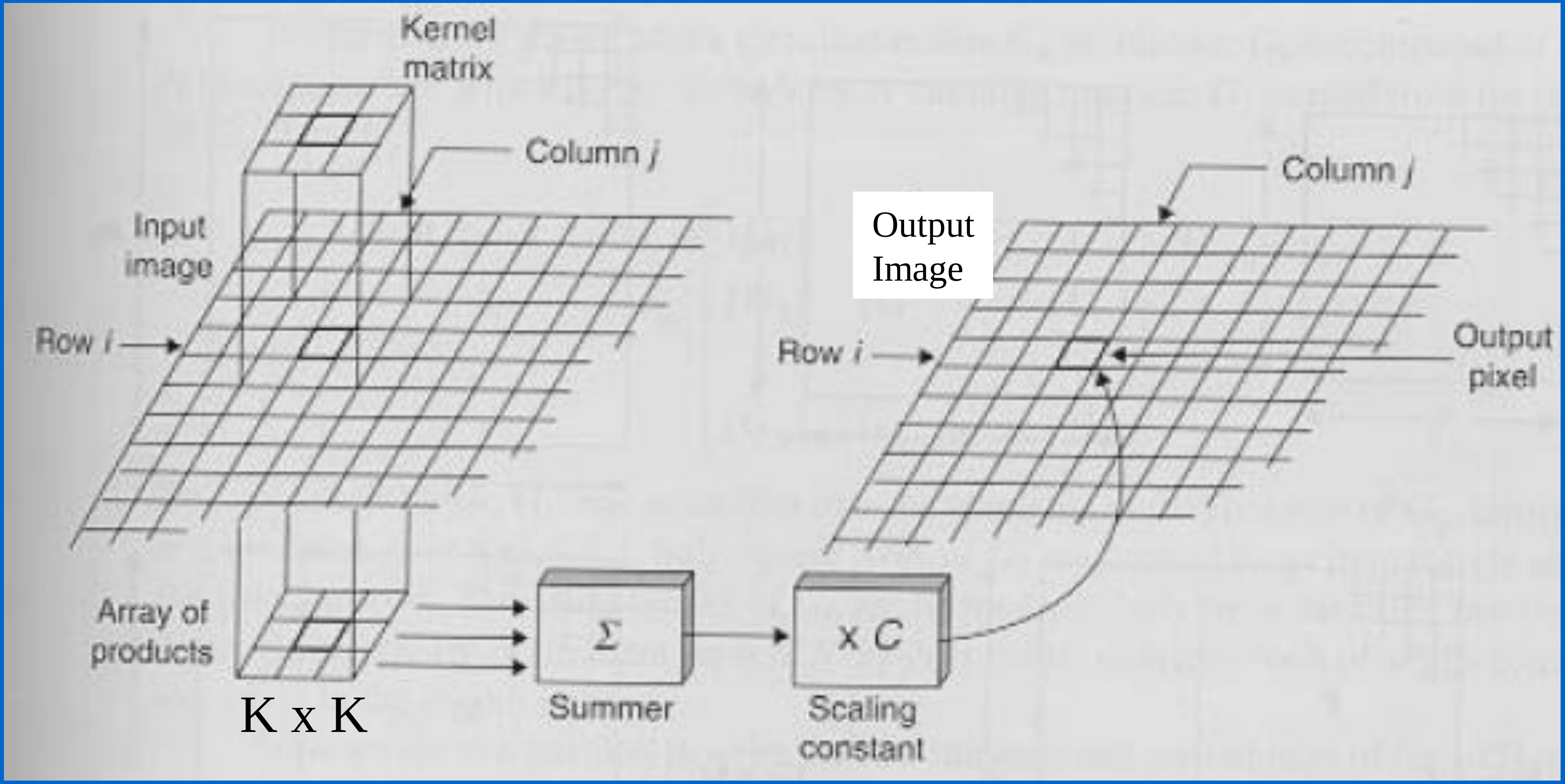
Convolution

CS474/674 – Prof. Bebis

Section 3.4, 4.2

Correlation - Review



$$g(x,y) = w(x,y) \bullet f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x+s, y+t)$$

Convolution – Review

- Same as correlation except that the mask is flipped both horizontally and vertically.

$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x-s, y-t)$$

- Note that if w(x,y) is symmetric, that is w(x,y)=w(-x,-y), then convolution is equivalent to correlation!

1D Continuous Convolution

- Convolution is defined as follows:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(a)g(x-a)da$$

1. Flip g(a) → g(-a)
2. Shift g(x-a) $-\infty < x < \infty$
3. Compute area overlap
   f(a)g(x-a)

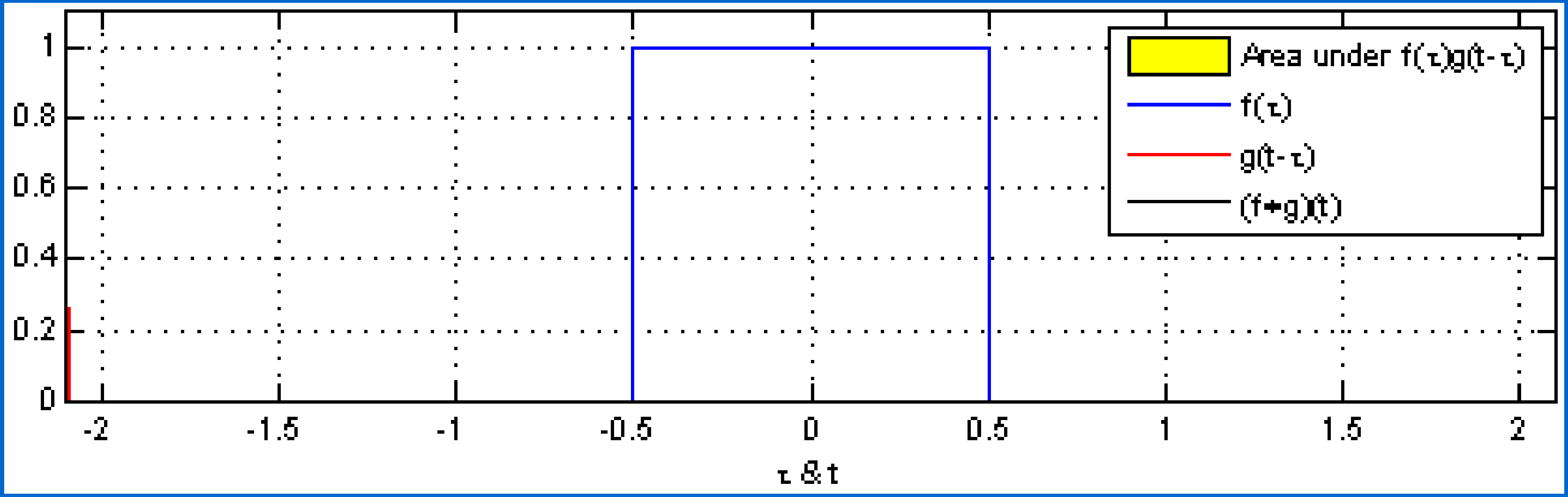
- Convolution is commutative:

