Definition: A ***discrete-time system*** is a device that operates on one discrete-time signal (the input) to produce another d.t. signal (the output).

$$x[n] \longrightarrow \boxed{\mathcal{T}} \longrightarrow y[n] = \mathcal{T}\{x[n]\}$$

input signal (excitation) — output signal (response)

The information depicted in the diagram above can also be conveyed by the notation: $\boxed{\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}}$ which is read as: *the system $\mathcal{T}$ maps the discrete-time signal $x[n]$ to the discrete-time signal $y[n]$*. A system definition must also convey the mapping from input to output. This is often accomplished via an input-output equation which relates $y[n]$ to $x[n]$. Given the system definition, you should be able to determine the system output for any input.

Example: Given that $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$. Find the output, $y[n]$, for the input sequence $x[n]$, when the input-output relationship of $\mathcal{T}$ is as stated below and:

$$x[n] = \begin{cases} |n|, & |n| \le 2 \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \boxed{x[n] = \{\ldots, 0, 0, 2, 1, \underset{\uparrow}{0}, 1, 2, 0, 0, \ldots\}}$$

a) $y[n] = x[n]$ $\Rightarrow y[n] = \{\ldots, 0, 0, 2, 1, \underset{\uparrow}{0}, 1, 2, 0, 0, \ldots\}$

b) $y[n] = x[n-1]$ $\Rightarrow y[n] = \{\ldots, 0, 0, 0, 2, \underset{\uparrow}{1}, 0, 1, 2, 0, \ldots\}$

c) $y[n] = x[n+1]$ $\Rightarrow y[n] = \{\ldots, 0, 2, 1, 0, \underset{\uparrow}{1}, 2, 0, 0, 0, \ldots\}$

d) $y[n] = x[n] - x[n-1]$ $\Rightarrow y[n] = \{\ldots, 0, 0, 2, -1, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\Rightarrow y[n] = \{\ldots, 0, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$

f) $y[n] = x^2[n]$ $\Rightarrow y[n] = \{\ldots, 0, 0, 4, 1, \underset{\uparrow}{0}, 1, 4, 0, 0, \ldots\}$

g) $y[n] = x[n^2]$ $\Rightarrow y[n] = \{\ldots, 0, 0, \underline{\quad}, \underline{\quad}, \underset{\uparrow}{0}, \underline{\quad}, \underline{\quad}, 0, 0, \ldots\}$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\Rightarrow y[n] = \{\ldots, 0, 0, 2, \underline{\quad}, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$

The ***impulse response*** of a system is the response of the system to a Kronecker Delta function (*i.e.*, to a unit impulse). $\boxed{\delta[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}}$

Example: Given the input-output relationships below, determine the impulse response, $h[n]$, of each associated system.

a) $y[n] = x[n]$ $\quad\Rightarrow\quad h[n] = \delta[n]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

b) $y[n] = x[n-1]$ $\quad\Rightarrow\quad h[n] = \delta[n-1]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{0}}, \underline{1}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

c) $y[n] = x[n+1]$ $\quad\Rightarrow\quad h[n] = \delta[n+1]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{0}}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

d) $y[n] = x[n] - x[n-1]$ $\quad\Rightarrow\quad h[n] = \delta[n] - \delta[n-1]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{}, \underset{\uparrow}{\underline{}}, \underline{}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\quad\Rightarrow\quad h[n] = \max(\delta[n+1], \delta[n], \delta[n-1])$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{}, \underset{\uparrow}{\underline{}}, \underline{}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

f) $y[n] = x^2[n]$ $\quad\Rightarrow\quad h[n] = \delta^2[n]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{}, \underset{\uparrow}{\underline{}}, \underline{}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

g) $y[n] = x\left[n^2\right]$ $\quad\Rightarrow\quad h[n] = \delta[n^2]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{}, \underset{\uparrow}{\underline{}}, \underline{}, \underline{0}, \underline{0}, \underline{0}, \ldots\}$$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\quad\Rightarrow\quad h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$$\Rightarrow h[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{}, \underset{\uparrow}{\underline{}}, \underline{}, \underline{}, \underline{}, \underline{}, \ldots\}$$

The ***step response*** of a system is the response of the system to the unit step function, $u[n]$.

$$\boxed{u[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}}$$

Example: Given the input-output relationships below, determine the step response, $y_{\text{step}}[n]$, of each associated system.

