

# Constrained optimal reduced-order models from input/output data

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**Abstract**—Model reduction by moment matching does not preserve, in a systematic way, the transient response of the system to be reduced, thus limiting the use of this model reduction technique in control problems. With the final goal of designing reduced-order models which can effectively be used (not just for analysis but also) for control purposes, we determine, using a data-driven approach, an estimate of the moments and of the transient response of an unknown system. We compute the unique, up to a change of coordinates, reduced-order model which possesses the estimated transient and, simultaneously, achieves moment matching at the prescribed interpolation points. The error between the output of the system and the output of the reduced-order model is minimized and we show that the resulting system is a *constrained optimal* (in a sense to be specified) reduced-order model. The results of the paper are illustrated by means of a simple numerical example.

## I. INTRODUCTION

The model reduction problem consists loosely speaking in finding a simplified mathematical description of a dynamical system in specific operating conditions, preserving at the same time specific properties. The problem has fundamental importance in many engineering fields since the increased computational power available nowadays is matched with the increased complexity of modern models and control laws [1]. For linear systems, the problem has been addressed with the use of Hankel operators [2]–[4], the theory of balanced realizations [5]–[8], and the interpolation theory [9]–[17]. For further detail and an extensive list of references see the monograph [18].

This paper is set in the framework introduced by [19] and the subsequent papers [20]–[24], and addresses the problem of determining *constrained optimal reduced-order models* by moment matching (with respect to a norm to be specified). In particular, the problem consists in finding a model which is constrained to possess a pre-assigned steady-state output response and approximates, optimally with respect to a certain criterion, the transient response of the system. Many researchers have studied the problem of optimal  $\mathcal{H}_2$  (or other norms) model reduction, see *e.g.* [17], [25]–[35]. However, to the best of the authors' knowledge, this is the first time that the reduced-order model to be optimized is *constrained*

to interpolate a given set of points. On the contrary, classical unconstrained<sup>1</sup> optimal  $\mathcal{H}_2$  model reduction uses also the interpolation points as parameters of the minimization, *i.e.* the steady-state is not constrained to have zero error for specific input signals but, as for the transient response, it is only required, for instance, to minimize the frequency response error along all the frequencies. The reason that justifies the interest in the constrained problem is that in many applications (*e.g.* power systems and converters [36], [37]), the system is excited by a specific class of input signals (which are connected with the interpolation points). It is then desirable that the steady-state error for this class of signals be identically equal to zero.

The rest of the paper is organized as follows. In Section II, we recall briefly the definition of moment and the model reduction techniques developed in [19] (Section II-A). Then, we formulate the optimization problem addressed in the paper (Section II-B). In Section III we present the main results of the paper: a method for the estimation of the steady-state response (Section III-A), a method for the estimation of the transient response (Section III-B) and a constrained optimal reduced-order model (Section III-C). In Section IV, we discuss how to select the initial condition of the reduced-order model (Sections IV-A) and we provide a procedural overview of the results of the paper (IV-B). Section V presents a simple academic example. Finally, Section VI contains some concluding remarks and future directions of investigation.

**Notation.** We use standard notation.  $\mathbb{R}_{\geq 0}$  denotes the set of non-negative real numbers;  $\mathbb{C}_{<0}$  denotes the set of complex numbers with negative real part;  $\mathbb{C}_0$  denotes the set of complex numbers with zero real part. The symbol  $I$  denotes the identity matrix and  $\sigma(A)$  denotes the spectrum of the matrix  $A \in \mathbb{R}^{n \times n}$ . The symbol  $\text{tr}(A)$  indicates the trace, *i.e.* the sum of the elements on the main diagonal of  $A$ . The symbol  $\otimes$  indicates the Kronecker product. The vectorization of a matrix  $A \in \mathbb{R}^{n \times m}$ , denoted by  $\text{vec}(A)$ , is the  $nm \times 1$  vector obtained by stacking the columns of the matrix  $A$  one on top of the other, namely  $\text{vec}(A) = [a_1^\top, a_2^\top, \dots, a_m^\top]^\top$ , where  $a_i \in \mathbb{R}^n$  denotes the  $i$ -th column of  $A$  and the superscript  $\top$  denotes the transpose. The symbol  $\epsilon_k$  indicates a vector with the  $k$ -th element equal to 1 and with all the other elements equal to 0. Given some finite data points set  $X = \{x_i \in \mathbb{R} : i = 1, \dots, m\}$ , the discrete  $\ell_p$  norm of a function  $f$  is defined as  $\|f\|_{\ell_p} = (\sum_{i=1}^m |f(x_i)|^p)^{\frac{1}{p}}$ , with  $1 \leq p < \infty$ , and the  $\ell_\infty$  norm is defined as  $\|f\|_{\ell_\infty} =$

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<sup>1</sup>The reduced-order model is not constrained to interpolate some prescribed moments but it still has to be a stable system.

$$\sup_{x_i \in X} |f(x_i)|.$$

## II. PRELIMINARIES

In this section, we first recall the notion of moment for linear systems as presented in [19]. Then, we state formally the problem of designing reduced-order models which approximate “optimally” the transient response of an unknown system.

