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# State–space solution to the $\mathscr{H}_-/\mathscr{H}_\infty$ fault-detection problem

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SUMMARY

In this paper we give an optimal state–space solution to the $\mathscr{H}_-/\mathscr{H}_\infty$ fault-detection (FD) problem for linear time invariant dynamic systems. An optimal $\mathscr{H}_-/\mathscr{H}_\infty$ FD filter minimizes the sensitivity of the residual signal to disturbances while maintaining a minimum level of sensitivity to faults. We provide a state–space realization of the optimal filter in an observer form using the solution of a linear matrix inequalities optimization problem. We also show that, through the use of weighting filters, the detection performance can be enhanced and some assumptions can be removed. Two numerical examples are given to illustrate the algorithm. Copyright © 2011 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

Observer-based fault-detection (FD) schemes are a class of model-based FD schemes that use dedicated state/output observers (or banks of such observers) to generate residual signals to provide fault signatures [1, 2]. The observer effectively cancels the process dynamics and is sensitive only to disturbances and faults. One of the most important issues in FD is therefore to indicate occurring faults in the presence of disturbances.

The FD problem can thus be regarded as a disturbance attenuation problem. The filter design objective is then to minimize the sensitivity to disturbances while maintaining a certain level of sensitivity to faults. Early work in this area includes frequency domain analysis [3] and $\mathscr{H}_\infty$-bounded filter design [4], however, a single $\mathscr{H}_\infty$-norm is not sufficient for separating faults and disturbances in the residual and multi-objective criteria involving different norms are required.

In this paper, we propose a state–space approach in a linear matrix inequalities (LMI) framework to develop an optimal FD scheme which is maximally insensitive to disturbances for a given level of sensitivity to faults. This so-called mixed $\mathscr{H}_-/\mathscr{H}_\infty$ filtering problem has been previously considered in [5–13], where partial solutions were given. Although a matrix factorization analysis of this problem has been presented in [14, 15], a complete solution in state–space is required to obtain an observer-based design.

After describing our notation, Section 2 gives a formulation of the FD problem. Section 3 transforms the formulation into a state–space framework under certain restrictive rank assumptions, whereas Section 4 provides an LMI solution. In Section 5, we give a model-matching interpretation of our solution. In Section 6 we show that the FD performance can be improved and the rank

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assumptions removed, by incorporating weighting filters in our design. Finally, two illustrative examples are given in Section 7 and Section 8 summarizes our results and gives some future directions.

The notation we use is mostly standard and is summarized next for convenience. The set of real (complex) $n \times m$ matrices is denoted by $\mathscr{R}^{n\times m}$ ($\mathscr{C}^{n\times m}$). For $A\in\mathscr{C}^{n\times m}$ we use the notation $A^{\mathrm{T}}$ and $A'$ to denote the transpose and complex conjugate transpose, respectively. For $A=A'\in\mathscr{C}^{n\times n}$, $\bar{\lambda}(A)$ denotes the largest and $\underline{\lambda}(A)$ the smallest eigenvalue of $A$, respectively, and $A\prec 0$ denotes that $A$ is negative definite (that is, $\bar{\lambda}(A)<0$). For $A\in\mathscr{C}^{n\times m}$, $\bar{\sigma}(A):=\sqrt{\bar{\lambda}(AA')}$ denotes the largest, and $\underline{\sigma}(A):=\sqrt{\underline{\lambda}(AA')}$ the smallest, singular values of $A$, respectively. The $n\times n$ identity matrix is denoted as $I_n$ and the $n\times m$ null matrix is denoted as $0_{n,m}$ with the subscripts normally dropped if they can be inferred from context.

$\mathscr{R}(s)^{m\times p}$ denotes the set of all $m\times p$ real rational matrix functions of $s$. $\mathscr{L}_\infty^{m\times p}$ denotes the space of $m\times p$ matrix functions with entries bounded on the extended imaginary axis $\mathrm{j}\mathscr{R}_e$. The subspace $\mathscr{H}_\infty^{m\times p}\subset\mathscr{L}_\infty^{m\times p}$ denotes matrix functions analytic in the closed right-half of the complex plane. A prefix $\mathscr{R}$ denotes a real rational function, so that $\mathscr{R}\mathscr{H}_\infty^{m\times p}$ denotes the set of all $m\times p$ stable real rational matrix functions of $s$. If $G(s)=D+C(sI-A)^{-1}B$ is a transfer matrix function we use the notation $G(s)\stackrel{s}{=}(A,B,C,D)$ and

$$G(s)\stackrel{s}{=}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

If $G_1(s)\stackrel{s}{=}(A,B_1,C,D_1)$ and $G_2(s)\stackrel{s}{=}(A,B_2,C,D_2)$, we write

$$[G_1(s)\ \ G_2(s)]\stackrel{s}{=}\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right].$$

For $G(s)\in\mathscr{R}\mathscr{L}_\infty^{m\times p}$, we define $G^{\sim}(s)=G^{\mathrm{T}}(-s)$ to be the para-Hermitian complex conjugate transpose of $G(s)$. A square matrix function $G(s)\in\mathscr{R}\mathscr{H}_\infty^{m\times m}$ is called inner if $G^{\sim}(s)G(s)=I_m$. For $G(s)\in\mathscr{R}\mathscr{L}_\infty^{m\times p}$ we define

$$\|G\|_\infty=\sup_{\omega\in\mathscr{R}}\bar{\sigma}(G(\mathrm{j}\omega)),\qquad \|G\|_-=\inf_{\omega\in\mathscr{R}}\underline{\sigma}(G(\mathrm{j}\omega)).$$

Note that $\|\cdot\|_-$ is not a true norm but we use this notation since it has become standard in the literature. See [9, 13] for a full discussion.

## 2. PROBLEM FORMULATION

Consider a linear time-invariant dynamic system subject to disturbances and process, sensor and actuator faults modelled as

$$\dot{x}(t)=Ax(t)+Bu(t)+B_f f(t)+B_d d(t),$$
$$y(t)=Cx(t)+Du(t)+D_f f(t)+D_d d(t),$$

where $x(t)\in\mathscr{R}^n$, $u(t)\in\mathscr{R}^{n_u}$ and $y(t)\in\mathscr{R}^{n_y}$ are the process state, input and output vectors, respectively, and where $d(t)\in\mathscr{R}^{n_d}$ and $f(t)\in\mathscr{R}^{n_f}$ are the disturbance and fault vectors, respectively. Here, $B_f\in\mathscr{R}^{n\times n_f}$ and $D_f\in\mathscr{R}^{n_y\times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d\in\mathscr{R}^{n\times n_d}$ and $D_d\in\mathscr{R}^{n_y\times n_d}$ are the corresponding disturbance distribution matrices [1]. By taking Laplace transforms, the system input/output behavior can be described by

$$y(s)=G(s)u(s)+G_d(s)d(s)+G_f(s)f(s),$$



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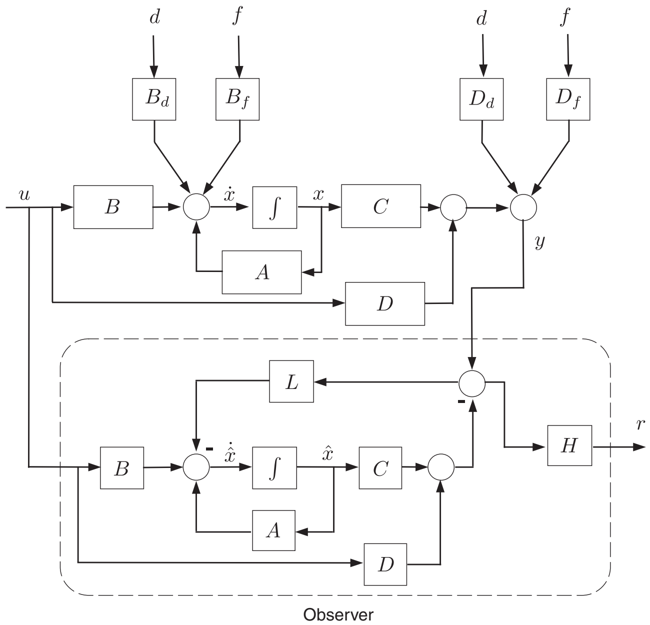


Figure 1. Observer-based FD scheme.

where

$$\begin{array}{cccc} & n_u & n_f & n_d \\ n_y & [G(s) & G_f(s) & G_d(s)] \end{array} \overset{s}{=} \left[\begin{array}{c|ccc} A & B & B_f & B_d \\ \hline C & D & D_f & D_d \end{array}\right] \tag{1}$$

are the process, disturbance and fault transfer matrices, respectively.