a) $y[n] = x[n]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u[n]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$$

b) $y[n] = x[n-1]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u[n-1]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{0}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$$

c) $y[n] = x[n+1]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u[n+1]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$$

d) $y[n] = x[n] - x[n-1]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u[n] - u[n-1]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\quad\Rightarrow\quad y_{\text{step}}[n] = \max(u[n+1], u[n], u[n-1])$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{1}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$$

f) $y[n] = x^2[n]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u^2[n]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underset{\uparrow}{\underline{1}}, \underline{1}, \underline{1}, \underline{1}, \underline{1}, \ldots\}$$

g) $y[n] = x\left[n^2\right]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = u[n^2]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$$

h) $y[n] = \displaystyle\sum_{k=-\infty}^{n} x[k]$ $\quad\Rightarrow\quad y_{\text{step}}[n] = \displaystyle\sum_{k=-\infty}^{n} u[k]$

$$\Rightarrow \; y_{\text{step}}[n] = \{\ldots, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underset{\uparrow}{\underline{\quad}}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \ldots\}$$

Classification of Discrete-Time Systems

1. ***Static vs. Dynamic:*** A discrete-time system is *static* or *memoryless* if its output at time n depends only on the input value at time n; it is *dynamic* if the output value at time n is influenced by input values at times other than time n.

   If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to n (for some finite $N \geq 0$), the system is said to have *finite memory of length* N, whereas if $N = \infty$, the system is said to have *infinite memory*.

Example

Determine which of the input-output equations listed below are characteristic of static systems.

a) $y[n] = x[n]$

b) $y[n] = x[n-1]$

c) $y[n] = x[n+1]$

d) $y[n] = x[n] - x[n-1]$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$

f) $y[n] = x^2[n]$

g) $y[n] = x[n^2]$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$

Classification of Discrete-Time Systems

1. ***Static vs. Dynamic:*** A discrete-time system is *static* or *memoryless* if its output at time n depends only on the input value at time n; it is *dynamic* if the output value at time n is influenced by input values at times other than time n.

   If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to n (for some finite $N \geq 0$), the system is said to have *finite memory of length* N, whereas if $N = \infty$, the system is said to have *infinite memory*.

2. ***Time-invariant vs. Time-varying:*** A relaxed system, $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$, is said to be *time-invariant* if and only if the fact that $y[n]$ is the response of the system to $x[n]$ implies that the response of the system to $x[n-k]$ will be $y[n-k]$, *i.e.,* if and only if: $\mathcal{T}\{x[n]\} = y[n]$ implies that $\mathcal{T}\{x[n-k]\} = y[n-k]$ for every input signal $x[n]$ and every time shift k.

   A procedure for determining whether or not a system, $\mathcal{T}$, is time-invariant, is detailed in steps (a)-(e) below:

   a) Define the notation you'll be using. For example, let $x_1[n]$ denote an unspecified discrete-time signal and let k denote an unspecified integer. Let the signals $x_2[n]$, $y_1[n]$, and $y_2[n]$ be defined in terms of the signal, $x_1[n]$, and the integer, k, as follows:
   $$y_1[n] = \mathcal{T}\{x_1[n]\},\ x_2[n] = x_1[n-k], \text{ and } y_2[n] = \mathcal{T}\{x_2[n]\}$$

   b) Use the system's input-output relationship to obtain one equation relating $y_1[n]$ to $x_1[\cdot]$ and another equation relating $y_2[n]$ to $x_2[\cdot]$.

   c) Use the relation that $x_2[n] = x_1[n-k]$ together with the equation of part (b) relating $y_2[n]$ to $x_2[\cdot]$ in order to obtain an equation relating $y_2[n]$ to $x_1[\cdot]$.

   d) Replace n by $n - k$ on both sides of the equation of part (b) which relates $y_1[n]$ to $x_1[\cdot]$ to obtain an equation relating $y_1[n-k]$ to $x_1[\cdot]$.

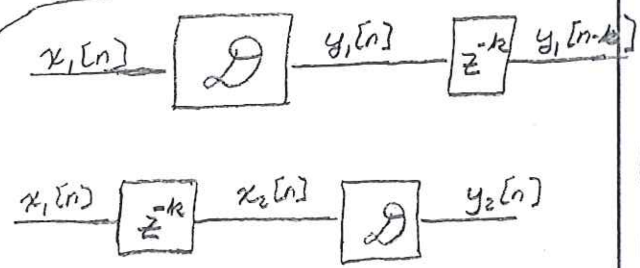
   e) Compare equations obtained in parts (c) and (d) to determine whether $y_2[n]$ is equal to $y_1[n-k]$. If $y_2[n] = y_1[n-k]$, conclude the system is time-invariant; otherwise, conclude the system is not time-invariant.