### A. Model reduction by moment matching - Recalled

Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad (1)$$

with  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}$ ,  $y(t) \in \mathbb{R}$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$  and  $C \in \mathbb{R}^{1 \times n}$ . Let  $W(s) = C(sI - A)^{-1}B$  be the associated transfer function and assume that (1) is minimal, *i.e.* controllable and observable.

*Definition 1:* Let  $s_i \in \mathbb{C} \setminus \sigma(A)$ . The 0-moment of system (1) at  $s_i$  is the complex number  $\eta_0(s_i) = W(s_i)$ . The  $k$ -moment of system (1) at  $s_i$  is the complex number  $\eta_k(s_i) = \frac{(-1)^k}{k!} \left[ \frac{d^k}{ds^k} W(s) \right]_{s=s_i}$ , with  $k \geq 1$  integer.

In [19], exploiting the observation that the moments of system (1) can be characterized in terms of the solution of a Sylvester equation (see [14], [15]), it has been noted that the moments are in one-to-one relation with the well-defined steady-state output response of the interconnection between a specific signal generator and system (1).

*Theorem 1:* [19] Consider system (1) and suppose  $s_i \in \mathbb{C} \setminus \sigma(A)$ , for all  $i = 1, \dots, \eta$ . Consider any non-derogatory matrix  $S \in \mathbb{R}^{\nu \times \nu}$  with characteristic polynomial  $p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i}$ , where  $\nu = \sum_{i=1}^{\eta} k_i$ . Then there exists a one-to-one relation between the moments  $\eta_0(s_1), \dots, \eta_{k_1-1}(s_1), \dots, \eta_0(s_\eta), \dots, \eta_{k_\eta-1}(s_\eta)$  and

- the matrix  $C\Pi$ , where  $\Pi$  is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S, \quad (2)$$

for any  $L \in \mathbb{R}^{1 \times \nu}$  such that the pair  $(L, S)$  is observable;

- the steady-state response, provided  $\sigma(A) \subset \mathbb{C}_{<0}$ , of the output  $y$  of the interconnection of system (1) with the system

$$\dot{\omega} = S\omega, \quad u = L\omega, \quad (3)$$

for any  $L$  and  $\omega(0)$  such that the triple  $(L, S, \omega(0))$  is minimal.

*Remark 1:* The minimality of the triple  $(L, S, \omega(0))$  implies the observability of the pair  $(L, S)$  and the “controllability” of the pair  $(S, \omega(0))$ . This last condition, called *excitability* of the pair  $(S, \omega(0))$ , is a geometric characterization of the property that the signals generated by (3) are persistently exciting, see [38]. The excitability of the pair  $(S, \omega(0))$  can be also connected to the notion of exploration noise used in the data-driven dynamic programming problem, see [39]–[41]. However, note that while it is not clear how

to select the exploration noise in optimal control problems, the excitability condition provides a precise characterization of the frequencies that have to be present in the excitation signal.

In this framework a family of reduced-order models is characterized as follows.

*Definition 2:* [19] Consider system (1) and let  $S$  and  $L$  be such that the hypotheses of Theorem 1 are satisfied. Then the system

$$\dot{\xi} = F\xi + Gu, \quad \psi = H\xi, \quad (4)$$

is a model of system (1) at  $S$  if there exists a unique solution  $P$  of the equation

$$FP + GL = PS, \quad (5)$$

such that

$$HP = C\Pi, \quad (6)$$

where  $\Pi$  is the unique solution of (2).

Finally, as shown in [19], the family of systems

$$\dot{\xi} = (S - GL)\xi + Gu, \quad \psi = C\Pi\xi, \quad (7)$$

with  $G$  any matrix such that  $\sigma(S) \cap \sigma(S - GL) = \emptyset$ , belongs to the family (4) and contains all the models of dimension  $\nu$  interpolating the moments of system (1) at the eigenvalues of the matrix  $S$ .

### B. Problem formulation

With the long-term goal of designing data-driven reduced-order models by moment matching that can be used for control problems, in this paper we want to solve the problem of providing a systematic method to approximate the transient response of a, possibly, unknown system using moment matching techniques. To achieve this goal, we need to observe a few facts about the model reduction theory presented in the previous section.

The family of models (7) is built on three ideas: avoiding to solve equation (5), selecting  $P = I$ ; copying the dynamics of the signal generator (3), *i.e.* the relation  $\xi = \omega$  holds for the steady-state response (provided it exists) of the interconnection of system (7) with (3); and having a convenient parametrization for the family of reduced-order models for which additional constraints can be imposed. Moreover, observe that the moment matching method preserves, systematically, only the steady-state response of the system. The transient response is not retained in the matrix  $\Pi$ . Thus, it is worth investigating the possibility of embedding some type of information about the transient response of system (1) in (7) or (4). In this paper, we solve this problem, which can be formulated as follows.

