An Introduction to Modelling and Control of Flexible and Soft Robots

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QUESTION 1. Write the expression of the potential energy for a rigid-link model with $n = 4$ elastic joints.
Assume the following:
- the system is planar
- the stiffness k is the same for all joints, thus $K = \text{diag}\{k\}$
- the system is not subject to gravity (the potential energy V is only elastic)

$$\frac{k\,(q_1{}^2 + q_2{}^2 + q_3{}^2 + q_4{}^2)}{2}$$

QUESTION 2. Write the expression of the kinetic energy for a rigid-link model with $n = 4$ elastic joints. Use Matlab Mupad for symbolic computation.
Assume the following:
- all links have equal length l and equal mass m
- the mass of each link is concentrated in its midpoint

Hint: see Example 1 in the lecture notes.
*Bonus points: write the expression of the inertia matrix.

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

position

```
x1:=l*sin(q_1)/2
```

$$\frac{l\,\sin(q_1)}{2}$$

```
y1:=l*cos(q_1)/2
```

$$\frac{l\,\cos(q_1)}{2}$$

```
x2:=l*sin(q_1)+l*sin(q_1+q_2)/2
```

$$\frac{l\,\sin(q_1+q_2)}{2} + l\,\sin(q_1)$$

```
y2:=l*cos(q_1)+l*cos(q_1+q_2)/2
```

$$\frac{l\,\cos(q_1+q_2)}{2} + l\,\cos(q_1)$$

```
x3:=l*sin(q_1)+l*sin(q_1+q_2)+l*sin(q_1+q_2+q_3)/2
```

$$l\,\sin(q_1+q_2) + l\,\sin(q_1) + \frac{l\,\sin(q_1+q_2+q_3)}{2}$$

```
y3:=l*cos(q_1)+l*cos(q_1+q_2)+l*cos(q_1+q_2+q_3)/2
```

$$l\cos(q_1+q_2)+l\cos(q_1)+\frac{l\cos(q_1+q_2+q_3)}{2}$$

```
x4:=l*sin(q_1)+l*sin(q_1+q_2)+l*sin(q_1+q_2+q_3)+l*sin(q_1+q_2
+q_3+q_4)/2
```

$$\frac{l\sin(q_1+q_2+q_3+q_4)}{2}+l\sin(q_1+q_2)+l\sin(q_1)+l\sin(q_1+q_2+q_3)$$

```
y4:=l*cos(q_1)+l*cos(q_1+q_2)+l*cos(q_1+q_2+q_3)+l*cos(q_1+q_2
+q_3+q_4)/2
```

$$\frac{l\cos(q_1+q_2+q_3+q_4)}{2}+l\cos(q_1+q_2)+l\cos(q_1)+l\cos(q_1+q_2+q_3)$$

velocity

```
x1_dot:=diff(x1, q_1)*dq_1+diff(x1, q_2)*dq_2+diff(x1,
q_3)*dq_3+diff(x1, q_4)*dq_4
```

$$\frac{\mathrm{dq}_1\, l\cos(q_1)}{2}$$

```
y1_dot:=diff(y1, q_1)*dq_1+diff(y1, q_2)*dq_2+diff(y1,
q_3)*dq_3+diff(y1, q_4)*dq_4
```

$$-\frac{\mathrm{dq}_1\, l\sin(q_1)}{2}$$

```
x2_dot:=diff(x2, q_1)*dq_1+diff(x2, q_2)*dq_2+diff(x2,
q_3)*dq_3+diff(x2, q_4)*dq_4
```

$$\mathrm{dq}_1\left(\frac{l\cos(q_1+q_2)}{2}+l\cos(q_1)\right)+\frac{\mathrm{dq}_2\, l\cos(q_1+q_2)}{2}$$

```
y2_dot:=diff(y2, q_1)*dq_1+diff(y2, q_2)*dq_2+diff(y2,
q_3)*dq_3+diff(y2, q_4)*dq_4
```

$$-\mathrm{dq}_1\left(\frac{l\sin(q_1+q_2)}{2}+l\sin(q_1)\right)-\frac{\mathrm{dq}_2\, l\sin(q_1+q_2)}{2}$$

```
x3_dot:=diff(x3, q_1)*dq_1+diff(x3, q_2)*dq_2+diff(x3,
q_3)*dq_3+diff(x3, q_4)*dq_4
```

$$\mathrm{dq}_1\left(l\cos(q_1+q_2)+l\cos(q_1)+\frac{l\cos(q_1+q_2+q_3)}{2}\right)+\mathrm{dq}_2\left(l\cos(q_1+q_2)+\frac{l\cos(q_1+q_2+q_3)}{2}\right)+\frac{\mathrm{dq}_3\, l\cos(q_1+q_2+q_3)}{2}$$

```
y3_dot:=diff(y3, q_1)*dq_1+diff(y3, q_2)*dq_2+diff(y3,
q_3)*dq_3+diff(y3, q_4)*dq_4
```

$$-\mathrm{dq}_1\left(l\sin(q_1+q_2)+l\sin(q_1)+\frac{l\sin(q_1+q_2+q_3)}{2}\right)-\mathrm{dq}_2\left(l\sin(q_1+q_2)+\frac{l\sin(q_1+q_2+q_3)}{2}\right)-\frac{\mathrm{dq}_3\, l\sin(q_1+q_2+q_3)}{2}$$

```
x4_dot:=diff(x4, q_1)*dq_1+diff(x4, q_2)*dq_2+diff(x4,
q_3)*dq_3+diff(x4, q_4)*dq_4
```

$$\mathrm{dq}_3\,(\sigma_1 + l\cos(q_1+q_2+q_3)) + \mathrm{dq}_1\,(\sigma_1 + l\cos(q_1+q_2) + l\cos(q_1) + l\cos(q_1+q_2+q_3)) + \mathrm{dq}_2\,(\sigma_1 + l\cos(q_1+q_2) + l\cos(q_1+q_2+q_3)) + \frac{\mathrm{dq}_4\, l\cos(q_1+q_2+q_3+q_4)}{2}$$

where

$$\sigma_1 = \frac{l\cos(q_1+q_2+q_3+q_4)}{2}$$

```
y4_dot:=diff(y4, q_1)*dq_1+diff(y4, q_2)*dq_2+diff(y4,
q_3)*dq_3+diff(y4, q_4)*dq_4
```

$$-\mathrm{dq}_3\,(\sigma_1 + l\sin(q_1+q_2+q_3)) - \mathrm{dq}_1\,(\sigma_1 + l\sin(q_1+q_2) + l\sin(q_1) + l\sin(q_1+q_2+q_3)) - \mathrm{dq}_2\,(\sigma_1 + l\sin(q_1+q_2) + l\sin(q_1+q_2+q_3)) - \frac{\mathrm{dq}_4\, l\sin(q_1+q_2+q_3+q_4)}{2}$$

where

$$\sigma_1 = \frac{l\sin(q_1+q_2+q_3+q_4)}{2}$$

the formula of the kinetic energy T

kinetic energy

```
T:=simplify(1/2*m*(x1_dot^2+y1_dot^2)+1/2*m*(x2_dot^2+y2_dot^2
)+1/2*m*(x3_dot^2+y3_dot^2)+1/2*m*(x4_dot^2+y4_dot^2))
```