   To prove that a system is not time-invariant, you may follow the same procedure above or you may simply provide a counter example: *i.e.,* you may provide a specific choice of waveform $x_1[n]$ and integer k for which $y_2[n] \neq y_1[n-k]$ where the signals $y_1[n]$ and $y_2[n]$ are as defined in part (a).

Example which illustrates procedure to determine if a d.t. system, $\mathfrak{D}$, is time-invariant:

Let $\mathcal{D}: x[n] \longmapsto y[n]$ according to $y[n] = x[n] - x[n-1]$

check for time-invariance



if $y_2[n]$ is equal to $y_1[n-k]$ for any choice of $x_1[n]$ and any choice of k, then $\mathcal{D}$ is time-invariant; otherwise, it is not time-invariant

Is $\mathcal{D}$ time-invariant?

(a) Define notation:

Let $y_1[n] = \mathcal{D}\{x_1[n]\}$

Let $x_2[n] = x_1[n-k]$ (1)

Let $y_2[n] = \mathcal{D}\{x_2[n]\}$

(b) Apply the system definitions:

$y_1[n] = x_1[n] - x_1[n-1]$ (2)

$y_2[n] = x_2[n] - x_2[n-1]$ (3)

* Always apply the system operator to $x_2[n]$ before using the relationship in (1).

(c) Substituting (1) in (3) so as to express $y_2[n]$ in terms of $x_1[\cdot]$

$y_2[n] = x_1[n-k] - x_1[n-k-1]$ (4)

(d) We can replace n by $n-k$ on both sides of (2) to find an expression for $y_1[n-k]$ in terms of $x_1[\cdot]$

$y_1[n-k] = x_1[n-k] - x_1[n-k-1]$ (5)

(e) Comparing (4) + (5), we find:

$y_2[n] = y_1[n-k]$

$\therefore$ we may conclude that the system is time-invariant.

Let $\mathcal{T}: \{x[n]\} \longmapsto \{y[n]\}$ according to $y[n] = x[-n]$

Determine whether $\mathcal{T}$ is time-invariant.

Solution 1

Let $y_1[n] = \mathcal{T}\{x_1[n]\} \xrightarrow{\text{sys. defn.}} y_1[n] = x_1[-n] \implies y_1[n-k] = x_1[-(n-k)]$ (1)

Let $x_2[n] = x_1[n-k] \implies x_2[-n] = x_1[-n-k]$

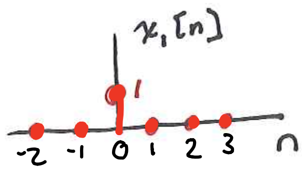
Let $y_2[n] = \mathcal{T}\{x_2[n]\} \implies y_2[n] = x_2[-n] = x_1[-n-k]$ (2)

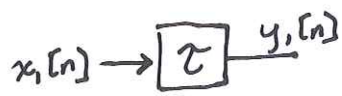
Since $y_2[n] \neq y_1[n-k]$ (compare (1) + (2)), $\mathcal{T}$ is not time-invariant

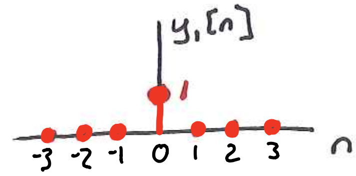
Solution 2

provide a counter-example to show that $\mathcal{T}$ is not time-invariant

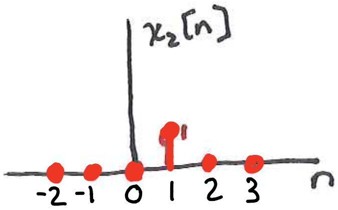
Let $x_1[n] = \delta[n]$ and let $y_1[n] \equiv \mathcal{T}\{x_1[n]\} \Rightarrow y_1[n] = x_1[-n] = \delta[-n]$
$= \delta[n]$