In general, a residual signal in an FD system should represent the inconsistency between the actual system variables and the mathematical model, and respond to faults and disturbances only. A standard observer-based approach to generate a residual is the use of system duplication to cancel the input dynamics [1, 16]. A general observer-based FD scheme is illustrated in Figure 1. Here, $\hat{x}$ is the state estimate. The observer and residual gains $L \in \mathscr{R}^{n \times n_y}$ and $H \in \mathscr{R}^{n_f \times n_y}$, respectively, may be adjusted to achieve the FD objectives.

By defining the state $e(t) = x(t) - \hat{x}(t)$, the closed-loop form of the residual dynamics is given by

$$r(s) = T_{rf}(s) f(s) + T_{rd}(s) d(s),$$

where

$$\begin{array}{ccc} & n_f & n_d \\ n_f & [T_{rf}(s) & T_{rd}(s)] \end{array} \overset{s}{=} \left[\begin{array}{c|cc} A+LC & B_f+LD_f & B_d+LD_d \\ \hline HC & HD_f & HD_d \end{array}\right] \tag{2}$$

are the transfer matrices from faults and disturbances to residuals, respectively.

The observer-based scheme is equivalent to a filter-based scheme, as

$$r(s) = F(s)\left(G_f(s) f(s) + G_d(s) d(s)\right),$$

where

$$F(s) \overset{s}{=} (A+LC, L, HC, H) \in \mathscr{RH}_\infty^{n_f \times n_y}$$



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is a free stable post-filter with two degrees of freedom $L$ and $H$ to stabilize the residual dynamics and improve the detection performance. Thus

$$[T_{rf}(s)\;\; T_{rd}(s)] = F(s)[G_f(s)\;\; G_d(s)]. \tag{3}$$

With these preliminaries the problem under consideration can be formulated as follows.

*Problem 2.1*
Let $G(s) \in \mathscr{RH}_{\infty}^{n_y \times n_u}$, $G_d(s) \in \mathscr{RH}_{\infty}^{n_y \times n_d}$ and $G_f(s) \in \mathscr{RH}_{\infty}^{n_y \times n_f}$ have state–space realizations given in (1). Assume that:

A1. The pair $(A, C)$ is detectable (see Remark 2.1).
A2. $G_f(s)$ has no zeros on the extended imaginary axis (see Remark 2.2).
A3. $n_y \geqslant n_f$ (see Remark 2.3).

Find

$$\gamma_o := \inf\{\|FG_d\|_{\infty} : \|FG_f\|_{-} \geqslant 1,\, F(s) \stackrel{s}{=} (A+LC, L, HC, H) \in \mathscr{RH}_{\infty}^{n_f \times n_y}\}, \tag{4}$$

and an optimal filter $F(s) \stackrel{s}{=} (A+LC, L, HC, H) \in \mathscr{RH}_{\infty}^{n_f \times n_y}$ that achieves the minimum.

*Remark 2.1*
Assumption A1 is needed to guarantee the existence of at least one $L$ such that $A+LC$ is stable. □

*Remark 2.2*
The assumption on the zeros of $G_f(s)$ is necessary for $\|FG_f\|_{-} > 0$ and $F(s)$ stable. Note that the assumption implies that $G_f(\infty) = D_f$ has full column rank. There are two ways for relaxing this assumption. In the first, the definition of the $\|\cdot\|_{-}$ index is restricted to a frequency range for which $G_f(\mathrm{j}\omega)$ has full rank using the Generalized KYP Lemma [17]. See [18, 19] for more details. In the second, frequency weights are introduced to cancel the zeros on the imaginary axis. See [9, 13, 20] for more details. In Section 6 we adopt the second approach to remove this assumption by introducing weights and redefining the constraints in (4).

*Remark 2.3*
In the case that $n_y < n_f$, the assumptions A1 and A2, together with Theorem 13.32 in [21] imply that $G_f(s) = \hat{G}_f(s) G_i(s)$ where $\hat{G}_f(s) \in \mathscr{RL}_{\infty}^{n_y \times n_y}$ and $G_i(s) \in \mathscr{RL}_{\infty}^{n_y \times n_f}$ such that $G_i(s) G_i^{\sim}(s) = I_{n_y}$ and furthermore, $\hat{G}_f(s)$ can be chosen to have the same $A$ and $C$ matrices as $G_f(s)$. Thus, we may replace $G_f(s)$ by $\hat{G}_f(s)$ since $\|FG_f\|_{-} = \|F\hat{G}_f G_i\|_{-} = \|F\hat{G}_f\|_{-}$ and so there is no loss of generality in assuming that $n_f \leqslant n_y$.

Different versions of Problem 2.1 were considered in [3] (with the $\mathscr{H}_{-}$ index replaced by the $\mathscr{H}_{\infty}$–norm) and [2] (using $\mathscr{H}_2$–norms). Problem 2.1 was also considered in [6] where a mixed linear/quadratic matrix inequality-based suboptimal solution was given. In [5], a Riccati equation-based solution is given in the special case that $G_d(s)$ has full row rank over the extended imaginary axis. Problem 2.1 was also considered in [11], where they give an LMI formulation for $\|FG_f\|_{-} > \beta$ and another for $\|FG_d\|_{\infty} < \gamma$ in terms of an auxiliary Lyapunov matrix $P$. However, their solution is suboptimal since, as is common practice in multi-objective LMI-based design methods, the same Lyapunov matrix $P$ is used for both LMIs. Some of this conservativeness is ameliorated in [13], where they give the optimal solution in the form of a bilinear matrix inequality optimization problem. It is also argued in [6, 22] that using the $\mathscr{H}_{-}$–index in the cost function may result in an FD filter which is more robust against process uncertainty. In [15], a matrix factorization approach is used to address the problem, but without imposing an observer structure on the FD filter. Robust state–space approaches for certain classes of inputs are proposed in [23, 24], although the fault dynamics considered are different from the ones used here. In this paper we derive a complete state–space solution to the optimization problem.

## 3. EQUIVALENT FORMULATIONS

In this section we derive a sequence of equivalent formulations of Problem 2.1 that successively remove the constraints and that will allow a simple LMI solution.

First, we give an equivalent formulation to Problem 2.1 which removes the constraint on the stability of $F(s)$. The next result shows that if an $F(s)\in\mathscr{RL}_\infty^{n_f\times n_y}$ achieves the minimum, we can always find a left coprime factorization $F(s)=(M(s))^{-1}\tilde{F}(s)$ such that $M(s)\in\mathscr{RH}_\infty^{n_f\times n_f}$ is inner and $\tilde{F}(s)\in\mathscr{RH}_\infty^{n_f\times n_y}$. The main difficulty in the proof is the case that the realization of $F(s)$ is not detectable.

*Lemma 3.1*
Let all variables be as defined in Problem 2.1. Then

$$\begin{aligned}\gamma_o &:= \inf\{\|\tilde{F}G_d\|_\infty : \|\tilde{F}G_f\|_- \geqslant 1,\ \tilde{F}(s)\stackrel{s}{=}(A+\tilde{L}C,\tilde{L},\tilde{H}C,\tilde{H})\in\mathscr{RH}_\infty^{n_f\times n_y}\}\\ &= \inf\{\|FG_d\|_\infty : \|FG_f\|_- \geqslant 1,\ F(s)\stackrel{s}{=}(A+LC,L,HC,H)\in\mathscr{RL}_\infty^{n_f\times n_y}\}=:\gamma.\end{aligned}\qquad(5)$$

*Proof*
Note first that $\gamma_o\geqslant\gamma$ since $\mathscr{RH}_\infty^{n_f\times n_y}\subset\mathscr{RL}_\infty^{n_f\times n_y}$. Let $F(s)\stackrel{s}{=}(A+LC,L,HC,H)\in\mathscr{RL}_\infty^{n_f\times n_y}$ be a minimizer for (5) so that

$$\|FG_d\|_\infty=\gamma,\quad \|FG_f\|_-\geqslant 1.$$

To prove $\gamma_o=\gamma$ we construct $\tilde{F}(s)\stackrel{s}{=}(A+\tilde{L}C,\tilde{L},\tilde{H}C,\tilde{H})\in\mathscr{RH}_\infty^{n_f\times n_y}$ such that $\|\tilde{F}G_f\|_-\geqslant 1$ and $\|\tilde{F}G_d\|_\infty=\gamma$. The rest of the proof depends on the detectability of the pair $(A+LC,HC)$.