$$f(x) * g(x) = g(x) * f(x)$$

Think of f(x) as the image and g(x) as the mask although you can reverse their roles!

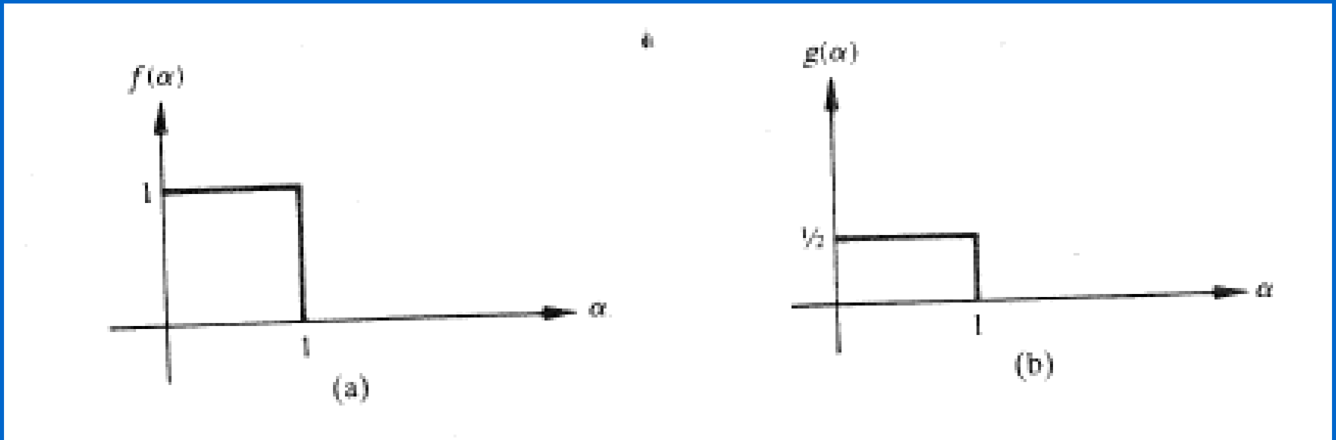
Example 1

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(a)g(x-a)da$$

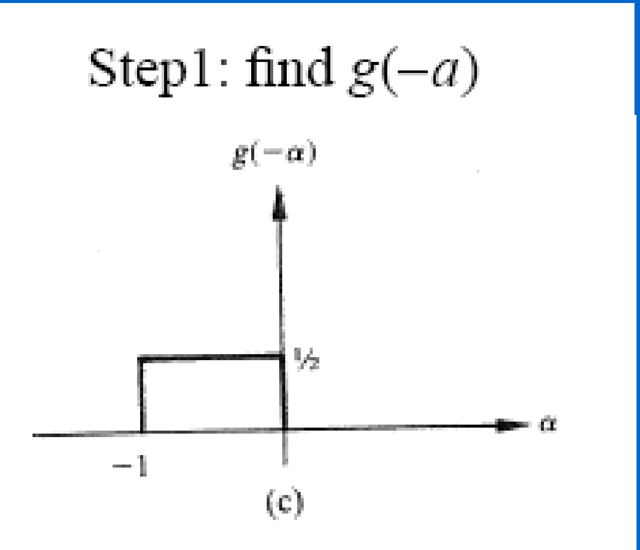


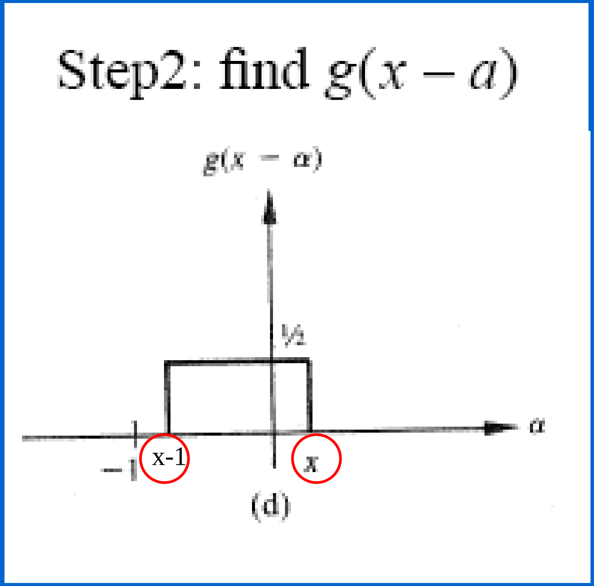
Example 2

- Compute the convolution of the following two functions:



Example 2 (cont'd)