$$\big(l^2 m\,(30\,\mathrm{dq}_1\mathrm{dq}_2 + 12\,\mathrm{dq}_1\mathrm{dq}_3 + 2\,\mathrm{dq}_1\mathrm{dq}_4 + 12\,\mathrm{dq}_2\mathrm{dq}_3 + 2\,\mathrm{dq}_2\mathrm{dq}_4 + 2\,\mathrm{dq}_3\mathrm{dq}_4 + 4\,\mathrm{dq}_1^2\sigma_1 + 28\,\mathrm{dq}_1^2 + 15\,\mathrm{dq}_2^2 + 6\,\mathrm{dq}_3^2 + \mathrm{dq}_4^2 + 12\,\mathrm{dq}_1^2\cos(q_2+q_3) + 4\,\mathrm{dq}_1^2\cos(q_3+q_4) + 4\,\mathrm{dq}_2^2\cos(q_3+q_4) + 20\,\mathrm{dq}_1^2\cos(q_2)$$
$$+ 12\,\mathrm{dq}_1^2\cos(q_3) + 4\,\mathrm{dq}_1^2\cos(q_4) + 12\,\mathrm{dq}_2^2\cos(q_3) + 4\,\mathrm{dq}_2^2\cos(q_4) + 4\,\mathrm{dq}_3^2\cos(q_4) + 4\,\mathrm{dq}_1\mathrm{dq}_2\sigma_1 + 4\,\mathrm{dq}_1\mathrm{dq}_3\sigma_1 + 4\,\mathrm{dq}_1\mathrm{dq}_4\sigma_1 + 12\,\mathrm{dq}_1\mathrm{dq}_2\cos(q_2+q_3) + 12\,\mathrm{dq}_1\mathrm{dq}_3\cos(q_2+q_3) + 8\,\mathrm{dq}_1\mathrm{dq}_2\cos(q_3+q_4)$$
$$+ 4\,\mathrm{dq}_1\mathrm{dq}_3\cos(q_3+q_4) + 4\,\mathrm{dq}_1\mathrm{dq}_4\cos(q_3+q_4) + 4\,\mathrm{dq}_2\mathrm{dq}_3\cos(q_3+q_4) + 4\,\mathrm{dq}_2\mathrm{dq}_4\cos(q_3+q_4) + 20\,\mathrm{dq}_1\mathrm{dq}_2\cos(q_2) + 24\,\mathrm{dq}_1\mathrm{dq}_2\cos(q_3) + 8\,\mathrm{dq}_1\mathrm{dq}_2\cos(q_4) + 12\,\mathrm{dq}_1\mathrm{dq}_3\cos(q_3) + 8\,\mathrm{dq}_1\mathrm{dq}_3\cos(q_4)$$
$$+ 12\,\mathrm{dq}_2\mathrm{dq}_3\cos(q_3) + 4\,\mathrm{dq}_1\mathrm{dq}_4\cos(q_4) + 8\,\mathrm{dq}_2\mathrm{dq}_3\cos(q_4) + 4\,\mathrm{dq}_2\mathrm{dq}_4\cos(q_4) + 4\,\mathrm{dq}_3\mathrm{dq}_4\cos(q_4))\big)/8$$

where

$$\sigma_1 = \cos(q_2+q_3+q_4)$$

Inertia matrix

```
m11:=simplify(2*T|dq_1=1|dq_2=0|dq_3=0|dq_4=0)
```

$$\frac{l^2 m\,(4\cos(q_2+q_3+q_4) + 12\cos(q_2+q_3) + 4\cos(q_3+q_4) + 20\cos(q_2) + 12\cos(q_3) + 4\cos(q_4) + 28)}{4}$$

```
m22:=simplify(2*T|dq_1=0|dq_2=1|dq_3=0|dq_4=0)
```

$$\frac{l^2 m\,(4\cos(q_3+q_4) + 12\cos(q_3) + 4\cos(q_4) + 15)}{4}$$

```
m33:=simplify(2*T|dq_1=0|dq_2=0|dq_3=1|dq_4=0)
```

$$\frac{l^2 m\,(4\cos(q_4) + 6)}{4}$$

```
m44:=simplify(2*T|dq_1=0|dq_2=0|dq_3=0|dq_4=1)
```

$$\frac{l^2 m}{4}$$

```
m12:=simplify(2*T-m11-m22|dq_1=1|dq_2=1|dq_3=0|dq_4=0)/2
```

$$\frac{l^2 m\,(2\cos(q_2+q_3+q_4) + 6\cos(q_2+q_3) + 4\cos(q_3+q_4) + 10\cos(q_2) + 12\cos(q_3) + 4\cos(q_4) + 15)}{4}$$

```
m13:=simplify(2*T-m11-m33|dq_1=1|dq_2=0|dq_3=1|dq_4=0)/2
```

$$\frac{l^2\, m\, (\cos(q_2+q_3+q_4)+3\cos(q_2+q_3)+\cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3)}{2}$$

```
m14:=simplify(2*T-m11-m44|dq_1=1|dq_2=0|dq_3=0|dq_4=1)/2
```

$$\frac{l^2\, m\, (2\cos(q_2+q_3+q_4)+2\cos(q_3+q_4)+2\cos(q_4)+1)}{4}$$

```
m23:=simplify(2*T-m22-m33|dq_1=0|dq_2=1|dq_3=1|dq_4=0)/2
```

$$\frac{l^2\, m\, (\cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3)}{2}$$

```
m24:=simplify(2*T-m22-m44|dq_1=0|dq_2=1|dq_3=0|dq_4=1)/2
```

$$\frac{l^2\, m\, (2\cos(q_3+q_4)+2\cos(q_4)+1)}{4}$$

```
m34:=simplify(2*T-m33-m44|dq_1=0|dq_2=0|dq_3=1|dq_4=1)/2
```

$$\frac{l^2\, m\, (2\cos(q_4)+1)}{4}$$

The expression of the inertia matrix M

```
M:=matrix([[m11,m12,m13,m14],[m12,m22,m23,m24],[m13,m23,m33,m3
4],[m14,m24,m34,m44]])
```

$$\begin{pmatrix} \frac{l^2 m (4\sigma_8+12\cos(q_2+q_3)+\sigma_7+20\cos(q_2)+12\cos(q_3)+4\cos(q_4)+28)}{4} & \sigma_3 & \sigma_4 & \sigma_5 \\ \sigma_3 & \frac{l^2 m(\sigma_7+12\cos(q_3)+4\cos(q_4)+15)}{4} & \sigma_1 & \sigma_6 \\ \sigma_4 & \sigma_1 & \frac{l^2 m(4\cos(q_4)+6)}{4} & \sigma_2 \\ \sigma_5 & \sigma_6 & \sigma_2 & \frac{l^2 m}{4} \end{pmatrix}$$

where

$$\sigma_1 = \frac{l^2 m (\cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3)}{2}$$

$$\sigma_2 = \frac{l^2 m (2\cos(q_4)+1)}{4}$$

$$\sigma_3 = \frac{l^2 m (2\sigma_8+6\cos(q_2+q_3)+\sigma_7+10\cos(q_2)+12\cos(q_3)+4\cos(q_4)+15)}{4}$$

$$\sigma_4 = \frac{l^2 m (\sigma_8+3\cos(q_2+q_3)+\cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3)}{2}$$

$$\sigma_5 = \frac{l^2 m (2\sigma_8+\sigma_9+2\cos(q_4)+1)}{4}$$

$$\sigma_6 = \frac{l^2 m (\sigma_9+2\cos(q_4)+1)}{4}$$

$$\sigma_7 = 4\cos(q_3+q_4)$$

$$\sigma_8 = \cos(q_2+q_3+q_4)$$

$$\sigma_9 = 2\cos(q_3+q_4)$$

```
T0:=1/2*matrix([[dq_1,dq_2,dq_3,dq_4]])*M*matrix([dq_1,dq_2,dq_3,dq_4])
```

$$\left[\left[\frac{dq_3\left(\frac{dq_2 l^2 m\sigma_1}{2}+\frac{dq_1 l^2 m\sigma_4}{2}+\frac{dq_4 l^2 m\sigma_2}{4}+\frac{dq_3 l^2 m(4\cos(q_4)+6)}{4}\right)}{2}+\frac{dq_4\left(\frac{dq_4 l^2 m}{4}+\frac{dq_1 l^2 m\sigma_5}{4}+\frac{dq_3 l^2 m\sigma_2}{4}+\frac{dq_2 l^2 m\sigma_6}{4}\right)}{2}+\frac{dq_2\left(\frac{dq_3 l^2 m\sigma_1}{2}+\frac{dq_1 l^2 m\sigma_3}{4}+\frac{dq_4 l^2 m\sigma_6}{4}+\frac{dq_2 l^2 m(\sigma_7+12\cos(q_3)+4\cos(q_4)+15)}{4}\right)}{2}\right.\right.$$
$$\left.\left.+\frac{dq_1\left(\frac{dq_4 l^2 m\sigma_5}{4}+\frac{dq_2 l^2 m\sigma_3}{4}+\frac{dq_1 l^2 m(4\sigma_8+12\cos(q_2+q_3)+\sigma_7+20\cos(q_2)+12\cos(q_3)+4\cos(q_4)+28)}{4}+\frac{dq_3 l^2 m\sigma_4}{2}\right)}{2}\right]\right]$$

where

$\sigma_1 = \cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3$

$\sigma_2 = 2\cos(q_4)+1$

$\sigma_3 = 2\sigma_8+6\cos(q_2+q_3)+\sigma_7+10\cos(q_2)+12\cos(q_3)+4\cos(q_4)+15$