(a) Suppose that the pair $(A+LC,HC)$ is detectable. Then it follows from Theorem 13.34 in [21] that there exists a left coprime factorization $F(s)=(M(s))^{-1}\tilde{F}(s)$ such that $M(s)\in\mathscr{RH}_\infty^{n_f\times n_f}$ is inner and $\tilde{F}(s)\in\mathscr{RH}_\infty^{n_f\times n_y}$ where

$$[M(s)\ \ \tilde{F}(s)]\stackrel{s}{=}\left[\begin{array}{c|cc} A+LC+L_0HC & L_0 & L+L_0H\\ \hline HC & I_{n_f} & H\end{array}\right]\stackrel{s}{=}\left[\begin{array}{c|cc} A+\tilde{L}C & L_0 & \tilde{L}\\ \hline HC & I_{n_f} & H\end{array}\right],$$

where

$$\tilde{L}=L+L_0H,\qquad L_0=-XC^{\mathrm{T}}H^{\mathrm{T}}\qquad(6)$$

and where $X=X^{\mathrm{T}}\in\mathscr{R}^{n\times n}$ is the stabilizing solution of the algebraic Riccati equation

$$(A+LC)X+X(A+LC)^{\mathrm{T}}-XC^{\mathrm{T}}H^{\mathrm{T}}HCX=0.\qquad(7)$$

Then $\|\tilde{F}G_d\|_\infty=\|MFG_d\|_\infty=\gamma$ and $\|\tilde{F}G_f\|_-=\|MFG_f\|_-\geqslant 1$ since $M(s)$ is inner. Thus $\gamma_o=\gamma$.

(b) Suppose that the pair $(A+LC,HC)$ is not detectable. Then, introducing a similarity transformation on the data if necessary,

$$F(s)\stackrel{s}{=}\left[\begin{array}{c|c} A+LC & L\\ \hline HC & H\end{array}\right]\stackrel{s}{=}\left[\begin{array}{cc|c} A_{11}+L_1C_1 & A_{12}+L_1C_2 & L_1\\ 0 & A_{22}+L_2C_2 & L_2\\ \hline 0 & HC_2 & H\end{array}\right]\qquad(8)$$

where $A_{11}+L_1C_1$ is antistable (that is, all its poles are in the open right half-plane), the pair $(A_{22}+L_2C_2,HC_2)$ is detectable, $HC_1=0$ and $A_{21}+L_2C_1=0$ and where the partitioning of $A$ and $C$ is induced by the partitioning of the realization of $F(s)$ in (8). Next we prove



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that the pair $(A_{11}, C_1)$ is detectable. Since we assume that the pair $(A, C)$ is detectable, it follows that

$$\begin{bmatrix} A_{11}-sI & A_{12} \\ A_{21} & A_{22}-sI \\ C_1 & C_2 \end{bmatrix}$$

has full column rank for all $s$ such that $s+s'>0$. Using the fact that $A_{21}=-L_2C_1$, it follows that

$$\begin{bmatrix} A_{11}-sI & A_{12} \\ 0 & A_{22}+L_2C_2-sI \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & L_2 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{11}-sI & A_{12} \\ A_{21} & A_{22}-sI \\ C_1 & C_2 \end{bmatrix}$$

has full column rank for all $s$ such that $s+s'>0$. Thus, the pair $(A_{11}, C_1)$ is detectable. It follows that there exists $\tilde{L}_1$ such that $A_{11}+\tilde{L}_1C_1$ is stable. Define $\tilde{F} \overset{s}{=} (A+\tilde{L}C, \tilde{L}, HC, H)$ where $\tilde{L}=[\tilde{L}_1^{\mathrm{T}} \;\; L_2^{\mathrm{T}}]^{\mathrm{T}}$. Then

$$\tilde{F}(s) \overset{s}{=} \left[\begin{array}{cc|c} A_{11}+\tilde{L}_1C_1 & A_{12}+\tilde{L}_1C_2 & \tilde{L}_1 \\ 0 & A_{22}+L_2C_2 & L_2 \\ \hline 0 & HC_2 & H \end{array}\right]$$

is detectable. Furthermore, removing a nonminimal part of the realizations of $F(s)$ and $\tilde{F}(s)$ we get $\tilde{F}(s)=F(s)$. Then $\|\tilde{F}G_d\|_\infty=\gamma$ and $\|\tilde{F}G_f\|_- \geqslant 1$. Thus it follows from Part (a) that $\gamma_o=\gamma$. □

The next result shows that we may restrict $F(s)$ to the set of all (not necessarily stable) left inverses of $G_f(s)$.

*Lemma 3.2*
Let all variables be as defined in Problem 2.1. Then

$$\begin{aligned} \gamma_o &= \inf\{\|\tilde{F}G_d\|_\infty : \|\tilde{F}G_f\|_- \geqslant 1, \tilde{F}(s) \overset{s}{=} (A+\tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H}) \in \mathscr{RL}_\infty^{n_f\times n_y}\} \\ &= \inf\{\|FG_d\|_\infty : F(s)G_f(s)=I_{n_f}, F(s) \overset{s}{=} (A+LC, L, HC, H) \in \mathscr{RL}_\infty^{n_f\times n_y}\} =: \gamma. \end{aligned} \tag{9}$$

*Proof*
Note first that $\gamma_o \leqslant \gamma$ since $\|I_{n_f}\|_- \geqslant 1$. Let $\tilde{F}(s) \overset{s}{=} (A+\tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H})$ be a minimizer for (9) and define $Q(s)=\tilde{F}(s)G_f(s)$ so that $\|Q\|_- \geqslant 1$ and $\|\tilde{F}G_d\|_\infty=\gamma_o$. Then $Q^{-1} \in \mathscr{RL}_\infty^{n_f\times n_f}$ and $\|Q^{-1}\|_\infty \leqslant 1$. Let $F(s)=(Q(s))^{-1}\tilde{F}(s) \in \mathscr{RL}_\infty^{n_f\times n_f}$. Then a state–space manipulation verifies that $F(s) \overset{s}{=} (A+LC, L, HC, H)$ where $L=\tilde{L}-(B_f+\tilde{L}D_f)(\tilde{H}D_f)^{-1}\tilde{H}$ and $H=(\tilde{H}D_f)^{-1}\tilde{H}$. Furthermore, $F(s)G_f(s)=I_{n_f}$. Now, $\|FG_d\|_\infty=\|Q^{-1}\tilde{F}G_d\|_\infty \leqslant \|\tilde{F}G_d\|_\infty=\gamma_0$ since $\|Q^{-1}\|_\infty \leqslant 1$. This proves that $\gamma \leqslant \gamma_0$ and since $\gamma \geqslant \gamma_o$ it follows that $\gamma_o=\gamma$. □

*Remark 3.1*
An interpretation of the lemma is that the optimal cost function is the minimum, among all left inverses $F(s)$ of $G_f(s)$, of $\|FG_d\|_\infty$. In [5], a solution to Problem 2.1 is provided which is essentially equivalent to evaluating $1/\|FG_f\|_-$ for *one* left inverse of an outer factor $G_{do}(s)$ of $G_d(s)$. Their solution is therefore, in general, suboptimal (unless $G_{do}(s)$ is square, or, equivalently, unless $G_d(s)$ has full row rank over the extended imaginary axis) since it remained to minimize with respect to *all* left inverses of $G_{do}(s)$.