*Problem 1:* Consider system (1), model (4) and the input  $u$  described by the signal generator (3) with a given matrix  $S$ . Determine the initial condition  $\xi(0)$  and the matrices  $F$ ,  $G$  and  $H$  such that system (4) is a *constrained optimal model* of system (1), *i.e.* solves the minimization problem

$$\min_{F, G, H, \xi(0)} \|y(t) - \psi(t)\|_\ell \quad (8)$$

with  $\ell$  a discrete norm to be specified, subject to the constraint that equations (5) and (6) have a unique solution  $P$  for the given matrix  $S$ .

Since the focus of the paper is on data-driven model reduction in which the data are time-domain measurements, it is natural to formulate the problem in terms of a discrete  $\ell$  norm. Note that considering time-domain norms has also the advantage that the approach does not require stability of the system. In fact, since in practice the data points collected are finite in number, the norms that we are considering are truncated  $\ell$  norms, which are well-defined also for unstable linear systems.

*Remark 2:* The optimality conditions of the unconstrained  $\mathcal{H}_2$  problem, *i.e.* the problem in which also the interpolation points are parameters of the minimization, are known since the '60s and are given in [25]. In [17] the IRKA algorithm is proposed: the output of the algorithm converges to an optimal (not necessarily stable) model. In [35] a variation of the IRKA algorithm that guarantees stability is given.

*Remark 3:* The constrained  $\ell$  optimal model reduction problem offers the possibility of obtaining the best approximation in terms of a specific  $\ell$  norm and in addition to preserve, with zero error, a prescribed steady-state response. The choice of solving the constrained or unconstrained problem depends, as always in engineering, on the specific application. If the interest of the designer is to have the best approximation along all the frequencies, then the unconstrained problem should be used. On the other hand if the designer knows that the system is driven by a specific class of input signals (as it is desirable when moment matching is preferred with respect to other reduction methods), for instance like in the case of the reduction of power systems [36], [42], then the constrained problem should be solved.

*Remark 4:* Since the aim of the paper is to devise a reduced-order model from input/output data, it is expected that the input of the unknown system be periodic. Since a necessary optimality condition for the unconstrained problem is that  $\sigma(F) = \sigma(-S)$ , the optimal model of the resulting constrained  $\ell$  problem cannot be an optimal solution of the unconstrained  $\mathcal{H}_2$  problem.

### III. A REDUCED-ORDER MODEL PRESERVING THE TRANSIENT RESPONSE

This section contains the main result of the paper: the construction of constrained  $\ell$  optimal reduced-order models. We begin noting that that if  $\Pi$  is the unique solution of the Sylvester equation (2), which is the case if  $A$  and  $S$  do not have common eigenvalues, then the output of the interconnection of system (1) with (3) can be written, see [19], as

$$y(t) = C\Pi\omega(t) + Ce^{At}(x(0) - \Pi\omega(0)). \quad (9)$$

If the system is asymptotically stable then  $y_{ss}(t) = C\Pi\omega(t)$  corresponds to the steady-state response, whereas  $y_{tr}(t) = Ce^{At}(x(0) - \Pi\omega(0))$  corresponds to the transient response. If the system is not asymptotically stable, we cannot define the steady-state and the transient response, however, the

output response still satisfies (9). Thus, if the system is asymptotically stable, we can estimate the moments  $C\Pi$  from input-output data as described in Section III-A. If the system is not asymptotically stable, we assume that the matrices  $A$ ,  $B$  and  $C$  are known and, thus, that we have computed the moments  $C\Pi$  solving the Sylvester equation (2).

*Remark 5:* The assumption that the system matrices are known in the unstable case is made for ease of exposition and it can be relaxed. For instance, for particular classes of uncertain linear systems it is possible to determine a stabilizing control law [43], [44]. Then, after applying this stabilizing feedback, we can use the results of the paper for the asymptotically stable case. Note that to obtain a reduced order model of the original open-loop system we can exploit the open-loop model reduction technique given in [22].

#### A. Estimation of the steady-state response for stable systems

In [45] an algorithm to determine the moments of a linear stable system described by equation (1) without assuming any knowledge on the matrices  $A$ ,  $B$  and  $C$  has been given. The algorithm is based on the following result that we report to make the paper self-contained.

*Theorem 2:* [45] Assume that  $\sigma(S) \subset \mathbb{C}_0$ ,  $\sigma(A) \subset \mathbb{C}_{<0}$ , and that the triples  $(C, A, B)$  and  $(L, S, \omega(0))$  are minimal. Then there exists a selection of time instants  $0 = t_0 < t_1 < \dots < t_{k-w} < \dots < t_k < \dots < t_q$ , with  $w > 0$  and  $q \geq w$  such that defining the time-snapshots  $\tilde{R}_k \in \mathbb{R}^{w \times \nu}$  and  $\tilde{Y}_k \in \mathbb{R}^w$  as

$$\tilde{R}_k = [\omega(t_{k-w+1}) \quad \dots \quad \omega(t_{k-1}) \quad \omega(t_k)]^\top$$

