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Nonlinear Model Reduction by Moment Matching

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Abstract

Mathematical models are at the core of modern science and technology. An accurate description of behaviors, systems and processes often requires the use of complex models which are difficult to analyze and control. To facilitate analysis of and design for complex systems, model reduction theory and tools allow determining "simpler" models which preserve some of the features of the underlying complex description. A large variety of techniques, which can be distinguished depending on the features which are preserved in the reduction process, has been proposed to achieve this goal. One such a method is the moment matching approach.

This monograph focuses on the problem of model reduction by moment matching for nonlinear systems. The central idea of the method is the preservation, for a prescribed class of inputs and under some technical assumptions, of the steady-state output response of the system to be reduced. We present the moment matching approach from this vantage point, covering the problems of model reduction for nonlinear systems, nonlinear time-delay systems, data-driven model reduction for nonlinear systems and model reduction for "discontinuous" input signals. Throughout the monograph linear systems, with their simple structure and strong properties, are used as a paradigm to facilitate understanding of the theory and provide foundation of the terminology and notation. The text is enriched by several numerical examples, physically motivated examples and with connections to well-established notions and tools, such as the phasor transform.

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Introduction

The availability of mathematical models is essential for the analysis, control and design of modern technological devices. As the computational power has advanced, the complexity of these mathematical descriptions has increased. This has maintained the computational needs at the top or above the available possibilities. A solution to this problem is represented by the use of reduced order models, which are exploited in the prediction, analysis and control of a wide class of behaviors. For instance, reduced order models are used to simulate weather forecast models and design very large scale integrated circuits and networked dynamical systems. The model reduction problem can be informally formulated as the problem of finding a simplified description of a dynamical system in specific operating conditions, preserving at the same time specific properties, *e.g.* stability. For linear systems, the problem has been addressed from several perspectives which can be divided into two main groups: singular value decomposition (SVD) approximation methods and Krylov approximation methods. The theory of balanced realizations, the use of Hankel operators and of proper orthogonal decomposition (POD) belong to the first group, whereas the use of interpolation theory belongs to the latter.

The additional difficulties of the reduction of nonlinear systems carry the need to develop different or "enhanced" techniques. Several methods which extend balancing and proper orthogonal decomposition to nonlinear systems have been proposed. Reduction of special classes of nonlinear systems and local reduction (for instance around a limit cycle) represent another approach. Although many results and efforts have been made, at present there is no complete theory of model reduction for nonlinear systems or, at least, not as complete as the theory developed for linear systems.

In this chapter we briefly recall the main model reduction methods which have been presented in the literature. We then establish the objective of this monograph and summarize its contribution and content. The chapter continues with a section in which the notation used throughout the monograph is gathered and is concluded with some bibliographical remarks on the methods described in this introduction.

1.1 Main Methods of Model Reduction for Linear Systems

Since the order of a dynamical system is usually defined as the number of states that the system has, model reduction methods require the elimination of some state variables. If we want that the reduced order model preserves some sort of "likeness" to the system to be reduced, then the elimination of the states cannot be arbitrary. To render precise this problem formulation two questions need to be answered.

Q1. What are the characteristics and properties that the reduction method aims to preserve?

Q2. What is lost in the reduction process and how can we quantify/alleviate this loss?

Depending on how these two questions are answered we obtain a multitude of different reduction techniques. It is important to stress from the onset that there is no "perfect" or "best" reduction method. In fact, the problem of model reduction epitomizes typical engineering problems in which there exists a trade off between the accuracy or performance achieved and the cost required to achieve it. In the following we briefly recall the main ideas behind the most common model reduction methods.

1.1.1 Singular value decomposition methods

Balancing and balanced approximations

With the objective of economizing on the order of the system, we wonder which states should be eliminated. It seems reasonable that unobservable and uncontrollable states should be the first candidates in the elimination process since they do not contribute to the input-output behavior of the system. This implies that if our objective is to economize on the order of the systems, these modes should be eliminated by a sensible method. The information on the degree of controllability and observability of a state is given by the controllability Gramian and observability Gramian, respectively. In particular, a difficult to control state is a state which requires high control energy to be steered to zero. However, a problem arises when we have mixed situations, such as a state which may be difficult to control but easy to observe. To be able to rank all the states with respect to a common criterion, it is fundamental to introduce the concept of balancing. From a mathematical viewpoint, balancing methods consist in the simultaneous diagonalization, by means of a singular value decomposition, of the reachability and observability Gramians. In this way we can identify the states that are simultaneously the least controllable and least observable. Then the reduction simply consists in eliminating these states. Moreover, balanced truncation methods preserve stability and naturally provide an upper bound on the approximation error in terms of the $\mathcal{H}_\infty$-norm. This quantifies what is lost in the reduction process. Finally, note that since the Gramians are related to the solutions of Riccati equations, variations of the balanced truncation method can be obtained using variations of the Riccati equations. Among these variations we mention stochastic balancing, bounded real balancing and positive real balancing. All these methods share the same answer to question Q1, namely the characteristics on which we base the reduction are the observability and the controllability Gramians, however, they differ in the answer to question Q2, namely in the properties and the type of approximation error of the reduced order model.

Hankel-norm approximation

While balanced truncation provides a bound on the approximation error, the reduced order model obtained is not optimal with respect to any given norm. An alternative method, still based on a singular value decomposition, is represented by the optimal approximation in the Hankel-norm. With this method, an optimal reduced order model is sought with respect to a 2-induced norm of the Hankel operator of the system. Similarly to balancing, the Hankel-norm approximation yields a stable reduced order model and an upper bound on the $\mathcal{H}_\infty$ norm which depends on the neglected Hankel singular values. However, the main characteristic of the method is that the model obtained is optimal with respect to an optimality criterion, *i.e.* with respect to the Hankel-norm. Note that the optimal model in the Hankel-norm is not optimal in the $\mathcal{H}_\infty$ norm. As a consequence, with respect to this last norm, balancing may offers a better approximation.

Proper orthogonal decomposition

Proper orthogonal decomposition is a method which is widely applied in practice since it does not necessarily require a high order model to begin with. In the proper orthogonal decomposition method a cloud of state measurements is obtained at several instants of time. These measurements are collected in data matrices, known as time-snapshots, which are then decomposed along orthonormal directions in a linear fashion. A reduced order model is obtained truncating the number of orthonormal directions used with respect to some optimality criterion (often a 2-induced norm). Proper orthogonal decomposition is strictly linked to other singular value decomposition methods, such as balancing, however, POD has the important advantage with respect to other SVD methods of operating on data clouds instead of on the matrices of the systems. As a consequence the method can be attempted also on systems which are not described by linear differential equations.

1.1.2 Model reduction using Krylov methods

Model reduction using Krylov methods, also known as moment matching methods, or interpolation methods, belongs to a different category of reduction techniques with respect to the SVD methods. The interpolation theory relies on the notion of moment. Note that a linear differential system which is observable and controllable is fully described by its transfer function. Given a set of complex interpolation points (which have to be selected with respect to some criterion), we determine the coefficients of the Laurent series expansion of the transfer function at these interpolation points. These coefficients are called moments. The moment matching method consists in determining a lower order model which has a transfer function that, at the same interpolation points, possesses the same coefficients of the Laurent expansion (up to a certain order). In other words, in moment matching a reduced order model is such that its transfer function (and derivatives of this) takes the same values of the transfer function (and derivatives of this) of the system to be reduced at the same interpolation points. This is graphically represented in Fig. 1.1 in which the magnitude (top) and phase (bottom) of the transfer function of a reduced order model (dashed/red line) matches the respective quantities of a given system (solid/blue line) at 30 rad/s.

The advantage of moment matching over the SVD methods is that the numerical implementation is much more efficient. Since only matrix-vector multiplications are used, *i.e.* no matrix factorizations or inversions are needed, the number of operations required to compute a reduced order model of order ν given a system of order $n >> \nu$ is $\mathcal{O}(\nu n^2)$. This is to be compared with a $\mathcal{O}(n^3)$ computational complexity of balancing and Hankel-norm approximations. On the other hand, among the drawbacks of moment matching methods there are the difficulty in preserving important properties of the original system, such as stability, and the lack of a bound on the estimation error.

Note, finally, that model reduction of linear systems is an active area of research in various domains of engineering and mathematics, and many variations and improvements have been proposed for all of these methods. For instance, mixed singular value decomposition and

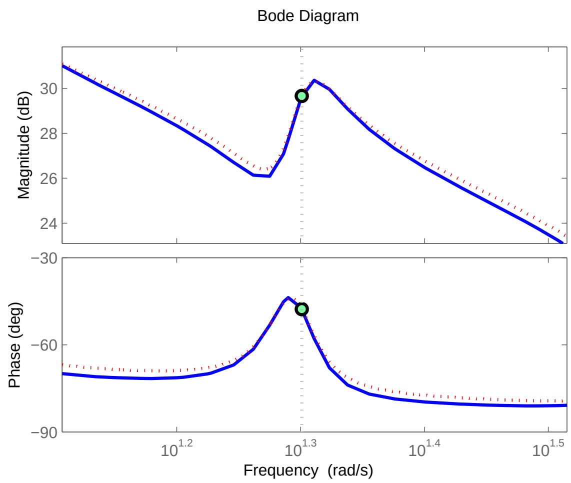

Figure 1.1: Diagrammatic illustration of the interpolation approach. Magnitude (top graph) and phase (bottom graph) plots of a given system (solid/blue line) and of a reduced order model (dashed/red line). The green circle represents the interpolation point.

Krylov methods are capable of yielding reduced order models that simultaneously maintain some of the properties of the system to be reduced and are determined in a computationally efficient manner.

1.2 Contents of the Monograph

The goal of this monograph is to present, in a uniform and complete way, moment matching techniques for nonlinear systems. The focus is on the so-called "steady-state" notion of moment. The moment is defined using the steady-state output response of the system interconnected with an interpolating signal generator. While the theory and the techniques are developed from a pure nonlinear perspective, throughout the monograph we point out several connections with the interpolation theory and the classical "interpolation-based" notion of moment. This justifies the terminology used and improves the understanding of the nonlinear theory. The chapters are enriched with examples and conclude with bibliographical notes. The monograph comprises:

Chapter 2 begins with a very general formulation of the problem that we call “model reduction by moment matching”. The problem is formulated from a “general-system perspective” and not from a linear system point of view. We then specialize the problem to nonlinear differential systems and we introduce the notions of steady-state and of moment for this class of systems, clarifying the nature of the relation between these two objects. We also relate the introduced notions with classical interpolation theory. We then present families of nonlinear reduced order models and we study the possibility of achieving specific properties, such as assigning prescribed zero dynamics. We specialize these results to linear systems, proposing families of linear reduced order models which preserve specific properties (properties which are usually difficult to maintain in the interpolation-based approach). We conclude the chapter with a selection of additional topics regarding systems in special form.

Chapter 3 deals with the problem of model reduction for nonlinear time-delay systems. The center manifold theory for time-delay systems is used to extend the definition of moment to nonlinear time-delay systems and a family of systems achieving moment matching for nonlinear time-delay systems is given. The possibility of interpolating multiple moments increasing the number of delays but maintaining the number of equations is investigated and the problem of obtaining a reduced order model of an unstable system is discussed. Similarly to the previous chapter, the results are also specialized to linear time-delay systems. Several examples illustrate the theory.

Chapter 4 presents a theoretical framework and a collection of techniques to obtain reduced order models by moment matching from input/output data for nonlinear, possibly time-delay, systems. We begin providing algorithms for the determination of an approximation of the moment which converges asymptotically to the actual moment of the nonlinear system. The computational complexity is discussed and the results are also specialized to linear systems. Several examples illustrate the theory.

Chapter 5 investigates the limitations of the characterization of moment based on a signal generator described by differential equations. With the final aim of solving the model reduction problem for a class of input signals generated by exogenous systems which do not have an implicit (differential) form, a time-varying parametrization of the steady-state of the system is used to extend, exploiting an integral matrix equation, the definition of moment to this class of input signals. The equivalence of the new definition and the classical interpolation-based notion of moment is proved under specific conditions. Special attention is given to periodic signals due to the wide range of practical applications where these are encountered. Reduced order models matching the steady-state response to explicit signal generators are given for linear systems and several connections with the classical reduced order models are drawn.

1.3 Notation

Standard notation has been adopted, most of which is defined in this section and used throughout the remainder of the monograph. When additional notation (not included in this section) is introduced, this is defined in the relevant parts of the monograph.

The symbol $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers; $\mathbb{C}_{<0}$ denotes the set of complex numbers with strictly negative real part; $\mathbb{C}_0$ denotes the set of complex numbers with zero real part and $\mathbb{D}_{<1}$ the set of complex numbers with modulo less than one.

The symbol I denotes the identity matrix and $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The symbol $\otimes$ indicates the Kronecker product and $||A||$ indicates the induced Euclidean matrix norm. Given a list of n elements a_i, $\mathrm{diag}(a_i)$ indicates a diagonal matrix with diagonal elements equal to the a_i's. The vectorization of a matrix $A \in \mathbb{R}^{n\times m}$, denoted by $\mathrm{vec}(A)$, is the $nm \times 1$ vector obtained by stacking the columns of the matrix A one on top of the other, namely $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \dots, a_m^\top]^\top$, where $a_i \in \mathbb{R}^n$ are the columns of A and the superscript $\top$ denotes the transposition operator. The superscript $*$ indicates the conjugate transpose operator. The symbol $\mathrm{adj}(A)$ denotes the adjugate (known also as classical adjoint or adjunct) of A, namely the transpose of its cofactor matrix.

The symbol $\Re[z]$ indicates the real part of the complex number z, $\Im[z]$ denotes its imaginary part and ι denotes the imaginary unit. The symbol ϵ_k indicates a vector with the k-th element equal to 1 and with all the other elements equal to 0. Given a function f, $\overline{F}$ represents its phasor at ω, whereas $< f(t) >$ indicates its time average.

Given a set of delays $\{\tau_j\}$, the symbol $\mathfrak{R}_T^n = \mathfrak{R}_T^n([-T, 0], \mathbb{R}^n)$, with $T = \max_j\{\tau_j\}$, indicates the set of continuous functions mapping the interval $[-T, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The subscripts "τ_j" and "χ_j" denote the translation operator, *e.g.* $x_{\tau_j}(t) = x(t - \tau_j)$.

Let $\bar{s} \in \mathbb{C}$ and $A(s) \in \mathbb{C}^{n\times n}$. Then $\bar{s} \notin \sigma(A(s))$ means that $\det(\bar{s}I - A(\bar{s})) \neq 0$. $\sigma(A(s)) \subset \mathbb{C}_{<0}$ means that for all $\bar{s}$ such that $\det(\bar{s}I - A(\bar{s})) = 0$, $\bar{s} \in \mathbb{C}_{<0}$.

The symbol $\mathcal{L}(f(t))$ denotes the Laplace transform of the function f (provided that f is Laplace transformable) and $\mathcal{L}^{-1}\{F(s)\}$ denotes the inverse Laplace transform of $F(s)$ (provided it exists). With some abuse of notation, $\sigma(\mathcal{L}(f(t)))$ denotes the set of poles of $\mathcal{L}(f(t))$. The symbol $\delta_0(t)$ denotes the Dirac δ-function.

Given two functions, $f : Y \to Z$ and $g : X \to Y$, with $f \circ g : X \to Z$ we denote the composite function $(f \circ g)(x) = f(g(x))$ which maps all $x \in X$ to $f(g(x)) \in Z$. $L_f h(x)$ denotes the Lie derivative of the smooth function h along the smooth vector field f, *i.e.* $L_f h(\cdot) = \frac{\partial h}{\partial x} f(x)$. Given a function $y : \mathbb{R} \to \mathbb{R}$ the symbol $y^{(k)}$ denotes the k-th order time derivative of y (provided it exists). Given a scalar function $V : \mathbb{R}^n \to \mathbb{R} : x \mapsto V(x)$, the symbols V_x and V_{xx} denote, respectively, the gradient and the Hessian matrix of the function V, provided they exist.

1.4 Bibliographical Notes

To report all the developments and results on model reduction and to give credit to all the researchers who have contributed to the field would be a titanic effort (if at all possible) considering the enormous research activity which has contributed to this field. The following references should not be considered at all exhaustive but should be seen as a possible starting point for the interested reader.

1.4.1 Model reduction for linear systems

Several survey papers have been written on the topic of model reduction for linear systems. Here we list a few examples known to the authors. For a survey paper on balanced truncation see, *e.g.* Gugercin and Antoulas [2004]. For survey papers on model reduction based on Krylov subspaces see, *e.g.* Bai [2002] and Freund [2003]. Other survey papers on model reduction of linear systems are, for instance, Fortuna et al. [1992], Antoulas et al. [2001] and Baur et al. [2014]. For further detail and an extensive list of references on the problem of model reduction for linear systems see the monograph Antoulas [2005].

Balanced approximations and Hankel-norm approximations

Balanced truncation has been originally introduced by Moore [1981], which recognizes that the idea is closely related to the "principal axis realization" proposed by Mullis and Roberts [1976]. Almost immediately it has been shown that the method possesses the property of preserving the stability of the system, see Pernebo and Silverman [1982], and provides a computable error bound, see Enns [1984] and Glover [1984]. Modifications of the standard method to achieve preservation of passivity have been proposed in *e.g.* Phillips et al. [2003], Saraswat et al. [2005], Yan et al. [2007] and Reis and Stykel [2010]. An efficient and numerically robust implementation of balanced truncation is the square-root method, see Laub et al. [1987] and Tombs and Postlethwaite [1987], which is based on the Cholesky factorizations of the Gramians. The Schur method proposed by Safonov and Chiang [1989] enhances some of the robustness properties of the square-root method. The balancing-free square-root method proposed in Varga [1992] combines the square-root method and the Schur method. Another class of methods based on the Gramians is the family of Cross-Gramian methods given in Fernando and Nicholson [1983, 1984], Aldhaheri [1991], Antoulas et al. [2001], Sorensen and Antoulas [2002] and Baur and Benner [2008], which have properties similar to balanced truncation (preservation of stability and computable error bound). Numerical efficient implementations of the balanced truncation methods have been proposed in Rabiei and Pedram [1999], Van Dooren [2000], Benner et al. [2000, 2003], Benner [2004], Gugercin and Li [2005] and Baur and Benner [2008]. Other numerically efficient methods related to balanced truncation are the singular perturbation approximation, see Liu and Anderson [1986], Varga [1992] and Benner et al. [2000], frequency weighted balanced truncation, see Enns [1984], Gawronski and Juang [1990] and Gugercin and Antoulas [2004], fractional balanced reduction, see Meyer [1990], and balanced stochastic truncation, see Benner et al. [2001]. The numerical stability of the balanced truncation methods often relies upon the stability of the system. Generalizations of the method to unstable systems have been proposed in Zhou et al. [1999]. Extensions to time-varying systems have been given in Lall and Beck [2003], Sandberg and Rantzer [2004] and Sandberg [2006]. Several approximated versions of the balanced truncation method have been presented. Willcox and Peraire [2002] have proposed a method which they interpreted as frequency-domain POD, and that later has been called Poor Man's Truncated Balanced Reduction method in Phillips and Silveira [2005]. The dominant subspace projection method is another heuristic balanced-free method which approximates, in a certain sense, balanced truncation. see Penzl [2006] (see also Li and White [2001] for another version). Finally, some results on model reduction for linear systems based on the notion of Hankel operators are given in Adamjan et al. [1971], Glover [1984], Safonov et al. [1990], Kavranoğlu and Bettayeb [1993] and Benner et al. [2004].

Krylov methods

The origin of this approach can be traced back to the related problems of Nevanlinna-Pick interpolation and of partial realization of covariance sequences, see Georgiou [1983], Kimura [1983, 1986], Antoulas et al. [1990], Byrnes et al. [1995], Georgiou [1999] and Byrnes et al. [2001]. An early result based on Krylov methods is the asymptotic waveform evaluation method proposed in Pillage and Rohrer [1990]. This method computes the moments explicitly and, consequently, is numerically unstable and inefficient. The problem of numerical instability has been tackled in several works, starting with the Lanczos Padé method proposed in Gallivan et al. [1994] and Feldmann and Freund [1995], and the "passive reduced-order interconnect macromodeling algorithm" presented in Odabasioglu et al. [1998]. Techniques to double the number of interpolated points have been firstly proposed in Grimme [1997] and are referred to as dual rational Arnoldi and Lanczos methods. In general Krylov methods do not preserve stability and passivity. Stability of the reduced order model can be enforced using the restarting techniques given in Grimme et al. [1995] or the restarted dual Arnoldi method proposed by Jaimoukha and Kasenally [1997]. Other techniques to preserve these and other structural properties are presented in, *e.g.* Bai and Freund [2001], Freund [2004], Li and Bai [2005], Beattie and Gugercin [2008], Polyuga and Van der Schaft [2010, 2011, 2012] and Gugercin et al. [2012]. An important problem for Krylov methods is the selection of the interpolation points. Early results on this aspect are given in Chiprout and Nakhla [1995], where the complex frequency hopping, which is based on a binary search, is proposed. Another approach using a binary search is given in Achar and Nakhla [2001]. Recent results on the problem of selecting the interpolation points are given in Chu et al. [2006] and Gugercin et al. [2008]. In this last paper the iterative rational Krylov algorithm (IRKA) is prosposed, which is becoming increasingly popular. While stability is not guaranteed in all instances, the method is numerical effective and solves first-order necessary conditions of optimality with respect to the $\mathcal{H}_2$ norm. Several modifications of this method have been proposed in Gugercin et al. [2008], Van Dooren et al. [2008] and Bunse-Gerstner et al. [2010] for MIMO systems, and in Flagg et al. [2013] for the $\mathcal{H}_\infty$ case. Another adaptive algorithm, more efficient than IRKA, but less precise, has been presented in Druskin and Simoncini [2011]. An algorithm less efficient than IRKA, but that allows to select the order of the reduced order model adaptively is given in Panzer et al. [2013a]. A data-driven Krylov approach has been presented under the name of Loewner framework in Mayo and Antoulas [2007]. A drawback of the Krylov methods is the lack of an error bound. This problem is addressed in several works in which results for systems in special form are obtained, see *e.g.* Grimme [1997], Bai et al. [1999], Bechtold et al. [2004], Panzer et al. [2013b] and Konkel et al. [2014].

1.4.2 Model reduction for nonlinear systems

From the '90s, considerable research effort has been dedicated to the problem of model reduction for nonlinear systems. The problem of model reduction for special classes of systems, such as differential-algebraic systems, bilinear systems and mechanical/Hamiltonian systems has been studied in Al-Baiyat et al. [1994], Lall et al. [2003], Soberg et al. [2007] and Fujimoto [2008]. Several results rely on approximating the nonlinearity with a polynomial, see *e.g.* Chen [1999], Phillips [2000, 2003], Rewienski and White [2003] and Benner [2004], or the ability of transforming the system into a quadratic bilinear form, see *e.g.* Gu [2009, 2011], Benner and Breiten [2015] and Antoulas et al. [2016]. The first breakthrough on balancing for nonlinear systems has been made in Scherpen [1993]. This paper originated subsequent results (sometimes referred to as energy-based methods, see Scherpen and Gray [2000]) which exploit balancing, see Scherpen and Van der Schaft [1994] and Gray and Mesko [1997], or the notion of Hankel operator, see Gray and Scherpen [2001], Scherpen and Gray [2002] and Fujimoto and Scherpen [2005, 2010]. Techniques based on the reduction around a limit cycle or a manifold have been presented in Verriest and Gray [1998] and Gray and Verriest [2006]. Model reduction methods based on proper orthogonal decomposition have been developed for linear and nonlinear systems, see *e.g.* Kunisch and Volkwein [1999], Willcox and Peraire [2002], Hinze and Volkwein [2005], Grepl et al. [2007], Kunisch and Volkwein [2008] and Astrid et al. [2008]. Finally, some computational aspects have been investigated in Lall et al. [2002], Willcox and Peraire [2002], Gray and Verriest [2006] and Fujimoto and Tsubakino [2008].

Moment matching for nonlinear systems

A fundamental preliminary result for the development of model reduction by moment matching for nonlinear systems has been to recognize that the problem of determining the moments of a system corresponds to the problem of solving a particular Sylverster equation, see Gallivan et al. [2004a] and Gallivan et al. [2006]. Exploiting this property, the notion of moment has been revisited for linear systems and extended to nonlinear systems firstly in Astolfi [2007b] and then in Astolfi [2007a, 2008, 2010]. The method presented in this monograph is based on the framework formalized in Astolfi [2010] and subsequent papers. We invite the reader to consult the bibliographical notes of the following chapters for further references on nonlinear model reduction by moment matching.

2

Model Reduction by Moment Matching

The purpose of this chapter is to present the problem of model reduction by moment matching from a nonlinear perspective. We begin with a general formulation of the problem and of the notion of moment. We then specialize this notion to nonlinear systems described by differential equations. Throughout the chapter we provide links to the linear framework which are instrumental to understand and interpret the theory and to connect it to the current literature.

2.1 The Problem of Model Reduction by Moment Matching

In this section we introduce a few basic definitions and we provide a general formulation of the problem addressed in this monograph. When considering a mathematical model representing a dynamical system we often think of a set of ordinary differential equations. However, it is worth stressing at the onset that ordinary differential equations are only one possible representation that can be used to describe a dynamical system. More general mathematical representations exist. This motivates us to give the formal definition of model of a dynamical system which we use in the monograph.

Definition 2.1. Let $x(t) \in \mathbb{R}^n$ be the *state* of a dynamical system Σ. We call n the *order* of the system. Let u, with $u(t) \in \mathbb{R}^m$, be the *input* of Σ. Let t_0 and $x_0 = x(t_0)$ be the initial time and the initial state, respectively. If there exists a function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ such that

$$x(t) = \Phi(t, t_0, x_0, u_{[t_0,t)}), \tag{2.1}$$

for all $t \geq t_0$, we call equation (2.1) the *representation in explicit form*, or the *explicit model*, of Σ. Assume $\Phi(t, t_0, x_0, u_{[t_0,t)})$ has a continuous derivative with respect to t for every t_0, x_0 and u, and there exists a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, continuous for each t over $\mathbb{R}^n \times \mathbb{R}^m$, such that

$$\dot{x} = f(x, u). \tag{2.2}$$

We call the differential equation (2.2) the *representation in implicit form*, or the *implicit model*, of Σ. We call the function of time x the *motion* of the system or, equivalently, the *response* of the system to u or, for implicit models, the *solution* of (2.2). We call the time-ordered collection of states visited by the motion of the system starting from a given initial condition a *trajectory* of the system. The function Φ is called *flow* which, for implicit models, we indicate with the symbol $\Phi_t^f(x_0)$ to highlight the fact that $x(t) = \Phi_t^f(x_0)$ solves equation (2.2) with initial condition $x(0) = x_0$.

While we focus most of our attention on implicit models described by equations of the form (2.2), we also study in Chapter 3 the problem of model reduction for implicit models described by more general classes of differential equations (namely time-delay differential equations). In addition, in Chapter 5 the problem of model reduction is partially extended to systems in explicit form.

The object we are interested in for the purpose of model reduction is the so-called steady-state response of a system to a certain input. To make the concept formally precise we assume that $t_0 = 0$ and introduce a few definitions.

Definition 2.2. A set $\mathcal{X}$ is called *positively invariant* if, for all initial conditions $x_0 \in \mathcal{X}$, the motion x exists for all $t \geq 0$ and it is such that $x(t) \in \mathcal{X}$ for all $t \geq 0$.

Definition 2.3. The motion of system (2.1) is said to be *ultimately bounded* if there exists a bounded subset $\mathcal{B} \subset \mathbb{R}^n$ with the property that, for every compact subset $\mathcal{X}_0$ of a positively invariant set $\mathcal{X}$, there is a time $T > 0$ such that $x(t) \in \mathcal{B}$ for all $t \geq T$ and all $x_0 \in \mathcal{X}_0$.

Definition 2.4. Let $\mathcal{B} \subset \mathbb{R}^n$ and suppose $x(t)$ is defined for all $t \geq 0$ and all $x_0 \in \mathcal{B}$. The *omega limit set of the set* $\mathcal{B}$, denoted by $w(\mathcal{B})$, is the set of all points $x \in \mathbb{R}^n$ for which there exists a sequence of motions $\{x_k\}$ with $x_k(t) \in \mathcal{B}$ and a sequence $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that $\lim_{k\to\infty} x_k(t_k) = x$.

Definition 2.5. Suppose that the motions of system (2.1), with initial conditions in a closed and positively invariant set $\mathcal{X}$, are ultimately bounded with ultimate bounding subset $\mathcal{B}$. A *steady-state response* is any motion with initial condition $x_0 \in w(\mathcal{B})$.

Note that this notion of steady-state does not imply (purposely) uniqueness. We also provide another, more intuitive, characterization of the steady-state response in which the dependence from the input is highlighted.

Definition 2.6. Consider system (2.1). If $x^\star(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u^\star(t) \in \mathcal{U} \subset \mathbb{R}$ are such that

$$\lim_{t\to\infty} ||\Phi(t, t_0, x^\circ, u^\star_{[t_0,t)}) - \Phi(t, t_0, x^\star, u^\star_{[t_0,t)})|| = 0 \tag{2.3}$$

for every $x^\circ \in \mathcal{X}$, then any $x^{ss}(t) = \Phi(t, t_0, x^\star, u^\star_{[t_0,t)})$, bounded for all t, is called the *steady-state response* of system Σ to the input $u^\star$.

The advantage of Definitions 2.5 and 2.6 is that the steady-state response is well-defined, under a few hypotheses, for systems in explicit form, *i.e.* it is not necessary that the steady-state response be the solution of a differential equation. Based on the definition of steady-state response, we can now introduce the notion of moment.

Definition 2.7. Consider the explicit model (2.1), with an output map $h : \mathbb{R}^n \to \mathbb{R}$ and an input described by $l(\omega)$, with $\omega(t) \in W \subset \mathbb{R}^\nu$ and $l : \mathbb{R}^\nu \to \mathbb{R}$. Suppose that the steady-state response x^{ss} to the input

$l(\omega)$ can be described by a map $\pi : \mathbb{R}^\nu \times \mathbb{R} \to \mathbb{R}^n : (\omega, t) \mapsto \pi(\omega, t)$, *i.e.* $x^{ss}(t) = \pi(\omega(t), t)$. The *moment of system* Σ *at* l is defined as the mapping $h \circ \pi$.

Note that the steady-state output response and the moment of systems Σ are not the same object, although they are strictly related through the mapping $h \circ \pi$. We clarify the nature of this relation in the next section. We can now formulate an abstract version of the problem of model reduction by moment matching.

Problem 1. Given a system Σ of order n equipped with the moment $h \circ \pi$ at l, the problem of *model reduction by moment matching* consists in determining a system $\widetilde{\Sigma}$ of order $\nu < n$ equipped with the moment $\kappa \circ p$ at l such that $h(\pi(\omega, t)) = \kappa(p(\omega, t))$ for all $\omega \in W$ and for all $t \geq 0$.

While we postpone the justification of the terminology used to a later stage, it is worth dedicating now some effort to establish why a reduction technique based on the preservation of the steady-state mapping $h \circ \pi$ makes sense at all. To this end we provide an example which shows that the ideas of approximating a mathematical description exploiting the steady-state response is used in other domains of engineering and mathematics.

Example 2.1. Consider a linear electrical circuit with currents and voltages described by linear differential equations of the form

$$a_n \frac{d^n}{dt^n} x + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} x + \cdots + a_0 x = u, \tag{2.4}$$

where $x(t) \in \mathbb{R}$ represents a current or voltage, $u(t) \in \mathbb{R}$ is a current or voltage input and $a_i \in \mathbb{R}$, with $i = 0, \ldots, n$, with $n \geq 0$. Without loss of generality we assume that $a_n \neq 0$. Since the transient response of electrical circuits decays rapidly to zero, the steady-state analysis of electrical devices is of paramount importance for their design. However, a trade-off between the level of the approximation and the difficulty of the analysis has usually to be made. Among the techniques that have been developed for this analytic task, the phasor transform represents a powerful and flexible mathematical tool which has been used for the study of the steady-state behavior of circuits driven by sinusoidal sources. If the input u is a sinusoidal signal of angular frequency ω and phase φ, then the steady-state analysis of (2.4) can be carried out by means of the phasor of $x(t)$. This is usually introduced through the inverse phasor transform which is defined in its simplest form as

$$x(t) = \Re\left[\overline{X} e^{\iota(\omega t+\varphi)}\right], \tag{2.5}$$

where[1] $\overline{X} : \mathbb{C} \to \mathbb{C}$ is called the phasor of $x(t)$. The phasor transform greatly simplifies the analysis of the circuit because it changes integro-differential equations in algebraic equations, which are computationally and analytically more easily solvable.

Example 2.1 shows that in circuit analysis the idea of simplifying a mathematical description looking only at the steady-state behavior is not new. Moreover, the example also suggests that there exists a relation between moments and phasors which we formalize at the end of this chapter.

The phasor analysis is not the only example in which the idea of simplifying a mathematical representation utilizing only its steady-state description is applied. Another example is given by the geometric analysis of singular perturbation systems, in which the steady-state response of the so-called "fast subsystem" is used instead of the full dynamics.

2.2 The Notion of Moment for Nonlinear Systems

In this section we focus on the characterization of moments for systems described by nonlinear ordinary differential equations. Consider a nonlinear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = f(x,u), \qquad y = h(x), \tag{2.6}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and f and h analytic mappings, a signal generator described by the equations

$$\dot{\omega} = s(\omega), \qquad u = l(\omega), \tag{2.7}$$

[1] Phasors are frequency dependent. We omit the argument ω from the phasor $\overline{X}(\omega)$ for ease of notation.

with $\omega(t) \in \mathbb{R}^v$, and s and l analytic mappings, and the interconnected system

$$\dot{\omega} = s(\omega), \qquad \dot{x} = f(x, l(\omega)), \qquad y = h(x). \tag{2.8}$$

In addition, suppose that $f(0,0) = 0$, $s(0) = 0$, $l(0) = 0$, $h(0) = 0$, *i.e.* zero is an equilibrium point of system (2.8), and that the vector fields f and s are complete, *i.e.* all their flow curves exist for all $t \in \mathbb{R}$. From the discussion of the previous section we are interested in characterizing the steady-state response of system (2.8). The interconnection of system (2.6) with the signal generator (2.7) captures the idea that we are interested in studying the behavior of the system only for specific input signals, *i.e.* the ones generated by (2.7). We expect that the system and these input signals have to satisfy certain properties to make the problem well-posed. Borrowing from the theory of output regulation, the right tool to characterize the steady-state response of the interconnection (2.8) is the center manifold theory. In what follows we discuss the main assumptions to have a well-defined center manifold and steady-state response.

The first assumption that we need is that the zero equilibrium of system (2.6) be locally exponentially stable. This assumption is essential to have the property that the trajectories of the system converge towards an invariant manifold (dependent upon the signal generator). However, exponential stability of the zero equilibrium alone is not sufficient to characterize the steady-state. We also need some assumptions on the signal generator. Since we are interested in the steady-state response produced by specific inputs, we expect that these inputs do not decay asymptotically to zero. In fact, the steady-state, if it exists, would be identically equal to zero for any signal generator (2.7) which has an asymptotically stable zero equilibrium. In this instance, we would have no information regarding the actual signal generator we have used. On the other hand, we also expect that the input does not grows unbounded, otherwise the response would not be forward bounded (see Definition 2.6). To formalize this property of the signal generator we introduce the notions of Poisson stability and neutral stability.

Definition 2.8. Consider the signal generator (2.7). A point $\omega \in W$ is said to be *Poisson stable* if the flow $\Phi_t^s(\omega)$ of the vector field s is defined for every $t \in \mathbb{R}$, and for every neighborhood $\mathcal{N}$ of ω and every constant $T \in \mathbb{R}_{\geq 0}$ there exist constants $t_- < -T$ and $t_+ > T$ such that $\Phi^s_{t_-}(\omega^\circ) \in \mathcal{N}$ and $\Phi^s_{t_+}(\omega^\circ) \in \mathcal{N}$. The signal generator (2.7) is Poisson stable if there exists a dense set of Poisson stable initial points on W.

Definition 2.9. The signal generator (2.7) is neutrally stable if it is Poisson stable and the zero equilibrium is stable.

Neutral stability is a standard hypothesis when characterizing the steady-state of the interconnection (2.8). It can easily be seen as a generalization of the property of periodicity. In fact, under the neutral stability assumption the trajectories of the signal generator (2.7) are not periodic, but they are required to pass arbitrarily close to every already visited Poisson stable point, both forward and backward in time.

We note, however, that neutral stability is a property of the state ω. If, for instance, we were to select an output map $l(\omega) \equiv 0$, then the steady-state of the interconnection (2.8) would still be zero. Similarly, if we are interested in characterizing also the steady-state response of the *output* of system (2.6) we need to protect us from a similar issue also with respect to h. For instance, if the output map h is such that $h(x) \equiv 0$, it is obvious that the steady-state output response is identically equal to zero even though the steady-state response of x may be not zero. Note also that since we are looking for a map from ω to x or y we also need to avoid cases in which, for instance, $\frac{\partial f}{\partial u} = 0$ for all x, *i.e.* the input does not influence the state x. All these issues are avoided if we consider minimal realizations of the system and the signal generator. A minimal realization is a model which has some "observability" and "controllability" property as defined below.

Definition 2.10. Two states x_1 and x_2 are said to be indistinguishable for system (2.6) if for every input function u the output function $t \mapsto h(\Phi(t, 0, x_1, u_{[t_0,t)}))$ of the system for the initial state $x(0) = x_1$ and the output function $t \mapsto h(\Phi(t, 0, x_2, u_{[t_0,t)}))$ of the system for the initial state $x(0) = x_2$ are identical on their common domain of definition. The *system is observable* if it has the property that if the states x_1 and x_2 are indistinguishable then $x_1 = x_2$.

Definition 2.11. Let V be a neighborhood of $x(0)$ and $R^V(x(0), T)$ be the *reachable set* from $x(0)$ at time $T \geq 0$ following trajectories which remain in the neighborhood V of $x(0)$ for $t \leq T$, *i.e.* all points x for which there exists an input u such that the evolution of system (2.6) from $x(0)$ satisfies $x(t) \in V$, for $0 \leq t \leq T$, and $x(T) = x$. Let $R^V_T(x(0)) = \bigcup_{\tau \leq T} R^V(x(0), \tau)$. A *system is locally accessible from* $x(0)$ if $R^V_T(x(0))$ contains a non-empty open subset of $\mathcal{X} \subset \mathbb{R}^n$ for all non-empty neighborhoods V of $x(0)$ and all $T > 0$. If this holds for all $x(0) \in \mathcal{X}$ then the *system is locally accessible.*

Definition 2.12. System (2.6) is *minimal* if it is observable and accessible.

If system (2.6) is not minimal we can find a local change of coordinates such that the system is decomposed in four subsystems, one of which is observable and accessible. We can then apply the theory presented in this monograph to this subsystem. Note also that if we apply the results of the monograph to a non-minimal system, the reduced order model is a model of the observable and accessible subsystem. We can then assume that the system (2.6) is minimal. We highlight that this assumption is not restrictive. In fact, since our objective is to determine a reduced order model, namely to economize on the dimension of the representation of our model, it is sensible that the first candidate to elimination is that part of the system which is not observable and/or accessible, *i.e.* that part that does not contribute to the input-output behavior of the system.

Similarly, we assume that the signal generator (2.7) is minimal, *i.e.* observable, because the signal generator does not have an input. In fact, since we are using the signal generator to characterize inputs to our system that we want to study, it is natural to construct the signal generator in such a way that the inputs that we generate are observable.

The previous discussion can be summarized in the following assumption.

Assumption 2.1. System (2.6) is minimal and the zero equilibrium is locally exponentially stable. The signal generator (2.7) is observable and neutrally stable.

Exploiting this assumption, we can characterize the steady-state of system (2.8) by means of a partial differential equation.

Lemma 2.1. Consider system (2.6) and the signal generator (2.7). Suppose Assumption 2.1 holds. Then there is a mapping π, locally defined in a neighborhood of $\omega = 0$, with $\pi(0) = 0$, which solves the partial differential equation

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), l(\omega)), \tag{2.9}$$

for all $\omega \in W$. In addition, the steady-state response of system (2.8) is $x^{ss}(t) = \pi(\omega(t))$ for any $x(0)$ and $\omega(0)$ sufficiently small.

Proof. The Jacobian matrix of the interconnected system (2.8) at the zero equilibrium is

$$\begin{bmatrix} \tilde{S} & 0 \\ \tilde{B} & \tilde{A} \end{bmatrix},$$

where $\tilde{S} = \frac{\partial s}{\partial \omega}\Big|_{\omega=0}$, $\tilde{A} = \frac{\partial f}{\partial x}\Big|_{x=0}$ and $\tilde{B} = \frac{\partial f}{\partial \omega}\Big|_{\omega=0}$. By Assumption 2.1, $\sigma(\tilde{A}) \subset \mathbb{C}_{<0}$, $\sigma(\tilde{S}) \subset \mathbb{C}_0$ and the eigenvalues of S are simple. Thus, by the center manifold theory, the interconnected system has a center manifold at $(\omega, x) = (0, 0)$ described by the equation $x = \pi(\omega)$, where π is a mapping satisfying equation (2.9). In addition the center manifold is locally exponentially attractive and, for all pairs $(x^\circ, \omega^\star)$ in some neighborhood of $(0, 0)$,

$$||x(t) - \pi(\omega(t)|| \leq Ke^{-\alpha t}||x^\circ - \pi(\omega^\star)||,$$

for all $t \geq 0$ and for some $K > 0$ and $\alpha > 0$. Since the graph of $x = \pi(\omega)$ is an invariant manifold we have that $x(t, \pi(\omega^\star), u^\star_{[t_0,t)}) = \pi(\omega(t))$. As a consequence

$$\lim_{t\to\infty} ||x(t, x^\circ, u^\star_{[t_0,t)}) - x(t, \pi(\omega^\star), u^\star_{[t_0,t)})|| = 0.$$

Finally, note that given the stability assumptions of the zero equilibrium of system (2.8), for any $x(0)$ and $\omega(0)$ sufficiently small, the responses are bounded for all t. This concludes the proof, *i.e.* $x^{ss}(t) = \pi(\omega(t))$ is the steady-state response of system (2.8) according to Definition 2.6. $\square$

Remark 2.1. In general the center manifold is not unique. For instance, consider the system

$$\dot{y} = -y, \qquad \dot{z} = -z^3,$$

which has the center manifold $\pi(z) = ce^{-\frac{1}{2}z^{-2}}$ if $z \neq 0$, $\pi(z) = 0$ if $z = 0$, for every value of $c \in \mathbb{R}$. However, given a system with a non-unique center manifold, for every center manifold it is possible to construct a modified system for which this center manifold is unique (and global). Moreover, solutions which stay in a neighborhood of zero are not modified and are therefore contained in every center manifold. Thus, the modified system has the same local dynamical properties of the original system. Hence, in the following we assume that the mapping π solving equation (2.9) is unique.

Exploiting this characterization of steady-state and Definition 2.7, the notion of moment can be specialized to nonlinear systems described by equations of the form (2.6). Before providing the definition, it is useful to introduce a weaker version of Assumption 2.1.

Assumption 2.2. System (2.6) is minimal, the signal generator (2.7) is observable and equation (2.9) has a solution π.

Definition 2.13 (Steady-state notion of moment)**.** Consider system (2.6) and the signal generator (2.7). Suppose Assumption 2.2 holds. The mapping $h \circ \pi$ is the *moment of system (2.6) at* (s, l).

