FAULT DETECTION AND ISOALTION

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FDP Statement

- Consider the LTI system with faults and disturbances:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_f f(t) + B_d d(t) \quad (1)$$
$$y(t) = Cx(t) + Du(t) + D_f f(t) + D_d d(t) \quad (2)$$

- We seek a residual signal $r(t)$ such that, ideally,

$$r(t) \neq 0 \text{ if and only if } f(t) \neq 0.$$

- In practice we settle for: (subject to stability)

$$\min_{\|T_{rf}\|_{-} \geq 1} \|T_{rd}\|_{\infty}.$$

- Here T_{rd} denotes the transfer matrix from d to r and T_{rf} denotes the transfer matrix from f to r,

$$\|T_{rd}\|_{\infty} = \max_{\omega \in \mathcal{R}} \bar{\sigma}[T_{rd}(j\omega)] = \max_{\|d\|_2 = 1} \|T_{rd} d\|_2$$
$$\|T_{rf}\|_{-} = \min_{\omega \in \mathcal{R}} \underline{\sigma}[T_{rf}(j\omega)] = \min_{\|f\|_2 = 1} \|T_{rf} f\|_2.$$

- $\bar{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ are largest & smallest singular values.

- $\|.\|_2^2$ denotes the total signal energy.

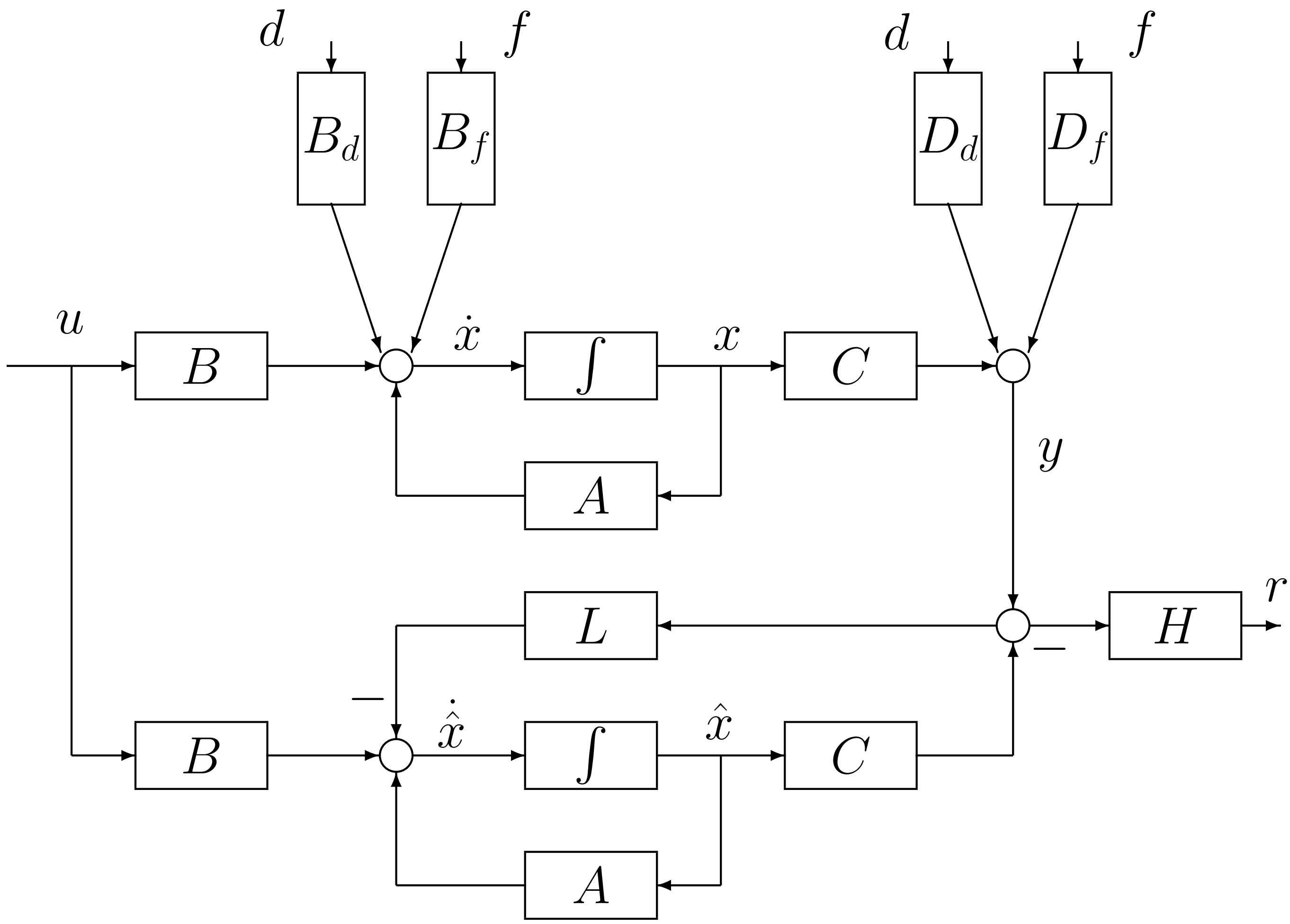
Residual Generation using an Observer–based Analytic Redundancy Approach

- Define the state–estimation error $e(t) = x(t) - \hat{x}(t)$,

$$\begin{aligned} \dot{e} &= (A + LC)e + (B_f + LD_f)f + (B_d + LD_d)d \\ r &= HCe + HD_f f + HD_d d \end{aligned}$$

- It follows that

$$\left[\begin{array}{cc} T_{rf} & T_{rd} \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A + LC & B_f + LD_f & B_d + LD_d \\ \hline HC & HD_f & HD_d \end{array}\right]$$



First FDP Reformulation

- The figure below gives an equivalent generator with

$$\begin{bmatrix} T_{rf} & T_{rd} \end{bmatrix} = F \begin{bmatrix} G_f & G_d \end{bmatrix}$$

  where

$$F \overset{s}{=} \left[\begin{array}{c|c} A+LC & L \\ \hline HC & H \end{array}\right], \quad \begin{bmatrix} G & G_f & G_d \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|ccc} A & B & B_f & B_d \\ \hline C & 0 & D_f & D_d \end{array}\right]$$

