Dr Giordano Scarciotti

# Model order Reduction

Given a system $\Sigma_n$ of order $n$ with property (P)

Find another system $\Sigma_\nu$ of order $\nu \ll n$ with the same property (P) + additional properties

(P) = <u>moments</u> MOR by moment matching

$$\Sigma_n : \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \quad (1) \quad x \in \mathbb{R}^n \quad u \in \mathbb{R} \quad A \in \mathbb{R}^{n \times n} \quad B \in \mathbb{R}^{n \times 1} \quad C \in \mathbb{R}^{1 \times n}$$

$$W(s) = C(sI - A)^{-1} B$$

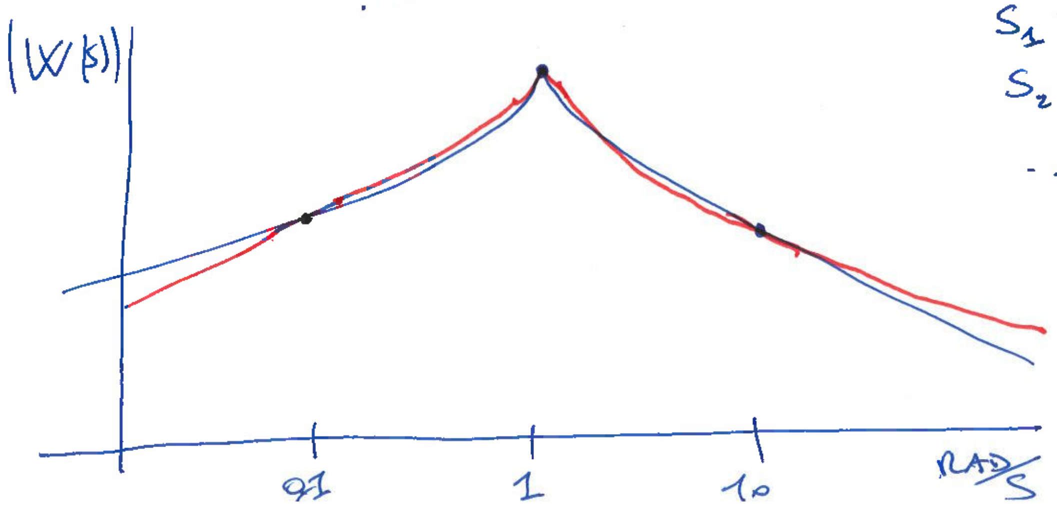
Assumption: $\Sigma_n$ is minimal, i.e. $\Sigma_n$ is observable and reachable

Definition 1: Let $s_i \in \mathbb{C} \setminus \sigma(A)$. The 0-moment of system (1) at $s_i$ is

$$\eta_0(s_i) = W(s)\Big|_{s=s_i}$$

The k-moment of system (1) at $s_i$ is

$$\eta_k(s_i) = \frac{(-1)^k}{k!} \left[ \frac{d^k}{ds^k} W(s) \right]_{s=s_i}$$




$s_1 = +i \quad s_3 = 10i$

$s_2 = -i \quad s_4 = -10i$

...

**Lemma 1** Let $s_i \in \mathbb{C} \setminus \sigma(A)$. Consider

$$C \tilde{\Pi} \Psi_k = [\eta_0(s_i) \ \eta_1(s_i) \ \dots \ \eta_k(s_i)]$$

$$\Psi_k = \mathrm{diag}(1, -1, 1, \dots, (-1)^k) \in \mathbb{R}^{(k+1) \times (k+1)}$$

$\tilde{\Pi}$ is the unique solution of the Sylvester equation

$$\tilde{\Pi} \Sigma_k = A \tilde{\Pi} + B L_k$$

$$L_k = [1 \ 0, \dots \ 0] \in \mathbb{R}^{k+1}$$

$$\Sigma_k = \begin{bmatrix} s_i & 1 & 0 & \dots & \\ 0 & s_i & 1 & 0 & \dots \\ & & \ddots & \ddots & 1 \\ & & & & s_i \end{bmatrix} \in \mathbb{C}^{(k+1) \times (k+1)}$$

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**Lemma 2** Consider the pair $(S, L)$ where $S$ has characteristic polynomial

$$p(s) = \prod_{i=1}^{\bar{k}} (s - s_i)^{k_i}$$

$V = \sum_{i=1}^{\bar{k}} k_i$ $(S \in \mathbb{R}^{V \times V})$, and $L \in \mathbb{R}^{1 \times V}$ is such that $(S, L)$ is observable. Assume $\sigma(S) \cap \sigma(A) = \emptyset$