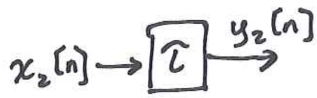


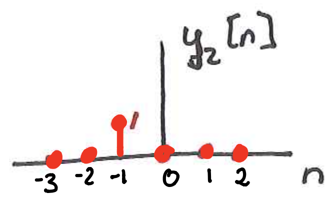




Let $x_2[n] = x_1[n-1] = \delta[n-1]$ + let $y_2[n] = \mathcal{T}\{x_2[n]\} \Rightarrow y_2[n] = x_2[-n]$







$\therefore$ $\mathcal{T}$ is not time-invariant since a shift in the input did not result in the same shift to the output

i.e. $x_2[n] = x_1[n-1]$ but $y_2[n] \neq y_1[n-1]$

Determine which of the input-output equations below describe time-invariant systems.

a) $y[n] = x[n]$

b) $y[n] = x[n - 1]$

c) $y[n] = x[n + 1]$

d) $y[n] = x[n] - x[n - 1]$

e) $y[n] = \max(x[n + 1], x[n], x[n - 1])$

f) $y[n] = x^2[n]$

g) $y[n] = x[n^2]$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$

3. ***Linear vs. Nonlinear:*** A relaxed system, $\mathcal{T} : \{x[n]\} \mapsto \{y[n]\}$, is said to be *linear* if and only if the response of the system to a weighted sum of signals is equal to the corresponding weighted sum of responses to each of the individual signals. That is, a system, $\mathcal{T}$, is linear if and only if:

$$\mathcal{T}\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 \mathcal{T}[x_1[n]] + a_2 \mathcal{T}[x_2[n]]$$

for arbitrary input sequences, $x_1[n]$ and $x_2[n]$, and arbitrary constants, a_1 and a_2.

A procedure for determining whether or not a system, $\mathcal{T}$, is linear, is detailed in steps (a)-(e) below:

a) Define your notation. For example, let $x_1[n]$ and $x_2[n]$ denote unspecified discrete-time signals and let a_1 and a_2 denote unspecified constants. Let the signals $x_3[n]$, $y_1[n]$, $y_2[n]$, and $y_3[n]$ be defined in terms of the signals, $x_1[n]$ and $x_2[n]$, and the constants, a_1 and a_2, as follows:
$$y_1[n] = \mathcal{T}\{x_1[n]\}, \quad y_2[n] = \mathcal{T}\{x_2[n]\}$$
$$x_3[n] = a_1 x_1[n] + a_2 x_2[n], \quad \text{and} \quad y_3[n] = \mathcal{T}\{x_3[n]\}$$

b) Use the system's input-output relationship to obtain three equations: one relating $y_1[n]$ to $x_1[\cdot]$, a second relating $y_2[n]$ to $x_2[\cdot]$, and a third relating $y_3[n]$ to $x_3[\cdot]$.

c) Use the relation that $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$ together with the equation of part (b) relating $y_3[n]$ to $x_3[\cdot]$ in order to obtain an equation relating $y_3[n]$ to $x_1[\cdot]$, $x_2[\cdot]$, a_1, and a_2.

d) Multiply both sides of the first equation of part (b) by the constant a_1; multiply both sides of the second equation of part (b) by the constant a_2; then sum the two resulting equations, thus obtaining an equation which relates the signal $(a_1 y_1[n] + a_2 y_2[n])$ to $x_1[\cdot]$, $x_2[\cdot]$, a_1, and a_2.

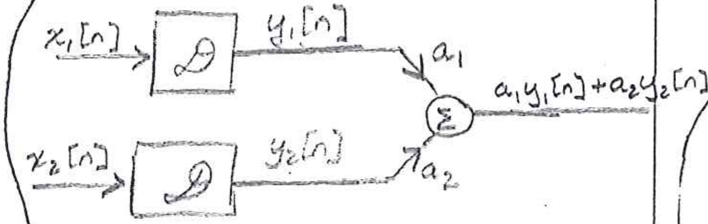
e) Compare equations obtained in parts (c) and (d) to determine whether $y_3[n]$ is equal to $(a_1 y_1[n] + a_2 y_2[n])$. If $y_3[n] = a_1 y_1[n] + a_2 y_2[n]$, conclude the system is linear; otherwise, conclude the system is nonlinear.