Note that Assumption 2.2 in Definition 2.13 is implied by Assumption 2.1 needed for the result in Lemma 2.1. In fact, by Lemma 2.1, if Assumption 2.1 holds, then equation (2.9) has a solution π and the moment is well-defined according to Definition 2.13. Moreover, the moment $h \circ \pi$ computed along a particular trajectory $\omega(t)$ coincides with the steady-state response of system (2.6) driven by (2.7). This characterization of moment is illustrated in Fig. 2.1, in which the moment is represented, with abuse of notation, as the steady-state response of the interconnected system (2.8). However, if the equilibrium $x = 0$ of the system $\dot{x} = f(x, 0)$ is unstable, it is still possible to define the moment of system (2.6) at (s, l) in terms of the mapping $h \circ \pi$ in accordance with Definition 2.13. In general it is not required that the zero equilibrium

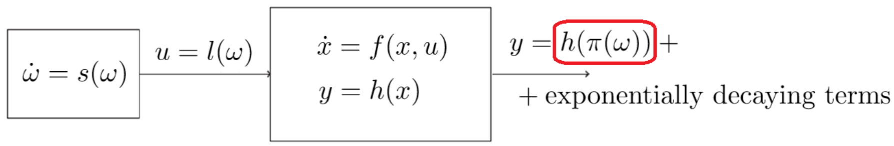

Figure 2.1: Diagrammatic illustration of the characterization of moment given in Definition 2.13. The term denoting the steady-state response is circled.

be hyperbolic. In fact, equation (2.9) may have a well-defined solution π even though the linearized system around the zero equilibrium has eigenvalues on the imaginary axis. Similarly, neutral stability is not required by Definition 2.13 and the moment can be determined also when the signal generator produces diverging or decaying trajectories. However, whenever the more restrictive Assumption 2.1 does not hold, it is not possible to establish a relation with the steady-state response of the interconnected system (2.8).

2.2.1 The special case of linear systems

To clarify the reasons behind the approach and terminology used it is of interest, before proceeding with the definition of reduced order models by moment matching, to specialize to linear systems the concepts and results presented so far. Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{2.10}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$, and a signal generator described by the equations

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{2.11}$$

with $\omega(t) \in \mathbb{R}^\nu$, $S \in \mathbb{R}^{\nu\times\nu}$ and $L \in \mathbb{R}^{\nu\times 1}$. Using this notation, it is convenient to rewrite Assumption 2.1 for linear systems.

Assumption 2.3. The triple (A, B, C) is minimal and $\sigma(A) \subset \mathbb{C}_{<0}$. The pair (S, L) is observable, $\sigma(S) \subset \mathbb{C}_0$ and the eigenvalues of S are simple.

We can now reformulate Lemma 2.1 for linear systems.

Lemma 2.2. Consider system (2.10) and the signal generator (2.11). Suppose Assumption 2.3 holds. Then there is a unique matrix $\Pi \in \mathbb{R}^{n\times\nu}$ which solves the Sylvester equation

$$\Pi S = A\Pi + BL. \tag{2.12}$$

In addition, the steady-state response of system (2.10) is $x^{ss}(t) = \Pi\omega(t)$ for any $x(0)$ and $\omega(0)$.

Proof. The proof follows from Lemma 2.1 once noted that $\frac{\delta\pi}{\delta\omega} = \Pi$, $s(\omega) = S\omega$ and $f(\pi(\omega), l(\omega)) = A\Pi\omega + BL\omega$. □

A straightforward consequence of Lemma 2.2 is a special way to write the response of the interconnection of system (2.10) and the signal generator (2.11). This result is exploited in Chapter 4.

Corollary 2.3. Consider system (2.10) and the signal generator (2.11). Suppose Assumption 2.3 holds. Then for any $x(0)$ and $\omega(0)$ and for all $t \in \mathbb{R}$,

$$x(t) = \Pi\omega(t) + e^{At}(x(0) - \Pi\omega(0)). \tag{2.13}$$

Proof. Note that

$$\overbrace{x - \Pi\omega}^{\cdot} = Ax + BL\omega - \Pi S\omega = Ax + (BL - \Pi S)\,\omega.$$

Substituting (2.12) in the right-hand side of the last equation yields

$$\overbrace{x - \Pi\omega}^{\cdot} = A(x - \Pi\omega),$$

from which (2.13) follows. □

Lemma 2.2 is not simply the linear version of Lemma 2.1: it has deeper implications. In fact, even though we have defined the mapping $h \circ \pi$ (which is $C\Pi\omega$ in the linear case) as the moments of the system, as anticipated in Chapter 1 the word "moment" is usually used in the literature of model reduction in relation to the transfer function. To clarify the relation between the two notions let

$$W(s) = C(sI - A)^{-1}B$$

be the transfer function of system (2.10). The k-moment of system (2.10) at s_i is usually defined as the k-th coefficient of the Laurent series expansion of the transfer function $W(s)$ at $s_i \in \mathbb{C}$, provided it exists.

Definition 2.14 (Interpolation notion of moment)**.** Let $s_i \in \mathbb{C} \setminus \sigma(A)$ and assume that the triple (A, B, C) is minimal. The 0-*moment of system (2.10) at* s_i is the complex number

$$\eta_0(s_i) = C(s_i I - A)^{-1} B.$$

The k-*moment of system (2.10) at* s_i is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!} \left[\frac{d^k}{ds^k} (C(sI - A)^{-1} B) \right]_{s=s_i},$$

with $k \geq 1$ integer.

With the following series of results we prove that the moments of linear systems defined as the scalars $\eta_k(s_i)$ are related to the moment defined via the matrix $C\Pi$ and, under stronger assumptions, to the steady-state output response of the interconnection of system (2.10) with the signal generator (2.11). In establishing this relation we also provide a justification of the terminology and tools we have introduced in the nonlinear setting.

We begin with a result showing that the moments $\eta_k(s_i)$ can be rewritten as the elements of the solution of a Sylvester equation.

Lemma 2.4. Let $s_i \in \mathbb{C} \setminus \sigma(A)$ and assume that the triple (A, B, C) is minimal. Consider system (2.10), then

$$[\eta_0(s_i) \ \ \ldots \ \ \eta_k(s_i)] = C\widetilde{\Pi}\Psi_k,$$

where

$$\Psi_k = \mathrm{diag}(1, -1, 1, \ldots, (-1)^k) \in \mathbb{R}^{(k+1)\times(k+1)}$$

and $\widetilde{\Pi} \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester equation

$$\widetilde{\Pi}\Sigma_k = A\widetilde{\Pi} + BL_k, \tag{2.14}$$

with $L_k = [1\ 0\ \dots\ 0] \in \mathbb{R}^{(k+1)}$ and

$$\Sigma_k = \begin{bmatrix} s_i & 1 & 0 & \dots & 0 \\ 0 & s_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s_i & 1 \\ 0 & \dots & \dots & 0 & s_i \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}.$$

Proof. Let $\widetilde{\Pi} = [\widetilde{\Pi}_0\ \widetilde{\Pi}_1\ \dots\ \widetilde{\Pi}_k]$ and note that (2.14) can be rewritten as

$$\begin{array}{rcl} \widetilde{\Pi}_0 s_i & = & A\widetilde{\Pi}_0 + B, \\ \widetilde{\Pi}_1 s_i + \widetilde{\Pi}_0 & = & A\widetilde{\Pi}_1, \\ & \vdots & \\ \widetilde{\Pi}_k s_i + \widetilde{\Pi}_{k-1} & = & A\widetilde{\Pi}_k. \end{array}$$

As a result

$$\begin{aligned} \widetilde{\Pi}_0 &= (s_i I - A)^{-1} B, \\ \widetilde{\Pi}_1 &= -(s_i I - A)^{-1}(s_i I - A)^{-1} B = \left[\frac{d}{ds}(sI - A)^{-1} B\right]_{s=s_i}, \\ &\ \ \vdots \\ \widetilde{\Pi}_k &= \frac{1}{k!}\left[\frac{d^k}{ds^k}(sI - A)^{-1} B\right]_{s=s_i}, \end{aligned}$$

which proves the claim. □

Equation (2.14) can be written eliminating the fact that Σ_k and L_k have a special structure. To this end we first recall the definition of non-derogatory matrix, which is used to prove the next result.

Definition 2.15. A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

Lemma 2.5. Consider the signal generator (2.11) and let $S \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\bar{k}} (s - s_i)^{k_i}, \tag{2.15}$$

where $\nu = \sum_{i=1}^{\bar{k}} k_i$, and let L be such that the pair (S, L) is observable. Consider system (2.10), suppose $\sigma(S) \cap \sigma(A) = \emptyset$ and that the triple (A, B, C) is minimal. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_{\bar{k}}), \ldots, \eta_{k_{\bar{k}}-1}(s_{\bar{k}})$ and the matrix $C\Pi$, where Π is the unique solution of the Sylvester equation (2.12).

Proof. Consider the case $\bar{k} = 1$. By observability of the pair (S, L) there is a unique invertible matrix T such that $S = T^{-1}\Sigma_k T$ and $L = L_k T$. Then equation (2.12) becomes

$$\Pi T^{-1}\Sigma_k T = BL_k T + A\Pi T^{-1}T.$$

The claim follows defining $\widetilde{\Pi} = \Pi T^{-1}$, recalling that $e^{T^{-1}XT} = T^{-1}e^X T$ and that the moments are coordinates invariant. The proof for $\bar{k} > 1$ is similar to the case $\bar{k} = 1$. The only difference is that the matrix S has $\bar{k}$ Jordan blocks, *i.e.* one, and only one, for each distinct eigenvalue (because S is non-derogatory). Hence, the claim follows repeating the same argument for each block. □

Remark 2.2. Observability of (S, L) implies that S is non-derogatory. For example, consider the derogatory matrix

$$S = \begin{bmatrix} s_1 & 0 \\ 0 & s_1 \end{bmatrix}.$$

with $s_1 \in \mathbb{C}$. The pair (S, L) is unobservable for any vector $L \in \mathbb{R}^{2\times 1}$. Consider now the non-derogatory matrix

$$S = \begin{bmatrix} s_1 & 1 \\ 0 & s_1 \end{bmatrix}.$$

There exists (infinitely many) vectors $L \in \mathbb{R}^{2\times 1}$ such that the pair (S, L) is observable. From the previous proof it is evident that the $k_i - 1$ moment at s_i can be computed solving a Sylvester equation in which a matrix S which is non-derogatory has k_i eigenvalues equal to s_i. If the matrix S is derogatory, then having k_i eigenvalues equal to s_i does not guarantee that we can determine the $k_i - 1$ moment at s_i. In the following we sometimes stress that S is non-derogatory even though the pair (S, L) is assumed to be observable.

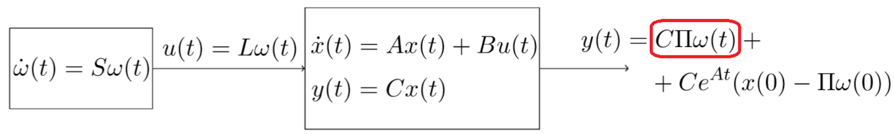

Figure 2.2: Diagrammatic illustration of Theorem 2.6. The term denoting the steady-state response is circled.

The result of the lemma can be given the following interpretation. The matrices A, B, C and the roots of (2.15) identify the moments; then, given any observable pair (S, L) with S a non-derogatory matrix with characteristic polynomial (2.15), there exists an invertible matrix $T \in \mathbb{R}^{\nu\times\nu}$ such that the elements of the vector $C\Pi T^{-1}$ are equal to the moments. It is clear that the importance of this formulation is that it establishes, through the Sylvester equation (2.12), a relation between the moments and the steady-state response of the output of the system.

This relation between the notion of moment and the steady-state output response is formalized in the following result and illustrated in Fig. 2.2.

Theorem 2.6. Consider system (2.10) and the signal generator (2.11). Suppose Assumption 2.3 holds and let $\omega(0)$ be such that the pair $(S, \omega(0))$ is reachable. Then there exists a one-to-one relation between the moments $\eta_0(s_1)$, $\eta_0(s_2)$, $\ldots$, $\eta_0(s_{\bar{k}})$, with $s_i \in \sigma(S)$ for all $i = 1, \ldots, \bar{k}$, and the steady-state response of the output y of the interconnection of system (2.10) with the signal generator (2.11).

Proof. First of all note that with the expression "there exists a one-to-one relation between the moments and the steady-state output response" we mean that the moments are uniquely determined by the steady-state response of $y(t)$ and *vice versa*. The equivalence between the numbers $\eta_0(s_1)$, $\eta_0(s_2)$, $\ldots$, $\eta_0(s_{\bar{k}})$ and $C\Pi$ has been proved in Lemma 2.5. It suffices to prove that under the stated assumptions we can uniquely determine $C\Pi$ from $C\Pi\omega(t)$ (the converse is trivial). This is possible if we can select ν sample times t_i, with $i = 1, \ldots, \nu$, such that

$$\operatorname{rank}\begin{bmatrix} \omega(t_1) & \ldots & \omega(t_\nu) \end{bmatrix} = \nu. \tag{2.16}$$

By hypothesis, the pair $(S, \omega(0))$ is reachable. Therefore, by a standard result of sampled-data theory for linear systems, it is always possible to select $\delta \in \mathbb{R}_{>0}$ such that $\delta(s_l - s_h) \neq \pm 2\pi\iota k$ for all $k \in \mathbb{Z}_{\geq 0}$, all $l = 1, \ldots, \nu$ and all $h = 1, \ldots, \nu$, with $l \neq h$. Finally, setting $t_i = i\delta$ for every $i = 1, \ldots, \nu$ guarantees that condition (2.16) holds. This concludes the proof. $\square$

Remark 2.3. The reachability of the pair $(S, \omega(0))$, also known as *excitability* of the pair $(S, \omega(0))$, is (with additional technical assumptions) a geometric characterization of the property that all signals generated by (2.11) are persistently exciting (as also evident from the proof of Theorem 2.6).

As already stressed in the nonlinear case, moments are well-defined even though the steady-state response does not exist. However, the system to be reduced and the signal generator have to possess *strong* stability properties to establish a connection between these two objects. While for linear systems this may be restrictive (since moments are justified by the interpolation interpretation given in Definition 2.14), for nonlinear systems Assumption 2.1 provides easy to check conditions that guarantee the existence of the moment and, at the same time, the foundations of a meaningful interpretation of the notion of moment.

In the following it becomes important to distinguish when we are, or not, exploiting the relation with the steady-state response. To streamline the presentation, we formulate, similarly to the nonlinear case, the assumptions required by Lemma 2.5 explicitly.

Assumption 2.4. The triple (A, B, C) is minimal, the pair (S, L) is observable and $\sigma(S) \cap \sigma(A) = \emptyset$.

Thus, in the following when we use Assumptions 2.1 or 2.3 we are exploiting the relation between moments and steady-state, whereas when we use the less restrictive Assumptions 2.2 or 2.4 we are using just the notion of moment.

Remark 2.4. Since we have established a one-to-one relation between moments and the steady-state output response of an interconnected system, one could wonder if it is possible to devise an algorithm that, given the signal ω and the output y, retrieves the moments of a system with unknown matrices A, B and C. This issue is addressed in Chapter 4.

2.3 Families of Reduced Order Models for Nonlinear Systems

We now formally define what we intend for "reduced order models by moment matching" for the class of nonlinear systems described by equations of the form (2.6).

Definition 2.16. Consider system (2.6) and the signal generator (2.7). The system described by the equations

$$\dot{\xi} = \phi(\xi, u), \qquad \psi = \kappa(\xi), \tag{2.17}$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$, and ϕ and κ analytic mappings, is a *model of system (2.6) at* (s, l) if system (2.17) has the same moment at (s, l) as system (2.6). In this case, system (2.17) is said to *match* the moment of system (2.6) at (s, l). Furthermore, system (2.17) is a *reduced order model of system (2.6) at* (s, l) if $\nu < n$.

From this definition we can easily derive a general result.

Lemma 2.7. Consider system (2.6) and the signal generator (2.7). Suppose Assumption 2.2 holds. Then system (2.17) matches the moments of system (2.6) at (s, l) if the equation

$$\frac{\partial p}{\partial \omega} s(\omega) = \phi(p(\omega), l(\omega)) \tag{2.18}$$

has a unique solution p such that

$$h(\pi(\omega)) = \kappa(p(\omega)), \tag{2.19}$$

where π is the unique solution of equation (2.9).

Proof. The claim is a direct consequence of Definition 2.16 and the definition of moment. In fact, note that equation (2.18) defines the moments of the reduced order model, whereas equation (2.19) gives the matching condition between the moments of the system to be reduced and the moments of the reduced order model. $\square$

The conditions of Lemma 2.7 have a simple interpretation. In fact, equation (2.18) identifies the mapping p which describes the moment at (s, l) of system (2.17). Equation (2.19) expresses the matching condition, *i.e.* the moment of system (2.6) at (s, l) must be equal to the moment of system (2.17) at (s, l). Moreover, if Assumption 2.1 holds and the zero equilibrium of system (2.17) is locally exponentially stable, then the steady-state output response $y^{ss}(t) = h(\pi(\omega(t)))$ of system (2.6) is equal to the steady-state output response $\psi^{ss}(t) = \kappa(p(\omega(t)))$ of system (2.17).

Note that there are many tunable degrees of freedom in the model (2.17). For instance, we can select the mappings ϕ and κ such that equations (2.18) and (2.19) hold. A particularly convenient selection is possible if we make the following assumption.

Assumption 2.5. There exist mappings κ and p such that $\kappa(0) = 0$, $p(0) = 0$, p is locally continuously differentiable, equation (2.19) holds and p has a local inverse p^{-1}.

Proposition 2.1. Consider system (2.6) and the signal generator (2.7). Suppose Assumptions 2.2 and 2.5 hold. Then the system described by the equations

$$\dot{\xi} = s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u, \qquad \psi = h(\pi(\xi)), \tag{2.20}$$

where π is the unique solution of equation (2.9), is a model of system (2.6) at (s, l) if δ is an arbitrary mapping such that the partial differential equation

$$\frac{\partial p}{\partial \omega} s(\omega) = s(p(\omega)) - \delta(p(\omega))l(p(\omega)) + \delta(p(\omega))l(\omega), \tag{2.21}$$

has the unique solution $p(\omega) = \omega$.

Proof. If there exist mappings κ and p for which Assumption 2.5 is satisfied then, consistently with Lemma 2.7, a family of models that achieves moment matching at (s, l) is described by

$$\begin{aligned} \dot{\xi} &= \bar{\phi}(\xi) + \delta(\xi)u, \\ \psi &= \kappa(\xi), \end{aligned}$$

with

$$\bar{\phi}(\xi) = \left[\frac{\partial p(\omega)}{\partial \omega} s(\omega) - \delta(p(\omega))l(\omega)\right]_{\omega=p^{-1}(\xi)},$$

where p is the unique solution of (2.18) and δ is a free mapping. A choice of the mappings κ and p that have the required properties is given by $p(\omega) = \omega$ and $\kappa(\omega) = h(\pi(\omega))$. This yields a family of models described by the equations (2.20) with δ any arbitrary mapping such that equation (2.21) has the unique solution $p(\omega) = \omega$. □

The family of models (2.20) is parametrized in δ, which can be used to achieve specific properties of the reduced order model. In the remaining of the section we study several properties which are of interest for nonlinear systems and that the designer may want to impose upon the reduced order model. For instance, we may want to preserve a property of the system to be reduced, or we may want to achieve a special representation of the reduced order model which is particularly useful for nonlinear systems. In what follows, we always assume that the δ achieving a specific property is such that equation (2.21) has the unique solution $p(\omega) = \omega$.

Matching with asymptotic stability

We begin with the problem of determining a reduced order model described by equations of the form (2.20) which has an asymptotically stable equilibrium at zero. This problem can be solved if it is possible to select the mapping δ such that the zero equilibrium of the system $\dot{\xi} = s(\xi) - \delta(\xi)l(\xi)$ is locally asymptotically stable. We observe that the solution of this problem boils down to a property of the signal generator. For example, it is sufficient (but not necessary) that the pair $\left(\left.\frac{\partial s(\xi)}{\partial \xi}\right|_{\xi=0}, \left.\frac{\partial l(\xi)}{\partial \xi}\right|_{\xi=0}\right)$ be observable or detectable. Another sufficient condition is that the pair (s, l) is observable and there exists a continuously differentiable non-negative function $V : \mathbb{R}^\nu \to \mathbb{R}$, with $V(0) = 0$, such that $L_s V(\xi) = 0$. In fact, if this is the case the selection $\delta(\xi) = V_\xi^\top l(\xi)$ solves the problem.

Matching with prescribed relative degree and zero dynamics

The notion of relative degree is of fundamental importance for nonlinear systems. In fact, many problems as, for instance, exact linearization, asymptotic stabilization and output tracking can be easily solved for nonlinear systems that have a well-defined relative degree. To streamline the presentation of the results we recall the definition of relative degree. Consider the simplified system

$$\dot{\xi} = s(\xi) + \delta(\xi)u, \qquad \psi = h(\pi(\xi)). \tag{2.22}$$

As it is clear from the following definition, system (2.20) has relative degree r at ξ_0 if and only if system (2.22) has relative degree r at ξ_0.

Definition 2.17. System (2.22) is said to have *relative degree* r *at* ξ_0 if

$$L_\delta h(\pi(\xi)) = L_\delta L_s h(\pi(\xi)) = \cdots = L_\delta L_s^{r-2} h(\pi(\xi)) = 0, \tag{2.23}$$

and

$$L_\delta L_s^{r-1} h(\pi(\xi_0)) \neq 0, \tag{2.24}$$

for all ξ in a neighborhood of ξ_0.

We consider the problem of constructing a reduced order model which has a given relative degree $r \in [1, \nu]$ at some point ξ_0. If we can construct a reduced order model with relative degree $r = \nu$, then there exists a change of coordinates and a state feedback such that the model can be exactly linearized. In the case that $r < \nu$, then the model can be only partially linearized. In particular, there exists a change of coordinates such that the model is described by the equations

$$\begin{array}{rcl} \dot{z}_1 & = & z_2, \\ & \vdots & \\ \dot{z}_{r-1} & = & z_r, \\ \dot{z}_r & = & a(\tilde{\zeta}, \zeta) + a_u(\tilde{\zeta}, \zeta)u, \\ \dot{\zeta} & = & q(\tilde{\zeta}, \zeta) + q_u(\tilde{\zeta}, \zeta)u. \end{array} \tag{2.25}$$

where $\tilde{\zeta} = \begin{bmatrix} z_1 & \cdots & z_r \end{bmatrix}^\top$, $\zeta = \begin{bmatrix} z_{r+1} & \cdots & z_\nu \end{bmatrix}^\top$, a, a_u, q and q_u are functions, with $a_u(0, \zeta(t)) \neq 0$ for all $t \in \mathbb{R}$ and for $\zeta(t)$ sufficiently close to zero. The subsystem $\dot{\zeta} = q(0,\zeta) - q_u(0,\zeta)\frac{a(0,\zeta)}{a_u(0,\zeta)}$ is the so-called *zero dynamics*. We would like to find families of reduced order models for which we can set the properties of the zero dynamics, *e.g.* make its zero equilibrium asymptotically stable.

It is well-known that the key for solving these problems lies in the properties of a certain codistribution.

Assumption 2.6. The codistribution

$$d\mathcal{O}_\nu = \mathrm{span}\{dh(\pi(\xi)), \cdots, dL_s^{\nu-1}h(\pi(\xi))\} \tag{2.26}$$

has dimension ν at ξ_0.

From a classical results of nonlinear systems we know that if Assumption 2.6 holds then system (2.20) is locally observable at ξ_0. Moreover, we can use this assumption to give a preliminary lemma which establishes the conditions for which model (2.20) has a well-defined relative degree.

Lemma 2.8. There exists a δ such that system (2.20) has relative degree r at ξ_0, for all $r \in [1, \nu]$, if and only if Assumption 2.6 holds.

Proof. (*Necessity*). Equations (2.23) and (2.24) can be rewritten as

$$\begin{bmatrix} dh(\pi(\xi)) \\ \vdots \\ dL_s^{r-2}h(\pi(\xi)) \\ dL_s^{r-1}h(\pi(\xi)) \end{bmatrix} \delta(\xi) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma(\xi) \end{bmatrix},$$

for some function γ such that $\gamma(\xi_0) \neq 0$. This equation has a solution for all $r \in [1, \nu]$ if and only if the matrix in the left hand side has rank r at ξ_0. Setting $r = \nu$ yields that the codistribution $d\mathcal{O}_\nu$ has dimension ν.

(*Sufficiency*). Since $d\mathcal{O}_\nu$ has rank ν and it is spanned by ν differentials, for all $r \in [1, \nu]$ the distribution

$$d\mathcal{O}_i = \mathrm{span}\{dh(\pi(\xi)), \cdots, dL_s^{i-1}h(\pi(\xi))\}$$

has dimension i, which implies that the equations (2.23) and (2.24) have a solution δ locally well-defined around ξ_0, hence the claim. □

Corollary 2.9. If system (2.8) is locally observable at $[\xi_0^\top, \pi(\xi_0)^\top]^\top$, then system (2.20) is locally observable at ξ_0.

Proof. Suppose, by contradiction, that system (2.20) is not observable at ξ_0. Then there is a point ξ_a (in a neighborhood of ξ_0) which is indistinguishable from ξ_0. Consider now system (2.8) and the points $[\xi_0^\top, \pi(\xi_0)^\top]^\top$ and $[\xi_a^\top, \pi(\xi_a)^\top]^\top$. Simple computations, recalling that the manifold $x = \pi(\xi)$ is invariant, show that these points are not distinguishable, hence system (2.8) is not locally observable at $[\xi_0^\top, \pi(\xi_0)^\top]^\top$, which yields a contradiction. □

This last result gives guarantees on the observability of a reduced order model of an observable system. Note that, as shown in the next section, we can achieve stronger results for linear systems.

Lemma 2.8 provides the condition for the existence of the solution to the problem of achieving moment matching with a prescribed relative degree. However, we still do not know how to achieve this. To solve this problem and the problem of obtaining a zero dynamics with specific properties we need a preliminary lemma.

Lemma 2.10. Consider system (2.20). Suppose Assumption 2.6 holds. Then there exists a coordinates transformation $\chi = \Xi(\xi)$, locally defined around ξ_0, such that, in the new coordinates, system (2.20) is described by equations of the form

$$\begin{array}{rcl} \dot{\chi}_1 & = & \chi_2 + \tilde{\delta}_1(\chi)(v - \tilde{l}(\chi)), \\ \dot{\chi}_2 & = & \chi_3 + \tilde{\delta}_2(\chi)(v - \tilde{l}(\chi)), \\ & \vdots & \\ \dot{\chi}_\nu & = & \tilde{f}(\chi) + \tilde{\delta}_\nu(\chi)(v - \tilde{l}(\chi)), \\ \psi & = & \chi_1, \end{array} \tag{2.27}$$

where $[\tilde{\delta}_1(\chi), \cdots, \tilde{\delta}_\nu(\chi)]^\top = \delta(\Xi^{-1}(\chi))$, $\tilde{l}(\chi) = l(\Xi^{-1}(\chi))$, and $\tilde{f}(\chi) = L_s^\nu h(\pi(\Xi^{-1}(\chi)))$.

Proof. By Assumption 2.6 the functions $h \circ \pi$, ..., $L_s^{\nu-1} h \circ \pi$ qualify as local coordinates around ξ_0. In these coordinates the system (2.20) is trivially described by equations of the form (2.27). □

We are now in the position to determine reduced order models with a given relative degree and with a zero dynamics which is parameterized by a set of free functions.

Proposition 2.2. Consider system (2.20). Suppose Assumption 2.6 holds and ξ_0 is an equilibrium of system (2.20). Then, for all $r \in [1, \nu - 1]$, there is a mapping δ such that system (2.20) has relative degree r and its zero dynamics have a locally exponentially stable equilibrium. In addition, there is a coordinate transformation, locally defined around ξ_0, such that in the new coordinates the zero dynamics of system (2.20) are described by equations of the form

$$\begin{array}{rcl} \dot{z}_{r+1} & = & z_{r+2} + \hat{\delta}_1(\zeta) z_{r+1}, \\ \dot{z}_{r+2} & = & z_{r+3} + \hat{\delta}_2(\zeta) z_{r+1}, \\ & \vdots & \\ \dot{z}_{\nu} & = & \hat{f}(\zeta) + \hat{\delta}_{\nu-r}(\zeta) z_{r+1}, \end{array} \tag{2.28}$$

where the $\hat{\delta}_i$ are free functions and

$$\hat{f}(\zeta) = \tilde{f}(\chi)\Big|_{\chi=[0,\cdots,0,z_{r+1},\cdots,z_\nu]^\top}.$$

Proof. By Lemma 2.10, Assumption 2.6, and invariance of the zero dynamics with respect to coordinates transformation, we can consider the system described by equations of the form (2.27). Fix now $r \in [1, \nu - 1]$ and note that for $r = \nu$ the system does not have zero dynamics. Select $\tilde{\delta}_i$ such that $\tilde{\delta}_r(\chi_0) \neq 0$ and $\tilde{\delta}_j(\chi) = 0$, for all $j \in [1, r-1]$ (note that if $r = 1$ the set of such $\tilde{\delta}_i$ is empty). The resulting system has relative degree r. Consider now the r-th equation in (2.27), namely

$$\dot{\chi}_r = \chi_{r+1} + \tilde{\delta}_r(\chi)(v - \tilde{l}(\chi))$$

and, by definition of zero dynamics, we obtain

$$v - \tilde{l}(\chi) = -\frac{\chi_{r+1}}{\tilde{\delta}_r(\chi)}.$$

Thus, the zero dynamics are described by the last $\nu - r$ equations of system (2.27) with $v = \tilde{l}(\chi) - \frac{\chi_{r+1}}{\tilde{\delta}_r(\chi)}$, $\chi_1 = \ldots = \chi_r = 0$, $z_i =$

χ_{r+i} and $\hat{\delta}_i = -\frac{\tilde{\delta}_{r+i}(\chi)}{\tilde{\delta}_r(\chi)}\Big|_{\chi=[0,\cdots,0,z_{r+1},\cdots,z_\nu]^\top}$ which have the form given in equation (2.28). To complete the proof recall that ξ_0 is (by assumption) an equilibrium, hence, the system (2.28) has an equilibrium which can be rendered locally exponentially stable by a proper selection of the functions $\hat{\delta}_1, \cdots, \hat{\delta}_{\nu-r}$. □

Matching with a passivity constraint

Consider now the problem of selecting the mapping δ such that system (2.20) is lossless or passive. The property of passivity is desirable because it provides a powerful framework to solve the problem of global asymptotic stabilization for interconnected systems. We define passive systems through the characterization given by the so-called Kalman-Yacubovitch-Popov (KYP) lemma.

Definition 2.18. System (2.20) has the KYP property if there exists a continuously differentiable non-negative function $V : \mathbb{R}^\nu \to \mathbb{R}$, with $V(0) = 0$, such that

$$L_{s-\delta l}V(\xi) \leq 0, \qquad L_\delta V(\xi) = h(\pi(\xi)), \tag{2.29}$$

for each $\xi \in \mathbb{R}^\nu \subset \mathbb{R}^\nu$.

Definition 2.19. System (2.20) is *passive*, with continuously differentiable *storage function* V, if it has the KYP property. If the first relation in (2.29) holds as an equality, then system (2.20) is *lossless*.

Theorem 2.11. The family of reduced order models (2.20) contains, locally around ξ_0, a lossless (passive, respectively) system with a continuously differentiable storage function V if there exists a differentiable function V, locally positive definite around ξ_0, such that

$$L_sV(\xi) = h(\pi(\xi))l(\xi), \quad (L_sV(\xi) \leq h(\pi(\xi))l(\xi) \text{ resp.}), \tag{2.30}$$

locally around ξ_0, and

$$V_{\xi\xi}(\xi_0) > 0. \tag{2.31}$$

Proof. We provide the proof for the lossless case, since the claims for the passivity case follows identical arguments. Equation (2.31) implies that the equation $L_\delta V(\xi) = h(\pi(\xi))$, in the unknown δ, has a (unique) solution $\delta^\star$, which is locally well-defined and continuous around ξ_0. Consider now system (2.20) with $\delta = \delta^\star$. Then $L_{s-\delta^\star l} V(\xi) = 0$, and this, together with the definition of $\delta^\star$, proves the claim. □

The conditions in the previous theorem are sufficient but not necessary. In fact, by Definition 2.19, if the family of systems (2.20) contains, locally around ξ_0, a lossless system with a differentiable storage function V then there exists δ such that $L_{s-\delta l} V(\xi) = 0$ and $L_\delta V(\xi) = h(\pi(\xi))$, locally around ξ_0. Replacing the second equality in the first yields $L_s V(\xi) - h(\pi(\xi)) l(\xi) = 0$, *i.e.* condition (2.30), with $V_{\xi\xi}(\xi_0) \geq 0$. Thus, condition (2.31) is not necessarily implied.

2.3.1 Reduced order models for linear systems

We now specialize our analysis to the case in which both the system to be reduced and the signal generator are linear systems, namely they are described by equations of the form (2.10) and (2.11), respectively, obtaining families of linear reduced order models. Similarly to what discussed for the nonlinear case, we want then to investigate if we can, and how to, exploit potential free parameters to achieve specific properties of the reduced order model.

Definition 2.20. Consider system (2.10) and the signal generator (2.11). The system described by the equations

$$\dot{\xi} = F\xi + Gu, \qquad \psi = H\xi, \tag{2.32}$$

with $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}$, $F \in \mathbb{R}^{\nu \times \nu}$, $G \in \mathbb{R}^{\nu \times 1}$, $H \in \mathbb{R}^{1 \times \nu}$ is a *model of system (2.10) at* (S, L) if system (2.32) has the same moments at (S, L) as system (2.10). System (2.32) is a *reduced order model of system (2.10) at* (S, L) if $\nu < n$.

Note that Definition 2.20 is consistent with the more general formulation in Definition 2.16. We now give a linear equivalent of Lemma 2.7.

Lemma 2.12. Consider system (2.10) and the signal generator (2.11). Suppose Assumption 2.4 holds. Then system (2.32) is a model of system (2.10) at (S, L) if there exists a unique solution P of the equation

$$FP + GL = PS, \tag{2.33}$$

such that

$$HP = C\Pi, \tag{2.34}$$

where Π is the unique solution of (2.12).

Proof. The claim is a direct consequence of Definition 2.20 and the definition of moment. □

The first step to determine a reduced order model is to solve the Sylvester equation (2.12) associated to the system to be reduced. Several methods and algorithms to solve Sylvester equations are known. However, note that the determination of the matrix Π may be computationally expensive because one of its dimensions is the order of the system to be reduced. Notably, all model reduction techniques in the literature *avoid* solving any Sylvester equation. In fact, model reduction by moment matching is among the computationally most efficient methods of model reduction available. This fact inspires an alternative approach. Consider system (2.10) and the interpolation points $s_i \in \sigma(S)$. Construct a reduced order model achieving moment matching at the points s_i with any efficient algorithm, *e.g.* Lanczos algorithm, Arnoldi algorithm or the iterative rational Krylov algorithm. This yields a reduced order model for system (2.10) described by equations of the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad \tilde{y} = \tilde{C}\tilde{x}, \tag{2.35}$$

where $\tilde{x}(t) \in \mathbb{R}^{\nu}$, $\tilde{y}(t) \in \mathbb{R}$, $\tilde{A} \in \mathbb{R}^{\nu\times\nu}$, $\tilde{B} \in \mathbb{R}^{\nu\times 1}$ and $\tilde{C} \in \mathbb{R}^{1\times\nu}$. At this point it is sufficient to apply the model reduction techniques presented herein considering the reduced order model (2.35) as the system to be reduced. This time the corresponding matrix $\tilde{\Pi}$ has dimensions $\nu \times \nu$. Note also that an alternative approach is discussed in Chapter 4, in which an algorithm to determine reduced order models in a computationally efficient way, namely without solving the Sylvester equation (2.12) or the partial differential equation (2.9), is presented.

On the other hand, the computation of the solution P of the Sylvester equation (2.33) of the reduced order model can be avoided altogether, as shown in the following result.

Proposition 2.3. Consider system (2.10) and the signal generator (2.11). Suppose Assumption 2.4 holds. Then the system described by the equations

$$\dot{\xi} = (S - GL)\xi + Gu, \qquad \psi = C\Pi\xi, \tag{2.36}$$

where Π is the unique solution of (2.12), is a model of system (2.10) at (S, L) for any G such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$.

Proof. Consider system (2.32) and set $F = S - GL$ and $H = C\Pi$. Substituting these matrices in equations (2.33) and (2.34) we obtain $(S - GL)P + GL = PS$ and $C\Pi P = C\Pi$. It is straightforward to see that $P = I$ is a solution of these two equations. To conclude the proof note that equation (2.33) has $P = I$ as unique solution if and only if $\sigma(S) \cap \sigma(S - GL) = \emptyset$. □

It can be observed that the family of models (2.36) is built on three ideas: avoiding to solve equation (2.33), selecting the solution as I; copying (asymptotically) the dynamics of the signal generator (2.11), *i.e.* the relation $\xi = \omega$ holds for the steady-state of ξ, if it exists; and having a convenient parametrization for the family of reduced order models for which additional constraints can be easily imposed. These are also the principles upon which the family of nonlinear models (2.20) was constructed.

For the family of models (2.36) we can prove a result of *completeness* of the parameterization, *i.e.* we show that the family (2.36) contains all the models of order ν achieving moment matching at (S, L).

Proposition 2.4. Consider the family of systems (2.36). Consider a model of system (2.10) at (S, L) of dimension ν, and let $K_G(s)$ be its transfer function. Then there exists a unique G such that $K_G(s) = C\Pi(sI - (S - GL))^{-1}G$.

Proof. Let $K_G(s) = \frac{N(s)}{D(s)}$ and select G such that $D(s) = \det(sI - (S - GL))$. Under the stated assumptions there is a unique G satisfying this condition. To complete the proof we need to show that $N(s) = C\Pi \operatorname{adj}(sI - (S - GL))G$. The condition of moment matching implies ν independent equality conditions on the polynomials $N(s)$ and $C\Pi\,\mathrm{adj}(sI-(S-GL))G$, which are both of degree $\nu-1$, hence these polynomials are identical, which proves the claim. □

With the expression "the family (2.32) contains all the models of dimension ν achieving moment matching at $\sigma(S)$" we mean that all the models that can be obtained using interpolation/projection approaches belong to the family of systems (2.32). In the interpolation approach, which is the standard in the literature, the matching of the moments is achieved by means of two oblique projectors, called Krylov projectors. By Proposition 2.4 the models obtained with these other moment matching approaches belong to the family of systems (2.32). However, the parametrization given in (2.32) offers a substantial advantage. The relation between Krylov projectors and the properties of the reduced order models, such as asymptotic stability, zeros and relative degree, is often nontrivial. On the other hand, the advantage of the presented formulation is that the family of systems (2.32) is parametrized in G, which allows to set with ease several properties of the reduced order model, as shown in the following. For instance, setting the eigenvalues of the reduced order model is an easy task, whereas with classical Krylov methods this is rather difficult.

We now investigate the possibility of using the matrix G to achieve specific properties of the reduced order model. To this end we implicitly assume that the G achieving a specific property is such that $\sigma(S)\cap\sigma(S-GL)=\emptyset$. Note that while some problems which are relevant for nonlinear systems are relevant also for linear systems, we also investigate problems that are interesting only for the latter.

Maximal pole-zero cancellation

Note that the assumption that $\sigma(S)\cap\sigma(S-GL)=\emptyset$ implies that the pair $(S-GL,G)$ is controllable. However, the pair $(C\Pi,S-GL)$ is not necessarily observable. As a consequence, the family of models (2.32) may be not minimal, *i.e.* there are pole-zero cancellations in the transfer function. Thus, we now determine the matrix G which maximizes the number of pole-zero cancellations. The resulting model is a model with the lowest order achieving matching of a prescribed number of moments. The next theorem is stated without proof.

Theorem 2.13. Let Σ_G be a reduced order model belonging to the family (2.32) that matches ν moments of system (2.10) and let $K_G(s) = C\Pi(sI - S + GL)^{-1}G$ be its transfer function. Assume Σ_G is not minimal. Let $z = [z_1 \cdots z_k]^\top \in \mathbb{C}^k$, $z_i \neq z_j$, with $i = 1, \ldots, k$ and $j = 1, \ldots, k$, be such that $z_i \notin \sigma(S)$. Then the following statements hold.

(PZ1) Assume $v = 2k$. Let

$$M(z) = \begin{bmatrix} L(z_1 I - S)^{-1} \\ L(z_2 I - S)^{-1} \\ \vdots \\ L(z_k I - S)^{-1} \end{bmatrix} \in \mathbb{C}^{k \times 2k}, \quad N(z) = \begin{bmatrix} C\Pi(z_1 I - S)^{-1} \\ C\Pi(z_2 I - S)^{-1} \\ \vdots \\ C\Pi(z_k I - S)^{-1} \end{bmatrix} \in \mathbb{C}^{k \times 2k},$$
$$T(z) = \begin{bmatrix} N(z) \\ M(z) \end{bmatrix} \in \mathbb{C}^{2k \times 2k}. \tag{2.37}$$

Then there exists a unique $G = T^{-1}(z) \begin{bmatrix} -\mathbf{1} \\ 0 \end{bmatrix}$, with $\mathbf{1} = [1 \cdots 1]^\top$, such that the numbers z_i are both zeros and poles of $K_G(s)$. Moreover, setting $[\theta_1\ \theta_2] = C\Pi T^{-1}(z)$, the cancellations yield a model of minimal order k given by

$$\hat{K}(s) = \frac{\displaystyle\sum_{i=1}^{k} \theta_{1i} \prod_{j \neq i}^{k} (s - z_j)}{\displaystyle\prod_{j=1}^{k} (s - z_i) + \sum_{i=1}^{k} \theta_{2i} \prod_{j \neq i}^{k} (s - z_j)}, \tag{2.38}$$

where $\theta_i = [\theta_{i1}\, \theta_{i2} \cdots \theta_{ik}]^\top$, $i = 1, 2$.

(PZ2) Assume $\nu = 2k + 1$. Then there exists a parametrized matrix $G(\alpha) \in \mathbb{C}^{2k+1}$, $\alpha \in \mathbb{C}$, such that the numbers z_i are both zeros and poles of $K_G(s)$, yielding a subclass of models $\Sigma_{G(\alpha)}$ of minimal order $k+1$, described by $K_{G(\alpha)}$ as in (2.38), with $[\theta_1(\alpha)\ \theta_2(\alpha)] = C\Pi(\alpha)T^{-1}(z, \alpha)$, $T(z, \alpha) \in \mathbb{C}^{(2k+2)\times(2k+2)}$.

Connections with the Georgiou-Kimura parametrization and the Nevanlinna-Pick interpolation problem

It is of interest to establish connections with the so-called Georgiou-Kimura parametrization and the classical Nevanlinna-Pick interpolation problem.

The Georgiou-Kimura parameterization arises in the solution of the covariance extension problem. This problem can be recast as the problem of matching k moments at 0. The solution of this problem is given by the rational function

$$W_{GK}(s) = \frac{\hat{\psi}_n(s) + \gamma_1 \hat{\psi}_{n-1}(s) + \cdots + \gamma_n \hat{\psi}_0(s)}{\widetilde{\psi}_n(s) + \gamma_1 \widetilde{\psi}_{n-1}(s) + \cdots + \gamma_n \widetilde{\psi}_0(s)},$$

where $\gamma_i \in \mathbb{R}$, for $i \in [1, n]$, and $\widetilde{\psi}_i$ and $\hat{\psi}_i$ are the Szegö orthogonal polynomial of the first and second kind, respectively. It is worth comparing the above parameterization with the parameterization proposed in this work, which is given in terms of the vector G. To this end, note that, in the Georgiou-Kimura parameterization, the parameters γ_i enter linearly both in the numerator and in the denominator, whereas in the proposed representation the entries of the parameter G enter nonlinearly in both cases. However, note that it is possible to relate directly the parameter G with some systems theoretic properties of the reduced order model (which solves the covariance extension problem), whereas it is very hard to derive similar interpretations for the parameters γ_i. Note finally that, by Proposition 2.4, the set of all rational transfer functions given by the Georgiou-Kimura parameterization coincides with the set characterized by the family (2.36). We illustrate the above discussion with an example.

Example 2.2. Consider a reduced order model described by the equations (2.36), with

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{k \times k}, \tag{2.39}$$

$k \geq 1$ and

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad\qquad C\Pi = \begin{bmatrix} \eta_0 & \eta_1 & \cdots & \eta_{k-1} \end{bmatrix}. \quad (2.40)$$

Thus, the reduced order model matches k moments at zero, solving the covariance extension problem. Note that, independently from the selection of G, the transfer function $W(s) = C\Pi(sI - (S - GL))^{-1}G$ of the reduced order model is $W(s) = \eta_0 + \eta_1 s + \cdots + \eta_{k-1}s^{k-1} + \ldots$.

The main difference between the model reduction problem and the classical Nevanlinna-Pick interpolation problem is that in the latter the number of interpolation points is equal to the order of the approximating system plus one. As a result, the interpolating problem can be regarded as a model reduction problem by moment matching in which a model of order ν should match $\nu + 1$ moments.

