Indian Institute of Science Education and Research Kolkata

Department of Physical Sciences

Summer Project Report
May - July 2021

Fermions in 3+1 dimensions: Dirac, Weyl and Majorana Fields

Submitted by
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on
15/07/2021
under the supervision of
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Declaration

I declare here that the report included in this project entitled “Fermions in 3+1 dimensions: Dirac, Weyl and Majorana Fields” is the summer internship carried out by me under the guidance of Dr. Aradhya Shukla in the Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata, India from May 16 to July 15, 2021 under the supervision of Prof. Prasanta K. Panigrahi.

In keeping with general practice of reporting scientific observations, due acknowledgements have been made wherever the work described is based on the findings of other investigators.

S. Kundu.	Raipur	15/07/2021
Student’s e-Signature	**Place**	**Date**

Certificate

It is certified that the summer research work included in the project report entitled "**Fermions in 3+1 dimensions: Dirac, Weyl and Majorana Fields**" has been carried out by **Mr. Surjakanta Kundu** under my supervision and guidance. The content of this project report has not been submitted elsewhere for the award of any academic and professional degree.

July 15, 2021
IISER Kolkata

Signature of Supervisor
(Prof. Prasanta K. Panigrahi)
Project Supervisor

Acknowledgements

I, Surjakanta Kundu (19MS009), a 2nd year BS-MS student from IISER Kolkata, express my gratitude for completing my project work on “Fermions in 3+1 dimensions: Dirac, Weyl and Majorana Fields”. Firstly, I would like to express my special thanks to Prof. Prasanta K. Panigrahi for giving me the opportunity to work and learn under his supervision. I would also like to extend my gratitude to Dr. Aradhya Shukla for guiding me throughout this project and helping me understand the topics. I sincerely thank IISER Kolkata for giving me the opportunity to carry out my summer internship.

S. Kundu.	Raipur	15/07/2021
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Notation and Convention

We have used the following notation and convention throughout the report –

$$\vec{x} = (x_1, x_2, x_3)$$

$$\vec{J} = (J_1, J_2, J_3)$$

$$\vec{A} = (A_1, A_2, A_3)$$

An arbitrary vector $\vec{v}$ is understood as (v_1, v_2, v_3).

We have used **Einstein's convention of summation over repeated indices,** i.e.

$$\vec{A} \cdot \vec{B} = \sum A_i \cdot B_i = A_i B_i = \delta_{ij} A_i B_j,$$

where $\delta_{ij} = \delta^{ij}$ is the **Kronecker Delta in the Euclidean space.**

In the Minkowski space, the vectors have four components and are classified as **contravariant and covariant vectors** (on the basis of transformation properties under Lorentz transformations).

$$\text{Contravariant vectors} = A^\mu = \left(A^0, \vec{A}\right)$$

$$\text{Covariant vectors} = A_\mu = \left(A^0, -\vec{A}\right)$$

Moreover, these vectors are related to each other through the metric tensor of Minkowski space as

$$A_\mu = g_{\mu\nu} A^\nu \text{ and } A^\mu = g^{\mu\nu} A_\nu,$$

where $\boldsymbol{g^{\mu\nu}}$ **is the metric tensor** of the Minkowski space.

We adopt the following signature of $g^{\mu\nu}$ –

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}$$

Moreover, we use the following relations –

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\nu$$

$$g^{\mu\nu} = g^{\nu\mu} \text{ and } g_{\mu\nu} = g_{\nu\mu}$$

We define the invariant scalar product of two 4-vectors as

$$A \cdot B = A^\mu B_\mu = A_\mu B^\mu = g^{\mu\nu} A_\mu B_\nu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B}$$

We also define

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \text{ and } \Box = \partial^2 = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

The two totally antisymmetric fourth rank tensors, $\epsilon^{\mu\nu\lambda\rho}$ and $\epsilon_{\mu\nu\lambda\rho}$, are constructed in the Minkowski space as

$$\epsilon^{0123} = 1 = -\epsilon_{0123} \text{ and } \epsilon^{0ijk} = \epsilon_{ijk} = -\epsilon_{0ijk},$$

where ϵ_{ijk} is the **three-dimensional Levi-Civita tensor** with $\epsilon_{123} = 1$.

Hence,

$$\epsilon^i_{jk} = g^{il} \epsilon_{ljk}$$

Moreover, the notation '$\mathbb{1}$' denotes the identity matrix of the suitable size and dimensions, such as

$$\mathbb{1}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contents

Fermions in 3+1 dimensions: Dirac, Weyl and Majorana Fields

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Abstract

We discuss the Dirac, Weyl and Majorana fermion fields in 3+1 dimensions and explore the solutions of the Klein-Gordon Equation as well as the Dirac Equation. The properties of Dirac Field under Majorana and Weyl conditions have also been addressed. The effects of Lorentz Transformations on fermion fields have been described. We derive the expressions of various operators that can be formed using Dirac's γ-matrices. Fermionic fields can be classified on the basis of chirality or helicity. We differentiate between these critical parameters of the classification of fermions. We investigate how one can obtain Dirac and Majorana fermionic fields using the irreducible forms of Weyl fermionic fields. After expanding the solutions of Dirac Equation in the two-component notation, the behaviour of Fourier modes, in both non-relativistic and ultra-relativistic limits, have been examined. Finally, we shed light on the different discrete symmetries that the fermions exhibit and discuss the important CPT theorem. The equality of mass and inversion of charge for particles and antiparticles, which are the consequences of the CPT theorem, have been derived.

1 Klein-Gordon Equation and its Solutions

The Time Dependent Schrodinger Equation states

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi \tag{1}$$

Considering relativistic models and using Einstein's energy-momentum relation, we obtain

$$E^2 = \vec{p}^2 + m^2$$

$$E^2 - \vec{p}^2 = p^\mu p_\mu = m^2 \tag{2}$$

Where $p^\mu p_\mu = p_0^2 - \vec{p}^2$ in Minkowski spacetime coordinates. In order to obtain a relativistic quantum mechanical equation, we promote these variables to operators acting on ϕ.

$$p^\mu p_\mu \phi = m^2\phi$$

$$(i\hbar\partial^\mu)(i\hbar\partial_\mu)\phi = m^2\phi$$

$$-\hbar^2\Box\phi = m^2\phi$$

Where $\Box = \partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \overrightarrow{\nabla}^2$

Keeping $\hbar = 1$, we obtain

$$(\Box + m^2)\phi = 0 \tag{3}$$

This is the Klein-Gordon equation and it is invariant under Lorentz Transformations. The equation has plane wave solutions of the form $e^{\pm ikx}$.

These plane wave solutions form a complete basis and any general solution can be expressed using these plane wave basis.

Consider $\phi(x)$ to be such a general solution. Expanding $\phi(x)$ on the basis of plane waves,

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ikx} \tilde{\phi}(k) d^4k \tag{4}$$

Where $\phi(x)$ and $\tilde{\phi}(k)$ are Fourier transforms of each other and the constant has been introduced for later convenience. Since $\phi(x)$ is a solution of Klein-Gordon Equation,

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = 0 \tag{5}$$

Using (4), we obtain

$$\frac{1}{(2\pi)^{\frac{3}{2}}} (\partial^\mu \partial_\mu + m^2) \int e^{-ikx} \tilde{\phi}(k) d^4k = 0$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int (m^2 - k^2)\tilde{\phi}(k) e^{-ikx} = 0 \tag{6}$$

Unless $m^2 - k^2 = 0$, $\tilde{\phi}(k) = 0$. Hence, $\tilde{\phi}(k)$ has non-zero contribution only when the following condition is satisfied

$$k^2 = (k^0)^2 - \vec{k}^2 = m^2$$

$$\therefore \tilde{\phi}(k) = \delta(m^2 - k^2) a(k) \tag{7}$$

Using this expression of $\tilde{\phi}(k)$ in (6),

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ikx} \delta(m^2 - k^2) a(k) d^4k \tag{8}$$

We observe that when

$$k^2 = m^2$$

$$k^0 = \pm\sqrt{\vec{k}^2 + m^2} = \pm E_k \tag{9}$$

Where E_k is the energy of a body with magnitude of momentum $= k$.

Since $k^2 - m^2 = 0 = (k^0)^2 - E_k^2$,

$$\delta(m^2-k^2) = \delta((k^0)^2 - E_k^2)$$

$$= \tfrac{1}{2|k^0|}(\delta((k^0) - E_k) -(\ \delta\Big((k^0) + E_k\Big)$$

$$= \tfrac{1}{2E_k}(\delta((k^0) - E_k) -(\ \delta\Big((k^0) + E_k\Big) \tag{10}$$

Using (10) in (8),

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{-ik^0x^0+i\vec{k}\cdot\vec{x}}\frac{1}{2E_k}(\delta((k^0) - E_k) - (\ \delta\Big((k^0) + E_k\Big)a\Big(k^0,\vec{k}\Big)dk^0d^3k$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{2E_k}\Big(e^{-iE_kx^0+i\vec{k}\cdot\vec{x}}a\Big(E_k,\vec{k}\Big) + e^{iE_kx^0+i\vec{k}\cdot\vec{x}}a\Big(-E_k,\vec{k}\Big)\Big)\,d^3k$$

We change $\vec{k}$ to $-\vec{k}$ in the second term,

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{2k_0}\Big(e^{-ikx}a(k) + e^{ikx}a(-k)\Big)d^3k \tag{11}$$

For a real solution $\phi^*(x) = \phi(x)$,

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{2k_0}\Big(e^{ikx}a(k) + e^{-ikx}a(-k)\Big)d^3k$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{2k_0}\Big(e^{-ikx}a(k) + e^{ikx}a(-k)\Big)d^3k$$

Hence, we obtain

$$a^*(k) = a(-k) \text{ and } a^*(-k) = a(k)$$

$$\therefore \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{2k_0}\Big(e^{-ikx}a(k) + e^{ikx}a^*(k)\Big)d^3k \tag{12}$$

Where $k^0 = E_k = \sqrt{\vec{k}^2 + m^2} > 0$.

Since, k^0 is a constant, $a(k)$ is a function of $\vec{k}$. Using this property, we can write

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{1}{2k_0} \left(e^{-ikx} a(\vec{k}) + e^{ikx} a^*(\vec{k}) \right) d^3k \tag{13}$$

For quantum fields, $a(k)$ and $a^*(k)$ become operators, which are Hermitian conjugates of the other.

2 The Dirac Equation and its solutions

2.1 Dirac Equation

The negative energy solutions of the Klein-Gordon equation led to negative probability. Hence, Dirac proposed the following

$$E^2 - \vec{p}^2 = m^2$$

$$p^2 = p^\mu p_\mu = m^2 \tag{14}$$

Where (14) is a $n \times n$ matrix relation. We also consider the existence of four linearly dependent $n \times n$ matrices γ^μ ($\mu = 0,1,2,3$), such that

$$\gamma^\mu p_\mu \ is\ the\ square\ root\ of\ p^2. \tag{15}$$

Promoting the variables to operators, we obtain from equations (14)(9) and (15),

$$\gamma^\mu p_\mu \psi = m\psi$$

$$(\gamma^\mu p_\mu - m)\psi = 0 \tag{16}$$

(16) is the **Dirac Equation**.

Keeping $\hbar = 1$ in (16), we obtain

$$\begin{aligned}(\gamma^\mu p_\mu - m)\psi &= (i\gamma^\mu \partial_\mu - m)\psi = 0 \\ &= \left(i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \overrightarrow{\nabla} - m\right)\psi = 0\end{aligned}$$

Multiplying γ_0 from left, we obtain

$$(i\partial_0 + i\gamma_0 \cdot \vec{\gamma} - \gamma_0 m)\psi = 0$$

$$i\frac{\partial\psi}{\partial t} = \left(-i\gamma_0\vec{\gamma}\cdot\overrightarrow{\nabla} + \gamma_0 m\right)\psi \tag{17}$$

Where we have used $\gamma_0^2 = 1$. Since $-i\overrightarrow{\nabla}= \vec{p}$,

$$-i\vec{\gamma}\cdot\overrightarrow{\nabla}= \vec{\gamma}\cdot\vec{p} = \gamma^i p^i \tag{18}$$

Using (18) in (17), we obtain

$$i\frac{\partial\psi}{\partial t} = \gamma_0(\gamma^i p^i + m)\psi \tag{19}$$

On comparing (19) with the Time Dependent Schrodinger Equation, we obtain the following Hamiltonian

$$H = \gamma_0(\gamma^i p^i + m)\psi \tag{20}$$

We can also derive the Dirac Equation from the following Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Where $\bar{\psi} = \psi^\dagger\gamma_0$. In general **the Euler-Lagrange equation** is given by

$$\frac{\partial L}{\partial\phi(x)} - \partial_\mu\frac{\partial L}{\partial\left(\partial\phi(x)\right)} = 0 \tag{21}$$

Since $\frac{\partial L}{\partial\bar{\psi}} = i\gamma^\mu\partial_\mu\psi - m\psi = (i\gamma^\mu\partial_\mu - m)\psi$ and $\frac{\partial L}{\partial(\partial_\mu\bar{\psi})} = 0$, the Euler-Lagrange equation for this given Lagrangian gives

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

$(i\gamma^\mu\partial_\mu - m)\psi = 0$, which is the Dirac Equation.

γ^μ or the **Dirac Matrices** satisfy the following conditions –

$$[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{22}$$

$$[\gamma_0\gamma_\mu\gamma_0]_+ = \gamma_\mu^\dagger \tag{23}$$

These matrices need to satisfy the **Clifford Algebra**. Hence, there are infinite choices for Dirac Matrices. In **the Pauli-Dirac representation**, the explicit forms of the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{24}$$

Where $\mathbb{1}_{2\times 2}$ represents 2×2 **identity matrix** and σ_i are the **Pauli matrices**.