$\sigma_4 = \sigma_8+3\cos(q_2+q_3)+\cos(q_3+q_4)+3\cos(q_3)+2\cos(q_4)+3$

$\sigma_5 = 2\sigma_8+\sigma_9+2\cos(q_4)+1$

$\sigma_6 = \sigma_9+2\cos(q_4)+1$

$\sigma_7 = 4\cos(q_3+q_4)$

$\sigma_8 = \cos(q_2+q_3+q_4)$

$\sigma_9 = 2\cos(q_3+q_4)$

```
verify:=simplify(T-T0)
```

$$(0)$$

QUESTION 3. Write the equations of motion of the system in Port-Hamiltonian form. Provide an explicit expression of ∇qH and ∇pH. Use Matlab Mupad.
Assume the following:

- the damping matrix is diagonal with equal elements $R = \mathrm{diag}\{b\}$

Hint: see Equation (1) in the lecture notes. When computing ∇pH, recall that $p = M\dot{q}$

The port-Hamiltonian $H = T + V$

Hamiltonian

```
H:=T+V
```

$$\Big(m\,\big(30\,dq_1\,dq_2 + 12\,dq_1\,dq_3 + 2\,dq_1\,dq_4 + 12\,dq_2\,dq_3 + 2\,dq_2\,dq_4 + 2\,dq_3\,dq_4 + 4\,dq_1^2\,\sigma_1 + 28\,dq_1^2 + 15\,dq_2^2 + 6\,dq_3^2 + dq_4^2 + 12\,dq_1^2\cos(q_2+q_3) + 4\,dq_1^2\cos(q_3+q_4) + 4\,dq_2^2\cos(q_3+q_4) + 20\,dq_1^2\cos(q_2)$$
$$+ 12\,dq_1^2\cos(q_3) + 4\,dq_1^2\cos(q_4) + 12\,dq_2^2\cos(q_3) + 4\,dq_2^2\cos(q_4) + 4\,dq_3^2\cos(q_4) + 4\,dq_1\,dq_2\,\sigma_1 + 4\,dq_1\,dq_3\,\sigma_1 + 4\,dq_1\,dq_4\,\sigma_1 + 12\,dq_1\,dq_2\cos(q_2+q_3) + 12\,dq_1\,dq_3\cos(q_2+q_3) + 8\,dq_1\,dq_2\cos(q_3+q_4)$$
$$+ 4\,dq_1\,dq_3\cos(q_3+q_4) + 4\,dq_1\,dq_4\cos(q_3+q_4) + 4\,dq_2\,dq_3\cos(q_3+q_4) + 4\,dq_2\,dq_4\cos(q_3+q_4) + 20\,dq_1\,dq_2\cos(q_2) + 24\,dq_1\,dq_2\cos(q_3) + 8\,dq_1\,dq_2\cos(q_4) + 12\,dq_1\,dq_3\cos(q_3) + 8\,dq_1\,dq_3\cos(q_4)$$
$$+ 12\,dq_2\,dq_3\cos(q_3) + 4\,dq_1\,dq_4\cos(q_4) + 8\,dq_2\,dq_3\cos(q_4) + 4\,dq_2\,dq_4\cos(q_4) + 4\,dq_3\,dq_4\cos(q_4)\big)\,l^2\Big)/8 + \frac{k\,(q_1^2+q_2^2+q_3^2+q_4^2)}{2}$$

where

$\sigma_1 = \cos(q_2+q_3+q_4)$

dH_q ∇qH

```
dH_q:=matrix([diff(H, q_1),diff(H, q_2),diff(H, q_3),diff(H,
q_4)])
```

$$\begin{pmatrix} k\,q_1 \\ k\,q_2 - \frac{l^2\,m\,(20\,dq_1^2\sin(q_2)+\sigma_{11}+\sigma_1+20\,dq_1\,dq_2\sin(q_2)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_5+\sigma_4)}{8} \\ k\,q_3 - \frac{l^2\,m\,(12\,dq_1^2\sin(q_3)+12\,dq_2^2\sin(q_3)+\sigma_{11}+\sigma_1+\sigma_3+\sigma_2+24\,dq_1\,dq_2\sin(q_3)+12\,dq_1\,dq_3\sin(q_3)+12\,dq_2\,dq_3\sin(q_3)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_5+\sigma_4+\sigma_6+\sigma_{10}+\sigma_9+\sigma_8+\sigma_7)}{8} \\ k\,q_4 - \frac{l^2\,m\,(4\,dq_1^2\sin(q_4)+4\,dq_2^2\sin(q_4)+4\,dq_3^2\sin(q_4)+\sigma_{11}+\sigma_3+\sigma_2+8\,dq_1\,dq_2\sin(q_4)+8\,dq_1\,dq_3\sin(q_4)+4\,dq_1\,dq_4\sin(q_4)+8\,dq_2\,dq_3\sin(q_4)+4\,dq_2\,dq_4\sin(q_4)+4\,dq_3\,dq_4\sin(q_4)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_6+\sigma_{10}+\sigma_9+\sigma_8+\sigma_7)}{8} \end{pmatrix}$$

where

$\sigma_1 = 12\,dq_1^2\sin(q_2+q_3)$

$\sigma_2 = 4\,dq_2^2\sin(q_3+q_4)$

$\sigma_3 = 4\,dq_1^2\sin(q_3+q_4)$

$\sigma_4 = 12\,dq_1\,dq_3\sin(q_2+q_3)$

$\sigma_5 = 12\,dq_1\,dq_2\sin(q_2+q_3)$

$\sigma_6 = 8\,dq_1\,dq_2\sin(q_3+q_4)$

$\sigma_7 = 4\,dq_2\,dq_4\sin(q_3+q_4)$

$\sigma_8 = 4\,dq_2\,dq_3\sin(q_3+q_4)$

$\sigma_9 = 4\,dq_1\,dq_4\sin(q_3+q_4)$

$\sigma_{10} = 4\,dq_1\,dq_3\sin(q_3+q_4)$

$\sigma_{11} = 4\,dq_1^2\,\sigma_{12}$

$\sigma_{12} = \sin(q_2+q_3+q_4)$

dH_p = q_dot ∇pH

```
dH_p = matrix([dq_1,dq_2,dq_3,dq_4])
```

$$dH_p = \begin{pmatrix} dq_1 \\ dq_2 \\ dq_3 \\ dq_4 \end{pmatrix}$$

dT_q (kinetic energy)

```
dT_q:=matrix([diff(T, q_1),diff(T, q_2),diff(T, q_3),diff(T, q_4)])
```

$$\begin{pmatrix} k\,q_1 \\ k\,q_2 - \frac{l^2 m\left(20\,dq_1^2 \sin(q_2)+\sigma_{11}+\sigma_1+20\,dq_1\,dq_2 \sin(q_2)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_5+\sigma_4\right)}{8} \\ k\,q_3 - \frac{l^2 m\left(12\,dq_1^2 \sin(q_3)+12\,dq_2^2 \sin(q_3)+\sigma_{11}+\sigma_1+\sigma_3+\sigma_2+24\,dq_1\,dq_2 \sin(q_3)+12\,dq_1\,dq_3 \sin(q_3)+12\,dq_2\,dq_3 \sin(q_3)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_5+\sigma_4+\sigma_6+\sigma_{10}+\sigma_9+\sigma_8+\sigma_7\right)}{8} \\ k\,q_4 - \frac{l^2 m\left(4\,dq_1^2 \sin(q_4)+4\,dq_2^2 \sin(q_4)+4\,dq_3^2 \sin(q_4)+\sigma_{11}+\sigma_3+\sigma_2+8\,dq_1\,dq_2 \sin(q_4)+8\,dq_1\,dq_3 \sin(q_4)+4\,dq_1\,dq_4 \sin(q_4)+8\,dq_2\,dq_3 \sin(q_4)+4\,dq_2\,dq_4 \sin(q_4)+4\,dq_3\,dq_4 \sin(q_4)+4\,dq_1\,dq_2\,\sigma_{12}+4\,dq_1\,dq_3\,\sigma_{12}+4\,dq_1\,dq_4\,\sigma_{12}+\sigma_6+\sigma_{10}+\sigma_9+\sigma_8+\sigma_7\right)}{8} \end{pmatrix}$$