Matching with prescribed eigenvalues

While for nonlinear systems we have investigated the problem of determining reduced order models which have an asymptotically stable zero equilibrium, for linear systems we want to achieve the stronger result of setting the eigenvalues of the reduced order model.

Given a set $\Theta = \{\lambda_1, \cdots, \lambda_\nu\}$ of ν values $\lambda_i \in \mathbb{C}$, with $\Theta \cap \sigma(S) = \emptyset$, we consider the problem of determining G such that $S - GL$ has spectrum equal to Θ. The problem can be solved simply selecting G such that

$$\sigma(S - GL) = \Theta, \quad (2.41)$$

By observability of the pair (S, L), there is a unique matrix G such that this is possible. This is illustrated in the next example.

Example 2.3. Consider a reduced order model described by the equations (2.36), with S, L and $C\Pi$ as in equations (2.39) and (2.40), and the problem of assigning k eigenvalues at $-\lambda$, with $\lambda \in \mathbb{R}_{>0}$. The problem is solved by the selection

$$G = \begin{bmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix} \lambda & \begin{pmatrix} k \\ 2 \end{pmatrix} \lambda^2 & \cdots & \begin{pmatrix} k \\ k \end{pmatrix} \lambda^k \end{bmatrix}^\top,$$

which yields a reduced order model with k eigenvalues at $-\lambda$ and with transfer function

$$
\begin{aligned}
W(s) = & \frac{G}{(s+\lambda)^k}\left(\begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_k \end{bmatrix} s^k + \begin{bmatrix} 0 \\ \eta_0 \\ \vdots \\ \eta_{k-1} \end{bmatrix} s^{k-1} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \eta_0 \end{bmatrix}\right) = \\
= & \eta_0 + \eta_1 s + \cdots + \eta_{k-1} s^{k-1} + \ldots .
\end{aligned}
$$

Matching with prescribed relative degree and zeros

We now study the linear version of the problem of setting the relative degree and assigning a prescribed zero dynamics to the reduced order model. In other words, consider system (2.36) and the problem of selecting G such that the system has a given relative degree $r \in [1, \nu]$, and/or has prescribed zeros. Before providing the solution to these problems we recall here the definition of relative degree and zeros for linear systems.

Definition 2.21. System (2.36) has *relative degree* r if

$$
\begin{bmatrix} C\Pi \\ \vdots \\ C\Pi S^{r-2} \\ C\Pi S^{r-1} \end{bmatrix} G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma \end{bmatrix}, \tag{2.42}
$$

for some non-zero γ.

Definition 2.22. The *zeros* of system (2.36) are the roots of the polynomial

$$
\det \begin{bmatrix} sI - (S - GL) & G \\ C\Pi & 0 \end{bmatrix}.
$$

Theorem 2.14. The following statements are equivalent.

(RD) Equation (2.42) has a solution G for all $r \in [1, \nu]$.

(O1) The system

$$
\dot{\xi} = S\xi, \qquad \psi = C\Pi\xi, \tag{2.43}
$$

is observable.

(O2) The system

$$\dot{\omega} = S\omega, \quad \dot{x} = Ax + BL\omega, \quad y = Cx, \tag{2.44}$$

is observable.

(Z1) The zeros of system (2.36) can be arbitrarily assigned selecting G.

(Z2) The zeros of the system

$$\dot{\xi} = S\xi + Gu, \qquad \psi = C\Pi\xi, \tag{2.45}$$

can be arbitrarily assigned selecting G.

Proof. (RD)$\Rightarrow$(O1). If equation (2.42) has a solution G for all $r \in [1, \nu]$ then $C\Pi \neq 0$, and setting $r = 2$ yields that $C\Pi S$ is linearly independent from $C\Pi$, *i.e.*

$$\text{rank} \begin{bmatrix} C\Pi \\ C\Pi S \end{bmatrix} = 2.$$

Using the same arguments we infer that, for each $r \in [1, \nu]$,

$$\text{rank} \begin{bmatrix} C\Pi \\ C\Pi S \\ \vdots \\ C\Pi S^{r-1} \end{bmatrix} = r, \tag{2.46}$$

hence for $r = \nu$ we conclude observability of system (2.43).

(O1)$\Rightarrow$(RD). Observability of system (2.43) implies equation (2.46), for all $r \in [1, \nu]$, and hence solvability of equation (2.42) for all $r \in [1, \nu]$.

(O2)$\Rightarrow$(O1). We prove this implication by contradiction. Suppose system (2.43) is not observable. Then there exist a (possibly complex-valued) vector v and a scalar λ such that $Sv = \lambda v$ and $C\Pi v = 0$. Let

$$w = \begin{bmatrix} v \\ \Pi v \end{bmatrix} = \begin{bmatrix} I \\ \Pi \end{bmatrix} v$$

and note that

$$\begin{bmatrix} S & 0 \\ BL & A \end{bmatrix} w = \begin{bmatrix} Sv \\ (BL + A\Pi)v \end{bmatrix} = \begin{bmatrix} I \\ \Pi \end{bmatrix} Sv = \lambda w,$$

and $\begin{bmatrix} 0 & C \end{bmatrix} w = 0$, hence system (2.44) is not observable, which proves the claim.

(O1)$\Rightarrow$(O2). We prove this implication by contradiction. Suppose system (2.44) is not observable. Then there exist a (possibly complex-valued) vector $w = [w_1' \ w_2']'$ and a scalar λ such that

$$\begin{bmatrix} S & 0 \\ BL & A \end{bmatrix} w = \lambda w \qquad\qquad \begin{bmatrix} 0 & C \end{bmatrix} w = 0.$$

This implies $Sw_1 = \lambda w_1$ and

$$\begin{array}{rcl} BLw_1 + Aw_2 & = & \Pi S w_1 - A\Pi w_1 + Aw_2 = \lambda \Pi w_1 - A\Pi w_1 + Aw_2 \\ & = & \lambda w_2, \end{array}$$

hence $(A - \lambda I)(w_2 - \Pi w_1) = 0$. As a result, since λ is an eigenvalue of S and S and A do not have common eigenvalues $w_2 = \Pi w_1$, which implies $C\Pi w_1 = 0$, hence system (2.43) is not observable, which is a contradiction that proves the claim.

(Z1)$\Leftrightarrow$(Z2). To prove the claim we simply show that the zeros of the two systems coincide. For, note that

$$\begin{bmatrix} sI - (S - GL) & G \\ C\Pi & 0 \end{bmatrix} = \begin{bmatrix} sI - S & G \\ C\Pi & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ L & 1 \end{bmatrix},$$

which proves the claim.

(Z2)$\Rightarrow$(O1). The zeros of system (2.45) are the roots of the polynomial

$$\det \begin{bmatrix} sI - S & G \\ C\Pi & 0 \end{bmatrix}.$$

Observability of system (2.43) implies that

$$\text{rank} \begin{bmatrix} sI - S \\ C\Pi \end{bmatrix} = \nu,$$

for all $s \in \mathbb{C}$. Let $z_i(s)$ be the determinant of the minor of order ν obtained eliminating from the observability pencil the i-th row of the matrix $sI - S$. By observability, the ν polynomials $z_1(s), \cdots, z_\nu(s)$ do not have common roots, *i.e.* are independent polynomials of degree $\nu - 1$.

This implies that such polynomials form a basis for the space of polynomials of degree $\nu - 1$. Note now that (setting $G = [G_1 \; G_2 \; \cdots \; G_\nu]^\top$)

$$\begin{array}{rcl} Z(s) & = & \det \left[\begin{array}{cc} sI - S & G \\ C\Pi & 0 \end{array} \right] \\ & = & (-1)^\nu \left(G_1 z_1(s) - \cdots + (-1)^{\nu-1} G_\nu z_\nu(s) \right), \end{array} \tag{2.47}$$

which implies that it is possible to arbitrarily assign the polynomial which defines the zeros of the system.

(O1)$\Rightarrow$(Z2). If the zeros of system (2.45) can be arbitrarily assigned, then the polynomial $Z(s)$ in equation (2.47) can be arbitrarily assigned, and this implies that the polynomials $z_i(s)$ are independent, hence observability of system (2.43). $\square$

Remark 2.5. If system (2.43) or system (2.44) are not observable, it may still be possible to assign the relative degree of the model for some $r \in [1, \nu]$. Note that if it is possible to assign a relative degree $\tilde{r}$, then it is possible to assign any relative degree $r \in [1, \tilde{r}]$.

The matrix G assigning the relative degree to system (2.36) is not unique. This degree of freedom may be exploited to partly assign the eigenvalues or the zeros of the reduced order model. We illustrate this idea in the following examples.

Example 2.4. Consider a reduced order model described by the equations (2.36), with

$$S = \left[\begin{array}{cc} \alpha & \bar{\omega} \\ -\bar{\omega} & \alpha \end{array} \right],$$

$L = \left[\begin{array}{cc} 1 & 0 \end{array} \right]$ and $C\Pi = \left[\begin{array}{cc} \eta_{0s} & \eta_{0c} \end{array} \right]$, with $\bar{\omega} > 0$, $\alpha \in \mathbb{R}$ and $\eta_{0s}^2 + \eta_{0c}^2 \neq 0$. The reduced order model (2.36) obtained with these matrix selection matches the 0-moment at $\alpha \pm \iota\bar{\omega}$.

All matrices G such that the reduced order model has relative degree one are described by

$$G = \tilde{\gamma} \left[\begin{array}{c} \eta_{0s} \\ \eta_{0c} \end{array} \right] + \delta \left[\begin{array}{c} -\eta_{0c} \\ \eta_{0s} \end{array} \right],$$

with $\tilde{\gamma} \neq 0$ and $\delta \in \mathbb{R}$. Note that such G does not depend explicitly on the values of α and $\bar{\omega}$ (although η_{0c} and η_{0s} depends on the values of α and $\bar{\omega}$). The transfer function $W(s)$ of the reduced order model is given by

$$W(s) = \frac{(\tilde{\gamma}s - \alpha\tilde{\gamma} + \bar{\omega}\delta)(\eta_{0s}^2 + \eta_{0c}^2)}{s^2 + (\tilde{\gamma}\eta_{0s} - \delta\eta_{0c} - 2\alpha)s + \alpha(\alpha + \tilde{\gamma}\eta_{0s} - \delta\eta_{0c}) + \bar{\omega}(\bar{\omega} + \delta\eta_{0s} + \tilde{\gamma}\eta_{0c})}, \tag{2.48}$$

and the parameters δ and $\tilde{\gamma}$ can be used to assign its poles, or the zero and the DC-gain, or to obtain a reduced order model which is asymptotically stable and minimum phase, *i.e.* it is passive.

All matrices G such that the reduced order model has relative degree two are described by

$$G = \frac{\tilde{\gamma}}{\bar{\omega}} \begin{bmatrix} -\eta_{0c} \\ \eta_{0s} \end{bmatrix},$$

with $\tilde{\gamma} \neq 0$. Note that such G does not depend explicitly on the value of α. The transfer function $W(s)$ of the reduced order model is given by

$$W(s) = \frac{\tilde{\gamma}(\eta_{0s}^2 + \eta_{0c}^2)}{s^2 - (2\alpha + \eta_{0c}\frac{\tilde{\gamma}}{\bar{\omega}})s + \alpha^2 - \frac{\tilde{\gamma}}{\bar{\omega}}\alpha\eta_{0c} + \bar{\omega}^2 + \tilde{\gamma}\eta_{0s}}, \tag{2.49}$$

and the parameter $\tilde{\gamma}$ can be selected to ensure asymptotic stability of the reduced order model provided $\eta_{0c} \neq 0$. It is worth noting that, consistently with the theory, the value of the transfer functions in equation (2.48) and (2.49) for $s = \alpha \pm \iota\bar{\omega}$ depends only upon η_{0s} and η_{0c}.

Example 2.5. Consider a reduced order model described by the equations (2.36), with S, L and $C\Pi$ as in equations (2.39) and (2.40). Let $\eta_0 \neq 0$. By direct inversion of (2.42), all matrices G such that the reduced order model has relative degree ν are described by

$$G = \begin{bmatrix} \eta_0 & \eta_1 & \cdots & \eta_{k-1} \\ 0 & \eta_0 & \cdots & \eta_{k-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \eta_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma \end{bmatrix},$$

with $\gamma \neq 0$. For instance, for $\nu = 4$,

$$G = \tilde{\gamma} \begin{bmatrix} \frac{\eta_1^3 - 2\eta_0\eta_1\eta_2 + \eta_0^2\eta_3}{\eta_0^4} & \frac{\eta_1^2 - \eta_0\eta_2}{\eta_0^3} & -\frac{\eta_1}{\eta_0^2} & \frac{1}{\eta_0} \end{bmatrix}^{\top},$$

with $\tilde{\gamma} \neq 0$. The transfer function of the reduced order model is given by

$$W(s) = \frac{\tilde{\gamma}\eta_0^4}{\eta_0^4 s^4 + (\eta_1^3 - 2\eta_0\eta_1\eta_2 + \eta_0^2\eta_3)\tilde{\gamma}s^3 + (\eta_1^2 - \eta_0\eta_2)\eta_1\tilde{\gamma}s^2 - \eta_0\eta_1^2\tilde{\gamma}s + \eta_0^3\tilde{\gamma}} \tag{2.50}$$

and the parameter $\tilde{\gamma}$ can be used to assign, for example, the high-frequency gain.

Example 2.6. Consider a reduced order model described by the equations (2.36), with

$$S = \begin{bmatrix} 0 & \bar{\omega} & 0 \\ -\bar{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$L = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ and $C\Pi = \begin{bmatrix} \eta_{0s} & \eta_{0c} & \eta_0 \end{bmatrix}$. We assume that $\eta_0 \neq 0$ and $\eta_{0s}^2 + \eta_{0c}^2 \neq 0$ which imply that the pair $(S, C\Pi)$ is observable. Hence, the reduced order model matches the 0-moment at $\pm\bar{\omega}\iota$ and at zero and it has relative degree 3 for all matrices G described by

$$G = \begin{bmatrix} \dfrac{\gamma\eta_{0s}}{\bar{\omega}^2(\eta_{0s}^2 + \eta_{0c}^2)} & \dfrac{\gamma\eta_{0c}}{\bar{\omega}^2(\eta_{0s}^2 + \eta_{0c}^2)} & -\dfrac{\eta_{0s}^2 + \eta_{0c}^2}{\eta_0} \end{bmatrix}^\top,$$

with $\gamma \neq 0$. Since there exists such a G, by Theorem 2.14 all the zeros of the reduced order model can be arbitrarily assigned by means of (possibly another) G. For the sake of simplicity, let $\bar{\omega} = 1$. Then

$$Z(s) = \eta_0 G_3 + (\eta_{0s}G_2 - \eta_{0c}G_1)s + (\eta_{0s}G_1 + \eta_{0c}G_2 + \eta_0 G_3)s^2,$$

and this can be arbitrarily assigned provided $\eta_0(\eta_{0s}^2 + \eta_{0c}^2) \neq 0$. Finally note that this also implies, and is implied by, the observability of the system $\dot{\xi} = S\xi$, $\psi = C\Pi\xi$.

2.4 Additional Topics for Linear and Nonlinear Systems

In this section we present a series of topics regarding systems in special form. We begin introducing the notion of moment at infinity for linear and nonlinear systems. Then we investigate the problem of model reduction of a linear system driven by a nonlinear signal generator and of model reduction of a nonlinear system driven by a linear signal generator. We conclude the chapter clarifying the nature of the relation between moments and phasors introduced in Example 2.1.

2.4.1 Moment at infinity

The descriptions of moment given in Definitions 2.13 and 2.14 do not characterize the moments at $s_i = +\infty$. Since the k moments of a linear system at s_i are defined as the first k coefficients of the Laurent series expansion of the transfer function $W(s)$ at $s_i \in \mathbb{C}$, in a similar way the k moments at infinity can be computed by evaluating the expansion at infinity of the transfer function. In addition, by using the final value theorem, the 0-, 1-, $\ldots$, k-moments correspond to the first, second, ..., $k+1$-th coefficients of the expansion at $t = 0^+$ of the impulse response. Thus, for the linear system (2.10) the k moments at infinity are defined as $\eta_k(\infty) = CA^{k-1}B$ (with $\eta_0(\infty) = 0$), *i.e.* the first $k+1$ moments at infinity coincide with the first $k+1$ Markov parameters of the system. Recall also that

$$CA^kB = \frac{d^k}{dt^k}(Ce^{At}B)\bigg|_{t=0} = y_I^{(k)}(0^+), = y_{F,B}^{(k)}(0^+),$$

where $y_I^{(k)}$ denotes k-th order time derivative of the impulse response of the system and $y_{F,B}^{(k)}$ denotes the k-th order time derivative of the free output response from $x(0) = B$.

Consider now a nonlinear affine system[2] described by equations of the form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x), \tag{2.51}$$

[2] We focus on affine systems since for non-affine systems the impulse response, and its derivatives, may not be well-defined functions (*e.g.* they may be distributions). To illustrate this consider the system $\dot{x} = u^2$, $y = x$, with $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$. Setting $u(t) = \delta_0(t)$ and integrating yields

$$y_I(t) = \int_0^t \delta_0^2(\tau)d\tau,$$

which does not exists.

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and f, g and h analytic mappings. Integrating the first of equations (2.51), with $x(0) = 0$ and $u(t) = \delta_0(t)$, evaluating for $t = 0^+$, and substituting in the second equation yields $y_I(0^+) = y_I^0(0^+) = h(g(0^+))$ and $y_I^{(k)}(0^+) = L_f^k h \circ g(0^+)$, for $k \geq 0$. It is therefore natural to define the k-moment at infinity as follows.

Definition 2.23. For $k > 0$, the *moment at infinity of the nonlinear system (2.51)* is $\eta_k(\infty) = y_I^{(k-1)}(0^+) = y_{F,g(0)}^{(k-1)}(0^+)$ (with $\eta_0(\infty) = 0$), where $y_{F,g(0)}^{(k)}(t)$ denotes the k-th order time derivative of the free output response of the system from $x(0) = g(0)$.

These considerations allow to derive a reduced order model which matches the 0-, $\cdots$, k-moments at infinity of system (2.51). For, consider a linear system described by equations of the form (2.10), with $\xi(t) \in \mathbb{R}^k$, and F, G and H such that $HF^iG = y_I^{(i)}(0)$, for $i = 0, \cdots, k-1$ (note that the matrices F, G and H can be computed using standard realization algorithms, *e.g.* the Ho-Kalman realization algorithm. In addition, these matrices are not uniquely defined, hence it is possible to achieve additional properties as, for example, to assign the eigenvalues of the reduced order model). As a consequence of the discussion above, the linear system thus constructed is a model of the nonlinear system achieving moment matching at infinity. Note that the computation of such a reduced order model does not require the solution of any partial differential equation, but simply regularity of the nonlinear system. The above discussion is illustrated in the next example.

Example 2.7. Consider the model of an inverted pendulum on a cart, described by the equations

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{mlx_4^2 \sin x_3 - mg \cos x_3 \sin x_3 + u}{M + m \sin^2 x_3}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= \frac{g(m+M) \sin x_3 - ml \cos x_3 \sin x_3 x_4^2 - \cos x_3 u}{l(M + m \sin^2 x_3)}, \\
y &= x_1,
\end{aligned}
$$

where $x_1(t)$ is the position of the cart, $x_3(t)$ is the angle of the pendulum (with respect to an upward vertical), $x_2(t)$ and $x_4(t)$ are the corresponding, linear and angular, velocities, $u(t)$ is an external input force, and m, M, l and g are positive constants. The first six moments of this system at infinity can be directly calculated, using the expression above, yielding $\eta_0(\infty) = 0$, $\eta_1(\infty) = 0$, $\eta_2(\infty) = \frac{1}{M}$, $\eta_3(\infty) = 0$, $\eta_4(\infty) = \frac{m(glM^2-1)}{l^2M^4}$, $\eta_5(\infty) = 0$. From these expressions it is possible to derive a reduced order (linear) model. For example, a reduced order model, matching the first three moments, is described by equations of the form (2.32), with $\xi(t) \in \mathbb{R}^2$, and

$$F = \begin{bmatrix} 0 & 1 \\ -f_{21} & -f_{22} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} \dfrac{1}{M} & 0 \end{bmatrix}.$$

The coefficients f_{21} and f_{22} can be selected to assign the eigenvalues of the matrix F, or to achieve other interpolation conditions. For example, selecting $f_{21} = -m\frac{glM^2-1}{M^3l^2}$ and $f_{22} = 0$ ensures matching of the first five moments at infinity. A four-dimensional reduced order (linear) model achieving matching of the first five moments at infinity, and with the same eigenvalues of the linearization of the nonlinear system around $x = 0$, *i.e.* 0, 0, $\frac{\sqrt{(M+m)Mlg}}{Ml}$, $-\frac{\sqrt{(M+m)Mlg}}{Ml}$, is described by equations of the form (2.32), with $\xi(t) \in \mathbb{R}^4$, and

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g\dfrac{M+m}{Ml} & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} -\dfrac{m+glM^3}{l^2M^4} & 0 & \dfrac{1}{M} & 0 \end{bmatrix}.$$

One may wonder if the obtained reduced order model is equivalent to the linearized system around zero. Since the transfer function of this system is

$$H(sI-F)^{-1}G = \frac{ls^2 - g - \frac{m}{M^3l}}{s^2(Mls^2 - g(M+m))},$$

this last reduced order model does not coincide with the linearized model around the zero equilibrium which is given by

$$A_\ell = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g\dfrac{M+m}{Ml} & 0 \end{bmatrix} \qquad B_\ell = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix}$$

$$C_\ell = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

and has transfer function

$$C_\ell(sI - A_\ell)^{-1}B_\ell = \frac{ls^2 - g}{s^2(Mls^2 - g(M+m))}.$$

2.4.2 Matching for linear systems at (s, l)

In this section we consider the problem of model reduction for linear systems at (s, l), *i.e.* we consider the case in which the system to be reduced is linear whereas the signal generator is a nonlinear system. For such a problem, under suitable assumptions, it is possible to obtain in an explicit way a formal description of reduced order models.

Proposition 2.5. Consider the linear system (2.10) and the signal generator (2.7), with $l(\omega) = L\omega$. Suppose Assumption 2.2 holds and that s and π can be expressed, locally around $\omega = 0$, as formal power series, *i.e.* $s(\omega) = \sum_{i=1}^{\infty} s^{[i]}(\omega)$ and $\pi(\omega) = \sum_{i=1}^{\infty} \pi^{[i]}(\omega)$, where $s^{[i]}$ and $\pi^{[i]}$ denote polynomial vector fields which are homogeneous of degree i. Suppose in addition that $s^{[1]}(\omega) = 0$. Then a family of reduced order models achieving moment matching at (s, l) is formally described by the equations

$$\dot{\xi} = s(\xi) - \delta(\xi)L\xi + \delta(\xi)u, \qquad \psi = C\pi(\xi),$$

with δ a free mapping,

$$\pi^{[1]}(\xi) = -A^{-1}BL\xi \tag{2.52}$$

and

$$\pi^{[k]}(\xi) = A^{-1}\sum_{i=1}^{k-1} \frac{\partial \pi^{[i]}(\xi)}{\partial \xi} s^{[k-i+1]}(\xi), \tag{2.53}$$

for $k \geq 2$.

Proof. Note that equation (2.9) can be written as $A\pi(\omega) + BL\omega = \frac{\partial \pi}{\partial \omega} s(\omega)$. In addition, by the stated assumptions, this equation has a unique solution locally around $\omega = 0$. Then, by a direct computation, we conclude that the solution π of this equation admits the formal power series description given by equations (2.52) and (2.53). The result then follows from Proposition 2.3. $\square$

The next example illustrates this result.

Example 2.8. Consider the linear system (2.10) with $\sigma(A) \subset \mathbb{C}_{\leq 0}$ and the nonlinear signal generator (2.7) with $\omega = [\omega_1, \omega_2, \omega_3]^\top$,

$$s(\omega) = \begin{bmatrix} \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \\ \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \end{bmatrix}, \tag{2.54}$$

$I_1 > 0$, $I_2 > 0$, $I_3 > 0$, $I_i \neq I_j$, for $i \neq j$, and

$$l(\omega) = L\omega = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \omega, \tag{2.55}$$

with $L_1 L_2 L_3 \neq 0$. This signal generator, which describes the evolution of the angular velocities of a free rigid body in space, is neutrally stable and, under the stated assumption on L, observable. Moreover, $s^{[2]} = \dot{\omega}$, $s^{[j]} = 0$ for all $j \in \mathbb{N} \setminus \{2\}$ and $\frac{\partial s^{[j]}}{\partial \omega} s^{[2]} = \frac{d^j \omega}{dt^j}$ for all $j \geq 1$. Hence,

$$\pi^{[i]}(\omega) = -A^{-i} BL \frac{d^{i-1}\omega}{dt^{i-1}},$$

and the moment of system (2.10) at (s, l) is given, formally, by

$$C\pi(\omega) = -CA^{-1}\left(BL\omega + A^{-1}BL\dot{\omega} + \cdots + A^{-i+1}BL\frac{d^{i-1}\omega}{dt^{i-1}} + \cdots \right),$$

which is a polynomial series in ω. Note that a recursive expression for the $(i+1)$-th derivative of ω is given by

$$\frac{d^{i+1}\omega}{dt^{i+1}} = \left[\begin{array}{c} \dfrac{I_2 - I_3}{I_1} \displaystyle\sum_{k=0}^{i} \left(\begin{array}{c} n \\ i \end{array} \right) \frac{d^{i-k}\omega_2}{dt^{i-k}} \frac{d^k\omega_3}{dt^k} \\ \dfrac{I_3 - I_1}{I_2} \displaystyle\sum_{k=0}^{i} \left(\begin{array}{c} n \\ i \end{array} \right) \frac{d^{i-k}\omega_3}{dt^{i-k}} \frac{d^k\omega_1}{dt^k} \\ \dfrac{I_1 - I_2}{I_3} \displaystyle\sum_{k=0}^{i} \left(\begin{array}{c} n \\ i \end{array} \right) \frac{d^{i-k}\omega_1}{dt^{i-k}} \frac{d^k\omega_2}{dt^k} \end{array} \right].$$

We infer that a reduced order model for a linear asymptotically stable system at the (s, l) given in equations (2.54) and (2.55) is described by

$$\begin{array}{rcl} \dot{\xi} & = & s(\xi) - \delta(\xi)L\xi + \delta(\xi)u, \\ \psi & = & -CA^{-1} \left[BL\omega + A^{-1}BL\dot{\omega} + \cdots + A^{-i+1}BL\dfrac{d^{i-1}\omega}{dt^{i-1}} + \cdots \right]_{\omega=\xi}. \end{array}$$

Simulations have been run selecting $I_1 = 1$, $I_2 = 2$, $I_3 = 3$, $L_1 = 1$, $L_2 = 1/2$, $L_3 = 1/3$. Note that $s(\xi) - \delta(\xi)L\xi$ is independent of the system to be reduced and that the selection $\delta = \mathrm{diag}(1/I_1, 1/I_2, 1/I_3)L^\top$ yields a reduced order model with a globally asymptotically stable equilibrium at $\xi = 0$. The linear system to be reduced is a randomly selected asymptotically stable system of dimension 50. The initial conditions of the signal generator have been selected as $\omega(0) = \frac{1}{5}\,[1\ 1\ 1]^\top$.

The linear system and the reduced order model, both driven by the signal generator, have been numerically integrated from zero initial conditions. Fig. 2.3 (top) displays the output $y(t)$ (solid line) of the linear system when driven by the signal generator, and the signals $\psi^{[1]}$ (dotted line), $\psi^{[2]}$ (dash-dotted line) and $\psi^{[6]}$ (dashed line), obtained by truncating the formal power series defining ψ to the first, second and sixth order terms, respectively. Fig. 2.3 (bottom) displays the approximation errors $y - \psi^{[1]}$ (dotted line), $y - \psi^{[2]}$ (dash-dotted line), $y - \psi^{[6]}$ (dashed line). Note that, in steady-state,

$$\begin{array}{c} \max(|y(t) - \psi^{[1]}(t)|) \approx 0.0630 > \\ \max(|y(t) - \psi^{[2]}(t)|) \approx 0.0272 > \\ \max(|y(t) - \psi^{[3]}(t)|) \approx 0.0126 > \\ \max(|y(t) - \psi^{[4]}(t)|) \approx 0.0091 > \\ \max(|y(t) - \psi^{[5]}(t)|) \approx 0.0058 > \\ \max(|y(t) - \psi^{[6]}(t)|) \approx 0.0056, \end{array}$$

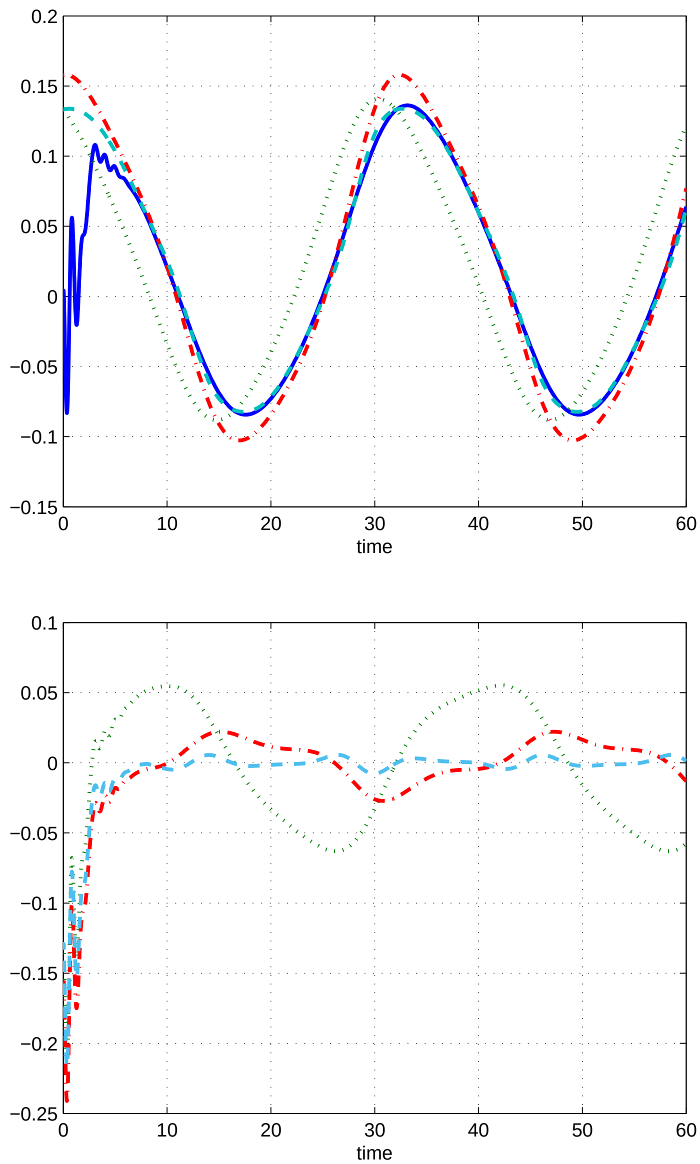

Figure 2.3: Top: time histories of the output of the driven linear system and of the approximating outputs of the driven reduced order model: y (solid line), $\psi^{[1]}$ (dotted line), $\psi^{[2]}$ (dash-dotted line) and $\psi^{[6]}$ (dashed line). Bottom: time histories of the approximation errors: $y-\psi^{[1]}$ (dotted line), $y-\psi^{[2]}$ (dash-dotted line), $y-\psi^{[6]}$ (dashed line).

which shows that the approximation error decreases by adding terms in the formal power series defining the output of the reduced order model.

2.4.3 Matching for nonlinear systems at $S\omega$

In this section we consider the problem of model reduction for nonlinear systems at $s(\omega) = S\omega$, *i.e.* we consider the case in which the signal generator is a linear system. This problem is of particular interest since it is easy to infer that the reduced order models have a very simple description, *i.e.* a family of reduced order models is given by the equations

$$\dot{\xi} = (S - \delta(\xi)L)\xi + \delta(\xi)u, \qquad \psi = h(\pi(\omega)),$$

where δ is a free mapping. In particular, selecting $\delta = G$, for some constant matrix G, we have that the family of reduced order models is described by a linear differential equation with a nonlinear output map. This structure has two main advantages. The former is that the matrix G can be selected to achieve additional goals, such as to assign the eigenvalues or the relative degree of the reduced order model (provided additional assumptions on the output map holds). The latter is that the computation of the reduced order model boils down to the computation of the output map $h \circ \pi$. For instance, this can be approximated with the method presented in Chapter 4.

We consider now the special case of model reduction with 0-moment matching at $s^\star = 0$, *i.e.* the model reduction problem at $s(\omega) = 0$. This problem can be solved, under specific assumptions, without solving any partial differential equation, as detailed in the following statement.

Proposition 2.6. Consider system (2.6) and the signal generator $\dot{\omega} = 0$, $u = \omega$. Suppose Assumption 2.1 holds. Then the zero moment at $s^\star = 0$ of system (2.6) is (locally) well-defined and given by $h \circ \pi$, with π the unique solution of the algebraic equation $f(\pi(\omega), \omega) = 0$. Finally, a reduced order model, for which the zero equilibrium is locally asymptotically stable is given by

$$\dot{\xi} = -\delta(\xi)(\xi - u), \qquad \psi = h(\pi(\xi)),$$

with $\xi(t) \in \mathbb{R}$ and δ such that $\delta(0) > 0$.

Proof. We simply need to show that equation (2.18) has a unique solution. For, note that in this case, equation (2.18) rewrites as $-\delta(p(\omega))\,(p(\omega)-\omega)) = 0$, which, by positivity of $\delta(0)$, has (locally) the unique solution $p(\omega) = \omega$. □

This result is illustrated in the next section in which a reduced order model for a DC-to-DC converter is determined.

Averaged model of the DC-to-DC Ćuk converter

The averaged model of the DC-to-DC Ćuk converter is given by the equations

$$\begin{aligned} L_1 \frac{d}{dt} i_1 &= -(1-u)v_2 + E_1, \\ C_2 \frac{d}{dt} v_2 &= (1-u)i_1 + u i_3, \\ L_3 \frac{d}{dt} i_3 &= -u v_2 - v_4, \\ C_4 \frac{d}{dt} v_4 &= i_3 - G_4 v_4, \\ y &= v_4, \end{aligned} \tag{2.56}$$

where $i_1(t) \in \mathbb{R}_{\geq 0}$ and $i_3(t) \in \mathbb{R}_{\leq 0}$ describe currents, $v_2(t) \in \mathbb{R}_{\geq 0}$ and $v_4(t) \in \mathbb{R}_{\leq 0}$ voltages, L_1, C_2, L_3, E_1 and G_4 positive parameters and $u(t) \in (0,1)$ a continuous control signal which represents the slew rate of a pulse width modulation circuit used to control the switch position in the converter. The 0-moment of the system at $s = 0$ is

$$h(\pi(\omega)) = \frac{\omega}{\omega - 1} E_1, \tag{2.57}$$

and a locally asymptotically stable reduced order model achieving moment matching at $s^\star = 0$ is

$$\dot{\xi} = -\delta(\xi)(\xi - u), \qquad \psi = E_1 \frac{\xi}{\xi - 1}, \tag{2.58}$$

with $\delta(0) > 0$, which is well-defined if $\xi \neq 1$. This is consistent with the fact that the 0-moment at $s^\star = 0$ is defined for $\omega \neq 1$.

Table 2.1: Parameters of the model of the DC-to-DC Ćuk converter.

$L_1 = 10\ mH$	$L_3 = 10\ mH$	$C_2 = 22\ \mu F$
$C_4 = 22\ \mu F$	$E = 12\ V$	$G = 0.0447\ S$

The parameters of the simulations are reported in Table 2.1. The input signal is selected piecewise constant with jumps every 0.05 seconds. The reduced order model is described by equations (2.58), where the function δ depends upon the input signal u and it is equal to the real part of the slowest eigenvalue of the system (2.56) (which is a linear system for constant u).

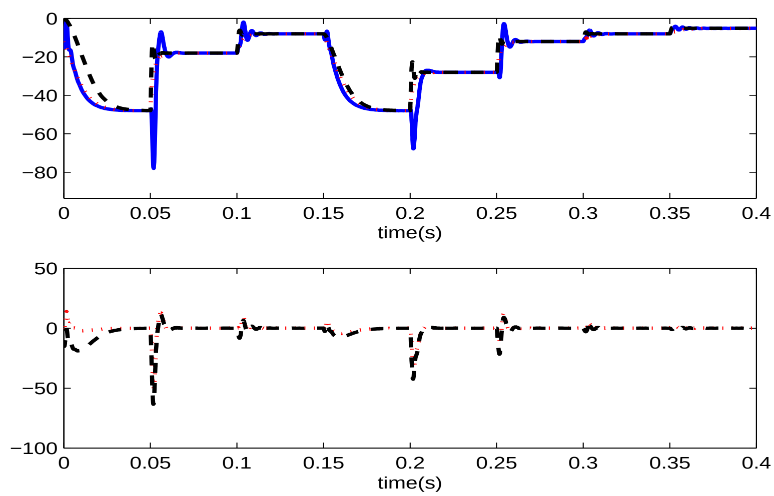

Figure 2.4: Time histories of the output of the averaged model of the Ćuk converter and of the *approximating* outputs of the reduced order models (top): y (solid line), ψ (dotted line), ψ_2 (dashed line). Time histories of the approximation errors (bottom): $y - \psi$ (dotted line), $y - \psi_2$ (dashed line).

Fig. 2.4 (top) displays the output y (solid line) of the averaged model of the Ćuk converter and of the output of the reduced order model ψ (dotted line). The figure shows that the reduced order model provides a good *static* approximation of the behavior of the system but does not capture its dynamic (under-damped) behavior. The dynamic behavior can be captured constructing a second order model, which is (in the spirit of the model (2.58)) a linear system with a nonlinear output map. Since such a model is required to match only one moment, it is possible to assign its eigenvalues at the location of the dominant modes of system (2.56) with u fixed. The output ψ_2 of this two dimensional reduced order model is also displayed in Fig. 2.4 (dashed line). Note that this signal may provide a better approximation of y, as shown in Fig. 2.4 (bottom), in which the errors $y-\psi$ (dotted line) and $y-\psi_2$ (dashed line) have been plotted.

2.4.4 Phasors as moments at $\iota\hat{\omega}$

We now formally clarify the relation between moments and phasors which has been hinted in Example 2.1, in particular, we show that the phasors of an electric circuit are the moments at $\iota\hat{\omega}$ of the linear system describing the circuit. Note that the discussion is presented applying the Kirchhoff's Voltage Law: we assume that the sources are voltage sources and the state variables and the output are currents. As a consequence we show that the moments are the phasors of the currents. An equivalent analysis based on the Kirchhoff's Current Law can be derived. We begin introducing two preliminary definitions.

Definition 2.24. Consider system[3] (2.10) with $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}$, $y(t) \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$ and $C \in \mathbb{C}^{1\times n}$. The *phasor transform of the linear system (2.10)* for the source $u(t) = a_u e^{\iota(\hat{\omega}t+\varphi)}$, with $a_u \in \mathbb{R}$, $\hat{\omega} \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, is

$$\overline{X}\iota\hat{\omega}e^{\iota\hat{\omega}t} = A\overline{X}e^{\iota\hat{\omega}t} + Ba_u e^{\iota\varphi}e^{\iota\hat{\omega}t}, \qquad \overline{Y}e^{\iota\hat{\omega}t} = C\overline{X}e^{\iota\hat{\omega}t}. \tag{2.59}$$

To streamline the presentation we introduce the following definition.

Definition 2.25. The system (2.10) and the generator (2.11) are said to be in the *real convention* if the matrices A, B, C, S and L have real entries. They are said to be in the *mixed convention* if the matrices A, B and C have real entries and the matrices S and L have complex

[3]While the state of a dynamical system normally represents real quantities, in this section we use the complex domain because the quantities involved in the phasor analysis are complex valued.

entries. They are said to be in the *complex convention* if the matrices A, B, C, S and L have complex entries.

Note that in the real convention and in the mixed convention, for all integers k with $2 \leq 2k \leq n$, the component x_{2k} of x is a current i_k, whereas the component x_{2k-1} of x is the integral $\int_{t_0}^{t} i_k(\tau)d\tau$. Furthermore, we assume that $C = \epsilon_k$ for some $1 \leq k \leq n$, *i.e.* the output of the system is a current or the integral of a current.

With the next result we show that writing the phasor transform of a linear electric circuit is equivalent to writing the associated Sylvester equation. Moreover, the components of the solution of this Sylvester equation written in the mixed convention are the phasors of all the currents (and of the integrals of the currents) in the circuit.

Proposition 2.7. Consider the source $u(t) = a_u e^{\iota(\hat{\omega}t+\varphi)}$, with $a_u \in \mathbb{R}$, $\hat{\omega} \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, and assume $\iota\hat{\omega} \notin \sigma(A)$. The phasor transform of system (2.10) (see Definition 2.24) written in the mixed convention coincides with the Sylvester equation (2.12) with $S = \iota\hat{\omega}$ and $L = a_u e^{\iota\varphi}$. The components of Π, which is the unique solution of equation (2.12), are the phasors of the currents and of the integrals of the currents in the circuit.

Proof. We first compute the phasor transform of system (2.10) for the source $u(t) = a_u e^{\iota(\hat{\omega}t+\varphi)}$, namely

$$\overline{X}\iota\hat{\omega}e^{\iota\hat{\omega}t} = A\overline{X}e^{\iota\hat{\omega}t} + Ba_u e^{\iota\varphi}e^{\iota\hat{\omega}t}, \qquad \overline{Y}e^{\iota\hat{\omega}t} = C\overline{X}e^{\iota\hat{\omega}t},$$

by Definition 2.24. Since $e^{\iota\hat{\omega}t} \neq 0$ for all $t \in \mathbb{R}$, it can be canceled out yielding

$$\overline{X}\iota\hat{\omega} = A\overline{X} + Ba_u e^{\iota\varphi}, \qquad \overline{Y} = C\overline{X}.$$

Thus the phasors of all the currents (and their integrals) in the circuit and the phasor of the output current are given by

$$\overline{X} = (\iota\hat{\omega}I - A)^{-1}Ba_u e^{\iota\varphi}, \qquad \overline{Y} = C(\iota\hat{\omega}I - A)^{-1}Ba_u e^{\iota\varphi},$$

respectively. Consider now the signal generator (2.11) with

$$S = \iota\hat{\omega}, \quad L = a_u e^{\iota\varphi}.$$

The associated Sylvester equation (2.12) is

$$A\Pi + Ba_u e^{\iota\varphi} = \Pi\iota\hat{\omega},$$

which, if $\iota\hat{\omega} \notin \sigma(A)$, has the unique solution

$$\Pi = (\iota\hat{\omega}I - A)^{-1}Ba_u e^{\iota\varphi},$$

which concludes the proof. □

Corollary 2.15. The phasor of the output response y of system (2.10) is the moment of the system at $\iota\hat{\omega}$, namely $\overline{Y} = C\Pi$. The *inverse phasor transform* of the output current y of system (2.10) is

$$y(t) = \Re\left[C\Pi e^{St}\right].$$

Proof. The first claim follows noting that the phasor of the output response of system (2.10) is given by

$$\overline{Y} = C\Pi = C(\iota\hat{\omega}I - A)^{-1}Ba_u e^{\varphi}.$$

To prove the second claim we note that

$$\Re\left[C\Pi e^{St}\right] = \Re\left[\overline{Y}e^{\iota\hat{\omega}t}\right],$$

which is the inverse phasor transform of the output response of system (2.10). □

Remark 2.6. The higher order derivatives $\frac{d^n x}{dt^n}$ and the integral $\int_{t_0}^{t} x(\tau)d\tau$ are transformed in the phasor domain into $(\iota\hat{\omega})^n\overline{X}e^{\iota\hat{\omega}}$ and $\frac{1}{\iota\hat{\omega}}\overline{X}e^{\iota\hat{\omega}}$, respectively. Similarly they are transformed in the "moment domain" into $\Sigma^n\Pi e^{\iota\hat{\omega}}$ and $\Sigma^{-1}\Pi e^{\iota\hat{\omega}}$, respectively.