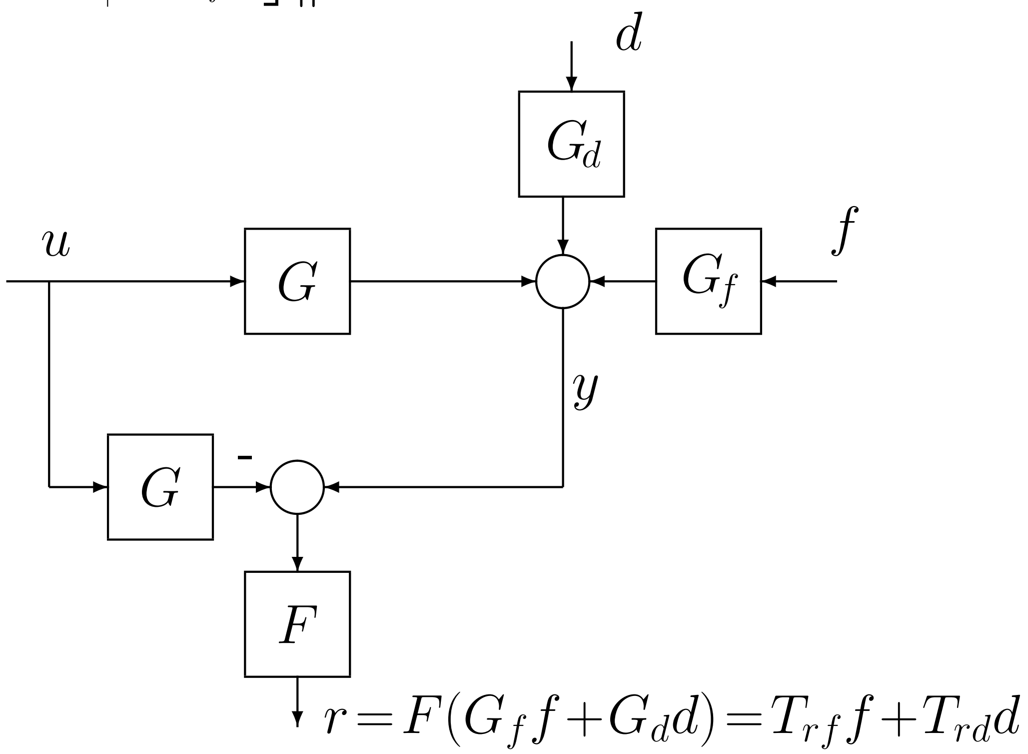
- To ensure the existence of stable proper F assume

$$\text{The pair } (A, C) \text{ is detectable} \qquad (3)$$

$$G_f \text{ has full column rank on } j\mathcal{R}_e \qquad (4)$$

- FDP: Find L & H with $A + LC$ stable that achieve

$$\gamma_o := \min_{\left\|\left[\begin{array}{c|c} A+LC & B_f+LD_f \\ \hline HC & HD_f \end{array}\right]\right\|_- \geq 1} \left\|\left[\begin{array}{c|c} A+LC & B_d+LD_d \\ \hline HC & HD_d \end{array}\right]\right\|_\infty = \min_{\|FG_f\|_- \geq 1} \|FG_d\|_\infty$$



Allpass Transfer Matrices

- Let $M(s) = D + C(sI - A)^{-1}B$. Define $M^{\sim}(s) = M(-s)^T$. Then $M^{\sim}$ is the complex conjugate transpose of M for imaginary s.

- If $M(s)M^{\sim}(s) = M^{\sim}(s)M(s) = I$, M is called allpass.

- If $M(s)$ is allpass, the singular values of any $N(s)$ are the same as those of $M(s)N(s)$ for imaginary s:
$$\|MN\|_{-} = \|N\|_{-}, \qquad \|MN\|_{\infty} = \|N\|_{\infty}.$$

- **Example** $M(s) = \frac{s-1}{s+1}$:
$$M(s)M^{\sim}(s) = \left(\frac{s-1}{s+1}\right)\left(\frac{-s-1}{-s+1}\right) = 1, \;\; |M(j\omega)| = |\frac{j\omega-1}{j\omega+1}| = 1.$$

- **Theorem:** Let $M(s) = D + C(sI - A)^{-1}B$. If $\exists X$ s.t.
$$DD^T = I, \quad AX + XA^T - BB^T = 0, \quad XC^T - BD^T = 0$$
then $M(s)$ is allpass. **Proof:**
$$MM^{\sim} \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right] \overset{s}{=} \left[\begin{array}{cc|c} A & -BB^T & BD^T \\ 0 & -A^T & C^T \\ \hline C & -DB^T & DD^T \end{array}\right]$$
$$\overset{s}{=} \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & -A^T & C^T \\ \hline C & 0 & I \end{array}\right] \equiv I \text{ using similarity } T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.$$

Spectral Factorization

- **Theorem:** Let $F \stackrel{s}{=} \left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right]$ (not necessarily stable) with (A_F, C_F) detectable. Then there exists a stabilizing solution to the algebraic Riccati equation

$$A_F X + X A_F^T - X C_F^T C_F X = 0, \quad A_F - X C_F^T C_F \text{ stable}$$

Define $L_F = -X C_F^T$ and

$$M \stackrel{s}{=} \left[\begin{array}{c|c} A_F & -L_F \\ \hline C_F & I \end{array}\right], \quad \hat{F} \stackrel{s}{=} \left[\begin{array}{c|c} A_F + L_F C_F & B_F + L_F D_F \\ \hline C_F & D_F \end{array}\right]$$

Then

1. $F = M\hat{F}$ (verified by direct calculation)
2. M is allpass since

$$A_F X + X A_F^T - (-L_F)(-L_F^T) = 0, \qquad X C_F^T - (-L_F) = 0$$

3. $\hat{F}$ (called a spectral factor of F) is stable (since X is chosen to be the stabilizing solution)

- Example: $\overbrace{\frac{s+1}{(s+2)(s-3)}}^{F(s)} = \overbrace{\frac{s+3}{s-3}}^{M(s)} \; \overbrace{\frac{s+1}{(s+2)(s+3)}}^{\hat{F}(s)}$

- It follows that

$$\|F G_f\|_- = \left\|M \hat{F} G_f\right\|_- = \left\|\hat{F} G_f\right\|_-, \; \|F G_d\|_\infty = \left\|\hat{F} G_d\right\|_\infty$$

- Generically, (A_F, C_F) is detectable and we can drop the requirement that $F(s)$ is stable.