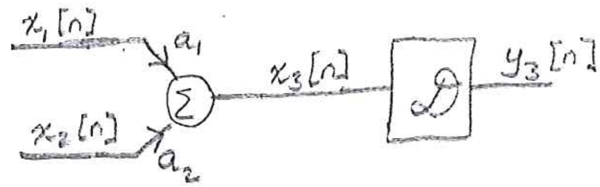
To prove that a system is not linear, you may follow the same procedure above or you may simply provide a counter example (*i.e.,* you may provide a specific choice of waveforms: $x_1[n]$ and $x_2[n]$, and constants: a_1 and a_2 for which $y_3[n] \neq a_1 y_1[n] + a_2 y_2[n]$ where the signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ are as defined in part (a)).

Example which illustrates procedure to determine if a d.t. system, $\mathfrak{D}$, is linear

Let $\mathcal{D}: x[n] \longmapsto y[n]$ according to $y[n] = x[n] - x[n-1]$

check for linearity





if $y_3[n]$ is equal to $a_1 y_1[n] + a_2 y_2[n]$, for any choice of $x_1[n]$, $x_2[n]$, a_1 + a_2 then the system is linear; otherwise, it is not linear.

Is $\mathcal{D}$ linear?

(a)
Let $y_1[n] = \mathcal{D}\{x_1[n]\}$
let $y_2[n] = \mathcal{D}\{x_2[n]\}$
Let $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$ (1)
Let $y_3[n] = \mathcal{D}\{x_3[n]\}$

(b)
By system definition:

$$y_1[n] = x_1[n] - x_1[n-1] \quad (2)$$
$$y_2[n] = x_2[n] - x_2[n-1] \quad (3)$$
$$y_3[n] = x_3[n] - x_3[n-1] \quad (4)$$

Using (1) to rewrite (4) in terms of $x_1[\cdot]$ and $x_2[\cdot]$ yields:

(c)
$$y_3[n] = \left(a_1 x_1[n] + a_2 x_2[n]\right) - \left(a_1 x_1[n-1] + a_2 x_2[n-1]\right)$$
$$= a_1\left(x_1[n] - x_1[n-1]\right) + a_2\left(x_2[n] - x_2[n-1]\right) \quad \text{(regrouping terms)}$$

(d)
$$= a_1 y_1[n] + a_2 y_2[n] \quad \text{(see eqns. (2) + (3))}$$

(e) Since $y_3[n] = a_1 y_1[n] + a_2 y_2[n]$, we may conclude that <u>the system is linear</u>

Example to check linearity of system

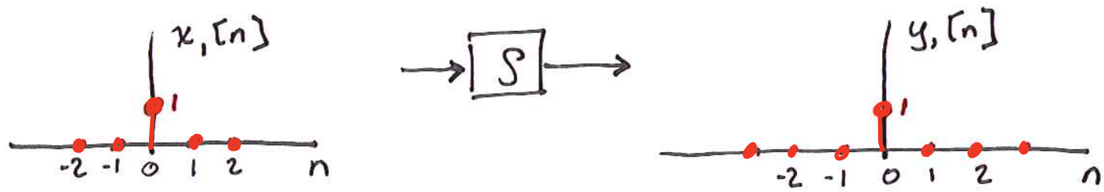
Let $S: \{x[n]\} \longmapsto \{y[n]\}$ accordg to $y[n] = x^2[n]$

Determine whether S is linear.

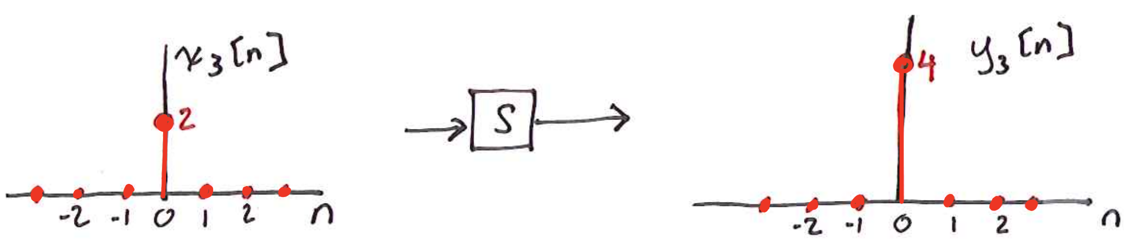
Soln

doesn't seem like it would be linear, so we will try to find a counter example

Let $x_1[n] = \delta[n]$ and let $y_1[n] \equiv S\{x_1[n]\} \Rightarrow y_1[n] = \delta^2[n]$



Let $x_3[n] = 2x_1[n]$ and let $y_3[n] \equiv S\{x_3[n]\} \Rightarrow y_3[n] = x_3^2[n]$



Conclusion: The system is not linear.

