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ldots\}$$

h) $y[n] = \sum_{k=-\infty}^{n} x[k] \quad \Rightarrow \quad y_{\text{step}}[n] = \sum_{k=-\infty}^{n} u[k]$

$$\Rightarrow y_{\text{step}}[n] = \{\ldots,\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underset{\uparrow}{\underline{\quad}},\underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ldots\}$$

Classification of Discrete-Time Systems

1. ***Static vs. Dynamic:*** A discrete-time system is *static* or *memoryless* if its output at time n depends only on the input value at time n; it is *dynamic* if the output value at time n is influenced by input values at times other than time n.

   If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to n (for some finite $N \geq 0$), the system is said to have *finite memory of length* N, whereas if $N = \infty$, the system is said to have *infinite memory*.

2. ***Time-invariant vs. Time-varying:*** A relaxed system, $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$, is said to be *time-invariant* if and only if the fact that $y[n]$ is the response of the system to $x[n]$ implies that the response of the system to $x[n-k]$ will be $y[n-k]$, *i.e.,* if and only if: $\mathcal{T}\{x[n]\} = y[n]$ implies that $\mathcal{T}\{x[n-k]\} = y[n-k]$ for every input signal $x[n]$ and every time shift k.

   A procedure for determining whether or not a system, $\mathcal{T}$, is time-invariant, is detailed in steps (a)-(e) below:

   a) Define the notation you'll be using. For example, let $x_1[n]$ denote an unspecified discrete-time signal and let k denote an unspecified integer. Let the signals $x_2[n]$, $y_1[n]$, and $y_2[n]$ be defined in terms of the signal, $x_1[n]$, and the integer, k, as follows:
   $y_1[n] = \mathcal{T}\{x_1[n]\}$, $x_2[n] = x_1[n-k]$, and $y_2[n] = \mathcal{T}\{x_2[n]\}$

   b) Use the system's input-output relationship to obtain one equation relating $y_1[n]$ to $x_1[\cdot]$ and another equation relating $y_2[n]$ to $x_2[\cdot]$.

   c) Use the relation that $x_2[n] = x_1[n-k]$ together with the equation of part (b) relating $y_2[n]$ to $x_2[\cdot]$ in order to obtain an equation relating $y_2[n]$ to $x_1[\cdot]$.

   d) Replace n by $n - k$ on both sides of the equation of part (b) which relates $y_1[n]$ to $x_1[\cdot]$ to obtain an equation relating $y_1[n-k]$ to $x_1[\cdot]$.

   e) Compare equations obtained in parts (c) and (d) to determine whether $y_2[n]$ is equal to $y_1[n-k]$. If $y_2[n] = y_1[n-k]$, conclude the system is time-invariant; otherwise, conclude the system is not time-invariant.

   To prove that a system is not time-invariant, you may follow the same procedure above or you may simply provide a counter example: *i.e.,* you may provide a specific choice of waveform $x_1[n]$ and integer k for which $y_2[n] \neq y_1[n-k]$ where the signals $y_1[n]$ and $y_2[n]$ are as defined in part (a).

3. ***Linear vs. Nonlinear:*** A relaxed system, $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$, is said to be *linear* if and only if the response of the system to a weighted sum of signals is equal to the corresponding weighted sum of responses to each of the individual signals. That is, a system, $\mathcal{T}$, is linear if and only if:

$$\mathcal{T}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{T}[x_1[n]] + a_2\mathcal{T}[x_2[n]]$$

for arbitrary input sequences, $x_1[n]$ and $x_2[n]$, and arbitrary constants, a_1 and a_2.

A procedure for determining whether or not a system, $\mathcal{T}$, is linear, is detailed in steps (a)-(e) below:

a) Define your notation. For example, let $x_1[n]$ and $x_2[n]$ denote unspecified discrete-time signals and let a_1 and a_2 denote unspecified constants. Let the signals $x_3[n]$, $y_1[n]$, $y_2[n]$, and $y_3[n]$ be defined in terms of the signals, $x_1[n]$ and $x_2[n]$, and the constants, a_1 and a_2, as follows:

$$y_1[n] = \mathcal{T}\{x_1[n]\}, \quad y_2[n] = \mathcal{T}\{x_2[n]\}$$
$$x_3[n] = a_1x_1[n] + a_2x_2[n], \quad \text{and} \quad y_3[n] = \mathcal{T}\{x_3[n]\}$$

b) Use the system's input-output relationship to obtain three equations: one relating $y_1[n]$ to $x_1[\cdot]$, a second relating $y_2[n]$ to $x_2[\cdot]$, and a third relating $y_3[n]$ to $x_3[\cdot]$.

c) Use the relation that $x_3[n] = a_1x_1[n] + a_2x_2[n]$ together with the equation of part (b) relating $y_3[n]$ to $x_3[\cdot]$ in order to obtain an equation relating $y_3[n]$ to $x_1[\cdot]$, $x_2[\cdot]$, a_1, and a_2.

d) Multiply both sides of the first equation of part (b) by the constant a_1; multiply both sides of the second equation of part (b) by the constant a_2; then sum the two resulting equations, thus obtaining an equation which relates the signal $(a_1y_1[n] + a_2y_2[n])$ to $x_1[\cdot]$, $x_2[\cdot]$, a_1, and a_2.

e) Compare equations obtained in parts (c) and (d) to determine whether $y_3[n]$ is equal to $(a_1y_1[n] + a_2y_2[n])$. If $y_3[n] = a_1y_1[n] + a_2y_2[n]$, conclude the system is linear; otherwise, conclude the system is nonlinear.

To prove that a system is not linear, you may follow the same procedure above or you may simply provide a counter example (*i.e.,* you may provide a specific choice of waveforms: $x_1[n]$ and $x_2[n]$, and constants: a_1 and a_2 for which $y_3[n] \neq a_1y_1[n] + a_2y_2[n]$ where the signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ are as defined in part (a)).

4. ***Linear Time-invariant (LTI)***: A system is said to be linear time-invariant if and only if it is both linear and time-invariant. Most of the systems we will analyze in this class are LTI systems. It can be shown that *LTI systems are completely characterized by their impulse response*; that is, given the impulse response of an LTI system, one can find the output to any given input without the need for an input-output relationship.

5. ***Causal vs. Noncausal:*** A system is *causal* if and only if the output of the system at time n depends only on present and past inputs at time n, *i.e.,* if:
$$y[n] = g(x[n], x[n-1], x[n-2], \ldots)$$
Although a real-time noncausal system is unrealizable, noncausal systems are often used in non real-time applications where the signal is recorded and processed off-line (such as with image processing) or in applications where a small amount of delay is tolerable.

   It can be shown that an LTI system is causal if and only if its impulse response, $h[n]$, is causal, *i.e.*, if and only if:
$$h[n] = 0\,, \quad n < 0$$

6. ***Stable vs. Unstable:*** A relaxed system is said to be bounded-input bounded-output (BIBO) stable if and only if every bounded input signal produces a bounded output signal. (Note: a sequence, $x[n]$, is bounded if there exist some finite number, M_x, such that $|x[n]| \leq M_x < \infty$ for all n.)

   It can be shown that an LTI system is BIBO stable if and only if its impulse response, $h[n]$, is absolutely summable, *i.e.,* if and only if:
$$\left[\sum_{n=-\infty}^{\infty} |h[n]|\right] < \infty$$

7. ***FIR vs. IIR:*** A system is termed as a ***finite impulse response (FIR) system*** if the impulse response is of finite length; otherwise, it is termed an ***infinite impulse response (IIR) system***.