Constraint Simplification

- Suppose that $F \stackrel{s}{=} \left[\begin{array}{c|c} A{+}LC & L \\ \hline HC & H \end{array}\right]$ satisfies the constraint so that $\|FG_f\|_- \geq 1$ and suppose that $\|FG_d\|_\infty \leq \gamma$.

- Note that $\|FG_f\|_- \geq 1$ implies that $(FG_f)^{-1}$ exists and $\left\|(FG_f)^{-1}\right\|_\infty \leq 1$.

- Define $\hat{F} = (FG_f)^{-1}F$. Then

$$\hat{F}G_f = \ (FG_f)^{-1}FG_f = I$$

and

$$\left\|\hat{F}G_d\right\|_\infty = \left\|(FG_f)^{-1}FG_d\right\|_\infty \leq \left\|(FG_f)^{-1}\right\|_\infty \|FG_d\|_\infty \leq \gamma.$$

- Furthermore, a state–space calculation shows that

$$\hat{F} \stackrel{s}{=} \left[\begin{array}{c|c} A + \hat{L}C & \hat{L} \\ \hline \hat{H}C & \hat{H} \end{array}\right]$$

where $\hat{H} = (HD_f)^{-1}H$ and $\hat{L} = L - (B_f + LD_f)\hat{H}$.

- So we can replace the constraint $\|FG_f\|_- \geq 1$ by the constraint $FG_f = I$.

Second FDP Reformulation

- Thus the problem becomes: Find L & H that achieve

$$\gamma_o = \min_{\left[\begin{array}{c|c} A+LC & B_f+LD_f \\ \hline HC & HD_f \end{array}\right] \equiv I} \left\| \left[\begin{array}{c|c} A+LC & B_d+LD_d \\ \hline HC & HD_d \end{array}\right] \right\|_\infty = \min_{FG_f=I} \|FG_d\|_\infty$$

- Note that we impose no stability requirements on $A+LC$.

- Consider the constraint

$$\left[\begin{array}{c|c} A+LC & B_f+LD_f \\ \hline HC & HD_f \end{array}\right] \equiv I$$

- We certainly need

$$HD_f = I.$$

Also, the condition

$$B_f + LD_f = 0$$

is sufficient. It can be shown that, if the pair $(A+LC, HC)$ is observable (which generically is the case), then this latter condition is also necessary.

- It follows that we can replace the constraint by

$$HD_f = I, \qquad LD_f = -B_f$$

- Suppose that D_f has a singular value decomposition

$$D_f = U\Sigma V^T = \left[\, U_1 \;\; U_2 \,\right] \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V^T, \quad \Sigma_1 > 0 \tag{5}$$

- Define

$$D_f^\dagger = V\Sigma_1^{-1}U_1^T, \qquad D_f^\perp = U_2^T. \tag{6}$$

- Then

$$D_f^\dagger D_f = I, \quad D_f^\perp D_f = 0$$

- It follows that all H and L satisfying the constraint are given by

$$H = D_f^\dagger + SD_f^\perp, \qquad L = -B_f D_f^\dagger + RD_f^\perp \tag{7}$$

where R and S are free parameters.

- Thus the problem becomes: Find R & S that achieve

$$\gamma_o = \min_{R,\, S} \left\| \left[\begin{array}{c|c} \overbrace{(A - B_f D_f^\dagger C)}^{A_1} + R\,\overbrace{D_f^\perp C}^{C_2} & \overbrace{(B_d - B_f D_f^\dagger D_d)}^{B_1} + R\,\overbrace{D_f^\perp D_d}^{D_2} \\ \hline \underbrace{D_f^\dagger C}_{C_1} + S\,\underbrace{D_f^\perp C}_{C_2} & \underbrace{D_f^\dagger D_d}_{D_1} + S\,\underbrace{D_f^\perp D_d}_{D_2} \end{array} \right] \right\|_\infty \tag{8}$$

or

$$\gamma_o = \min_{R,\, S} \left\| \left[\begin{array}{c|c} A_1 + RC_2 & B_1 + RD_2 \\ \hline C_1 + SC_2 & D_1 + SD_2 \end{array} \right] \right\|_\infty$$

LMIs and the Bounded Real Lemma (BRL)

- Consider the following state–space representation

$$\dot{x}=Ax+Bu, \quad y=Cx+Du, \quad x(0)=0$$

  with A stable and let $G(s) = D + C(sI - A)^{-1}B$.

- If $\|u\|_2$ is bounded, then $\lim\limits_{t\to\infty} x(t) = 0$.

- Let $\gamma > 0$ be given. Then

$$\|G\|_\infty < \gamma \iff J := \int_0^\infty [y^T y - \gamma^2 u^T u] dt < 0, \ \|u\|_2 < \infty$$

- For $P = P^T \in \mathcal{R}^{n\times n}$,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x^T P x] dt = x(\infty)^T P x(\infty) - x(0)^T P x(0) = 0$$

- So

$$0 = \int_0^\infty \dot{x}^T P x + x^T P \dot{x} dt = \int_0^\infty [(Ax+Bu)^T P x + x^T P (Ax+Bu)] dt$$
$$= \int_0^\infty [x^T (A^T P + PA) x + x^T P B u + u^T B^T P x] dt$$

- Use $y=Cx+Du$ and add the last expression to J

$$J = \int_0^\infty [x^T (A^T P + PA + C^T C) x + x^T (PB + C^T D) u +$$
$$u^T (B^T P + D^T C) x + u^T (D^T D - \gamma^2 I) u] dt$$
$$= \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

- We have proved the 'if' part of the following lemma.