Using the expressions for $F(s)G_d(s)$ and $F(s)G_f(s)$ in (2) and (3), and with a slight abuse of notation to clarify dependence on $L$ and $H$, Lemma 3.2 asserts that

$$\gamma_o = \inf\left\{ \|\underbrace{(A+LC, B_d+LD_d, HC, HD_d)}_{F(s)G_d(s)}\|_\infty : \begin{bmatrix} L \\ H \end{bmatrix} \in \mathscr{R}^{(n+n_f)\times n_y}, \right.$$
$$\left. \underbrace{(A+LC, B_f+LD_f, HC, HD_f)}_{F(s)G_f(s)} \equiv I_{n_f} \right\}.$$

Consider the constraint

$$F(s)G_f(s) \stackrel{s}{=} (A+LC, B_f+LD_f, HC, HD_f) \equiv I_{n_f}.$$

We are concerned with the set of all $L$ and $H$ such that $F(s)G_f(s) \equiv I_{n_f}$. It follows that we certainly need $HD_f = I_{n_f}$ and, furthermore, $B_f + LD_f = 0$ is sufficient for $F(s)G_f(s) \equiv I_{n_f}$. Next we show that, without loss of generality, we may restrict $L$ such that $B_f + LD_f = 0$.

*Lemma 3.3*
Let all variables be as defined in Problem 2.1. Then, in a previously introduced notation,

$$\gamma_o = \inf\{\|(A+\tilde{L}C, B_d+\tilde{L}D_d, \tilde{H}C, \tilde{H}D_d)\|_\infty : \begin{bmatrix} \tilde{L} \\ \tilde{H} \end{bmatrix} \in \mathscr{R}^{(n+n_f)\times n_y},$$
$$(A+\tilde{L}C, B_f+\tilde{L}D_f, \tilde{H}C, \tilde{H}D_f) \equiv I\} \tag{10}$$
$$= \inf\{\|(A+LC, B_d+LD_d, HC, HD_d)\|_\infty : \begin{bmatrix} L \\ H \end{bmatrix} \in \mathscr{R}^{(n+n_f)\times n_y},$$
$$B_f + LD_f = 0, HD_f = I_{n_f}\} =: \gamma. \tag{11}$$

*Proof*
Note first that the argument given just before the lemma shows that $\gamma_o \leqslant \gamma$. Let $\tilde{L} \in \mathscr{R}^{n\times n_y}$ and $\tilde{H} \in \mathscr{R}^{n_f \times n_y}$ be optimal for (10). Then $\tilde{H}D_f = I_{n_f}$. Suppose now that the pair $(A+\tilde{L}C, \tilde{H}C)$ is observable. Then the constraint in (10) implies that $B_f + \tilde{L}D_f = 0$ and we are done. Suppose on the other hand that the pair $(A+\tilde{L}C, \tilde{H}C)$ is not observable. Then, introducing a similarity transformation on the state–space data if necessary,

$$\left[\begin{array}{c|c} A+\tilde{L}C & B_f+\tilde{L}D_f \\ \hline \tilde{H}C & \tilde{H}D_f \end{array}\right] = \left[\begin{array}{cc|c} A_{11}+\tilde{L}_1C_1 & A_{12}+\tilde{L}_1C_2 & B_{f_1}+\tilde{L}_1D_f \\ 0 & A_{22}+\tilde{L}_2C_2 & B_{f_2}+\tilde{L}_2D_f \\ \hline 0 & \tilde{H}C_2 & I_{n_f} \end{array}\right]$$

with the pair $(A_{22}+\tilde{L}_2C_2, HC_2)$ observable and where the partitioning on $A, \tilde{L}, C$ and $B_f$ is induced. Thus, $B_{f_2}+\tilde{L}_2D_f = 0$ is necessary for the constraint in (10). Furthermore,

$$\gamma_o = \left\|\left[\begin{array}{c|c} A+\tilde{L}C & B_d+\tilde{L}D_d \\ \hline \tilde{H}C & \tilde{H}D_d \end{array}\right]\right\|_\infty = \left\|\left[\begin{array}{cc|c} A_{11}+\tilde{L}_1C_1 & A_{12}+\tilde{L}_1C_2 & B_{d_1}+\tilde{L}_1D_d \\ 0 & A_{22}+\tilde{L}_2C_2 & B_{d_2}+\tilde{L}_2D_d \\ \hline 0 & \tilde{H}C_2 & \tilde{H}D_d \end{array}\right]\right\|_\infty$$
$$= \left\|\left[\begin{array}{c|c} A_{22}+\tilde{L}_2C_2 & B_{d_2}+\tilde{L}_2D_d \\ \hline \tilde{H}C_2 & \tilde{H}D_{d_2} \end{array}\right]\right\|_\infty$$



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by removing the unobservable part. Let $L_2=\tilde{L}_2$ and choose $L_1$ such that $B_{f_1}+L_1D_f=0$. This is possible since $D_f$ has full column rank by assumption. Now define $H=\tilde{H}$ and $L=[L_1^{\mathrm{T}}\ \ L_2^{\mathrm{T}}]^{\mathrm{T}}$. Then $HD_f=I_{n_f}$ and $B_f+LD_f=0$. Furthermore,

$$\left\|\left[\begin{array}{c|c} A+LC & B_d+LD_d \\ \hline HC & HD_d \end{array}\right]\right\|_\infty = \left\|\left[\begin{array}{cc|c} A_{11}+L_1C_1 & A_{12}+L_1C_2 & B_{d_1}+L_1D_d \\ 0 & A_{22}+L_2C_2 & B_{d_2}+L_2D_d \\ \hline 0 & HC_2 & HD_d \end{array}\right]\right\|_\infty$$

$$= \left\|\left[\begin{array}{c|c} A_{22}+\tilde{L}_2C_2 & B_{d_2}+\tilde{L}_2D_d \\ \hline \tilde{H}C_2 & \tilde{H}D_{d_2} \end{array}\right]\right\|_\infty = \gamma_o$$

by removing the unobservable part and since $L_2=\tilde{L}_2$. Thus $\gamma_o=\gamma$. □

## 4. LMI SOLUTION

Lemma 3.3 reduces the FD problem to the solution of the optimization problem in (11). In this section, we use a parameterization of all $L$ and $H$ such that $B_f+LD_f=0$ and $HD_f=0$, as well as a generalized version of the bounded real lemma (where we do not impose stability constraints, [25]) to solve Problem 2.1.

Since $D_f\in\mathscr{R}^{n_y\times n_f}$ has full column rank, there exists $D_f^{\dagger}\in\mathscr{R}^{n_f\times n_y}$ and $D_f^{\perp}\in\mathscr{R}^{(n_y-n_f)\times n_y}$ (which can be obtained, say, from a singular value decomposition of $D_f$) such that

$$\begin{bmatrix} D_f^{\dagger} \\ D_f^{\perp} \end{bmatrix} D_f = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}, \quad \operatorname{rank}\left(\begin{bmatrix} D_f^{\dagger} \\ D_f^{\perp} \end{bmatrix}\right) = n_y. \tag{12}$$

It follows that all $L$ and $H$ satisfying the constraints in (11) are given by

$$L=-B_fD_f^{\dagger}+RD_f^{\perp}, \quad H=D_f^{\dagger}+SD_f^{\perp}, \tag{13}$$

where $R\in\mathscr{R}^{n\times(n_y-n_f)}$ and $S\in\mathscr{R}^{n_f\times(n_y-n_f)}$ are free parameters.

The next result gives a non-linear matrix inequality solution to the optimization problem in (11) and follows from the application of a Schur complement to the generalized bounded real lemma of [25] and the parameterization of $L$ and $H$ in (13). Note that, for clarity, we indicate all variables by bold type.

*Theorem 4.1*
Let all variables be as defined in Lemma 3.3. Then

$$\gamma_o=\inf\left\{\gamma:\boldsymbol{R}\in\mathscr{R}^{n\times(n_y-n_f)},\boldsymbol{S}\in\mathscr{R}^{n_f\times(n_y-n_f)},\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}\in\mathscr{R}^{n\times n},\right.$$

$$\left.\begin{bmatrix} (A-B_fD_f^{\dagger}C+\boldsymbol{R}D_f^{\perp}C)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(A-B_fD_f^{\dagger}C+\boldsymbol{R}D_f^{\perp}C) & \star & \star \\ (B_d-B_fD_f^{\dagger}D_d+\boldsymbol{R}D_f^{\perp}D_d)^{\mathrm{T}}\boldsymbol{P} & -\gamma I & \star \\ D_f^{\dagger}C+\boldsymbol{S}D_f^{\perp}C & D_f^{\dagger}D_d+\boldsymbol{S}D_f^{\perp}D_d & -\gamma I \end{bmatrix}\prec 0\right\} \tag{14}$$

where $\star$ denotes terms readily inferred from symmetry.

