## Initial Value Theorem

If $x[n]$ is a causal signal, then $x[0] = \lim_{z \to \infty} X(z)$.

### Exercises

a) Prove the Initial Value Theorem.

Soln: By definition: $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

If $x[n]$ is a causal signal, then:

$X(z) =$

Hence:

$\lim_{z \to \infty} X(z) =$

b) Given: $x[n] \overset{Z.T.}{\longleftrightarrow} X(z) = \frac{1}{1 - az^{-1}}$, $|z| > |a|$

Find: $x[0]$

Soln: $x[0] \overset{\text{by I.V.T.}}{=} \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{1}{1 - az^{-1}} =$

c) Given: $x[n] \overset{Z.T.}{\longleftrightarrow} X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$, $|z| > |a|$

Find: $x[0]$.

Soln: $x[0] \overset{\text{by I.V.T.}}{=} \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{az^{-1}}{(1 - az^{-1})^2} =$

## Convolution in time domain $\Longleftrightarrow$ Multiplication in Z domain

Given that $x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$ with $ROC_1$

$x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$ with $ROC_2$

If $x[n] = x_1[n] * x_2[n]$

Then: $x[n] \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$ with $ROC_x \supseteq ROC_1 \cap ROC_2$

**Example 1:** Let $x_1[n] = \{1 \ \underset{\uparrow}{2} \ 3\}$, $x_2[n] = \{\underset{\uparrow}{0} \ 1 \ 2\}$ and let $x[n] = x_1[n] * x_2[n]$.

Find $X_1(z)$, $X_2(z)$ and $X(z)$.

**Solution**

$x_1[n] = \{1 \ \underset{\uparrow}{2} \ 3\} \Longrightarrow X_1(z) = z + 2 + 3z^{-1}$, $ROC_1: 0<|z|<\infty$

$x_2[n] = \{\underset{\uparrow}{0} \ 1 \ 2\} \Longrightarrow X_2(z) = z^{-1} + 2z^{-2}$, $ROC_2: |z|>0$

$$
\begin{array}{r}
z + 2 + 3z^{-1} \\
z^{-1} + 2z^{-2} \\
\hline
2z^{-1} + 4z^{-2} + 6z^{-3} \\
1 + 2z^{-1} + 3z^{-2} \quad\quad \\
\hline
\end{array}
$$

$X(z) = X_1(z)X_2(z) = 1 + 4z^{-1} + 7z^{-2} + 6z^{-3}$, $ROC_x: |z|>0$

note: $ROC_x = \{z: |z|>0\} \supseteq ROC_1 \cap ROC_2 = \{z: 0<|z|<\infty\}$

## Ex. 2

Let $x_1[n] = u[n] \longleftrightarrow X_1(z) = \frac{1}{1-z^{-1}}, \ |z| > 1$

Let $x_2[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1]$

$$\Rightarrow X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}, \ |z| > \frac{1}{2}$$

Given that $x[n] = x_1[n] * x_2[n]$, find $X(z)$ and its ROC.

By convolution property, we have that:

$$X(z) = X_1(z) X_2(z) = \frac{1}{1-z^{-1}} \times \frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}} = \underline{\frac{1}{1-\frac{1}{2}z^{-1}}}$$

note that due to a pole-zero cancellation, $X(z)$ has only one pole at $z = \frac{1}{2}$. Hence the region of convergence for $X(z)$ is $|z| > \frac{1}{2}$.

$$\therefore \ ROC_X = \{z : |z| > \tfrac{1}{2}\} \supseteq (ROC_1 \cap ROC_2)$$

$$= \{z : |z| > 1\} \cap \{z : |z| > \tfrac{1}{2}\}$$

$$= \{z : |z| > 1\}$$

## Proof of Convolution-in-time property

$$x[n] = x_1[n] * x_2[n] \Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \overbrace{\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]}^{x[n]}\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n}\, \underbrace{z^{k} z^{-k}}_{1}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k}}_{X_1(z)}\ \underbrace{\sum_{n=-\infty}^{\infty} x_2[\overbrace{n-k}^{m}]\, z^{-(\overbrace{n-k}^{m})}}_{X_2(z)}$$

$$= X_1(z)\, X_2(z)$$

In general, the ROC for $X(z)$ will be the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$. In some cases, the ROC of $X(z)$ may be bigger than the intersection. This may happen as a result of pole-zero cancellations or issues of causality.