and

$$\tilde{Y}_k = [y(t_{k-w+1}) \quad \dots \quad y(t_{k-1}) \quad y(t_k)]^\top,$$

the matrix  $\tilde{R}_k$  is full-rank and the estimate

$$\text{vec}(\widetilde{C\Pi}_k) = (\tilde{R}_k^\top \tilde{R}_k)^{-1} \tilde{R}_k^\top \tilde{Y}_k, \quad (10)$$

is an approximation of  $C\Pi$ , *i.e.* the sequence  $\{t_k\}$  is such that

$$\lim_{k \rightarrow \infty} \widetilde{C\Pi}_k = C\Pi.$$

Given a small strictly positive number  $\eta$ , which has the role of threshold, an algorithm has been devised in [45] which gives an approximation of  $C\Pi$  with an arbitrary small error, in the sense of

$$\left( \widetilde{C\Pi}_k - \widetilde{C\Pi}_{k-1} \right) \left( \widetilde{C\Pi}_k - \widetilde{C\Pi}_{k-1} \right)^\top \leq \frac{\eta}{t_k - t_{k-1}}, \quad (11)$$

iterating (10) for subsequent times  $t_i$  in the sequence  $\{t_k\}$ .

*Definition 3:* Let  $\widetilde{C\Pi}_\eta = \widetilde{C\Pi}_k$  for a  $k$  such that (11) holds. The elements of  $\widetilde{C\Pi}_\eta$  are called *estimated moments* of system (1) at  $S$ .

### B. Estimation of the “transient response”

This section is written as if system (1) were asymptotically stable. However, if the system is not asymptotically stable we assume we know  $A$ ,  $B$ ,  $C$ , we have determined  $\Pi$  as the unique solution of (2) and we set  $\widetilde{C\Pi}_\eta = C\Pi$ . In fact, the results in this section hold also in the unstable case (with minor changes of terminology since we cannot use the expression “steady-state” and “transient” responses). Consider the time series  $\{y(t_i)\}$  and  $\{\omega(t_i)\}$  that have been used to obtain the estimated moments and assume that  $x(0) \neq \Pi\omega(0)$ . From this time series, we can construct another series  $\{\tilde{y}_{tr}(t_i)\}$  given by

$$\tilde{y}_{tr}(t_i) = y(t_i) - \widetilde{C\Pi}_\eta \omega(t_i), \quad (12)$$

that, by comparison with equation (9), represents an approximation of the term  $Ce^{At}(x(0) - \Pi\omega(0))$ , which is the transient output response of system (1). Note that the assumption that  $x(0) \neq \Pi\omega(0)$  implies that the time series  $\{\tilde{y}_{tr}(t_i)\}$  is not identically equal to zero. If the assumption  $x(0) \neq \Pi\omega(0)$  does not hold, then a different  $\omega(0)$  can be selected to generate new input/output data which satisfy it. Let  $\chi(t) = x(t) - \Pi\omega(t)$  and consider the system

$$\dot{\chi} = A\chi \quad z_f = C\chi. \quad (13)$$

The transient output response  $y_{tr}(t)$  of system (1) corresponds to the free output response  $z_f(t)$  of system (13). Thus, the estimated time series  $\{\tilde{y}_{tr}(t_i)\}$  is equal to the estimated time series  $\{\tilde{z}_f(t_i)\}$ . We now briefly recall the *prediction-error identification* problem as presented in [46]. Consider the time series  $\{\tilde{z}_f(t_i)\}$  (note that the input is identically zero since this is a free response) and the problem of determining the parameters

$$\tilde{\theta} = [\tilde{F}, \tilde{H}],$$

where  $\tilde{F} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$  and  $\tilde{H} \in \mathbb{R}^{1 \times \tilde{n}}$ , with  $\tilde{n} \leq n$ , are approximations of the matrices  $A$  and  $C$ , respectively, of system (13). Define the prediction error as  $e(t, \theta) = \tilde{z}_f(t) - \hat{z}_f(t, \tilde{\theta})$ , where  $\hat{z}_f(t, \tilde{\theta})$  is the predicted  $\tilde{z}_f(t)$  using the estimated parameters  $\tilde{\theta}$ . The prediction-error identification problem consists in determining

$$\tilde{\theta} = \arg \min_{\theta} \frac{1}{N+1} J_l^N(\theta), \quad (14)$$

with

$$J_l^N(\theta) = \sum_{t=t_0=0}^{t_N} l(\tilde{z}_f(t) - \hat{z}_f(t, \theta)), \quad (15)$$

where  $l : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  and  $t_N > 0$ . Many different algorithms to solve this problem have been presented (see [46] for an overview) and some of them are readily available in several programming languages. For instance the command “*ssest*” of MATLAB<sup>TM</sup> solves the problem (14) for a squared or linear error norm, *i.e.*

$$J_{\ell_2}^N(\theta) = \frac{1}{2} \sum_{t=t_0=0}^{t_N} |e(t, \theta)|^2 = \frac{1}{2} \|e\|_{\ell_2}^2, \quad (16)$$

and

$$J_{\ell_1}^N(\theta) = \sum_{t=t_0=0}^{t_N} |e(t, \theta)| = \|e\|_{\ell_1}. \quad (17)$$

*Remark 6:* The command *ssest* computes the Hankel singular values from  $\{\tilde{z}_f(t_i)\}$  and provides an estimation of the order  $\tilde{n}$  of the identified system. If it is possible to generate new data, this information can be used to change the order of the reduced-order model and to redesign the signal generator to match the order  $\tilde{n}$ . If this is not possible, the information on the singular values is ignored and the order  $\tilde{n} = \nu$  is selected.