- **Bounded Real Lemma** Let $G(s) = D + C(sI - A)^{-1}B$ and $\gamma > 0$ be given. Then A is stable and $\|G\|_\infty < \gamma$ if and only if there exists $P = P^T > 0$ s.t.

$$\underbrace{\begin{bmatrix} A^TP+PA+C^TC & PB+C^TD \\ B^TP+D^TC & D^TD-\gamma^2 I \end{bmatrix}}_{\mathcal{L}(P)} < 0\cdot$$

- Let $V(x) = x^TPx$. We demonstrate the stability of A by showing that V defines a Lyapunov function.
  1. $P = P^T > 0$ implies that $V(x) > 0$ for all $x \neq 0$.
  2. When $u=0$, $\dot{V}(x) = \dot{x}^TPx + x^TP\dot{x} = x^T(A^TP+PA)x < 0$ for all $x \neq 0$ since the (1,1) block of the linear matrix inequality (LMI) implies $A^TP+PA < 0$.

- It follows that

$$\|G\|_\infty^2 = \inf\{\gamma^2 : P = P^T > 0,\ \mathcal{L}(P) < 0\}$$

- The LMI optimization problem is convex since if $P_1 = P_1^T > 0$, $\mathcal{L}(P_1) < 0$ and $P_2 = P_2^T > 0$, $\mathcal{L}(P_2) < 0$, then for all $0 \leq \alpha \leq 1$ with $P_\alpha = \alpha P_1 + (1-\alpha)P_2$ we have $P_\alpha > 0$ and $\mathcal{L}(P_\alpha) < 0$.

- There exist powerful numerical techniques for the solution of LMI optimization problems.

Extensions of BRL

- **Schur complement** Suppose that

$$X = X^T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$$

Then

$$X < 0 \Longleftrightarrow \left(X_{22} < 0 \quad \& \quad \overbrace{X_{11} - X_{12}X_{22}^{-1}X_{12}^T}^{S} < 0 \right).$$

- The result follows from the observation that

$$\begin{bmatrix} I & -X_{12}X_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -X_{22}^{-1}X_{12}^T & I \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & X_{22} \end{bmatrix}$$

S is called the Schur complement of X.

- Now,

$$\overbrace{\mathcal{L}(P)}^{S} = \overbrace{\begin{bmatrix} A^TP+PA & PB \\ B^TP & -\gamma^2 I \end{bmatrix}}^{X_{11}} - \overbrace{\begin{bmatrix} C^T \\ D^T \end{bmatrix}}^{X_{12}} (\overbrace{-I}^{X_{22}^{-1}}) \overbrace{\begin{bmatrix} C & D \end{bmatrix}}^{X_{12}^T}$$

- It follow that

$$\mathcal{L}(P) < 0 \Leftrightarrow \overbrace{\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma^2 I & D^T \\ C & D & -I \end{bmatrix}}^{\hat{\mathcal{L}}(P)}$$

- If we are not concerned with stability:

$$\max_{\omega \in \boldsymbol{R}} \bar{\sigma}[G(j\omega)]^2 = \inf\{\gamma^2 : P = P^T, \ \hat{\mathcal{L}}(P) < 0\}$$

Matrix Inequality Solution of the FDP

- Applying the (last) version of the BRL to the (last) version of the FDP (setting $A := A_1 + RC_2,\ B := B_1 + RD_2,\ C := C_1 + SC_2,\ D := D_1 + SD_2$): Find $\boldsymbol{R}$ and $\boldsymbol{S}$ that achieve

$$\gamma_o^2 = \inf\ \boldsymbol{\gamma}^2$$

subject to $\boldsymbol{P} = \boldsymbol{P}^T$ and

$$\begin{bmatrix} (A_1+\boldsymbol{R}C_2)^T\boldsymbol{P}+\boldsymbol{P}(A_1+\boldsymbol{R}C_2) & \boldsymbol{P}(B_1+\boldsymbol{R}D_2) & (C_1+\boldsymbol{S}C_2)^T \\ (B_1+\boldsymbol{R}D_2)^T\boldsymbol{P} & -\boldsymbol{\gamma}^2 I & (D_1+\boldsymbol{S}D_2)^T \\ C_1+\boldsymbol{S}C_2 & D_1+\boldsymbol{S}D_2 & -I \end{bmatrix} < 0$$

- For clarity I have shown the variables in bold type.

- Unfortunately, the solution is given in the form of a nonlinear matrix inequality.

- Note, however, that:
  1. The only nonlinearities occur in the term $\boldsymbol{PR}$.
  2. These are the only terms that $\boldsymbol{R}$ appears in.

- This suggests introducing a change of variable

$$\boldsymbol{Z} = \boldsymbol{PR}.$$

LMI Solution of the FDP

- With all variables and assumptions as defined above: Find $\boldsymbol{Z}$ and $\boldsymbol{S}$ that achieve

$$\gamma_o^2 = \inf_{\boldsymbol{P},\, \boldsymbol{Z},\, \boldsymbol{S},\, \boldsymbol{\gamma}^2} \boldsymbol{\gamma}^2$$

subject to

$$\boldsymbol{P} = \boldsymbol{P}^T$$

and

$$\begin{bmatrix} A_1^T\boldsymbol{P}+\boldsymbol{P}A_1+\boldsymbol{Z}C_2+C_2^T\boldsymbol{Z}^T & \boldsymbol{P}B_1+\boldsymbol{Z}D_2 & C_1^T+C_2^T\boldsymbol{S}^T \\ B_1^T\boldsymbol{P}+D_2^T\boldsymbol{Z}^T & -\boldsymbol{\gamma}^2 I & D_1^T+D_2^T\boldsymbol{S}^T \\ C_1+\boldsymbol{S}C_2 & D_1+\boldsymbol{S}D_2 & -I \end{bmatrix} < 0 \qquad (9)$$

- It can be shown that, even if $\boldsymbol{P}$ is singular, the equation $\boldsymbol{PR} = \boldsymbol{Z}$ always has a solution $\boldsymbol{R}$.

- This now an LMI optimization problem whose solution can be easily obtained.

Algorithm for the Solution of the FDP

- **Initial Data:** Let all matrices be as given in (1)–(2).

- **Assumptions:** Assume that (3) and (4) are satisfied.

- **Initial computation:** Effect the SVD in (5).

- **Defined Data:**
  1. Define $D_f^{\dagger}$ and $D_f^{\perp}$ as in (6).
  2. Define $A_1,\ B_1,\ C_1,\ C_2,\ D_1$ and D_2 as in (8).

- **Main Computation:** Use the LMI solver in Matlab (the function *mincx*) to compute $\boldsymbol{P},\ \boldsymbol{Z},\ \boldsymbol{S}$ and γ_o.