where

$\sigma_1 = 12\,dq_1^2 \sin(q_2+q_3)$

$\sigma_2 = 4\,dq_2^2 \sin(q_3+q_4)$

$\sigma_3 = 4\,dq_1^2 \sin(q_3+q_4)$

$\sigma_4 = 12\,dq_1\,dq_3 \sin(q_2+q_3)$

$\sigma_5 = 12\,dq_1\,dq_2 \sin(q_2+q_3)$

$\sigma_6 = 8\,dq_1\,dq_2 \sin(q_3+q_4)$

$\sigma_7 = 4\,dq_2\,dq_4 \sin(q_3+q_4)$

$\sigma_8 = 4\,dq_2\,dq_3 \sin(q_3+q_4)$

$\sigma_9 = 4\,dq_1\,dq_4 \sin(q_3+q_4)$

$\sigma_{10} = 4\,dq_1\,dq_3 \sin(q_3+q_4)$

$\sigma_{11} = 4\,dq_1^2\,\sigma_{12}$

$\sigma_{12} = \sin(q_2+q_3+q_4)$

dM

```
dM:=diff(M, q_1)*dq_1+diff(M, q_2)*dq_2+diff(M, q_3)*dq_3+diff(M, q_4)*dq_4
```

$$\begin{pmatrix} -\frac{dq_4\,l^2 m\,(4\,\sigma_{11}+\sigma_8+4\sin(q_4))}{4} - \frac{dq_2\,l^2 m\,(4\,\sigma_{11}+\sigma_3+20\sin(q_2))}{4} - \frac{dq_3\,l^2 m\,(4\,\sigma_{11}+\sigma_3+\sigma_8+12\sin(q_3))}{4} & \sigma_4 & \sigma_5 & \sigma_6 \\ \sigma_4 & -\frac{dq_4\,l^2 m\,(\sigma_8+4\sin(q_4))}{4} - \frac{dq_3\,l^2 m\,(\sigma_8+12\sin(q_3))}{4} & \sigma_1 & \sigma_7 \\ \sigma_5 & \sigma_1 & -\sigma_2 & -\frac{\sigma_2}{2} \\ \sigma_6 & \sigma_7 & -\frac{\sigma_2}{2} & 0 \end{pmatrix}$$

where

$\sigma_1 = -\frac{dq_3\,l^2 m\,(\sin(q_3+q_4)+3\sin(q_3))}{2} - \frac{dq_4\,l^2 m\,(\sin(q_3+q_4)+2\sin(q_4))}{2}$

$\sigma_2 = dq_4\,l^2\,m \sin(q_4)$

$\sigma_3 = 12\sin(q_2+q_3)$

$\sigma_4 = -\frac{dq_4\,l^2 m\,(2\,\sigma_{11}+\sigma_8+4\sin(q_4))}{4} - \frac{dq_2\,l^2 m\,(2\,\sigma_{11}+\sigma_9+10\sin(q_2))}{4} - \frac{dq_3\,l^2 m\,(2\,\sigma_{11}+\sigma_9+\sigma_8+12\sin(q_3))}{4}$

$\sigma_5 = -\frac{dq_4\,l^2 m\,(\sigma_{11}+\sin(q_3+q_4)+2\sin(q_4))}{2} - \frac{dq_3\,l^2 m\,(\sigma_{11}+\sigma_{10}+\sin(q_3+q_4)+3\sin(q_3))}{2} - \frac{dq_2\,l^2 m\,(\sigma_{11}+\sigma_{10})}{2}$

$\sigma_6 = -\frac{dq_2\,l^2 m\,\sigma_{11}}{2} - \frac{dq_4\,l^2 m\,(2\,\sigma_{11}+\sigma_{12}+2\sin(q_4))}{4} - \frac{dq_3\,l^2 m\,(2\,\sigma_{11}+\sigma_{12})}{4}$

$\sigma_7 = -\frac{dq_3\,l^2 m \sin(q_3+q_4)}{2} - \frac{dq_4\,l^2 m\,(\sigma_{12}+2\sin(q_4))}{4}$

$\sigma_8 = 4\sin(q_3+q_4)$

$\sigma_9 = 6\sin(q_2+q_3)$

$\sigma_{10} = 3\sin(q_2+q_3)$

$\sigma_{11} = \sin(q_2+q_3+q_4)$

$\sigma_{12} = 2\sin(q_3+q_4)$

dMt

```
dMt:=diff(M*matrix([dq_1,dq_2,dq_3,dq_4]),
q_1)*dq_1+diff(M*matrix([dq_1,dq_2,dq_3,dq_4]),
q_2)*dq_2+diff(M*matrix([dq_1,dq_2,dq_3,dq_4]),
q_3)*dq_3+diff(M*matrix([dq_1,dq_2,dq_3,dq_4]), q_4)*dq_4
```

$$\Bigg[\Bigg[-\mathrm{dq}_3\left(\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_5}{4}+\frac{\mathrm{dq}_1\, l^2\, m\,(4\,\sigma_{16}+\sigma_4+\sigma_{15}+12\sin(q_3))}{4}+\frac{\mathrm{dq}_3\, l^2\, m\,\sigma_6}{2}+\frac{\mathrm{dq}_4\, l^2\, m\,(2\,\sigma_{16}+\sigma_{14})}{4}\right)-\mathrm{dq}_4\left(\frac{\mathrm{dq}_4\, l^2\, m\,\sigma_8}{4}+\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_{11}}{4}+\frac{\mathrm{dq}_1\, l^2\, m\,(4\,\sigma_{16}+\sigma_{15}+4\sin(q_4))}{4}+\frac{\mathrm{dq}_3\, l^2\, m\,\sigma_{10}}{2}\right)$$

$$-\mathrm{dq}_2\left(\frac{\mathrm{dq}_4\, l^2\, m\,\sigma_{16}}{2}+\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_7}{4}+\frac{\mathrm{dq}_1\, l^2\, m\,(4\,\sigma_{16}+\sigma_4+20\sin(q_2))}{4}+\frac{\mathrm{dq}_3\, l^2\, m\,(\sigma_{16}+\sigma_{12})}{2}\right)\Bigg],$$

$$\Bigg[-\mathrm{dq}_4\left(\frac{\mathrm{dq}_1\, l^2\, m\,\sigma_{11}}{4}+\frac{\mathrm{dq}_3\, l^2\, m\,\sigma_2}{2}+\frac{\mathrm{dq}_4\, l^2\, m\,\sigma_9}{4}+\frac{\mathrm{dq}_2\, l^2\, m\,(\sigma_{15}+4\sin(q_4))}{4}\right)-\mathrm{dq}_3\left(\frac{\mathrm{dq}_1\, l^2\, m\,\sigma_5}{4}+\frac{\mathrm{dq}_3\, l^2\, m\,\sigma_1}{2}+\frac{\mathrm{dq}_4\, l^2\, m\sin(q_3+q_4)}{2}+\frac{\mathrm{dq}_2\, l^2\, m\,(\sigma_{15}+12\sin(q_3))}{4}\right)-\frac{\mathrm{dq}_1\,\mathrm{dq}_2\, l^2\, m\,\sigma_7}{4}\Bigg],$$

$$\Bigg[-\mathrm{dq}_3\left(\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_1}{2}+\frac{\mathrm{dq}_1\, l^2\, m\,\sigma_6}{2}\right)-\mathrm{dq}_4\left(\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_2}{2}+\frac{\mathrm{dq}_1\, l^2\, m\,\sigma_{10}}{2}+\sigma_3+\frac{\mathrm{dq}_4\, l^2\, m\sin(q_4)}{2}\right)-\frac{\mathrm{dq}_1\,\mathrm{dq}_2\, l^2\, m\,(\sigma_{16}+\sigma_{12})}{2}\Bigg],$$