### C. Constrained optimal reduced-order model

Now that estimates of the steady-state response and of the transient response are available we present a result which allows to obtain a reduced-order model by moment matching that possesses these characteristics.

*Proposition 1:* Suppose the pair  $(\tilde{F}, \tilde{H})$  is observable,  $\sigma(\tilde{F}) \subset \mathbb{C}_{<0}$  and  $\sigma(\tilde{F}) \cap \sigma(S) = \emptyset$ . Then there exist unique matrices  $\tilde{G}$  and  $\tilde{P}$  solving the equations

$$\tilde{F}\tilde{P} - \tilde{P}S = -\tilde{G}L, \quad \tilde{H}\tilde{P} = \widetilde{C\Pi}_\eta. \quad (18)$$

In addition, the system described by the equations

$$\dot{\xi} = \tilde{F}\xi + \tilde{G}u, \quad \psi = \tilde{H}\xi, \quad (19)$$

is a model of system (1) at  $S$ .

*Proof:* Using the vectorization operator and the properties of the Kronecker product, equations (18) can be rewritten as

$$\begin{bmatrix} I \otimes \tilde{F} - S^\top \otimes I & L^\top \otimes I \\ I \otimes \tilde{H} & 0 \end{bmatrix} \begin{bmatrix} \text{vec}(\tilde{P}) \\ \text{vec}(\tilde{G}) \end{bmatrix} = \begin{bmatrix} 0 \\ \text{vec}(\widetilde{C\Pi}_\eta) \end{bmatrix}.$$

By observability of the pairs  $(S, L)$  and  $(\tilde{F}, \tilde{H})$  and the fact that  $\sigma(\tilde{F}) \cap \sigma(S) = \emptyset$ , the solution

$$\begin{bmatrix} \text{vec}(\tilde{P}) \\ \text{vec}(\tilde{G}) \end{bmatrix} = \begin{bmatrix} I \otimes \tilde{F} - S^\top \otimes I & L^\top \otimes I \\ I \otimes \tilde{H} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \text{vec}(\widetilde{C\Pi}_\eta) \end{bmatrix} \quad (20)$$

is unique. The claim that the resulting system is a model of (1) at  $S$  follows from the fact that this system belongs to the family of models (4) given in Definition 2 (if we allow the relaxation  $C\Pi = \widetilde{C\Pi}_\eta$ ). ■

*Remark 7:* The proof of Proposition 1 gives a constructive method to determine the matrices  $\tilde{G}$  and  $\tilde{P}$ .

*Remark 8:* Model (19) belongs to the family of models (7) obtained fixing  $P$  equal to  $I$ . In fact, the family of models (7) contains all the models of order  $\nu$  and, thus, there exists a change of coordinates for which model (19) achieves moment matching with  $P = I$ .

*Remark 9:* There are no free parameters in the model (19). Thus, additional properties, such as prescribing eigenvalues or zeros, cannot be imposed.

*Corollary 1:* System (19), with  $\tilde{G}$  selected as in (20), is a constrained  $\ell$  optimal reduced-order model which interpolates the moments of system (1) at  $S$ .

*Proof:* The state space model (19) has  $\nu^2 + 2\nu$  parameters (the matrices  $\tilde{F}$ ,  $\tilde{G}$  and  $\tilde{H}$ ).  $\nu$  parameters (the matrix  $\tilde{G}$ ) are used to satisfy the  $\nu$  constraints which guarantee that the model matches the moments of system (1) at the prescribed interpolation points. All the remaining parameters, namely the matrices  $\tilde{F}$  and  $\tilde{H}$ , are obtained with a prediction-error identification algorithm that, as shown in [46], solves the optimization problem (14). ■

#### IV. ERROR BOUNDS AND SELECTION OF $\xi(0)$

Now that we have obtained an optimal reduced-order model we also would like to determine an “optimal” initial condition  $\xi(0)$  of the reduced-order model. First of all note that, even though we have ignored the initial condition for ease of exposition, the selection of the initial condition is usually included in the minimization of (14). In fact, the function `ssest` of MATLAB<sup>TM</sup> returns the optimal initial condition. However, once the optimal model is obtained we would like to be able to determine the optimal initial condition given a new series of data without the need of repeating the identification procedure for  $\tilde{F}$  and  $\tilde{H}$ .