The explicit forms of the Dirac matrices specify the following Hermiticity properties –

$$\gamma^0 = \gamma_0^\dagger \text{ and } \gamma^i = -\gamma_i^\dagger \tag{25}$$

2.2 Solutions of the Dirac Equation

We consider a four-component wave function –

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \text{where } \psi_\alpha(x) = e^{-ipx}u_\alpha(p) \text{ and } \alpha = 1,2,3,4. \tag{26}$$

Here, $u_\alpha(p)$ are **the basis spinors** and $\Psi(x)$ is a solution of the Dirac Equation. Hence,

$$\begin{gathered}(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \\ \left(i\gamma^\mu\partial_\mu(e^{-ipx}) - m(e^{-ipx})\right)u(p) = 0 \\ (i\gamma^\mu(-ip_\mu) - m)u(p) = 0 \\ (\gamma^\mu p_\mu - m)u(p) = 0\end{gathered} \tag{27}$$

We confine the motion along the z-axis to simplify the problem, i.e. $p_1 = p_x = 0$ and $p_2 = p_y = 0$. Hence, (27) becomes

$$(\gamma^0 p_0 + \gamma^3 p_3 - m)u(p) = 0 \tag{28}$$

Using the Pauli Dirac representation of γ^μ matrices, we obtain

$$\begin{pmatrix} p_0 - m & 0 & p_3 & 0 \\ 0 & p_0 - m & 0 & -p_3 \\ -p_3 & 0 & -(p_0+m) & 0 \\ 0 & p_3 & 0 & -(p_0+m) \end{pmatrix} \begin{pmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{pmatrix} = 0 \tag{29}$$

Since a system of linear homogenous equations requires the determinant of the coefficient matrix to be zero, for a non-trivial solution,

$$\begin{vmatrix} p_0 - m & 0 & p_3 & 0 \\ 0 & p_0 - m & 0 & -p_3 \\ -p_3 & 0 & -(p_0+m) & 0 \\ 0 & p_3 & 0 & -(p_0+m) \end{vmatrix} = 0$$

$$(p_0 - m)[(p_0 - m)(p_0 + m)^2 - p_3^2(p_0 + m)] + p_3[p_3^3 + (p_0 - m)\big(-p_3(p_0 + m)\big)] = 0$$

$$(p_0^2 - m^2)^2 - 2p_3^2(p_0^2 - m^2) + p_3^4 = 0$$

$$(p_0^2 - p_3^2 - m^2) = 0$$

$$p_0^2 - p_3^2 - m^2 = 0 \tag{30}$$

Hence, we obtain a non-trivial plane wave solution of the Dirac equation for the energy values

$$p_0 = \pm E = \pm\sqrt{p_3^2 + m^2}$$

Before proceeding to determine the solutions the solutions of the Dirac equation, we consider

$$u(p) = \begin{pmatrix} \tilde{u}(p) \\ \tilde{v}(p) \end{pmatrix}$$

where $\tilde{u}(p) = \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}$ and $\tilde{v}(p) = \begin{pmatrix} v_1(p) \\ v_2(p) \end{pmatrix}$

for positive energy solutions, i.e.,

$$p_0 = E_+ = \sqrt{p_3^2 + m^2}$$

(29) becomes

$$\begin{pmatrix} (E_+ - m)\mathbb{1}_{2\times 2} & \sigma_3 p_3 \\ -\sigma_3 p_3 & -(E_+ + m)\mathbb{1}_{2\times 2} \end{pmatrix} \begin{pmatrix} \tilde{u}(p) \\ \tilde{v}(p) \end{pmatrix} = 0$$

$$(E_+ - m)\tilde{u}(p) + \sigma_3 p_3 \tilde{v}(p) = 0 \;\; and \;\; \sigma_3 p_3 \tilde{u}(p) + \;(E_+ + m)\tilde{v}(p) \tag{31}$$

Expressing $\tilde{v}(p)$ in terms of $\tilde{u}(p)$,

$$\tilde{v}(p) = -\frac{\sigma_3 p_3}{E_+ + m}\tilde{u}(p) \tag{32}$$

We continue similarly for negative energy solutions

$$p_0 = E_- = -\sqrt{p_3^2 + m^2}$$

and (29) becomes

$$(E_- - m)\tilde{u}(p) + \sigma_3 p_3 \tilde{v}(p) = 0 \;\; and \;\; \sigma_3 p_3 \tilde{u}(p) + \;(E_- + m)\tilde{v}(p) = 0 \tag{33}$$

Also (33) yields

$$\tilde{u}(p) = \frac{-\sigma_3 p_3}{E_- - m}\tilde{v}(p)$$

Therefore, $u_+(p) = \begin{pmatrix} \tilde{u}(p) \\ -\frac{\sigma_3 p_3}{E_+ + m}\tilde{u}(p) \end{pmatrix}$ and $u_-(p) = \begin{pmatrix} \frac{-\sigma_3 p_3}{E_- - m}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix}$

when $p_1 \neq p_2 \neq 0$, the solutions in the general form are

$$u_+(p) = \begin{pmatrix} \tilde{u}(p) \\ -\dfrac{\vec{\sigma}\cdot\vec{p}}{E_+ + m}\tilde{u}(p) \end{pmatrix}, \qquad u_-(p) = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_- - m}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix} \tag{34}$$

The sign changes since the index of p_i is raised to p^i.

After normalization (34) becomes

$$u_+(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \tilde{u}(p) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_+ + m}\tilde{u}(p) \end{pmatrix}, \qquad u_-(p) = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_- - m}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix} \tag{35}$$

However, this normalization does not work for massless spinors.

We adopt the following normalization for massless spinors –

$$u_+(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \tilde{u}(p) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_+}\tilde{u}(p) \end{pmatrix}, \qquad u_-(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_-}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix} \tag{36}$$

2.3 Real solutions of the Dirac equation

We need to choose a certain representation of γ-matrices to make the Dirac Equation real to find real solutions of the equation. Majorana found this representation which is as follows –

$$\begin{aligned} \gamma_M^0 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma_M^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} \\ \gamma_M^2 &= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \qquad \gamma_M^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \end{aligned} \tag{37}$$

Since σ^2 is imaginary and γ_m^1 and γ_m^3 have imaginary non-zero elements,

$$(\gamma_M^\mu)^* = -\gamma_M^\mu \tag{38}$$

These γ-matrices also satisfy the Clifford algebra. Hence according to Pauli's Fundamental Theorem, two sets of γ-matrices satisfying the Clifford algebra must be related by a similarity transformation. For γ-matrices Majorana representation the similarity transformation is

$$\gamma_M^\mu = S_M \gamma^\mu S_M^{-1} \tag{39}$$

where $S_M = \frac{1}{\sqrt{2}}\gamma^0(\mathbb{1}_{4\times 4} + \gamma^2)$.

If we use this particular set of γ_m matrices in the Dirac equation then the solutions found would be real. These solutions represent Majorana Fermions.

There is yet another kind of representation of the γ-matrices known as the Weyl representation. The similarity transformation between Pauli Dirac representation and Weyl representation is

$$\gamma_W^\mu = S_W \gamma^\mu S_W^{-1} \tag{40}$$

where $S_W = \frac{1}{\sqrt{2}}(\mathbb{1}_{4\times 4} + \gamma_5 \gamma^0)$.

The solutions found from the Dirac equation using γ_w matrices are used to represent the massless fermions or Weyl fermions. Let γ^μ and γ'^μ be the Dirac matrices in two different representations and they are related as follows

$$\gamma^\mu = U \gamma'^\mu U^{-1} \tag{41}$$

where U is the suitable unitary matrix.

Considering ψ and ψ' as the solutions of the Dirac equation in two different representations,

$$\psi = U\psi' \tag{42}$$

If ψ corresponds to the Majorana representation, then,

$$\begin{aligned} \psi' &= (\psi')^* \\ U^{-1}\psi &= (U^{-1}\psi)^* \\ \psi &= U(U^\dagger \psi)^* \\ \therefore \psi &= UU^T \psi^* \end{aligned} \tag{43}$$

Since $UU^\dagger = \mathbb{1}$ and U^T is also unitary. We keep

$$UU^T = \gamma_0 C \tag{44}$$

Hence, (42) becomes

$$\psi = \gamma_0 C \psi^* \tag{45}$$

Denoting $\gamma_0 C\psi^*$ as $\hat{\psi}$ for an arbitrary solution ψ, the following relation holds true for a Majorana fermion –

$$\psi = \hat{\psi}$$

We proceed to discuss the properties of the new matrix C in (43). Using (43), we can derive

$$\begin{aligned} C^{-1}\gamma_\mu C &= U^*U^\dagger\gamma_0\gamma_\mu\gamma_0 UU^T = U^*U^\dagger\gamma_\mu^\dagger UU^T \\ &= U^*(U^\dagger\gamma_\mu U)^\dagger U^T = U^*{\gamma'_\mu}^\dagger U^T = (U^*{\gamma'_\mu}^* U^\dagger)^T \end{aligned} \tag{46}$$

Where we have used (41) and (23) to derive (46). Now, if we proceed to use (38), we obtain

$$C^{-1}\gamma_\mu C = -(U\gamma'_\mu U^\dagger)^T = -\gamma_\mu^T \tag{47}$$

The above expression is the definition of matrix C, which does not use the Majorana representation. Moreover, since $\gamma_0 C = UU^T$ and UU^T is unitary,

$$\begin{aligned} UU^T(UU^T)^* &= \mathbb{1} \\ \gamma_0 C(\gamma_0 C)^* &= \mathbb{1} \\ C^{-1}\gamma_0 C(\gamma_0 C)^* &= C^{-1} \\ -\gamma_0^T(\gamma_0 C)^* &= C^{-1} \\ -\gamma_0^T(\gamma_0^* C^*) &= C^{-1} \\ -(\gamma_0^T)^2 C^* &= C^{-1} \\ -(\gamma_0)^2 C^* &= C^{-1} \\ -CC^* &= \mathbb{1} \end{aligned} \tag{48}$$

Where we have used the relations, $\gamma_0^\dagger = \gamma_0$ and $\gamma_0^2 = \mathbb{1}$, as well as (47). (48) can be expressed as

$$C^T = -C, \tag{49}$$

since C is unitary. Hence, ***C* is an antisymmetric matrix irrespective of the representation of the Dirac matrices**.

2.4 Fourier expansion of the solution

The general form of the Dirac Field is

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \sqrt{\frac{m}{(2\pi)^3 k_0}} \left(e^{-ik\cdot x} c(\vec{k},s) u_\alpha(k,s) + e^{ik\cdot x} d^\dagger(\vec{k},s) v_\alpha(k,s)\right) d^3k \quad (50)$$

where $c(\vec{k},s)$ and $d^+(\vec{k},s)$ are anti-community operators, and, $u_\alpha(k,s)$ and $v_\alpha(k,s)$ denote the positive and negative energy spinors in (35).

For a Majorana Fermion,

$$\psi_M(x) = \psi_M^*(x)$$

$\therefore$ We expand the solution using Fourier representation quite similar to the expression in (50).

$$\psi_M(x) = \sum_s \int (a_s(\vec{p}) u_s^M(p) e^{-ipx} + \ a_s^\dagger(\vec{p}) u_s^{M^*}(p) e^{ipx})\, d^3p \quad (51)$$

The two terms in this expression are conjugates of each other and hence, yields a real $\psi_M(x)$.

We represent the solution in an arbitrary notation using a suitable unitary matrix U.

The basis spinors u_s^M and $u_s^{M^*}$ transform as

$$u_s(p) = U u_s^M(p)$$

$$\text{and } u_s^*(p) = U u_s^{M^*}(p) = U\left(U^\dagger u_s(p)\right)^* = UU^T u_s^*(p) = \gamma_0 c u_s^*(p) = v_s(p)$$

Hence, (51) becomes

$$\psi(x) = \sum_s \int (a_s(\vec{p}) u(p) e^{-ipx} + \ a_s^\dagger(\vec{p}) v_s(p) e^{ipx})\, d^3p \quad (52)$$

We proceed to check if (46) remains invariant under Lorentz transformation.

A fermion field under infinitesimal Lorentz transformation

$$x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu \ and \ \psi'(x') = \exp(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu})\psi(x) \tag{53}$$

$$\text{where } \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{54}$$

Multiplying $\gamma_0 c$ and taking conjugates of both sides of (53), we obtain

$$\gamma_0 C\psi^*(x') = \hat{\psi}'(x') = \gamma_0 C \exp(\frac{i}{4}\omega^{\mu\nu}\sigma^*_{\mu\nu})\psi^*(x)$$

$$= \gamma_0 C \exp(\frac{i}{4}\omega^{\mu\nu}\sigma^*_{\mu\nu})(\gamma_0 C)^{-1}\gamma_0 c\psi^*(x)$$

$$= \gamma_0 C \exp(\frac{i}{4}\omega^{\mu\nu}\sigma^*_{\mu\nu})(\gamma_0 C)^{-1}\hat{\psi}(x) \tag{55}$$

We notice

$$\gamma^*_\mu = (\gamma^\dagger_\mu)^T = \gamma^T_0 \gamma^T_\mu \gamma^T_0 = -(\gamma_0 C)^{-1}\gamma_\mu(\gamma_0 C) \tag{56}$$

and,

$$\gamma_0 C\sigma^*_{\mu\nu}(\gamma_0 C)^{-1} = -\gamma_0 C\frac{i}{2}(\gamma^*_\mu\gamma^*_\nu - \gamma^*_\nu\gamma^*_\mu)(\gamma_0 C)^{-1}$$

$$= -\frac{i}{2}\{\gamma_0 C(\gamma^*_\mu\gamma^*_\nu)(\gamma_0 C)^{-1} - (\gamma_0 C)(\gamma^*_\nu\gamma^*_\mu)(\gamma_0 C)^{-1}$$

$$= -\frac{i}{2}\{(\gamma_0 c)\gamma^*_\mu(\gamma_0 C)^{-1}(\gamma_0 C)\gamma^*_\nu(\gamma_0 C)^{-1} - (\gamma_0 C)\gamma^*_\nu(\gamma_0 C)^{-1}(\gamma_0 C)\gamma^*_\mu(\gamma_0 C)^{-1}\} \tag{57}$$

Using (56) in (57), we obtain,

$$\gamma_0 C\sigma^*_{\mu\nu}(\gamma_0 C)^{-1} = -\frac{i}{2}\{(-\gamma_\mu)(-\gamma_\nu) - (-\gamma_\nu)(-\gamma_\mu)\}$$

$$\gamma_0 C\sigma^*_{\mu\nu}(\gamma_0 C)^{-1} = -\frac{i}{2}[\gamma_\mu, \gamma_\nu] = -\sigma_{\mu\nu} \tag{58}$$

Using (58) in (55), we obtain

$$\hat{\psi}'(x') = \exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\hat{\psi}(x) \tag{59}$$

Hence, **the reality or Majorana condition is invariant under Lorentz transformations**. Since both sides transform similarly under Lorentz transformations '$\hat{\psi}$' is called the **Lorentz-covariant conjugate** of ψ.