$$\Bigg[-\mathrm{dq}_3\left(\frac{\mathrm{dq}_2\, l^2\, m\sin(q_3+q_4)}{2}+\frac{\mathrm{dq}_1\, l^2\, m\,(2\,\sigma_{16}+\sigma_{14})}{4}\right)-\mathrm{dq}_4\left(\frac{\mathrm{dq}_1\, l^2\, m\,\sigma_8}{4}+\frac{\mathrm{dq}_2\, l^2\, m\,\sigma_9}{4}+\frac{\sigma_3}{2}\right)-\frac{\mathrm{dq}_1\,\mathrm{dq}_2\, l^2\, m\,\sigma_{16}}{2}\Bigg]\Bigg]$$

where

$\sigma_1 = \sin(q_3+q_4)+3\sin(q_3)$

$\sigma_2 = \sin(q_3+q_4)+2\sin(q_4)$

$\sigma_3 = \mathrm{dq}_3\, l^2\, m\sin(q_4)$

$\sigma_4 = 12\sin(q_2+q_3)$

$\sigma_5 = 2\,\sigma_{16}+\sigma_{13}+\sigma_{15}+12\sin(q_3)$

$\sigma_6 = \sigma_{16}+\sigma_{12}+\sin(q_3+q_4)+3\sin(q_3)$

$\sigma_7 = 2\,\sigma_{16}+\sigma_{13}+10\sin(q_2)$

$\sigma_8 = 2\,\sigma_{16}+\sigma_{14}+2\sin(q_4)$

$\sigma_9 = \sigma_{14}+2\sin(q_4)$

$\sigma_{10} = \sigma_{16}+\sin(q_3+q_4)+2\sin(q_4)$

$\sigma_{11} = 2\,\sigma_{16}+\sigma_{15}+4\sin(q_4)$

$\sigma_{12} = 3\sin(q_2+q_3)$

$\sigma_{13} = 6\sin(q_2+q_3)$

$\sigma_{14} = 2\sin(q_3+q_4)$

$\sigma_{15} = 4\sin(q_3+q_4)$

$\sigma_{16} = \sin(q_2+q_3+q_4)$

QUESTION 4. Write the expression of the matrix $G^{\perp}$ for $n = 4$ and the corresponding potential-energy PDE. Use Matlab Mupad.
Hint: see Example 4 in the lecture notes. Note that different choices of $G^{\perp}$ are possible. Verify that $G^{\perp}G = 0$ and that $\mathrm{rank}\{G^{\perp}\} = n - 1$.

PDEs

```
G:=matrix([1,1,1,1])
```

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

```
G_orth:=matrix([[1,0,0,-1],[0,1,0,-1],[0,0,1,-1]])
```

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

```
G_orth*G
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
linalg::rank(G_orth)
```

$$3$$

```
dV:=matrix([diff(V, q_1),diff(V, q_2),diff(V, q_3),diff(V,
q_4)])
```

$$\begin{pmatrix} k\, q_1 \\ k\, q_2 \\ k\, q_3 \\ k\, q_4 \end{pmatrix}$$

```
dV_d:=matrix([dV_d1,dV_d2,dV_d3,dV_d4])
```

$$\begin{pmatrix} \mathrm{dV_d1} \\ \mathrm{dV_d2} \\ \mathrm{dV_d3} \\ \mathrm{dV_d4} \end{pmatrix}$$

```
PDEs:=G_orth*(dV-k_m*dV_d)
```

$$\begin{pmatrix} \mathrm{dV_d4}\, k_m - \mathrm{dV_d1}\, k_m + k\, q_1 - k\, q_4 \\ \mathrm{dV_d4}\, k_m - \mathrm{dV_d2}\, k_m + k\, q_2 - k\, q_4 \\ \mathrm{dV_d4}\, k_m - \mathrm{dV_d3}\, k_m + k\, q_3 - k\, q_4 \end{pmatrix}$$

So, expression for PDEs

$$\begin{pmatrix} dV_d4\, k_m - dV_d1\, k_m + k\, q_1 - k\, q_4 \\ dV_d4\, k_m - dV_d2\, k_m + k\, q_2 - k\, q_4 \\ dV_d4\, k_m - dV_d3\, k_m + k\, q_3 - k\, q_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

QUESTION 5. Compute the solution of the potential-energy PDE for $n = 4$. The solution should have a minimizer at $q = q*$. Use Maple to solve the PDE and Matlab Mupad for verification.

Use Maple to solve:

$PDEsys := [diff(Vd(q1, q2, q3, q4), q1) \cdot km - diff(Vd(q1, q2, q3, q4), q4) \cdot km = k \cdot q1 - k \cdot q4, diff(Vd(q1, q2, q3, q4), q2) \cdot km - diff(Vd(q1, q2, q3, q4), q4) \cdot km = k \cdot q2 - k \cdot q4, diff(Vd(q1, q2, q3, q4), q3) \cdot km - diff(Vd(q1, q2, q3, q4), q4) \cdot km = k \cdot q3 - k \cdot q4]$

$$PDEsys := \left[\left(\frac{\partial}{\partial q1} Vd(q1, q2, q3, q4)\right) km - \left(\frac{\partial}{\partial q4} Vd(q1, q2, q3, q4)\right) km = k\, q1 - k\, q4, \left(\frac{\partial}{\partial q2} Vd(q1, q2, q3, q4)\right) km - \left(\frac{\partial}{\partial q4} Vd(q1, q2, q3, q4)\right) km = k\, q2 - k\, q4, \left(\frac{\partial}{\partial q3} Vd(q1, q2, q3, q4)\right) km - \left(\frac{\partial}{\partial q4} Vd(q1, q2, q3, q4)\right) km = k\, q3 - k\, q4 \right] \quad (1)$$

$Vd0 := pdsolve(PDEsys)$

$$Vd0 := \left\{ Vd(q1, q2, q3, q4) = \frac{\_F1(q3 + q2 + q1 + q4)\, km - k\,((q2 + q3 + q4)\, q1 + (q3 + q4)\, q2 + q3\, q4)}{km} \right\} \quad (2)$$

$pdetest(Vd0, PDEsys)$

$$[0, 0, 0] \quad (3)$$

the strict minimizer conditions

$$\nabla_q V_d(q^*) = 0$$

$$\nabla_q^2 V_d(q^*) > 0$$

Then, the solution for $n = 4$

$$V_d = \frac{k}{2k_m} \sum_{i=1}^{4} q_i^2 - \frac{1}{4}\left(\sum_{i=1}^{4} q_i\right)^2 + \frac{k_p}{2}\left(\sum_{i=1}^{4} q_i - \sum_{i=1}^{4} q_i^*\right)^2$$

$$Vd1 := \left\{ Vd(q1, q2, q3, q4) = \frac{k \cdot \left(q1^2 - \frac{(q1 + q2 + q3 + q4)^2}{4} + q2^2 + q3^2 + q4^2\right)}{2\, km} + \frac{kp(q1 + q2 + q3 + q4 - 4 \cdot qd)^2}{2} \right\}$$

$$Vd1 := \left\{ Vd(q1, q2, q3, q4) = \frac{k \left(q1^2 - \frac{(q3 + q2 + q1 + q4)^2}{4} + q2^2 + q3^2 + q4^2\right)}{2\, km} + \frac{kp(q1 + q2 + q3 + q4 - 4\, qd)^2}{2} \right\}$$

$pdetest(Vd1, PDEsys)$

$$[0, 0, 0]$$

Use Mupad to check

Solution of the PDEs

```
V_d:=k/(2*k_m)*((q_1^2+q_2^2+q_3^2+q_4^2)-
(q_1+q_2+q_3+q_4)^2/4)+k_p/(2)*(q_1+q_2+q_3+q_4-4*q_D)^2
dV_dtest:=matrix([diff(V_d, q_1),diff(V_d, q_2),diff(V_d,
q_3),diff(V_d, q_4)])
```

$$\frac{k_p\,(q_1+q_2+q_3+q_4-4\,q_D)^2}{2}+\frac{k\left(q_1{}^2-\frac{(q_1+q_2+q_3+q_4)^2}{4}+q_2{}^2+q_3{}^2+q_4{}^2\right)}{2\,k_m}$$