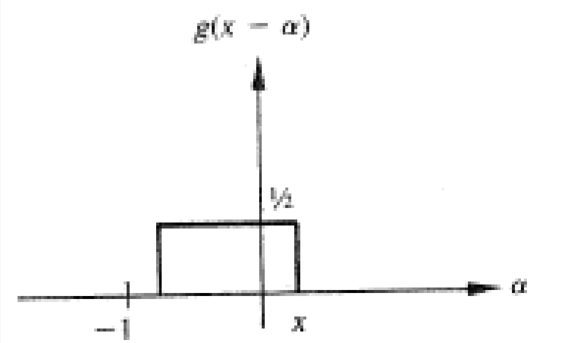




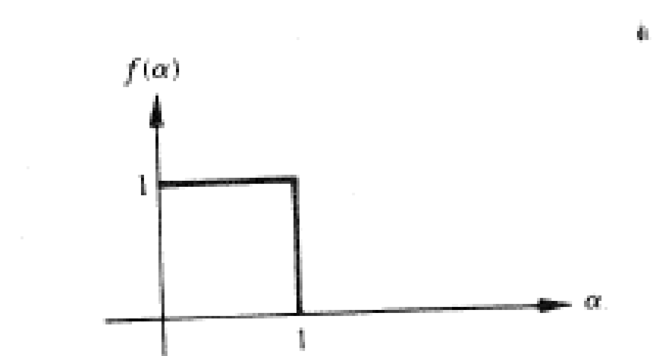
Example 2 (cont'd)

Step 3: Compute the integral for $-\infty < x < \infty$

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(a)g(x-a)da$$



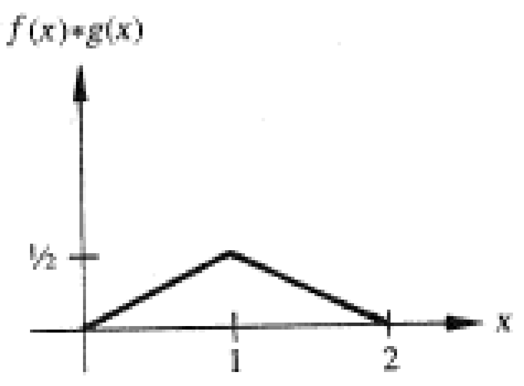
(d)



(a)

Example 2 (cont'd)

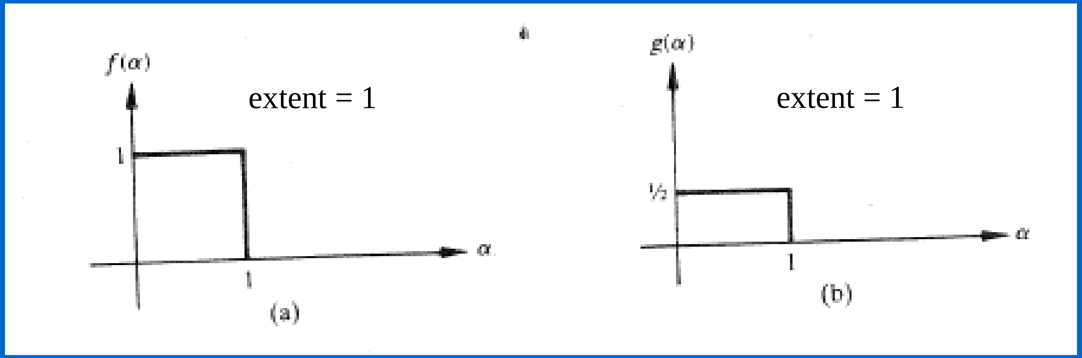
$$f(x) * g(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ 1 - x/2 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

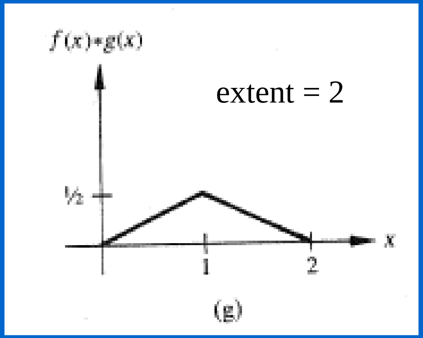


(g)

Important Observations

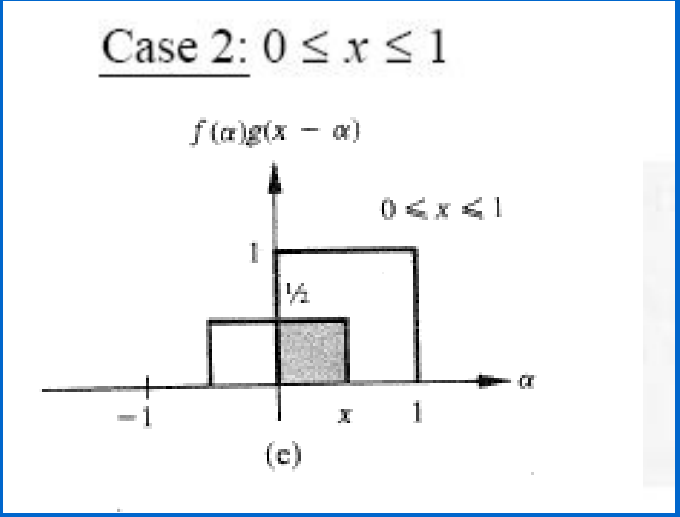
- The extent of f(x) * g(x) is equal to the extent of f(x) plus the extent of g(x)