An inspection of the matrix inequality in (14) shows that the only non-linear term is the product $\boldsymbol{PR}$. The following result shows that, by defining a new variable $\boldsymbol{Z}=\boldsymbol{PR}$, the matrix inequality can be linearized. The main difficulty is the proof that we can always choose a nonsingular $\boldsymbol{P}$ (to ensure a unique solution $\boldsymbol{R}$ to $\boldsymbol{PR}=\boldsymbol{Z}$) and that $A-B_fD_f^{\dagger}C+\boldsymbol{R}D_f^{\perp}C$, which is equal to $A+LC$, has no imaginary axis eigenvalues thus ensuring that $F(s)\in\mathscr{RL}_{\infty}^{n_f\times n_y}$, that is, $F(s)$ has no imaginary axis poles, which is a requirement of Lemma 3.1.

*Theorem 4.2*
Let all variables be as defined in Theorem 4.1. Then

$$\gamma_o=\inf\left\{\gamma:\boldsymbol{Z}\in\mathscr{R}^{n\times(n_y-n_f)},\boldsymbol{S}\in\mathscr{R}^{n_f\times(n_y-n_f)},\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}\in\mathscr{R}^{n\times n},\right.$$

$$\left.\begin{bmatrix}(A-B_fD_f^{\dagger}C)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(A-B_fD_f^{\dagger}C)+\boldsymbol{Z}D_f^{\perp}C+(D_f^{\perp}C)^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} & \star & \star\\ (B_d-B_fD_f^{\dagger}D_d)^{\mathrm{T}}\boldsymbol{P}+(D_f^{\perp}D_d)^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} & -\gamma I & \star\\ D_f^{\dagger}C+\boldsymbol{S}D_f^{\perp}C & D_f^{\dagger}D_d+\boldsymbol{S}D_f^{\perp}D_d & -\gamma I\end{bmatrix}\prec 0\right\}. \tag{15}$$

Furthermore, $\boldsymbol{P}$ can be chosen to be nonsingular. Finally, $A-B_fD_f^{\dagger}C+\boldsymbol{R}D_f^{\perp}C$ has no eigenvalues on the imaginary axis, so that the optimal filter $F(s)\stackrel{s}{=}(A+LC,L,HC,H)\in\mathscr{RL}_{\infty}^{n_f\times n_y}$, where $L$ and $H$ are defined in (13).

*Proof*
We can write (14) as

$$\gamma_o=\inf\left\{\gamma:\boldsymbol{R}\in\mathscr{R}^{n\times(n_y-n_f)},\boldsymbol{Z}\in\mathscr{R}^{n\times(n_y-n_f)},\boldsymbol{S}\in\mathscr{R}^{n_f\times(n_y-n_f)},\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}\in\mathscr{R}^{n\times n},\boldsymbol{Z}=\boldsymbol{PR},\right.$$

$$\left.\begin{bmatrix}A_1^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}A_1+\boldsymbol{Z}C_2+C_2^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} & \star & \star\\ B_1^{\mathrm{T}}\boldsymbol{P}+D_2^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} & -\gamma I & \star\\ C_1+\boldsymbol{S}C_2 & D_1+\boldsymbol{S}D_2 & -\gamma I\end{bmatrix}\prec 0\right\}, \tag{16}$$

where for notational convenience we have defined

$$\left[\begin{array}{c|c}A_1 & B_1\\ \hline C_1 & D_1\\ C_2 & D_2\end{array}\right]=\left[\begin{array}{c|c}A-B_fD_f^{\dagger}C & B_d-B_fD_f^{\dagger}D_d\\ \hline D_f^{\dagger}C & D_f^{\dagger}D_d\\ D_f^{\perp}C & D_f^{\perp}D_d\end{array}\right]. \tag{17}$$

Thus, the result follows if we can show that $\boldsymbol{P}$ can always be chosen to be nonsingular. Suppose $\boldsymbol{P}$ is singular and denote the partitioned matrix in (16) by $\mathscr{L}(\boldsymbol{P})$. Then $\mathscr{L}(\boldsymbol{P}+\varepsilon I)=\mathscr{L}(\boldsymbol{P})+\varepsilon\mathscr{L}_1$ where

$$\mathscr{L}_1=\begin{bmatrix}A_1^{\mathrm{T}}+A_1 & B_1 & 0\\ B_1^{\mathrm{T}} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$



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Since $\mathscr{L}(\boldsymbol{P}) \prec 0$ it is clear that $\varepsilon$ can be chosen such that $\boldsymbol{P}+\varepsilon I$ is nonsingular and $\mathscr{L}(\boldsymbol{P}+\varepsilon I) \prec 0$. Replacing $\boldsymbol{P}$ by $\boldsymbol{P}+\varepsilon I$, defining $\boldsymbol{R}=\boldsymbol{P}^{-1}\boldsymbol{Z}$ and comparing with (14) prove (16).

To prove the final part, assume for contradiction that $(A-B_f D_f^{\dagger} C+\boldsymbol{R} D_f^{\perp} C)x=\lambda x$ with $x\neq 0$ and $\lambda+\lambda'=0$. Then pre- and post-multiplying the (1,1) entry of the matrix in (14) by $x'$ and $x$, respectively, gives $(\lambda+\lambda')x'\boldsymbol{P}x=0$ which contradicts negative definiteness. □

*Remark 4.1*
We have opted for a strict LMI in (14) to avoid numerical difficulties associated with optimality. Note that the existence of nonsingular $\boldsymbol{P}$ and the fact that $A+LC$ has no imaginary axis eigenvalues depend crucially on the strict character of the LMI. It follows that the solution may be badly conditioned near optimality, which is a common issue for LMI problems [26].

*Remark 4.2*
In [13] an optimal solution to Problem 2.1 is given, but in terms of a bilinear matrix inequality optimization problem, whose solution is in general intractable. A suboptimal solution is then derived by linearizing these matrix inequalities. In contrast, Theorem 4.2 gives the optimal solution of Problem 2.1 in the form of an LMI optimization problem.

By way of summarizing our results we give the following algorithm for the solution of Problem 2.1

*Algorithm 4.1*
Let $G_d(s)\in\mathscr{RL}_\infty^{n_y\times n_d}$ and $G_f(s)\in\mathscr{RL}_\infty^{n_y\times n_f}$ be as given in (1). Assume that $n_y\geqslant n_f$, $(A,C)$ is detectable and that $G_f(s)$ has no extended imaginary axis zeros.

1. Define $D_f^{\dagger}$ and $D_f^{\perp}$ such that (12) is satisfied.
2. Define $A_1, B_1, C_1, C_2, D_1$ and $D_2$ from (17).
3. Find $\gamma_o, \boldsymbol{P}, \boldsymbol{Z}$ and $\boldsymbol{S}$ that solve the LMI optimization in (15).
4. If $\boldsymbol{P}$ is singular, perturb $\boldsymbol{P}$ as described in Theorem 4.2 such that $\boldsymbol{P}$ is nonsingular.
5. Define $\boldsymbol{R}=\boldsymbol{P}^{-1}\boldsymbol{Z}$.
6. Define $L$ and $H$ as in (13).
   (a) If $A+LC$ is stable, we are done.
   (b) Else, if the pair $(A+LC, HC)$ is detectable, let $X=X^{\mathrm{T}}\in\mathscr{R}^{n\times n}$ be the stabilizing solution of the algebraic Riccati equation (7) and redefine $L:=L-XC^{\mathrm{T}}H^{\mathrm{T}}H$.
   (c) Else, redefine $L$ using the procedure in part (2) of the proof of Lemma 3.1 so that the pair $(A+LC, HC)$ is detectable and go to Step 6(b).
7. The optimal FD filter is given by $F(s)\stackrel{s}{=}(A+LC, L, HC, H)\in\mathscr{RH}_\infty^{n_f\times n_y}$.