Moments of a wireless power transfer system

We now provide a worked out example which illustrates the results of the previous section. Fig. 2.5 illustrates a wireless power transfer system consisting of two coils. We assume that a sinusoidal voltage source with an amplitude of V_s and an angular frequency of $\hat{\omega}$ is applied to the transmitter coil on the input side. A load resistor R_L is connected

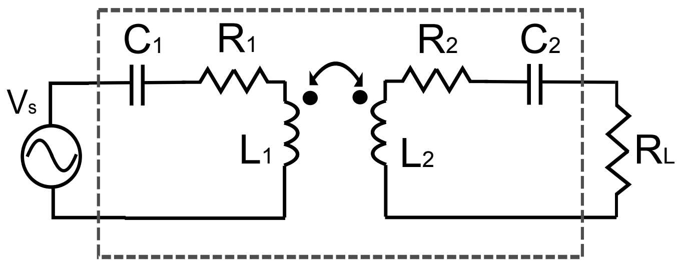

Figure 2.5: Equivalent circuit of a wireless power transfer system.

to the receiving coil on the output side. By applying the Kirchhoff's Voltage Law to the two coils we obtain the equations

$$\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1} \int i_1 dt + M_{12} \frac{di_2}{dt} &= u(t), \\ M_{21} \frac{di_1}{dt} + R_{2L} i_2 + L_2 \frac{di_2}{dt} + \frac{1}{C_2} \int i_2 dt &= 0, \end{aligned} \tag{2.60}$$

where i_1 and i_2 are the currents flowing in the coils 1 and 2, respectively, R_1 and R_2 are the resistances, $R_{2L} = R_2 + R_L$, L_1 and L_2 are the self-inductances, C_1 and C_2 are the capacitances and $M_{12} = M_{21}$ are the mutual inductances between the two coils. We are interested in determining the amplitude and phase of the steady-state current in the receiving coil, *i.e.* the phasor $\overline{I}_2$.

We start by solving the problem with the phasor transform approach. Transforming the differential equations (2.60) we obtain the complex algebraic system

$$Z_1 \overline{I}_1 + \iota\hat{\omega} M_{12} \overline{I}_2 = V_s, \qquad \iota\hat{\omega} M_{21} \overline{I}_1 + Z_2 \overline{I}_2 = 0,$$

where $Z_1 = R_1 + \iota\hat{\omega} L_1 - \iota\frac{1}{\hat{\omega} C_1}$ and $Z_2 = R_{2L} + \iota\hat{\omega} L_2 - \iota\frac{1}{\hat{\omega} C_2}$. Solving with respect to $\overline{I}_2$ yields

$$\overline{I}_2 = \frac{-\iota\hat{\omega} M_{21}}{Z_1 Z_2 + \hat{\omega}^2 M_{21} M_{12}} V_s.$$

Now we compute the moment of the differential system (2.60) at (2.11). We begin using the "mixed convention" that, as highlighted later, is the most useful among the three representations. Thus, consider $L = V_s$ and $S = \iota\hat{\omega}$ and the state

$$x_1(t) = \int_{t_0}^{t} i_1(\tau)\, d\tau, \quad x_2(t) = i_1(t), \quad x_3(t) = \int_{t_0}^{t} i_2(\tau)\, d\tau, \quad x_4(t) = i_2(t).$$

System (2.60) can be represented by the first order system of differential equations (2.10) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{L_2}{C_1\hat{L}} & -\dfrac{R_1L_2}{\hat{L}} & \dfrac{M_{12}}{C_2\hat{L}} & \dfrac{M_{12}R_{2L}}{\hat{L}} \\ 0 & 0 & 0 & 1 \\ \dfrac{M_{21}}{C_1\hat{L}} & \dfrac{R_1M_{21}}{\hat{L}} & -\dfrac{L_1}{C_2\hat{L}} & -\dfrac{L_1R_{2L}}{\hat{L}} \end{bmatrix}, \quad \begin{array}{l} B = \begin{bmatrix} 0 & \dfrac{L_2}{\hat{L}} & 0 & -\dfrac{M_{21}}{\hat{L}} \end{bmatrix}^\top, \\ C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \end{array} \tag{2.61}$$

where $\hat{L} = L_1L_2 - M_{12}M_{21} \neq 0$ by definition of mutual inductance. The solution of the Sylvester equation (2.12) is given by

$$\Pi = (\iota\hat{\omega} - A)^{-1}BV_s,$$

and the moment of the system is given by

$$C\Pi = \epsilon_4\Pi = \frac{\iota\hat{\omega}^3M_{21}}{D\hat{L}}V_s,$$

with

$$D = -\frac{\hat{\omega}^2}{\hat{L}}\left(Z_1Z_2 + \hat{\omega}^2M_{12}M_{21}\right)$$

the determinant of the matrix $(\iota\hat{\omega}I - A)$. In a similar way we can show that

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{bmatrix}^\top = \begin{bmatrix} \dfrac{1}{\iota\hat{\omega}}\bar{I}_1 & \bar{I}_1 & \dfrac{1}{\iota\hat{\omega}}\bar{I}_2 & \bar{I}_2 \end{bmatrix}^\top.$$

It is interesting to explore which relation between phasors and moments holds when the "real convention" is used. Consider system (2.10) with the matrices given in (2.61) and the matrices of the signal generator (2.11) given by

$$L = \begin{bmatrix} V_s & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & \hat{\omega} \\ -\hat{\omega} & 0 \end{bmatrix}.$$

The input of the system is $u = V_s\cos(\hat{\omega}t)$ instead of $u = V_s\cos(\hat{\omega}t) + \iota V_s\sin(\hat{\omega}t)$. It comes with no surprises that the phasors are related to the solution of the Sylvester equation by the relations

$$\begin{bmatrix} \Pi_{11} + \imath\Pi_{12} \\ \Pi_{21} + \imath\Pi_{22} \\ \Pi_{31} + \imath\Pi_{32} \\ \Pi_{41} + \imath\Pi_{42} \end{bmatrix} = \begin{bmatrix} \frac{1}{\imath\hat{\omega}}\bar{I}_1 \\ \bar{I}_1 \\ \frac{1}{\imath\hat{\omega}}\bar{I}_2 \\ \bar{I}_2 \end{bmatrix}.$$

Finally, we investigate the use of the "complex convention". Consider the coordinates $x_1(t) = i_1(t)$, $x_2(t) = i_2(t)$ and the signal generator (2.11) with $L = V_s$ and $S = \imath\hat{\omega}$. System (2.60) can be represented by a system of integro-differential equations given by

$$\dot{x}_1 = \frac{1}{\hat{L}}\left(-R_1 L_2 x_1 - \frac{L_2}{C_1}\int x_1 dt + M_{12}R_{2L}x_2 + \frac{M_{12}}{C_2}\int x_2 dt + L_2 u\right),$$
$$\dot{x}_2 = \frac{1}{\hat{L}}\left(-R_{2L} L_1 x_2 - \frac{L_1}{C_2}\int x_2 dt + M_{21}R_1 x_1 + \frac{M_{21}}{C_1}\int x_1 dt - M_{21} u\right).$$

Exploiting Remark 2.6, we can write the Sylvester equation and find its solution, namely

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} \imath\hat{\omega}\hat{L} + R_1 L_2 + \frac{L_2}{\imath\hat{\omega}C_1} & -M_{12}R_{2L} - \frac{M_{12}}{\imath\hat{\omega}C_2} \\ -M_{21}R_1 - \frac{M_{21}}{\imath\hat{\omega}C_1} & \imath\hat{\omega}\hat{L} + L_1 R_{2L} + \frac{L_1}{\imath\hat{\omega}C_2} \end{bmatrix}^{-1} \begin{bmatrix} L_2 V_s \\ -M_{21}V_s \end{bmatrix}$$

showing easily that

$$\begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}^\top = \begin{bmatrix} \bar{I}_1 & \bar{I}_2 \end{bmatrix}^\top.$$

The relation between moments and phasors is further investigated in Chapter 5.

2.5 Bibliographical Notes

The concept of system used in this monograph and the terminology "representation in implicit and explicit form" is adapted from Zadeh and Desoer [1963] and Kalman et al. [1969]. The notion of steady-state response in terms of omega limit set of a set is defined in Isidori and

Byrnes [2008], see also Bhatia and Szegö [1970], Hale et al. [2002] and Sell and You [2002]. The characterization of steady-state in term of converging trajectories has been adapted from Isidori [1995, p.387]. Note that we have added the property of forward and backward boundedness to make it equivalent to Pavlov et al. [2006, Definition 2.14]. For a general introduction regarding the center manifold theory we refer the reader to Carr [1981] or to [Isidori, 1995, Appendix B]. A discussion on the uniqueness of the center manifold and the modification to make the center manifold unique can be found in Sijbrand [1985]. For more detail on differential geometry and manifolds, see Boothby [2003]. For other classical concepts regarding nonlinear systems, such as observability, relative degree, normal forms, see Isidori [1995] and Nijmeijer and van der Schaft [1990]. The problem of determining a minimal realization of a nonlinear system is discussed concisely in Sussmann [1973] (see also the references therein). Techniques to approximate the mapping π have been presented in *e.g.* Krener [1992] and [Huang, 2004, Section 4.2 and 4.3].

The relation between moments and the steady-state response of the interconnection of the system to be reduced and a signal generator has been recognized in Astolfi [2007b]. Based on this relation, the nonlinear enhancement of moment and the concepts of nonlinear moment matching have been introduced in Astolfi [2007b, 2008]. These results are further investigated, extended and presented in an organized way in Astolfi [2010] upon which a large part of this chapter is based. The results for linear systems can be extended to multi-input, multi-output systems following the discussion given in Ionescu and Astolfi [2013], which is based on the theory of tangential interpolation presented in Gallivan et al. [2004b]. Other results for linear and nonlinear systems, such as the minimality analysis of the reduced order models (see Theorem 2.13) and connections with the interpolation theory, can be found in Ionescu et al. [2014] and Ionescu and Astolfi [2016]. For an extension to differential-algebraic equations see Scarciotti [2015a, 2016, 2018]. Note that the relation between moments and the solution of a Sylvester equation has been already pointed out in Gallivan et al. [2004a, 2006]. The excitation role of the initial condition $\omega(0)$ has been first recognized in Scarciotti and Astolfi [2015a,b] and formalized, for linear and nonlinear systems, in Padoan et al. [2016a,b, 2017]. Efficient algorithms for the determination of reduced order models have been proposed in Feldmann and Freund [1995], Grimme et al. [1995], Jaimoukha and Kasenally [1997] and are thoroughly described in Antoulas [2005]. The iterative rational Krylov algorithm, being able to determine optimal reduced order models without computing the moments, has been proposed in Gugercin et al. [2008]. The Georgiou-Kimura parametrization has been presented in the works of Georgiou [1983] and Kimura [1983, 1986], whereas a description of the classical Nevanlinna-Pick interpolation problem can be found in Doyle et al. [1992]. The problem of moment matching with prescribed eigenvalues has been solved, using Krylov-type projection tools, in Antoulas [2009], whereas model reduction by moment matching and with a stability constraint has been discussed in Gugercin and Willcox [2008]. For additional references on these topics we direct the reader to Section 1.4.

Some of the examples in this chapter have various sources. More detail on the Euler equations can be found in Nijmeijer and van der Schaft [1990], Astolfi [1999] and Aeyels and Szafranski [1988], whereas some additional detail on the Ćuk converter can be found in Rodriguez et al. [2005]. A textbook presentation of the phasor analysis is given, for instance, in Davis [1998] and Nilsson and Riedel [2008]. Finally, the relation between moments and phasor has been characterized in Scarciotti and Astolfi [2016c,d].

3

Model Reduction of Neutral Systems with Discrete and Distributed Delays

In this chapter we extend the theory of model reduction by moment matching to nonlinear time-delay systems. This extension is based on the fact that the conditions and properties of the center manifold hold for time-delay systems as for finite dimensional systems. Similarly to the previous chapter we dedicate considerable attention to the linear case for which stronger results can be obtained. This chapter is organized to parallel the development of Chapter 2: the results are presented in the same order, minimizing the differences in the formulation to facilitate understanding of this more advanced material.

3.1 The Notion of Moment for Nonlinear Time-Delay Systems

Time-delay systems are a class of infinite dimensional systems which have been extensively studied in the literature. From a practical point of view every dynamical system presents delays of some extent. Thus, many examples of time-delay systems arise from biology, chemistry, physics and engineering. The interesting and important aspect of time-delay systems is that delays in closed-loop systems can generate unexpected behaviors: for instance "small" delays may be destabilizing, while "large" delays may be stabilizing.

The problem of model reduction of time-delay systems is a classic topic in control theory. The problem has been tackled in different ways. The majority of the results concern linear time-delay systems and are based on rational interpolations and operators.

In this section we derive an extension of the notion of moment for nonlinear differential time-delay systems. To keep the notation simple we consider, without loss of generality, only delays in the state and in the input, *i.e.* the output is delay-free.

Consider a nonlinear, single-input, single-output, continuous-time, time-delay system described by the equations

$$\begin{aligned} \dot{x} &= f(x_{\tau_0}, \dots, x_{\tau_\varsigma}, u_{\tau_{\varsigma+1}}, \dots, u_{\tau_\mu}), \qquad & y &= h(x), \\ x(\theta) &= \beta(\theta), & -T &\leq \theta \leq 0, \end{aligned} \tag{3.1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\beta \in \mathfrak{R}^n_T$, $\tau_0 = 0$, $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, \dots, \mu$ and f and h analytic mappings. We recall from the notation that x_τ indicates the delayed quantity $x(t-\tau)$ and that in this case the quantity τ is called "discrete delay". Note that the results presented in the following hold also for the so-called "distributed delay" $x_\tau = \int_{t-\tau}^{t} x(\bar{\tau}) d\bar{\tau}$ and for this reason in this section we abuse the notation and we do not specify if x_τ indicates a discrete or distributed delay. Consider, now, a signal generator described by the equations

$$\dot{\omega} = s(\omega), \qquad u = l(\omega), \tag{3.2}$$

with $\omega(t) \in \mathbb{R}^\nu$, s and l analytic mappings, and the interconnected system

$$\begin{aligned} \dot{\omega} &= s(\omega), \\ \dot{x} &= f(x_{\tau_0}, \dots, x_{\tau_\varsigma}, l(\omega_{\tau_{\varsigma+1}}), \dots, l(\omega_{\tau_\mu})), \\ y &= h(x). \end{aligned} \tag{3.3}$$

Suppose that $f(0, \dots, 0, 0, \dots, 0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$, *i.e.* zero is an equilibrium point of the interconnected system. Similarly to the delay-free case we need to restrict the class of systems and signal generators under consideration: assuming that the system to be reduced has some stability property and the signal generator has some excitation property. In the present framework Assumption 2.1 is replaced as follows.

Assumption 3.1. The zero equilibrium of the system $\dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\varsigma}, 0, \ldots, 0)$ is locally exponentially stable. The signal generator (3.2) is observable and neutrally stable.

Note that, differently from Assumption 2.1, Assumption 3.1 does not require the minimality of (3.1). The reason is that the characterization of the minimal realization of nonlinear time-delay systems is still an open problem. However, from the discussion in Chapter 2 we expect that the reduced order model obtained is a model of the "minimal", whatever the exact meaning is, subsystem of system (3.1). We are now ready to extend Lemma 2.1 to the present framework.

Lemma 3.1. Consider system (3.1) and the signal generator (3.2). Suppose Assumption 3.1 holds. Then there exists a mapping $\pi(\omega)$, locally defined in a neighborhood of $\omega = 0$, with $\pi(0) = 0$, which solves the partial differential equation

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\bar{\omega}_{\tau_0}), \ldots, \pi(\bar{\omega}_{\tau_\varsigma}), l(\bar{\omega}_{\tau_{\varsigma+1}}), \ldots, l(\bar{\omega}_{\tau_\mu})), \tag{3.4}$$

where $\bar{\omega}_{\tau_i} = \Phi^s_{-\tau_i}(\omega)$ for discrete delays and $\bar{\omega}_{\tau_i} = \int_{-\tau_i}^0 \Phi^s_{\bar{\tau}}(\omega) d\bar{\tau}$ for distributed delays, with $i = 0, \ldots, \mu$. In addition, the steady-state response of system (3.3) is $x^{ss}(t) = \pi(\omega(t))$ for any $x(0)$ and $\omega(0)$ sufficiently small.

Proof. It is well-known that the center manifold theory holds for time-delay systems as it holds for delay-free systems. In particular, as for finite dimensional systems, if the linearized system has q eigenvalues on the imaginary axis then there exists a q-dimensional local integral manifold (which is in fact a center manifold) for the original system. In addition, the well-defined restriction of the dynamics of the system to the manifold is finite dimensional. Thus, under the stated assumptions it is possible to compute the Jacobian matrix of the interconnected system (3.3) at the zero equilibrium point which is partitioned in an asymptotically stable part and a marginally stable part. The rest of the proof is similar to the one of Lemma 2.1. □

Remark 3.1. The partial differential equation (3.4) is independent of time (as the corresponding equations given in Chapter 2), *e.g.* if $s(\omega) = S\omega$ then $\bar{\omega}_{\tau_i} = e^{-S\tau_i}\omega$ for discrete delays or $\bar{\omega}_{\tau_i} = S^{-1}(I - e^{-S\tau_i})\omega$ for distributed delays.

Similarly to the delay-free case, we can define the moment of system (3.1) exploiting the mapping π. While the existence of such a mapping is guaranteed by Assumption 3.1, a weaker assumption can be identified.

Assumption 3.2. The signal generator (2.7) is observable and equation (3.4) has a solution π.

Assumption 3.2 is the time-delay equivalent of Assumption 2.2. We can now define the moment for time-delay systems.

Definition 3.1 (Steady-state notion of moment)**.** Consider system (3.1) and the signal generator (3.2). Suppose Assumption 3.2 holds. The mapping $h \circ \pi$ is the *moment of system (3.1) at* (s, l).

Similarly to Definition 2.13, if the stronger Assumption 3.1 holds, then the moment $h \circ \pi$ computed along a particular trajectory $\omega(t)$ coincides with the steady-state response of system (3.1) driven by (3.2). If Assumption 3.1 does not hold, it is still possible to determine the moment according to Definition 3.1.

3.1.1 The special case of linear time-delay systems

In this section we specialize the previous discussion to linear differential time-delay systems. As for the delay-free case, we can achieve better understanding and provide more intuition behind the theory. Moreover, since linear time-delay systems are more easily tractable, we can obtain stronger results. To keep the notation as simple as possible we begin with the class of systems with discrete-delays, *i.e.* we postpone the analysis for systems with distributed-delays to Section 3.1.2.

Consider a linear, single-input, single-output, continuous-time, time-delay system with constant discrete-delays described by the equations

$$\begin{aligned} \dot{x} &= \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j}, \qquad y = \sum_{j=0}^{\varsigma} C_j x_{\tau_j}, \\ x(\theta) &= \beta(\theta), \qquad -T \le \theta \le 0, \end{aligned} \tag{3.5}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\beta \in \mathfrak{R}^n_T$, $A_j \in \mathbb{R}^{n\times n}$ and $C_j \in \mathbb{R}^{1\times n}$ with[1] $j = 0, \dots, \varsigma$, $B_j \in \mathbb{R}^{n\times 1}$ with $j = \varsigma + 1, \dots, \mu$, $\tau_0 = 0$ and $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, \dots, \mu$. Consider a signal generator described by the equations

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{3.6}$$

with $\omega(t) \in \mathbb{R}^\nu$, $S \in \mathbb{R}^{\nu\times\nu}$, $L \in \mathbb{R}^{\nu\times 1}$. Let

$$W(s) = \bar{C}(s)(sI - \bar{A}(s))^{-1}\bar{B}(s), \tag{3.7}$$

with

$$\begin{aligned} \bar{A}(s) &= \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j}, \quad \bar{B}(s) = \textstyle\sum_{j=\varsigma+1}^{\mu} B_j e^{-s\tau_j}, \\ \bar{C}(s) &= \sum_{j=0}^{\varsigma} C_j e^{-s\tau_j}, \end{aligned} \tag{3.8}$$

be the associated transfer function. With the notation introduced for the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, it is straightforward to define the moments of system (3.5) at some $s_i \in \mathbb{C}$ based on the Laurent series expansion of the transfer function $W(s)$ at $s_i \in \mathbb{C}$.

Definition 3.2 (Interpolation notion of moment)**.** Let $s_i \in \mathbb{C} \setminus \sigma(\bar{A}(s))$. The 0-*moment of system (3.5) at* s_i is the complex number

$$\eta_0(s_i) = \bar{C}(s_i)(s_i I - \bar{A}(s_i))^{-1}\bar{B}(s_i).$$

The k-*moment of system (3.5) at* s_i is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}\bar{C}(s)(sI - \bar{A}(s))^{-1}\bar{B}(s)\right]_{s=s_i},$$

with $k \ge 1$ integer.

[1] The delays of A_j and C_j are taken, without loss of generality, equal to ease the notation.

Similarly to the delay-free case, we can show that there exists a one-to-one relation between the moments and the (unique) solution of a Sylvester-like equation (also known as generalized Sylvester equation). To this end we need a preliminary result on the uniqueness of the solution of such equation.

Lemma 3.2. The equation

$$\sum_{j=0}^{\varsigma} A_j \Pi e^{-S\tau_j} - \Pi S = -\sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j}, \tag{3.9}$$

with $S \in \mathbb{R}^{\nu\times\nu}$ and $L \in \mathbb{R}^{1\times\nu}$, has a unique solution if and only if $\sigma(S) \cap \sigma(\bar{A}(s)) = \emptyset$.

Proof. Suppose, without loss of generality, that the matrix S is in complex Jordan form. Then the matrices $S^\top$ and $e^{-S^\top\tau_j}$ are lower triangular and their i-th eigenvalue is s_i and $e^{-s_i\tau_j}$, respectively. We recall that the eigenvalues of the sum of lower triangular matrices are the sums of the eigenvalues. Equation (3.9) can be rewritten using the vectorization operator and the Kronecker product as

$$\left(\sum_{j=0}^{\varsigma}\left(e^{-S^\top\tau_j}\otimes A_j\right) - S^\top \otimes I\right)\mathrm{vec}\,(\Pi) = \mathrm{vec}\left(-\sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j}\right).$$

Thus equation (3.9) has a unique solution if and only if

$$\det\left(\sum_{j=0}^{\varsigma}\left(e^{-S^\top\tau_j}\otimes A_j\right) - S^\top \otimes I\right) \neq 0,$$

which holds if and only if

$$\prod_{i=1}^{\bar{k}} \det\left(\sum_{j=0}^{\varsigma} A_j e^{-s_i\tau} - s_i I\right) \neq 0,$$

where $\bar{k}$ is the number of distinct eigenvalues s_i. This proves the claim. □

The result in Lemma 2.4 is extended to time-delay systems as follows.

Lemma 3.3. Let $s_i \in \mathbb{C} \setminus \sigma(\bar{A}(s))$. Consider system (3.5), then

$$\begin{bmatrix} \eta_0(s_i) & \dots & \eta_k(s_i) \end{bmatrix} = \sum_{j=0}^{\varsigma} C_j \widetilde{\Pi} e^{-\Sigma_k \tau_j} \Psi_k,$$

where $\Psi_k = \mathrm{diag}(1, -1, 1, \dots, (-1)^k) \in \mathbb{R}^{(k+1)\times(k+1)}$ and $\widetilde{\Pi} \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester-like equation

$$\sum_{j=0}^{\varsigma} A_j \widetilde{\Pi} e^{-\Sigma_k \tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j L_k e^{-\Sigma_k \tau_j} = \widetilde{\Pi} \Sigma_k, \tag{3.10}$$

with $L_k = [1\ 0\ \dots\ 0] \in \mathbb{R}^{(k+1)}$ and

$$\Sigma_k = \begin{bmatrix} s_i & 1 & 0 & \dots & 0 \\ 0 & s_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s_i & 1 \\ 0 & \dots & \dots & 0 & s_i \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}.$$

Proof. The proof is computationally involved but it follows the same steps of the proof of Lemma 2.4. First of all note that since $s_i \in \mathbb{C} \setminus \sigma(\bar{A}(s))$, by Lemma 3.2, equation (3.10) has a unique solution $\widetilde{\Pi}$. Let $\widetilde{\Pi} = [\widetilde{\Pi}_0\ \widetilde{\Pi}_1\ \dots\ \widetilde{\Pi}_k]$. Since Σ_k is in Jordan form then

$$e^{-\Sigma_k \tau_j} = e^{-s_i \tau_j} \begin{bmatrix} 1 & -\tau_j & \frac{(-\tau_j)^2}{2} & \dots & \frac{(-\tau_j)^{k-1}}{(k-1)!} \\ 0 & 1 & -\tau_j & \dots & \frac{(-\tau_j)^{k-2}}{(k-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\tau_j \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

Thus, the first column of equation (3.10) can be rewritten as

$$\sum_{j=0}^{\varsigma} A_j \widetilde{\Pi}_0 e^{-s_i \tau_j} + \textstyle\sum_{j=\varsigma+1}^{\mu} B_j e^{-s_i \tau_j} = \widetilde{\Pi}_0 s_i, \tag{3.11}$$

the second column can be rewritten as

$$\sum_{j=0}^{\varsigma} A_j e^{-s_i \tau_j} \widetilde{\Pi}_1 + \sum_{j=0}^{\varsigma} -\tau_j A_j e^{-s_i \tau_j} \widetilde{\Pi}_0 - \sum_{j=\varsigma+1}^{\mu} \tau_j B_j e^{-s_i \tau_j} = \widetilde{\Pi}_1 s_i + \widetilde{\Pi}_0, \tag{3.12}$$

and so on until the last column

$$\sum_{l=k}^{0}\sum_{j=0}^{\varsigma} A_j\widetilde{\Pi}_{k-l}\frac{(-\tau_j)^l}{l!}e^{-s_i\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j\frac{(-\tau_j)^k}{k!}e^{-s_i\tau_j} = \widetilde{\Pi}_k s_i + \widetilde{\Pi}_{k-1}. \tag{3.13}$$

As a result, $\widetilde{\Pi}_0$ can be determined from equation (3.11) as

$$\widetilde{\Pi}_0 = \left(s_iI - \sum_{j=0}^{\varsigma} A_je^{-s_i\tau_j}\right)^{-1}\sum_{j=\varsigma+1}^{\mu} B_je^{-s_i\tau_j} = \left(s_iI - \bar{A}(s_i)\right)^{-1}\bar{B}(s_i),$$

$\widetilde{\Pi}_1$ from equation (3.12) and $\widetilde{\Pi}_0$ as

$$\begin{aligned}\widetilde{\Pi}_1 = & -\left(s_iI - \sum_{j=0}^{\varsigma} A_je^{-s_i\tau_j}\right)^{-1}\left(I + \sum_{j=1}^{\varsigma}\tau_jA_je^{-s_i\tau_j}\right)\left(s_iI - \sum_{j=0}^{\varsigma} A_je^{-s_i\tau_j}\right)^{-1}\times \\ & \times\sum_{j=\varsigma+1}^{\mu} B_je^{-s_i\tau_j} - \left(s_iI - \sum_{j=0}^{\varsigma} A_je^{-s_i\tau_j}\right)^{-1}\sum_{j=\varsigma+1}^{\mu}\tau_jB_je^{-s_i\tau_j} = \\ & = \left[\frac{d}{ds}\left((sI - \bar{A}(s))^{-1}\bar{B}(s)\right)\right]_{s=s_i}.\end{aligned}$$

Iterating for all k, yields

$$\widetilde{\Pi}_k = \frac{1}{k!}\left[\frac{d^k}{ds^k}\left((sI - \bar{A}(s))^{-1}\bar{B}(s)\right)\right]_{s=s_i}.$$

Finally, exploiting the columns of $\widetilde{\Pi}$, the moments can be written as

$$\begin{aligned}\sum_{j=0}^{\varsigma} C_j\widetilde{\Pi}e^{-\Sigma_k\tau_j} &= \left[\sum_{j=0}^{\varsigma} C_j\widetilde{\Pi}_0e^{-s_i\tau_j} \quad \dots \quad \sum_{l=k}^{0}\sum_{j=0}^{\varsigma} C_j\widetilde{\Pi}_{k-l}\frac{(-\tau_j)^l}{l!}e^{-s_i\tau_j}\right] \\ &= \left[\bar{C}(s_i)\widetilde{\Pi}_0 \quad \dots \quad \sum_{l=k}^{0}\frac{1}{l!}\frac{d^l}{ds^l}\left[\bar{C}(s)\right]_{s=s_i}\widetilde{\Pi}_{k-l}\right] \\ &= \left[\eta_0(s_i) \quad \dots \quad (-1)^k\eta_k(s_i)\right],\end{aligned}$$

which proves the claim. □

Equation (3.10) can be written eliminating the fact that Σ_k and L_k have a special structure. As a result the following holds.

Lemma 3.4. Consider the signal generator (3.6) and let $S \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\bar{k}} (s - s_i)^{k_i}, \tag{3.14}$$

where $\nu = \sum_{i=1}^{\bar{k}} k_i$, and let L be such that the pair (S, L) is observable. Consider system (3.5) and suppose $\sigma(S) \cap \sigma(\bar{A}(s)) = \emptyset$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_{\bar{k}}), \ldots, \eta_{k_{\bar{k}}-1}(s_{\bar{k}})$ and the matrix $\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j}$, where Π is the unique solution of the Sylvester-like equation (3.9).

Proof. It is sufficient to prove the claim for $\bar{k} = 1$. By observability of the pair (S, L) there is a unique invertible matrix T such that $S = T^{-1}\Sigma_k T$ and $L = L_k T$. Then equation (3.9) becomes

$$\sum_{j=0}^{\varsigma} A_j \Pi e^{-(T^{-1}\Sigma_k T)\tau_j} - \Pi T^{-1}\Sigma_k T = -\sum_{j=\varsigma+1}^{\mu} B_j L e^{-(T^{-1}\Sigma_k T)\tau_j}.$$

The claim follows defining $\widetilde{\Pi} = \Pi T^{-1}$, recalling that $e^{T^{-1}XT} = T^{-1}e^X T$ and that the moments are coordinates invariant. $\square$

To conclude the discussion and justify the definition of moment for nonlinear time-delay systems, it remains to characterize the relation between the notion of moment and the steady-state response of the system. To this end, it is convenient to specialize Assumption 3.1 to linear systems.

Assumption 3.3. The pair (S, L) is observable, $\sigma(S) \subset \mathbb{C}_0$, the eigenvalues of S are simple and $\sigma(A) \subset \mathbb{C}_{<0}$.

Theorem 3.5. Consider system (3.5) and the signal generator (3.6). Suppose Assumption 3.3 holds and let $\omega(0)$ be such that the pair $(S, \omega(0))$ is reachable. Then there exists a one-to-one relation between the moments $\eta_0(s_1)$, $\eta_0(s_2)$, $\ldots$, $\eta_0(s_{\bar{k}})$, with $s_i \in \sigma(S)$ for all $i = 1, \ldots, \bar{k}$, and the steady-state response of the output y of the interconnection of system (3.5) with the signal generator (3.6).

Proof. Consider the interconnection of system (3.5) with system (3.6). By the assumptions on $\sigma(\bar{A}(s))$ and $\sigma(S)$, the interconnected system has a globally well-defined invariant manifold given by $\mathcal{M} = \{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi\omega\}$, with Π the unique solution of the Sylvester-like equation (3.9). We prove now that $\mathcal{M}$ is attractive. Consider the equation

$$\begin{aligned}\overset{\cdot}{\overbrace{x - \Pi\omega}} &= \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j L \omega_{\tau_j} - \Pi S \omega \\ &= \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \left(\sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j} - \Pi S \right) \omega\end{aligned}$$

in which we used the fact that $\omega(t-\tau_j) = e^{-S\tau_j}\omega(t)$. Substituting (3.9) in the right-hand side of the last equation, yields

$$\overset{\cdot}{\overbrace{x - \Pi\omega}} = \sum_{j=0}^{\varsigma} A_j (x_{\tau_j} - \Pi\omega_{\tau_j}).$$

Computing the Laplace transform on both sides yields

$$s(X(s) - \Pi\Omega(s)) - (x(0) - \Pi\omega(0)) = \left(\sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} \right) (X(s) - \Pi\Omega(s))$$

and, by the assumptions on $\sigma(\bar{A}(s))$, we have

$$X(s) - \Pi\Omega(s) = (sI - \bar{A}(s))^{-1}(x(0) - \Pi\omega(0)).$$

Finally, computing the inverse Laplace transform yields

$$x(t) - \Pi\omega(t) = \mathcal{L}^{-1}\{(sI - \bar{A}(s))^{-1}(x(0) - \Pi\omega(0))\}.$$

Since $\sigma(\bar{A}(s)) \subset \mathbb{C}_{<0}$, $\mathcal{M}$ is attractive. As a result

$$\begin{aligned}y(t) &= \sum_{j=0}^{\varsigma} C_j (x_{\tau_j} - \Pi\omega_{\tau_j}) + \sum_{j=0}^{\varsigma} C_j \Pi\omega_{\tau_j} \\ &= \sum_{j=0}^{\varsigma} C_j \Pi\omega_{\tau_j} + \sum_{j=0}^{\varsigma} C_j \mathcal{L}^{-1}\{(sI - \bar{A}(s))^{-1}(x(0) - \Pi\omega(0))\} e^{-S\tau_j} \\ &= \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j}\omega + \varepsilon(t),\end{aligned}$$

where $\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j}\omega(t)$ describes the steady-state response, whereas

$$\varepsilon(t) = \sum_{j=0}^{\varsigma} C_j \mathcal{L}^{-1}\{(sI - \bar{A}(s))^{-1}(x(0) - \Pi\omega(0))\} e^{-S\tau_j},$$

describes the transient response which vanishes exponentially. The rest of the proof follows the same steps of the proof of Theorem 2.6. □

Similarly to the delay-free case we point out that moments are well-defined even though the steady-state response does not exist. The weaker assumption (already used in Lemma 3.4) for which this may happen is now formulated explicitly. This is the linear equivalent of Assumption 3.2.

Assumption 3.4. The pair (S, L) is observable and $\sigma(S) \cap \sigma(\bar{A}(s)) = \emptyset$.

Note that Assumptions 3.1 and 3.3 imply, but are not implied by, Assumptions 3.2 and 3.4. Whenever we use Assumptions 3.1 and 3.3 in the following, we are exploiting the relation between moments and steady-state response.

3.1.2 A general class of linear time-delay systems

All the results presented for discrete-delays can be generalized to linear neutral[2] differential time-delay systems with distributed-delays. Consider a linear, single-input, single-output, continuous-time, neutral time-delay system with discrete-delays and distributed-delays described by the equations

$$\begin{aligned}
\dot{x} &= \sum_{j=1}^{q} D_j \dot{x}_{c_j} + \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta, \\
y &= \sum_{j=0}^{\varsigma} C_j x_{\tau_j}, \\
x(\theta) &= \beta(\theta), \qquad -T \le \theta \le 0,
\end{aligned} \tag{3.15}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\beta \in \mathfrak{R}^n_T$, $A_j \in \mathbb{R}^{n\times n}$ and $C_j \in \mathbb{R}^{1\times n}$ with $j = 0, \ldots, \varsigma$, $B_j \in \mathbb{R}^{n\times 1}$ with $j = \varsigma + 1, \ldots, \mu$, $D_j \in \mathbb{R}^{n\times n}$ with $j = 1, \ldots, q$, $G_j \in \mathbb{R}^{n\times n}$ and $H_j \in \mathbb{R}^{n\times 1}$ with $j = 1, \ldots, r$, $\tau_0 = 0$, $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, \mu$, $c_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, q$ and $h_j \in \mathbb{R}_{>0}$

[2]The term "neutral time-delay" refers to systems in which delayed $\dot{x}$ terms appear in the equations. It is not related to neutral stability.

with $j = 1, \ldots, r$. The transfer function $W(s)$ is defined by (3.7) with

$$\begin{aligned} \bar{A}(s) &= \sum_{j=1}^{q} D_j s e^{-sc_j} + \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} + \sum_{j=1}^{r} G_j \frac{1-e^{-sh_j}}{s}, \\ \bar{B}(s) &= \sum_{j=\varsigma+1}^{\mu} B_j e^{-s\tau_j} + \sum_{j=1}^{r} H_j \frac{1-e^{-sh_j}}{s}, \\ \bar{C}(s) &= \sum_{j=0}^{\varsigma} C_j e^{-s\tau_j}. \end{aligned} \tag{3.16}$$

Lemma 3.4 can be extended to the present framework as follows.

Lemma 3.6. Assume $0 \notin \sigma(S)$. Lemma 3.4 holds, with the same assumptions, for system (3.15) replacing equation (3.9) with

$$\begin{aligned} \sum_{j=0}^{\varsigma} A_j \Pi e^{-S\tau_j} + \sum_{j=1}^{r} G_j \Pi S^{-1}(I - e^{-Sh_j}) + \sum_{j=1}^{q} D_j \Pi S e^{-Sc_j} - \Pi S = \\ = - \sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j} - \sum_{j=1}^{r} H_j L S^{-1}(I - e^{-Sh_j}). \end{aligned} \tag{3.17}$$

Proof. The proof is similar to that of Lemma 3.4, hence it is omitted. The additional assumption that S is invertible is necessary because in equation (3.17) the distributed-delays generate terms in S^{-1}. □

Similarly, we can extend Theorem 3.5 as follows. For the next result we need to introduce a notion of stability known as "formal stability".

Assumption 3.5. The zero equilibrium point of the difference equation

$$x(t) + \sum_{j=1}^{q} D_j x(t - c_j) = 0,$$

with $D_j \in \mathbb{R}^{n\times n}$ and $c_j \in \mathbb{R}$, is asymptotically stable, *i.e.* system (3.15) is formally stable.

Theorem 3.7. Assume $0 \notin \sigma(S)$ and Assumption 3.5 holds. Theorem 3.5 holds, with the same assumptions, for system (3.15).

Proof. The claim can be proved noting that $\sigma(\bar{A}(s)) \subset \mathbb{C}_{<0}$, with the definitions in (3.16), and Assumption 3.5 guarantee that the zero equilibrium of system (3.15) is exponentially stable. □

These two results extend the theory presented so far to the class of neutral differential time-delay systems with distributed-delays, *i.e.* we have provided the conditions for which the moments are well-defined and these can be related to the steady-state response (when it exists) of the system.

Remark 3.2. Many systems can be described by the equations (3.15) with, in most cases, a single neutral delay, *i.e.* $q = 1$. In this case Assumption 3.5 holds if $\sigma(D_1) \subset \mathbb{D}_{<1}$.

Remark 3.3. Since hyperbolic partial differential equations can be locally expressed as neutral time-delay systems and, conversely, any time-delay $y(t) = u(t-\tau)$ can be represented by a transport equation, the techniques presented in this chapter can be used to establish a model reduction theory for some classes of partial differential equations.

To keep the notation light only the discrete-delay case is considered in the remaining of the chapter. However, the extension of the following results to system (3.15) is straightforward.

3.2 Families of Reduced Order Models for Nonlinear Time-Delay Systems

In this section we provide a family of reduced order models and we study how some specific properties can be maintained by the reduced order model. We begin with a formal definition of reduced order model for nonlinear time-delay systems.

Definition 3.3. Consider system (3.1) and the signal generator (3.2). The system described by the equations

$$\dot{\xi} = \phi(\xi_{\chi_0}, \ldots, \xi_{\chi_\varrho}, u_{\chi_{\varrho+1}}, \ldots, u_{\chi_\rho}), \qquad \psi = \kappa(\xi), \tag{3.18}$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$, $\chi_0 = 0$, $\chi_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, \rho$, and ϕ and κ analytic mappings, is a *model of system (3.1) at* (s, l) if system (3.18) has the same moment at (s, l) as system (3.1). In this case, system (3.18) is said to *match* the moment of system (3.1) at (s, l). Furthermore, system (3.18) is a *reduced order model of system (3.1) at* (s, l) if $\nu < n$, or if $\varrho < \varsigma$, or if $\rho < \mu$.

By Definition 3.3, we consider a model of a system to be reduced if it is described by a smaller number of equations or it possesses a smaller number of delays.

Lemma 3.8. Consider system (3.1) and the signal generator (3.2). Suppose Assumption 3.2 holds. Then system (3.18) matches the moments of system (3.1) at (s, l) if the equation

$$\frac{\partial p}{\partial \omega} s(\omega) = \phi(p(\bar{\omega}_{\chi_0}), \ldots, p(\bar{\omega}_{\chi_\varrho}), l(\bar{\omega}_{\chi_{\varrho+1}}), \ldots, l(\bar{\omega}_{\chi_\rho})), \tag{3.19}$$

where $\bar{\omega}_{\chi_i} = \Phi^s_{-\chi_i}(\omega)$, with $i = 0, \ldots, \rho$, has a unique solution p such that

$$h(\pi(\omega)) = \kappa(p(\omega)), \tag{3.20}$$

where π is the unique solution of (3.4).

Proof. The claim follows from Definition 3.3 and the definition of moment. □

Similarly to the delay-free case, we are now in a position of providing (under an additional assumption) a family of reduced order models with tunable parameters which satisfies the conditions of Lemma 3.8.

Assumption 3.6. There exist mappings κ and p such that $\kappa(0) = 0$, $p(0) = 0$, p is locally continuously differentiable, equation (3.20) holds and p has a local inverse p^{-1}.

Proposition 3.1. Consider system (3.1) and the signal generator (3.2). Suppose Assumptions 3.2 and 3.6 hold. Then the system described by the equations

$$\begin{aligned} \dot{\xi} \;&= s(\xi) - \textstyle\sum_{j=\varrho+1}^{\rho} \delta_j(\xi) l(\bar{\xi}_{\chi_j}) - \gamma(\bar{\xi}_{\chi_1}, \ldots, \bar{\xi}_{\chi_\varrho}) \\ &\qquad + \gamma(\xi_{\chi_1}, \ldots, \xi_{\chi_\varrho}) + \textstyle\sum_{j=\varrho+1}^{\rho} \delta_j(\xi) u_{\chi_j}, \\ \psi \;&= h(\pi(\xi)), \end{aligned} \tag{3.21}$$

where π is the unique solution of (3.4), is a model of system (3.1) at (s, l) if δ_j and γ are arbitrary mappings such that the partial differential equation

$$\begin{aligned} \frac{\partial p}{\partial \omega} s(\omega) \;&= s(p(\omega)) - \textstyle\sum_{j=\varrho+1}^{\rho} \delta_j(p(\omega)) l(p(\bar{\omega}_{\chi_l})) - \gamma(p(\bar{\omega}_{\chi_1}), \ldots, p(\bar{\omega}_{\chi_\varrho})) \\ &\quad + \gamma(p(\omega_{\chi_1}), \ldots, p(\omega_{\chi_\varrho})) + \textstyle\sum_{j=\varrho+1}^{\rho} \delta_j(p(\omega)) l(\omega_{\chi_j}), \end{aligned}$$

has the unique solution $p(\omega) = \omega$.

Proof. The proof is analogous to the one of Proposition 2.1, hence it is omitted. □

The nonlinear model (3.21) has several free design parameters, namely δ_j, γ, χ_j, ϱ and ρ. We note that selecting $\gamma = 0$, $\varrho = 0$, $\rho = 1$ and $\chi_1 = 0$ (in this case we define $\delta = \delta_1$), yields a reduced order model with no delays. This reduced order model coincides with the one in Chapter 2 (equation (2.20)) and all results therein are directly applicable: the mapping δ can be selected to achieve matching with asymptotic stability, matching with prescribed relative degree, etc. However, as it is clarified later in the chapter, the choice of eliminating the delays is likely to destroy some important feature of the model.

Remark 3.4. The results of this section can be extended to more general classes of time-delay systems provided that, for such systems, the center manifold theory applies. In particular, one can consider the class of nonlinear neutral differential time-delay systems described by equations of the form

$$\begin{aligned} &d(\dot{x}_{\tau_0}, \ldots, \dot{x}_{\tau_{\varsigma_1}}) = f(x_{\tau_{\varsigma_1+1}}, \ldots, x_{\tau_{\varsigma_2}}, u_{\tau_{\varsigma_2+1}}, \ldots, u_{\tau_\mu}), \\ &y = h(x), \end{aligned} \tag{3.22}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\tau_0 = 0$, $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, \mu$ and d, f and h analytic mappings. The center manifold theory does not hold for this class of systems for a general mapping d. However, as already pointed out for linear neutral time-delay systems, for the simple case

$$\begin{aligned} \dot{x} + D\dot{x}_{\tau_1} &= f(x_{\tau_2}, \ldots, x_{\tau_{\varsigma_1}}, u_{\tau_{\varsigma_1+1}}, \ldots, u_{\tau_\mu}), \\ y &= h(x), \end{aligned} \tag{3.23}$$

with $D \in \mathbb{R}^{n\times n}$, the center manifold theory holds as for standard time-delay systems if the matrix D is such that $\sigma(D) \subset \mathbb{D}_{<1}$.