**Then** there exists a one-to-one relation between the moments $\eta_0(s_0) \dots \eta_{k_0-1}(s_0), \eta_0(s_1) \dots \eta_{k_1-1}(s_1) \dots$ with the matrix $C\Pi$ where $\Pi$ is the unique solution of

$$A\Pi - \Pi S = -BL$$

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**Definition** The moments of (1) at $(S, L)$ are the elements of $C\Pi$

LEMMA 3 | CONSIDER AN AUXILIARY SYSTEM

$$\dot{w} = Sw \qquad u = Lw$$

$w \in \mathbb{R}^v$, $S \in \mathbb{R}^{v \times v}$, $L \in \mathbb{R}^{1 \times v}$ $(S, L)$ IS OBSERVABLE.
ASSUME $\sigma(S) \subset \mathbb{C}_0$ AND THE EIGENVALUES OF $S$ ARE SIMPLE. CONSIDER THE SYSTEM

$$\dot{x} = Ax + Bu \quad \cdot \quad y = Cx$$

ASSUME THAT $\sigma(A) \subset \mathbb{C}_{<0}$. THEN

$$y(t) = C \cdot \Pi \cdot w(t) + C \cdot e^{At}(x(0) - \Pi w(0))$$

AND THE STEADY-STATE IS

$$y_{ss}(t) = C\Pi \, w(t)$$

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PROOF

$$A\Pi - \Pi S = -BL \qquad BL - \Pi S = -A\Pi$$

$$z(t) = x(t) - \Pi w(t)$$

$$\dot{z} = \dot{x} - \Pi \dot{w} = Ax + BLw - \Pi Sw =$$

$$= Ax + (BL - \Pi S)w$$

$$= Ax + (-A\Pi)w$$

$$= A(x - \Pi w) = Az$$

$$\dot{z} = Az \qquad z(t) = e^{At} z(0)$$

$$x(t) - \Pi w(t) = e^{At}(x(0) - \Pi w(0))$$

$$x(t) = \Pi w(t) + e^{At}(x(0) - \Pi w(0))$$

$$y = Cx = C\Pi w + Ce^{At}(x(0) - \Pi w(0))$$

$$y_{ss}(t) = C\Pi w(t)$$

DEFINITION | GIVEN (1) AND $(S, L)$. THE SYSTEM

$$\boxed{\begin{aligned} \dot{\xi} &= F\xi + Gu \\ y_{RED} &= H\xi \end{aligned}} \qquad (2)$$

$\xi \in \mathbb{R}^{\nu}$, $F \in \mathbb{R}^{\nu \times \nu}$, $G \in \mathbb{R}^{\nu \times 1}$, $H \in \mathbb{R}^{1 \times \nu}$

IS A REDUCED ORDER MODEL BY MOMENT MATCHING IF (2) HAS THE SAME MOMENTS OF (1) AT $(S, L)$

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(P) $\quad \Pi \quad A\Pi - \Pi S = -BL \rightarrow C\Pi$

$\quad P \quad FP - PS = -GL \rightarrow HP$

IF $P$ IS THE UNIQUE SOLUTION OF

$$FP - PS = -GL$$

THEN THE MOMENTS OF (2) ARE $HP$

(2) IS A ROM BY MM. $\Longleftrightarrow$ $\begin{cases} FP - PS = -GL \\ C\Pi = HP \end{cases}$

FREE PARAMETERS $\nu^2 + 2\nu$

$\underline{C\Pi \in \mathbb{R}^{\nu}}$ CONSTRAINTS $\nu$

SET $P = I$

$$F - S = -GL \Rightarrow F = S - GL$$

$$C\Pi = H \Rightarrow H = C\Pi$$

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$$\begin{aligned} \dot{\xi} &= (S - GL)\xi + Gu \\ y_{RED} &= C\Pi\xi \end{aligned}$$

FOR ANY $G$ SUCH THAT $\sigma(F = S - GL) \cap \sigma(S) = \emptyset$

$FP = PS - GL$ $\quad C\Pi = HP$

$$\boxed{\begin{aligned} F &= \underbrace{PSP^{-1}}_{\tilde{S}} - G\underbrace{LP^{-1}}_{\tilde{L}} \\ H &= C\underbrace{\Pi P^{-1}}_{\tilde{\Pi}} \end{aligned}}$$

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PROBLEM ASSIGN THE EIGENVALUES OF $S - GL$ USING $G$

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SIDE NOTE $(\tilde{A}, \tilde{B})$ REACHABLE $\Rightarrow \exists\, \tilde{K}$ SUCH THAT

$$\sigma(\tilde{A} - \tilde{B}\tilde{K}) = \{\lambda_1, \lambda_2, \dots\}$$ WHERE $\lambda_i$ IS ANY COMPLEX NUMBER