3 Helicity and Chirality

3.1 Helicity

The relative orientation of a particle's momentum and angular momentum can be used to define the 'handedness' of the particle. Hence, **helicity** is defined as

$$h_p \equiv 2\vec{J} \cdot \frac{\vec{p}}{|\vec{p}|} \tag{60}$$

Where $\vec{J}$ 0is the total angular momentum of the particle and $\vec{p}$ is the 3-momentum of the particle. Since the orbital part of the angular momentum is perpendicular to the momentum of the particle, the only contribution is from the spin angular momentum. Hence, (60) becomes

$$h_p \equiv 2\vec{S} \cdot \frac{\vec{p}}{|\vec{p}|} \tag{61}$$

Where $\vec{S}$ is the spin operator and is defined as

$$S_i = \tfrac{1}{2}\tilde{\alpha}_i \text{ where } \tilde{\alpha}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{62}$$

We note that

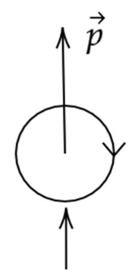
$$h_p^2 = 4\left(\vec{S} \cdot \frac{\vec{p}}{|\vec{p}|}\right)\left(\vec{S} \cdot \frac{\vec{p}}{|\vec{p}|}\right) = \mathbb{1} \tag{63}$$

Hence, the eigenvalues of h_p are ± 1 only. A fermion satisfying the Dirac equation has a helicity

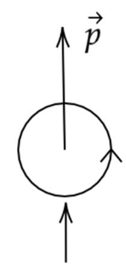
$$h_p \equiv \vec{\alpha} \cdot \frac{\vec{p}}{|\vec{p}|} \, , \tag{64}$$

where $\alpha_i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk}$ and σ^{jk} are the space-space components of $\sigma^{\mu\nu}$ in (54) .

Eigenstates of h_p with eigenvalue -1 are called ‘left-handed’ and eigenstates with eigenvalue +1 are called ‘right-handed’. Considering the spin to be caused due to circular motion,



Left-handed



Right-handed

We observe that

$$[\alpha_i, H] = -2i\epsilon_{ijk}\alpha_j p_k \tag{65}$$

$$\text{and } [p_i, H] = [p_i, \vec{\alpha} \cdot \vec{p} + \beta m] = 0 \tag{66}$$

$$\therefore [\alpha_i p_i, H] = [\alpha_i, H]p_i = -2i\epsilon_{ijk}\alpha_j p_k p_i = 0 \tag{67}$$

Hence, h_p commutes with the Dirac Hamiltonian and does not change with time for a free particle. Moreover, h_p remains invariant under rotation due to the inner product present in the definition. However, h_p does not remain invariant under boosts since direction of momentum changes in different frames whereas spin does not. Hence, helicity depends on the frame of the observer. This problem only exists for massive fermions. For massless spinors , helicity remains invariant under boosts since they travel at the speed of light in every frame.

3.2 Chirality

From (36), we note that

$$u_+(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \tilde{u}(p) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_+}\tilde{u}(p) \end{pmatrix} \; and \; u_-(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_-}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix}$$

We consider $u_+(p)$ as $u(p)$ and $u_-(-p)$ as $v(p)$.

Then,

$$u(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \tilde{u}(p) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_+}\tilde{u}(p) \end{pmatrix} \; and \; v(p) = \sqrt{\frac{E}{2}} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_-}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix}$$

$$u(p) = \sqrt{\frac{|\vec{p}|}{2}} \begin{pmatrix} \tilde{u}(p) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\tilde{u}(p) \end{pmatrix} \; and \; v(p) = \sqrt{\frac{|\vec{p}|}{2}} \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\tilde{v}(p) \\ \tilde{v}(p) \end{pmatrix}, \tag{68}$$

since for a massless spinor, $E^2 = p^2$.

We introduce the new Dirac matrix –

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{69}$$

It has the following properties –

$$\gamma_5^\dagger = \gamma_5 \text{ and } (\gamma_5)^2 = \mathbb{1} \tag{70}$$

γ_5 also commutes with Dirac Hamiltonian for massless fermions. Hence, $u(p)$ and $v(p)$ are also eigenstates of γ_5. Since $(\gamma_5)^2 = \mathbb{1}$, γ_5 has ± 1 as its eigenvalues.

Spinors with $+1$ eigenvalue satisfy

$$\gamma_5 u_R(p) = u_R(p) \text{ and } \gamma_5 v_R(p) = v_R(p) \tag{71}$$

Such spinors are called **positive-chiral** spinors.

Similarly, spinors corresponding to -1 eigenvalue satisfy

$$\gamma_5 u_L(p) = -u_L(p) \text{ and } \gamma_5 v_L(p) = -v_L(p) \tag{72}$$

These spinors are called **negative-chiral** spinors.

We proceed to define the following operators

$$P_R = \tfrac{1}{2}(\mathbb{1} + \gamma_5) \text{ and } P_L = \tfrac{1}{2}(\mathbb{1} - \gamma_5), \tag{73}$$

which act as **Projection Operators.** A general ψ can be projected to its positive and negative chiral states, $\psi = \psi_L + \psi_R$, where $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$.

We observe that

$$\begin{aligned}
[\gamma_5, \sigma_{\mu\nu}] = \gamma_5\sigma_{\mu\nu} - \sigma_{\mu\nu}\gamma_5 &= \frac{i}{2}\{\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_5\} \\
&= \frac{i}{2}\{\gamma_5(\gamma_\mu\gamma_\nu) - \gamma_5(\gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu)\gamma_5 + (\gamma_\nu\gamma_\mu)\gamma_5\} \\
&= \frac{i}{2}\{\gamma_5(\gamma_\mu\gamma_\nu) - \gamma_5(\gamma_\nu\gamma_\mu) - \gamma_5(\gamma_\mu\gamma_\nu) + \gamma_5(\gamma_\nu\gamma_\mu)\} = 0,
\end{aligned} \tag{74}$$

since

$$[\gamma_5, \gamma_\mu]_+ = 0 \tag{75}$$

(74) suggests that chiral projections (for massless spinors) are Lorentz covariant. However, chiral projections fail for massive particles, since γ_5 does not commute with the mass term present in the massive Dirac Hamiltonian. Hence, none of the above properties can describe the "handedness" of a massive fermion.

4 Weyl Fermions

4.1 Irreducible fermion fields

Lorentz algebra is isomorphic to the direct sum of two angular momentum algebras, i.e.

$$SO(3{,}1) \simeq SO(3) \oplus SO(3) \simeq SU(2) \oplus SU(2) \tag{76}$$

The operator that implements finite transformations on wave functions is given by

$$D^{(j_A,j_B)}(\Lambda) = e^{-i\left(\theta_A^i A_i^{j_A} + \theta_B^i B_i^{j_B}\right)}$$

$$= e^{-i\left(\theta^i J_i^{(J_A,J_B)} + \delta^i K_i^{(K_A,K_B)}\right)}, \tag{77}$$

where θ^i and δ^i are finite parameters of rotation and boost. They are given by

$$\theta^i = \frac{1}{2}(\theta_A^i + \theta_B^i) \text{ and } \delta^i = \frac{i}{2}(\theta_A^i - \theta_B^i) \tag{78}$$

Here

$$A_i = \tfrac{1}{2}(J_i + iK_i) \text{ and } B_i = \tfrac{1}{2}(J_i - K_i) \tag{79}$$

and, j_A and j_B are the eigenvalues of the **Casimir operators $\vec{A}^2$ and $\vec{B}^2$**.

The eigenvalues j_A and j_B can take the following range of values –

$$\begin{aligned} j_A &= 0, \frac{1}{2}, 1, \frac{3}{2} ... \\ j_B &= 0, \frac{1}{2}, 1, \frac{3}{2} ... \end{aligned} \tag{80}$$

The **dimensionality of a representation** labelled by (j_A, j_B) is given by

$$Dimensionality\ \left(D^{(j_A,j_B)}\right) = (2j_A + 1)(2j_B + 1) \tag{81}$$

and the values of spin of a representation can lie between $|j_A - j_B|$ and $j_A + j_B$, i.e.

$$j = |j_A - j_B|, ..., j_A + j_B \tag{82}$$

Keeping $j_A = \frac{1}{2}$ and $j_B = 0$ in (81) and (82), we obtain

$$D^{(\frac{1}{2},0)}, dimensionality = 2\ and\ j = \frac{1}{2}$$

Similarly, keeping $j_A = 0$ and $j_B = \frac{1}{2}$ in (81) and (82), we obtain

$$D^{(0,\frac{1}{2})}, dimensionality = 2\ and\ j = \frac{1}{2}$$

They represent the left-handed and right-handed Weyl fermions and are irreducible. However, one can construct reducible fermion field using the above fields.

4.2 Relation between helicity and chirality for massless particles

Let ψ satisfy the massless Dirac equation. Then,

$$(\gamma^0|\vec{p}| - \vec{\gamma}\cdot\vec{p})\psi = 0 \tag{83}$$

Multiplying both sides by γ^0 and dividing by $|\vec{p}|$,

$$\left(\mathbb{1} - \gamma_0\vec{\gamma}\cdot\frac{\vec{p}}{|\vec{p}|}\right)\psi = 0 \tag{84}$$

We proceed to find $\gamma_0\gamma_i$. We begin with $[\gamma_\mu, \gamma_\nu]$, and $[\gamma_\mu, \gamma_\nu]_+$,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{85}$$

$$\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu = -2i\sigma_{\mu\nu} \tag{86}$$

Substituting (lxxxv) and (lxxxvi), we obtain

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu} \tag{87}$$

We note that

$$\epsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu}\gamma_5 = 2i\sigma^{\lambda\rho} \tag{88}$$

$$\therefore \frac{i}{2}\epsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu} = \sigma^{\lambda\rho}\gamma_5 \tag{89}$$

Multiplying (87) with γ_5,

$$\gamma_\mu\gamma_\nu\gamma_5 = g_{\mu\nu}\gamma_5 - i\sigma_{\mu\nu}\gamma_5 \tag{90}$$

Using (89) in (90),

$$\gamma_\mu \gamma_\nu \gamma_5 = g_{\mu\nu} \gamma_5 - \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \sigma^{\mu\nu} \tag{91}$$

On considering $\mu = 0$ (time) and $(\nu, \lambda, \rho) = (i, j, k)$ (space), (91) becomes

$$\gamma_0 \gamma_i \gamma_5 = g_{0i} \gamma_5 - \frac{1}{2} \epsilon_{0ijk} \sigma^{jk} \tag{92}$$

According to the adopted convention,

$$\epsilon_{0ijk} = -\epsilon_{ijk} \tag{93}$$

and we further observe that

$$\gamma_0 \gamma_i = \gamma_i^\dagger \gamma_0^{-1} \tag{94}$$

$$\begin{aligned} \text{and } \gamma_0 \gamma_i + \gamma_i \gamma_0 &= 2 g_{0i} \\ \gamma_i^\dagger \gamma_0^{-1} + \gamma_i \gamma_0 &= 2 g_{0i} \\ (-\gamma_0^{-1} + \gamma_0) \gamma_i &= 2 g_{0i} \\ \therefore g_{0i} &= 0, \end{aligned} \tag{95}$$

since $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$.

Using (93) and (95) in (92),

$$\begin{aligned} \gamma_0 \gamma_i \gamma_5 &= \frac{1}{2} \epsilon_{ijk} \sigma^{jk} \\ \gamma_0 \gamma_i \gamma_5 &= \alpha_i \\ \gamma_0 \gamma_i &= \gamma_5 \alpha_i \end{aligned} \tag{96}$$

Using (96) in (84), we obtain

$$\left(\mathbb{1} - \gamma_5 \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|} \right) \psi = 0 \tag{97}$$

On multiplying (97) with γ_5,

$$\gamma_5 \psi = \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|} \psi \tag{98}$$

Hence, **helicity and chirality is same for massless or Weyl spinors.**

4.3 Fourier expansion of the solutions

Let $\psi_{W,L}$ be a left-chiral Weyl fermion field.

Then,

$$P_L \psi_{W,L} = \psi_{W,L} \; and \; P_R \psi_{W,L} = 0 \tag{99}$$

Mimicking (50) and using the properties of Weyl fields, we expand $\psi_{W,L}$ as

$$\psi_{W,L}(x) = \int (a(p) u_L(p) \; e^{-ip\cdot x} + \hat{a}^\dagger(p) v_L(p) \; e^{ip\cdot x}) \; d^3p \tag{100}$$

Since we have chosen one particular chirality, there is no sum over different spin values.