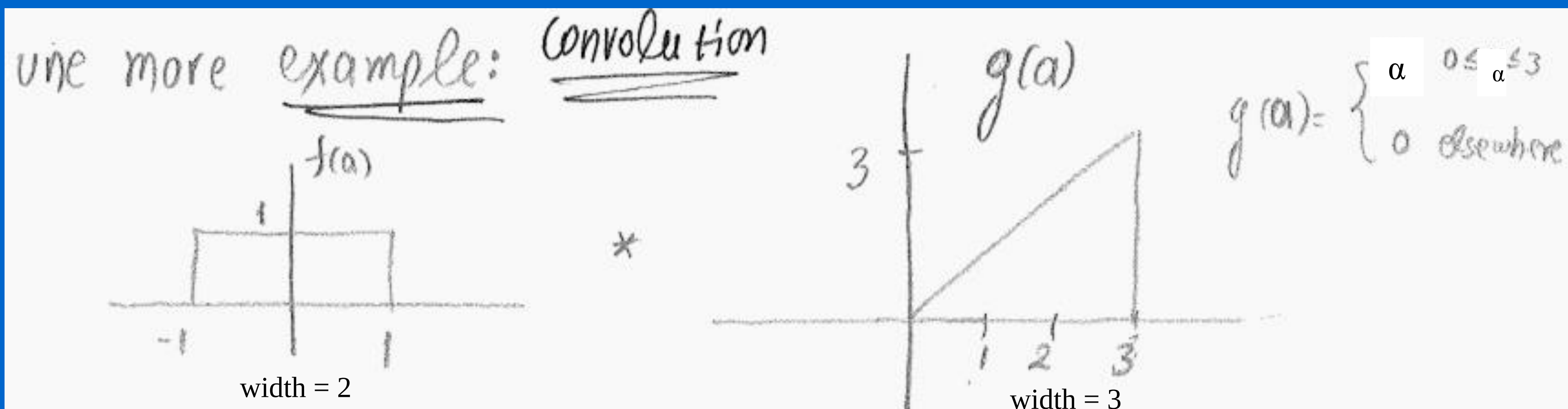
Important Observations (cont'd)

- For every x, the limits of the integral are determined as follows:
  - Lower limit: MAX (left limit of f(x), left limit of g(x-a))
  - Upper limit: MIN (right limit of f(x), right limit of g(x-a))



$$\int_{-\infty}^{\infty} f(a)g(x-a)da = \int_{0}^{x} 1\,\frac{1}{2}\,da = \frac{x}{2}$$

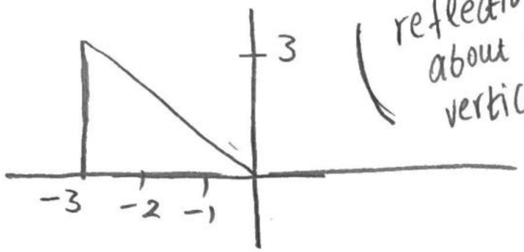
Example 3



one more example: Convolution
f(α)
1
-1
1
width = 2
*
g(α)
3
1
2
3
width = 3
g(α)= { α 0≤α≤3
0 elsewhere

Example 3 (cont'd)

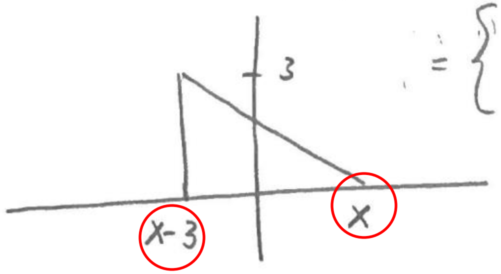
Step 1: find $g(-a)$



(reflection about the vertical axis)

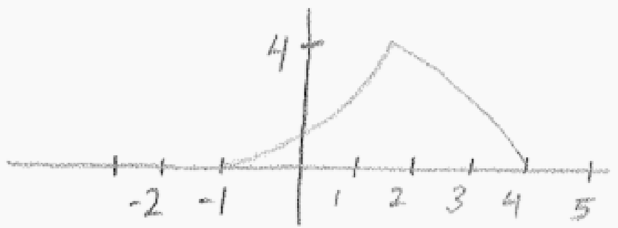
Step 2: find $g(x-a)$ =

$$= \begin{cases} x-a & 0 \le x-a \le 3 \\ 0 & \text{elsewhere} \end{cases}$$



Example 3 (cont'd)

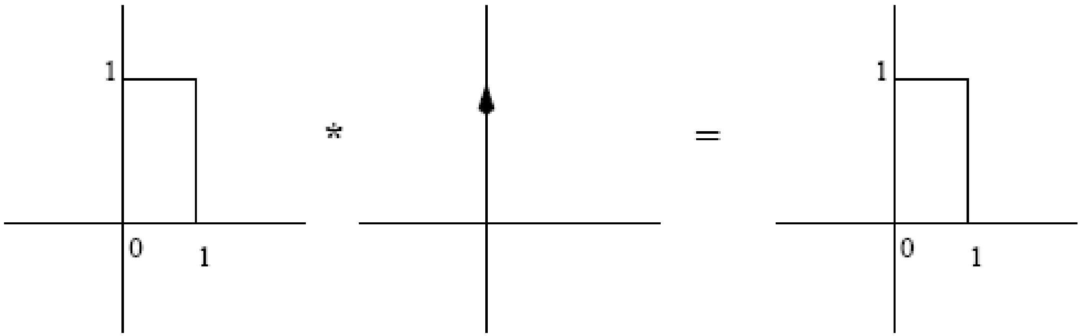
$$f(x)*g(x)=\begin{cases} \frac{1}{2}(x+1)^2 & -1\le x\le 1 \\ 2x & 1\le x\le 2 \\ 4+x-\frac{1}{2}x^2 & 2\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$



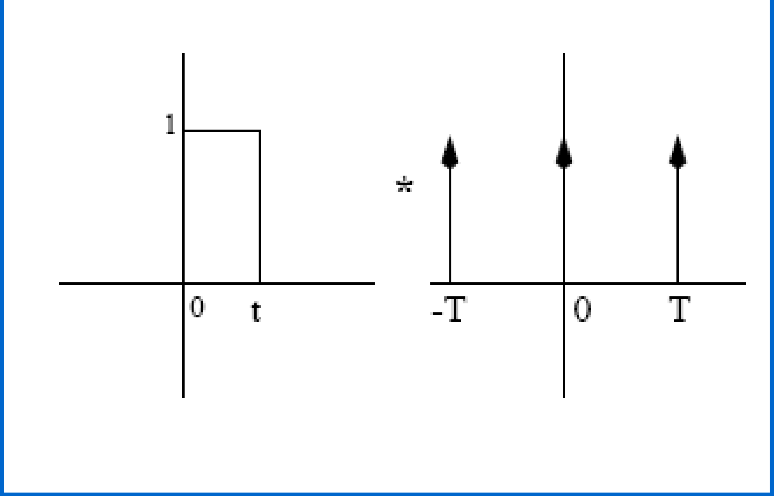
Convolution with an impulse (i.e., delta function)