##### A. The input of the system is $u = L\omega$

In this case, considering the approximation introduced by using the estimated moments  $C\tilde{\Pi}_\eta$  with the  $\ell_2$  optimal reduced-order model (19), it is easy to show that

$$\begin{aligned} \|y(t) - \psi(t)\|_{\ell_2} &\leq \tilde{\varepsilon}_\eta + \\ &+ \left\| C e^{At}(x(0) - \Pi\omega(0)) - \tilde{H} e^{\tilde{F}t}(\xi(0) - \tilde{P}\omega(0)) \right\|_{\ell_2}, \end{aligned} \quad (21)$$

where

$$\tilde{\varepsilon}_\eta = \left( C\Pi - \widetilde{C\Pi}_\eta \right) \left( C\Pi - \widetilde{C\Pi}_\eta \right)^\top \|\omega(t)\|_{\ell_\infty}.$$

*Remark 10:* Note that  $\tilde{\varepsilon}_\eta$  can be made arbitrarily small decreasing the threshold  $\eta$  in (11).

Thus, we can minimize the error between the two output responses selecting  $\xi(0)$  as the minimizer  $\tilde{\xi}$  of the function

$$V_{L\omega}(\xi(0)) = \left\| C e^{At}(x(0) - \Pi\omega(0)) - \tilde{H} e^{\tilde{F}t}(\xi(0) - \tilde{P}\omega(0)) \right\|_{\ell_2}, \quad (22)$$

i.e.

$$\tilde{\xi} = \arg \min_{\xi(0)} V_{L\omega}(\xi(0)). \quad (23)$$

*Theorem 3:* Define the time-snapshots  $\tilde{Y}_p \in \mathbb{R}^{p \times 1}$  and  $\tilde{\Sigma}_p \in \mathbb{R}^{p \times \nu}$  as

$$\tilde{Y}_p = \begin{bmatrix} y(0) - \widetilde{C\Pi}_\eta \omega(0) & \cdots & y(t_p) - \widetilde{C\Pi}_\eta \omega(t_p) \end{bmatrix}^\top,$$

and

$$\Sigma_p = \begin{bmatrix} \tilde{H} & \tilde{H} e^{\tilde{F}t_1} & \cdots & \tilde{H} e^{\tilde{F}t_p} \end{bmatrix}^\top.$$

If the matrix  $\Sigma_p$  is full-rank then

$$\tilde{\xi}_p = \tilde{z}_f(0) + \tilde{P}\omega(0), \quad (24)$$

with

$$\tilde{z}_f(0) = (\Sigma_p^\top \Sigma_p)^{-1} \Sigma_p^\top \tilde{Y}_p, \quad (25)$$

is an approximation of the optimal initial condition  $\tilde{\xi}$ , i.e.

$$\tilde{\xi} = \lim_{p \rightarrow \infty} \tilde{\xi}_p. \quad (26)$$

*Proof:* Note that  $V_{L\omega}(\tilde{\xi}) = J_{\ell_2}^N(\tilde{\theta})$ . Thus, recalling that  $\tilde{z}_f(t) = y(t) - \widetilde{C\Pi}_\eta \omega(t)$ , to compute  $\tilde{\xi}_p$  it is sufficient to compute  $\tilde{z}_f(0)$  from

$$\tilde{z}_f(t) = \tilde{H} e^{\tilde{F}t} \tilde{z}_f(0),$$

which yields (25). Equation (24) follows by comparison with equation (9) written for the reduced-order model. Note that assuming that the pair  $(\tilde{F}, \tilde{H})$  is observable (see Proposition 1) implies that the matrix  $\Sigma_p$  is full rank for  $p = \nu$ . The resulting  $\tilde{\xi}_\nu$  is such that  $y(0) - \psi(0) = 0$ . However, this does not minimize (22). To this end we should select  $p$  as large as possible to obtain the least-squares estimation of  $\tilde{\xi}$ , yielding (26). ■

*Corollary 2:* Consider system (1), model (19), the signal generator (3) and the optimal initial condition  $\tilde{\xi}$  computed as the solution of (23). Then

$$\|y(t) - \psi(t)\|_{\ell_2} \leq J_{\ell_2}^N(\tilde{\theta}) + \tilde{\varepsilon}_\eta. \quad (27)$$

*Proof:* The claim follows directly from equation (22) and the inequality (21). ■

*Remark 11:* If the moments  $C\Pi$  are known, then  $\tilde{\varepsilon}_\eta = 0$ . Hence, the norm of the error between the output responses of the two systems is, as expected, the optimal cost of the prediction-error problem.

Hence, we have constructively solved Problem 1, i.e. we have computed the unique, up to a change of coordinates, reduced-order model which possesses the estimated transient and, simultaneously, achieves moment matching at the prescribed interpolation points. Moreover, we have determined the optimal initial condition to minimize the error between the output of the system and the output of the reduced-order model.

##### B. Procedural overview

We can now summarize the results of the paper in a procedure to determine reduced-order models achieving moment matching at prescribed points that are also optimal with respect to a selected norm. The method can be applied to input/output measurements or to data generated from simulations. The procedure is summarized in the following steps.

- 1: Compute the estimated moments  $\widetilde{C\Pi}_\eta$  from equation (10).
- 2: Compute the time series  $\{\tilde{y}_{tr}(t_i)\}$  from equation (12).
- 3: Compute the minimizer of (14) using a prediction-error algorithm.
- 4: Compute  $\tilde{G}$  from equation (20).
- 5: **If**  $u = L\omega$  **then** compute the optimal initial condition  $\tilde{\xi}$  from equation (24).