- **Solution Construction:**
  1. Find $\boldsymbol{R}$ from the equation $\boldsymbol{P}\boldsymbol{R} = \boldsymbol{Z}$.
  2. Evaluate H and L from (7).
     (a) If $A + LC$ is stable we are done.
     (b) Else (assuming $(A + LC, HC)$ is detectable), let $X = X^T \geq 0$ be the solution to the ARE
     $$(A + LC)X + X(A + LC)^T - X(HC)^T(HC)X = 0$$
     such that $A + LC - X(HC)^T(HC)$ is stable. Redefine $L := L - X(HC)^T H$

Example

- **Initial Data:** Let

$$\left[\begin{array}{c||c|c|c} A & B & B_f & B_d \\ \hline\hline C & 0 & D_f & D_d \end{array}\right] = \left[\begin{array}{c||c|c|cc} -3 & 2 & 1 & 5 & -5 \\ \hline\hline 4 & 0 & 7 & -5 & 0 \\ -6 & 0 & 2 & 4 & 3 \end{array}\right]$$

- **Assumptions:** Assumption (3) and (4) are satisfied.

- **Initial computation:** Effecting the svd of D_f gives:
- **Defined Data:**
  1. $D_f^{\dagger} = [0.1321, 0.0377]$ and $D_f^{\perp} = [-0.2747, 0.9615]$.
  2. The matrices in (8) are given as

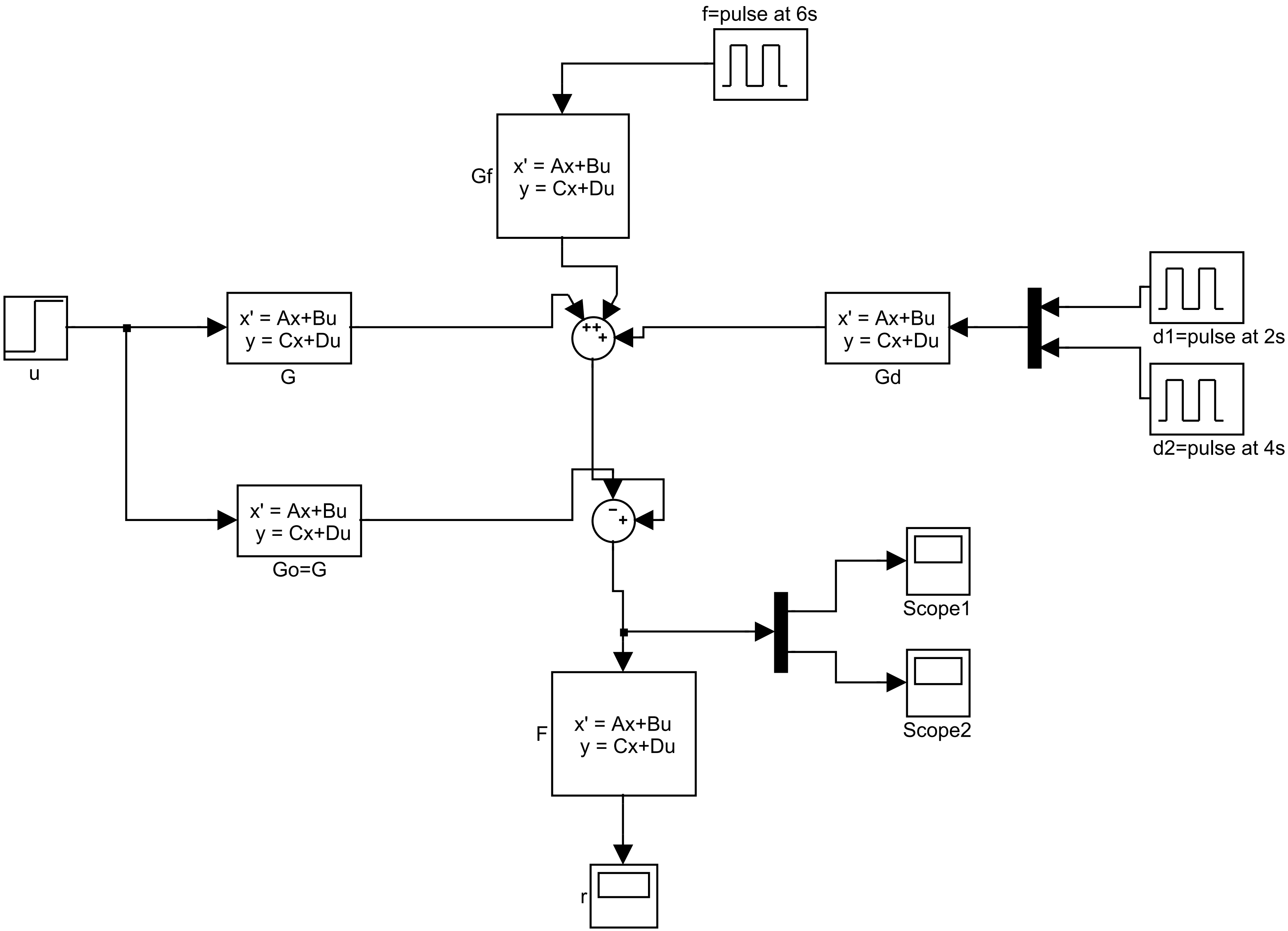
$$\left[\begin{array}{c||c} A_1 & B_1 \\ \hline\hline C_1 & D_1 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{c||cc} -3.3019 & 5.5094 & -5.1132 \\ \hline\hline 0.3019 & -0.5094 & 0.1132 \\ \hline -6.8680 & 5.2197 & 2.8846 \end{array}\right]$$