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(a)\delta(x-a)da = f(x)$$

(since $\delta(x-a) = 1$ if $a = x$)



Convolution with an “train” of impulses



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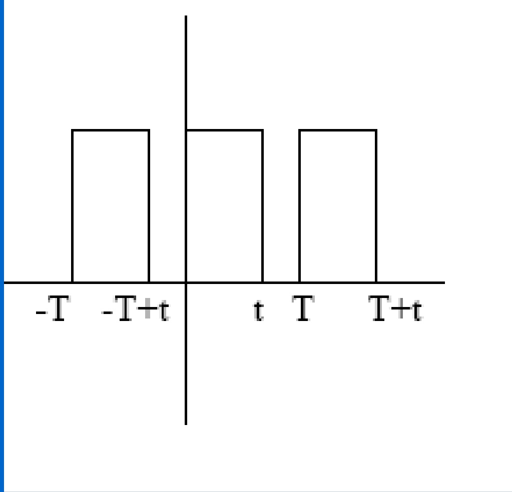


Convolution Theorem

- Convolution in the spatial domain is equivalent to multiplication in the frequency domain.

$$f(x) * g(x) <\!-\!-\!> F(u)G(u)$$

$f(x) \longleftrightarrow F(u)$

$g(x) \longleftrightarrow G(u)$

- Multiplication in the spatial domain is equivalent to convolution in the frequency domain.

$$f(x)g(x) <\!-\!-\!> F(u) * G(u)$$

Efficient computation of (f * g)

- 1. Compute $F(f(x)) = F(u)$ and $F(g(x)) = G(u)$
- 2. Multiply them: $F(u)G(u)$ (i.e., element-wise complex multiplication)
- 3. Compute the inverse FT: $F^{-1}(F(u)G(u)) = f(x) * g(x)$

We will later analyze computational complexity!

Discrete Convolution

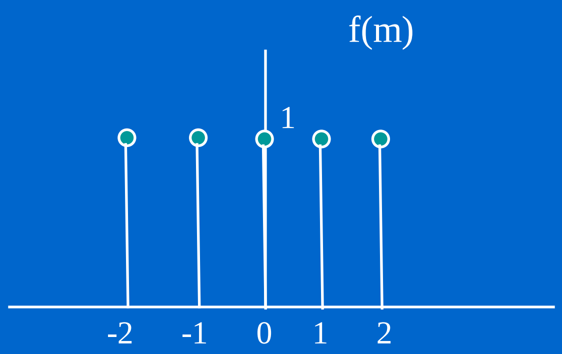
- Replace integral with summation
- Integration variable becomes an index.
- Displacements take place in discrete increments

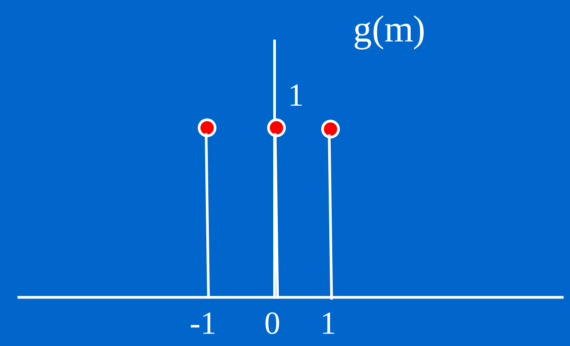
$$f(x) * g(x) = \sum_{m=-\infty}^{\infty} f(m)g(x-m), \quad -\infty < x < \infty$$

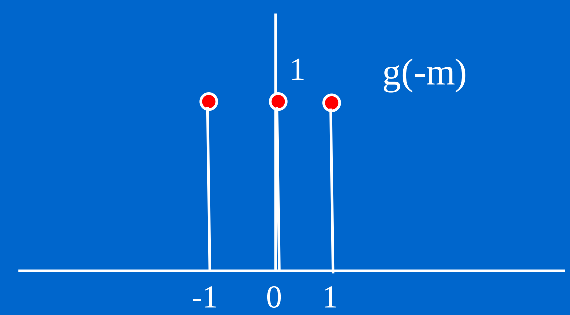
Example

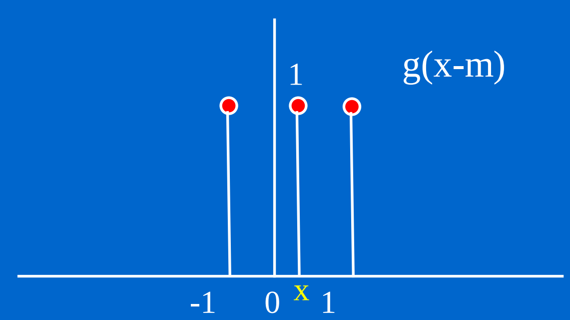
$$f(x) * g(x) = \sum_{m=-\infty}^{\infty} f(m) g(x-m), \ -\infty < x < \infty$$

Example (cont'd)







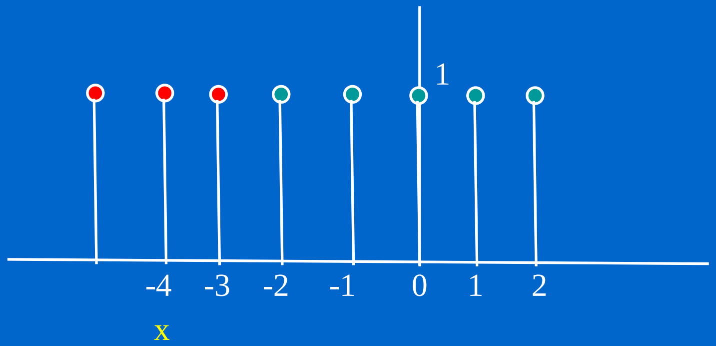


Compute the convolution assuming discrete values for x, $-\infty < x < \infty$

Example (cont'd)