We proceed to find the explicit forms of $a(p)$ and $\hat{a}^\dagger(p)$. Considering the value of the integral to be I_k and inverting the Fourier transformation, we obtain

$$\begin{aligned} a(k) &= \frac{1}{(2\pi)^3 I_k N_k} \int e^{ik\cdot x} u_L^\dagger(K) \psi_{W,L}(x) d^3x \\ \text{and } \hat{a}^\dagger(k) &= \frac{1}{(2\pi)^3 I_k N_k} \int e^{-ik\cdot x} v_L^\dagger(K) \psi_{W,L}(x) d^3x \end{aligned} \tag{101}$$

where we have used the following relations –

$$\begin{gathered} u_s^\dagger(\vec{p}) u_{s'}(\vec{p}) = N_P \delta_{ss'} \; , v_s^\dagger(\vec{p}) v_{s'}(\vec{p}) = N_P \delta_{ss'} \\ \text{and } u_s^\dagger(p) v_{s'}(-\vec{p}) = \; v_s^\dagger(p) u_{s'}(-\vec{p}) = 0 \end{gathered} \tag{102}$$

From (101), we obtain

$$a^\dagger(k) = \frac{1}{(2\pi)^3 I_k N_k} \int e^{-ik\cdot x} \psi_{W,L}^\dagger(x) u_L(k) d^3x \tag{103}$$

A general $\psi(x)$ satisfying the Dirac equation also satisfies the following relation –

$$[\ \psi(x), J_{\mu\nu}] = \Big(i(x_\mu d_\nu - x_\nu d_\mu) + \frac{1}{2}\sigma_{\mu\nu}\Big)\,\psi(x) \tag{104}$$

The first term on the right-hand side arises due to the orbital angular momentum. Since, orbital angular momentum does not contribute to helicity and taking spatial components of $J_{\mu\nu}$,

$$[\psi(x), h_p] = \frac{1}{2}\sigma_{\mu\nu}\psi(x) = \frac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\psi(x) \tag{105}$$

Hermitian conjugate of the above relation gives

$$[\psi^\dagger(x), h_p] = -\psi^\dagger(x)\frac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|} \tag{106}$$

The negative sign arises due to the Hermitian conjugate of σ_{jk} present in α_i, i.e.

$$\sigma_{jk} = [\gamma_j, \gamma_k] = \gamma_j\gamma_k - \gamma_k\gamma_j$$

$$\sigma^\dagger_{jk} = (\gamma_j\gamma_k)^\dagger - (\gamma_k\gamma_j)^\dagger = \gamma^\dagger_k\gamma^\dagger_j - \gamma^\dagger_j\gamma^\dagger_k$$

$$= \gamma_k\gamma_j - \gamma_j\gamma_k = -\sigma_{jk}$$

(106) is true for Weyl fields $\psi_{W,L}(x)$. We continue to find

$$[a^\dagger(k), h_k] = \frac{1}{(2\pi)^3 I_k N_k}\int e^{-ik\cdot x}[\psi_{W,L}(x), h_k]u_L(k)d^3x \tag{107}$$

Using (106) in (107), we obtain,

$$[a^\dagger(k), h_k] = \frac{-1}{(2\pi)^3 I_k N_k}\int e^{-ik\cdot x}\psi^\dagger_{W,L}(x)\frac{\vec{\alpha}\cdot\vec{k}}{|\vec{k}|}u_L(k)\ d^3x \tag{108}$$

Since, we are considering a massless field, using (98),

$$[a^\dagger(k), h_k] = \frac{-1}{(2\pi)^3 I_k N_k}\int e^{-ik\cdot x}\psi^\dagger_{W,L}(x)\gamma^\dagger_5 u_L(k)\ d^3x \tag{109}$$

and since $\gamma_5\psi_{W,L}(x) = -\psi_{W,L}(x)$ as $\psi_{W,L}$ is a left chiral Weyl field, (109) becomes

$$[a^\dagger(k), h_k] = \frac{1}{(2\pi)^3 I_k N_k} \int e^{-ik\cdot x} \psi^\dagger_{W,L}(x) u_L(k)\ d^3x = a^\dagger(k) \tag{110}$$

Hence, (110) can be written as

$$a^\dagger(k) h_k - h_k a^\dagger(k) = a^\dagger(k) \tag{111}$$

Applying both sides of (111) to the vacuum state where no momentum is present, we obtain,

$$a^\dagger(k) h_k |0\rangle - h_k a^\dagger(k)|0\rangle = a^\dagger(k)|0\rangle$$

The first term reduces to zero since $k = 0$ at ground state,

$$h_k a^\dagger(k)|0\rangle = -a^\dagger(k)|0\rangle \tag{112}$$

We observe from (112) that helicity of state $a^\dagger(k)|0\rangle$ is -1. Similarly, it can be shown that state $\hat{a}(k)|0\rangle$ has helicity +1. Hence, the Weyl field operator $(\psi_{W,L}(x))$ annihilates a negative helicity particle and creates a positive helicity antiparticle.

5 Majorana Fermions from Weyl Fermions

As mentioned earlier, any fermion field can be represented using the two irreducible representations of Weyl fermion fields. We proceed to construct a Majorana fermion field using Weyl fields.

Since a Majorana fermion is massive, it should have both left and right chiral components. Chirality is not conserved for massive fermions. Hence, both left chiral and right chiral. Weyl fields are required for a Majorana field. It is also necessary that the Weyl fields satisfy the Majorana condition.

We consider a left chiral Weyl field ϕ_L. We find that

$$P_R \phi_L = 0$$

$$(\mathbb{1} + \gamma_5)\phi_L = 0 \tag{113}$$

Taking conjugates of both sides of (113) and multiplying by $\gamma_0 c$ to the left, we obtain

$$\gamma_0 c\ (\mathbb{1} + \gamma_5^*)\phi_L^* = 0$$

$$\gamma_0 c\ (\mathbb{1} + \gamma_5^T)\phi_L^* = 0 \tag{114}$$

Using (47) and (69), we find

$$C\gamma_5 C = iC^{-1}(\gamma^0\gamma^1\gamma^2\gamma^3)C$$

$$= i(C^{-1}\gamma^0 C)(C^{-1}\gamma^1 C)(C^{-1}\gamma^2 C)(C^{-1}\gamma^3 C)$$

$$= i(-\gamma^0)^T(-\gamma^1)^T(-\gamma^2)^T(-\gamma^3)^T$$

$$\therefore C^{-1}\gamma_5 C = i\gamma_0^T\gamma_1^T\gamma_2^T\gamma_3^T \tag{115}$$

$$C^{-1}\gamma_5 C = i(\gamma^3\gamma^2\gamma^1\gamma^0)^T$$

$$C^{-1}\gamma_5 C = i(\gamma_5^{-1})^T$$

$$C^{-1}\gamma_5 C = i\gamma_5^T$$

$$\therefore \gamma_5 C = C\gamma_5^T \tag{116}$$

Using (116) in (114), we get

$$(\gamma_0 C + \gamma_0\gamma_5 C)\phi_L^* = 0$$

$$\gamma_0(\mathbb{1} + \gamma_5)C\phi_L^* = 0$$

$$(\mathbb{1} - \gamma_5)\gamma_0 C\phi_L^* = 0$$

$$(\mathbb{1} - \gamma_5)\hat{\phi}_L^* = 0 \tag{117}$$

(117) states that if ϕ_L is a left-chiral Weyl field, then, $\hat{\phi}_L^* = \gamma_0 C\phi_L^*$ is a right-chiral Weyl field. Hence, ϕ_L and $\hat{\phi}_L^*$ can be used to construct a Majorana fermion field, since both of the Weyl fields have opposite chirality and also complex conjugate of each other, which makes the resultant Majorana field real.

On observing the Dirac Lagrangian, we find that the mass term is of the form $\bar{\psi}\psi$. Its chiral projection can be written as $\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L$. Moreover, a term of the form $\phi_L^T C^{-1}\phi_L$ of a specific chirality cannot be used to express mass since this term is not Hermitian. The Lagrangian should have the Hermitian conjugate of the term, which is of the form $\hat{\phi}_L C^{-1} \hat{\phi}_L$ and $\hat{\phi}_L$ is a right chiral Weyl field. Hence, both kinds of chirality are required to express mass. Hence, a Weyl field with only one chirality does not have any mass.

6 Dirac fermions from Weyl fermions

Dirac fermions are massive and hence, require Weyl fields of both chirality. However, these fermions need not satisfy the reality condition or the Majorana condition. Hence, a Dirac field can be defined as

$$\psi(x) = \phi_1(x) + \hat{\phi}_2(x), \tag{118}$$

Where $\phi_1(x)$ and $\phi_2(x)$ are two independent left-chiral Weyl fields and $\hat{\phi}_2(x)$ is the Lorentz covariant conjugate of $\phi_2(x)$. Thus, $\hat{\phi}_2(x)$ is a right-chiral Weyl field. Hence, the Dirac field is the most general solution of the Dirac equation. Imposing the reality condition gives the Majorana field, whereas imposing the condition of the requirement of a single chirality yields a Weyl field.

Moreover, one cannot impose both the conditions simultaneously. We proceed to prove this below –

The γ-matrices are purely imaginary in the Majorana representation. Hence, γ_5 is purely imaginary in this representation. Let us consider a Weyl field ϕ_W, which is also a Majorana field. Hence, $\phi_W$is a real field. Since $\phi_W$is a Majorana field, it should follow the below relation with γ_5^M in the Majorana representation –

$$\gamma_5^M \phi_W = \pm \phi_W \tag{119}$$

However, as stated earlier, γ_5^M is purely imaginary and hence, requires ϕ_W to be imaginary for real eigenvalues. This requirement contradicts our assumption of ϕ_W being a real Majorana field. Hence, a Weyl field of a specific chirality cannot be a Majorana field simultaneously.

7 Two-component representation of fermions

7.1 Weyl Fermions

For Weyl fields, the four-component solution of the Dirac equation only has two independent components. This can easily be obtained if we use the Weyl representation of the Dirac matrices. Using (40), we obtain the following Dirac matrices in the Weyl representation

$$\breve{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \breve{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{120}$$

where $\breve{\gamma}$ denotes the Dirac matrices in the Weyl representation. We also obtain

$$\breve{\gamma}_5 = i\breve{\gamma}^0\breve{\gamma}^1\breve{\gamma}^2\breve{\gamma}^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{121}$$

Hence, the projection operators P_L and P_R in this representation become

$$\breve{P}_L = (\mathbb{1} - \breve{\gamma}_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \breve{P}_R = (\mathbb{1} + \breve{\gamma}_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{122}$$

We consider ϕ_L, a left chiral Weyl field. In the Weyl representation –

$$\begin{aligned} \breve{P}_R\phi_L &= 0 \\ \therefore (\mathbb{1} + \breve{\gamma}_5)\phi_L &= 0 \end{aligned} \tag{123}$$

On observing the explicit forms of $\breve{P}_R$ in (122), we note that its action on ϕ_L would reduce the lower two components to zero. Hence, we ignore the lower components that reduce to zero in the two-component notation. In the case of a right chiral field, the top two components vanish.

We extend this notation to include the Lagrangian and the equations of motion. The Lagrangian of the Dirac equation for a particle with no mass is

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi \\ &= i\psi^\dagger(\gamma^0\gamma^0\partial_0 + \gamma^0\gamma^i\partial_i)\psi \\ &= i\psi^\dagger(\partial_0 + \alpha^i\partial_i)\psi, \end{aligned} \tag{124}$$

where $\alpha^i = \gamma^0\gamma^i$. We consider an arbitrary Weyl field $\breve{\phi}$ such that

$$\breve{\phi} = \begin{pmatrix} p_t \\ p_b \end{pmatrix}, \tag{125}$$

where p_t and p_b are the two-component fields of $\breve{\phi}$ in the chiral representation. In the chiral representation, the explicit form of α^i is

$$\breve{\alpha}^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{126}$$

Since the Lagrangian holds true in any representation, we keep $\breve{\phi}$ instead of ψ and $\breve{\alpha}^i$ in (124), we obtain

$$\mathcal{L} = ip_t^\dagger(\partial_0 - \sigma^i\partial_i)p_t + ip_b^\dagger(\partial_0 - \sigma^i\partial_i)p_b \tag{127}$$

If $\breve{\phi}$ is a right chiral field, then $p_t = 0$ as seen in (123). Hence, (127) reduces to

$$\mathcal{L} = ip_b^\dagger(\partial_0 - \sigma^i\partial_i)p_b \tag{128}$$

$$\mathcal{L} = ip_b^\dagger\sigma^\mu\partial_\mu p_b, \tag{129}$$

$$\text{where } \sigma^\mu = (\mathbb{1}, \vec{\sigma}) \tag{130}$$

Similarly, for a Left chiral field

$$\mathcal{L} = ip_t^\dagger\overline{\sigma}^\mu\partial_\mu p_t, \tag{131}$$

$$\text{where } \overline{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma}) \tag{132}$$

Thus, we can write the Lagrangian in two-component notation.