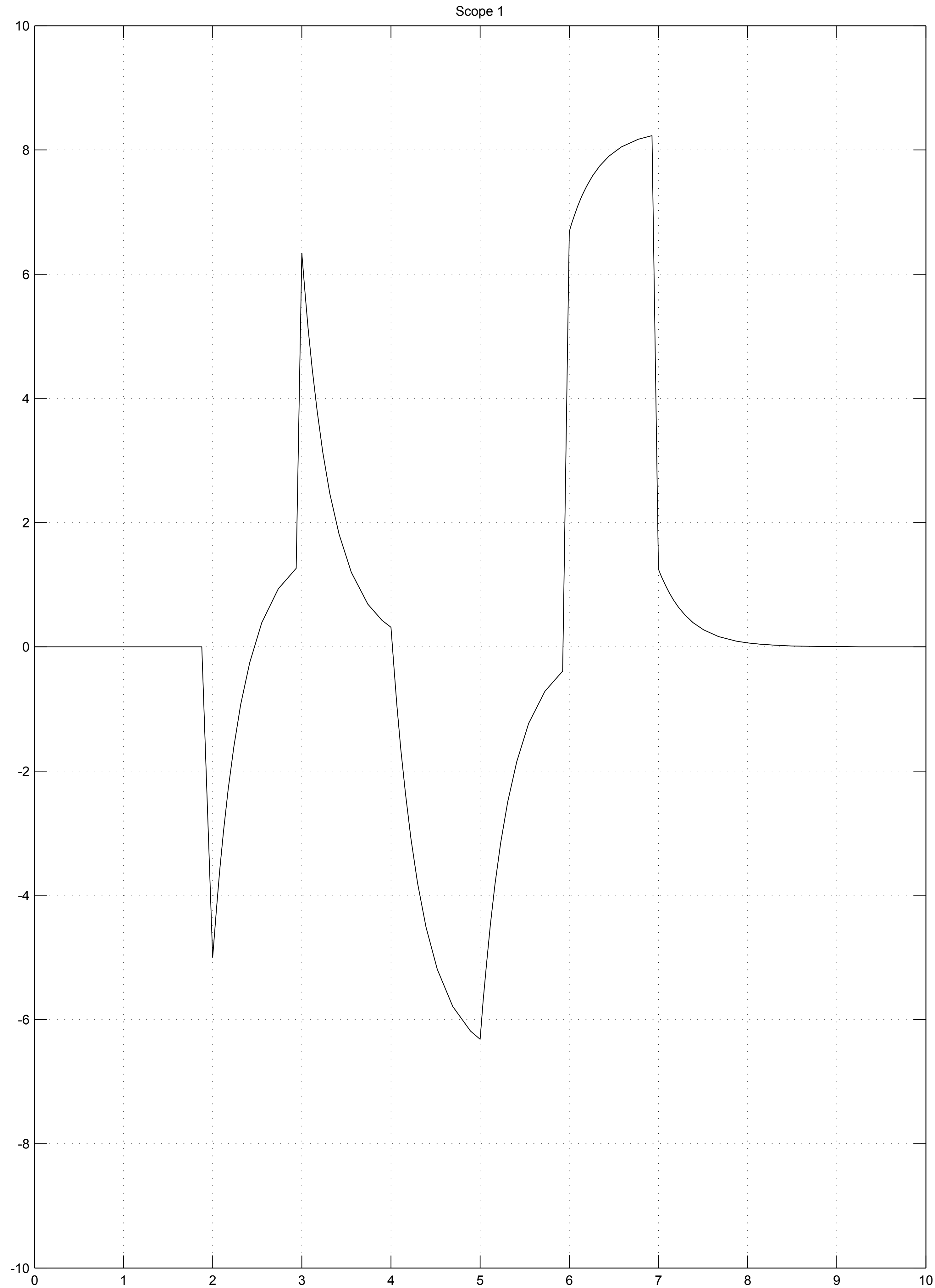
- **Main Computation:** The LMI solver (*mincx*) gives

$$[P, Z, R, S] = [-0.0072, 0.0259, -3.6007, 0.0656], \quad \gamma_o = 0.3455.$$

- Finally,

$$L = [33.7638, 25.5805], \quad H = [0.1141, 0.1008].$$


f=pulse at 6s
Gf
x' = Ax+Bu
y = Cx+Du
u
x' = Ax+Bu
y = Cx+Du
G
x' = Ax+Bu
y = Cx+Du
Gd
d1=pulse at 2s
d2=pulse at 4s
x' = Ax+Bu
y = Cx+Du
Go=G
Scope1
Scope2
F
x' = Ax+Bu
y = Cx+Du
r



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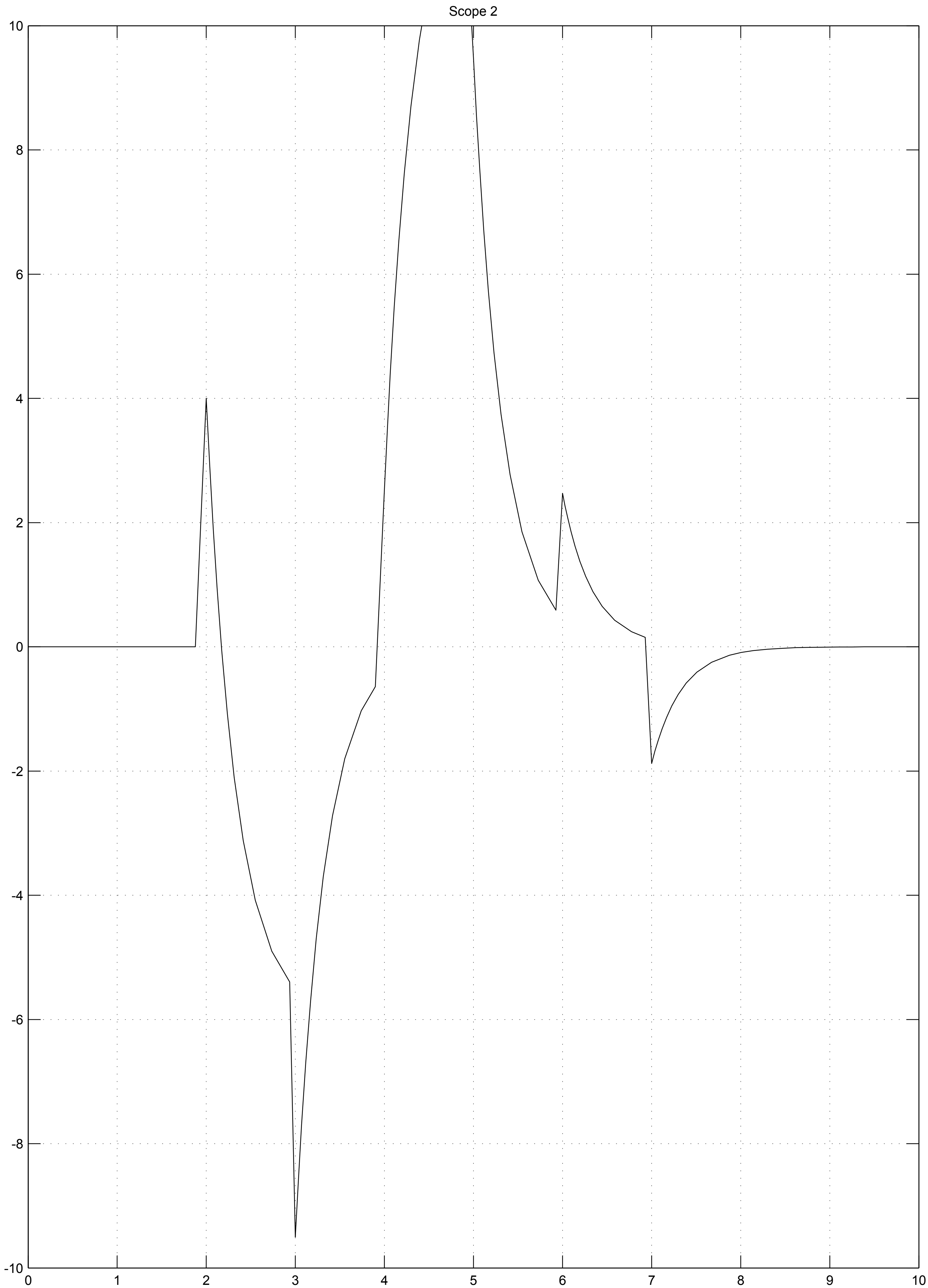