*Remark 4.3*
Suppose that $n_y=n_f$, that is, suppose that the number of faults is equal to the number of measurement. It follows from (13) that the optimal $L$ and $H$ are given simply by $L=-B_f D_f^{-1}$ and $H=D_f^{-1}$ if $A+LC$ is stable. (If $A+LC$ is unstable, we replace $L$ by $\tilde{L}$ using (6) and (7).) Furthermore, it follows from Theorem 4.2 and the definitions in (17) that

$$\gamma_o=\inf\left\{\gamma:\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}\in\mathscr{R}^{n\times n},\ \begin{bmatrix} A_1^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}A_1 & \star & \star \\ B_1^{\mathrm{T}}\boldsymbol{P} & -\gamma I & \star \\ C_1 & D_1 & -\gamma I \end{bmatrix}\prec 0\right\}=\|(A_1,B_1,C_1,D_1)\|_\infty. \tag{18}$$

*Remark 4.4*
Apart from the issues of the possible singularity of $P$ (Step 4) and the loss of detectability of the pair $(A+LC, HC)$ (Step 6(c)), both of which are pathological, the implementation of Algorithm 4.1 involves a singular value decomposition (Step 1), the solution of an LMI optimization (Step 3)

and, in case $A+LC$ is unstable, the solution of an algebraic Riccati equation (Step 6(b)). Thus, generically, the algorithm is straightforward to implement.

*Remark 4.5*
Suppose that Algorithm 4.1 terminates at Step 6(a). Then we have that $F(s)G_f(s)=I_{n_f}$ and $F(s)$ is stable. It follows that our optimal FD filter also achieves fault isolation, wherein the transfer matrix from the fault to the residual signals is the identity matrix.

## 5. MODEL MATCHING FORMULATION

Theorem 4.2 gives an attractive LMI-based numerical solution of the FD problem. From a system theoretic point of view, it is interesting to give a formulation of the FD problem in terms of the solution of a model-matching problem.

Consider the expression for $\gamma_o$ in (9). By defining

$$M(s)=\begin{bmatrix} G_f^{\dagger}(s) \\ G_f^{\perp}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A-B_f D_f^{\dagger} C & -B_f D_f^{\dagger} \\ \hline D_f^{\dagger} C & D_f^{\dagger} \\ D_f^{\perp} C & D_f^{\perp} \end{array}\right]$$

it is easy to check that $M(s)G_f(s)=[I_f \;\; 0]^{\mathrm{T}}$. It follows that all $F(s)$ such that $F(s)G_f(s)=I_{n_f}$ are given as

$$F(s)=G_f^{\dagger}(s)+Q(s)G_f^{\perp}(s)=[I_f \;\; Q(s)]M(s) \tag{19}$$

for arbitrary $Q(s)$. It follows that $F(s)G_d(s)=T_{11}(s)+Q(s)T_{21}(s)$ where

$$T(s)=\begin{bmatrix} T_{11}(s) \\ T_{21}(s) \end{bmatrix}=M(s)G_d(s).$$

Furthermore, a state–space calculation shows that $T(s)$ has the state–space realization given in (17). A further state–space calculation using (19) shows that $F(s)$ has an observer form $F(s)\stackrel{s}{=}(A+LC, L, HC, H)$ if and only if $Q(s)$ has a corresponding observer form (relative to $T(s)$) $Q(s)\stackrel{s}{=}(A_1+RC_2, R, C_1+SC_2, S)$. It follows that Problem 2.1 has an equivalent model-matching formulation

$$\gamma_o=\inf\{\|T_{11}+QT_{21}\|_\infty : Q(s)\stackrel{s}{=}(A_1+RC_2, R, C_1+SC_2, S)\}$$

whose solution is, of course, given by Theorem 4.2.

The model-matching formulation above allows us to highlight two special cases in which the solution is particularly interesting. Since $T_{21}(s)\in\mathscr{RL}_\infty^{(n_y-n_f)\times n_d}$, it follows that, in the case that $n_y=n_f$, we have that $\gamma_o=\|G_f^{-1}G_d\|_\infty$, which agrees with (18) since $(G_f(s))^{-1}G_d(s)\stackrel{s}{=}(A_1, B_1, C_1, D_1)$. Furthermore, in the general case, $\gamma_o=0$ if and only if the equation $T_{11}(s)+Q(s)T_{21}(s)=0$ has a solution $Q(s)\in\mathscr{RL}_\infty^{n_f\times(n_y-n_f)}$. This result is consistent with Theorem 5.1 and Lemma 3.1 in [3] since

$$M(s)[G_f(s) \;\; G_d(s)]=\begin{bmatrix} I_f & T_{12}(s) \\ 0 & T_{21}(s)\end{bmatrix}$$

and $M(s)$ has full column rank.

## 6. WEIGHTING FILTER DESIGN

Although Theorem 4.2 gives a complete solution to Problem 2.1, the design based on the result suffers from two drawbacks. The first concerns the restrictive assumption on the zeros of $G_f(s)$, namely, that $G_f(s)$ has no extended imaginary-axis zeros, and in particular the requirement that $D_f$ has full column rank. The second concerns the cost functions in the problem formulation, namely, the use of the whole frequency range for the optimization. Effectively, we require the largest singular value of $T_{rd}(\mathrm{j}\omega)$ to be small compared with the smallest singular value of $T_{rf}(\mathrm{j}\omega)$ over the whole frequency range. This requirement translates into the requirement that $\gamma_o$ is small compared with 1, which may not always be met.

In principle, for FD, and through the use of appropriate post-filters, we only need the largest singular value of $T_{rd}(\mathrm{j}\omega)$ to be small compared with the smallest singular value of $T_{rf}(\mathrm{j}\omega)$ on a small frequency range, or even at a single frequency. This also facilitates the design of a threshold to avoid missing faults or false alarms.

To tackle both these issues, we outline a procedure for introducing a set of weighting filters to further enhance the performance of the original filter $F(s)$. The basic idea is to choose a suitable bandpass weighting filter $W_f(s)$ that ensures the sensitivity to faults to be maintained around a certain frequency $\omega_0$. At the same time, another weighting filter $W_d(s)$ can be selected to suppress the sensitivity to disturbances in the same frequency range.

Furthermore, the weighting filter $W_f(s)$ is chosen such that it shares the same extended imaginary-axis zeros as $G_f(s)$, including direction; all other poles and zeros being in the open left half of the complex plane, i.e. such that $G_f(s)(W_f(s))^{-1}$ has full rank over the extended imaginary-axis and has detectable realization if $G_f(s)$ does. One way of achieving this is to effect a factorization $G_f(s)=G_o(s)G_z(s)$ where $G_o(s)$ has no extended imaginary-axis zeros and all the zeros of $G_z(s)$ are on the extended imaginary-axis. Then $W_f(s)$ is chosen of the form $W_f(s)=W_o(s)G_z(s)$ where $W_o(s)$ is stable and minimum phase. Of course, we still need the assumption that $G_f(s)$ has full normal rank over the imaginary axis, that is, it has full rank except for a finite number of points on the extended imaginary-axis.

The following results achieve such a factorization. The procedure is a standard application of linear algebra and therefore we will only sketch the proof.

*Lemma 6.1*

Let $G(s)=D+C(sI-A)^{-1}B\in\mathscr{RL}_\infty^{p\times m}$ with $p\geqslant m$. Assume that $G(s)$ has full normal column rank over $\mathrm{j}\mathscr{R}_e$ and that the pair $(A, C)$ is detectable. Then $G(s)=G_1(s)G_2(s)$ where $G_1(s)\in\mathscr{RL}_\infty^{p\times m}$ has full rank over $\mathrm{j}\mathscr{R}_e$ and $G_2(s)\in\mathscr{RL}_\infty^{m\times m}$ has all its zeros on $\mathrm{j}\mathscr{R}_e$. Furthermore, the poles of $G_2(s)$ may be chosen anywhere in the complex plane. Finally, $G_1(s)$ may be chosen such that it has the same $A$ and $C$ matrices as $G(s)$.