3.2.1 Open-loop reduced order model

We have seen in Section 3.1 that to establish a relation between moments and steady-state response, both for time-delay and delay-free systems, a strong assumption on the stability of the signal generator and of the system is required. While we can consider the assumption on the signal generator not restrictive (because we can chose the interpolating signals), the assumption on the system may be restrictive since the equilibrium point may be unstable. To avoid this problem we can apply feedback to stabilize the equilibrium point of the closed-loop system. However, while the closed-loop system satisfies the assumptions, we are still interested in obtaining a reduced order model of the original open-loop system. Thus, we consider now the problem of obtaining a reduced order model of an open-loop system from the closed-loop system. For ease of notation we assume that there are no delays on the input u.

Consider a closed-loop, nonlinear, single-input, single-output, continuous-time, time-delay system described by the equations

$$\dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_{\mu_1}}, u), \quad u = g(x_{\epsilon_0}, \ldots, x_{\epsilon_{\mu_2 - 1}}) + v_{\tau_{\mu_2}}, \quad y = h(x), \tag{3.24}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $v(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\tau_0 = 0$, $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, \mu_1$, $\epsilon_0 = 0$, $\epsilon_j \in \mathbb{R}_{>0}$ with $j = 1, \ldots, \mu_2$ and f, g and h analytic mappings. Consider the signal generator (3.2) and the interconnected system

$$\begin{aligned} \dot{\omega} &= s(\omega), \\ \dot{x} &= f(x_{\tau_0}, \ldots, x_{\tau_{\mu_1}}, g(x_{\epsilon_0}, \ldots, x_{\epsilon_{\mu_2 - 1}}) + l(\omega_{\tau_{\mu_2}})), \\ y &= h(x). \end{aligned} \tag{3.25}$$

Suppose that $f(0, \ldots, 0, 0) = 0$, $g(0, \ldots, 0, 0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$. Exploiting the results of the previous sections, a solution to the problem of determining open-loop reduced order models is given as follows.

Definition 3.4. Consider system (3.24) and the signal generator (3.2). The system described by the equations

$$\dot{\xi} = \phi(\xi_{\chi_0}, \ldots, \xi_{\chi_\rho}, u), \qquad \psi = \kappa(\xi), \tag{3.26}$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\chi_0 = 0$, $\chi_j \in \mathbb{R}_{>0}$, with $j = 1, \ldots, \rho$, and ϕ and κ analytic mappings, is an *open-loop model of system (3.24) at*

(s,l) if the system

$$
\begin{aligned}
\dot{\xi} &= \phi(\xi_{\chi_0}, \ldots, \xi_{\chi_\rho}, u), \\
u &= g(\pi(p^{-1}(\xi_{\epsilon_0})), \ldots, \pi(p^{-1}(\xi_{\epsilon_{\mu_2-1}}))) + v_{\chi_{\mu_2}}, \\
\psi &= \kappa(\xi),
\end{aligned}
\tag{3.27}
$$

is a model of the (closed-loop) system (3.25) at (s,l), where $v(t) \in \mathbb{R}$, π is the unique solution of the equation

$$
\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\bar{\omega}_{\tau_0}), \ldots, \pi(\bar{\omega}_{\tau_\mu}), g(\pi(\bar{\omega}_{\epsilon_0}), \ldots, \pi(\bar{\omega}_{\epsilon_{\mu_2-1}})) + l(\bar{\omega}_{\tau_{\mu_2}})),
\tag{3.28}
$$

where $\bar{\omega}_{\tau_i} = \Phi^s_{-\tau_i}(\omega)$, with $i = 0, \ldots, \mu_1$, and $\bar{\omega}_{\epsilon_i} = \Phi^s_{-\epsilon_i}(\omega)$, with $i = 0, \ldots, \mu_2$, and p is invertible and the unique solution of the equation

$$
\frac{\partial p}{\partial \omega} s(\omega) = \phi(p(\bar{\omega}_{\chi_0}), \ldots, p(\bar{\omega}_{\chi_\rho}), l(\bar{\omega}_{\chi_{\mu_2}})),
\tag{3.29}
$$

where $\bar{\omega}_{\chi_i} = \Phi^s_{-\chi_i}(\omega)$, with $i = 0, \ldots, \rho$, such that

$$
h(\pi(\omega)) = \kappa(p(\omega)).
\tag{3.30}
$$

Obtaining a reduced order model of an open-loop system given the closed-loop system simplifies the solution of the problem of the reduction of nonlinear systems when their zero equilibrium is not locally exponentially stable. In fact, as explained in Chapter 4, if the steady-state response of the system is well-defined, then we can exploit it to approximate the solution of the partial differential equation associated to π (which is otherwise difficult to determine). We illustrate this point with an example.

Model of an oilwell drillstring

Consider a neutral type model of the torsional dynamics of an oilwell drillstring. In recent years there has been increasing interest on the modeling and analysis of oilwell drilling vibrations because of its economic consequences. In drilling operations the nonlinear interaction between the bit of the drillstring and the bottom of the hole originates the stick-slip phenomenon. This consists in the undesired event that a constant rotational velocity applied at the top of the string does not translate to a steady speed at the bottom of the hole. In particular the bit undergoes intervals where it is completely blocked and intervals where the accumulated energy is released and the rotational speed becomes larger than the prescribed value. Among several models of the dynamics of the system which have been presented, we consider the model described by the neutral differential time-delay system

$$\begin{aligned} \dot{x} &= \bar{\Upsilon}\dot{x}_{\tau_1} - \Psi x - \Psi\bar{\Upsilon}x_{\tau_1} - \tfrac{1}{I_B}T(x) + \tfrac{1}{I_B}\bar{\Upsilon}T(x_{\tau_1}) + 2\tfrac{\Psi c_a}{\Lambda}\Omega_{\tau_2}, \\ y &= x, \end{aligned} \tag{3.31}$$

with

$$\bar{\Upsilon} = \frac{c_a - \sqrt{IG_sJ}}{c_a + \sqrt{IG_sJ}}, \quad \Psi = \frac{\sqrt{IG_J}}{I_B}, \quad \Lambda = c_a - \sqrt{IG_sJ},$$

where $x(t)$ is the angular velocity at the bottom of the string, $y(t)$ is the output of the system, $\Omega(t)$ is the input variable, I is the inertia, J is the geometrical moment of inertia, G_s is the shear modulus, I_B is the lumped inertia representing the block at the bottom, c_a is a constant related to the local torsion of the drillstring, $\tau_2 = \Gamma$, $\tau_1 = 2\Gamma$, $\Gamma = L_s\sqrt{\frac{I}{G_sJ}}$ and L_s is the length of the string. The nonlinear function T describes the bit-rock interaction. Several models of this function have been proposed. We consider the continuous function

$$T(x) = c_b x + W_{ob}R_b\left[\mu_{cb} + (\mu_{sb} - \mu_{cb})e^{-\frac{\gamma_b}{v_f}|x|}\right]\tanh(t_g x), \tag{3.32}$$

where μ_{sb}, μ_{cb} are, respectively, the static and Coulomb friction coefficients, W_{ob} is the weight on the bit, R_b is the bit radius, γ_b is a positive constant defining the decaying velocity of the exponential, v_f is a constant velocity introduced to have appropriate units, c_b is the viscous damping coefficient and t_g is the gain of the hyperbolic tangent.

The parameters used in the simulation are listed in Table 3.1. System (3.31), with (3.32), has been simulated using the MATLAB solver *ddensd* with relative and absolute tolerances of 10^{-5} and 10^{-6}, respectively. Fig. 3.1 shows the time histories of the state of system (3.31), with (3.32) and $\Omega(t) = r(t) = \text{const}$, for different values of the desired angular velocity r. The typical behavior of the interaction between bit and rock can be seen, *i.e.* for high values of r the steady-state is constant whereas for low values of r the stick-slip phenomenon is present.

Table 3.1: Parameters of the model of the oilwell drillstring

$G_s = 79.3 \cdot 10^9\ N/m^2$,	$I = 0.095\ kg \cdot m$,	$L_s = 1172\ m$,
$J = 1.19 \cdot 10^{-5}\ m^4$,	$R_b = 0.155575$,	$v_f = 1$,
$W_{ob} = 97347\ N$,	$I_B = 89\ kg \cdot m^2$,	$\mu_{cb} = 20\ rad/s$,
$c_a = 2000\ N \cdot m \cdot s$,	$\mu_{sb} = 0.8$,	$\gamma_b = 0.9$,
$c_b = 0.03\ N \cdot m \cdot s/rad$,	$t_g = 10.$	$\zeta_1 = 6.96 \cdot 10^3$
$\zeta_2 = 0.09$,		

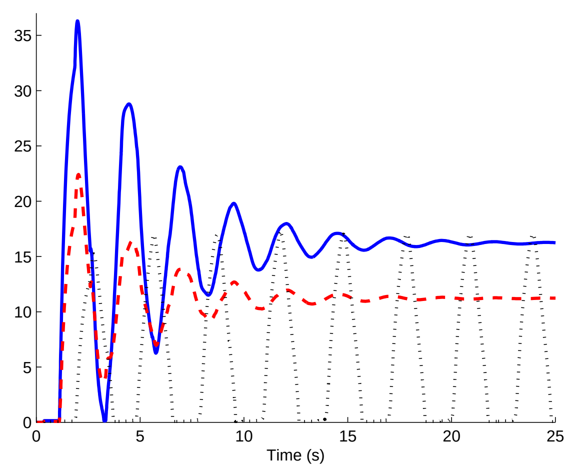

Figure 3.1: Angular speed of the system (3.31), with (3.32), for different values of r: 20 rad/s (solid line), 15 rad/s (dashed line) and 10 rad/s (dotted line). Note the stick-slip phenomenon for $r(t) = 10\ rad/s$.

System (3.31), with (3.32), is a nonlinear neutral differential time-delay system for which the origin is not exponentially asymptotically stable. However, several feedbacks have been proposed to asymptotically stabilize the origin of the system. For example, we apply the

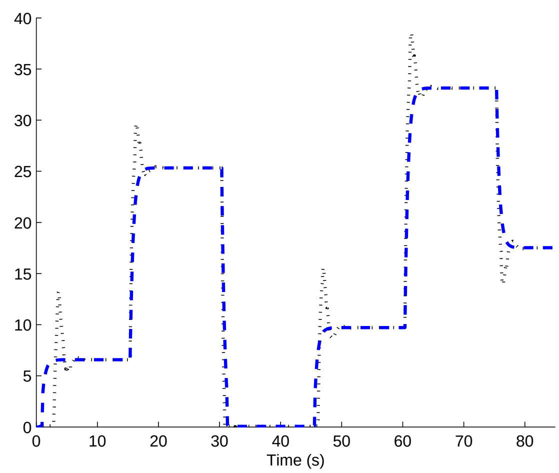

Figure 3.2: Time histories of the output of system (3.31), with (3.32) and (3.33), (dotted line) and the output ψ of the reduced order model (3.34), with $\delta = 2$, (dashed line) for various desired velocities.

feedback control law

$$\Omega(t) = k_1\dot{x}(t - \tau_2) + k_2 x(t - \tau_2) + r(t), \tag{3.33}$$

with $k_1 = -0.05$ and $k_2 = 0.36$. For the closed-loop system (3.31), with (3.32) and (3.33), we compute numerically the solution of equation (3.4). The function π is approximated by the piecewise continuous function

$$\pi(\omega) = \begin{cases} 0, & \omega \leq 5.7, \\ 1.5633\omega - 5.9250, & \omega > 5.7. \end{cases}$$

A simple reduced order model achieving moment matching at $\dot{\omega} = s(\omega) = 0$ and belonging to the family of models (3.21) is described by the equations

$$\dot{\xi} = -\delta(\xi)\left[\xi - r_{\tau_2}\right], \quad \psi = \pi(\xi). \tag{3.34}$$

Fig. 3.2 shows a comparison between the output of system (3.31), with (3.32) and (3.33), and the output of the reduced order model (3.34), with $\delta = 2$, for various desired angular velocities.

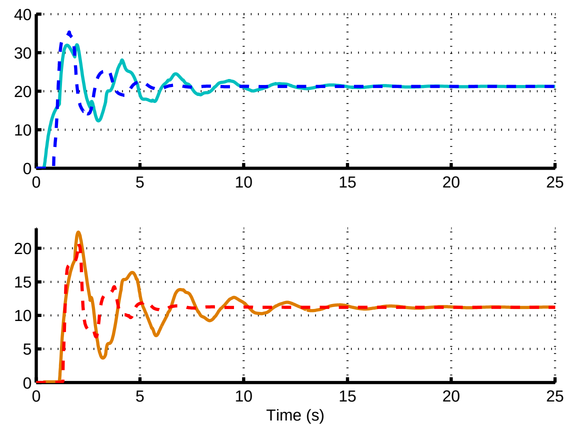

Figure 3.3: Time histories of the output of system (3.31), with (3.32), (solid line) and of system (3.35) (dashed line) with $\delta(z) = qz^2 + \varepsilon$ for $r = 25$, $q = 0.0333$, $\varepsilon = 0.3$ (top) and $r = 15$, $q = 0.125$, $\varepsilon = 0.01$ (bottom).

We are now interested, using system (3.34), in obtaining a model of the *open-loop system* (3.31) with (3.32). An open-loop model of system (3.31), with (3.32), achieving moment matching at $\dot{\omega} = 0$ is given by

$$\dot{\xi} = -\delta(\xi)\left[\xi - \mu_{\tau_2}\right], \quad \psi = \pi(\xi), \tag{3.35}$$

with $\mu = -k_1\pi(\dot{\xi}_{\tau_2}) - k_2\pi(\xi_{\tau_2}) + r$. Fig. 3.3 shows the time histories of the output of the open-loop system (3.31), with (3.32), and of the model (3.35), with $\delta(z) = qz^2 + \varepsilon$, for $r = 25$, $q = 0.0333$, $\varepsilon = 0.3$ (top) and $r = 15$, $q = 0.125$, $\varepsilon = 0.01$ (bottom). We can see that the model (3.35) and the open-loop system (3.31), with (3.32), have the same steady-state value and that, using the free mapping δ, the transient behavior can be partially recovered.

3.2.2 Reduced order model for linear time-delay systems

In this section a family of linear time-delay systems achieving moment matching is presented and the possibility of interpolating a larger number of points maintaining the same "number of equations" is investigated. First of all, we specialize Definition 3.3 to linear systems.

Definition 3.5. Consider system (3.5) and the signal generator (3.6). The system described by the equations

$$\dot{\xi} = \sum_{j=0}^{\varrho} F_j \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_j u_{\chi_j}, \qquad \psi = \sum_{j=0}^{d} H_j \xi_{\chi_j}, \tag{3.36}$$

with $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}$, $F_j \in \mathbb{R}^{\nu \times \nu}$ for $j = 0, \dots, \varrho$, $G_j \in \mathbb{R}^{\nu \times 1}$ for $j = \varrho + 1, \dots, \rho$, $H_j \in \mathbb{R}^{1 \times \nu}$ for $j = 0, \dots, k$, $\chi_0 = 0$ and $\chi_j \in \mathbb{R}_{>0}$ for $j = 1, \dots, \max\{\rho, d\}$, is a *model of system (3.5) at* (S, L), if system (3.36) has the same moments at (S, L) as system (3.5). System (3.36) is a *reduced order model of system (3.5) at* (S, L) if $\nu < n$, or if $\varrho < \varsigma$, or if $\rho < \mu$, or if $d < \varsigma$.

Lemma 3.8 can be rewritten as follows.

Lemma 3.9. Consider system (3.5) and the signal generator (3.6). Suppose Assumption 3.4 holds. Then the system (3.36) is a model of system (3.5) at (S, L) if there exists a unique solution P of the equation

$$\sum_{j=0}^{\varrho} F_j P e^{-S\chi_j} - PS = -\sum_{j=\varrho+1}^{\rho} G_j L e^{-S\chi_j}, \tag{3.37}$$

such that

$$\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} = \sum_{j=0}^{d} H_j P e^{-S\chi_j}, \tag{3.38}$$

where Π is the unique solution of (3.9).

Proof. The claim is a consequence of Definition 3.5 and the definition of moment. □

Similarly to the delay-free case, the computation of P can be avoided. The next result is a specialization of Proposition 3.1 to the linear framework.

Proposition 3.2. Consider system (3.5) and the signal generator (3.6). Suppose Assumption 3.4 holds. Then the system described by the equations

$$
\begin{aligned}
\dot{\xi} &= \left(S - \sum_{j=\varrho+1}^{\rho} G_j L e^{-S\chi_j} - \sum_{j=1}^{\varrho} F_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{\varrho} F_j \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_j u_{\chi_j}, \\
\psi &= \left(\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} - \sum_{j=1}^{d} H_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{d} H_j \xi_{\chi_j},
\end{aligned}
\tag{3.39}
$$

where Π is the unique solution of (3.9), is a model of system (3.5) at (S, L) for any G_j and F_j such that $\sigma(S) \cap \sigma(\bar{F}(s)) = \emptyset$, where $\bar{F}(s) = \sum_{j=0}^{\varrho} F_j e^{-s\chi_j}$.

Proof. Consider system (3.36). Set

$$
\begin{aligned}
F_0 &= S - \sum_{j=\varrho+1}^{\rho} G_j L e^{-S\chi_j} - \sum_{j=1}^{\varrho} F_j e^{-S\chi_j}, \\
H_0 &= \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} - \sum_{j=1}^{d} H_j e^{-S\chi_j},
\end{aligned}
\tag{3.40}
$$

and note that this selection is such that equations (3.37) and (3.38) holds for $P = I$. To conclude the proof note that equation (3.37) has $P = I$ as unique solution if and only if $\sigma(S) \cap \sigma(\bar{F}(s)) = \emptyset$. □

The proposed model has several free design parameters, namely G_j, F_j, H_j, χ_j, ϱ, ρ and d. Selecting $\varrho = 0$, $\rho = 1$, $d = 0$ and $\chi_1 = 0$ (in this case we define $G = G_1$), yields a reduced order model with no delays. In other words, we reduce an infinite dimensional system to a finite dimensional one of dimension ν. This reduced order model coincides with the one in Chapter 2 (equation (2.36)) and all results therein are directly applicable: the parameter G can be selected to achieve matching with prescribed eigenvalues, matching with prescribed relative degree, etc. On the other hand, the choice of eliminating the delays is likely to destroy some underlying dynamics of the model. With this in mind, a possible choice is to keep F_j, G_j and H_j free with $\varrho = d = \varsigma$ and $\rho = \mu$. In this case we can use the matrices F_j, G_j and H_j, with $\tau_j = \chi_j$, to maintain some important physical properties of the delay structure of the system. This concept is illustrated in the next example.

Example 3.1. To illustrate the above idea consider the model of a LC transmission line described by the linear neutral differential time-delay system

$$\begin{aligned} \dot{x} &= Ax + D\dot{x}_\tau + Bu, \\ y &= Cx, \end{aligned} \tag{3.41}$$

with

$$A = -\frac{1}{C_1}\begin{bmatrix} \frac{1}{R_1} + \sqrt{\frac{C_0}{L_0}} & 0 \\ -C_1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -\frac{2}{C_1}\sqrt{\frac{C_0}{L_0}}\alpha \\ 0 & \alpha \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^\top, \quad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$

$$\alpha = \frac{1 - R_0\sqrt{\frac{C_0}{L_0}}}{1 + R_0\sqrt{\frac{C_0}{L_0}}}, \quad \tau = 2\sqrt{L_0 C_0},$$

in which $C_1 \in \mathbb{R}_{>0}$, $R_1 \in \mathbb{R}_{>0}$, $C_0 \in \mathbb{R}_{>0}$, $L_0 \in \mathbb{R}_{>0}$, $R_0 \in \mathbb{R}_{>0}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$. The system is such that if $R_0\sqrt{C_0/L_0} = 1$ the delay part of the system disappears (a phenomenon called line-matching) and the model can be described by a system of ordinary differential equations. In the reduced order model it is desirable to maintain this property to preserve the physical structure of the system. For simplicity suppose $S = 1$ and $L = 1$. Then the vector Π can be computed from equation (3.17), that in this case is

$$A\Pi + D\Pi e^{-\tau} - \Pi = -B,$$

which has a unique solution if

$$-\frac{1}{C_1}\left(\frac{1}{R_1} + \sqrt{\frac{C_0}{L_0}}\right) \neq 1, \qquad e^{-\tau}\frac{1 - R_0\sqrt{\frac{C_0}{L_0}}}{1 + R_0\sqrt{\frac{C_0}{L_0}}} \neq 1.$$

Hence, a family of reduced order models, parameterized in G, is described by the equations

$$\begin{aligned} \dot{\xi} &= (1 - e^{-\tau}\alpha - G)\xi + \alpha\dot{\xi}_\tau + Gu, \\ \psi &= C\Pi\xi. \end{aligned} \tag{3.42}$$

Both equations (3.41) and (3.42) describe linear neutral differential time-delay systems when $R_0\sqrt{C_0/L_0} \neq 1$ and linear delay-free systems otherwise.

The matrices F_j, G_j and H_j can also be used to preserve the structure of the model. This is illustrated in the next section.

Reduction of a platoon of vehicles

The platooning problem consists in controlling a group of vehicles tightly spaced following a leader, all moving in a longitudinal direction. The advantages of the automatic cruise control are twofold. First, the use of automatic control to replace human drivers and their low-predictable reaction time with respect to traffic problems (spacing of around $30\,m$ at $60\,km/h$) can reduce the spacing distance between vehicles, consequently decreasing the traffic congestion. Second, the automatic control reduces the human error factor and then increases safety. In recent years successful experiments involving autonomous vehicles have been carried out (*e.g.* the Google driver-less cars), and the use of this technology may be possible in the immediate future. However, when a large number of vehicles is considered, it can be computationally demanding to study the dynamics of the whole platoon to guarantee individual vehicle stability and avoid slinky-type effects (*i.e.* the amplification of the spacing errors between subsequent vehicles as the vehicle "index" increases).

In what follows we use a model well-studied, for which the solution of the platooning problem is known, to illustrate the results of the section. In particular, we are interested in reducing the number of vehicles to only a leader and a follower. Let $x_i(t)$ be the position of the i-th vehicle with respect to some well-defined reference, $v_i(t)$ its speed, $a_i(t)$ its acceleration and denote with $e_i = x_{i+1} - x_i - \ell_i$ the spacing error, with $\ell_i > 0$ the minimum separation distance, for $i = 1, \dots, N$. The resulting model is described by the equations ($i \in \{1, \dots n-1\}$)

$$\begin{aligned}
\dot{e}_i(t) &= v_{i+1}(t) - v_i(t), \\
\dot{v}_i(t) &= a_i(t), \\
\dot{a}_i(t) &= -\frac{a_i(t)}{c} + \frac{1}{c}[k_s e_i(t-\tau) + k_v(v_{i+1}(t-\tau) - v_i(t-\tau))],
\end{aligned} \tag{3.43}$$

where $c > 0$ is the engine time constant, $\tau > 0$ is the total delay (including fueling and transport) for each vehicle, and k_s and k_v are the transmission gains between the vehicles. To this platoon we add a leader car with dynamics described by the equations

$$\begin{aligned}
\dot{v}_n(t) &= a_n(t), \\
\dot{a}_n(t) &= -\frac{a_n(t)}{c} + \frac{1}{c} k_v \left(u(t) - v_n(t)\right),
\end{aligned} \tag{3.44}$$

where $u(t)$ is a desired velocity imposed on the leader with no delay. We select as output of the system the sum of all the spacing errors, namely the distance between the first and the last vehicle. We rewrite the system in compact form as

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\
y(t) &= Cx(t),
\end{aligned} \tag{3.45}$$

with

$$A_0 = \frac{1}{c} \left[\begin{array}{ccccc|c}
A_0^1 & A_0^2 & 0 & \dots & 0 & 0 \\
0 & A_0^1 & A_0^2 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & 0 & A_0^1 & A_0^2 & 0 \\
0 & \dots & \dots & 0 & A_0^1 & A_0^3 \\
\hline
0 & \dots & \dots & \dots & 0 & A_0^4
\end{array}\right], \qquad A_0^1 = \begin{bmatrix} 0 & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_0^2 = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_0^3 = \begin{bmatrix} c & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_0^4 = \begin{bmatrix} 0 & c \\ -k_v & -1 \end{bmatrix},$$

$$A_1 = \frac{1}{c}\left[\begin{array}{ccccc|c} A_1^1 & A_1^2 & 0 & \dots & 0 & 0 \\ 0 & A_1^1 & A_1^2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & A_1^1 & A_1^2 & 0 \\ 0 & \dots & \dots & 0 & A_1^1 & A_1^3 \\ \hline 0 & \dots & \dots & \dots & \dots & 0 \end{array}\right],$$

$$A_1^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_s & -k_v & 0 \end{bmatrix}, \; A_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_v & 0 \end{bmatrix}, \; A_1^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_v & 0 \end{bmatrix},$$

$$B = \left[\begin{array}{ccccccccccccccc} 0 & 0 & 0 & | & 0 & 0 & 0 & | & \dots & | & 0 & 0 & 0 & | & 0 & \frac{k_v}{c} \end{array}\right]^T,$$

$$C = \left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & | & 1 & 0 & 0 & | & \dots & | & 1 & 0 & 0 & | & 0 & 0 \end{array}\right].$$

Consider $n = 8$ identical vehicles with $c = 0.25\,s$, $k_s = 0.875\,s^{-2}$, $k_v = 2.5\,s^{-1}$ and $\tau = 0.005\,s$. We propose two reduced order models that match the 0-moments at $s_1 = 0$, $s_{2,3} = \pm\pi/5$, $s_{4,5} = \pm\pi/30$, with $u(t) = L\omega(t)$, $\dot{\omega}(t) = S\omega(t)$,

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\pi}{5} & 0 & 0 \\ 0 & -\frac{\pi}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{30} \\ 0 & 0 & 0 & -\frac{\pi}{30} & 0 \end{bmatrix},$$

$L = [1\ 0\ 1\ 0\ 1]^\top$, and described by the equations

$$\begin{aligned} \dot{\xi}(t) &= F_0\xi(t) + F_1\xi(t-\tau) + Gu(t), \\ \psi(t) &= C\Pi\xi(t), \end{aligned} \tag{3.46}$$

with F_0 defined as in (3.40) and F_1 free. Note that the number of equations decreases from $3n-1$ to $\nu = 5$. We denote with ψ_I the output of the system (3.46) when F_1 is defined as

$$F_1 = \frac{1}{c}\begin{bmatrix} A_1^1 & A_1^3 \\ 0 & 0 \end{bmatrix}. \tag{3.47}$$

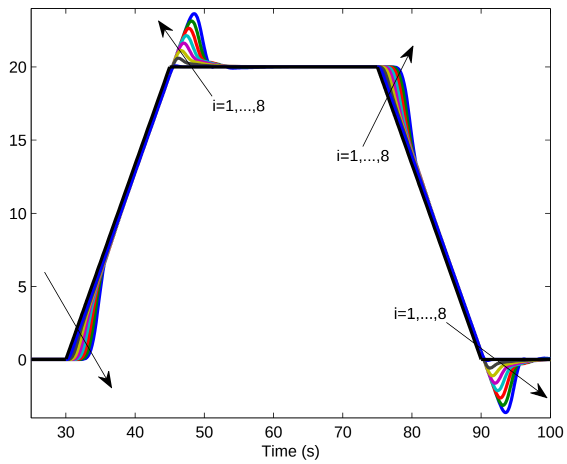

Figure 3.4: Speed of the eight vehicles.

Note that F_1 has the same structure of A_1. We denote with ψ_0 the output of the system (3.46) when $F_1 = 0$. In the latter case all the eigenvalues of the matrix F_0 have been placed at $-\frac{1}{2}$. The input given to the system consists of a speed increase from 0 to $20\,m/s = 72\,km/h$ in $15\,s$, a constant speed of $20\,m/s$ for $30\,s$ and a deceleration to $0\,m/s$ in $15\,s$. The speed of the vehicles are shown in Fig. 3.4. Fig. 3.5 shows the time histories of the output signals $y(t)$ (solid line), $\psi_I(t)$ (dashed line), $\psi_0(t)$ (dotted line). Fig. 3.6 shows the absolute errors between $y(t)$ and $\psi_I(t)$ (dashed line), and between $y(t)$ and $\psi_0(t)$ (dotted line). We see that the output is similar in the three cases and that the reduced order model with delays is tighter to the system, *i.e.* the ratio between the area under the error curve of the model with delays and the area under the error curve of the model with no delays is 0.799.

The matrices F_j, G_j and H_j can also be used to achieve other modeling objectives, as shown in the next section.

Reduced order model interpolating at $(\varrho + 1)\nu$ points

In this section we show how to exploit the matrices F_j and H_j in (3.39) to achieve moment matching at more than ν points, still maintaining

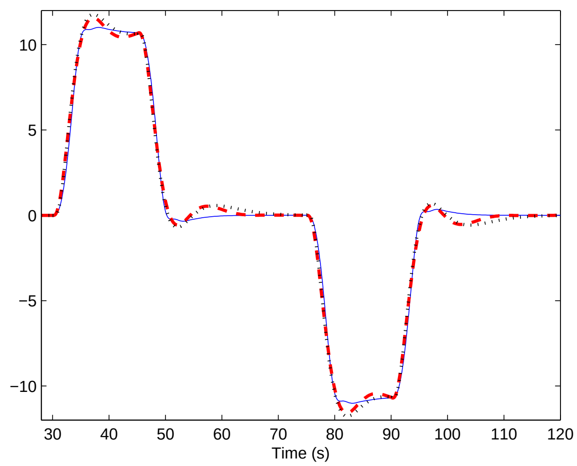

Figure 3.5: Output signals $y(t)$ (solid line), $\psi_I(t)$ (dashed line), and $\psi_0(t)$ (dotted line).

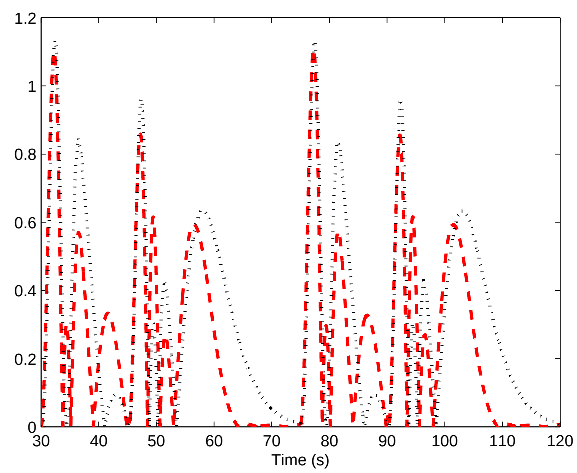

Figure 3.6: Absolute errors between $y(t)$ and $\psi_I(t)$ (dashed line), and between $y(t)$ and $\psi_0(t)$ (dotted line).

the same dimension ν of the matrix F_0. We analyze the case in which $\varrho = 1$, $\rho = 3$ and $d = 1$ (F_1, G_2, G_3 and H_1 are the free parameters), for ease of notation. We further assume without loss of generality that there are no delays in the equation of the output y of system (3.5). The general case can be analyzed in a similar way. The problem is solved in the next proposition.

Proposition 3.3. Let $S_a \in \mathbb{R}^{\nu\times\nu}$ and $S_b \in \mathbb{R}^{\nu\times\nu}$ be two non-derogatory matrices such that $\sigma(S_a) \cap \sigma(S_b) = \emptyset$ and let L_a and L_b be such that the pairs (L_a, S_a) and (L_b, S_b) are observable. Let $\Pi_a = \Pi$ be the unique solution of equation (3.9), with $L = L_a$ and $S = S_a$, and let $\Pi_b = \Pi$ be the unique solution of equation (3.9), with $L = L_b$ and $S = S_b$. Consider F_0 and H_0 as in (3.40) with $\chi_2 = 0$, $S = S_a$ and $L = L_a$.

- If $d = 1$ and $L = L_a = L_b$, system (3.36) with the selection

$$
\begin{aligned}
F_1 &= (S_b - S_a - G_3(e^{-S_b\chi_3} - e^{-S_a\chi_3}))(e^{-S_b\chi_1} - e^{-S_a\chi_1})^{-1}, \\
F_0 &= S_a - G_2 L - G_3 L e^{-S_a\chi_3} - F_1 e^{-S_a\chi_1}, \\
H_1 &= (C\Pi_b - C\Pi_a)(e^{-S_b\chi_1} - e^{-S_a\chi_1})^{-1}, \\
H_0 &= C\Pi_a - H_1 e^{-S_a\chi_1},
\end{aligned}
\tag{3.48}
$$

  belongs to the family (3.39) and is a reduced order model of system (3.5) achieving moment matching at S_a and S_b, for any G_2 and G_3 such that $s_i \notin \sigma(\bar{F}(s))$, for all $i = 1, \dots, \bar{k}$.

- If $d = 0$, the family (3.39) with

$$
\begin{aligned}
F_1 = (&P_b S_b - S_a P_b + G_2 L_a P_b + G_3 L_a e^{-S_b\chi_3} P_b \\
&- G_2 L_b - G_3 L_b e^{-S_b\chi_3})(P_b e^{-S_b\chi_1} - e^{-S_a\chi_1} P_b)^{-1},
\end{aligned}
\tag{3.49}
$$

  is, for any P_b such that $C\Pi_a P_b = C\Pi_b$, a reduced order model of system (3.5) achieving moment matching at S_a and S_b, for any G_2 and G_3 such that $s_i \notin \sigma(\bar{F}(s))$, for all $i = 1, \dots, \bar{k}$.

Proof. We begin with the case $d = 1$. Easy computations show that the matrices

$$
\begin{aligned}
F_0 &= S_a - G_2 L_a - G_3 L_a e^{-S_a\chi_3} - F_1 e^{-S_a\chi_1}, \\
H_0 &= C\Pi_a - H_1 e^{-S_a\chi_1},
\end{aligned}
\tag{3.50}
$$

defined in (3.40), solve the equations

$$\begin{aligned} &F_0 P_a + F_1 P_a e^{-S_a \chi_1} - P_a S_a = -G_2 L_a - G_3 L_a e^{-S_a \chi_3}, \\ &C\Pi_a = H_0 P_a + H_1 P_a e^{-S_a \chi_1}, \end{aligned} \tag{3.51}$$

with $P_a = I$. The matrix F_1 given in (3.49) solves the equation

$$F_0 P_b + F_1 P_b e^{-S_b \chi_1} - P_b S_b = -G_2 L_b - G_3 L_b e^{-S_b \chi_3},$$

for any invertible P_b. Substituting H_0 in

$$C\Pi_b = H_0 P_b + H_1 P_b e^{-S_b \chi_1}, \tag{3.52}$$

yields

$$H_1 = (C\Pi_b - C\Pi_a P_b)(P_b e^{-S_b \chi_1} - e^{-S_a \chi_1} P_b)^{-1}.$$

The matrices F_0, F_1, H_0, H_1 are such that the resulting reduced order model achieves moment matching at S_a and S_b and selecting $L = L_a = L_b$ and $P_b = I$ they yield (3.48). If $d = 0$, equation (3.52) reduces to

$$C\Pi_b = H_0 P_b$$

for some P_b. We then have to prove that there always exists a P_b such that $C\Pi_a P_b = C\Pi_b$ and F_1 is well-defined. To prove the first claim note that the condition consists in finding ν^2 parameters to solve ν equations. If $C\Pi_a \neq 0$ there exist always such a P_b, full rank and upper triangular (possibly after a change of coordinates). Finally note that by the hypotheses on the system and the signal generator there exists at least a component of $C\Pi_a$ which is not zero. To prove the second claim we have to show that

$$\text{rank}\left\{ P_b e^{-S_b \chi_1} - e^{-S_a \chi_1} P_b \right\} = \nu. \tag{3.53}$$

Note now that selecting S_a and S_b in complex Jordan form implies that the matrices in equation (3.53) are all upper triangular. Condition (3.53) can be rewritten as

$$\pi_{bi} \left(e^{-s_{bi} \chi_1} - e^{-s_{ai} \chi_1} \right) \neq 0, \qquad \forall\, i = 1, \ldots, \nu,$$

with π_{bi}, s_{ai} and s_{bi} the eigenvalues of P_b, S_a and S_b, respectively. Since $\sigma(S_a) \cap \sigma(S_b) = \emptyset$ then $\sigma(e^{-S_a \chi_1}) \cap \sigma(e^{-S_b \chi_1}) = \emptyset$, hence the claim follows. $\square$

The family of systems characterized in Proposition 3.3 achieves moment matching at 2ν interpolation points. Note that the matrices G_2 and G_3 remain free parameters and they can be used to achieve the properties discussed in Chapter 2. For instance G_2 and G_3 can be used to set both the eigenvalues of F_0 and F_1. In addition, note that G_j has exactly ν free parameters. Hence, for instance, to assign the eigenvalues of j matrices F_j, j matrices G_j are needed.

Remark 3.5. Although it is possible to interpolate at several different points s_i maintaining the same dimension ν, the order of interpolation at s_i cannot exceed ν because it is limited, by definition, by the dimension of the matrix S_j.

Remark 3.6. The result can be generalized to $\varrho > 1$ delays, obtaining reduced order models that interpolate at $(\varrho + 1)\nu$ points. This result can be used also when the system to be reduced is not a time-delay system. In other words, a system described by ordinary differential equations can be reduced to a system described by time-delay differential equations with an arbitrary number of delays ϱ achieving moment matching at $(\varrho+1)\nu$ points. This property of interpolating an arbitrary large number of points comes to the cost that the reduced order model becomes an infinite dimensional system.

The next example illustrates the idea of approximating delay-free systems with time-delay systems exploiting the additional degrees of freedom to increase the number of interpolation points.

Example 3.2. Consider a single-input, single-output system of order $n = 1006$, which has a Bode plot with three peaks, described by the equations

$$\dot{x} = Ax + Bu, \qquad\qquad y = Cx,$$

where $A = \mathrm{diag}(A_1, A_2, A_3, A_4)$, with

$$A_1 = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 20 \\ -20 & -1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -1 & 40 \\ -40 & -1 \end{bmatrix},$$

and

$$A_4 = \mathrm{diag}(-1, -2, \dots, -1000), \qquad B^\top = C = [\,\underbrace{10 \dots 10}_{6\text{ times}} \;\; \underbrace{1 \dots 1}_{1000\text{ times}}\,].$$

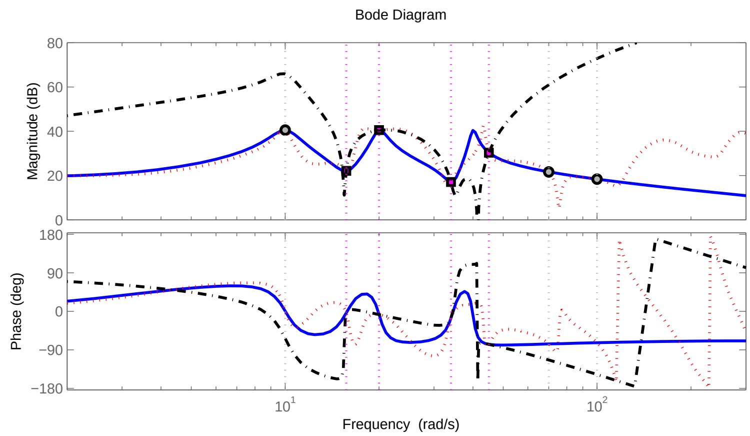

Figure 3.7: Bode plot of the system (solid line), of the delay-free reduced order model (dash-dotted line) and of the time-delay reduced order model (dotted line). The squares indicate the first set of interpolation points, whereas the circles indicate the second set.

We start obtaining a linear delay-free reduced order model of order $\nu = 8$. The matrices of the signal generator (3.6) have been selected as $S = S_a = \mathrm{diag}(S_2, S_3, S_4, S_5)$, with $S_2 = 1.57\,S_1$, $S_3 = 2\,S_1$, $S_4 = 3.4\,S_1$, $S_5 = 4.5\,S_1$, where $S_1 = A_1 + I$, and L randomly generated, to interpolate the moments close to the three peaks. The delay-free model (3.39) has been constructed with the technique presented in Chapter 2 assigning the eigenvalues of F_0 such that $\sigma(F_0) \subset \sigma(A)$. Fig. 3.7 shows the Bode plot of the system to be reduced (solid line) and of the reduced order model (dash-dotted line). The interpolation points are indicated by four squares. Note that the reduced order model approximates poorly the system because few interpolation points have been used. To improve the approximation, we apply the technique presented in Proposition 3.3. The matrix S_b has been selected as $S_b = \mathrm{diag}(S_6, S_7, S_8, S_1)$, with $S_7 = 7\,S_1$, $S_8 = 10\,S_1$ and

$$S_6 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Selecting $\chi = \chi_1 = \chi_3 = 0.05$, yields

$$F_0 = S_a - (S_b - S_a)(e^{-S_b\chi} - e^{-S_a\chi})^{-1}e^{-S_a\chi} - G_2L,$$

and

$$F_1 = (S_b - S_a)(e^{-S_b\chi} - e^{-S_a\chi})^{-1} - G_3 L.$$

Note that, because of the selection $\chi_1 = \chi_3$, F_0 does not depend upon G_3. Thus, the eigenvalues of both F_0 and F_1 have been assigned such that $\sigma(F_0) = \sigma(F_1) \subset \sigma(A)$. Fig. 3.7 shows the Bode plot of this reduced order model (dotted line). The three additional interpolation points are indicated with the circles. In addition the plot shows clearly that the model interpolates at zero. Thus, the addition of one delay improved the quality of the approximation of the system without increasing the size of the matrices. However, note that a delay-free model with $\nu = 16$ would be a better approximation because the introduction of the delay is, at the same time, detrimental (in particular at high frequencies).

3.3 Additional Topics for Linear and Nonlinear Time-delay Systems

Similarly to what we have done in Chapter 2, we now present a series of results for systems in special form.

3.3.1 Moment at infinity

As pointed out in Chapter 2, we can use the final value theorem to characterize the moments at $s_i = +\infty$. Note that for differential time-delay systems, the transfer function $W(s)$ is transcendental. This implies that the computation of the limit at $s = +\infty$ has to be done with care. It is fundamental to determine what is the meaning of the limit at infinity because the result (if well-defined) would depend upon which direction at infinity is considered. In this context the limit has to be taken along the positive real axis. Then we have the following results[3].

Theorem 3.10. Let Υ be the set of values of $j = \varsigma + 1, \ldots, \mu$ such that $\tau_j = 0$. Consider system (3.5).

[3]To be coherent with equation (3.1) we ignore the delays in the equation of the output y of the linear system (3.5).

- If $\Upsilon \neq \emptyset$ then the k moments at infinity are $\eta_k(\infty) = \sum_{j\in\Upsilon} CA_0^{k-1}B_j$, with $\eta_0(\infty) = 0$.
- If $\Upsilon = \emptyset$ then all the moments at infinity are identically zero.

Consider system (3.1).

- If $\Upsilon \neq \emptyset$ then the k moments at infinity are $\eta_k(\infty) = y_I^{(k-1)}(0^+)$, with $\eta_0(\infty) = 0$.
- If $\Upsilon = \emptyset$ then all the moments at infinity are identically zero.

Proof. By the equivalence between the moments at infinity and the impulse response at $t = 0^+$, it follows that if $\tau_j = 0$ for some $j = \varsigma + 1, \ldots, \mu$ the behavior of the systems at 0^+ is the same as the corresponding delay-free system (because no delay on the state has "kicked in" at $t = 0^+$). If $\tau_j \neq 0$ for all $j = \varsigma + 1, \ldots, \mu$ then the impulse response is delayed and it follows that the response at 0^+ is zero together with its time derivative. Once established that the behavior is as for delay-free systems, the claim follows from the discussion in Chapter 2. □

From Theorem 3.10 it appears that a finite dimensional system is sufficient to characterize the moments at infinity and we can use it to match the moments at infinity as described in Chapter 2. However, note that some properties of the transfer function are lost with a finite dimensional system. In fact if, for instance, $\tau_j \neq 0$ for some $j = 1, \ldots, \varsigma$ and $\tau_j = 0$ for some $j = \varsigma + 1, \ldots, \mu$ the expansion at infinity along the *negative* real axis is identically zero, while for finite dimensional systems the expansion is the same as along the positive real axis. Or, if $\tau_1 = \tau_{\varsigma+1} > 0$ and $\tau_j = 0$ for $i = 2, \ldots, \varsigma, \varsigma + 2, \ldots, \mu$, then the first coefficient of the expansion at infinity along the negative real axis is $-CB_1$ for time-delay systems, while zero (*i.e.* the same as along the positive axis) for finite dimensional systems. This suggests that a finite dimensional model that matches the moments of the system at infinity may not be a good approximation of the dynamics of the system far from $t = 0$ and that to preserve the properties of the transcendental transfer function it is necessary to choose a model with the same delay structure as the original system.