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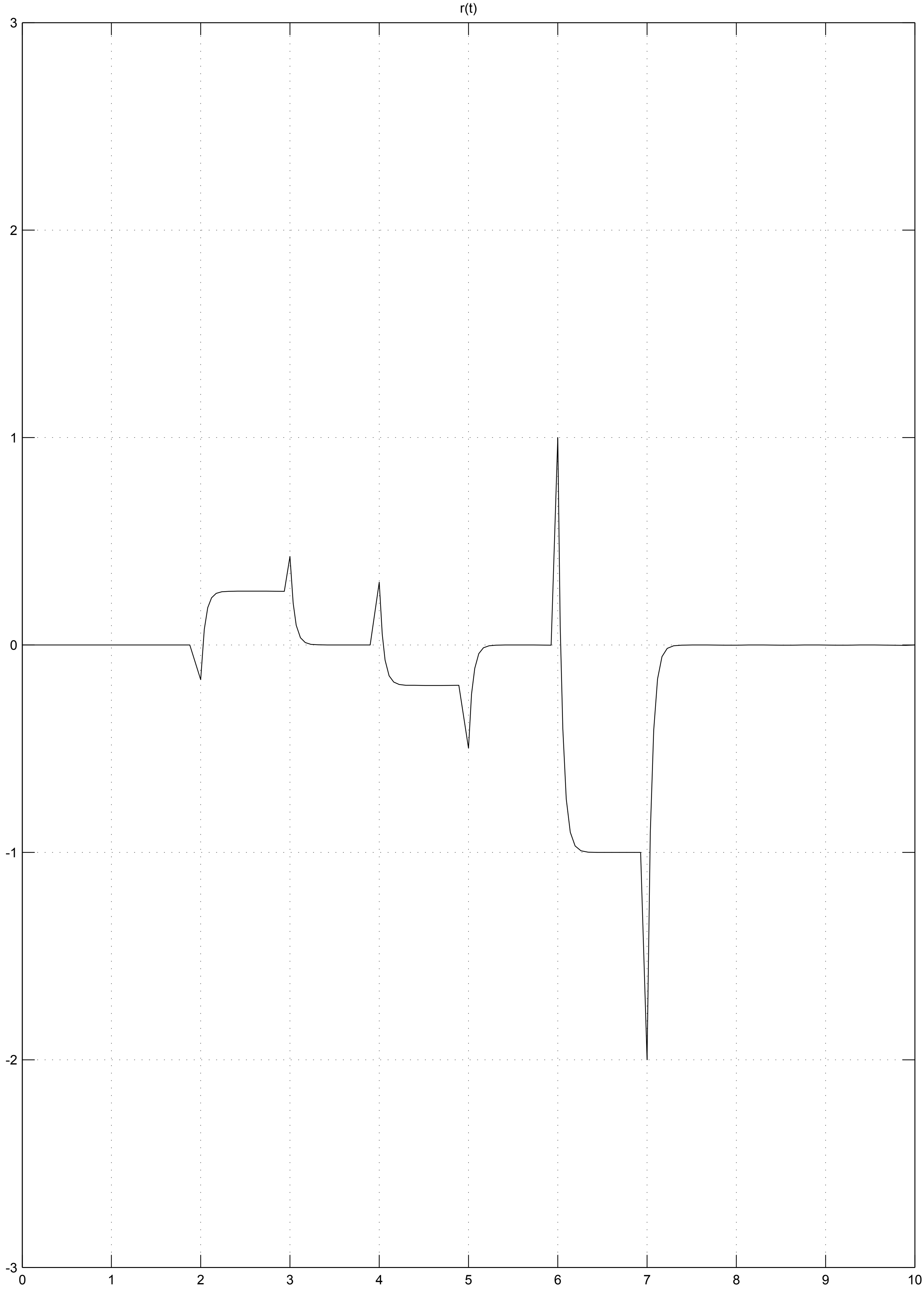

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Frequency Weighted FDP

- We extend the FDP by introducing frequency weights.

- We have hitherto defined singular value requirements on $T_{rf}(s)$ and $T_{rd}(s)$ over the whole frequency range.

- That is, we require $\underline{\sigma}\,[T_{rf}(j\omega)] > 1$ and $\bar{\sigma}\,[T_{rd}(j\omega)] < \gamma$ for all ω.

- Suppose that $W(s)$ is a perfect passband filter:

$$\begin{cases} W(j\omega) = I, \ \omega \in [\omega_1, \omega_2] \\ W(j\omega) = 0, \ \text{otherwise} \end{cases}$$

- Then an alternative requirement is

$$\min_{\|T_{rf}W\|_{-} \geq 1} \|T_{rd}W\|_{\infty}.$$

- This will ensure that T_{rf} is 'large' and T_{rd} is 'small' over the frequency range $[\omega_1, \omega_2]$.