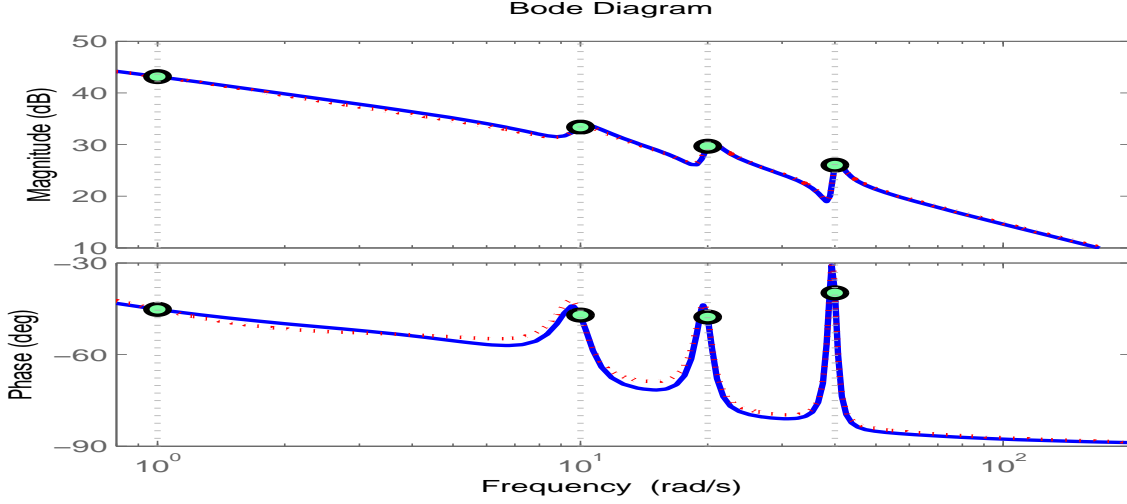


Fig. 1. Bode plot of system (1) (solid/blue line) and of the reduced-order model, with  $\eta = 5.1181 \cdot 10^{-9}$  and *ssest* (dotted/red line). The circles indicate the interpolation points.

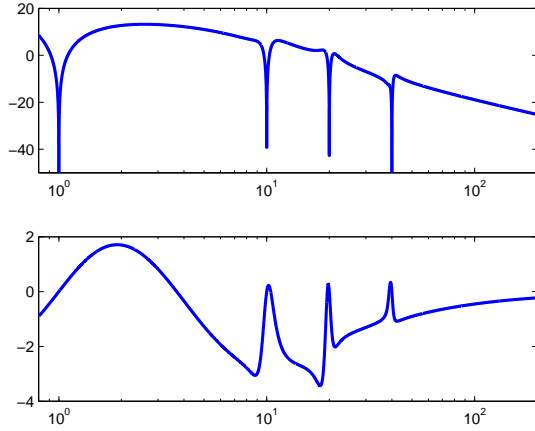


Fig. 2. Absolute errors between the Bode plot of system (1) and the Bode plot of the reduced-order model. The top graph is in decibel whereas the bottom graph is in degrees.

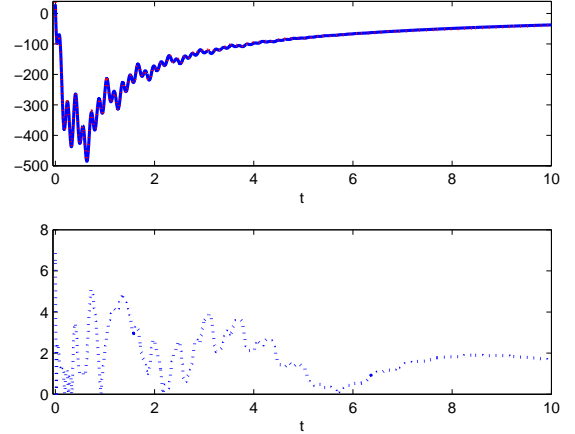


Fig. 3. Case  $u = L\omega$ . Time histories of the transient output response of system (1) (solid/blue line) and of the transient output response of the reduced-order model (dotted/red line). The bottom graph shows the corresponding absolute error.

## V. SIMULATIONS

In this section we illustrate the results of the paper by means of an academic example.

We reduce a system constructed along the line of the one given in [18], [48]. The system is single-input, single-output and of order  $n = 86$  and presents a frequency response with three peaks. The state-space matrices of system (1) are given by  $A = \text{diag}(A_1, A_2, A_3, A_4)$ , with

$$A_1 = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 20 \\ -20 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 40 \\ -40 & -1 \end{bmatrix}, \quad A_4 = \text{diag}(-0.1, -0.2, \dots, -8),$$

and  $B^\top = C = [-4.2 \quad -4.1 \quad \dots \quad 4.2 \quad 4.3]$ .