We can also derive the two-component notation of the massless Dirac equation from (129) and (131). We consider (129)

$$\begin{aligned} \mathcal{L} &= ip_b^\dagger\sigma^\mu\partial_\mu p_b \\ \mathcal{L} &= ip_b^\dagger\breve{\gamma}_0\breve{\gamma}_0\sigma^\mu\partial_\mu p_b \\ \mathcal{L} &= i\bar{p}_b\breve{\gamma}_0\sigma^\mu\partial_\mu p_b \end{aligned} \tag{133}$$

We consider the Euler-Lagrange equation to find the Dirac equation in two-component notation –

$$\frac{\partial \mathcal{L}}{\partial \bar{p}_b} - \frac{\partial \mathcal{L}}{\partial(\partial \bar{p}_b)} = 0 \tag{134}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{p}_b} = i\breve{\gamma}_0 \sigma^\mu \partial_\mu p_b \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial \bar{p}_b)} = 0$$

Hence, we obtain

$$\breve{\gamma}^0 \sigma^\mu \partial_\mu p_b = 0$$

$$\breve{\gamma}^0 (\breve{\gamma}^0 \breve{\gamma}^0 \partial_0 + \breve{\gamma}^0 \breve{\gamma}^i \partial_i) p_b = 0$$

$$(\breve{\gamma}^0 \breve{\gamma}^0 \breve{\gamma}^0 \partial_0 + \breve{\gamma}^0 \breve{\gamma}^0 \breve{\gamma}^i \partial_i p_b) = 0$$

$$(\breve{\gamma}^0 \partial_0 + \breve{\gamma}^0 \breve{\gamma}^0 \breve{\gamma}^i \partial_i) p_b = 0 \tag{135}$$

Multiplying $\breve{\gamma}^0$ to the right side,

$$(\breve{\gamma}^0 \breve{\gamma}^0 \partial_0 + \breve{\gamma}^0 \breve{\gamma}^0 \breve{\gamma}^i \breve{\gamma}^0 \partial_i) p_b = 0$$

$$(\partial_0 + \breve{\gamma}^0 \breve{\gamma}^{i\dagger} \partial_i) p_b = 0$$

$$(\partial_0 - \breve{\gamma}^0 \breve{\gamma}^i \partial_i) p_b = 0$$

$$\therefore \overline{\sigma}^\mu \partial_\mu p_b = 0 \tag{136}$$

Similarly, we can obtain the other Dirac equation from (131), which is given by

$$\sigma^\mu \partial_\mu p_t = 0 \tag{137}$$

Thus, the Dirac equation for massless fermions can be represented in two-component notation.

In the chiral representation, the $\sigma^{\mu\nu}$ matrix in (54) becomes

$$
\begin{aligned}
\breve{\sigma}^{0k} &= \frac{i}{2}[\breve{\gamma}^0, \breve{\gamma}^k] \\
&= \frac{i}{2}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \\
&= \begin{pmatrix} i\sigma^k & 0 \\ 0 & -i\sigma^k \end{pmatrix}
\end{aligned} \tag{138}
$$

$$
\begin{aligned}
\breve{\sigma}^{ij} &= \frac{i}{2}[\breve{\gamma}^i, \breve{\gamma}^j] \\
&= \frac{i}{2}\left\{\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right\} \\
&= \frac{i}{2}\begin{pmatrix} \sigma^j\sigma^i - \sigma^i\sigma^j & 0 \\ 0 & \sigma^j\sigma^i - \sigma^i\sigma^j \end{pmatrix} \\
&= \frac{i}{2}\begin{pmatrix} -2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -2i\epsilon_{ijk}\sigma_k \end{pmatrix} \\
&= \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}
\end{aligned} \tag{139}
$$

In this representation, α^i matrices in (64) reduces to $\sigma^{\mu\nu}$ matrices while acting on left-chiral or right-chiral Weyl fields.

7.2 Majorana Fermions

A Majorana field needs four components in the Majorana representation. However, it can be represented using two components if one allows for complex components. From (39) and (40), we note that

$$\gamma^\mu_W = S_W \gamma^\mu_D S_W^{-1},$$

$$\text{where } S_W = \frac{1}{\sqrt{2}}(\mathbb{1} + \gamma_5\gamma^0)$$

$$\text{and } \gamma^\mu_M = S_M \gamma^\mu_D S_M^{-1},$$

$$\text{where } S_M = \frac{1}{\sqrt{2}}\gamma^0(\mathbb{1} + \gamma^2).$$

(γ^μ_D represent the Dirac matrices in the Pauli-Dirac representation) and obtain

$$S_M S_W^{-1} \gamma_W^\mu S_W S_M^{-1} = \gamma_M^\mu \tag{140}$$

Keeping $S_W S_M^{-1} = U'$, we obtain the relation between the chiral and Majorana representation. Using the explicit forms of S_W and S_M from (39) and (40), we find

$$U' = \frac{1}{2}\begin{pmatrix} 1 & -i & 1 & i \\ i & 1 & -i & 1 \\ 1 & i & -1 & i \\ -i & 1 & -i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\sigma^2 & 1-\sigma^2 \\ 1-\sigma^2 & -1-\sigma^2 \end{pmatrix} \tag{141}$$

Let $\widetilde{\Psi}_M$ be a Majorana field with components $\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3$ and $\tilde{\psi}_4$. We find Ψ', which is the chiral representation of $\widetilde{\Psi}_M$. Using (141)

$$\Psi' = U\text{'}\widetilde{\Psi}_M$$

$$\begin{aligned} \Psi' &= \frac{1}{2}\begin{pmatrix} 1 & -i & 1 & i \\ i & 1 & -i & 1 \\ 1 & i & -1 & i \\ -i & 1 & -i & -1 \end{pmatrix}\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \\ \tilde{\psi}_4 \end{pmatrix} \\ \therefore\ \Psi' &= \frac{1}{2}\begin{pmatrix} \tilde{\psi}_1 + \tilde{\psi}_3 + i(-\tilde{\psi}_2 + \tilde{\psi}_4) \\ \tilde{\psi}_2 + \tilde{\psi}_4 + i(\tilde{\psi}_1 - \tilde{\psi}_3) \\ \tilde{\psi}_1 - \tilde{\psi}_3 + i(\tilde{\psi}_2 + \tilde{\psi}_4) \\ \tilde{\psi}_2 - \tilde{\psi}_4 + i(-\tilde{\psi}_1 - \tilde{\psi}_3 \end{pmatrix} \end{aligned} \tag{142}$$

One may notice that all information about the components of the Majorana field can be obtained from the top two components of Ψ'. We proceed to calculate each component of the Majorana field from (142). We obtain

$$\begin{aligned} \tilde{\psi}_1 &= \mathfrak{Re}(\psi_1') + \mathfrak{Im}(\psi_2') \\ \tilde{\psi}_2 &= \mathfrak{Re}(\psi_2') - \mathfrak{Im}(\psi_1') \\ \tilde{\psi}_3 &= \mathfrak{Re}(\psi_1') - \mathfrak{Im}(\psi_2') \\ \tilde{\psi}_4 &= \mathfrak{Re}(\psi_2') + \mathfrak{Im}(\psi_1'), \end{aligned} \tag{143}$$

where ψ_1' and ψ_2' are the top two components of Ψ'.

We continue to calculate $\breve{\gamma}^0 \breve{C}$ in the chiral representation. Using (44), we obtain

$$\breve{\gamma}^0 \breve{C} = U' U'^T = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \tag{144}$$

Let Ψ', the Majorana field in chiral representation be considered such that

$$\Psi' = \begin{pmatrix} q_t \\ q_b \end{pmatrix} \tag{145}$$

Such a Majorana field requires

$$\begin{aligned} \Psi' = \widehat{\Psi}' &= \breve{\gamma}^0 \breve{C} (\Psi')^* \\ \begin{pmatrix} q_t \\ q_b \end{pmatrix} &= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} q_t^* \\ q_b^* \end{pmatrix} \end{aligned} \tag{146}$$

From (146), we obtain

$$q_t = \sigma^2 q_b^* \text{ and } q_b = -\sigma^2 q_t^* \tag{147}$$

Hence, (145) can be written as

$$\Psi' = \begin{pmatrix} q_t \\ -\sigma^2 q_t^* \end{pmatrix} \tag{148}$$

Thus, the top two components (q_t) is sufficient to represent a Majorana field in the chiral representation.

We can extend this two-component notation to the Lagrangian and the equation of motion. The Lagrangian of a Majorana field in the four-component notation is

$$\mathcal{L} = \frac{1}{2} (i \overline{\Psi} \gamma^\mu \partial_\mu \Psi - m \overline{\Psi} \Psi) \tag{149}$$

The factor of $\frac{1}{2}$ ensures a consistent normalization for self-conjugate field (Majorana fields).

We proceed to derive the Lagrangian in the chiral representation using (120) and (121). The term with the derivatives within the parentheses is the same as the Lagrangian in (124). Hence, we are going to follow a similar procedure to find the two-component notation. We use (147) in (149) after substituting Ψ in (149) with q_t. We obtain for the derivative term within the parentheses –

$$i\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi = iq_t^{\dagger}\sigma^{\mu}\partial_{\mu}q_t + iq_t^T\sigma^2\overline{\sigma}^{\mu}\sigma^2\partial_{\mu}q_t^* \tag{150}$$

We observe the second term on the right-hand side of (150). Let us consider s and t as arbitrary column matrices and X as an arbitrary square matrix.

Then,

$$s^T Xt = s_{\alpha}X_{\alpha\beta}t_{\beta},$$

where s_{α}, $X_{\alpha\beta}$ and t_{β} are arbitrary elements of the s, X and t matrices.

If s_{α} and t_{β} are components of a field, then they anti-commute.

Hence, $s_{\alpha}X_{\alpha\beta}t_{\beta} = s_{\alpha}t_{\beta}X_{\alpha\beta} = -t_{\beta}s_{\alpha}X_{\alpha\beta} = -t_{\beta}X_{\alpha\beta}s_{\alpha} = -t_{\beta}X^T_{\beta\alpha}s_{\alpha}$.

In matrix form,

$$s^T Xt = -t^T X^T s \tag{151}$$

Using (151) and the relation

$$(\sigma^2\overline{\sigma}^{\mu}\sigma^2)^T = \sigma^{\mu}, \tag{152}$$

The second term present in the Right-hand side of (150) becomes

$$iq_t^T\sigma^2\overline{\sigma}^{\mu}\sigma^2\partial_{\mu}q_t^* = -i(\partial_{\mu}q_t^{\dagger})\sigma^{\mu}q_t \tag{153}$$

We note that the term in the RHS of (153) is the Hermitian conjugate of the first term in (150).

Now, we consider the mass term present in (149). We find

$$\begin{aligned} m\Psi'^{\dagger}\breve{\gamma}^0\Psi' &= m\begin{pmatrix} q_t^{\dagger} & -q_t^T\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_t \\ -\sigma^2 q_t^* \end{pmatrix} \\ &= -\, mq_t^{\dagger}\sigma^2 q_t^* - mq_t^T\sigma^2 q_t \end{aligned} \tag{154}$$

We note that the second term is the Hermitian conjugate of the first term. Hence, the Lagrangian using the chiral representation is

$$\mathcal{L} = \frac{1}{2}\left(iq_t^\dagger \sigma_\mu (\partial_\mu q_t) - i(\partial_\mu q_t^T)\sigma_\mu q_t - mq_t^\dagger \sigma^2 q_t^* - mq_t^T \sigma^2 q_t\right) \tag{155}$$

$$\mathcal{L} = \frac{1}{2}\left(iq_t^\dagger \sigma_\mu (\partial_\mu q_t) - mq_t^T \sigma^2 q_t\right) + \mathrm{H.\,c.}, \tag{156}$$

where H. c. denotes the Hermitian conjugates of the terms present within the parentheses in (156). Thus, the Lagrangian of a Majorana field in chiral representation only requires two components as evident from (156).

7.3 Solutions and their Fourier expansion

Since an arbitrary Weyl and Majorana field can be expressed using two-component notation, the Fourier expansion of the solution can also be represented using two components. Recalling (100), we have for a left chiral Weyl field,

$$\phi_{W,L} = \int (a(p)u_L(p)e^{-ipx} + \hat{a}^\dagger(p)v_L(p)e^{ipx})d^3p$$

Since chirality and helicity coincides for Weyl spinors,

$$\gamma_5 u_L = -u_L = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}u_L \tag{157}$$

$$and\ \gamma_5 v_L = -v_L = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}v_L \tag{158}$$

Hence, keeping $u_L = v_L = r_-$, i.e., we adopt r_- *and* r_+ as the basis spinors such that

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}r_\pm = \pm r_\pm \tag{159}$$

The Fourier expansion of the solution for a left chiral Weyl field becomes

$$\phi_{W,L} = \int (a(p)r_-(p)e^{-ipx} + \hat{a}^\dagger(p)r_-(p)e^{ipx})d^3p \tag{160}$$

The explicit forms of $r_-$and r_+ are

$$r_- = \begin{pmatrix} e^{-i\phi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix} \, and \; r_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ -ie^{i\theta}\sin\frac{\theta}{2} \end{pmatrix} \tag{161}$$

where the components of 3-momentum ($\vec{p}$) are

$$p_x = |p|\sin\theta\cos\phi\,, p_y = |p|\sin\theta\sin\phi\,, p_z = |p|\cos\theta \tag{162}$$

Phases have been adjusted such that

$$r_- = \sigma^2 r_+^* \; and \; r_+ = -\sigma^2 r_-^* \tag{163}$$

For a Majorana field, the solution is similar to (51),

$$\phi_M = \sum_{n=1,2} \int (a_n(p)\chi_n(p)e^{-ipx} + a_n^\dagger(p)\eta_n(p)e^{ipx})d^3p \tag{164}$$

where we use $\chi_n(p)$ and $\eta_n(p)$ as the two-component basis spinors and the two values of n corresponds to the two different helicities necessary to describe a Majorana Field.