3.3.2 Matching at $h \circ \pi_a$ and $h \circ \pi_b$

We conclude the chapter with a nonlinear version of the result yielding the interpolation at 2ν points, namely we show how to exploit the free parameters to achieve moment matching at two moments $h \circ \pi_a$ and $h \circ \pi_b$ maintaining the same number of equations describing the reduced order model. Consider system (3.1) and, to simplify the exposition, the signal generators described by the linear equation

$$\dot{\omega} = S_a \omega, \qquad\qquad u = L\omega.$$

As highlighted in Section 2.4.3, considering the model reduction problem for nonlinear systems when the signal generator is a linear system is of particular interest since the reduced order models have a very simple description. This observation holds true also in the case of time-delay systems, namely a nonlinear time-delay system can be approximated by a linear time-delay equation with a nonlinear output map. Hence, a reduced order model of system (3.1) at (S_a, L) is given by the family of systems

$$\begin{aligned} \dot{\xi} &= F_0\xi + F_1\xi_\chi + G_2 u + G_3 u_\chi, \\ \psi &= \kappa_0(\xi) + \kappa_1(\xi_\chi), \end{aligned} \tag{3.54}$$

with κ_0 and κ_1 analytic mappings, if there exists a unique matrix P_a such that

$$\begin{aligned} &F_0 P_a + F_1 P_a e^{-S_a\chi} - P_a S_a = -G_2 L - G_3 L e^{-S_a\chi}, \\ &h(\pi_a(\omega)) = \kappa_0(P_a\omega) + \kappa_1(P_a e^{-S_a\chi}\omega), \end{aligned}$$

Consider now another signal generator described by the linear equation

$$\dot{\omega} = S_b \omega, \qquad\qquad u = L\omega,$$

and the problem of selecting F_0, F_1, G_2, G_3, κ_0 and κ_1 such that the reduced order model (3.54) matches the moments of system (3.1) at (S_a, L) and (S_b, L). The problem presented in this section is solved by the next proposition.

Proposition 3.4. Let $S_a \in \mathbb{R}^{\nu\times\nu}$ and $S_b \in \mathbb{R}^{\nu\times\nu}$ be two non-derogatory matrices such that $\sigma(S_a) \cap \sigma(S_b) = \emptyset$ and let L be such that the pairs (L, S_a) and (L, S_b) are observable. Let $\pi_a = \pi$ be the unique solution of (3.4), with $L = L$ and $S = S_a$, and let $\pi_b = \pi$ be the unique solution of (3.4), with $L = L$ and $S = S_b$. Then system (3.54) with the selection

$$
\begin{aligned}
F_1 &= (S_b - S_a - G_3(e^{-S_b\chi} - e^{-S_a\chi_3}))(e^{-S_b\chi} - e^{-S_a\chi})^{-1}_{,} \\
F_0 &= S_a - G_2L - G_3Le^{-S_a\chi} - F_1e^{-S_a\chi}, \\
\kappa_0(\omega) &= h(\pi_a(\omega)) - \kappa_1(e^{-S_a\chi}\omega),
\end{aligned}
$$

and κ_1 a mapping such that

$$\kappa_1\left(e^{-S_b\chi}\omega\right) - \kappa_1\left(e^{-S_a\chi}\omega\right) = h(\pi_b(\omega)) - h(\pi_a(\omega)),$$

is a reduced order model of the nonlinear time-delay system (3.1) achieving moment matching at (S_a, L) and (S_b, L), for any G_2 and G_3 such that $\sigma(F_0 + F_1e^{-s\chi}) \cap (\sigma(S_a) \cup \sigma(S_b)) = \emptyset$.

Proof. As showed in the proof of Proposition 3.3, F_0 and F_1 solve the two Sylvester equations

$$
\begin{aligned}
F_0P_a + F_1P_ae^{-S_a\chi} - P_aS_a &= -G_2L - G_3Le^{-S_a\chi}, \\
F_0P_b + F_1P_be^{-S_b\chi} - P_bS_b &= -G_2L - G_3Le^{-S_b\chi},
\end{aligned}
$$

with $P_a = P_b = I$. It remains to determine the mappings κ_0 and κ_1 that solve the matching conditions

$$
\begin{aligned}
h(\pi_a(\omega)) &= \kappa_0(\omega) + \kappa_1\left(e^{-S_a\chi}\omega\right), \\
h(\pi_b(\omega)) &= \kappa_0(\omega) + \kappa_1\left(e^{-S_b\chi}\omega\right).
\end{aligned}
$$

Solving the first equation with respect to κ_0 and substituting the resulting expression in the second yields

$$\kappa_1\left(e^{-S_b\chi}\omega\right) - \kappa_1\left(e^{-S_a\chi}\omega\right) = h(\pi_b(\omega)) - h(\pi_a(\omega)),$$

from which the claim follows. $\square$

The family of linear time-delay systems with nonlinear output mapping characterized in Proposition 3.4 matches the moments $h \circ \pi_a$ and $h \circ \pi_b$ of the nonlinear system (3.1). Note that the matrices G_2 and G_3 remain free parameters and they can be used to achieve the properties discussed in Chapter 2.

Remark 3.7. Proposition 3.4 can be generalized to $\hat{\varrho} > 1$ delays, obtaining a reduced order model that matches $(\hat{\varrho}+1)\nu$ moments. The result can also be generalized to nonlinear generators $s_i(\omega)$ assuming that the flow $\Phi^{s_i}_{-\chi_i}(\omega)$ is known for all the delays χ_i and that $\gamma(\xi_{\chi_1}, \ldots, \xi_{\chi_{\hat{\varrho}}})$ in (3.21) is replaced by $\hat{\gamma}_1(\xi_{\chi_1}) + \cdots + \hat{\gamma}_{\hat{\varrho}}(\xi_{\chi_{\hat{\varrho}}})$.

Remark 3.8. Similarly to Proposition 3.3, the number of delays in (3.1) does not play a role in Proposition 3.4. Thus, this result can be applied to reduce a system with an arbitrary number of delays always obtaining a reduced order model with, for example, two delays.

3.4 Bibliographical Notes

This chapter is largely based upon the papers Scarciotti and Astolfi [2014a,b, 2016a] and the book chapter Scarciotti and Astolfi [2017b].

Time-delay systems are extensively studied in the literature, *e.g.* see the monographs Hale [1977], Stépán [1989], Hale and Verduyn Lunel [1993], Michiels and Niculescu [2007], Niculescu [2001], Zhong [2006], Bekiaris-Liberis and Krstic [2013], the survey papers Kharitonov [1998], Kolmanovskii et al. [1999], Richard [2003] and the papers that point out the effect on the stability of time-delays, *e.g.* Hale and Verduyn Lunel [2001], Niculescu et al. [1995], Zhang et al. [2003], Kolmanovskii et al. [1999], Olbrot [1984], Datko [1998] and MacDonald [1986], Beddington and May [1986], Abdallah et al. [1993] and Goubet et al. [1995].

The problem of reducing an infinite dimensional system to a finite dimensional one is not new in literature, see *e.g.* Curtain et al. [2010], Ionescu and Iftime [2012] and Iftime [2012], and is how the model reduction has been traditionally intended for time-delay systems, see *e.g.* Blondel and Megretski [2004], Mäkilä and Partington [1999a,b], Zhang et al. [2000], Al-Amer and Al-Sunni [2000], Banks and Kappel [1979], Gu et al. [1992], Glover et al. [1990], Glader et al. [1991], Ohta and Kojima [1999], Yoon and Lee [1997] and Artstein [1982], in which the problem of reducing the transfer function of a linear time-delay system to a rational function is studied. A variety of methods are used, *e.g.* Padé approximation, Taylor expansions, spline approximations and Hankel operator. However, it is well-known that the choice of eliminating the delays is likely to destroy some underlying dynamics of the model, see *e.g.* MacDonald [1986], Beddington and May [1986], Abdallah et al. [1993], Goubet et al. [1995] and Halevi [1996]. In fact, the optimal reduction of general time-delay systems is listed as an unsolved problem in systems theory in Blondel and Megretski [2004]. Note that the concept of minimality for time-delay systems is not uniquely defined, see for example Yamamoto [2008] for a discussion of the different characterizations of minimality for some classes of time-delay systems.

The center manifold theorem has been extended to infinite dimensional systems in Carr [1981]. A specific discussion for time-delay systems is given in [Hale, 1977, Section 10.4] (and references therein). Many other results on the analysis and control of time-delay systems are presented in Hale [1977], such as the stability result given in [Hale, 1977, Chapter 1, Theorem 6.2] which is used to prove some of the results of this chapter. A discussion regarding the property of formal stability for neutral time-delay systems can be found in Byrnes et al. [1984], Hale and Verduyn Lunel [2002] and Richard [2003]. Other specific cases for which the center manifold theory holds for neutral time-delay systems have been considered in Hale [1977, 1974], Byrnes et al. [1984] and references therein.

Since hyperbolic partial differential equations can be locally expressed as neutral time-delay systems and, conversely, any time-delay $y(t) = u(t - \tau)$ can be represented by a classical transport equation (see Kolmanovskii and Nosov [1986], Hale and Verduyn Lunel [1993] and Richard [2003]), the techniques presented in this chapter can be used to establish a model reduction theory for some classes of partial differential equations: the model of the torsional dynamics of the drill-string is, in fact, a neutral system originated from a partial differential equation.

The examples used in the chapter have been inspired by several different sources. The model of the LC transmission line is taken from Niculescu [2001]. For more detail on the problem of vehicle platooning see Ioannou and Chien [1993], Huang [1999], Middleton and Braslavsky [2010] and Niculescu [2001]. The system with dimension $n = 1006$ has been inspired by Penzl [2006] and Antoulas [2005]. The model of the torsional dynamics of an oilwell drillstring, the stabilising feedback and the parameters for the simulations are based upon the papers Saldivar et al. [2011a,b], Boussaada et al. [2012], Saldivar and Mondié [2013], Salvidar et al. [2014] and Navarro-López and Cortés [2007].

4

Data-Driven Model Reduction for Linear and Nonlinear Systems

The model reduction techniques that we have presented, as well as the majority of other model reduction methods presented in the literature, assume the knowledge of a state-space model of the system to be reduced (a notable exception is represented by the Loewner framework). In practice this model is not always available. In this chapter we propose data-driven on-line algorithms for the model reduction of linear and nonlinear, possibly time-delay, systems. Collecting *time-snapshots* of the input and output of the system at a given sequence of time instants t_k, two algorithms to define families of reduced order models at each instant of the iteration t_k are devised. The reduced order model asymptotically matches the moments of the unknown system to be reduced. These algorithms have several advantages with respect to an identification plus reduction technique: there is no need to identify the system, which is expensive both in terms of computational power and storage memory; since the reduced order model matches the moments of the unknown system, it is not just the result of a low-order identification but it actually retains some properties of the larger system; since the proposed algorithm determines directly the moment of a nonlinear system from the input and output data, it does not involve the computation of the solution of the partial differential equations (2.9) or (3.4), which is usually difficult. In addition, for linear systems the capability of determining reduced order models from data matrices (of order proportional to ν) is computationally efficient when compared with model reduction obtained by manipulating the system matrices (of order $n \gg \nu$). Thus, the method is of computational value, for both linear and nonlinear systems, even when the system to be reduced is known. The rest of the chapter is devoted to delay-free systems, however, the results can be easily extended to time-delay systems (see for instance Section 4.2.3).

4.1 Nonlinear Systems

For convenience we recall here the equations of system (2.6), namely

$$\dot{x} = f(x, u), \qquad y = h(x), \tag{4.1}$$

of the signal generator (2.7), namely

$$\dot{\omega} = s(\omega), \qquad u = l(\omega), \tag{4.2}$$

and the partial differential equation (2.9), namely

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), l(\omega)). \tag{4.3}$$

which are used throughout the chapter. The quantities appearing in the equations retain the original meaning given in Chapter 2.

4.1.1 On-line moment estimation from data

Solving equation (4.3) with respect to the mapping π is a difficult task even when there is perfect knowledge of the dynamics of the system, *i.e.* the mapping f. When f is not known equation (4.3) may be solved numerically requiring information on the state of the system. In practice, only measurements of the output y may be available which further complicates the task of determining π. In this section we develop an algorithm to determine the moment of nonlinear systems using measurements of the output. Note that all the results of the chapter exploit the relation between moments and steady-state response.

Recall that under the assumptions of Lemma 2.1, the equation

$$y(t) = h(\pi(\omega(t))) + \varepsilon(t), \tag{4.4}$$

where $\varepsilon(t)$ is an exponentially decaying signal, holds. We introduce the following standard assumption.

Assumption 4.1. The mapping $h \circ \pi$ belongs to the function space identified by the family of continuous basis functions $\varphi_j : \mathbb{R}^\nu \to \mathbb{R}$, with $j = 1, \ldots, M$ (M may be ∞), *i.e.* there exist $\Gamma_j \in \mathbb{R}$, with $j = 1, \ldots, M$, such that

$$h(\pi(\omega)) = \sum_{j=1}^{M} \Gamma_j \varphi_j(\omega),$$

for any ω.

To determine the family of basis functions in Assumption 4.1 we can implement a trial and error procedure starting, for instance, with the use of a polynomial expansion or with the use of an expansion based on functions belonging to the same class as the ones generated by the signal generator (*e.g.* sinusoids, for sinusoidal inputs).

Let

$$\begin{aligned} \Gamma &= \begin{bmatrix} \Gamma_1 & \Gamma_2 & \ldots & \Gamma_N \end{bmatrix}, \\ \Omega(\omega(t)) &= \begin{bmatrix} \varphi_1(\omega(t)) & \varphi_2(\omega(t)) & \ldots & \varphi_N(\omega(t)) \end{bmatrix}^\top, \end{aligned}$$

with $N \le M$. Using a weighted sum of basis functions, equation (4.4) can be written as

$$y(t) = \sum_{j=1}^{N} \Gamma_j \varphi_j(\omega(t)) + e(t) + \varepsilon(t) = \Gamma\Omega(\omega(t)) + e(t) + \varepsilon(t), \tag{4.5}$$

where $e(t) = \sum_{N+1}^{M} \Gamma_j \varphi_j(\omega(t))$ is the error resulting by stopping the summation at N. Let Γ_k be an on-line estimate of the matrix Γ computed at $T_k^w = \{t_{k-w+1}, \ldots, t_{k-1}, t_k\}$ with $0 \le t_0 < t_1 < \cdots < t_{k-w} < \cdots < t_k < \cdots < t_q$, with $w > 0$ and $q \ge w$, namely computed at the time t_k using the last w instants of time t_i assuming that $e(t)$ and $\epsilon(t)$ are known. Since this is not the case in practice, consider the approximation

$$y(t) \approx \sum_{j=1}^{N} \widetilde{\pi}_j \varphi_j(\omega(t)) = \widetilde{\Gamma}\Omega(\omega(t)), \tag{4.6}$$

which neglects the approximation error $e(t)$ and the transient error $\epsilon(t)$, and let $\widetilde{\Gamma}_k = \left[\begin{array}{cccc} \widetilde{\pi}_1 & \widetilde{\pi}_2 & \dots & \widetilde{\pi}_N \end{array} \right]$ be the approximation, in the sense of (4.6), of the estimate Γ_k. Finally we can compute this approximation as follows.

Lemma 4.1. Define the time-snapshots $\widetilde{U}_k \in \mathbb{R}^{w \times N}$ and $\widetilde{\Upsilon}_k \in \mathbb{R}^w$, with $w \geq \nu$, as

$$\widetilde{U}_k = \left[\begin{array}{cccc} \Omega(\omega(t_{k-w+1})) & \dots & \Omega(\omega(t_{k-1})) & \Omega(\omega(t_k)) \end{array} \right]^\top \tag{4.7}$$

and

$$\widetilde{\Upsilon}_k = \left[\begin{array}{cccc} y(t_{k-w+1}) & \dots & y(t_{k-1}) & y(t_k) \end{array} \right]^\top. \tag{4.8}$$

If $\widetilde{U}_k$ is full rank, then

$$\mathrm{vec}(\widetilde{\Gamma}_k) = (\widetilde{U}_k^\top \widetilde{U}_k)^{-1} \widetilde{U}_k^\top \widetilde{\Upsilon}_k, \tag{4.9}$$

is an approximation of the estimate Γ_k.

As expressed in the previous lemma, to ensure that the approximation is well-defined for all k, we need that the elements of T_k^w be selected such that $\widetilde{U}_k^\top \widetilde{U}_k$ is full column rank. This condition expresses a property of persistence of excitation that is guaranteed by the following assumption.

Assumption 4.2. The return times of the Poisson stable point $\omega(0)$ of system (4.2) have an upper bound (see Definition 2.8). In addition, the distribution

$$\mathcal{E} = \mathrm{span}\left\{\omega, \omega^{(1)}, \dots, \omega^{(k)}, \dots\right\}$$

has dimension ν at $\omega(0)$.

The condition on the dimension of the distribution $\mathcal{E}$ is called *excitation rank condition*. Note that for linear systems this condition reduces to the excitability of the pair $(S, \omega(0))$ (see Remark 2.3).

To ease the notation we introduce the following definition.

Definition 4.1. The *estimated moment* of system (4.1) at (s, l) is defined as

$$\widetilde{h \circ \pi}_{N,k}(\omega(t)) = \widetilde{\Gamma}_k \Omega(\omega(t)), \tag{4.10}$$

for all $t \in \mathbb{R}$, with $\widetilde{\Gamma}_k$ computed using (4.9).

Equation (4.9) is a classic least-square estimation formula: it is possible to obtain a recursive version of the formula. To this end we recall a standard result.

Lemma 4.2 (Matrix Inversion Lemma). Let A, B and $C^{-1} + DA^{-1}B$ be full rank matrices. Then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Proof. Direct computation yields

$$(A + BCD)(A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}) = I.$$

□

Theorem 4.3. Assume that $\Delta_k = (\widetilde{U}_k^\top \widetilde{U}_k)^{-1}$ and $\Psi_k = (\widetilde{U}_{k-1}^\top \widetilde{U}_{k-1} + \Omega(\omega(t_k))\Omega(\omega(t_k))^\top)^{-1}$ are full rank for all $t \geq t_r$ with $t_r \geq t_w$. Given $\mathrm{vec}(\widetilde{\Gamma}_r)$, Δ_r and Ψ_r, the recursive least-square formula

$$\begin{aligned} \mathrm{vec}(\widetilde{\Gamma}_k) = \mathrm{vec}(\widetilde{\Gamma}_{k-1}) + \Delta_k\Omega(\omega(t_k))\Big(y(t_k) - \Omega(\omega(t_k))^\top \mathrm{vec}(\widetilde{\Gamma}_{k-1})\Big) \\ -\Delta_k\Omega(\omega(t_{k-w}))\Big(y(t_{k-w}) - \Omega(\omega(t_{k-w}))^\top \mathrm{vec}(\widetilde{\Gamma}_{k-1})\Big), \end{aligned} \tag{4.11}$$

with

$$\begin{aligned} \Delta_k = \Psi_k - \\ -\Psi_k\Omega(\omega(t_{k-w}))\left(I + \Omega(\omega(t_{k-w}))^\top\Psi_k\Omega(\omega(t_{k-w}))\right)^{-1}\Omega(\omega(t_{k-w}))^\top\Psi_k \end{aligned} \tag{4.12}$$

and

$$\Psi_k = \Delta_{k-1} - \Delta_{k-1}\Omega(\omega(t_k))\Big(I + \Omega(\omega(t_k))^\top\Delta_{k-1}\Omega(\omega(t_k))\Big)^{-1}\omega(t_k)^\top\Delta_{k-1}. \tag{4.13}$$

holds for all $t \geq t_r$.

Proof. Note that

$$\begin{aligned} \Delta_k^{-1} &= \widetilde{U}_k^\top \widetilde{U}_k = \sum_{i=k-w+1}^{k} \Omega(\omega(t_i))\Omega(\omega(t_i))^\top = \\ &= \sum_{i=k-w}^{k-1} \Omega(\omega(t_i))\Omega(\omega(t_i))^\top + \Omega(\omega(t_k))\Omega(\omega(t_k))^\top - \Omega(\omega(t_{k-w}))\Omega(\omega(t_{k-w}))^\top \\ &= \Delta_{k-1}^{-1} + \Omega(\omega(t_k))\Omega(\omega(t_k))^\top - \Omega(\omega(t_{k-w}))\Omega(\omega(t_{k-w}))^\top \end{aligned}$$

and that rewriting equation (4.9) for $k-1$ yields

$$\widetilde{U}_{k-1}^{\top}\widetilde{\Upsilon}_{k-1}=\sum_{i=k-w}^{k-1}\Omega(\omega(t_i))y(t_i)=\Delta_{k-1}^{-1}\operatorname{vec}(\widetilde{\Gamma}_{k-1}).$$

Substituting the first equation in the second we obtain

$$\sum_{i=k-w}^{k-1}\Omega(\omega(t_i))y(t_i)=\Delta_k^{-1}\operatorname{vec}(\widetilde{\Gamma}_{k-1})-\Omega(\omega(t_k))\Omega(\omega(t_k))^{\top}\operatorname{vec}(\widetilde{\Gamma}_{k-1})+ \\ +\Omega(\omega(t_{k-w}))\Omega(\omega(t_{k-w}))^{\top}\operatorname{vec}(\widetilde{\Gamma}_{k-1}),$$

which substituted in (4.9), namely

$$\operatorname{vec}(\widetilde{\Gamma}_k)=\Delta_k\left(\sum_{i=k-w}^{k-1}\Omega(\omega(t_i))y(t_i)+\Omega(\omega(t_k))y(t_k)-\Omega(\omega(t_{k-w}))y(t_{k-w})\right)$$

yields equation (4.11). Finally equations (4.12) and (4.13) are obtained applying recursively the matrix inversion lemma to

$$\Delta_k=\left(\Psi_k^{-1}-\Omega(\omega(t_{k-w}))\Omega(\omega(t_{k-w}))^{\top}\right)^{-1}$$

and

$$\Psi_k=\left(\Delta_{k-1}^{-1}+\Omega(\omega(t_k))\Omega(\omega(t_k))^{\top}\right)^{-1}.$$

□

To start the iteration we can use dummy values to fill the initial entries of $\operatorname{vec}(\widetilde{\Gamma}_r)$, Δ_r and Ψ_r. Since the formulas "forget" the oldest measurements, after a sufficient number of iterations all the dummy measurements are forgotten.

We summarize the discussion of this section with the following algorithm.

Algorithm 1. Let k be a sufficiently large integer. Select $\widetilde{\tau}>0$ sufficiently small. Select $w\geq\nu$.

1: Construct the matrices $\widetilde{U}_k$ and $\widetilde{\Upsilon}_k$.

2: **If** rank $\left(\widetilde{U}_k^{\top}\widetilde{U}_k\right)=N$ **then** compute $\widetilde{\Gamma}_k$ solving equation (4.9) (or (4.11)).

Else increase w. **If** $k-w<0$ **then** restart the algorithm selecting a larger initial k.

3: **If**

$$\left\| \widetilde{\Gamma}_k - \widetilde{\Gamma}_{k-1} \right\| > \frac{\widetilde{\tau}}{t_k - t_{k-1}}, \tag{4.14}$$

then $k = k + 1$ **go to** 1.

4: **Stop**.

The convergence of the estimated moment is guaranteed by the next result.

Theorem 4.4. Suppose Assumptions 2.1, 4.1 and 4.2 hold. Then

$$\lim_{t\to\infty} \left(h(\pi(\omega(t))) - \lim_{N\to M} \widetilde{h\circ\pi}_{N,k}(\omega(t)) \right) = 0.$$

Proof. Assumption 4.2 guarantees that the times T_k^w can be selected such that the approximation $\widetilde{\Gamma}_k$ is well-defined for all k, whereas Assumption 2.1 guarantees that Lemma 2.1 holds and thus that $h \circ \pi$ is well-defined. The quantity $||\varepsilon(t_k)||$ vanishes exponentially to zero by Assumption 2.1. Hence, by Assumption 4.1, $\lim_{N\to M} \widetilde{h\circ\pi}_{N,k}(\omega(t)) - h(\pi(\omega(t))) = 0$. $\square$

In the next example we illustrate how to estimate the moment of a nonlinear system.

Example 4.1. A test subject runs on a treadmill starting from rest. In the first experiment a certain constant speed (from 1 to 10.5 km/h) has been set and the subject runs at that speed for a period of time sufficient to make his average heartbeat constant. This value corresponds to the steady-state response of his average heartbeat for a constant input speed, *i.e.* $\dot{\omega} = 0$. The experiment has been iterated five times and the measured steady-state average heartbeat values for different constant velocities are shown in the top graph of Fig. 4.1. In the second experiment, the incline (from option 0 to 8) of the treadmill has been considered as a constant input (for all the incline options the speed has been kept constant to 8 km/h). The steady-state average values of the heartbeat are shown in the bottom graph of Fig. 4.1. Also this

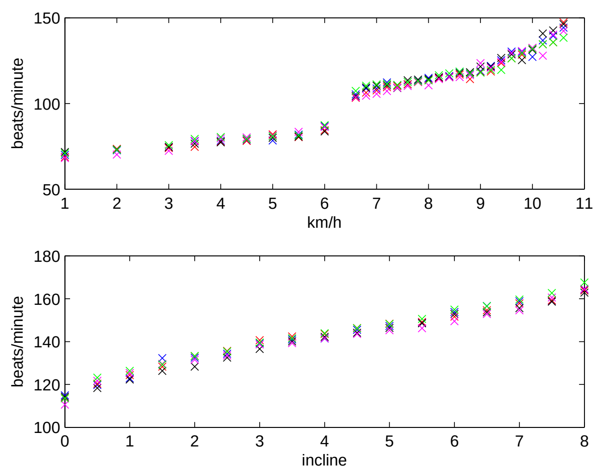

Figure 4.1: Steady-state average heartbeat for five iteration of the experiment for velocities from 1 to 10.5 km/h (top graph) and inclines from 0 to 8 (bottom graph).

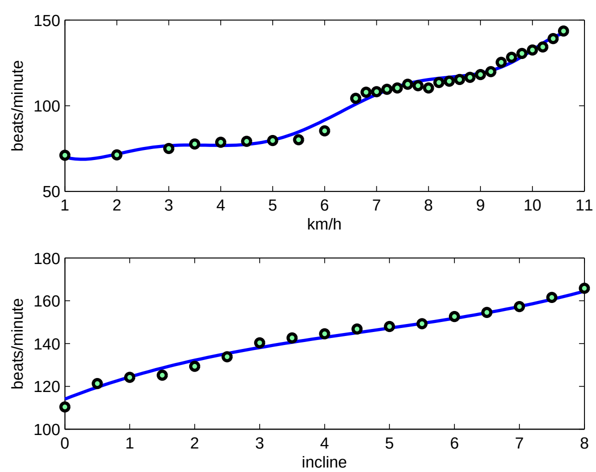

Figure 4.2: Estimated moment of the average heartbeat (solid line) and validation dataset (circles) for the velocity experiment (top graph) and for the incline experiment (bottom graph).

experiment has been iterated five times. A polynomial fitting of the data gives the estimated moment

$$\begin{aligned}\widetilde{h\circ\pi}_{9,\widetilde{\tau}}(\omega) = &-0.0003578\omega^8 + 0.01431\omega^7 - 0.2233\omega^6 + 1.682\omega^5 \\ &-5.852\omega^4 + 4.429\omega^3 + 24.44\omega^2 - 54.66\omega + 100.2,\end{aligned}$$

for the velocity experiment and

$$\widetilde{h\circ\pi}_{4,\widetilde{\tau}}(\omega) = 0.1169\omega^3 - 1.63\omega^2 + 11.86\omega + 114.1,$$

for the incline experiment. Note that the basis functions for the first experiment are the polynomials of order eight, whereas the basis functions for the second experiment are the polynomials of order three. Fig. 4.2 shows the estimated moment in blue/solid line and a validation set of data, indicated with circles, taken from a sixth iteration of the experiment for the velocity experiment (top graph) and the incline experiment (bottom graph). The estimated moment can be used to obtain a reduced order model as defined in the following section.

Up to this point we have always considered one trajectory $\omega(t)$. While this is sufficient in a linear setting, in which local properties are also global, it may be restrictive in the nonlinear setting. To this end, Algorithm 1 can be easily modified to operate with multiple trajectories. To this end, it suffices to implement the algorithm replacing the matrices $\widetilde{U}_k$ and $\widetilde{\Upsilon}_k$ with the matrices

$$\mathfrak{U}=\begin{bmatrix} \widetilde{U}_k^{1\top} & \widetilde{U}_k^{2\top} & \dots & \widetilde{U}_k^{q\top} \end{bmatrix}^\top, \quad \mathfrak{Y}=\begin{bmatrix} \widetilde{\Upsilon}_k^{1\top} & \widetilde{\Upsilon}_k^{2\top} & \dots & \widetilde{\Upsilon}_k^{q\top} \end{bmatrix}^\top, \tag{4.15}$$

respectively, where $\widetilde{U}_k^i$ and $\widetilde{\Upsilon}_k^i$ are the matrices in (4.7) and (4.8), respectively, sampled along the trajectory of system (4.2) starting from the initial condition $\omega(0) = \omega_0^i \in \mathbb{R}^\nu$, with $i = 1, \dots, q$, $q \geq 1$. We refer to this method as the *"$\mathfrak{U}/\mathfrak{Y}$" variation.*

4.1.2 Families of reduced order models

Using the approximation given by (4.10) a reduced order model of system (4.1) can be defined at each instant of time t_k as detailed in the following statements, the proofs of which are omitted since these are a straightforward consequence of the results of this chapter and of Chapter 2.

Definition 4.2. Consider system (4.1) and the signal generator (4.2). Then the system described by the equations

$$\dot{\xi} = \phi_k(\xi, u), \qquad \psi = \kappa_k(\xi), \tag{4.16}$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$ and ϕ_k and κ_k analytic mappings, is a *model of system (4.1) at* (s, l) *at time* t_k if system (4.16) has the moment equal to the estimated moment of system (4.1) at (s, l) at time t_k. In this case, system (4.16) is said to *match* the estimated moment of system (4.1) at (s, l) at time t_k. Furthermore, system (4.16) is a *reduced order model of system (4.1) at* (s, l) *at time* t_k if $\nu < n$.

Lemma 4.5. Consider system (4.1) and the signal generator (4.2). Suppose Assumptions 2.1, 4.1 and 4.2 hold. Then system (4.16) matches the estimated moment of system (4.1) at (s, l) at time t_k if the equation

$$\frac{\partial p_k}{\partial \omega} s(\omega) = \phi_k(p_k(\omega), l(\omega)), \tag{4.17}$$

has a unique solution p_k such that

$$\widetilde{h \circ \pi}_{N,k}(\omega) = \kappa_k(p_k(\omega)), \tag{4.18}$$

where $\widetilde{h \circ \pi}_{N,k}(\omega)$ is obtained with Algorithm 1.

Proposition 4.1. Consider system (4.1) and the signal generator (4.2). Suppose Assumptions 2.1, 2.5, 4.1 and 4.2 hold. Then the system described by the equations

$$\begin{aligned} \dot{\xi} &= s(\xi) - \delta_k(\xi) l(\xi) + \delta_k(\xi) u, \\ \psi &= \widetilde{h \circ \pi}_{N,k}(\xi), \end{aligned} \tag{4.19}$$

where $\widetilde{h \circ \pi}_{N,k}(\omega)$ is obtained using Algorithm 1, is a model of system (4.1) at (s, l) for all t_k if, for all $k \geq 0$, δ_k is an arbitrary mapping such that the partial differential equation

$$\frac{\partial p_k}{\partial \omega} s(\omega) = s(p_k(\omega)) - \delta_k(p_k(\omega)) l(p_k(\omega)) + \delta_k(p_k(\omega)) l(\omega),$$

has the unique solution $p_k(\omega) = \omega$.

Note that the conditions given in Chapter 2 to enforce additional properties (such as asymptotic stability and a prescribed relative degree) upon the reduced order model can be adapted to hold in the present scenario.

Approximated nonlinear model of the Ćuk converter

In this section we illustrate the results of the chapter by means of two examples based on the averaged model of the DC-to-DC Ćuk converter presented in Chapter 2. We begin obtaining a scalar reduced order model which achieves moment matching at zero. In this case the exact expression of the mapping $h \circ \pi$ is known, *i.e.* equation (2.57), and we can compare it directly with the approximation obtained with the polynomial expansion. In the second part of the section we obtain a planar reduced order model with nonlinear state dynamics using an input generated by a nonlinear mapping l. In this case the exact expression of the mapping π is not known and the results of the chapter are used to obtain an approximation of the mapping $h \circ \pi$.

Moment at zero: We have simulated system (2.56) with $u = \omega(0)$, where $\omega(0)$ switched every 0.05 s between the values of 0.2, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8. We have then applied the results of the chapter selecting the horizon length w equal to 1 and fixing the value of $\widetilde{h \circ \pi}(\omega)$ to the one just before the switching time. Then a cubic interpolation has been used to fit the seven points obtained from the simulations. Fig. 4.3 shows the function $h(\pi(\omega))$ given in (2.57) (solid line) and the approximation $\widetilde{h \circ \pi}(\omega)$ (dotted line) for values of $\omega \in (0, 0.9\,]$ (recall that for $u = 1$ the output of the converter becomes negatively unbounded). The seven data points are represented by squares. Note that the approximation is close to the actual moment of the system for $\omega \in (0, 0.85]$.

The reduced order model is chosen as a linear system of order $\nu = 1$ with eigenvalue equal to -230 (to yield a reduced order model with "response time" comparable to the one of the system). The top plot in Fig. 4.4 shows the time histories of the output of system (2.56) (solid line) and of the reduced order model (dotted line) for the set of inputs given by ω equal to 0.35, 0.85, 0.75, 0.55, 0.33, 0.15 and 0.45. Note that none of the values is one of the seven data points. The error between the two outputs is shown in the bottom graph of Fig. 4.4. The figure shows an overall good approximation of system (2.56). The error is comparable with the one shown in Fig. 2.4 and it is mainly due to the transient error $\varepsilon(t)$ caused by approximating a four dimensional nonlinear system with a scalar model.

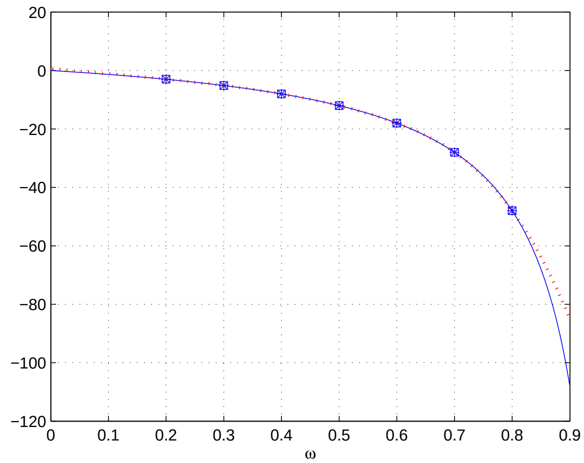

Figure 4.3: The functions $h(\pi(\omega))$ (solid line) and $\widetilde{h\circ\pi}(\omega)$ (dotted line) for $\omega \in (0, 0.9\,]$. The seven data points are represented by squares.

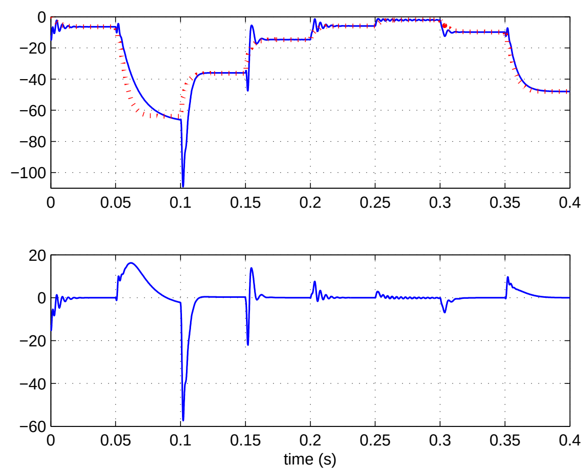

Figure 4.4: Top: time histories of the output of system (2.56) (solid line) and of the reduced order model (dotted line) for the set of input given by ω equal to 0.35, 0.85, 0.75, 0.55, 0.33, 0.15 and 0.45. Bottom: error between the two outputs.

Planar reduced order model: In the previous example the approximated moment of the system has been compared with the exact mathematical expression. However, the example is very simple since the generator is scalar and the input is constant. In this section we present the case in which the input is generated by the equations

$$\dot{\omega}_1 = -75\omega_2, \quad \dot{\omega}_2 = 75\omega_1, \quad u = \max\left(0.15, \frac{1}{2}\left(\omega_1\omega_2 + \omega_1^3 + \frac{1}{2}\right)\right).$$

Note that this produces a positive input signal with higher order harmonics.

Algorithm 1 has been applied with $t_j = j \cdot 10^{-4}$, $w = 838$ and $\widetilde{\tau} = 1.2 \cdot 10^{-7}$. The basis functions are selected as the polynomial surfaces of degree 2 in the first argument and 5 in the second, namely $N = 15$, $\varphi_1(\omega) = 1$, $\varphi_2(\omega) = \omega_1$, $\varphi_3(\omega) = \omega_2$, $\ldots$, $\varphi_{15}(\omega) = \omega_2^5$. To improve the quality of the approximation the "$\mathfrak{U}/\mathfrak{Y}$" variation of the algorithm has been implemented selecting $\omega_0^i \in W := [-0.6, 0.6] \times [-0.6, 0.6]$. These values of ω_0^i generate inputs u in the set $[0.15, 0.62]$. The final value of k is 4000, for which the algorithm stops with $\widetilde{\tau} = 1.186 \cdot 10^{-7}$. The resulting estimated moment is

$$\begin{aligned}\widetilde{h \circ \pi}(\omega)_{15,\widetilde{\tau}} = \; & -3.781 - 3.116\omega_1 + 0.641\omega_2 - 1.71\omega_1^2 - 9.964\omega_1\omega_2 \\ & -1.243\omega_2^2 - 12.23\omega_1^2\omega_2 + 5.185\omega_1\omega_2^2 - 1.764\omega_2^3 \\ & -6.876\omega_1^2\omega_2^2 - 3.026\omega_1\omega_2^3 + 0.8862\omega_2^4 + 14.63\omega_1^2\omega_2^3 \\ & -1.709\omega_1\omega_2^4 + 1.173\omega_2^5,\end{aligned} \tag{4.20}$$

which approximates the actual moment showed in Fig. 4.5. This polynomial approximation fits well for $u < 0.8$, whereas it does not decrease as fast as the actual output of the system when the input is close to 1. This suggests that the following results can be improved if other basis functions are used.

The reduced order model is chosen as in Proposition 4.1, with $\delta_{\widetilde{\tau}} = 220 \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, namely

$$\begin{aligned}\dot{\xi}_1 &= 75\xi_2 + 220\left(u - \max\left(0.15, \tfrac{1}{2}\left(\xi_1\xi_2 + \xi_1^3 + \tfrac{1}{2}\right)\right)\right), \\ \dot{\xi}_2 &= -75\xi_1 + 220\left(u - \max\left(0.15, \tfrac{1}{2}\left(\xi_1\xi_2 + \xi_1^3 + \tfrac{1}{2}\right)\right)\right), \\ \psi &= \widetilde{h \circ \pi}(\xi)_{15,\widetilde{\tau}}.\end{aligned}$$

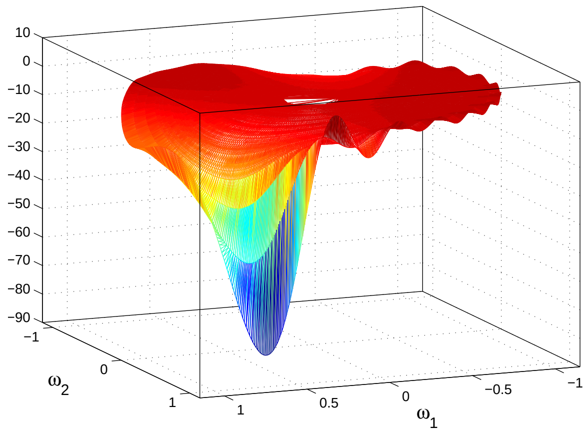

Figure 4.5: Surface obtained from the data points computed with $\omega(t)$ and $y(t)$.

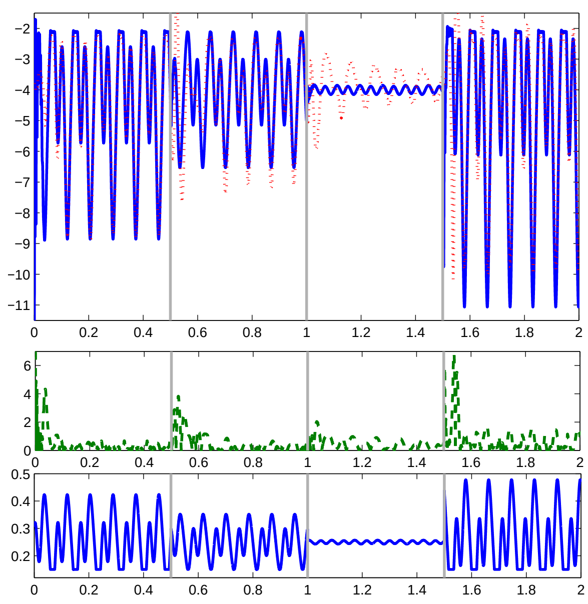

Figure 4.6: Top graph: time histories of the output of system (2.56) (solid/blue line) and of the nonlinear reduced order model (dotted/red line) for the input sequence represented in the bottom graph. The switching times are indicated by solid/gray vertical lines. Middle graph: absolute error (dashed/green line) between the two outputs.

The top graph in Fig. 4.6 shows the time histories of the output of system (2.56) (solid/blue line) and of the reduced order model (dotted/red line) for the input sequence represented in the bottom graph. The input is obtained switching $\omega(0)$ every 0.5s (the switching times are indicated by solid/gray vertical lines). $\omega(0)$ takes, in order, the values $(-0.45, -0.45)$, $(-0.25, -0.45)$, $(0.15, 0.05)$ and $(0.5, 0.5)$. The middle graph in Fig. 4.6 shows the absolute error (dashed/green line) between the two outputs. We note that the larger absolute error is in the third and fourth simulation. In the third, the error is due to poor transient performance and the problem could be alleviated with a selection of $\delta_{\widetilde{\tau}}$ as a function of ξ. The poor approximation for the fourth input is caused by the fact that the input signal lives at the edge of the area approximated by (4.20), where $\widetilde{h\circ\pi}(\omega)$ is not well-fitted.

4.2 Linear Systems

We now look for similar results for the special case of linear systems. As usual, in the linear framework stronger results can be obtained.

For convenience we recall here the equations of system (2.10), namely

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{4.21}$$

of the signal generator (2.11), namely

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{4.22}$$

and the Sylvester equation (2.12), namely

$$A\Pi + BL = \Pi S, \tag{4.23}$$

which are used throughout the chapter. The quantities appearing in the equations retain the original meaning given in Chapter 2.

4.2.1 A preliminary analysis

In this section we provide a preliminary analysis assuming the knowledge of the matrices A, B, C and of the state $x(0)$ in equation (4.21). This analysis is used in the following section for the development of an estimation algorithm which, this time, does not use the matrices A, B, C and the state $x(0)$.

Recall (see Corollary 2.3) that the output of system (4.21) driven by (4.22) is described by

$$x(t) = \Pi\omega(t) + e^{At}(x(0) - \Pi\omega(0)). \tag{4.24}$$

We evaluate equation (4.24) over a set of sample times $T_k^p = \{t_{k-p+1}, \ldots, t_{k-1}, t_k\}$ with $0 \leq t_0 < t_1 < \cdots < t_{k-p} < \cdots < t_k < \cdots < t_q$, with $p > 0$ and $q \geq p$. The set T_k^p represents a moving window of p sample times t_i, with $i = 0, \ldots, q$. We call Π_k the *estimate of the matrix* Π *at* T_k^p, namely the estimate computed at the time t_k using the last p instants of time $t_{k-p+1}, \ldots, t_k$. The estimate can be computed as follows.