The matrices of the signal generator (3) have been selected as

$S = \text{diag}(0, S_1, S_2, S_3, S_4)$ , with  $S_2 = A_1 + I$ ,  $S_1 = 0.1 S_2$ ,  $S_3 = 2 S_2$  and  $S_4 = 4 S_2$  to interpolate the moments at 0 and at the three frequency peaks. Exploiting Theorem 2 we compute the estimated moments with  $\eta = 5.1181 \cdot 10^{-9}$  and, using the MATLAB function *ssest* to implement the prediction-error algorithm, we determine a constrained optimal reduced-order model as defined in (19). Fig. 1 shows the Bode plot of system (1) in solid/blue line and of the optimal reduced-order model (19) in dotted/red line. The green circles indicate the interpolation points. We note that the approximation appears uniformly “good” along all the frequencies (although the model is not optimal with respect the  $\mathcal{H}_2$  norm). Fig. 2 shows the corresponding absolute error of the Bode plot. Note the small approximation error, especially in the phase graph.

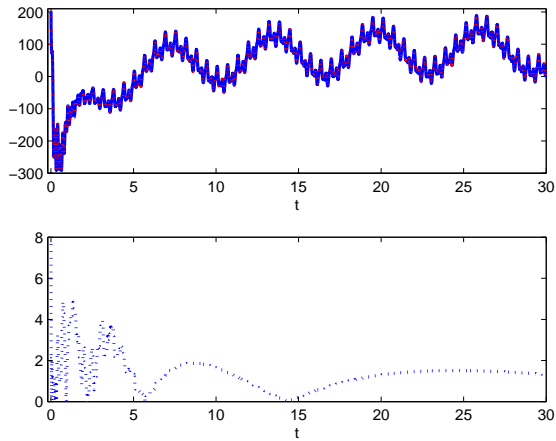


Fig. 4. Case  $u = L\omega$ . Time histories of the output response of system (1) (solid/blue line) and of the output response of the reduced-order model (dotted/red line). The bottom graph shows the corresponding absolute error.

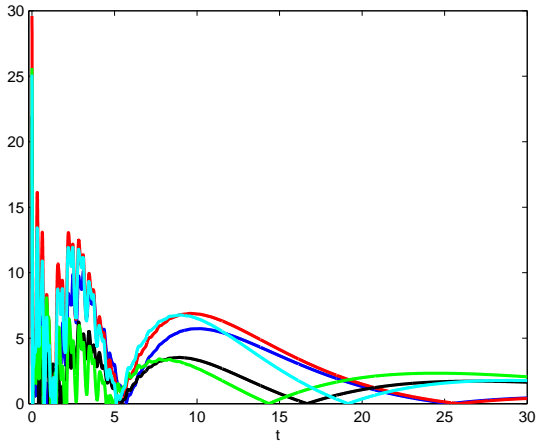


Fig. 5. Case  $u = L\omega$ . Time histories of absolute error between the output response of system (1) and of the output response of the reduced-order model for five different randomly generated  $x(0)$  and  $\omega(0)$ .

From equation (24) we compute the optimal initial condition  $\tilde{\xi}$  of the reduced-order model. As expected the obtained initial condition is equal to the one returned by the function *ssest*. Fig. 3, top graph, shows the transient output response of system (1) in solid/blue line and the transient output response of the reduced-order model in dotted/red line. The bottom graph shows the corresponding absolute error. Note that in the first 3 seconds the relative error is less than 3%. Fig. 4, top graph, shows the output response of system (1) in solid/blue line and the output response of the reduced-order model in dotted/red line. The bottom graph shows the corresponding absolute error. The error is almost identical to the one in Fig. 3 (the almost unnoticeable discrepancy is due to  $\tilde{\epsilon}_\eta$ ). To confirm that the matrices  $\tilde{A}$  and  $\tilde{C}$  are a good approximation of the system irrespectively of the initial conditions used to generate the input/output data we randomly generate the initial conditions  $x(0)$  and  $\omega(0)$

estimating  $\tilde{\xi}$  with equation (24) using the same matrices  $\tilde{F}$  and  $\tilde{H}$ . Fig. 5 shows the absolute errors of the output responses for five simulations. We note a similar pattern and we infer that the reduced-order model obtained is largely independent from the initial conditions of the input/output data used for the estimation.

## VI. CONCLUSION

In this paper we have studied the constrained optimal model reduction problem by moment matching. Using a data-driven approach we have determined an estimate of the moments and of the transient response of an unknown system. We have computed the unique reduced-order model which possesses the estimated transient and, simultaneously, achieves moment matching at the prescribed interpolation points. The resulting model is the optimal one in the family of reduced-order models by moment matching that possess the prescribed steady-state response. The discrepancy between the output of the system and the output of the reduced-order model has been characterized and a method to compute the optimal initial condition of the reduced-order model has been given. Finally, the results of the paper have been illustrated by means of simulations. Further research directions include the extension of these results to nonlinear systems, an analysis for the  $\mathcal{H}_2$  and special  $\ell_p$  error norms, the determination of the optimal initial state  $\xi$  when  $u \neq L\omega$  and the development of a theory of model reduction in which the models can be effectively used for control purposes.

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