The equation of the motion corresponding to the Lagrangian of a massive field in the chiral representation is

$$\overline{\sigma}^\mu i\partial_\mu\phi + m\sigma^2\phi = 0 \tag{165}$$

We note that

$$\begin{aligned}\overline{\sigma}^\mu\partial_\mu e^{\pm ipx} &= \pm i\overline{\sigma}^\mu p_\mu e^{\pm ipx} \\ &= \pm i(\mathbb{1}\cdot p_0 - \vec{\sigma}\cdot\vec{p})e^{\pm ipx} \\ &= \pm\, i(E + \vec{\sigma}\cdot\vec{p})e^{\pm ipx} \end{aligned}\tag{166}$$

We proceed by using (164) in (165) and then after equating the coefficient of $a_1(p)$ and $a_2(p)$, we obtain

$$\eta_n = \frac{E - \vec{\sigma}\cdot\vec{p}}{m} i\sigma^2\chi_n^* \tag{167}$$

$$\chi_n = -\frac{E - \vec{\sigma}\cdot\vec{p}}{m} i\sigma^2 \eta_n^* \tag{168}$$

where we have used the identities

$$\left(\frac{E - \vec{\sigma}\cdot\vec{p}}{m}\right)^{-1} = \left(\frac{E + \vec{\sigma}\cdot\vec{p}}{m}\right) \tag{169}$$

$$\text{and } \vec{\sigma}^* = \sigma^2\vec{\sigma}\sigma^2 \tag{170}$$

Hence, any linear independent set of χ_1 and χ_2 and their corresponding η_1 *and* η_2 can be used to describe the Fourier expansion of the Majorana field.

We proceed to examine the Fourier modes of the solution by making the following choices for η_2 and χ_1.

$$\chi_1 = r_- \text{ and } \eta_2 = r_- \tag{171}$$

We continue to find the values of χ_2 *and* η_1,

$$\begin{aligned}
\eta_1 &= \frac{E - \vec{\sigma}\cdot\vec{p}}{m} i\sigma^2 \chi_1^* \\
\eta_1 &= \frac{E - \vec{\sigma}\cdot\vec{p}}{m} i\sigma^2 r_-^* \\
\eta_1 &= -\frac{E - \vec{\sigma}\cdot\vec{p}}{m} r_+^* \\
\eta_1 &= -\left(\frac{E}{m} r_+^* - \frac{\vec{\sigma}\cdot\vec{p}}{m} r_-^*\right)
\end{aligned}$$

$$\therefore \eta_1 = -\frac{E - |\vec{p}|}{m} r_+ \tag{172}$$

where we have used (167) and (159).

Similarly, we can derive

$$\therefore \chi_2 = \frac{E - |\vec{p}|}{m} r_+ \tag{173}$$

According to Einstein's energy-momentum relation,

$$E^2 = p^2 + m^2$$

$$or, \frac{E-|\vec{p}|}{m} = \frac{m}{E+|\vec{p}|} \tag{174}$$

Hence, in the vanishing mass limit η_1 *and* χ_2 vanish too.

Using η_1, η_2, χ_1 *and* χ_2 we can write (164) as

$$\begin{aligned}\phi_m(x) = \int \Big(& a_-(p)r_-(p)e^{-ipx} + \frac{m}{E+|p|}a_+(p)r_+(p)e^{ipx} + a_+^\dagger(p)r_-(p)e^{ipx} \\ & + \frac{m}{E+|p|}a_-^\dagger(p)r_+(p)e^{-ipx}\Big)\, d^3p\end{aligned} \tag{175}$$

We note that the subscripts of $a^\dagger$ and those of r are opposite, since that term produces a state with helicity opposite to that of the basis spinor used.

Thus, the Fourier expansion of the Majorana solution can be represented using two-components.

In the non-relativistic limit $(|p| \approx 0, E \approx m$ *and* $\frac{m}{E+p} \approx 1$. In such cases, the two chirality/helicities are produced and annihilated with the same amplitude. In the ultra-relativistic limit, $E >> m$ and hence, the r_+ terms vanish and the Majorana field starts behaving like a chiral Weyl field with only one helicity (i.e., -1).

We can also construct a four-component notation of the Majorana field using (148). We keep $q_t(x) = \phi_m(x)$ in (164). Then, we proceed to find $q_b(x) = -\sigma^2 q_t^*(x)$. Keeping $q_t(x)$ on top of $-\sigma^2 q_t^*(x)$, we obtain the four components representation of the Majorana field. The spinors used in this representation would have four components that are given by

$$\breve{u}_n(p) = \begin{pmatrix} \chi_n(p) \\ -\sigma^2\eta_n^*(p) \end{pmatrix}, \qquad \breve{v}_n(p) = \begin{pmatrix} \eta_n(p) \\ -\sigma^2\chi_n^*(p) \end{pmatrix} \tag{176}$$

We check the conjugation relations between $\breve{u}_n(p)$ *and* $\breve{v}_n(p)$ and $\breve{\gamma}_0\breve{C}$ in chiral representation.

$$\breve{\gamma}_0\breve{C}\breve{u}_n^*(p) = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \chi_n(p) \\ -\sigma^2\eta_n^*(p) \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \chi_n^*(p) \\ \sigma^2\eta_n(p) \end{pmatrix}$$

$$= \begin{pmatrix} \eta_n(p) \\ -\sigma^2\chi_n^*(p) \end{pmatrix} = \breve{v}_n(p) \tag{177}$$

Similarly, it can be shown that

$$\breve{\gamma}_0\breve{C}\breve{v}_n^*(p) = \breve{u}_n(p) \tag{178}$$

Using the relation between $\chi_n$and η_n in (167) and (168), we can express $\breve{u}_n(p)$ and $\breve{v}_n(p)$ as

$$\breve{u}_n(p) = \begin{pmatrix} \chi_n(p) \\ \dfrac{E+\vec{\sigma}\cdot\vec{p}}{m}\chi_n(p) \end{pmatrix}, \breve{v}_n(p) = \begin{pmatrix} \eta_n(p) \\ -\dfrac{E+\vec{\sigma}\cdot\vec{p}}{m}\eta_n(p) \end{pmatrix} \tag{179}$$

Keeping $\chi_1 = r_-$ and $\eta_2 = r_-$, (179) becomes

$$\begin{aligned} \breve{u}_-(p) &= \begin{pmatrix} r_- \\ \dfrac{m}{E+|p|}r_- \end{pmatrix}, \breve{u}_+(p) = \begin{pmatrix} \dfrac{m}{E+|p|}r_+ \\ r_+ \end{pmatrix} \\ \breve{v}_+(p) &= \begin{pmatrix} -\dfrac{m}{E+|p|}r_+ \\ r_+ \end{pmatrix}, \breve{v}_-(p) = \begin{pmatrix} r_- \\ -\dfrac{m}{E+|p|}r_- \end{pmatrix} \end{aligned} \tag{180}$$

When $m \to 0$, these spinors become the eigenvalues of $\breve{\gamma}_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The Dirac field, however, cannot be represented using only two components. It has four independent, complex components that reduces to two when the Weyl condition or Majorana condition is imposed. It is convenient to use the four-component notation for Majorana and Weyl fermions, since the Dirac Matrices are of size 4×4 and fermions found in nature are not always Weyl or Majorana.

8 Discrete symmetries in Fermion fields

8.1 Parity ($\mathcal{P}$)

Under this transformation, the special coordinates are reflected through the origin and time *'t'* remains unchanged.

$$\vec{x} \xrightarrow{\mathcal{P}} -\vec{x} \tag{181}$$

Applying two parity transformations leads to the same coordinates.

$$\vec{x} \xrightarrow{\mathcal{P}} -\vec{x} \xrightarrow{\mathcal{P}} \vec{x}$$

Hence,

$$\mathcal{P}^2 = \mathbb{1} \tag{182}$$

Applying this transformation in quantum mechanics can be achieved using two different methods. In the first case, we consider $\mathcal{P}$ as the parity operator that acts on the quantum mechanical states.

$$|\psi\rangle \xrightarrow{\mathcal{P}} \mathcal{P}|\psi\rangle \tag{183}$$

If $\mathcal{P}$ acts on the coordinate bases

$$|x\rangle \xrightarrow{\mathcal{P}} \mathcal{P}|x\rangle = -|x\rangle \tag{184}$$

Using (184), we can obtain

$$\begin{aligned} \langle x^{\mathcal{P}}|y^{\mathcal{P}}\rangle &= \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \\ \langle -x|-y\rangle &= \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \\ \therefore\ \delta(x-y) &= \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \end{aligned} \tag{185}$$

Hence, $\mathcal{P}^\dagger\mathcal{P} = \mathbb{1}$ and $\mathcal{P}^2 = \mathbb{1}$. Therefore, $\mathcal{P}$ has two eigenvalues ± 1. As the eigenvalues of $\mathcal{P}$ are real, $\mathcal{P}$ is also Hermitian.

$$\mathcal{P}^\dagger = \mathcal{P} = \mathcal{P}^{-1} \text{ and } \mathcal{P}^2 = \mathbb{1} \tag{186}$$

A wave function $\Psi(x)$ transforms under parity $\mathcal{P}$ as

$$\Psi(x) \xrightarrow{\mathcal{P}} \Psi'(x) = \Psi(-x) \tag{187}$$

An eigenstate of parity has an associated wave function that is even or odd depending on the eigenvalue of the parity operator for that state.

An eigenstate of parity has an associated wave function that is even or odd depending on the eigenvalue of the parity operator for that state. A general state may not be a parity eigenstate. Hence, a general system may not show such symmetry.

The other approach assumes that the states do not change under transformation, rather the operators themselves transform. We consider O to be an arbitrary operator.

$$O \xrightarrow{\mathcal{P}} O^{\mathcal{P}} = \mathcal{P}^{\dagger} O \mathcal{P} \tag{188}$$

$$X \xrightarrow{\mathcal{P}} X^{\mathcal{P}} = -X \tag{189}$$

$$P \xrightarrow{\mathcal{P}} P^{\mathcal{P}} = -P \tag{190}$$

For an operator *O(X,P),*

$$\begin{gathered} O(X,P) \xrightarrow{\mathcal{P}} O^{\mathcal{P}}(X,P) = \mathcal{P}^{\dagger} O(X,P) \mathcal{P} = O(\mathcal{P}^{\dagger} X \mathcal{P}, \mathcal{P}^{\dagger} P \mathcal{P}) = \\ O(-X,-P) \end{gathered} \tag{191}$$

Parity is also conserved for an odd or even parity state. Relativistic quantum states require quantized fields that have multicomponent operators. Under a parity transformation, a multi-component object would transform as

$$\psi_{\alpha}(\vec{x},t) \xrightarrow{\mathcal{P}} \psi_{\alpha}^{\mathcal{P}}(\vec{x},t) = \eta_{\psi} S_{\alpha}^{\beta} \psi_{\beta}(-\vec{x},t), \tag{192}$$

where S_{α}^{β} can mix the different components of the object. Since two parity transformations lead to the same object, we obtain

$$\eta_{\psi}^{2} S^{2} = \mathbb{1} \tag{193}$$

η_{ψ} is the **intrinsic parity of field $\boldsymbol{\psi_{\alpha}}$** and denotes the parity eigenvalue of a particle state at rest.

We use (192) on a Dirac Field, which is described by a four component object.

$$\psi_\alpha(\vec{x},t) \xrightarrow{\mathcal{P}} \psi_\alpha^{\mathcal{P}}(\vec{x},t) = \eta_\psi S_\alpha^\beta \psi_\beta(-\vec{x},t), \text{where } \alpha = 1{,}2{,}3 \text{ and } 4. \tag{194}$$

In matrix notation

$$\Psi^{\mathcal{P}}(\vec{x},t) = \eta_\psi S \Psi(-\vec{x},t) \tag{195}$$

with $\eta_\psi^2 S^2 = \mathbb{1}$. Since the eigenvalues of $\mathcal{P}$ are ± 1, we consider

$$\eta_\psi = \pm 1 \text{ and } S^2 = \mathbb{1} \tag{196}$$

We observe that

$$\begin{aligned}\overline{\Psi}^{\mathcal{P}}(\vec{x},t) = (\Psi^{\mathcal{P}})^\dagger(\vec{x},t)\gamma^0 &= \eta_\psi^* \Psi^\dagger(-\vec{x},t) S^\dagger \gamma^0 \\ &= \eta_\psi^* \Psi^\dagger(-\vec{x},t)\gamma^0\gamma^0 S^\dagger \gamma^0 \\ &= \eta_\psi \overline{\Psi}(-\vec{x},t)\gamma^0 S^\dagger \gamma^0\end{aligned} \tag{197}$$

The conserved current of the Dirac system is

$$J^\mu(\vec{x},t) = \overline{\Psi}(\vec{x},t)\gamma^\mu \Psi(\vec{x},t) \tag{198}$$

Under Parity transformation

$$\begin{aligned}J^{\mu\mathcal{P}}(\vec{x},t) &= \overline{\Psi}^{\mathcal{P}}(\vec{x},t)\gamma^\mu \Psi^{\mathcal{P}}(\vec{x},t) \\ J^{\mu\mathcal{P}}(\vec{x},t) &= \eta_\psi^2 \overline{\Psi}(-\vec{x},t)\gamma^0 S^\dagger \gamma^0 \gamma^\mu S \Psi(-\vec{x},t) \\ J^{\mu\mathcal{P}}(\vec{x},t) &= \overline{\Psi}(-\vec{x},t)\gamma^0 S^\dagger \gamma^0 \gamma^\mu S \Psi(-\vec{x},t)\end{aligned} \tag{199}$$

We also note that

$$\begin{aligned}J^{0\mathcal{P}}(\vec{x},t) &= J^0(-\vec{x},t) \\ J^{i\mathcal{P}}(\vec{x},t) &= -J^i(-\vec{x},t)\end{aligned} \tag{200}$$