Theorem 4.6. Let the time-snapshots $Q_k \in \mathbb{R}^{np\times n\nu}$ and $\chi_k \in \mathbb{R}^{np}$, with $p \geq \nu$, be defined as

$$Q_k = \begin{bmatrix} \omega(t_{k-p+1})^\top \otimes I - \omega(0)^\top \otimes e^{At_{k-p+1}} \\ \vdots \\ \omega(t_{k-1})^\top \otimes I - \omega(0)^\top \otimes e^{At_{k-1}} \\ \omega(t_k)^\top \otimes I - \omega(0)^\top \otimes e^{At_k} \end{bmatrix},$$

and

$$\chi_k = \begin{bmatrix} x(t_{k-p+1}) - e^{At_{k-p+1}}x(0) \\ \vdots \\ x(t_{k-1}) - e^{At_{k-1}}x(0) \\ x(t_k) - e^{At_k}x(0) \end{bmatrix},$$

respectively. Assume the matrix Q_k has full rank, then

$$\text{vec}(\Pi) = (Q_k^\top Q_k)^{-1}Q_k^\top \chi_k. \tag{4.25}$$

Proof. Equation (4.24) can be rewritten as

$$\Pi\omega(t) - e^{At}\Pi\omega(0) = x(t) - e^{At}x(0). \tag{4.26}$$

Using the vectorization operator and the Kronecker product on equation (4.26) yields

$$\text{vec}(\Pi\omega(t)) - \text{vec}(e^{At}\Pi\omega(0)) = \text{vec}(x(t) - e^{At}x(0)),$$

and

$$(\omega(t)^\top \otimes I - \omega(0)^\top \otimes e^{At}) \operatorname{vec}(\Pi) = \operatorname{vec}(x(t) - e^{At}x(0)). \tag{4.27}$$

Computing equation (4.27) at all elements of T_k^p yields

$$Q_k \operatorname{vec}(\Pi) = \chi_k. \tag{4.28}$$

If the matrix Q_k has full rank, we can compute Π from the last equation yielding equation (4.25). □

As we pointed out in the nonlinear case, the selection of the set T_k^p can affect the *quality* of the data and the rank of the matrix Q_k. To avoid the issue, we explicitly formulate a linear equivalent of Assumption 4.2.

Assumption 4.3. The pair $(S, \omega(0))$ is excitable.

Recall, from the proof of Theorem 2.6, that if Assumption 4.3 holds, then the elements of T_k^ν can be selected such that $\operatorname{rank}\left(\begin{bmatrix} \omega(t_{k-\nu+1}) & \dots & \omega(t_k) \end{bmatrix}\right) = \nu$ for all k. In the following when we suppose that Assumption 4.3 holds, we assume that T_k^ν is chosen such that this rank condition holds. We can now provide the conditions under which the matrix Q_k is full rank.

Lemma 4.7. Suppose Assumptions 2.3 and 4.3 hold. If $p = \nu$, then Q_k is full rank.

Proof. Consider the i-block element of the matrix Q_k, namely

$$\omega(t_i)^\top \otimes I - \omega(0)^\top \otimes e^{At_i}.$$

Note that the properties of the Kronecker product yield

$$\begin{aligned}
\omega(t_i)^\top \otimes I - \omega(0)^\top \otimes e^{At_i} &= \omega(0)^\top e^{S^\top t_i} \otimes II - \omega(0)^\top I \otimes I e^{At_i} = \\
&= (\omega(0)^\top \otimes I)(e^{S^\top t_i} \otimes I) - (\omega(0)^\top \otimes I)(I \otimes e^{At_i}) = \\
&= (\omega(0)^\top \otimes I)(e^{S^\top t_i} \otimes I - I \otimes e^{At_i}).
\end{aligned}$$

Since $\sigma(A) \subset \mathbb{C}_{<0}$ and $\sigma(S) \subset \mathbb{C}_0$, the excitability of $(S, \omega(0))$ implies that the i-block element of the matrix Q_k is a $n \times n\nu$ matrix of rank n. Assumption 4.3 implies that ν of these blocks are linearly independent for any $t_i > 0$. As a result Q_k is a square full rank matrix. □

Remark 4.1. Since real data are affected by noise the assumptions of Lemma 4.7 may not hold. In this case p can be taken larger than $n\nu$ and, as well-known from linear algebra, the solution of equation (4.25) is the least squares solution of (4.28).

The discussion carried out so far has the drawback that we require information on the state of the system. In practice, this is usually not the case and only the output y is available. Thus, in the following, we reformulate Theorem 4.6 using the output in place of the state.

Theorem 4.8. Let the time-snapshots $R_k \in \mathbb{R}^{w\times n\nu}$ and $\Upsilon_k \in \mathbb{R}^{w}$, with $w \geq n\nu$, be defined as

$$R_k = \begin{bmatrix} (\omega(0)^\top \otimes C)(e^{S^\top t_{k-w+1}} \otimes I - I \otimes e^{At_{k-w+1}}) \\ \vdots \\ (\omega(0)^\top \otimes C)(e^{S^\top t_{k-1}} \otimes I - I \otimes e^{At_{k-1}}) \\ (\omega(0)^\top \otimes C)(e^{S^\top t_{k}} \otimes I - I \otimes e^{At_{k}}) \end{bmatrix},$$

and

$$\Upsilon_k = \begin{bmatrix} y(t_{k-w+1}) - Ce^{At_{k-w+1}}x(0) \\ \vdots \\ y(t_{k-1}) - Ce^{At_{k-1}}x(0) \\ y(t_k) - Ce^{At_k}x(0) \end{bmatrix},$$

respectively. Assume the matrix R_k has full rank, then

$$\mathrm{vec}(\Pi) = (R_k^\top R_k)^{-1} R_k^\top \Upsilon_k. \tag{4.29}$$

Proof. The result can be proved following the same steps used to obtain equation (4.25). □

Similarly to Lemma 4.7, we now provide the conditions under which R_k is full rank.

Lemma 4.9. Suppose Assumptions 2.3 and 4.3 hold. If $w = n\nu$, then R_k is full rank.

Proof. The proof is similar to the one of Lemma 4.7, although this time also the observability of (C, A) is used (which comes from the minimality of (A, B, C)). □

4.2.2 On-line moment estimation from data

Equation (4.25) contains terms that depend upon the matrix A and the initial states $x(0)$ and $\omega(0)$. Exploiting the fact that these terms enter the response as exponentially decaying functions of time, *i.e.* $\omega(0)^\top \otimes e^{At}$ and $e^{At}x(0)$, we present now an approximate version of the results of the previous section.

Definition 4.3. Let the time-snapshots $\widetilde{Q}_k \in \mathbb{R}^{np\times n\nu}$ and $\widetilde{\chi}_k \in \mathbb{R}^{np}$, with $p \geq \nu$, be

$$\widetilde{Q}_k = \left[\begin{array}{cccc} \omega(t_{k-p+1}) \otimes I & \dots & \omega(t_{k-1}) \otimes I & \omega(t_k) \otimes I \end{array}\right]^\top$$

and

$$\widetilde{\chi}_k = \left[\begin{array}{cccc} x(t_{k-p+1})^\top & \dots & x(t_{k-1})^\top & x(t_k)^\top \end{array}\right]^\top.$$

Assume the matrix $\widetilde{Q}_k$ has full rank, then following the same steps used to obtain equation (4.25), we define

$$\mathrm{vec}(\widetilde{\Pi}_k) = (\widetilde{Q}_k^\top \widetilde{Q}_k)^{-1}\widetilde{Q}_k^\top \widetilde{\chi}_k. \tag{4.30}$$

Note that if Assumption 4.3 holds and $p = \nu$, then $\widetilde{Q}_k$ is square and full rank (the proof of this fact is similar to the proof of Lemma 4.7, thus, it is omitted).

We now prove that $\widetilde{\Pi}_k$ converges to Π. To this end, we first present a preliminary result.

Lemma 4.10. Suppose Assumptions 2.3 and 4.3 hold. There exists a matrix $\bar{\Pi}$ such that $\lim\limits_{t_k\to\infty} \widetilde{\Pi}_k = \bar{\Pi}$.

Proof. By Assumption 2.3 there exists a matrix $\bar{\Pi}$ such that the steady-state response $x^{ss}(t)$ of the interconnection of system (4.21) and the generator (4.22) is described by the equation $x^{ss}(t) = \bar{\Pi}\omega(t)$. Then substituting $\widetilde{\chi}_k^{ss} = \widetilde{Q}_k \,\mathrm{vec}(\bar{\Pi})$ in equation (4.30), yields

$$\lim_{t_k\to\infty} \mathrm{vec}(\widetilde{\Pi}_k) = (\widetilde{Q}_k^\top \widetilde{Q}_k)^{-1}\widetilde{Q}_k^\top \widetilde{\chi}_k^{ss} = \mathrm{vec}(\bar{\Pi}),$$

which is well-defined by Assumption 4.3 if $p = \nu$. □

Theorem 4.11. Let Π be the solution of equation (4.23). Suppose Assumptions 2.3 and 4.3 hold. There exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} \widetilde{\Pi}_k = \Pi.$$

Proof. The matrix $\widetilde{\Pi}_k$ defined in equation (4.30) is such that

$$x(t_k) = \widetilde{\Pi}_k \omega(t_k), \tag{4.31}$$

whereas Π is such that

$$\dot{x}(t)|_{t=t_k} = \Pi S \omega(t_k) + A e^{A t_k}(x(0) - \Pi\omega(0)). \tag{4.32}$$

Consider the first equation of system (4.21) computed at t_k, namely

$$\dot{x}(t)|_{t=t_k} = A x(t_k) + BL\omega(t_k). \tag{4.33}$$

Substituting equations (4.31) and (4.32) in equation (4.33) yields

$$\Pi S \omega(t_k) + A e^{A t_k}(x(0) - \Pi\omega(0)) = A\widetilde{\Pi}_k \omega(t_k) + BL\omega(t_k)$$

and

$$\left(A\widetilde{\Pi}_k + BL - \Pi S\right)\omega(t_k) = A e^{A t_k}(x(0) - \Pi\omega(0)),$$

from which, using equation (4.23) and multiplying by A^{-1}, the equation

$$\left(\widetilde{\Pi}_k - \Pi\right)\omega(t_k) = e^{A t_k}(x(0) - \Pi\omega(0))$$

follows. By Assumption 2.3 there exists a sequence $\{t_k\}$, with $\lim\limits_{k\to\infty} t_k = \infty$, such that for any $t_i \in \{t_k\}$, $\omega(t_i) \neq 0$ and Assumption 4.3 holds. By Assumption 2.3

$$\lim_{k\to\infty}\left(\widetilde{\Pi}_k - \Pi\right)\omega(t_k) = \lim_{k\to\infty} e^{A t_k}(x(0) - \Pi\omega(0)) = 0,$$

and by Assumptions 4.3 and Lemma 4.10, $\lim\limits_{k\to\infty}\left(\widetilde{\Pi}_k - \Pi\right) = \lim\limits_{k\to\infty}\left(\bar{\Pi} - \Pi\right) = 0$. It follows that $\widetilde{\Pi}_k$ converges asymptotically to Π. $\square$

Remark 4.2. Equation (4.31) is reminiscent of the POD of the collection $\{x(t_i)\}$. However, the two concepts are quite different. In fact, the POD of $\{x(t_i)\}$ is

$$\left[\begin{array}{ccc} x(t_0) & \dots & x(t_q) \end{array}\right] = \underbrace{\left[\begin{array}{ccc} \mu(t_0) & \dots & \mu(t_q) \end{array}\right]}_{U} \mathcal{Q}$$

with $\mathcal{Q} \in \mathbb{R}^{q\times q}$ and $U^*U = I$. The dimensions of $\mathcal{Q}$ are related to the number of samples, whereas the dimensions of Π are related to the order of the system to be reduced and of the signal generator. In fact, the POD is a decomposition of the entire cloud of data $\{x(t_i)\}$ along the vectors $\mu(t_i)$, called principal directions of $\{x(t_i)\}$. Instead, in the technique proposed in this chapter the oldest data are discarded as soon as new data satisfying Assumption 4.3 are collected. As a consequence, while $\mathcal{Q}$ is built to describe the entire dynamics of $\{x(t_i)\}$, Π is built to describe only the steady-state response of the system to be reduced. The result is that the POD is usually used with the Petrov-Galerkin projection for a SVD-based approximation, whereas this technique is a moment matching method.

A similar discussion can be carried out for equation (4.29) that contains also terms which depend upon the matrix C. In this case note that equation (4.24) can be written as

$$y(t) = C\Pi\omega(t) + \varepsilon(t),$$

with $\varepsilon(t) = Ce^{At}(x(0) - \Pi\omega(0))$ an exponentially decaying signal.

Theorem 4.12. Define the time-snapshots $\widetilde{R}_k \in \mathbb{R}^{w\times\nu}$ and $\widetilde{\Upsilon}_k \in \mathbb{R}^{w}$, with $w \geq \nu$, as

$$\widetilde{R}_k = \left[\begin{array}{cccc} \omega(t_{k-w+1}) & \dots & \omega(t_{k-1}) & \omega(t_k) \end{array} \right]^\top$$

and

$$\widetilde{\Upsilon}_k = \left[\begin{array}{cccc} y(t_{k-w+1}) & \dots & y(t_{k-1}) & y(t_k) \end{array} \right]^\top.$$

Suppose Assumptions 2.3 and 4.3 hold. Then $\widetilde{R}_k$, for $w \geq \nu$ is full rank and

$$\mathrm{vec}(\widetilde{C\Pi}_k) = (\widetilde{R}_k^\top \widetilde{R}_k)^{-1}\widetilde{R}_k^\top \widetilde{\Upsilon}_k, \tag{4.34}$$

is an approximation of the on-line estimate $C\Pi_k$, namely there exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} \widetilde{C\Pi}_k = C\Pi.$$

Proof. Equation (4.34) can be derived following the same steps used to obtain equation (4.25). If Assumption 4.3 holds and $w = \nu$, then $\widetilde{R}_k$

is square and full rank (the proof of this fact is similar to the proof of Lemma 4.7, thus, it is omitted). The convergence of the sequence $\{\widetilde{C\Pi}_k\}$ to $C\Pi$ is proved repeating the proof of Theorem 4.11. $\square$

Since $\widetilde{R}_k$ is smaller than R_k, the determination of $\widetilde{C\Pi}_k$ is computationally less complex than the computation of $\widetilde{\Pi}_k$. Note also that, from equation (4.34) we are not able to retrieve the matrix $\widetilde{\Pi}_k$, but only $\widetilde{C\Pi}_k$. However, we only need $C\Pi$ to compute the reduced order model, *i.e.* Π is not explicitly required. Finally, as for nonlinear systems, we give the following definition.

Definition 4.4. The *estimated moments* of system (4.21) at (S, L) are defined as the elements of $\widetilde{C\Pi}_k$, computed as in (4.34).

An equivalent of Theorem 4.3 can be formulated in the linear framework. The recursive least-square algorithm is obtained with $\omega(t_k)$ playing the role of $\Omega(\omega(t_k))$, $\widetilde{R}_k$ playing the role of $\widetilde{U}_k$ and $\widetilde{C\Pi}_k$ playing the role of $\widetilde{\Gamma}_k$ (the proof is omitted because it is similar to the proof of Theorem 4.3).

Theorem 4.13. Assume that $\Delta_k = (\widetilde{R}_k^\top \widetilde{R}_k)^{-1}$ and $\Psi_k = (\widetilde{R}_{k-1}^\top \widetilde{R}_{k-1} + \omega(t_k)\omega(t_k)^\top)^{-1}$ are full rank for all $t \geq t_r$ with $t_r \geq t_w$. Given $\mathrm{vec}(\widetilde{C\Pi}_r)$, Δ_r and Ψ_r, the recursive least-square formula

$$\begin{aligned}\mathrm{vec}(\widetilde{C\Pi}_k) = {} & \mathrm{vec}(\widetilde{C\Pi}_{k-1}) + \Delta_k\omega(t_k)\Big(y(t_k) - \omega(t_k)^\top \mathrm{vec}(\widetilde{C\Pi}_{k-1})\Big) \\ & -\Delta_k\omega(t_{k-w})\Big(y(t_{k-w}) - \omega(t_{k-w})^\top \mathrm{vec}(\widetilde{C\Pi}_{k-1})\Big),\end{aligned} \tag{4.35}$$

with

$$\Delta_k = \Psi_k - \Psi_k\omega(t_{k-w})\left(I + \omega(t_{k-w})^\top \Psi_k \omega(t_{k-w})\right)^{-1} \omega(t_{k-w})^\top \Psi_k \tag{4.36}$$

and

$$\Psi_k = \Delta_{k-1} - \Delta_{k-1}\omega(t_k)\left(I + \omega(t_k)^\top \Delta_{k-1}\omega(t_k)\right)^{-1} \omega(t_k)^\top \Delta_{k-1}. \tag{4.37}$$

holds for all $t \geq t_r$.

For single-input, single-output systems the two matrix inversions in the definition of Δ_k and Ψ_k are two divisions. Equations (4.35)-(4.36)-(4.37) can be used to compute a fast, on-line, estimate of $\widetilde{C\Pi}_k$, since the computational complexity of updating (4.35) is $\mathcal{O}(1)$. Thus, the implementation of equations (4.35)-(4.36)-(4.37) to update $\widetilde{C\Pi}_k$ is preferred to equation (4.34), which has a computational complexity, when $w = \nu$, of $\mathcal{O}(\nu^3)$ at each iteration k. In comparison, the Arnoldi or Lanczos procedure for the model reduction by moment matching have a computational complexity of $\mathcal{O}(\nu n^2)$ (or $\mathcal{O}(\alpha\nu n)$ for a sparse matrix A, with α the average number of non-zero elements per row/column of A). In addition, note that these procedures require a model to be reduced and thus further expensive computation has to be considered for the identification of the original system.

The approximations $\widetilde{\Pi}_k$ and $\widetilde{C\Pi}_k$ can be computed with the following algorithm (this is obtained from Algorithm 1 selecting $N = \nu$ and $\varphi_j = \epsilon_j$).

Algorithm 2. Let k be a sufficiently large integer. Select $\widetilde{\tau} > 0$ sufficiently small. Select $w \geq \nu$.

1: Construct the matrices $\widetilde{R}_k$ and $\widetilde{\Upsilon}_k$.

2: **If** rank $\left(\widetilde{R}_k\right) = \nu$ **then** compute $\widetilde{C\Pi}_k$ solving equation (4.34), or (4.35).

Else increase w. **If** $k - w < 0$ **then** restart the algorithm selecting a larger initial k.

3: **If**

$$\left|\left|\widetilde{C\Pi}_k - \widetilde{C\Pi}_{k-1}\right|\right| > \frac{\widetilde{\tau}}{t_k - t_{k-1}}, \tag{4.38}$$

then $k = k + 1$ **go to** 1.

4: **Stop**.

Note, finally, that it is not always possible to arbitrarily select the input of the system to be reduced. For instance the input signal may be composed by several unwanted frequencies. Instead of system (4.22), consider the input described by the equations

$$\dot{\omega} = S\omega, \qquad u = L\omega + v,$$

with $v(t) \in \mathbb{R}^n$ an unknown signal. In this case the output response of system (4.21) is

$$y(t) = C\Pi\omega(t) + Ce^{At}(x(0) - \Pi\omega(0)) + \int_0^t e^{A(t-\tau)}Bv(\tau)d\tau,$$

which can be written as

$$y(t) = C\Pi\omega(t) + \varepsilon(t) + \mathfrak{v}(t),$$

with $\mathfrak{v}(t) = \int_0^t e^{A(t-\tau)}Bv(\tau)d\tau$ and $\varepsilon(t) = Ce^{At}(x(0) - \Pi\omega(0))$. Now we can filter out $\mathfrak{v}$ from y and u with a band-pass filter and apply the results of the chapter to the filtered y_f and u_f.

4.2.3 Families of reduced order models

Using the approximations given by Algorithm 2 a reduced order model of system (4.21) can be defined at each instant of time t_k as detailed in the following statements, the proofs of which are omitted since these are a straightforward consequence of the results of this chapter and of Chapter 2.

Definition 4.5. Consider system (4.21) and the signal generator (4.22). Then the system described by the equations

$$\dot{\xi} = F_k\xi + G_k u, \qquad \psi = H_k\xi, \tag{4.39}$$

with $\xi(t) \in \mathbb{R}^\nu$, $F_k \in \mathbb{R}^{\nu\times\nu}$, $G_k \in \mathbb{R}^{\nu\times 1}$, $H_k \in \mathbb{R}^{1\times\nu}$, is a *model of system (4.21) at* (S, L) *at time* t_k, if system (4.39) has the moments equal to the estimated moments of system (4.21) at (S, L) at time t_k. System (4.39) is a *reduced order model of system (4.21) at* (S, L) *at time* t_k if $\nu < n$.

Lemma 4.14. Consider system (4.21) and the signal generator (4.22). Suppose Assumptions 2.3 and 4.3 hold. Then system (4.39) is a model of system (4.21) at (S, L) at time t_k if there exists a unique solution P_k of the equation

$$F_kP_k + G_kL = P_kS, \tag{4.40}$$

such that

$$\widetilde{C\Pi}_k = H_kP_k, \tag{4.41}$$

where $\widetilde{C\Pi}_k$ is obtained with Algorithm 2.

Proposition 4.2. Consider system (4.21) and the signal generator (4.22). Suppose Assumptions 2.3 and 4.3 hold. Then the system described by the equations

$$\dot{\xi} = (S - G_k L)\xi + G_k u, \qquad \psi = \widetilde{C\Pi}_k \xi, \tag{4.42}$$

where $\widetilde{C\Pi}_k$ is obtained with Algorithm 2, is a model of system (4.21) at (S, L) for all t_k if, for all $k \geq 0$, G_k is any matrix such that $\sigma(S - G_k L) \cap \sigma(S) = \emptyset$.

Properties of the exponentially converging models

In Chapters 2 and 3 we have studied the problem of enforcing additional properties and constraints on the reduced order model. We now go through these properties to determine if, and under which conditions, they hold for the models given in this chapter.

Matching with prescribed eigenvalues: Consider system (4.42) and the problem of determining at every k the matrix G_k such that $\sigma(F_k) = \{\lambda_{1,k}, \ldots, \lambda_{\nu,k}\}$ for some prescribed values $\lambda_{i,k}$. The solution of this problem consists in selecting G_k such that

$$\sigma(S - G_k L) = \sigma(F_k).$$

This is possible for every k and for all $\lambda_{i,k} \notin \sigma(S)$ and note that G_k is independent from the estimate $\widetilde{C\Pi}_k$. Note also that by observability of (S, L), G_k is unique at every k.

Matching with interpolation at 2ν points: The results developed so far can be easily extended to linear time-delay systems renaming the moments of system (3.5) in Chapter 3 as $\overline{C\Pi} = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j}$. Theorem 4.11 holds for the linear time-delay system (3.5) replacing $\widetilde{C\Pi}_k$ with $\widetilde{\overline{C\Pi}}_k$ which is an approximation of $\overline{C\Pi}$ (the proof is a simple exercise). The following result extends Proposition 3.2 to the present framework and Proposition 4.2 to the time-delay framework.

Proposition 4.3. Consider system (3.5) and the signal generator (3.6). Suppose Assumptions 3.3 and 4.3 hold. Then the system described by the equations

$$
\begin{aligned}
\dot{\xi} &= \left(S - \sum_{j=\varrho+1}^{\rho} G_{j,k} L e^{-S\chi_j} - \sum_{j=1}^{\varrho} F_{j,k} e^{-S\chi_j} \right) \xi + \sum_{j=1}^{\varrho} F_{j,k} \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_{j,k} u_{\chi_j}, \\
\psi &= \left(\widetilde{C\Pi}_k - \sum_{j=1}^{d} H_{j,k} e^{-S\chi_j} \right) \xi + \sum_{j=1}^{d} H_{j,k} \xi_{\chi_j},
\end{aligned}
\tag{4.43}
$$

where $\widetilde{C\Pi}_k$ is obtained using Algorithm 2, is a model of system (3.5) at (S, L) for all t_k if, for all $k \geq 0$, $G_{j,k}$ and $F_{j,k}$ are any matrices such that $\sigma(S) \cap \sigma(\bar{F}_k(s)) = \emptyset$, where $\bar{F}_k(s) = \sum_{j=0}^{\varrho} F_{j,k} e^{-s\chi_j}$.

Let $S_a \in \mathbb{R}^{\nu \times \nu}$ and $S_b \in \mathbb{R}^{\nu \times \nu}$ be two non-derogatory matrices such that $\sigma(S_a) \cap \sigma(S_b) = \emptyset$ and let L be such that the pairs (L, S_a) and (L, S_b) are observable. Let $\widetilde{C\Pi}_{a,k} = \widetilde{C\Pi}_k$ be computed with Algorithm 2 with $L = L$ and $S = S_a$, and let $\widetilde{C\Pi}_{b,k} = \widetilde{C\Pi}_k$ be computed with Algorithm 2 with $L = L$ and $S = S_b$. Consider system (4.43) with $\varrho = 1$, $\rho = 3$, $d = 1$, $\chi_2 = 0$ and the problem of determining $F_{1,k}$ and $H_{1,k}$ such that system (4.43) is a model of system (3.5) at S_a and S_b at time t_k. This problem is solved for every k by the selection (see Proposition 3.3)

$$
\begin{aligned}
F_{1,k} &= (S_b - S_a - G_{3,k}(e^{-S_b\chi_3} - e^{-S_a\chi_3}))(e^{-S_b\chi_1} - e^{-S_a\chi_1})^{-1}, \\
F_{0,k} &= S_a - G_{2,k} L - G_{3,k} L e^{-S_a\chi_3} - F_{1,k} e^{-S_a\chi_1}, \\
H_{1,k} &= (\widetilde{C\Pi}_{b,k} - \widetilde{C\Pi}_{a,k})(e^{-S_b\chi_1} - e^{-S_a\chi_1})^{-1}, \\
H_{0,k} &= \widetilde{C\Pi}_{a,k} - H_{1,k} e^{-S_a\chi_1}.
\end{aligned}
\tag{4.44}
$$

The resulting model belongs to the family (4.43) for any $G_{2,k}$ and $G_{3,k}$ such that $\sigma(\bar{F}_k(s)) \cap \sigma(S) = \emptyset$.

Matching with prescribed relative degree and matching with prescribed zeros: These problems can be solved at each k as detailed in Chapter 2 if and only if

$$
\operatorname{rank} \begin{bmatrix} sI - S \\ \widetilde{C\Pi}_k \end{bmatrix} = n, \tag{4.45}
$$

for all $s \in \sigma(S)$ at k. Even though the asymptotic value of $\widetilde{C\Pi}_k$ satisfies this condition there is no guarantee that the condition holds for all k.

However, if the condition holds for the asymptotic value, there exists $\bar{k} > 0$ such that for all $k \geq \bar{k}$ equation (4.45) holds.

A system of order $n = 1010$

In this section we apply Algorithm 2 to a system similar to the one used in Section 3.2. The system has order $n = 1010$ and it has Bode plot with five peaks. The state space matrices of system (4.21) are given by $A = \mathrm{diag}(A_1, A_2, A_3, A_4, A_5, \bar{A})$, with

$$A_i = \begin{bmatrix} -1 & a_i \\ -a_i & -1 \end{bmatrix}, \quad \bar{A} = \mathrm{diag}(-1, -2, \ldots, -1000),$$

$a_1 = 50$, $a_2 = 100$, $a_3 = 150$, $a_4 = 200$, $a_5 = 400$, and $B^\top = C = [\underbrace{10 \ldots 10}_{10 \text{ times}} \;\; \underbrace{1 \ldots 1}_{1000 \text{ times}}]$.

The matrices of the signal generator (4.22) have been selected as $S = \mathrm{diag}(0, S_1, S_2, S_3, S_4, S_5, \bar{S})$, with $S_1 = A_1 + I$, $S_2 = 0.5\, S_1$, $S_3 = 1.5\, S_1$, $S_4 = 2\, S_1$, $S_5 = 4\, S_1$ and

$$\bar{S} = \begin{bmatrix} \frac{1}{5} S_1 & I \\ 0 & \frac{1}{5} S_1 \end{bmatrix},$$

to interpolate the moments at 0 and close to the five frequency peaks.

A reduced order model (4.42) at time t_k has been constructed assigning the eigenvalues of F_k. Fig. 4.7 shows the Bode plot of the system (solid/blue line), of the reduced order model at $t_{12803} = 8s$ (dotted/red line), of the reduced order model at $t_{22334} = 14s$ (dash-dotted/cyan line) and of the reduced order model at $t_{39873} = 25s$ (dashed/black line). Note that the frequencies of interest, indicated with circles, are interpolated already at $t_{12803} = 8s$. We see also that the error between the reduced order model and the system decreases as t_k increases and that at $t_{39873} = 25s$ the frequency responses of the reduced order model and the system match over a wide range of frequencies.

The surface in Fig. 4.8 (4.9, respectively) represents the magnitude (phase, respectively) of the transfer function of the reduced order model as a function of t_k, with $8 \leq t_k \leq 14\, s$. The solid/black (solid/red, respectively) line indicates the magnitude (phase, respectively) of the

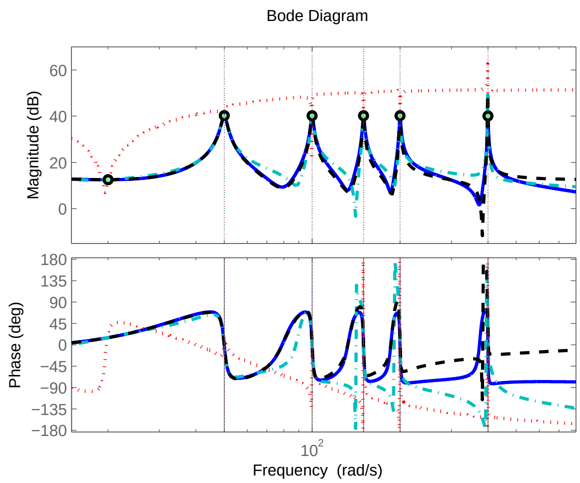

Figure 4.7: Bode plot of the system (solid/blue line), of the reduced order model at $t_{12803} = 8s$ (dotted/red line), of the reduced order model at $t_{22334} = 14s$ (dash-dotted/cyan line) and of the reduced order model at $t_{39873} = 25s$ (dashed/black line). The circles indicate the interpolation points.

transfer function of the reduced order model for the exact moments $C\Pi$. The figures show how the approximated magnitude and phase of model (4.42) at (S, L) of system (4.21) evolve over time and approach the respective quantities of the exact reduced order model as $t_k \to \infty$. The approximated and actual Bode plot are indistinguishable at $t_k = 25\,s$, but the figures are sliced at $t_k = 14$ to show the detail of the initial convergence, when the Bode plot changes more swiftly. Note also that we have used different color schemes to ease the individuation of the evolution of the two graphs: this explains the two different codings used in Fig. 4.8 and 4.9.

4.3 Bibliographical Notes

The results presented in this chapter have been adapted from Scarciotti and Astolfi [2015b,c, 2017a], Scarciotti et al. [2016] and Scarciotti et al. [2017a], see also Scarciotti [2015b, 2017a] for an application of the algorithms to the problem of model reduction of power systems. The results

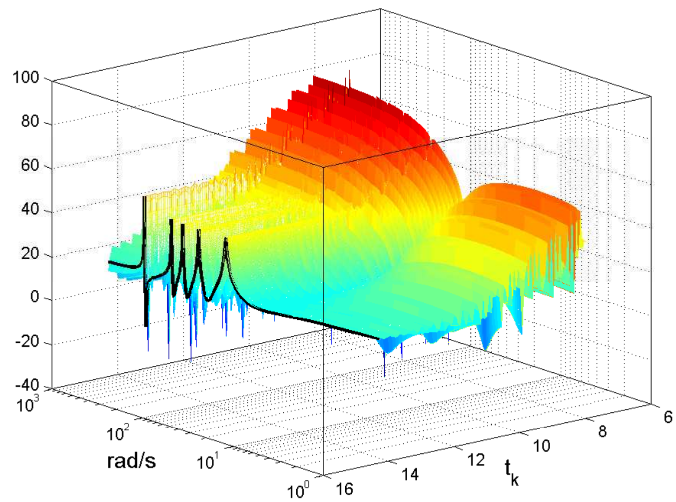

Figure 4.8: The colored mesh represents the magnitude of the transfer function of the reduced order model as a function of t_k, with $8 \leq t_k \leq 14\,s$. The solid/black line indicates the magnitude of the transfer function of the reduced order model for the exact moments $C\Pi$.

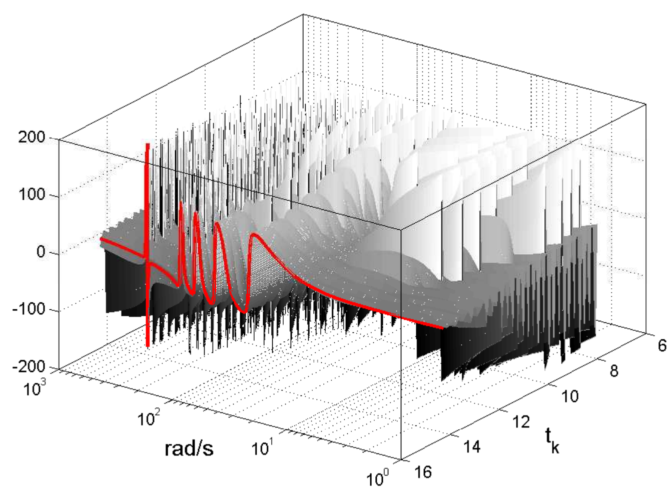

Figure 4.9: The grey mesh represents the phase of the transfer function of the reduced order model as a function of t_k, with $8 \leq t_k \leq 14\,s$. The solid/red line indicates the phase of the transfer function of the reduced order model for the exact moments $C\Pi$.

have been inspired by the learning algorithms given in Jiang and Jiang [2012], Bian et al. [2014] and Jiang and Jiang [2014] to solve a model-free adaptive dynamic programming problem (see also the references therein, *e.g.* Baird [1994] and Vrabie et al. [2009]).

The results of this chapter are clearly at the intersection of large and active research areas, such as model reduction and system identification. For instance, the use of time-snapshots and data matrices is reminiscent of the methods based on proper orthogonal decompositions, see *e.g.* Lorenz [1956], Antoulas [2005], Rowley et al. [2004], Noack et al. [2003] and Willcox and Peraire [2002], and on subspace identification, see *e.g.* van Overschee and de Moor [1996] and Verhaegen and Verdult [2007]. In model reduction, a data-driven moment matching approach for linear systems has been introduced under the name of Loewner framework in Mayo and Antoulas [2007]. This method constructs reduced order models by using matrices composed of frequency-domain measurements, which makes intrinsically difficult to extend the method to nonlinear systems. In system identification, various time-domain data-driven techniques have been presented, such as the eigensystem realization algorithm, see *e.g.* Cooper [1999], Houtzager et al. [2012], Majji et al. [2010] and Rebolho et al. [2014], and the dynamic mode decomposition, see Hemati et al. [2014].

The assumption that the mapping to be approximated can be represented by a family of basis functions is standard, see *e.g.* Toth [2010]. Some families of basis functions, for instance radial basis functions, benefit also of special properties such as "universal approximation", see Park and Sandberg [1991] and Rocha [2009].

The computational complexity $\mathcal{O}(\nu^3)$ of equation (4.34) corresponds to the complexity of the Gauss-Jordan elimination used to invert a ν by ν matrix (note that matrix multiplications can be reduced to a series of inversions). More efficient algorithms exist, such as the one given in Le Gall [2014] which has a computational complexity of $\mathcal{O}(\nu^{2.373})$. In comparison, the Arnoldi or Lanczos procedure for the model reduction by moment matching have a computational complexity of $\mathcal{O}(\nu n^2)$ [Antoulas, 2005, Section 14.1] (or $\mathcal{O}(\alpha\nu n)$ for a sparse matrix A.

The recursive least-square formulas given in Theorems 4.3 and 4.13 have been obtained adapting the results in Åström and Wittenmark [1995] (see also Greville [1960], Ben-Israel and Greville [2003] and Wang and Zhang [2011]) to the present scenario, in which at each step a new measurement is acquired and an old measurement is discarded. [Åström and Wittenmark, 1995, Chapter 11] also inspired the filtering technique illustrated after Algorithm 2.

As discussed also in Chapter 2, the excitation rank conditions have been introduced in Padoan et al. [2016a, 2017]. References regarding the examples can be found in the Bibliographical Notes of Chapters 2 and 3.

5

Model Reduction with Explicit Signal Generators

In this chapter we solves the problem of model reduction by moment matching for a general class of input signals. In Chapter 2 we have shown that, under specific assumptions, the moments are in one-to-one relation with the steady-state output response of the linear system interconnected with a signal generators $\dot{\omega} = S\omega$, where the matrix S has eigenvalues equal to the interpolation points. Now, we consider input signals generated by a linear exogenous system represented in explicit form (see Definition 2.1), *i.e.* an implicit (differential) form may not exist. This direction of investigation is motivated by a large number of applications in which standard operating conditions are associated to non-continuous or non-differential input signals. For instance, the input of a dynamical system describing a power electronic device can often be a pulse width modulated (PWM) wave which cannot be represented as the output of a system described by smooth differential equations.

The goal of the chapter is to extend the theory developed in Chapter 2 for linear signal generators with an implicit model to signal generators in explicit form.

5.1 Problem Formulation

For convenience we report here the equations of system (2.10), namely

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{5.1}$$

of the signal generator (2.11), namely

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{5.2}$$

and the Sylvester equation (2.12), namely

$$A\Pi + BL = \Pi S, \tag{5.3}$$

which are used throughout the chapter. The quantities appearing in the equations retain the original meaning given in Chapter 2.

To gain insight on the use of explicit models for signal generators, we look now at the linear generator in equation (5.2) making two trivial observations. The first one is that the interpolation points s_i, eigenvalues of S, are the poles of the Laplace transform of the free output response of system (5.2). The second one is that an input signal which is the sum of sinusoidal signals can be generated by equation (5.2) defining the matrix S such that the eigenvalues of S are in relation with the angular frequencies of the sinusoidal signals. We now try to reinterpret these two observations for a signal generator which does not have an implicit model. To begin with consider a square wave $\sqcap(t)$ defined as

$$\sqcap(t) = \mathrm{sign}(\sin(t)) = \begin{cases} 1, & (k-1)\pi < t < k\pi, \\ 0, & t = k\pi \text{ or } t = (k+1)\pi, \\ -1, & k\pi < t < (k+1)\pi, \end{cases} \tag{5.4}$$

with $\mathrm{sign}(0) = 0$, and $k = 1, 3, 5, \ldots, +\infty$. The Laplace transform of this function is

$$\mathcal{L}(\sqcap(t)) = \frac{1 - e^{-s\pi}}{s(1 + e^{-s\pi})},$$

which has the poles

$$\bar{s} = 0, \qquad \bar{s}_j = (2j+1)\iota,$$

with $j = -\infty, \ldots, -1, 0, 1, \ldots, +\infty$. We see, then, that if we "insist on" the relation between the implicit model and the Laplace transform and if we want to compute a reduced order model with the technique presented in Chapter 2, we have to interpolate at infinitely many points. This is suggested also by the second observation. Since the function $\sqcap(t)$ is periodic, it admits a Fourier series, namely

$$\sqcap(t) = \frac{4}{\pi} \sum_{k=1,3,5,\ldots,+\infty}^{\infty} \frac{1}{k} \sin(kt).$$

Then, consistently with the second observation we again have to interpolate at infinitely many points. Both these observations suggest that we could describe $\sqcap(t)$ by means of the infinite dimensional system

$$\dot{\omega} = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & +2\iota & 0 & & & & \\ & 0 & +\iota & 0 & & & \\ & & 0 & 0 & 0 & & \\ & & & 0 & -\iota & 0 & \\ & & & & 0 & -2\iota & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix} \omega,$$

with output $\sqcap = \hat{P}\omega$ for some "matrix" $\hat{P}$. This shows a limitation of the current approach. In fact, to interpolate at infinitely many points we need an "infinite dimensional mapping" Π and an infinite dimensional reduced order model.

To overcome these issues we consider signal generators in explicit form. Thus, consider a signal generators described by the equation

$$\omega(t) = \Lambda(t, t_0)\omega_0, \qquad u = L\omega, \tag{5.5}$$

with $\omega(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $L \in \mathbb{R}^{1\times\nu}$ and $\Lambda(t, t_0) \in \mathbb{R}^{\nu\times\nu}$ such that $\Lambda(t_0, t_0) = I$. Note that (5.5) provides a very general class of models which contains the implicit model (5.2), but that describes several other signal generators. For instance, it can represent signals generated by a time-varying system of the form

$$\dot{\omega} = S(t)\omega, \qquad u = L\omega, \tag{5.6}$$

with $S(t) \in \mathbb{R}^{\nu\times\nu}$, in which case Λ is (with additional assumptions, *e.g.* the semigroup property) the transition matrix associated to (5.6).

Equation (5.5) can also represent a signal generator described by some class of hybrid systems of the form

$$\begin{aligned} \dot{\omega}(t,k) &= S\omega(t,k), & u_c = L_c\omega, \\ \omega^+ &= \omega(t,k+1) = J\omega(t,k), & u_d = L_d\omega, \end{aligned} \tag{5.7}$$

with $J \in \mathbb{R}^{\nu\times\nu}$, $L_c \in \mathbb{R}^{1\times\nu}$, $L_d \in \mathbb{R}^{1\times\nu}$, $u_c(t) \in \mathbb{R}$ and $u_d(t) \in \mathbb{R}$, which jumps and flows on some hybrid time domain $\mathbb{R} \times \mathbb{Z}$. Note also that any periodic signal that can be written as the product of a function of time times the initial condition can be described by (5.5) adding the property

$$\Lambda(t,t_0) = \Lambda(t-T,t_0), \quad t \geq T + t_0, \tag{5.8}$$

where $T \in \mathbb{R}_{\geq 0}$ is the period of the signal u. Not only any periodic signal can be represented with (5.5), but (5.5) can be regarded as "the signal itself" generated by all the other representations. For instance, if we consider a square wave (5.4), then $\Lambda(t,t_0) = \sqcap(t-t_0)$, irrespective of the class of systems that generated the signal. For example, $\sqcap(t)$ can be generated by a nonlinear system, *i.e.* $\sqcap(t) = \text{sign}(\sin(t))$, or it can be generated by the hybrid system (5.7). In fact, considering the case of periodic jumps, with period T, yields

$$\sqcap(t) = J^{\lfloor \frac{t}{T} \rfloor} e^{St}. \tag{5.9}$$

It is evident that the characterization of the moments for the explicit signal generator (5.5) would solve the problem of model reduction by moment matching for many different classes of input signals.

Since the definition of moment given in this chapter is based on the existence of the steady-state response of system (5.1) driven by (5.5), we need to introduce further hypotheses on the class of input signals (5.5). First of all we specialize Assumption 2.1 to the present framework.

Assumption 5.1. The triple (A,B,C) is minimal and $\sigma(A) \subset \mathbb{C}_{<0}$. The signal generator (2.7) is observable (see Definition 2.10) and neutrally stable.

We point out that the neutral stability assumption does not require continuity of the flow of (2.7) and thus it can be used also with signal generators in explicit form. However, this class of generators requires further assumptions.

Assumption 5.2. The matrix valued function $\Lambda(t,t_0)$ is non-singular for all $t \geq t_0$.

Assumption 5.2 is essential to have uniqueness of the solution ω of (5.5). Note, in fact, that it is always satisfied by a generator of the form (5.2) and it is required for the uniqueness of the solution of system (5.6).

Assume now that there exists a set $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ in which $\Lambda(t,t_0)$ is differentiable with respect to t and consider the time-varying system described by the equation

$$\dot{z}(t) = \mathcal{A}(t)^\top z(t), \tag{5.10}$$

with $\mathcal{A}(t) = -\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}$. Let $\mathfrak{A}(t,t_0)$ be the transition matrix of system (5.10).