The solution in Mupad is same as the calculation of Maple

The first partial derivative of V_d

```
PDEstest:=simplify(G_orth*(dV-k_m*dV_dtest))
```

$$\begin{pmatrix} \sigma_1-\frac{k\left(\frac{q_2}{2}-\frac{3\,q_1}{2}+\frac{q_3}{2}+\frac{q_4}{2}\right)}{2\,k_m} \\ \sigma_1-\frac{k\left(\frac{q_1}{2}-\frac{3\,q_2}{2}+\frac{q_3}{2}+\frac{q_4}{2}\right)}{2\,k_m} \\ \sigma_1-\frac{k\left(\frac{q_1}{2}+\frac{q_2}{2}-\frac{3\,q_3}{2}+\frac{q_4}{2}\right)}{2\,k_m} \\ \sigma_1-\frac{k\left(\frac{q_1}{2}+\frac{q_2}{2}+\frac{q_3}{2}-\frac{3\,q_4}{2}\right)}{2\,k_m} \end{pmatrix}$$

where

$$\sigma_1=\frac{k_p\,(2\,q_1+2\,q_2+2\,q_3+2\,q_4-8\,q_D)}{2}$$

minimizer conditions $\nabla_q V_d(q^*)$ and the $\nabla_q^2 V_d(q^*)$

```
simplify(dV_dtest|q_1=q_D|q_2=q_D|q_3=q_D|q_4=q_D)
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
d2V_dtest:=matrix([[diff(dV_dtest[1], q_1),diff(dV_dtest[1],
q_2),diff(dV_dtest[1], q_3),diff(dV_dtest[1],
q_4)],[diff(dV_dtest[2], q_1),diff(dV_dtest[2],
q_2),diff(dV_dtest[2], q_3),diff(dV_dtest[2],
q_4)],[diff(dV_dtest[3], q_1),diff(dV_dtest[3],
q_2),diff(dV_dtest[3], q_3),diff(dV_dtest[3],
q_4)],[diff(dV_dtest[4], q_1),diff(dV_dtest[4],
q_2),diff(dV_dtest[4], q_3),diff(dV_dtest[4], q_4)]])
```

$$\begin{pmatrix} k_p + \frac{3k}{4k_m} & \sigma_1 & \sigma_1 & \sigma_1 \\ \sigma_1 & k_p + \frac{3k}{4k_m} & \sigma_1 & \sigma_1 \\ \sigma_1 & \sigma_1 & k_p + \frac{3k}{4k_m} & \sigma_1 \\ \sigma_1 & \sigma_1 & \sigma_1 & k_p + \frac{3k}{4k_m} \end{pmatrix}$$

where

$$\sigma_1 = k_p - \frac{k}{4k_m}$$

```
det(d2V_dtest)
```

$$\frac{4k^3 k_p}{k_m^3}$$

and its determinant is $\frac{4k^3k_p}{k_m^3} > 0$, the second minimizer condition hold. Which shows that the V_d is the solution of the PDE.

QUESTION 6. Write the expression of the control law u for $n = 4$ (use Matlab Mupad).
Assume the following:

• the damping injection is $D1 = GkvG\ T$, where $kv > 0$ is a scalar

Hint: see Example 5 in the lecture notes.

Control law

```
G_dag:=inverse(transpose(G)*G)*transpose(G)
```

$$\left(\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\right)$$

energy shaping control

```
u_e_s:=simplify(G_dag*(dV-k_m*dV_dtest))
```

$$\left(\frac{k\,q_1}{4}+\frac{k\,q_2}{4}+\frac{k\,q_3}{4}+\frac{k\,q_4}{4}-k_m\,k_p\,q_1-k_m\,k_p\,q_2-k_m\,k_p\,q_3-k_m\,k_p\,q_4+4\,k_m\,k_p\,q_D\right)$$

dampling injection control

```
u_d_1:=-k_v/k_m*transpose(G)*matrix([dq_1,dq_2,dq_3,dq_4])
```

$$\left(-\frac{\mathrm{dq}_1\,k_v}{k_m}-\frac{\mathrm{dq}_2\,k_v}{k_m}-\frac{\mathrm{dq}_3\,k_v}{k_m}-\frac{\mathrm{dq}_4\,k_v}{k_m}\right)$$

```
u:=u_e_s+u_d_1
```

$$\left(\frac{k\,q_1}{4}+\frac{k\,q_2}{4}+\frac{k\,q_3}{4}+\frac{k\,q_4}{4}-k_m\,k_p\,q_1-k_m\,k_p\,q_2-k_m\,k_p\,q_3-k_m\,k_p\,q_4+4\,k_m\,k_p\,q_D-\frac{\mathrm{dq}_1\,k_v}{k_m}-\frac{\mathrm{dq}_2\,k_v}{k_m}-\frac{\mathrm{dq}_3\,k_v}{k_m}-\frac{\mathrm{dq}_4\,k_v}{k_m}\right)$$

the control law u has two parts: u_{es} for energy shaping control and u_{di} for damping injection control.

$$u_{es} = \frac{k}{n}\left(\sum_{i=1}^{n} q_i\right) - k_p k_m\left(\sum_{i=1}^{n} q_i - nq^*\right)$$

$$u_{di} = -\frac{k_v}{k_m}\sum_{i=1}^{n} \dot{q}_i$$

QUESTION 7*. Write the expression of the payload compensation term $u*$ (use Matlab Mupad).

Assume the following:

• the payload δ is constant and known (i.e. torque affecting all joints in the same way)

Hint: see Example 6 in the lecture notes.

*this is a bonus question.

Payload / disturbance compensation

```
disturbance:=Delta0*matrix([1,1,1,1])
```

$$\begin{pmatrix} \mathrm{Delta0} \\ \mathrm{Delta0} \\ \mathrm{Delta0} \\ \mathrm{Delta0} \end{pmatrix}$$

```
Lambda0:=matrix([Lambda1,Lambda2,Lambda3,Lambda4])
```

$$\begin{pmatrix} \mathrm{Lambda1} \\ \mathrm{Lambda2} \\ \mathrm{Lambda3} \\ \mathrm{Lambda4} \end{pmatrix}$$

```
simplify(G_orth*(disturbance-k_m*Lambda0))
```

$$\begin{pmatrix} -k_m\,(\mathrm{Lambda1}-\mathrm{Lambda2}) \\ -k_m\,(\mathrm{Lambda2}-\mathrm{Lambda3}) \\ -k_m\,(\mathrm{Lambda2}-\mathrm{Lambda4}) \end{pmatrix}$$

```
Lambda1:=0
```

0

```
Lambda2:=0
```

0

```
Lambda3:=0
```

0

```
Lambda4:=0
```

0

```
simplify(G_orth*(disturbance-k_m*Lambda0))
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

payload compensation term $u*$

```
u_star:=G_dag*disturbance
```

$$\begin{pmatrix} \mathrm{Delta0} \end{pmatrix}$$

QUESTION 8. Program a simulation in Matlab using the ODE command and verify that the controller from Question 6 achieves the regulation goal $q = q\ *$.
Topics in Control 2019-2020 Further instructions:

• Use the following parameters for the simulation: $l = 0.025$, $m = 4$, $k = 5$, $b = 0.01$, $q\ * = \pi/6$, $kp = 0.1$, $km = 5$, $kv = 2$.

• Write a Matlab function called "Topics_in_control_ode_n4.m" and call the function using the Matlab script "Topics_in_control_simulations.m" provided.

Hints: copy the expression of ∇qH, ∇pH, M, $M-1$, and $\dot{p}$ from Mupad into the Matlab function. Recall that $\dot{p} = M\ddot{q} + \dot{M}\dot{q}$ if the inertia matrix is not constant. Provide an explicit expression of $M-1$ in the Matlab function rather than using the command "inv(M)".

Matlab code:
https://imperialcollegelondon.box.com/s/elyr5ej425a2cb2sp22bir3w7tlqqcb1

Inverse of inertia matrix