Comparing (199) and (200), we obtain for $\mu = 0$

$$\overline{\Psi}(-\vec{x},t)\gamma^0 S^\dagger S\Psi(-\vec{x},t) = \overline{\Psi}(-\vec{x},t)\gamma^0\Psi(-\vec{x},t)$$
$$\therefore S^\dagger S = \mathbb{1} \tag{201}$$

Since $S^2 = \mathbb{1}$, we obtain from (201)

$$S = S^\dagger = S^{-1} \tag{202}$$

For $\mu = i$, on comparing (199) and (200), we obtain

$$\overline{\Psi}(-\vec{x},t)\gamma^0 S^\dagger\gamma^0\gamma^i S\Psi(-\vec{x},t) = -\overline{\Psi}(-\vec{x},t)\gamma^i\Psi(-\vec{x},t)$$
$$\therefore S = \gamma^0 \tag{203}$$

Hence, for a Dirac field

$$\Psi(\vec{x},t) \xrightarrow{\mathcal{P}} \Psi^{\mathcal{P}}(\vec{x},t) = \eta_\psi\gamma^0\Psi(-\vec{x},t)$$
$$\text{and } \Psi(\vec{x},t) \xrightarrow{\mathcal{P}} \Psi^{\mathcal{P}}(\vec{x},t) = \eta_\psi\Psi(-\vec{x},t)\gamma^0 \text{ with } \eta_\psi = \pm 1. \tag{204}$$

We proceed to define $y^\mu = (t, -\vec{x})$. Then, the Dirac equation for a parity transformed $\Psi(\vec{x})$ is given by

$$\begin{aligned}\left(i\gamma^\mu\frac{\partial}{\partial y^\mu} - m\right)\Psi^{\mathcal{P}}(y) &= \left(i\gamma^0\frac{\partial}{\partial t} - i\gamma^i\frac{\partial}{\partial x^i} - m\right)\eta_\psi\gamma^0\Psi(x)\\ &= \eta_\psi\gamma^0\left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - m\right)\Psi(x)\\ &= \eta_\psi\gamma^0\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\Psi(x) = 0\end{aligned} \tag{205}$$

Hence, **space inversion is a symmetry of the Dirac Equation.** However, **two-component Weyl fermions violate Parity.**

In the case of fermions, **the relative intrinsic parity between Dirac particles and anti-particles is negative**. Moreover, fermions always appear in pairs of a particle and an anti-particle and hence, the relative parity of a pair of fermions is significant.

8.2 Charge conjugation ($\mathcal{C}'$)

The Maxwell Equations

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \rho \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \frac{\partial \vec{E}}{\partial t} + \vec{J}
\end{aligned}
\tag{206}
$$

These equations remain invariant under the following transformations:

$$
\rho \to -\rho, \vec{J} \to -\vec{J}, \vec{E} \to -\vec{E} \text{ and } \vec{B} \to -\vec{B} \tag{207}
$$

We note that flipping the sign of the charge (charge conjugation) would lead to the transformations in (207). We denote the transformation by $\mathcal{C}'$. Moreover, since

$$
\begin{aligned}
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \cdot A^0 \\
\text{and } \vec{B} &= \vec{\nabla} \times \vec{A}
\end{aligned}
\tag{208}
$$

Hence, classically charge conjugation ($\mathcal{C}'$) leads to

$$
A_\mu \xrightarrow{\mathcal{C}'} -A_\mu \text{ and } J_\mu \xrightarrow{\mathcal{C}'} -J_\mu \tag{209}
$$

In quantum theory, (209) becomes

$$
\begin{aligned}
A_\mu(x) &\xrightarrow{\mathcal{C}'} -A_\mu^{\mathcal{C}'}(x) = \eta_A A_\mu(x) = -A_\mu(x) \\
J_\mu(x) &\xrightarrow{\mathcal{C}'} -J_\mu^{\mathcal{C}'}(x) = \eta_J J_\mu(x) = -J_\mu(x)
\end{aligned}
\tag{210}
$$

where charge conjugation parity $(\eta_{AJ}) = -1$. In the quantum theory, particles and antiparticles are assigned opposite charges. Hence, charge conjugation flips every quantum number between particles and antiparticles and is also called particle-antiparticle conjugation.

We now discuss the effects of charge conjugation on a Dirac Field interacting with an electromagnetic field. In this case the equation of motion is

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi(x) = 0 \tag{211}$$

The adjoint equation is

$$\bar{\psi}(x)(i\gamma^\mu(\partial_\mu - ieA_\mu) + m) = 0 \tag{212}$$

On taking transpose of (207),

$$(i(\gamma^\mu)^T(\partial_\mu - ieA_\mu) + m)\bar{\psi}^T = 0 \tag{213}$$

A Dirac particle that carries the same mass as that of another Dirac particle but a different charge would satisfy

$$(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\psi^{\mathcal{C}'}(x) = 0, \tag{214}$$

where $\psi^{\mathcal{C}'}(x)$ describes an antiparticle.

If we consider a matrix $\mathcal{C}'$such that

$$\mathcal{C}'(\gamma^\mu)^T\mathcal{C}'^{-1} = -\gamma^\mu \tag{215}$$

and using $\mathcal{C}'$ on (213),

$$\mathcal{C}'(i(\gamma^\mu)^T(\partial_\mu - ieA_\mu) + m)\bar{\psi}^T = 0$$

$$(i\mathcal{C}'(\gamma^\mu)^T\mathcal{C}'^{-1}(\partial_\mu - ieA_\mu) + m)\mathcal{C}'\bar{\psi}^T = 0$$

$$(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\mathcal{C}'\bar{\psi}^T = 0 \tag{216}$$

On comparing (216) with (214), we obtain

$$\psi^{\mathcal{C}'}(x) = \eta_\psi\mathcal{C}'\bar{\psi}^T \tag{217}$$

Since $(\gamma^\mu)^T$ satisfies the Clifford algebra

$$[-(\gamma^\mu)^T, -(\gamma^\nu)^T]_+ = 2g^{\mu\nu}\mathbb{1} \tag{218}$$

A matrix $\mathcal{C}'$ exists. The Pauli's fundamental theorem requires a similarity transformation between $(-\gamma^\mu)^T$ and γ^μ. Moreover $\mathcal{C}'$ should be antisymmetric,

$$\mathcal{C}'^T = -\mathcal{C}' \tag{219}$$

$\mathcal{C}'$ is dependent on the Dirac matrices representation. In the Pauli Dirac Representation,

$$\mathcal{C}' = \gamma^0\gamma^2 \tag{220}$$

We note that

$$\mathcal{C}' = \mathcal{C}'^\dagger = \mathcal{C}'^{-1} = -\mathcal{C}'^T \tag{221}$$

Hence,

$$\psi^{\mathcal{C}'} = \eta_\psi \mathcal{C}' \bar{\psi}^T = \eta_\psi \gamma^0\gamma^2(\gamma^0)^T\psi^* = -\eta_\psi\gamma^2\psi^* \tag{222}$$

However, independent of any representation of γ-matrices, we can write

$$\begin{gathered}\psi^{\mathcal{C}'} = \eta_\psi \mathcal{C}' \bar{\psi}^T \\ \text{and } \bar{\psi}^{\mathcal{C}'} = -\eta_\psi^* \psi^T \mathcal{C}'^{-1} \\ \text{with } |\eta_\psi|^2 = 1 \text{ and } \mathcal{C}'^\dagger \mathcal{C}' = \mathbb{1}\end{gathered} \tag{223}$$

Recalling from (204), a Dirac particle transforms under parity as

$$\begin{gathered}\Psi(\vec{x},t) \xrightarrow{\mathcal{P}} \Psi^{\mathcal{P}}(\vec{x},t) = \eta_\psi\gamma^0\Psi(-\vec{x},t) \\ \text{and } \Psi(\vec{x},t) \xrightarrow{\mathcal{P}} \Psi^{\mathcal{P}}(\vec{x},t) = \eta_\psi^{\mathcal{P}}\overline{\Psi}(-\vec{x},t)\gamma^0\end{gathered} \tag{224}$$

From (223), an antiparticle is described by

$$\psi^{\mathcal{C}'}(\vec{x},t) = \eta_{psi}^{\mathcal{C}'}\mathcal{C}'\bar{\psi}^T(\vec{x},t) \tag{225}$$

(225) will transform under parity as

$$\psi^{\mathcal{C}'}(\vec{x},t) \xrightarrow{\mathcal{P}} (\psi^{\mathcal{C}'})^{\mathcal{P}}(\vec{x},t) = \eta_\psi^{\mathcal{C}'}\mathcal{C}'(\bar{\psi}^{\mathcal{P}})^T(-\vec{x},t)$$

$$= \eta_\psi^{\mathcal{C}'}\eta_\psi^{\mathcal{P}}\mathcal{C}'(\bar{\psi}(-\vec{x},t)\gamma^0)^T$$

$$= \eta_\psi^{\mathcal{C}'}\eta_\psi^{\mathcal{P}}\mathcal{C}'(\gamma^0)^T\mathcal{C}'^{-1}\mathcal{C}'(\gamma^0)^*\psi^*(-\vec{x},t)$$

$$= -\eta_\psi^{\mathcal{P}}\eta_\psi^{\mathcal{C}'}\gamma^0\mathcal{C}'(\gamma^0)^T\psi^*(-\vec{x},t)$$

$$= -\eta_\psi^{\mathcal{P}}\gamma^0\big(\eta_\psi^{\mathcal{C}'}\mathcal{C}'\bar{\psi}^T(-\vec{x},t)\big)$$

$$\therefore \left(\psi^{\mathcal{C}'}(\vec{x},t)\right)^{\mathcal{P}} = -\eta_\psi^{\mathcal{P}}\gamma^0\psi^{\mathcal{C}'}(-\vec{x},t) \tag{226}$$

Hence, if $\eta_\psi^{\mathcal{P}}$ is the intrinsic parity of a Dirac particle, then, the intrinsic parity of the corresponding Dirac antiparticle will be $-\eta_\psi^{\mathcal{P}}$ and the relative particle-antiparticle parity will be $-\eta_\psi^{\mathcal{P}} \times \eta_\psi^{\mathcal{P}} = -1$.

We also note that

$$\gamma_5\psi^{\mathcal{C}'} = \eta_\psi^{\mathcal{C}'}\gamma_5\mathcal{C}'\gamma^{0\,T}\psi^*$$

$$= \eta_\psi^{\mathcal{C}'}\mathcal{C}'\mathcal{C}^{-1}\gamma_5\mathcal{C}'\gamma^{0\,T}\psi^*$$

$$= \eta_\psi^{\mathcal{C}'}\gamma_5\mathcal{C}'\gamma^{0\,T}\psi^* = -\eta_\psi^{\mathcal{C}'}\mathcal{C}'\gamma^{0\,T}\gamma_5^T\psi^*$$

$$= -\eta_\psi^{\mathcal{C}'}\mathcal{C}'(\gamma^0)^T(\gamma_5^\dagger\psi)^* = -\eta_\psi^{\mathcal{C}'}\mathcal{C}'\gamma^{0\,T}(\gamma_5\psi)^*$$

$$= -\eta_\psi^{\mathcal{C}'}\mathcal{C}'\gamma^{0\,T}((\gamma_5\psi)^\dagger)^T = -\eta_\psi^{\mathcal{C}'}\mathcal{C}'(\overline{\gamma_5\psi})^T \tag{227}$$

Thus, if $\gamma_5\psi = \psi$,

$$\gamma_5\psi^{\mathcal{C}'} = -\eta_\psi^{\mathcal{C}'}\mathcal{C}'\bar{\psi}^T = -\psi^{\mathcal{C}'} \tag{228}$$

and if $\gamma_5\psi = -\psi$,

$$\gamma_5\psi^{\mathcal{C}'} = -\eta_\psi^{\mathcal{C}'}\mathcal{C}'(-\bar{\psi}^T) = \eta_\psi^{\mathcal{C}'}\mathcal{C}'\bar{\psi}^T = \psi^{\mathcal{C}'} \tag{229}$$

Hence, the handedness of an antiparticle of a particle will be opposite to that of the particle.

For a Majorana fermion,

$$\phi_M = \phi_M^{\mathcal{C}'} = \eta_\psi \mathcal{C}' \overline{\phi_M}^T \tag{230}$$

where ϕ_M satisfies the following Dirac equations –

$$(i\gamma^\mu \partial_\mu - m)\phi_M = (i\gamma^\mu \partial_\mu - m)\bar{\phi}_M = 0$$

In this case, a Majorana Fermion requires

$$\eta_\psi^{\mathcal{P}} = -(\eta_\psi^{\mathcal{P}})^*, \tag{231}$$

which is satisfied only when the intrinsic parity of the Majorana fermion is imaginary.

For a Weyl Fermion, charge conjugation ($\mathcal{C}'$) changes the helicity of the particle as seen in (228) to (229). Moreover, the $\mathcal{C}'\mathcal{P}$ conjugate of a left-handed particle is a right-handed anti-particle, since charge conjugation changes to particle to anti-particle and parity changes the helicity. A free Majorana particle is an eigenstate of charge conjugation and does not violate charge conjugation. However, a neutrino violates charge conjugation. However, it satisfies $\mathcal{CP}$ symmetry.

One cannot build a theory that is invariant under charge conjugation. The charge conjugate of a left chiral Weyl field is a left chiral field with opposite quantum numbers, whose existence is not guaranteed by a left-chiral field. On the other hand, the $\mathcal{CP}$ conjugate of a left-handed Weyl particle is a right-handed Weyl antiparticle which is the Lorentz covariant conjugate of a left chiral Weyl field. Hence, its existence is guaranteed and is also included in the theory.