- The residual signal is then sampled over $[\omega_1, \omega_2]$ only, or equivalently, we follow $F(s)$ by $W(s)$.

- In practice, we use a low order bandpass filter.

Removing the Rank Assumption on G_f

- Recall from (4) that we needed the assumption that $G_f(s)$ has full column rank over the extended imaginary axis $j\mathcal{R}_e$.

- This is restrictive, and excludes, for example, the case when $G_f(s)$ is strictly proper and / or has $j\omega$–axis zeros, e.g.

$$G_f(s) = \frac{s}{(s+1)(s+2)} .$$

- This needed since we require that

$$F(s)G_f(s) = 1,$$

which, in this case, is impossible for proper and stable $F(s)$.

- This problem may be addressed by the introduction of weighting filters.

- For example, for the given $G_f(s)$ we may choose

$$W(s) = \frac{(s+3)(s+4)}{s}$$

The Fault Detection and Isolation Problem (FDIP)

- In the FDIP we seek a residual signal $r(t)$ such that, ideally,

$$r_i(t) \neq 0 \text{ if and only if } f_i(t) \neq 0,\ i = 1, \ldots, n_f.$$

- In practice we settle for: (subject to stability)

$$\min_{F(s)G_f(s) = N + M(sI - \Lambda)^{-1}} \|FG_d\|_\infty$$

  where $N,\ M$ & Λ are diagonal with $M > 0$ and Λ stable.

- That is, in fault isolation $F(s)G_f(s)$ is required to be a given diagonal stable transfer matrix.

- This problem can be reduced to our second reformulated FDP by replacing $G_f(s)$ by

$$G_f(s)\left[N + M(sI - \Lambda)^{-1}\right]^{-1}.$$

- The details are rather intricate and will not be given here.

Robust FDP

- One crucial assumption we have made is that we have an accurate model of the system dynamics.

- Thus in the observer–based formulation, we assume that the system matrices for $G(s) \stackrel{s}{=} (A, B, C, D)$ are known.

- This allowed us to use the observer to cancel the system dynamics and define a residual signal $r(t)$ that depends only on the faults and disturbances.

- The subsequent FD filter design consisted in making the filter sensitive to faults and insensitive to disturbances

- In practice, we only have approximate models, say
$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|c} A_o + \Delta_A & B_o + \Delta_B \\ \hline C_o + \Delta_C & D_o + \Delta_D \end{array}\right]$$
where (A_o, B_o, C_o, D_o) is a nominal model and Δ_A, Δ_B, Δ_C and Δ_D represent uncertainties.

- The FDP for uncertain systems remains an open research problem.

TOPICS IN CONTROL

Fault Detection (FD) Coursework

- The aim of the coursework is to introduce you to the design of FD filters. All the background material you may need is found in the lecture notes and the information provided below.

- Your design task is to implement the algorithm in the lecture notes for the following data:

$$\begin{aligned}\begin{bmatrix} G(s) & G_f(s) & G_d(s) \end{bmatrix} &\overset{s}{=} \left[\begin{array}{c||c|c|c} A & B & B_f & B_d \\ \hline\hline C & 0 & D_f & D_d \end{array}\right] \\ &= \left[\begin{array}{r||c|c|rr} -2 & 1 & 1 & 4 & -4 \\ \hline\hline 4 & 0 & 6 & -4 & 0 \\ -5 & 0 & 1 & 3 & 2 \end{array}\right]\end{aligned}$$

  with

$$[\, n\ n_u\ n_f\ n_d\ n_y \,] = [1\ 1\ 1\ 2\ 2\,]$$

- That is, assuming that the pair (A, C) is detectable and that $G_f(s)$ has full column rank over the imaginary axis, find L and H such that, with

$$F(s) \overset{s}{=} \left[\begin{array}{c|c} A + LC & L \\ \hline HC & H \end{array}\right] \in \mathcal{RH}_\infty$$

  the following minimum

$$\gamma^2 := \min_{\|FG_f\|_- \geq 1} \|FG_d\|_\infty$$

  is achieved. The $\mathcal{H}_\infty$–norm measures the maximum sensitivity of the residual to disturbance (which should therefore be kept small), while the $\mathcal{H}_-$ index measures the minimum sensitivity of the residual to faults.

- Use *Simulink* to provide simulation diagrams for the same fault and disturbance scenarios as in the lecture notes.

- Also, experiment with difference scenarios to demonstrate the effectiveness of the FD algorithm.

- Use the following *Matlab* code for the LMI part of the algorithm ($g2$ stands for γ^2 and the matrices A_1, B_1, C_1, C_2, D_1 and D_2 are defined in the lecture notes and require the evaluation of a singular value decomposition of D_f using the *Matlab* command *svd*):

  - $setlmis([]);$
  - $g2 = lmivar(1, [1, 1]); P = lmivar(1, [n, 1]);$
  - $Z = lmivar(2, [n, n_y - n_f]); S = lmivar(2, [n_f, n_y - n_f]);$
  - $L = newlmi;$
  - $lmiterm([L, 1, 1, P], 1, A_1, 's'); lmiterm([L, 1, 1, Z], 1, C_2, 's');$
  - $lmiterm([L, 1, 2, P], 1, B_1); lmiterm([L, 1, 2, Z], 1, D_2);$
  - $lmiterm([L, 1, 3, 0], C_1'); lmiterm([L, 1, 3, -S], C_2', 1);$
  - $lmiterm([L, 2, 2, g2], -.5, 1, 's');$
  - $lmiterm([L, 2, 3, 0], D_1'); lmiterm([L, 2, 3, -S], D_2', 1);$
  - $lmiterm([L, 3, 3, 0], -1);$
  - $LMI = getlmis;$
  - $[g2, x] = mincx(LMI, eye(decnbr(LMI), 1));$
  - $P = dec2mat(LMI, x, P);$
  - $Z = dec2mat(LMI, x, Z); S = dec2mat(LMI, x, S);$

- Remember to check for the stability of $A+LC$. If unstable, L should be modified by solving a Riccati equation as specified in the notes.

- Your report should be handed in no later than one week after the end of the last exam.

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