ACKERMANN FORMULA

$\tilde{K} = \text{PLACE}(\cdot)$ MATLAB

$\tilde{A} = S^T \quad \tilde{B} = L^T \Rightarrow K^T = \text{PLACE}(S^T, L^T, \{\dots\})$

$$\sigma(S^T - L^T K^T) = \{\dots\}$$

$$(S^T - L^T K^T)^T = (S - \overset{G}{K} L)$$

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$$A\Pi - \Pi S = -BL \qquad \Pi = ?$$

SYLVESTER MATLAB

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \bar{X}_1 & \bar{X}_2 & \cdots & \bar{X}_n \\ | & | & & | \end{bmatrix}$$

$$\text{VEC}(X) = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \\ \vdots \\ \bar{X}_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \ddots \end{bmatrix} \in \mathbb{R}^{m \times m} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \\ \vdots & & \ddots \end{bmatrix} \in \mathbb{R}^{p \times q}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \\ \vdots & & \ddots \end{bmatrix} \in \mathbb{R}^{mp \times mq}$$

$$\bar{A}, \bar{B}, \bar{C}$$

$$\mathrm{vec}(\bar{A}\bar{B}\bar{C}) = (C^T \otimes A) \cdot \mathrm{vec}(B)$$

$$A\Pi - \Pi S = -BL$$

$$\mathrm{vec}(A\Pi I) - \mathrm{vec}(I\Pi S) = -\mathrm{vec}(BL)$$

$$(I \otimes A)\,\mathrm{vec}(\Pi) - (S^T \otimes I)\,\mathrm{vec}(\Pi) = -\mathrm{vec}(BL)$$

$\cancel{(I \otimes A)}$

MATRIX VECTOR VECTOR

$$(I \otimes A - S^T \otimes I)\,\mathrm{vec}(\Pi) = -\mathrm{vec}(BL)$$

$$Ax = b \implies x = A \backslash b = A^{-1} b$$

$\lambda_i \neq \sigma_j$

$$\boxed{\mathrm{vec}(\Pi) = (I \otimes A - S^T \otimes I)^{-1}\,\mathrm{vec}(-BL)}$$

$$\iff \sigma(A) \cap \sigma(S) = \emptyset$$

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RECALL $\quad y(t) = C\Pi w(t) + Ce^{At}(x(0) - \Pi w(0))$

$$y_{ss}(t) = \underline{C\Pi}\, w(t)$$

$$\begin{bmatrix} y_{ss}(t_1) = C\Pi\, w(t_1) \\ y_{ss}(t_2) = C\Pi\, w(t_2) \\ \vdots \\ y_{ss}(t_p) = C\Pi\, w(t_p) \end{bmatrix}$$

NOTE THAT $\quad Y_{ss}(r) = \overset{1\times v}{C\Pi}\ \overset{v\times 1}{w(r)}$

SO

$$\underbrace{[Y_{ss}(r_1)\ \ Y_{ss}(r_2)\ \dots\ Y_{ss}(r_p)]}_{\mathbb{Y}\in\mathbb{R}^{1\times p}} = \underbrace{C\Pi}_{1\times v}\ \underbrace{[w(r_1) \vdots w(r_2) \vdots \dots \vdots w(r_p)]}_{\mathbb{W}\in\mathbb{R}^{v\times p}}$$

MULTIPLY ON THE RIGHT BY $\mathbb{W}^T$

$$\mathbb{Y}\mathbb{W}^T = C\Pi \underbrace{\mathbb{W}\mathbb{W}^T}_{v\times v}$$

UNDER CERTAIN ASSUMPTIONS (SEE THEOREM 4.12 PAG. 133)

$\mathbb{W}\mathbb{W}^T$ IS INVERTIBLE

$$\Rightarrow \quad C\Pi = \mathbb{Y}\mathbb{W}^T(\mathbb{W}\mathbb{W}^T)^{-1}$$

NOTE THAT THIS IS CONSISTENT WITH THEOREM 4.12 PAG. 133. IN FACT

$$\mathrm{VEC}(C\Pi) = (C\Pi)^T = (\mathbb{W}\mathbb{W}^T)^{-1}\mathbb{W}\mathbb{Y}^T$$

LET $\tilde{R} = \mathbb{W}^T$ AND ~~[illegible]~~ $\Upsilon = \mathbb{Y}^T$

AND WE OBTAIN (4.34) AT PAG. 133