8.3 Time reversal ($\mathcal{T}$)

Under this transformation

$$\vec{x} \xrightarrow{\mathcal{T}} \vec{x} \text{ and } t \xrightarrow{\mathcal{T}} -t \tag{232}$$

The Maxwell's equations are invariant if

$$
\begin{aligned}
\rho(\vec{x},t) &\xrightarrow{\mathcal{T}} \rho^{\mathcal{T}}(\vec{x},t) = \rho(\vec{x},-t) \\
\vec{J}(\vec{x},t) &\xrightarrow{\mathcal{T}} \vec{J}^{\mathcal{T}}(\vec{x},t) = -\vec{J}(\vec{x},-t) \\
\vec{E}(\vec{x},t) &\xrightarrow{\mathcal{T}} \vec{E}^{\mathcal{T}}(\vec{x},t) = \vec{E}(\vec{x},-t) \\
\vec{B}(\vec{x},t) &\xrightarrow{\mathcal{T}} \vec{B}^{\mathcal{T}}(\vec{x},t) = -\vec{B}(\vec{x},-t)
\end{aligned} \tag{233}
$$

Also under time reversal

$$
\begin{aligned}
A^0(\vec{x},t) &\xrightarrow{\mathcal{T}} A^{0^{\mathcal{T}}}(\vec{x},t) = A^0(\vec{x},-t) \\
\vec{A}(\vec{x},t) &\xrightarrow{\mathcal{T}} \vec{A}^{\mathcal{T}}(\vec{x},t) = -\vec{A}(\vec{x},-t)
\end{aligned} \tag{234}
$$

In quantum mechanics, symmetry transformation is applied using linear operators, which are unitary. Considering the Time reversal operator $\mathcal{T}$

$$
\psi(\vec{x},t) = \langle\vec{x}|\psi(t)\rangle \xrightarrow{\mathcal{T}} \psi^{\mathcal{T}}(\vec{x},t) = \langle\vec{x}|\psi(-t)\rangle^* = \psi^*(\vec{x},-t) \tag{235}
$$

Classically, we note that

$$
\vec{x} \xrightarrow{\mathcal{T}} \vec{x} \text{ and } \vec{p} \xrightarrow{\mathcal{T}} -\vec{p} \tag{236}
$$

For quantum mechanical operators

$$
\begin{aligned}
\mathcal{T} X \mathcal{T}^\dagger &= \eta_X X = X \Rightarrow \eta_X = 1 \\
\mathcal{T} P \mathcal{T}^\dagger &= \eta_P P = -P \Rightarrow \eta_P = -1
\end{aligned} \tag{237}
$$

$$
\mathcal{T}[X_i P_j]\mathcal{T}^\dagger = \mathcal{T}(X_i P_j - P_j X_i)\mathcal{T}^\dagger = X_i(-P_j) - (-P_j)X_i = -[X_i P_j] \tag{238}
$$

Hence, **the canonical commutation rules remain invariant under time reversal.**

For Dirac Fields, the fields transform under time reversal as

$$
\Psi(\vec{x},t) \xrightarrow{\mathcal{T}} \Psi^{\mathcal{T}}(\vec{x},t) = \eta_\psi \mathcal{T}\Psi(\vec{x},-t), \tag{239}
$$

where $\mathcal{T}$ is the time-reversal operator.

(239) must satisfy the Dirac equation for time reversal to be a symmetry. The Dirac Equation is

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \tag{240}$$

Upon complex conjugation of both sides, (240) becomes

$$\begin{gathered}
(-i\gamma^{\mu *}\partial_\mu - m)\Psi^*(\vec{x}, t) = 0 \\
\left(-i\gamma^{0*}\frac{\partial}{\partial t} - i\gamma^{i*}\frac{\partial}{\partial x_i} - m\right)\Psi^*(\vec{x}, t) = 0 \\
\left(-i\gamma^{0*}\frac{\partial}{\partial(-t)} - i\gamma^{i*}\frac{\partial}{\partial x_i} - m\right)\Psi^*(\vec{x}, -t) = 0 \\
\left(i\gamma^{0*}\frac{\partial}{\partial t} - i\gamma^{i*}\frac{\partial}{\partial x_i} - m\right)\Psi^*(\vec{x}, -t) = 0
\end{gathered} \tag{241}$$

For $\Psi^{\mathcal{T}}(\vec{x}, t)$ to be a solution of the Dirac equation

$$\begin{gathered}
(i\gamma^\mu \partial_\mu - m)\Psi^{\mathcal{T}}(x) = 0 \\
\left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x_i} - m\right)\Psi^{\mathcal{T}}(\vec{x}, t) = 0 \\
\left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x_i} - m\right)\mathcal{T}\Psi^*(\vec{x}, -t) = 0
\end{gathered} \tag{242}$$

On comparing (241) and (242), we obtain

$$\mathcal{T}\gamma^{0*}\mathcal{T}^{-1} = \gamma^0 \text{ and } \mathcal{T}\gamma^{i*}\mathcal{T}^{-1} = -\gamma^i \tag{243}$$

Hence, $\mathcal{T}$ commutes with γ^0, but anticommutes with γ^i. Moreover, the time reversed wave function is

$$\Psi^{\mathcal{T}}(\vec{x}, t) = \eta_\psi \mathcal{T}\Psi^*(\vec{x}, -t) \tag{244}$$

and it also satisfies the Dirac equation.

The $\gamma^{\mu *}$ matrices should satisfy the Clifford algebra –

$$[(\gamma^\mu)^*, (\gamma^\nu)^*]_+ = 2g^{\mu\nu}\mathbb{1} \tag{245}$$

and we also consider $\tilde{\gamma}^\mu$ as

$$\tilde{\gamma}^\mu = (\gamma^\mu)^\dagger \tag{246}$$

$$\therefore \tilde{\gamma}^0 = \gamma^0 \text{ and } \tilde{\gamma}^i = -\gamma^i \tag{247}$$

Hence, $\tilde{\gamma}^\mu$ matrices satisfy the Clifford algebra (245).

According to Pauli's Fundamental Theorem, $\gamma^{\mu *}$ and $\tilde{\gamma}^\mu$ can be derived from the other using a similarity transformation

$$\mathcal{T}\gamma^{\mu *}\mathcal{T}^{-1} = \tilde{\gamma}^\mu, \tag{248}$$

which leads to

$$\begin{gathered} \mathcal{T}\gamma^{0^*}\mathcal{T}^{-1} = \gamma^0 \\ \therefore \gamma^{0^*} = \left(\gamma^{0^\dagger}\right)^T = \gamma^{0^T} = \mathcal{T}^{-1}\gamma^0\mathcal{T} \end{gathered} \tag{249}$$

and

$$\begin{gathered} \mathcal{T}\gamma^{i^*}\mathcal{T}^{-1} = -\gamma^i \\ \therefore -\gamma^{i^*} = -\left(\gamma^{i^\dagger}\right)^T = \gamma^{i^T} = \mathcal{T}^{-1}\gamma^i\mathcal{T} \end{gathered} \tag{250}$$

$$\therefore \mathcal{T}^{-1}\gamma^\mu\mathcal{T} = \gamma^{\mu T} \tag{251}$$

In the case of charge conjugation

$$\mathcal{C}'\gamma^{\mu T}\mathcal{C}^{-1} = -\gamma^\mu \tag{252}$$

On combining (251) and (252), we obtain

$$\mathcal{C}'\mathcal{T}^{-1}\gamma^\mu(\mathcal{C}'\mathcal{T}^{-1})^{-1} = -\gamma^\mu \tag{253}$$

Hence, $\mathcal{C}'\mathcal{T}^{-1}$ anticommutes with all γ^μ's and is a multiple of γ_5. If we choose

$$\mathcal{C}'\mathcal{T}^{-1} = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{254}$$

and use (220), we obtain

$$\mathcal{T} = i\gamma^1\gamma^3 \tag{255}$$

We also note that

$$\mathcal{T} = \mathcal{T}^\dagger = \mathcal{T}^{-1} \tag{256}$$

Moreover, the time reversal operator, similar to the charge conjugation operator, satisfies

$$\mathcal{T}^T = -\mathcal{T} \tag{257}$$

Using the time inversion transformations

$$\begin{aligned} \psi(\vec{x},t) \xrightarrow{\mathcal{T}} \psi^{\mathcal{T}}(\vec{x},t) = \eta_\psi \mathcal{T}\psi^*(\vec{x},-t) = \eta_\psi \mathcal{T}\gamma^{0\,T}\bar{\psi}^T(\vec{x},-t) \\ = \eta_\psi \mathcal{T}\gamma^{0\,T}\bar{\psi}^T(\vec{x},-t) \end{aligned} \tag{258}$$

$$\text{and } \bar{\psi}(\vec{x},t) \xrightarrow{\mathcal{T}} \bar{\psi}^{\mathcal{T}}(\vec{x},t) = \eta_\psi^* \psi^T(\vec{x},-t)T^{-1}\gamma^0 \text{ with } \left|\eta_\psi\right|^2 = 1. \tag{259}$$

The above expressions define the time reversal symmetry of the Dirac equation.

Time reversal does not change the helicity –

$$h_p = \vec{S}\cdot\frac{\vec{p}}{|\vec{p}|} \xrightarrow{\mathcal{T}} \left(\overrightarrow{-S}\right)\cdot\frac{(-\vec{p})}{|\vec{p}|} = \vec{S}\cdot\frac{\vec{p}}{|\vec{p}|} = h_p$$

Moreover, the chirality of a spinor does not change due to time inversion –

$$\gamma_5\psi^{\mathcal{T}} = \eta_\psi\gamma_5\mathcal{T}\psi^* = \eta_\psi\mathcal{T}\gamma_5^T\psi^* = \eta_\psi\mathcal{T}(\gamma_5^\dagger\psi)^* = \eta_\psi\mathcal{T}(\gamma_5\psi)^* \tag{260}$$

8.4 $\mathcal{CPT}$ theorem

In the case of weak interactions, some of the above discrete symmetries are violated. The $\mathcal{CPT}$ theorem states that any physical Hamiltonian would be invariant under the combined operation of $\mathcal{P}$, $\mathcal{C}'$ and $\mathcal{T}$ in any order, though the individual symmetries may not be a symmetry of the system.

We consider the state of a particle at rest that satisfies

$$H|\alpha, m, s\rangle = m|\alpha, m, s\rangle, \tag{261}$$

where m is the mass of the particle, s is the spin and α denotes the other quantum numbers. Under $\mathcal{CPT}$ transformation

$$|\alpha, m, s\rangle \overset{\mathcal{C}'\mathcal{PT}}{\longrightarrow} \eta\langle\bar{\alpha}, \overline{m}, -s|, \tag{262}$$

where $\bar{\alpha}, \overline{m}$ and $-s$ correspond to the parameters of the antiparticle. We note from (261)

$$\begin{gathered} m = \langle\alpha, m, s|H|\alpha, m, s\rangle \\ m = \langle\alpha, m, s|(\mathcal{C}'\mathcal{PT})^{-1}(\mathcal{C}'\mathcal{PT})H(\mathcal{C}'\mathcal{PT})^{-1}(\mathcal{C}'\mathcal{PT})|\alpha, m, s\rangle \\ m = \langle\bar{\alpha}, \overline{m}, -s|(\mathcal{C}'\mathcal{PT})H(\mathcal{C}'\mathcal{PT})^{-1}|\bar{\alpha}, \overline{m}, -s\rangle \end{gathered} \tag{263}$$

If the Hamiltonian is $\mathcal{CPT}$ invariant, then

$$m = \langle\bar{\alpha}, \overline{m}, -s|H|\bar{\alpha}, \overline{m}, -s\rangle = \overline{m} \tag{264}$$

Hence, **the equality of mass of the particle and the mass of the antiparticle is a consequence of the $\mathcal{CPT}$ invariance of the Hamiltonian**.

We proceed to define $\hat{Q}$, an operator that measures the charge of a state, as

$$\hat{Q} = \int \bar{\psi}\gamma^0\psi \; d^3x \;\text{ for a Dirac field} \tag{265}$$

Under different discrete transformations, $\hat{Q}$ transforms as

$$\hat{Q} \xrightarrow{\mathcal{P}} \hat{Q}, \hat{Q} \xrightarrow{\mathcal{C}\prime} -\hat{Q}, \text{and } \hat{Q} \xrightarrow{\mathcal{T}} \hat{Q} \tag{266}$$

$$\therefore \hat{Q} \xrightarrow{\mathcal{C}'\mathcal{P}\mathcal{T}} -\hat{Q} \tag{267}$$

For a particle state with charge q at rest

$$\begin{gathered} q = \left\langle \alpha, q, s \middle| \hat{Q} \middle| \alpha, q, s \right\rangle \\ q = \langle \alpha, q, s | (\mathcal{C}'\mathcal{P}\mathcal{T})^{-1} (\mathcal{C}'\mathcal{P}\mathcal{T}) \hat{Q} (\mathcal{C}'\mathcal{P}\mathcal{T})^{-1} (\mathcal{C}'\mathcal{P}\mathcal{T}) | \alpha, q, s \rangle \\ q = \left\langle \bar{\alpha}, \bar{q}, -s \middle| -\hat{Q} \middle| \bar{\alpha}, \bar{q}, -s \right\rangle = -\bar{q} \end{gathered} \tag{268}$$

Hence, the electric charge for particles is equal in magnitude but opposite in charge of the antiparticles. Similarly, all other quantum numbers can be shown to be equal in magnitude but opposite in sign for antiparticles as compared to those of the particles. If any field violates both $\mathcal{CP}$ and $\mathcal{CPT}$ symmetries, then it does not depend on any of the above discrete symmetries. However, the proper Lorentz group ensures the existence of a Lorentz covariant conjugate. The condition for Majorana fermions, $\phi_M = \hat{\phi}_M$, remains true for a free as well as for an interactive Majorana fermion.

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