*Proof*

First we deal with the finite zeros of $G(s)$. Recall that the (transmission) zeros of $G(s)$ are the generalized eigenvalues of the matrix pencil

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}.$$

Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\overbrace{\begin{bmatrix} X \\ U \end{bmatrix}}^{T}=\begin{bmatrix} X \\ 0 \end{bmatrix}\Lambda \tag{20}$$

for some $X, U$ and $\Lambda$ where the eigenvalues of $\Lambda$ are the finite zeros of $G(s)$. In addition, $T$ has full column rank since $G(s)$ has full normal column rank. Define

$$G_1(s)\stackrel{s}{=}\left[\begin{array}{c|c} A & B+XL \\ \hline C & D \end{array}\right], \quad G_2(s)\stackrel{s}{=}\left[\begin{array}{c|c} \Lambda+LU & L \\ \hline U & I_m \end{array}\right]$$

where $L$ is determined below and note that $G_1(s)$ shares the same $A$ and $C$ matrices as $G(s)$. Then a calculation using (20) shows that $G(s)=G_1(s)G_2(s)$. Furthermore, $(G_2(s))^{-1}\stackrel{s}{=}(\Lambda, -L, U, I)$ so that the zeros of $G_2(s)$ are the finite zeros of $G(s)$. It follows that $G_1(s)$ has no finite $\mathrm{j}\mathscr{R}_e$



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zeros. Next, we show that $L$ may be chosen so that the poles of $G_2(s)$ may be placed anywhere on the complex plane, that is, we show that the pair $(\Lambda, U)$ is observable. Assume on the contrary that $\Lambda z = \lambda z$ and $Uz = 0$ for some $z \neq 0$ and some $\lambda$ (which is a finite $\mathrm{j}\mathscr{R}_e$ zero of $G(s)$). Then post-multiplying (20) by $z$, and noting that $Xz \neq 0$ since $T$ has full column rank, we get $AXz = \lambda Xz$ and $CXz = 0$ contradicting the detectability of $(A, C)$.

Next, we deal with the zeros at $\infty$. Now $G_1(s)$ has no finite zeros on $\mathrm{j}\mathscr{R}_e$. It follows that, by introducing the change of variable $z = s^{-1}$, we have $G_1(s) = \hat{G}_1(z)$ where $\hat{G}_1(z)$ has full rank at $\infty$ and only finite zeros (at $z = 0$). We repeat the procedure in the first part and the result follows by reversing the transformation. Note that if $G(s) = D + C(sI - A)^{-1}B$ and $z = s^{-1}$ then $G(s) = \hat{G}(z)$ where $\hat{G}(z) = D - CA^{-1}B - CA^{-1}(zI - A^{-1})^{-1}A^{-1}B$. It follows that $G_1(s)$ shares the same $A$ and $C$ matrices as $G(s)$. □

We are now in a position to modify Problem 2.1 by introducing weights and removing the assumption that $G_f(s)$ has no $\mathrm{j}\mathscr{R}_e$ zeros.

*Problem 6.1*
Let

$$[\hat{G}_f(s) \;\; G_d(s)] \stackrel{s}{=} \left[\begin{array}{c|cc} A & \hat{B}_f & B_d \\ \hline C & \hat{D}_f & D_d \end{array}\right].$$

Assume that the pair $(A, C)$ is detectable and that $\hat{G}_f(s)$ has full normal column rank. Then $\hat{G}_f(s) = G_f(s)G_z(s)$ where $G_f(s) \stackrel{s}{=} (A, B_f, C, D_f)$ has full rank over $\mathrm{j}\mathscr{R}_e$ and

$$[G_f(s) \;\; G_d(s)] \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_f & B_d \\ \hline C & D_f & D_d \end{array}\right].$$

Given stable and minimum-phase weighting functions $W_f(s)$ and $W_d(s)$. Find

$$\begin{aligned}\gamma_o := \inf\{\gamma : \bar{\sigma}(F(\mathrm{j}\omega)G_d(\mathrm{j}\omega)) \leqslant \gamma \underline{\sigma}(W_d(\mathrm{j}\omega)), \underline{\sigma}(F(\mathrm{j}\omega)G_f(\mathrm{j}\omega)) \\ \geqslant \bar{\sigma}(W_f(\mathrm{j}\omega)) \forall \omega \in \mathscr{R}, F(s) \in \mathscr{RH}_\infty^{n_f \times n_y}\},\end{aligned}$$

and an optimal filter $F(s)$ that achieves the design.

It is clear that Problem 6.1 reduces to Problem 2.1 since $G_f W_f^{-1}$ has full column rank on the imaginary axis and since

$$\bar{\sigma}(F(\mathrm{j}\omega)G_d(\mathrm{j}\omega)) \leqslant \gamma \underline{\sigma}(W_d(\mathrm{j}\omega)) \;\; \forall \omega \in \mathscr{R} \Leftrightarrow \|FG_dW_d^{-1}\|_\infty \leqslant \gamma$$

and

$$\underline{\sigma}(F(\mathrm{j}\omega)G_f(\mathrm{j}\omega)) \geqslant \bar{\sigma}(W_f(\mathrm{j}\omega)) \;\; \forall \omega \in \mathscr{R} \Leftrightarrow \|FG_fW_f^{-1}\|_- \geqslant 1.$$

On the other hand

$$\underline{\sigma}(F(\mathrm{j}\omega)G_f(\mathrm{j}\omega)) \geqslant \bar{\sigma}(W_f(\mathrm{j}\omega)) \Leftrightarrow \underline{\sigma}(F(\mathrm{j}\omega)\hat{G}_f(\mathrm{j}\omega)) \geqslant \bar{\sigma}(W_f(\mathrm{j}\omega)G_z(\mathrm{j}\omega)).$$

It follows that if $\hat{G}_f(s)$ has zeros on the imaginary axis, then the Fd filter ensures that the singular values of the fault-to-residual dynamics are larger than the singular values of $W_fG_z$.

*Remark 6.1*
If we choose $W_f(s)$ to be a bandpass filter around a frequency $\omega_0$ and if we choose $W_d(s) = (W_f(s))^{-1}$, then if $\gamma_o < 1$, we have a frequency separation between the singular values of $T_{rf}(\mathrm{j}\omega)$

and $T_{rd}(j\omega)$ around $\omega_0$. That is, the smallest singular value of $T_{rf}(j\omega)$ will be larger than the largest singular value of $T_{rd}(j\omega)$ around $\omega_0$. It follows that, in this case, FD can be effected, through the use of appropriate bandpass filters, around the frequency $\omega_0$.

*Remark 6.2*
Note that, in general, the order of the realizations for the weighted system $[G_f(s)(W_f(s))^{-1} \;\; G_d(s)(W_d(s))^{-1}]$ will be larger than that of $[G_f(s) \;\; G_d(s)]$, which will lead to a degree inflation in the order of the filter.

*Remark 6.3*
A second approach for tackling the issue of imaginary-axis zeros of $G_f(s)$ is to carry out the filter design over a finite frequency range in which $G_f(s)$ has full rank using the Generalized KYP Lemma [17–19]. This procedure, however, cannot be readily incorporated into our approach. Recall from Lemma 3.2 that we require $F(s)G_f(s)=I_{n_f}$ to ensure the residual is sufficiently sensitive to the fault signal. It is not clear how this constraint can be imposed over a finite frequency range and this issue requires further research.

## 7. ILLUSTRATIVE EXAMPLES

To illustrate the application of the detection filter scheme in a real system two examples are considered in this section.

First a jet engine example is considered. The GE-21 jet engine state–space model [27–29] is given as

$$[G(s) \mid G_f(s) \mid G_d(s)] \overset{s}{=} \left[\begin{array}{c||c|c|c} A & B & B_f & B_d \\ \hline C & D & D_f & D_d \end{array}\right],$$

$$= \left[\begin{array}{cc||ccc|cc|cc} -3.370 & 1.636 & 0.586 & -1.419 & 1.252 & 0.586 & -1.419 & -0.2973 & 0.0467 \\ -0.325 & -1.896 & 0.410 & 1.118 & 0.139 & 0.410 & 1.118 & 0.0094 & -0.1814 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0.0865 & 0.1177 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0.3331 & -0.4366 \\ 0.731 & 0.786 & 0.267 & -0.025 & -0.146 & 0 & 0 & -0.0251 & 0.0273 \end{array}\right].$$

The system has $n=2$ and $n_y=n_u=3$. We suppose that this system is subjected to two potential faults and two disturbances ($n_f=2$ and $n_d=2$). Since the example in [28, 29] does not involve disturbances, our $G_d(s)$ has been randomly generated. Since $n_f<n_y$, we apply Algorithm 4.1 in Section 4. Following Theorem 4.2, we have

$$P=\begin{bmatrix} 8.0956 & 10.5902 \\ 10.5902 & 20.9228 \end{bmatrix}, \quad Z=\begin{bmatrix} -261.7886 \\ -368.5587 \end{bmatrix}, \quad S=\begin{bmatrix} -0.3219 \\ 14.1901 \end{bmatrix}, \quad R=\begin{bmatrix} -27.5071 \\ -3.6923 \end{bmatrix}$$

by solving (15). According to (13), the observer gains are given as

$$L=\begin{bmatrix} -0.586 & 1.419 & -27.5071 \\ -0.410 & -1.118 & -3.6923 \end{bmatrix}, \quad H=\begin{bmatrix} 1 & 0 & -0.3219 \\ 0 & 1 & 14.1901 \end{bmatrix},$$

and the corresponding optimal $\gamma_o=0.4995$. Note that since $A+LC$ is stable, $T_{rf}(s)=I_2$ and our filter achieves fault isolation (see Remark 4.5). Note also that since $n_y=3>n_d=2$, the technique in [5] is not applicable since it assumes $G_d(s)$ has full row rank over the extended imaginary axis (Remark 3.1).