For a linear system, if you double the amplitude of the input, the output will also double. However, for the example above, when we multiplied the input by 2, the output was multiplied by 4.

Determine which of the input-output equations below describe linear systems.

a) $y[n] = x[n]$

b) $y[n] = x[n-1]$

c) $y[n] = x[n+1]$

d) $y[n] = x[n] - x[n-1]$

e) $y[n] = \max(x[n+1], x[n], x[n-1])$

f) $y[n] = x^2[n]$

g) $y[n] = x[n^2]$

h) $y[n] = \sum_{k=-\infty}^{n} x[k]$

4. ***Linear Time-invariant (LTI)***: A system is said to be linear time-invariant if and only if it is both linear and time-invariant. Most of the systems we will analyze in this class are LTI systems. It can be shown that *LTI systems are completely characterized by their impulse response*; that is, given the impulse response of an LTI system, one can find the output to any given input without the need for an input-output relationship.

5. ***Causal vs. Noncausal:*** A system is *causal* if and only if the output of the system at time n depends only on present and past inputs at time n, *i.e.,* if:
$$y[n] = g(x[n], x[n-1], x[n-2], \ldots)$$
Although a real-time noncausal system is unrealizable, noncausal systems are often used in non real-time applications where the signal is recorded and processed off-line (such as with image processing) or in applications where a small amount of delay is tolerable.

   It can be shown that an LTI system is causal if and only if its impulse response, $h[n]$, is causal, *i.e.*, if and only if:
$$h[n] = 0\,, \quad n < 0$$

6. ***Stable vs. Unstable:*** A relaxed system is said to be bounded-input bounded-output (BIBO) stable if and only if every bounded input signal produces a bounded output signal. (Note: a sequence, $x[n]$, is bounded if there exist some finite number, M_x, such that $|x[n]| \le M_x < \infty$ for all n.)

   It can be shown that an LTI system is BIBO stable if and only if its impulse response, $h[n]$, is absolutely summable, *i.e.,* if and only if:
$$\left(\sum_{n=-\infty}^{\infty} |h[n]|\right) < \infty$$

7. ***FIR vs. IIR:*** A system is termed as a ***finite impulse response (FIR) system*** if the impulse response is of finite length; otherwise, it is termed an ***infinite impulse response (IIR) system***.

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Since an LTI system is completely characterized by its impulse response, there can be at most one LTI system with a given impulse response. Note that of the three systems below that have the same impulse response, only one of them is LTI.

From Page 2 of Handout on D.T. Systems:

Example: Given the input-output relationships below, determine the impulse response, $h[n]$, of each associated system.

LTI a) $y[n] = x[n]$ $\Rightarrow$ $h[n] = \delta[n]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0, 0, \ldots \}$ (*)

LTI b) $y[n] = x[n-1]$ $\Rightarrow$ $h[n] = \delta[n-1]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{0}, 1, 0, 0, 0, \ldots \}$

LTI c) $y[n] = x[n+1]$ $\Rightarrow$ $h[n] = \delta[n+1]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 1, \underset{\uparrow}{0}, 0, 0, 0, 0, \ldots \}$

LTI d) $y[n] = x[n] - x[n-1]$ $\Rightarrow$ $h[n] = \delta[n] - \delta[n-1]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{1}, -1, 0, 0, 0, \ldots \}$

not LTI e) $y[n] = \max(x[n+1], x[n], x[n-1])$ $\Rightarrow$ $h[n] = \max(\delta[n+1], \delta[n], \delta[n-1])$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 1, \underset{\uparrow}{1}, 1, 0, 0, 0, \ldots \}$

not LTI f) $y[n] = x^2[n]$ $\Rightarrow$ $h[n] = \delta^2[n] = \delta[n]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0, 0, \ldots \}$ (*)

not LTI g) $y[n] = x[n^2]$ $\Rightarrow$ $h[n] = \delta[n^2] = \delta[n]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0, 0, \ldots \}$ (*)

LTI h) $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\Rightarrow$ $h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$\Rightarrow h[n] = \{ \ldots, 0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, \ldots \}$

* note that systems (a), (f), and (g) all have the same impulse response.