Assumption 5.3. The function $\mathcal{A}(t)$ is piecewise continuous with respect to t. Moreover, there exist $T \geq t_0$ and a polynomial p such that $||\mathfrak{A}(t,t_0)^\top|| \leq p(t)$ for all $t \geq T$.

This last technical assumption guarantees that the norm of z in system (5.10) does not diverge to infinity exponentially and it is needed, as shown in the next section, to guarantee that the steady-state response x^{ss} of system (5.1) driven by (5.5) can be written as $x^{ss}(t) = \Pi(t)\omega(t)$ for some matrix valued function Π. Moreover, the piecewise continuity of $\mathcal{A}$ guarantees that the steady-state response is unique. Assumption 5.3 is a generalization of the condition $\sigma(S) \subset \mathbb{C}_0$ and of the fact that the signal generator does not have the same modes of an asymptotically stable linear system, *i.e.* $\sigma(S) \cap \sigma(A) = \emptyset$. In fact, in the smooth case $\mathcal{A}(t) = -S$ and if $\sigma(S) \subset \mathbb{C}_0$, the condition $||e^{-S(t-t_0)}|| \leq p(t)$ holds trivially. Note that Assumptions 5.2 and 5.3 are "mild" assumptions. For instance, they are satisfied by the general class of discontinuous periodic signals which are considered later in this chapter.

5.2 Integral Definition of Moment

In this section we give a definition of moment in the case in which the signal generator does not have an implicit model. We begin by showing that the interconnection of system (5.1) with the signal generator (5.5) possesses a steady-state response x^{ss} described by the relation $x^{ss}(t) = \Pi(t)\omega(t)$ for some matrix valued function Π. The following result holds.

Theorem 5.1. Consider system (5.1) and the signal generator (5.5). Suppose Assumptions 5.1, 5.2 and 5.3 hold, and Λ is almost everywhere differentiable. Let

$$\Pi(t) = \left(e^{A(t-t_0)}\Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)d\tau\right)\Lambda(t,t_0)^{-1}, \quad (5.11)$$

be a family of matrix valued functions parametrized in $\Pi(t_0) \in \mathbb{R}^{n\times\nu}$. Then there exists a unique $\Pi_\infty(t)$ given by

$$\Pi_\infty(t) = \lim_{\bar{t}\to-\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)}BL\Lambda(\tau,\bar{t})d\tau\,\Lambda(t,\bar{t})^{-1}, \quad (5.12)$$

such that, for any $\Pi(t_0)$, $\lim_{t\to+\infty} \Pi(t)-\Pi_\infty(t) = 0$, where $\Pi_\infty(t)$ is the solution of (5.11) with $\Pi(t_0) = \Pi_\infty(t_0)$. Moreover, if $x(t_0) = \Pi_\infty(t_0)\omega(t_0)$ then $x(t) - \Pi_\infty(t)\omega(t) = 0$ for all $t \geq t_0$, and the set $\mathcal{M}_\infty = \{(x,\omega) \in \mathbb{R}^{n+\nu} \,|\, x(t) = \Pi_\infty(t)\omega(t)\}$ is attractive.

Proof. Let $\mathcal{T} = [t_0, t_1) \subset \mathbb{R}_{\geq 0}$ be a set in which $\Lambda(t,t_0)$ is differentiable with respect to t. Differentiating both sides of equation (5.11) over $\mathcal{T}$ yields

$$\begin{aligned}\dot{\Pi}(t)\Lambda(t,t_0) + \Pi(t)\dot{\Lambda}(t,t_0) - Ae^{A(t-t_0)}\Pi(t_0) = \\ &= BL\Lambda(t,t_0) + A\int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)d\tau \\ &= BL\Lambda(t,t_0) + A\Pi(t)\Lambda(t,t_0) - Ae^{A(t-t_0)}\Pi(t_0).\end{aligned}$$

Then, since Assumption 5.2 holds, we have

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}. \quad (5.13)$$

Let Π_1 and Π_2 be the solutions of equation (5.13) with initial conditions

$\Pi_1(t_0)$ and $\Pi_2(t_0)$, respectively, and define the error $\hat{E} = \Pi_1 - \Pi_2$. Then

$$\begin{aligned}\dot{\hat{E}}(t) &= A\Pi_1(t) + BL - \Pi_1(t)\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1} - \\ &\qquad - \left(A\Pi_2(t) + BL - \Pi_2(t)\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}\right) \\ &= A\hat{E}(t) - \hat{E}(t)\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}\end{aligned}$$

which has the solution

$$\hat{E}(t) = e^{A(t-t_0)}\hat{E}(t_0)\mathfrak{A}(t,t_0)^\top.$$

Since $\mathfrak{A}(t,t_0)^\top$ is bounded by a polynomial in $\mathcal{T}$, by Assumption 5.3, $||\hat{E}(t_1)|| < ||\hat{E}(t_0)||$, *i.e.* $\Pi_1(t_0)$ and $\Pi_2(t_0)$ go closer to each other during the intervals in which Λ is differentiable. By Assumption 5.1 there exists a sequence of times $\{t_j\}$, with $\lim\limits_{j\to+\infty} t_j = \infty$, such that $||\mathfrak{A}(t_j,t_0) - \mathfrak{A}(t_1,t_0)|| \le \hat{\varepsilon}$, with $\hat{\varepsilon}$ an arbitrary small scalar, *i.e.* $\mathfrak{A}(t,t_0)$ is bounded also outside $\mathcal{T}$. Hence, since $\sigma(A) \subset \mathbb{C}_{<0}$, $\lim\limits_{t\to\infty} \hat{E}(t) = 0$. This implies that there exist motions $\Pi_\infty(t)$ to which the solutions of equation (5.13) converge, *i.e.* for any $\Pi(t_0)$ there exists a $\Pi_\infty(t_0)$ such that $\lim\limits_{t\to+\infty} \Pi(t) - \Pi_\infty(t) = 0$. Moreover, $\Pi_\infty(t)$ is unique for any $t \ge t_0$ by the piecewise continuity of $\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}$.

Since the steady-state response is the response of the system when an infinite amount of time has passed, we can equivalently define the steady-state response as the response from an initial time that approaches an infinitely negative time, obtaining the expression (5.12).

Recall that the unique solution of system (5.1) with input u defined by equation (5.5) is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)\omega(t_0)d\tau.$$

Let $x(t_0) = \Pi_\infty(t_0)\omega(t_0)$, straightforward computations show that

$$\begin{aligned}&x(t) - \Pi_\infty(t)\omega(t) = e^{A(t-t_0)}\Pi_\infty(t_0)\omega(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)\omega(t_0)d\tau \\ &- \left(e^{A(t-t_0)}\Pi_\infty(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)d\tau\right)\Lambda(t,t_0)^{-1}\Lambda(t,t_0)\omega(t_0) = 0,\end{aligned}$$

for all $t \ge t_0$. The attractivity of $\Pi_\infty(t)$ and the invariance of $x(t) = \Pi_\infty(t)\omega(t)$ imply that the set $\mathcal{M}_\infty$ is attractive. $\square$

Corollary 5.2. Under the assumptions of Theorem 5.1, the steady-state response of system (5.1) driven by (5.5) is $x^{ss}(t) = \Pi_\infty(t)\omega(t)$ for any $x(t_0)$ and $\omega(t_0)$.

The definition of the function $\Pi_\infty(t)$ can be given as in (5.12) or, alternatively, in every interval $[t_i, t_{i+1})$, $i = 0, 1, \dots,$ as the unique solution of

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t, t_i)\Lambda(t, t_i)^{-1}, \tag{5.14}$$

with the initial condition $\Pi(t_0) = \Pi_\infty(t_0)$ which can be computed (or practically, approximated) with the expression (5.12).

The integral equation (5.12) or the differential equation (5.14) play the role of the Sylvester equation (5.3). Unlike when we have an implicit model of the signal generator, the matrix $\Pi_\infty(t)$ is in general a function of time. In fact, as remarked in Section 5.1, infinitely many interpolation points arise when periodic, possibly discontinuous, signals are considered. Thus, a constant Π should have infinitely many rows and columns.

Similarly to what done in Chapters 2 and 3, we introduce a weaker assumption (with respect to Assumption 5.1) which allows defining the moments of system (5.1) for the signal generator (5.5).

Assumption 5.4. The triple (A, B, C) is minimal, the signal generator (2.7) is observable and there exists a unique Π_∞ given in (5.12).

Definition 5.1. Consider system (5.1) and the signal generator (5.5). Suppose Assumption 5.4 holds. The mapping $C\Pi_\infty$ is the *moment of system (5.1) at* (Λ, L).

Consistently with the rest of the monograph, we have given a definition of moment (Definition 5.1) which is not necessarily related to the steady-state response of system (5.1). In particular, the moment may be well-defined even though the steady-state response does not exist. We now make precise the relation between the moment $C\Pi_\infty$ defined by the integral equation (5.12) and the steady-state response of system (5.1) driven by (5.5). To this end we introduce an additional assumption.

Assumption 5.5. The vector ω defined in equation (5.5) has a strictly proper Laplace transform. The pair $(\Lambda, \omega(t_0))$ is excitable, *i.e.* $\omega(t_0)$ is such that $\sigma(\mathcal{L}(\Lambda(t,t_0)\omega(t_0))) = \sigma(\mathcal{L}(\Lambda(t,t_0)))$.

This condition is derived from the observation that the excitability of $(S, \omega(0))$ is equivalent to the condition $\sigma(\mathcal{L}(e^{St}\omega(0))) = \sigma(S)$. This condition guarantees, together with the minimality of (A, B, C), that all the modes of ω are excited and observable in the output, and plays the role of similar excitation assumptions that we have used throughout the monograph.

Corollary 5.3. Consider system (5.1) and the signal generator (5.5). Suppose Assumptions 5.1, 5.2, 5.3 and 5.5 hold. Then there exists a one-to-one relation between the moment of (5.1) at (Λ, L) and the steady-state response of the output y of the interconnection of system (5.1) with the signal generator (5.5).

Proof. By Assumptions 5.1, 5.2 and 5.3, Theorem 5.1 holds. Hence, the steady-state output response of (5.1) driven by (5.5) is well-defined and given by $y^{ss}(t) = C\Pi_\infty(t)\omega(t)$, where Π_∞ is given in (5.12). By Assumption 5.5, we can repeat the steps of the proof of Theorem 2.6 to show that it is possible to compute the moment $C\Pi_\infty$ from y^{ss}. □

The choice of defining the moment of (5.1) as in Definition 5.1 is justified by the equivalence, when an implicit model of (5.5) is available, between the new and the classical definition given in Chapter 2. In the next result we prove that, under certain hypotheses, the solutions of the Sylvester equation (5.3) and of the integral equation (5.12) are the same.

Theorem 5.4. Consider system (5.1) and the signal generator (5.2). Suppose Assumption 2.4 holds. Then the quantity Π_∞ defined in equation (5.12) coincides with the unique solution of the Sylvester equation (5.3).

Proof. Let $\tilde{\Pi}$ be the solution of the Sylvester equation $A\tilde{\Pi} + BL = \tilde{\Pi}S$, which is unique because $\sigma(A) \cap \sigma(S) = \emptyset$. Note that Π_∞ given in (5.12) is well-defined and unique because $\sigma(A) \cap \sigma(S) = \emptyset$. Computing the derivative of the error $E = \Pi_\infty - \tilde{\Pi}$ yields

$$\begin{aligned}\dot{E}(t) &= A\Pi_\infty(t) - \Pi_\infty(t)S + BL - 0 = \\ &= A\Pi_\infty(t) - \Pi_\infty(t)S - (A\tilde{\Pi} - \tilde{\Pi}S) = AE(t) - E(t)S,\end{aligned}$$

which has the solution

$$E(t) = e^{A(t-t_0)}E(t_0)e^{-S(t-t_0)}.$$

This implies that $\Pi_\infty(t) - \tilde{\Pi}$ converges to zero. Since $\tilde{\Pi}$ is constant and $\Pi_\infty(t)$ is the limiting solution of (5.11), it follows that $\Pi_\infty(t_0) = \tilde{\Pi}$, $E(t_0) = 0$ and then $E(t) = 0$ for all $t \geq t_0$, which proves the claim. □

As anticipated in the previous section, we now focus our interest on periodic signals.

Corollary 5.5. Consider system (5.1) and the signal generator (5.5). Assume Assumptions 5.1, 5.2 and 5.3 hold. If for (5.5) the property

$$\Lambda(t, t_0) = D(t)\Lambda(t - T, t_0), \quad t \geq T + t_0, \tag{5.15}$$

holds, with $D(t) \in \mathbb{R}^{\nu \times \nu}$ non-singular for all $t \in \mathbb{R}_{\geq 0}$ and $T \in \mathbb{R}_{>0}$, then equation (5.11) becomes

$$\begin{aligned}\Pi_\infty(t) = e^{AT}&\Pi_\infty(t-T)D(t)^{-1} + \\ &+ \left[\int_{t-T}^{t} e^{A(t-\tau)}BL\Lambda(\tau, t_0)d\tau\right]\Lambda(t-T, t_0)^{-1}D(t)^{-1}.\end{aligned} \tag{5.16}$$

If $D(t) = I$ then

$$\Pi_\infty(t) = (I - e^{AT})^{-1}\left[\int_{t-T}^{t} e^{A(t-\tau)}BL\Lambda(\tau, t_0)d\tau\right]\Lambda(t, t_0)^{-1}. \tag{5.17}$$

Proof. Equation (5.16) is obtained substituting

$$x^{ss}(t) = \Pi_\infty(t)\omega(t) = \Pi_\infty(t)D(t)\omega(t-T)$$

and $\omega(t - T) = \Lambda(t - T, t_0)\omega(t_0)$ in

$$x^{ss}(t) = e^{AT}x^{ss}(t-T) + \int_{t-T}^{t} e^{A(t-\tau)}BL\Lambda(\tau, t_0)\omega(t_0)d\tau.$$

If $D(t) \equiv I$, $\omega(t) = \omega(t-T)$ and the steady-state of $x(t)$ is periodic with period T. Then $\Pi_\infty(t)\omega(t) = x^{ss}(t) = x^{ss}(t-T) = \Pi_\infty(t-T)\omega(t-T) = \Pi_\infty(t-T)\omega(t)$ which implies that $\Pi_\infty(t) = \Pi_\infty(t-T)$ and equation (5.17) follows. □

Remark 5.1. Equation (5.15) is a generalized form of equation (5.5) with the property (5.8) and describes a wide class of signals, possibly non-periodic. To show this note, for instance, that (5.2) can always be written as (5.15). In fact, for any $T \in \mathbb{R}_{\geq 0}$,

$$\omega(t) = e^{ST}\omega(t-T), \qquad \Lambda(t,t_0) = e^{S(t-t_0)}.$$

Thus (5.2) can be described by (5.15) with $D = e^{ST}$ for all t.

For periodic signals equation (5.17) defines in a simple form the periodic matrix $\Pi_\infty(t)$. In addition, note that this definition of $\Pi_\infty(t)$ does not depend upon the initial condition. As a result equation (5.17) can be used to exactly determine the initial condition $\Pi_\infty(0)$ of equation (5.11), as shown in the next example. Furthermore, exploiting the periodicity of the steady-state, $\Pi_\infty(t)$ has to be computed only over a period. This can be done off-line and the obtained values can then be used on-line for any time interval.

Example 5.1. Consider the interconnection of system (5.1) and (5.5). The matrices of (5.1) have been randomly generated with the function *rss* of MATLAB. For the remaining of the chapter we use the selection

$$A = \begin{bmatrix} -2.439 & 2.337 & -1.776 \\ -2.933 & -1.096 & 4.221 \\ 0.09223 & -4.579 & -1.537 \end{bmatrix}, \quad \begin{aligned} B &= \begin{bmatrix} 0 & -0.7648 & -1.402 \end{bmatrix}^T, \\ C &= \begin{bmatrix} 0 & 0.4882 & 0 \end{bmatrix}, \\ L &= \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{aligned} \tag{5.18}$$

Let $t_0 = 0$ and consider the matrix of square waves

$$\Lambda_\sqcap(t,0) = \begin{bmatrix} \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) & -\sqcap\left(\frac{2\pi}{T}t\right) \\ \sqcap\left(\frac{2\pi}{T}t\right) & \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) \end{bmatrix}. \tag{5.19}$$

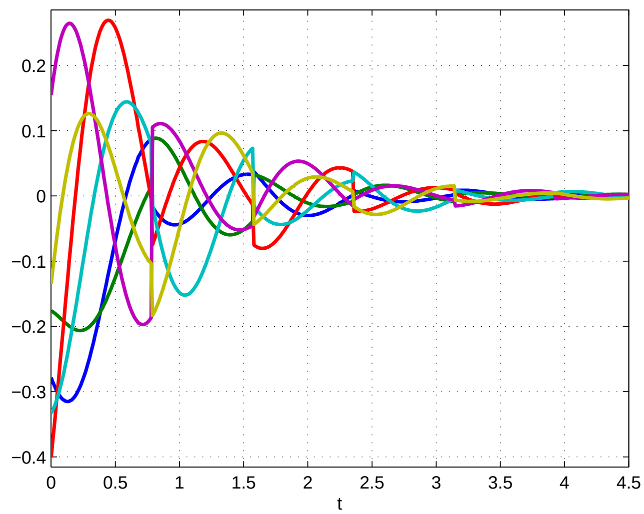

Figure 5.1: Time histories of the entries of the error $\varepsilon = \Pi_\infty - \tilde{\Pi}$, computed from equation (5.11) with initial condition $\Pi(0) = \Pi_\infty(0)$ and $\Pi(0) = \tilde{\Pi}(0) = 2\Pi_\infty(0)$, respectively.

Equation (5.17), computed for $t = 0$, yields

$$\Pi_\infty(0) = -A^{-1}(I - e^{AT})^{-1}\left[\left(e^{\frac{3}{4}AT} - e^{AT} + e^{\frac{1}{2}AT} - e^{\frac{1}{4}AT}\right)BL + \left(e^{\frac{1}{2}AT} - e^{\frac{3}{4}AT} + e^{\frac{1}{4}AT} - I\right)BL\Lambda_\sqcap\left(\frac{T}{4}, 0\right)\right], \tag{5.20}$$

which, for $T = \pi$, yields

$$\Pi_\infty(0) = \begin{bmatrix} 0.1760 & -0.2786 \\ 0.3327 & -0.4004 \\ 0.1349 & 0.1546 \end{bmatrix}.$$

Fig. 5.1 shows the time history of the error $\varepsilon(t) = \Pi_\infty(t) - \tilde{\Pi}(t)$, where $\Pi_\infty(t)$ is computed from equation (5.11) with initial condition $\Pi(0) = \Pi_\infty(0)$ defined in (5.20) and $\tilde{\Pi}(t)$ is computed from equation (5.11) with initial condition $\Pi(0) = \tilde{\Pi}(0) = 2\Pi_\infty(0)$. As proved in Theorem 5.1 the solution $\Pi_\infty(t)$ is exponentially attractive.

Example 5.2. Consider the interconnection of system (5.1) and (5.5), the selection (5.18), the matrix (5.19) and the two matrices

$$\Lambda_{\sim}(t,0) = \begin{bmatrix} \cos\left(\frac{2\pi}{T}t\right) & -\sin\left(\frac{2\pi}{T}t\right) \\ \sin\left(\frac{2\pi}{T}t\right) & \cos\left(\frac{2\pi}{T}t\right) \end{bmatrix}$$

and

$$\Lambda_{\wedge}(t,0) = \begin{bmatrix} \wedge\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) & -\wedge\left(\frac{2\pi}{T}t\right) \\ \wedge\left(\frac{2\pi}{T}t\right) & \wedge\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) \end{bmatrix},$$

where

$$\wedge(t) = \begin{cases} t \mod (\pi), & (k-1)\pi \le t < k\pi, \\ 1-(t \mod (\pi)), & k\pi \le t < (k+1)\pi, \end{cases}$$

with $k = 1, 3, 5, \ldots, +\infty$ and period $T = \pi$. Let $\Pi_{\sim}$, $\Pi_{\wedge}$ and $\Pi_{\sqcap}$ be the solutions of equation (5.17) and $y_{\sim}$, $y_{\wedge}$ and $y_{\sqcap}$ be the outputs of system (5.1) for $\Lambda_{\sim}$, $\Lambda_{\wedge}$ and $\Lambda_{\sqcap}$, respectively. Fig. 5.2 shows the time histories of the entries of the matrices $\Pi_{\sim}$ (top), $\Pi_{\wedge}$ (middle) and $\Pi_{\sqcap}$ (bottom). We note that $\Pi_{\sim}$ is constant, whereas $\Pi_{\wedge}$ and $\Pi_{\sqcap}$ are periodic. Fig. 5.3 shows the time history of the output (solid lines) $y_{\sim}$ (top), $y_{\wedge}$ (middle) and $y_{\sqcap}$ (bottom) and of the steady-state value of the output (dotted lines) computed as $C\Pi_{\sim}\omega$, $C\Pi_{\wedge}\omega$ and $C\Pi_{\sqcap}\omega$. We note that the outputs of the system approach the steady-state responses, as expected.

5.3 Reduced Order Models in Explicit Form

In this section, exploiting Definition 5.1, we present a family of systems achieving moment matching and we draw connections with the families of models given in Chapter 2.

Definition 5.2. Consider system (5.1) and the signal generator (5.5). The system described by the equations

$$\xi(t) = \widetilde{F}(t,t_0)\xi_0 + \int_{t_0}^{t} \widetilde{G}(t-\tau)u(\tau)d\tau, \qquad \psi(t) = \widetilde{H}(t)\xi(t), \quad (5.21)$$

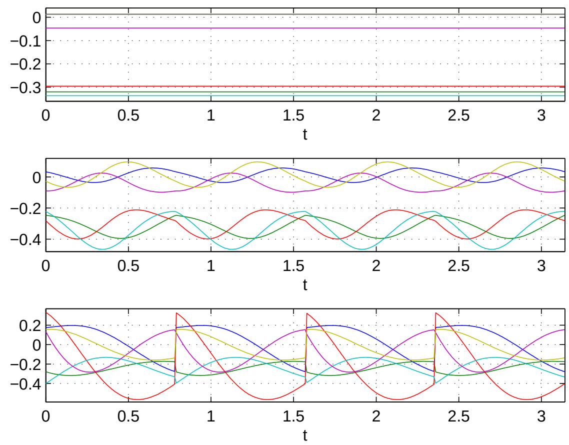

Figure 5.2: Time histories of the entries of the matrices $\Pi_{\sim}$ (top), $\Pi_{\wedge}$ (middle) and $\Pi_{\sqcap}$ (bottom).

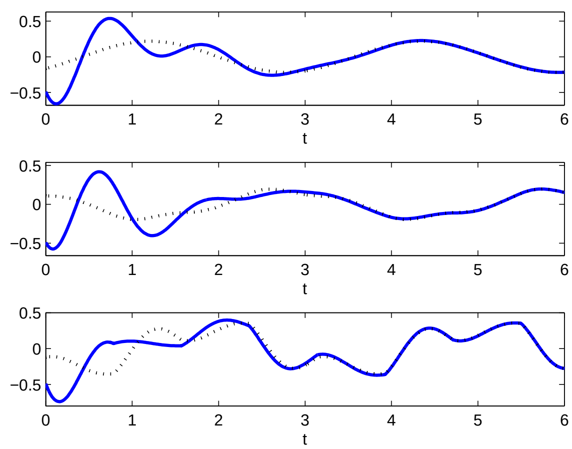

Figure 5.3: Time histories of the output (solid lines) $y_{\sim}$ (top), $y_{\wedge}$ (middle) and $y_{\sqcap}$ (bottom) and time histories of the steady-state of the output (dotted lines) computed as $C\Pi_{\sim}\omega$, $C\Pi_{\wedge}\omega$ and $C\Pi_{\sqcap}\omega$.

with $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}$, $\widetilde{F}(t,t_0) \in \mathbb{R}^{\nu\times\nu}$, $\widetilde{G}(t) \in \mathbb{R}^{\nu\times 1}$ and $\widetilde{H}(t) \in \mathbb{R}^{1\times\nu}$, is a *model of system (5.1) at* (Λ, L), if system (5.21) has the same moment at (Λ, L) as system (5.1). System (5.21) is a *reduced order model of system (5.1) at* (Λ, L) if $\nu < n$.

Lemma 5.6. Consider system (5.1) and the signal generator (5.5). Suppose Assumption 5.4 holds. Then system (5.21) is a model of system (5.1) at (Λ, L) if there exists a unique solution $P_\infty(t)$ of the equation

$$P_\infty(t) = \lim_{\bar{t}\to-\infty} \int_{\bar{t}}^{t} \widetilde{G}(t-\tau)L\Lambda(\tau,\bar{t})d\tau\, \Lambda^{-1}(t,\bar{t}), \tag{5.22}$$

such that

$$C\Pi_\infty(t) = \widetilde{H}(t)P_\infty(t), \tag{5.23}$$

where $\Pi_\infty(t)$ is the unique solution of (5.11)-(5.12).

Proof. The claim is a direct consequence of Definitions 5.1 and 5.2. □

Some of the conditions in the previous lemma can be satisfied if the following conditions (stronger but easier to verify) hold for the reduced order model (5.21). Firstly if the reduced order model is asymptotically stable, *i.e.* for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $||\xi(0)|| < \delta_\varepsilon$ then $||\xi(t)|| < \varepsilon$, and there exists a $\delta > 0$ such that if $||\xi(0)|| < \delta$ then $\lim_{t\to+\infty} \xi(t) = 0$, then the function P_∞ exists and describes a steady-state motion, *i.e.* the moment of system (5.21) are well-defined. Secondly if $P_\infty(t)$ is non-singular for all $t \in \mathbb{R}_{\geq 0}$ then a function $\widetilde{H}(t)$ which satisfies equation (5.23) exists for all $t \in \mathbb{R}_{\geq 0}$, *i.e.* the moments are matched.

Furthermore, to achieve moment matching with additional constraints we would like to have two additional properties.

Property 1: ideally we would like to have $P_\infty(t) = I$ (as in Chapters 2 and 3), which has proved to give a remarkable simplification in the definition of the reduced order model. We recall that the selection $P = I$ in Chapter 2 yields a family of systems, parametrized by a matrix $G \in \mathbb{R}^\nu$, which is complete, *i.e.* the family contains all systems of dimensions ν achieving moment matching.

Property 2: at the same time we would like to bring the first equation of (5.21) to a form for which we can easily enforce additional constraints. This form is described by the selection

$$\widetilde{F}(t,t_0) = e^{F(t-t_0)}, \qquad \widetilde{G}(t) = e^{Ft}G, \tag{5.24}$$

for some F and G, which makes the first equation of (5.21) a representation in explicit form of a linear time-invariant system which has an implicit model.

However, differently from the family of reduced order models given in Chapter 2, it is not possible for the reduced order model (5.21) to satisfy both properties, namely having an implicit model and be such that $P_\infty(t) = I$. Note, for instance, that if $\omega(t)$ belongs to the class of signals satisfying (5.8) and we use the selection (5.24), then the steady-state $\xi^{ss}(t)$, if it exists, is periodic and so it is $P_\infty(t)$ (see the proof of Corollary 5.5). Thus, this first route in which we simplify the problem fixing $P_\infty(t)$ brings the problem of determining $\widetilde{F}(t,t_0)$ and $\widetilde{G}(t)$ which satisfy equation (5.22) and, at the same time, enforce additional properties, *e.g.* stability, on the reduced order model. As a second route we can fix $\widetilde{F}(t,t_0)$ and $\widetilde{G}(t)$ with the selection given in (5.24) which, however, brings the problem of finding $P_\infty(t)$ and solving equation (5.23) with respect to $\widetilde{H}(t)$.

Both the choices are viable, however, it is easier to follow the second route, namely using the selection (5.24) and solving numerically (5.22) and (5.23). In this case the dynamics of the state ξ can be described by a linear system in implicit form for which the solution to the problem of selecting F and G such that additional properties are satisfied is given in Chapter 2.

Note also that both routes lead to models which are equivalent to the reduced order models given in Chapter 2 if $\Lambda(t,t_0) = e^{S(t-t_0)}$. Following the first route, easy computation shows that $\widetilde{F}(t,t_0) = e^{(S-GL)(t-t_0)}$ and $\widetilde{G}(t) = e^{(S-GL)t}G$, with G free, are a solution of equation (5.22) with $P_\infty(t) = I$. Following the second route, by Theorem 5.4, equation (5.22) is equivalent to the Sylvester equation associated to the general family of reduced order models given in Chapter 2.

Remark 5.2. As already discussed, if $\Lambda(t, t_0) = e^{S(t-t_0)}$ and $P_\infty(t) = P = I$, then in (5.24) $F = S - GL$ with G free. The family of models (5.21) reduces to the family parametrized in G given in Chapter 2. Note also that a different selection $\bar{P} \neq P = I$ yields the same class of models through a change of coordinates in the state space representation, namely $\bar{F} = \bar{P}(S - GL)\bar{P}^{-1}$, $\bar{G} = \bar{P}G$, $\bar{H} = C\Pi\bar{P}^{-1}$, and there is still only one free parameter G. On the other hand, in the selection (5.24), we have two free parameters, namely F and G which can be totally independent of each other.

We summarize these observations in the next statement in which we give a family of reduced order models in the special case of periodic input signals.

Proposition 5.1. Consider system (5.1) and the signal generator (5.5) with the property (5.8). Suppose Assumptions 5.1, 5.2 and 5.3 hold. Then the system described by the equations

$$\dot{\xi} = F\xi + Gu, \qquad \psi = C\Pi_\infty(t)P_\infty(t)^{-1}\xi, \tag{5.25}$$

with $F \in \mathbb{R}^{\nu\times\nu}$, $G \in \mathbb{R}^{\nu\times 1}$ and $\Pi_\infty(t)$ defined in (5.17), is a model of system (5.1) at (Λ, L) for any F and G such that $\sigma(F) \subset \mathbb{C}_{<0}$ and

$$P_\infty(t) = (I - e^{FT})^{-1}\left[\int_{t-T}^{t} e^{F(t-\tau)}GL\Lambda(\tau, t_0)d\tau\right]\Lambda(t, t_0)^{-1}, \tag{5.26}$$

is non-singular for all $t \in \mathbb{R}_{\geq 0}$.

Proof. In (5.21) select $\widetilde{F}(t, t_0)$ and $\widetilde{G}(t)$ as in (5.24). Since the signal generator (5.5) is periodic and $\sigma(F) \subset \mathbb{C}_{<0}$, equation (5.22) takes, by Corollary 5.5 the form given in (5.26). By assumption, $P_\infty(t)$ is non-singular and, thus, the matching condition (5.23) can be solved with respect to $\widetilde{H}(t)$, giving $\widetilde{H}(t) = C\Pi_\infty(t)P_\infty(t)^{-1}$. □

We conclude this section with a numerical example to illustrate this last result.

Example 5.3. Consider the interconnection of system (5.1) and (5.5) with Λ defined as in (5.19). The matrices of (5.1), with $n = 15$, have been selected as $A = \operatorname{diag}(-1/n, -2/n, \ldots, -1)$, $B =$

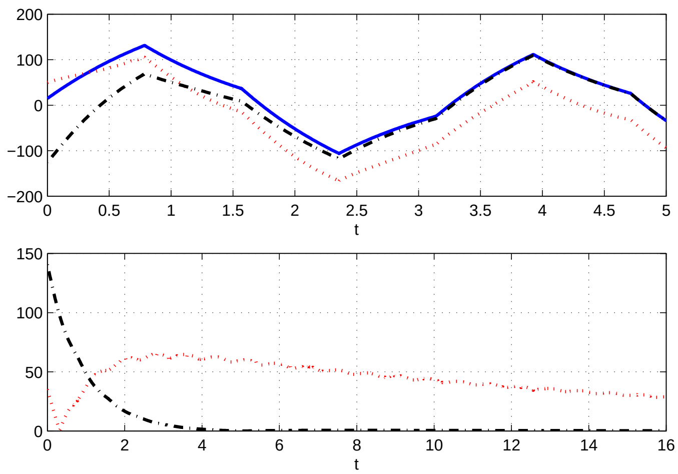

Figure 5.4: Top: time histories of the output y (solid line) of system (5.1), with (5.19), of the output ψ_I (dotted line) of system (5.25) and of the output ψ_{II} (dash-dotted line) of system (5.25) . Bottom: time histories of $|y - \psi_I|$ (dotted line) and of $|y - \psi_{II}|$ (dash-dotted line).

$[-1/n \;\; -2/n \;\; \dots \;\; -1]^\top$, $C = B^\top$, with the initial state $x(t_0) = [n/2 \;\; n/2{-}1 \;\; \dots \;\; -n/2{+}1]^\top$. Using the family (5.25), two reduced order models of system (5.1) have been computed. The first one is described by the selection $F = \text{diag}(-1, -0.0667)$ (which are the slowest and fastest modes of A) and $G = [0.0771 \;\; -0.0251]^\top$, while the second one by $F = \text{diag}(-0.9333, -1)$ (which are the two fastest modes of A) and $G = [0.0891 \;\; 0.008]^\top$. In both case G is such that $P_\infty(t)$ is non-singular for all $t \in \mathbb{R}_{\geq 0}$. The top graph of Fig. 5.4 shows the time histories of the output y (solid line), of the output ψ_I of the first reduced order model (dotted line) and of the output ψ_{II} of the second reduced order model (dash-dotted line). The bottom graph shows the time histories of $|y - \psi_I|$ (dotted line) and of $|y - \psi_{II}|$ (dash-dotted line). The figure shows that the output response of the second model converges to the output of the system much quicker than the one of the first model.

5.4 Discontinuous Phasor Transform

In Chapter 2 we have established a relation between moments and phasors. In particular we have shown that a phasor is a moment computed at the single point $\iota\hat{\omega}$. Exploiting this relation and the results of this chapter, we are able to extend the phasor analysis to more general classes of sources. Instead of considering sinusoidal sources we study any periodic source which can be represented with the explicit form (5.5). In particular, we are interested in square waves since these are usually used to control power converters. Since in the following we use the mixed convention which is more compact and results in more efficient computations, we discuss first how a square wave can be represented in the mixed convention. Let us consider as input a square wave with given angular frequency $\hat{\omega}$, *i.e.* $\sqcap(\hat{\omega}t+\pi/2)$. This signal does not satisfy Assumption 5.2. However, this issue can be easily solved considering the extended signal $\sqcap(\hat{\omega}t+\pi/2)+\iota\sqcap(\hat{\omega}t)$. This complex signal is never equal to zero and Assumption 5.2 is satisfied. Note that this is in line with the smooth case in which the source is described by $e^{\iota\hat{\omega}t} = \sin(\hat{\omega}t+\pi/2)+\iota\sin(\hat{\omega}t)$. Thus, a possible explicit signal generator in the mixed convention can be described by the equations

$$
\begin{aligned}
u(t) &= V_s\omega(t),\\
\omega(t) &= \sqcap\left(\hat{\omega}t+\frac{\pi}{2}\right)+\iota\sqcap(\hat{\omega}t),
\end{aligned}
\tag{5.27}
$$

with $V_s \in \mathbb{R}$. Note that the same signal can be represented in the real convention by the system

$$
\begin{aligned}
u(t) &= V_s\begin{bmatrix} 1 & 0 \end{bmatrix}\omega(t),\\
\omega(t) &= \begin{bmatrix} \sqcap\left(\hat{\omega}t+\dfrac{\pi}{2}\right) & -\sqcap(\hat{\omega}t) \\ \sqcap(\hat{\omega}t) & \sqcap\left(\hat{\omega}t+\dfrac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{aligned}
\tag{5.28}
$$

In fact, in the smooth case this matrix Λ would reduce to the usual rotation matrix and $u(t)=\cos(\hat{\omega}t)$.

With abuse of notation let $\Lambda(t)=\Lambda(t,0)$ and assume that the linear system is written in the mixed convention. The following definition is a natural generalization of the results of Chapter 2.

Definition 5.3. Consider system (5.1) and the signal generator (5.5). Suppose Assumptions 5.1, 5.2 and 5.3 hold, and $\Lambda(t)$ is almost everywhere differentiable. The components of the function

$$\Pi_\infty(t) = (I - e^{AT})^{-1} \left[\int_{t-T}^{t} e^{A(t-\tau)} BL\Lambda(\tau) d\tau \right] \Lambda(t)^{-1} \tag{5.29}$$

are the *discontinuous phasors* of all the currents and of all the integrals of the currents in system (5.1) for the source $\Lambda(t)$. The *discontinuous inverse phasor transform* of the steady-state output current $i(t)$ of system (5.1) is

$$i(t) = \Re \left[\overline{I}(t)\Lambda(t) \right], \tag{5.30}$$

with $\overline{I}(t) = \epsilon_k \Pi_\infty(t)$, for some $k \in \mathbb{Z}$.

Similarly to the sinusoidal case, the instantaneous currents are recovered multiplying the phasor with the source and taking the real part. However, differently from the sinusoidal case, the phasor $\overline{I}(t)$ is a time-dependent periodic function. Note that if $\Lambda(t)$ is sinusoidal, equation (5.29) defines the usual constant phasor and Π_∞ solves the Sylvester equation (5.3).

Now that we have defined the discontinuous phasor and the discontinuous inverse phasor transform we extend some of the properties of the phasor circuit analysis.

5.4.1 Inductance, capacitance and resistance

We now describe the v-i characteristics of some common components which constitute electronic devices. In this way on one hand we improve our understanding of this new tool and on the other hand we are able to compute the voltage across inductors, capacitors and resistors given the phasor of the current which flows through these components (which is useful for applications). This is particularly important to make this mathematical extension an accurate description of the physical quantities in the circuit. The expressions that relate voltage and current in an inductor, capacitor and resistor are, respectively,

$$v = L\frac{di}{dt}, \qquad v = \frac{1}{C}\int_0^t i \, d\tau, \qquad v = Ri, \tag{5.31}$$

where, differently from the rest of the monograph, C indicates the capacitance (instead of the output matrix of the system) and L the inductance (instead of the output matrix of the signal generator). Utilizing the classical phasor transform, it can be proved that the relations

$$\overline{V} = \iota\omega L\overline{I}, \qquad \overline{V} = \frac{1}{\iota\omega C}\overline{I}, \qquad \overline{V} = R\overline{I}, \tag{5.32}$$

hold. When the source is described by the generator (5.5), these relations may not hold anymore. Consider, for instance, the square wave (5.4). As pointed out at the beginning of this chapter this signal is described by infinitely many frequencies $\hat{\omega}_k$. We may then express the phasor as the sum of infinitely many frequencies. However, exploiting the discontinuous phasor transform we can express the phasor without approximations obtaining the following exact relations.

Theorem 5.7. Consider the first equation in (5.31). The relation

$$\overline{V}(t) = L\dot{\overline{I}}(t) + L\frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{I}(t), \tag{5.33}$$

holds.

Proof. Consider the first equation in (5.31). This is a scalar system with $\overline{I}(t) = \Pi_\infty(t) \in \mathbb{C}$, $A = 0$, $B = \frac{1}{L}$. The derivative of the current is

$$\frac{di}{dt} = \frac{d}{dt}\Re\left[\overline{I}(t)\Lambda(t)\right] = \Re\left[\frac{d}{dt}\left[\overline{I}(t)\Lambda(t)\right]\right] = \Re\left[\dot{\overline{I}}(t)\Lambda(t) + \overline{I}(t)\dot{\Lambda(t)}\right]$$

Substituting equation (5.13) in the last expression, yields

$$\Re\left[\left(A\overline{I}(t) + B\Gamma\right)\Lambda(t)\right] = \Re\left[\left(\dot{\overline{I}}(t) + \overline{I}(t)\dot{\Lambda}(t)\Lambda(t)^{-1}\right)\Lambda(t)\right]$$

from which we recognize, by comparison with (5.30), that $\dot{\overline{I}}(t) + \overline{I}(t)\dot{\Lambda}(t)\Lambda(t)^{-1}$ is the phasor. □

Note that if $\Lambda(t) = e^{\iota\omega t}$, then $\dot{\overline{I}}(t) = 0$, $\dot{\Lambda}(t)\Lambda(t)^{-1} = \iota\omega$ and (5.33) becomes the first relation in (5.32).

Theorem 5.8. Consider the second equation in (5.31). The relation

$$\dot{\overline{V}}(t) + \frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{V}(t) = \frac{1}{C}\overline{I}(t), \tag{5.34}$$

holds.

Proof. It is similar to the proof of Theorem 5.7. □

Since in the mixed convention the components with odd indeces of Π_∞, computed from (5.29), are those functions that multiplied by Λ give the steady-state of the integrals of the currents. The following result holds.

Corollary 5.9. In the mixed convention the components with odd indeces of Π_∞, computed with (5.29), are the phasors of the integrals of the currents in the circuit. Thus for the current i_k which flows in the capacitance C_k, the relation

$$\frac{\epsilon_{2k-1}^\top \Pi_\infty(t)}{C_k} = \overline{V}_k(t), \tag{5.35}$$

holds, where $\overline{V}_k$ is the phasor of the voltage across the capacitor C_k.

Corollary 5.9 provides a way to compute the phasor of the voltage across a capacitor. In fact, this value comes directly from solving (5.29).

Theorem 5.10. Consider the third equation in (5.31). The relation

$$\overline{V} = R\overline{I}(t), \tag{5.36}$$

holds.

Proof. The statement holds trivially noting that the multiplication by a real constant and the real part operator commutes. □

Fig. 5.5 shows the phasors equivalent of these circuit elements. Having defined the differential operator, the integral operator and the multiplicative operator, the v-i characteristics of other common circuits in the phasor domain can be easily obtained. For instance, transformers and gyrators, in which voltages and currents are related by a multiplication factor, show phasor equations similar to (5.36).

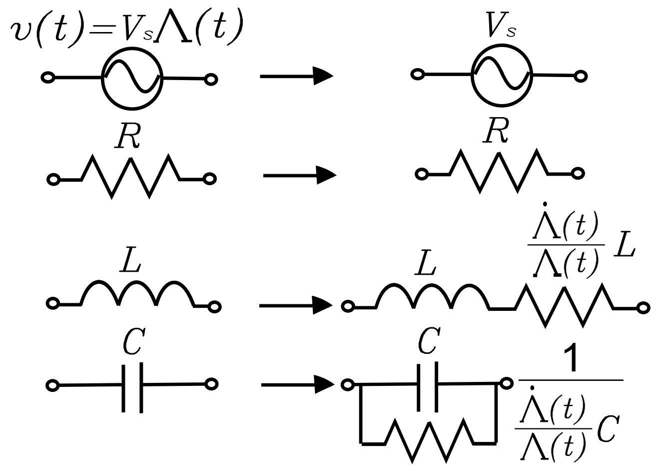

Figure 5.5: Phasor models for basic circuit elements.

5.5 Bibliographical Notes

The results on model reduction by matching the steady-state of explicit signal generators have been adapted from Scarciotti and Astolfi [2015a, 2016b]. Based on these results, several extensions have been proposed in Scarciotti and Astolfi [2016e,f], Scarciotti and Teel [2017a,b] and Scarciotti et al. [2017b] for hybrid systems, stochastic systems and differential inclusions.

Many of the results developed in this chapter assume that the reader has a basic understanding of time-varying systems. We refer the reader to Brockett [1970], in particular to Sections 3, 11 and 29. The uniqueness of $\Pi_\infty(t)$ derives from a standard result of ordinary differential equations, *i.e.* [Khalil, 2001, Theorem 3.2].

The discontinuous phasor transform has been introduced in Scarciotti and Astolfi [2016c], see also Scarciotti and Astolfi [2016e,f] and Scarciotti [2017b]. Other authors proposed some generalizations of the classical phasor transform, *i.e.* the inverse phasor transforms introduced in Rim and Cho [1990], Yin et al. [2003], Rim [2011]. These are specific cases of the more general phasor transform we have presented in this chapter (they can be recovered selecting $\Lambda(t) = e^{\iota\omega(t)}$).

6

Conclusions

In this monograph we have presented an organic set of notions, tools, algorithms and case studies for model reduction which rely upon the interpretation of the moment matching approximation problem as the problem of matching steady-state responses. We have shown that the moment matching approach allows determining reduced order models for several classes of systems, such as nonlinear systems, time-delay systems and systems which may not admit a differential description, thus extending the classical interpolation approach beyond the linear framework. In addition to providing the theoretical foundations of the model reduction by moment matching method we have also described algorithms to obtain reduced order models from data, thus blurring the boundaries between the areas of model reduction and system identification. Finally, we have established connections between the notion of moment and other classical notions, dedicating special attention to the relation between moments and the phasor transform.

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