```
M_inv:=inverse(M)
```

$$\left[\left[\frac{4\,(20\,\sigma_{39}-\sigma_{36}+\sigma_{38}+\sigma_{37}-45)}{\sigma_{30}},\ \sigma_6,\ \sigma_5,\ \sigma_4\right],\ \left[\sigma_6,\ \frac{8\left(18\,\sigma_{40}-50\cos(q_2)+10\,\sigma_{39}+\sigma_{35}+18\cos(q_3)^2+44\cos(q_4)^2+\sigma_{32}+10\,\sigma_{41}^2+\sigma_{33}-\sigma_{34}-\sigma_{31}-24\cos(q_3+q_4)\cos(q_3)\cos(q_4)-\sigma_8-55\right)}{\sigma_{30}},\ \sigma_3,\ \sigma_1\right],\right.$$

$$\left[\sigma_5,\ \sigma_3,\ \frac{8\left(18\,\sigma_{40}-78\cos(q_3)+26\,\sigma_{39}+50\cos(q_2)^2+26\cos(q_4)^2+28\,\sigma_{41}^2+\sigma_{18}+\sigma_{15}+\sigma_{16}-\sigma_{17}-\sigma_{14}-\sigma_8-\sigma_7-91\right)}{\sigma_{30}},\ \sigma_2\right],$$

$$\left.\left[\sigma_4,\ \sigma_1,\ \sigma_2,\ \frac{8\left(162\,\sigma_{40}-234\cos(q_4)+26\,\sigma_{39}+\sigma_{13}+300\cos(q_2)^2+234\cos(q_3)^2+\sigma_{10}+18\,\sigma_{41}^2+\sigma_{11}-\sigma_{12}-\sigma_9-360\cos(q_2+q_3)\cos(q_2)\cos(q_3)-\sigma_7-351\right)}{\sigma_{30}}\right]\right]$$

where

$$\sigma_1 = -\big(8\,\big(45\,\sigma_{41}-\sigma_{28}-65\cos(q_3+q_4)+39\cos(q_3)+\sigma_{27}+78\cos(q_3)\cos(q_4)+\sigma_{26}+36\,\sigma_{40}\cos(q_3+q_4)+50\,\sigma_{41}\cos(q_2)+\sigma_{29}-\sigma_{23}-12\,\sigma_{41}^2\cos(q_3)-30\cos(q_2+q_3)\cos(q_2)-\sigma_{21}-\sigma_{22}-26\cos(q_3+q_4)\cos(q_4)$$
$$+20\,\sigma_{41}\cos(q_2)\cos(q_4)+\sigma_{19}-60\cos(q_2+q_3)\cos(q_2)\cos(q_4)-\sigma_{24}+12\,\sigma_{41}\cos(q_2+q_3)\cos(q_3+q_4)+\sigma_{20}-36\,\sigma_{41}\cos(q_2+q_3)\cos(q_3)-\sigma_{25}\big)\big)/\sigma_{30}$$

$$\sigma_2 = -\big(4\,\big(130\cos(q_3+q_4)-78\cos(q_3)-234\cos(q_4)+52\,\sigma_{39}-156\cos(q_3)\cos(q_4)-72\,\sigma_{40}\cos(q_3+q_4)+\sigma_{13}+100\cos(q_2)^2-100\,\sigma_{41}\cos(q_2)+\sigma_{10}+36\,\sigma_{41}^2+\sigma_{18}+\sigma_{15}+\sigma_{11}+\sigma_{16}-\sigma_{12}-\sigma_{17}-\sigma_9$$
$$+120\cos(q_2+q_3)\cos(q_2)\cos(q_4)-\sigma_{14}+72\,\sigma_{41}\cos(q_2+q_3)\cos(q_3)-80\,\sigma_{41}\cos(q_3+q_4)\cos(q_2)-117\big)\big)/\sigma_{30}$$

$$\sigma_3 = -\frac{4\left(54\cos(q_2+q_3)-50\cos(q_2)-78\cos(q_3)+36\,\sigma_{40}-60\cos(q_2)\cos(q_3)-24\cos(q_2+q_3)\,\sigma_{39}+\sigma_{35}+52\cos(q_4)^2-36\,\sigma_{41}\cos(q_4)+\sigma_{32}+20\,\sigma_{41}^2+\sigma_{18}+\sigma_{15}+\sigma_{33}+\sigma_{16}-\sigma_{17}-\sigma_{34}-\sigma_{31}+40\cos(q_3+q_4)\cos(q_2)\cos(q_4)-\sigma_{14}-48\,\sigma_{41}\cos(q_2+q_3)\cos(q_4)+24\,\sigma_{41}\cos(q_3+q_4)\cos(q_3)-65\right)}{\sigma_{30}}$$

$$\sigma_4 = \frac{8\,(45\,\sigma_{41}-\sigma_{28}+\sigma_{27}+\sigma_{26}+\sigma_{29}-\sigma_{23}-\sigma_{21}-\sigma_{22}+\sigma_{19}-\sigma_{24}+\sigma_{20}-\sigma_{25})}{\sigma_{30}}$$

$$\sigma_5 = \frac{8\left(\sigma_{28}-25\cos(q_2)-\sigma_{27}-\sigma_{26}+10\,\sigma_{41}\cos(q_3+q_4)-\sigma_{29}+20\cos(q_2)\cos(q_4)^2+18\cos(q_2+q_3)\cos(q_3)-12\,\sigma_{41}\cos(q_3)\cos(q_4)-12\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_4)+\sigma_{24}+\sigma_{25}\right)}{\sigma_{30}}$$

$$\sigma_6 = -\frac{4\,(20\,\sigma_{39}-50\cos(q_2)+\sigma_{35}+\sigma_{38}+\sigma_{37}+\sigma_{32}+\sigma_{33}-\sigma_{34}-\sigma_{31}-\sigma_{36}-45)}{\sigma_{30}}$$

$\sigma_7 = 40\,\sigma_{41}\cos(q_3+q_4)\cos(q_2)$

$\sigma_8 = 24\,\sigma_{41}\cos(q_2+q_3)\cos(q_4)$

$\sigma_9 = 120\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_2)$

$\sigma_{10} = 200\cos(q_2)^2\cos(q_4)$

$\sigma_{11} = 156\cos(q_3+q_4)\cos(q_3)$

$\sigma_{12} = 120\,\sigma_{41}\cos(q_2)\cos(q_3)$

$\sigma_{13} = 108\,\sigma_{41}\cos(q_2+q_3)$

$\sigma_{14} = 24\,\sigma_{41}\cos(q_2+q_3)\cos(q_3+q_4)$