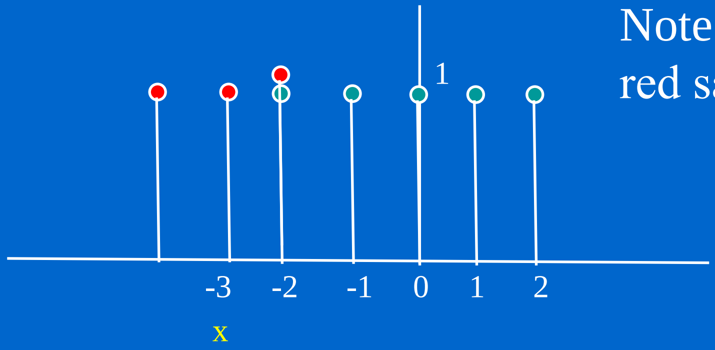
$x = -4 \, or \, x < -4$

no overlap



f *g = 0

$x = -3$

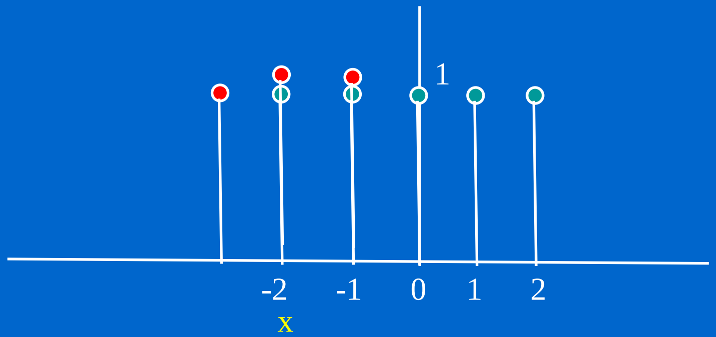


Note that I show some
red samples “taller” for clarity!

f *g = 1

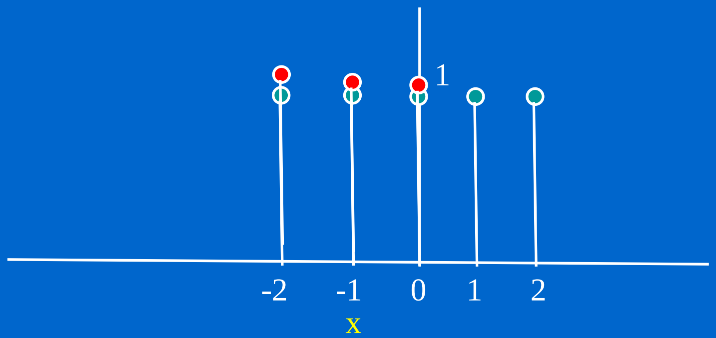
Example (cont'd)

$x = -2$



f *g = 2

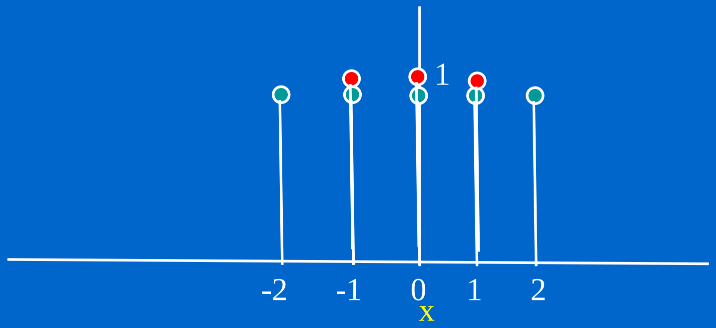
$x = -1$



f *g = 3

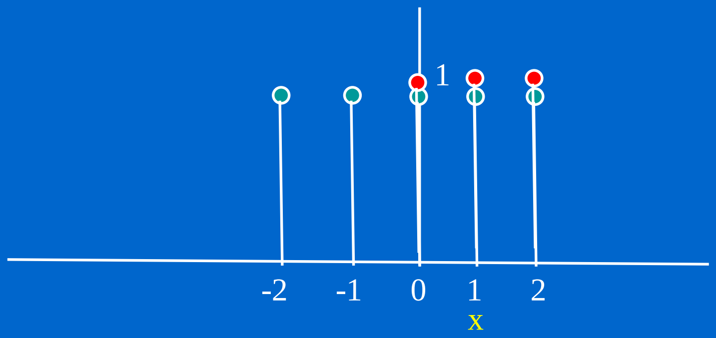
Example (cont'd)

$x = 0$



f *g = 3

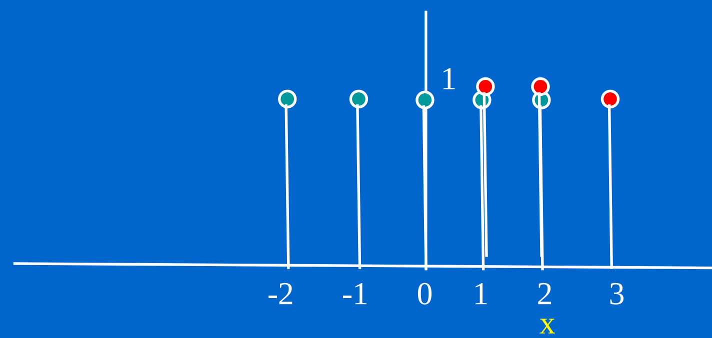
$x = 1$



f *g = 3

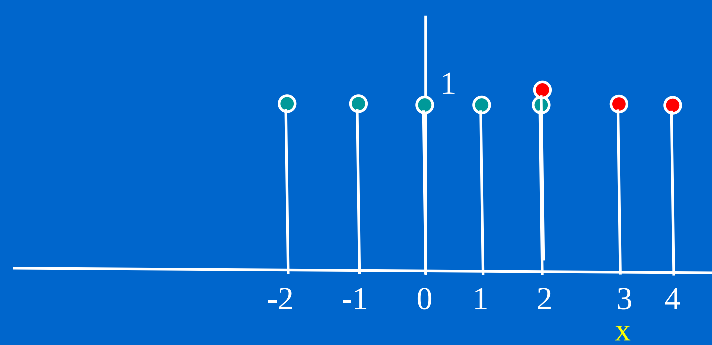
Example (cont'd)