## Conjugates Property of Z-transform

If $g[n] = x^*[n]$ then $G(z) = X^*(z^*)$, $ROC_G = ROC_x$

### Proof

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]\, z^{-n}$$ by definition of Z-transform

$$= \sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n}$$ using $g[n] = x^*[n]$

$$= \sum_{n=-\infty}^{\infty} \left( x[n] (z^*)^{-n} \right)^*$$ the complex conjugate of a product is the product of the complex conjugates

$$= \left( \sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right)^*$$ the complex conjugate of a sum is the sum of the complex conjugates

$$= \left( X(z^*) \right)^*$$ $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \Rightarrow X(z^*) = \sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n}$

$$= X^*(z^*)$$

The poles of $X^*(z^*)$ will have the same magnitude as the poles of $X(z)$. Hence $ROC_G = ROC_x$.

### Illustration of property

$$x[n] = e^{j2\pi F_0 n} u[n] \longleftrightarrow X(z) = \frac{1}{1 - e^{j2\pi F_0} z^{-1}}, \quad |z| > 1$$

Let $g[n] = x^*[n] = e^{-j2\pi F_0 n} u[n]$, then $G(z) = X^*(z^*)$, $|z| > 1$

note: $X(z^*) = \dfrac{1}{1 - e^{j2\pi F_0} (z^*)^{-1}} \Rightarrow X^*(z^*) = \dfrac{1}{1 - e^{-j2\pi F_0} (z)^{-1}}$

$$\Rightarrow G(z) = \frac{1}{1 - e^{-j2\pi F_0} (z)^{-1}}, \quad |z| > 1$$

## Exercise

**Given:** $x[n] \xleftrightarrow{Z.T.} X(z)$ and $y[n] \xleftrightarrow{Z.T.} Y(z)$

**Find:** $R_{xy}(z) \equiv \mathcal{Z}\{r_{xy}[n]\}$

**Soln:** - will express $r_{xy}[n]$ as a convolution of $x[n]$ and $y^*[-n]$. Can then make use of: the time-reversal, conjugates, and convolution properties of the Z-transform to find $R_{xy}(z)$.

## Exercise

Given $x[n] = a^n u[n]$, where $a$ is real-valued, find $R_{xx}(z)$, where $R_{xx}(z)$ is the Z.T. of $r_{xx}[n]$.

**Soln.** (will use the result from above with $Y(z) = X(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$)

$R_{xx}(z) =$

## Exercise (same problem - slightly different approach)

Given $x[n] = a^n u[n]$ where $a$ is real-valued,

find $R_{xx}(z)$, the Z-transform of $r_{xx}[n]$.

(The solution below starts from scratch, rather than starting from the more general expression for $R_{xy}(z)$ which allows for complex-valued $x[n]$ and $y[n]$.)

## Solution

$$x[n] = a^n u[n] \longleftrightarrow X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$r_{xx}[n] = x[n] * x^*[-n]$$ (using the relationship between autocorrelation and convolution)

$$\Rightarrow r_{xx}[n] = x[n] * \underbrace{x[-n]}_{g[n]}$$ (since $x[n]$ is real-valued)

Let $g[n] = x[-n]$, then by the time-negation property we know that:

$$G(z) = X(z^{-1}) = \frac{1}{1-az}, \quad |z^{-1}| > |a|$$

$$\Rightarrow G(z) = \frac{1}{1-az}, \quad |z| < \frac{1}{|a|}$$

Finally, since $r_{xx}[n] = x[n] * g[n]$, we know that:

$$R_{xx}(z) = X(z)G(z) = \left(\frac{1}{1-az^{-1}}\right)\left(\frac{1}{1-az}\right), \quad \underbrace{|a| < |z| < \frac{1}{|a|}}_{\text{intersection of } ROC_x \text{ and } ROC_g}$$