$\sigma_{15} = 60\cos(q_2+q_3)\cos(q_2)$

$\sigma_{16} = 52\cos(q_3+q_4)\cos(q_4)$

$\sigma_{17} = 40\,\sigma_{41}\cos(q_2)\cos(q_4)$

$\sigma_{18} = 24\,\sigma_{41}^2\cos(q_3)$

$\sigma_{19} = 36\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_3)$

$\sigma_{20} = 60\cos(q_2)\cos(q_3)\cos(q_4)$

$\sigma_{21} = 54\cos(q_2+q_3)\cos(q_4)$

$\sigma_{22} = 50\cos(q_3+q_4)\cos(q_2)$

$\sigma_{23} = 36\,\sigma_{41}\cos(q_3)^2$

$\sigma_{24} = 20\cos(q_3+q_4)\cos(q_2)\cos(q_4)$

$\sigma_{25} = 12\,\sigma_{41}\cos(q_3+q_4)\cos(q_3)$

$\sigma_{26} = 12\cos(q_2+q_3)\,\sigma_{39}$

$\sigma_{27} = 30\cos(q_2)\cos(q_3)$

$\sigma_{28} = 27\cos(q_2+q_3)$

$\sigma_{29} = 18\,\sigma_{41}\cos(q_4)$

$$\sigma_{30} = l^2\,m\,\big(-144\,\sigma_{41}^2\cos(q_3)^2+180\,\sigma_{41}^2+288\,\sigma_{41}\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_3)-432\,\sigma_{41}\cos(q_2+q_3)\cos(q_4)-400\,\sigma_{41}\cos(q_3+q_4)\cos(q_2)+480\,\sigma_{41}\cos(q_2)\cos(q_3)\cos(q_4)-144\,\sigma_{40}\,\sigma_{39}+324\,\sigma_{40}$$
$$+480\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_2)\cos(q_4)-720\cos(q_2+q_3)\cos(q_2)\cos(q_3)+260\,\sigma_{39}-624\cos(q_3+q_4)\cos(q_3)\cos(q_4)-400\cos(q_2)^2\cos(q_4)^2+500\cos(q_2)^2+468\cos(q_3)^2+468\cos(q_4)^2-585\big)$$

$\sigma_{31} = 24\cos(q_2+q_3)\cos(q_3+q_4)\cos(q_4)$

$\sigma_{32} = 40\cos(q_2)\cos(q_4)^2$

$\sigma_{33} = 36\cos(q_2+q_3)\cos(q_3)$

$\sigma_{34} = 24\,\sigma_{41}\cos(q_3)\cos(q_4)$

$\sigma_{35} = 20\,\sigma_{41}\cos(q_3+q_4)$

$\sigma_{36} = 48\cos(q_3+q_4)\cos(q_3)\cos(q_4)$

$\sigma_{37} = 36\cos(q_4)^2$

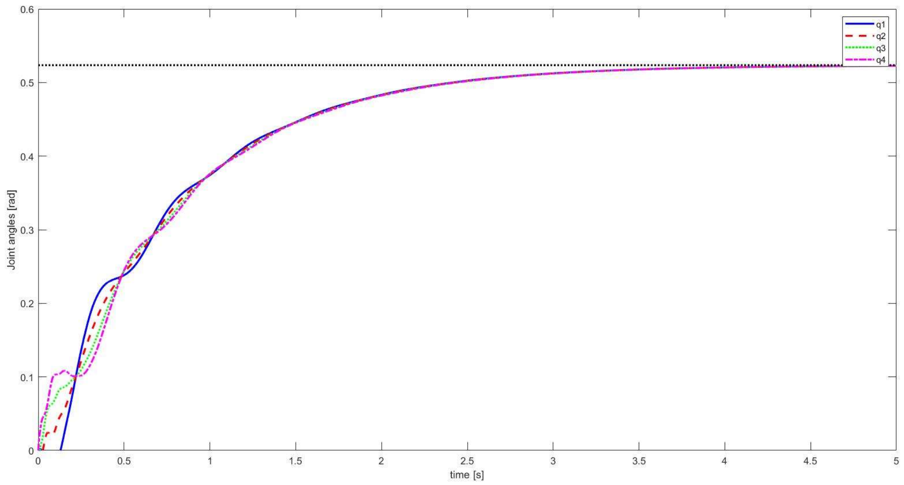
$\sigma_{38} = 36\cos(q_3)^2$

$\sigma_{39} = \cos(q_3+q_4)^2$

$\sigma_{40} = \cos(q_2+q_3)^2$

$\sigma_{41} = \cos(q_2+q_3+q_4)$

For no disturbance. The results are shown.

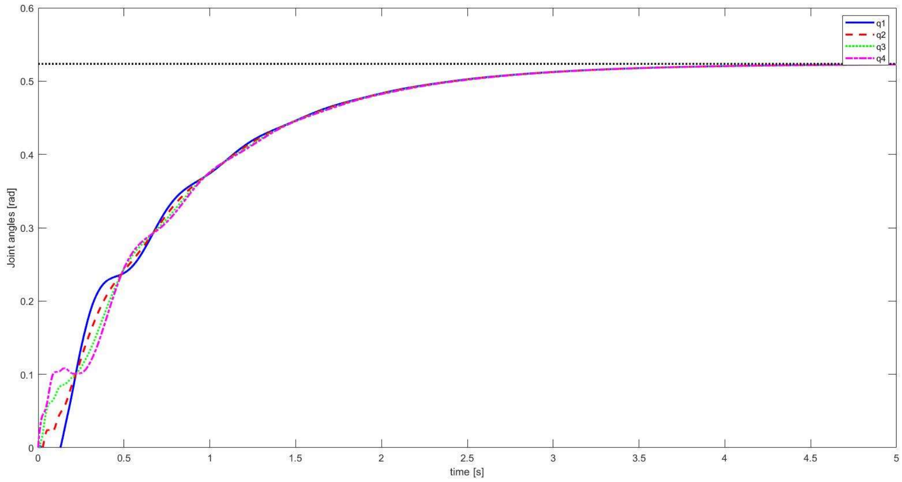


For $\delta = 0$, when $q = q^* = \frac{\pi}{6} = 0.5233$, every joint angle will converge to the minimizer value q^* and the total error (rad) is -0.0034 (T=5s), which means that the regulation goal can be achieved by obtained controller.

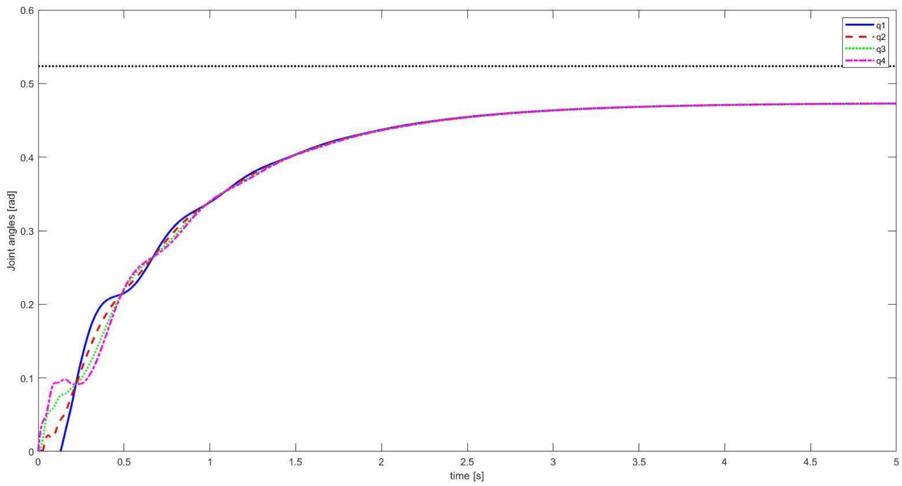
QUESTION 9*. Verify with a simulation in Matlab using the ODE command that the controller from Question 7 achieves the regulation goal $\theta = \theta *$ in the presence of a known constant payload $\delta = 0.1$.
*this is a bonus question.

Matlab code:
https://imperialcollegelondon.box.com/s/elyr5ej425a2cb2sp22bir3w7tlqqcb1

If the disturbance $\delta = 0.1$, the joint angles all converge to q^*, and the total error is also -0.0034 (T=5s) which is same as before (no disturbance).Which means the controller can achieve the regulation goal when the payload $\delta = 0.1$.

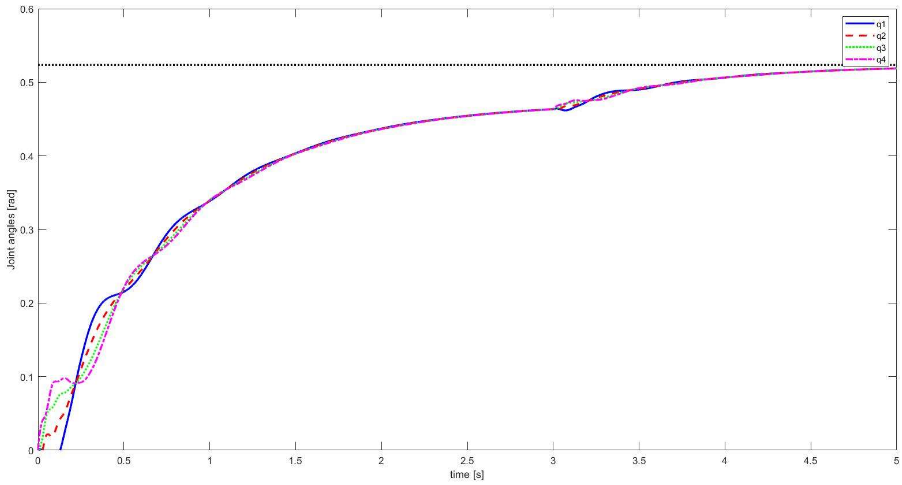


When the payload compensator removed



As shown in diagram, If the disturbance $\delta = 0.1$, there exists an offset between the final joint angles and the reference value, and the total error is -0.2031 (T=5s). So, the controller cannot achieve the regulation goal when the payload $\delta = 0.1$ without payload compensator.

Also consider if we add compensator again after 3s, the result shows below



Which is clear that the joint angles all converge to q^*again (after 3s), total error is -0.0188 at T=5s, which has a larger absolute value than the scenario which with compensator at initial (-0.0034). However, the controller can achieve the regulation goal when the payload $\delta = 0.1$ after adding a payoff compensator.

Appendix:
Matlab code:
https://imperialcollegelondon.box.com/s/elyr5ej425a2cb2sp22bir3w7tlqqcb1