Stevenson

To prove that an LTI system, $\mathcal{S}$, is completely characterized by its impulse response, $h[n]$, we need only show that knowledge of $h[n]$ allows us to find the system's response, $y[n]$, to any arbitrary input, $x[n]$.

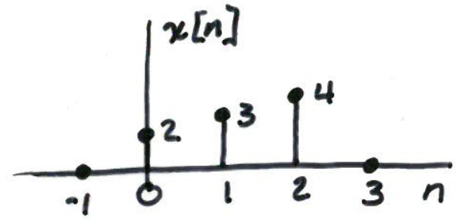
To show this, first note that:

Any discrete-time signal, $x[n]$, may be decomposed as a weighted sum of time-shifted Kronecker Delta functions:

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]\,\delta[n-k]}_{x_k[n]}$$

Example

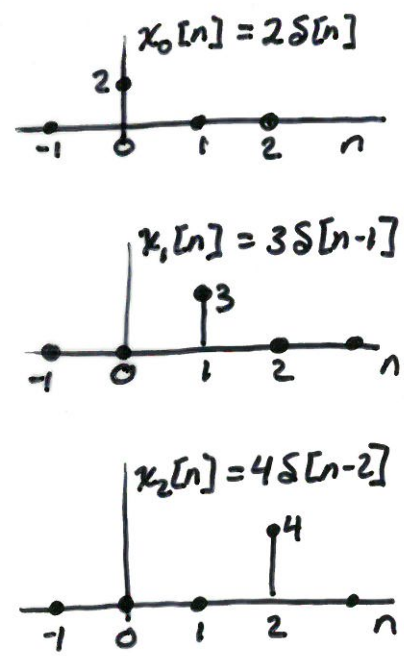
Let $x[n] = \{\underset{\uparrow}{2}\ 3\ 4\ 0\ 0 \ldots\}$



Then $x[n] = x_0[n] + x_1[n] + x_2[n]$

$$= x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2]$$

$$= 2\,\delta[n] + 3\,\delta[n-1] + 4\,\delta[n-2]$$



Therefore:

$$\mathcal{S}\{x[n]\} = \mathcal{S}\{2\delta[n] + 3\delta[n-1] + 4\delta[n-2]\}$$

$$= 2\,\mathcal{S}\{\delta[n]\} + 3\,\mathcal{S}\{\delta[n-1]\} + 4\,\mathcal{S}\{\delta[n-2]\}$$ by linearity of $\mathcal{S}$

$$= 2\,h[n] + 3\,h[n-1] + 4\,h[n-2]$$ by time-invariance of $\mathcal{S}$ and the fact that $h[n] = \mathcal{S}\{\delta[n]\}$

Stevenson

In general, if $\mathcal{S}$ is LTI with impulse response $h[n]$, then the response of $\mathcal{S}$ to an arbitrary signal $x[n]$ can be found as shown below.

$$\text{Let } x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]\,\delta[n-k]}_{x_k[n]} = \sum_{k=-\infty} x_k[n]$$

$$\text{Let } y[n] \equiv \mathcal{S}\{x[n]\} = \mathcal{S}\left\{\sum_{k=-\infty}^{\infty} x_k[n]\right\}$$

note that $\mathcal{S}$ is operating on a function of n

$$= \sum_{k=-\infty}^{\infty} \mathcal{S}\{x_k[n]\}$$

by linearity of $\mathcal{S}$.

$$= \sum_{k=-\infty}^{\infty} \mathcal{S}\{\overbrace{x[k]}^{\text{constant}}\ \overbrace{\delta[n-k]}^{\text{function of } n}\}$$

by definition of $x_k[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{S}\{\delta[n-k]\}$$

by linearity of $\mathcal{S}$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

by time-invariance of $\mathcal{S}$, coupled with the knowledge that $\mathcal{S}\{\delta[n]\} = h[n]$

Given an LTI system with impulse response $h[n]$, we may find the system response, $y[n]$, to an arbitrary input signal, $x[n]$, as the convolution of $x[n]$ with $h[n]$.

$$y[n] = x[n] \underset{\substack{\uparrow \\ \text{convolution operator}}}{*} h[n] \implies \begin{cases} y[n] = \displaystyle\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \\ \text{OR} \\ y[n] = \displaystyle\sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \end{cases}$$