$x = 2$



f *g = 2

$x = 3$



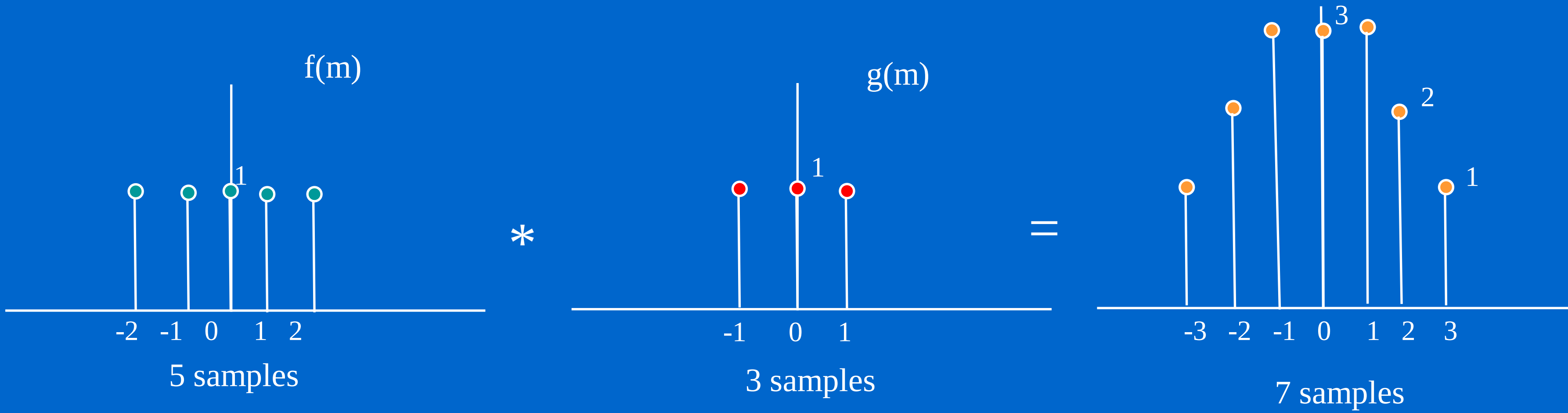
f *g = 1

$x \geq 4$

f *g = 0

no overlap

Example



length of f*g = length of f + length of g - 1

Convolution Theorem in Discrete Case

- Input sequences:

$$\{f(0), f(1), \ldots, f(A-1)\}, \{g(0), g(1), \ldots, g(B-1)\}$$

- Length of f*g sequence is: M=A+B-1

- Extended input sequences: make them length M by padding with zeroes:

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le A-1 \\ 0 & A \le x \le M-1 \end{cases} \qquad g_e(x) = \begin{cases} g(x) & 0 \le x \le B-1 \\ 0 & B \le x \le M-1 \end{cases}$$

Convolution Theorem in Discrete Case (cont'd)

- When dealing with discrete convolution, the convolution theorem holds true for the extended sequences only!

element-wise
complex multiplication

$$f_e(x) * g_e(x) \Longleftrightarrow F_e(u)\, G_e(u)$$

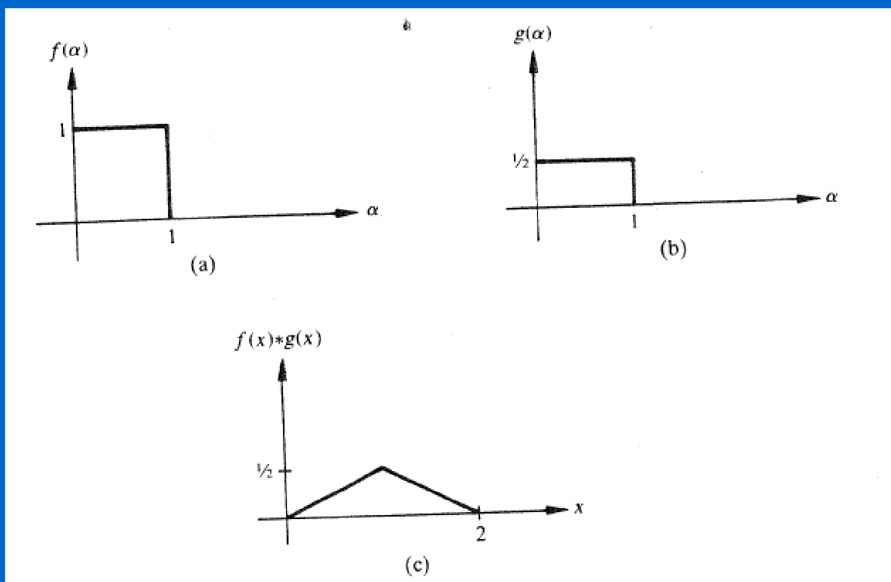
$$\text{where } f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

Note: in general, we can form $f_e(x)$ and $g_e(x)$ using:

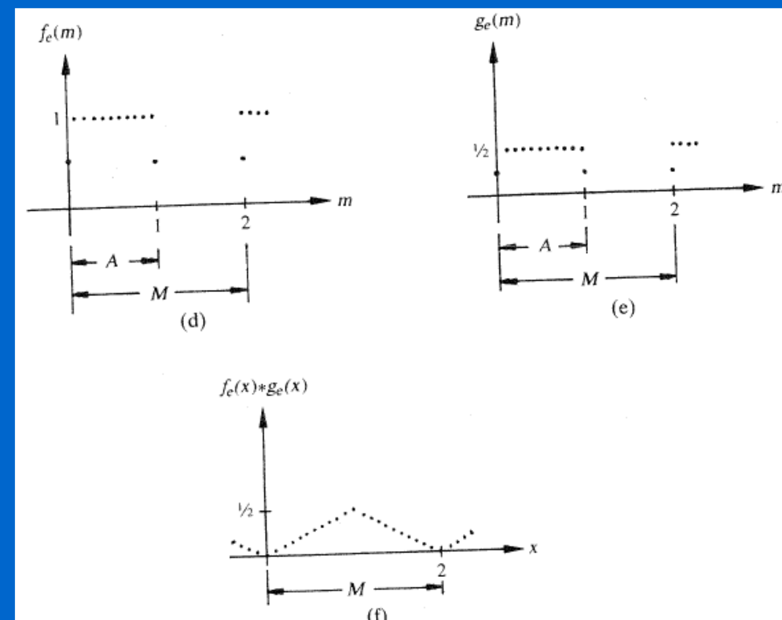
$$M \geq A + B - 1$$

Why do we need to consider the extended sequences ?

Continuous case

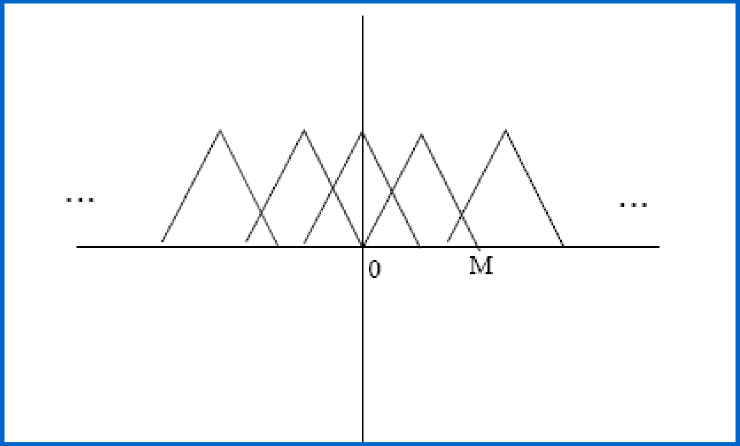


Discrete case

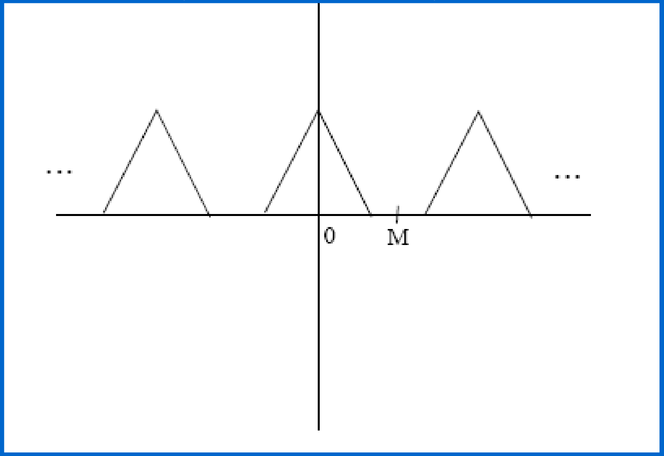


The discrete convolution is periodic (with period M=A+B-1) since DFT is periodic

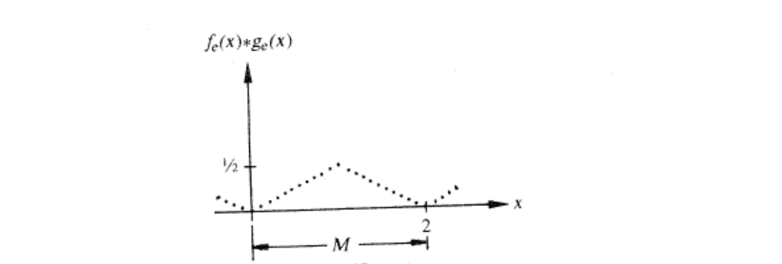
Why do we need to consider the extended sequences ? (cont'd)



If **M<A+B-1**, the periods move closer to each other (overlap)



If **M>A+B-1**, the periods move away from each other (no overlap)



(f)

Optimum:

M=A+B-1

2D Convolution

- Definition

$$f(x,y) * g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)g(x-a,y-b)\,da\,db$$

- 2D convolution theorem

element-wise
complex multiplication

$$f(x,y) * g(x,y) <---> F(u,v)\,G(u,v)$$

$$f(x,y)\,g(x,y) <---> F(u,v) * G(u,v)$$

Discrete 2D convolution

- Suppose f(x,y) is A x B and g(x,y) is C x D
- The size of f(x,y) * g(x,y) would be N x M where

  N=A+C-1 and M=B+D-1

- Form extended images (i.e., pad with zeroes):

$$f_e(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \text{ and } 0 \le y \le B-1 \\ 0 & A \le x \le M-1 \text{ and } B \le y \le N-1 \end{cases}$$

$$g_e(x) = \begin{cases} g(x,y) & 0 \le x \le C-1 \text{ and } 0 \le y \le D-1 \\ 0 & C \le x \le M-1 \text{ and } D \le y \le N-1 \end{cases}$$

Discrete 2D convolution (cont'd)

- The convolution theorem holds true for the extended images only!

$$f_e(x,y) * g_e(x,y) <\!-\!-\!- > F_e(u,v)G_e(u,v)$$

$$f_e(x,y) * g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x-m, y-n)$$

$$(x = 0, 1, \ldots, M-1,\ y = 0, 1, \ldots, N-1)$$

Example

1. We have mentioned in class several times that computing convolution in the time domain is expensive; however, it can be compute more efficiently in the frequency domain using the convolution theorem and Fast Fourier Transforms (FFTs). Let's verify this claim. **(a)** What is the complexity of convolving an N x N image with an M x M mask? Use big-O notation. **(b)** What is the complexity of 2D FFT? (*hint:* use the separability property of the FT) **(c)** How could we compute the convolution using FFTs? **(d)** What is the complexity of computing the convolution in the frequency domain? Use big-O notation.