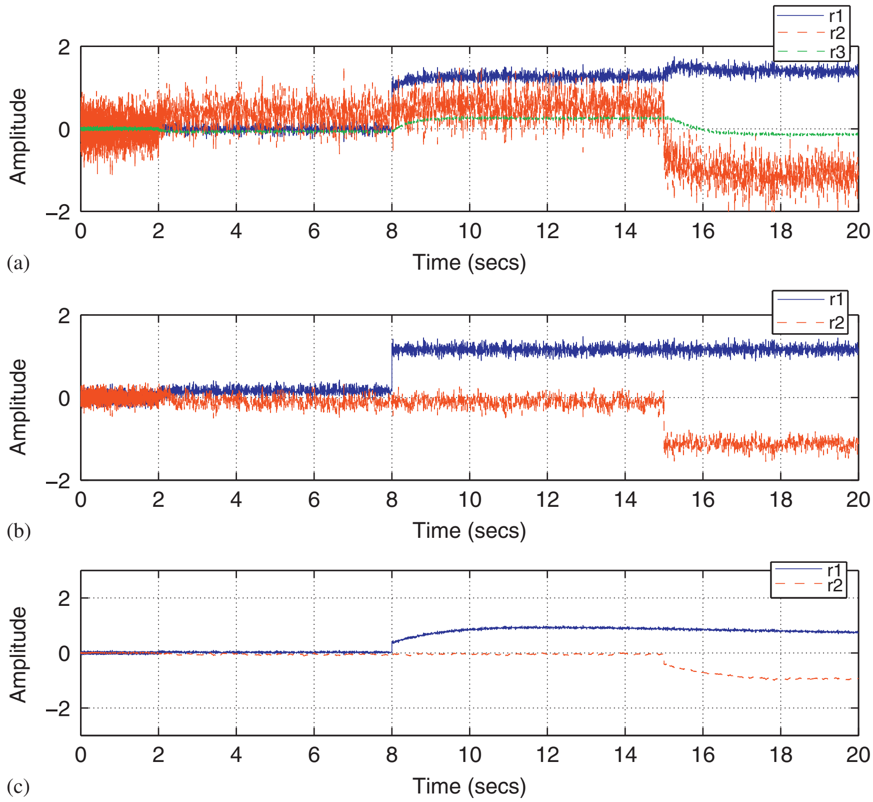
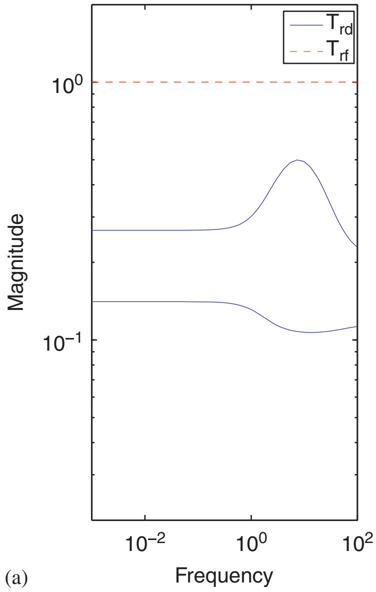



Figure 2. Response of the filtered fault-detection residual: (a) residual time responses before the filter; (b) residual time responses after the filter; and (c) weighted residual time responses.

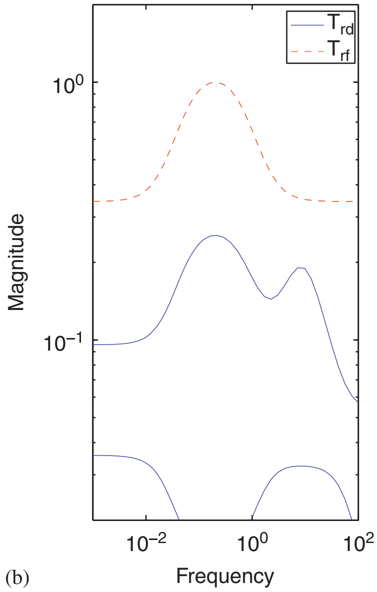
Assume that the disturbance 1 is a constant bias of amplitude 1 applied from the 2nd second while the disturbance 2 is a white noise with mean zero and standard deviation 1. Fault 1 and fault 2, simulated by an abrupt jump, are connected from the 8th second and 15th second, respectively. Figure 2(a) gives the residual response before the filter (i.e. of the signal $G_f(s)f(s)+G_d(s)d(s)$, obtained by setting $L=0$ and $H=I_{n_y}$). Figure 2(b) gives the filtered residual responses. Next, a bandpass filter centered around a frequency $\omega_0=0.2\,\text{rad/s}$ was used, with $W_d(s)=(W_f(s))^{-1}$ as introduced in Section 6. The optimal value of $\gamma$ is now $\gamma_o=0.2547$ with the corresponding residual given in Figure 2(c). The frequency responses are also compared in Figure 3. The frequency response of $W_f(s)$ is identical to that of $T_{rf}(s)$ shown in Figure 3(b). The result validates the employment of a weighting filter. Note that with the design incorporating the weights, it is easier to determine a residual threshold for FD given that the frequency property of the disturbance is known. The example makes it clear that the designed filter satisfies the performance requirement of rapid FD which is sufficiently robust against disturbances.

The second example is from [13] and has the data

$$
\left[\begin{array}{c||c|c} A & B_f & B_d \\ \hline C & D_f & D_d \end{array}\right]=\left[\begin{array}{cccc||cc|ccc}
-2.1210 & -0.5624 & -0.2651 & -0.2500 & 1 & 0 & 0.02 & -0.02 & 0 \\
4.0000 & 0 & 0 & 0 & 0 & 1 & 0.02 & 0.1 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 1 & 0.02 & -0.02 & 0 \\
0 & 0 & 0.2500 & 0 & 0 & 1 & 0.02 & 0.1 & 0 \\
\hline
-1.4140 & -0.4374 & -0.1768 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1
\end{array}\right].
$$



(a)



(b)

Figure 3. Frequency response of the residual: (a) optimal filter response and (b) weighting filter response.

Note that for this example, we have $n_y = n_f = 2$. It follows from Remark 4.3 that

$$L = -B_f D_f^{-1} = -\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \quad H = D_f^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{21}$$

are optimal observer gains. It can be verified that $A+LC$ is stable, and so we have that $T_{rf}(s) = I_2$. Thus, our FD filter also achieves fault isolation (Remark 4.5). The value of the optimal disturbance attenuation is $\gamma_o = 3.1707 = \frac{1}{0.3154}$ which agrees with the method of [5] and is slightly better than the solution of [13] ($\gamma_o = 3.1712 = \frac{1}{0.3153}$), after appropriate scaling. This verifies that the iterative method used in [13] gives a near optimal solution. Note that, compared with the solution given in [13], our solution is obtained directly as in (21) rather than using an iterative procedure. Furthermore, unlike the methods in [5, 13], our filter achieves fault isolation.

## 8. SUMMARY

We have given a full state–space solution to the observer-based optimal mixed $\mathscr{H}_-/\mathscr{H}_\infty$ FD problem in the form of an LMI optimization. By incorporating weighting functions, we have improved the design flexibility as well as removing a rank assumption on the fault dynamics [15].

Our FD algorithm is maximally robust against disturbances, and has some robustness properties against plant uncertainties [6]. Although these uncertainties may be recast as disturbances, this is often inadequate since, in many cases, model uncertainties appear as either parametric or unstructured uncertainties [28, 30, 31]. This may be a serious problem since model-based FD schemes operate in the open-loop [1]. It is hoped that robust $\mathscr{H}_\infty$-detection filtering approaches [20, 32, 33] can be incorporated into our framework.



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