Fortran 77 Routines for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Design of Continuous-Time Linear Control Systems [1]

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Abstract

We present Fortran 77 subroutines intended for state-space design of $\mathcal{H}_\infty$ (sub)optimal controllers and $\mathcal{H}_2$ optimal controllers for linear continuous-time control systems. The subroutines make use of LAPACK and BLAS libraries and produce estimates of the conditioning of the corresponding matrix algebraic Riccati equations. Modified formulae are implemented in the case of $\mathcal{H}_\infty$ design which allows to reduce the order of the inverted matrices. The subroutines will be included in the SLICOT library.

1 Introduction

In this report we present a set of Fortran 77 subroutines intended for state-space design of $\mathcal{H}_\infty$ (sub)optimal controllers and $\mathcal{H}_2$ optimal controllers for linear continuous-time control systems. The subroutines make use of LAPACK [1] and BLAS [8, 5, 4] libraries and implement a reliable Schur solver for the corresponding matrix algebraic Riccati equations which produces condition and error estimates [9]. In the case of $\mathcal{H}_\infty$ design we implement the Glover's-Doyle's result [6] with some modifications which allow to reduce the order of the matrices which are to be inverted in the solution of the Riccati equations.

The report is organised as follows. In Section 2 we give the Glover's-Doyle's formulae for $\mathcal{H}_\infty$ suboptimal design describing the modifications which are implemented to improve the numerical efficiency and present the formulae for $\mathcal{H}_2$ optimal design. In Section 3 we give detailed description of the algorithms and subroutines for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers design. In Section 4 a sixth-order example is given to illustrate the implementation of the routines and to demonstrate their features.

The subroutines for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ designs will be included in a future release of the SLICOT library

The following notations are used in the paper. $\mathcal{R}$ denotes the field of real numbers; $\mathcal{R}^{m\times n}$ – the space of $m \times n$ matrices $A = [a_{ij}]$ over $\mathcal{R}$; A^T – the transposed matrix A; $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ – the maximum and minimum singular values of A; $\|G(s)\|_2$ and $\|G(s)\|_\infty$ – the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm of a stable transfer function matrix $G(s)$, respectively, and I_n – the unit $n \times n$ matrix.

We denote by ε the roundoff unit of the machine arithmetic.

2 Formulae for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ design

Consider a generalised linear system, described by the equations

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t) \quad (2.1)$$

$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \quad (2.2)$$

$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t) \quad (2.3)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $w(t) \in \mathcal{R}^{m_1}$ is the exogenous input vector (the disturbance), $u(t) \in \mathcal{R}^{m_2}$ is the control input vector, $z(t) \in \mathcal{R}^{p_1}$ is the error vector, and $y(t) \in \mathcal{R}^{p_2}$ is the measurement vector, with $p_1 \geq m_2$ and $p_2 \leq m_1$. The transfer function matrix of the system will be denoted by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \quad (2.4)$$

$$= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad (2.5)$$

$$:= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (2.6)$$

The '$\mathcal{H}_\infty$ sub-optimal control problem' is to find an internally stabilising controller $K(s)$ such that, for a pre-specified positive value γ,

$$\|F_\ell(P,K)\|_\infty < \gamma \quad (2.7)$$

where $F_\ell(P,K)$ is a lower linear fractional transformation (LFT) on $K(s)$, equal to the closed-loop transfer function $T_{zw}(s)$ from w to z,

$$T_{zw}(s) := F_\ell(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Glover and Doyle [6] derived necessary and sufficient conditions for the existence of an $\mathcal{H}_\infty$ sub-optimal controller and further parametrized all such controllers. This result is obtained under the following assumptions.

A1 (A, B_2) is stabilisable and (C_2, A) is detectable;

A2 $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$;

A3 $\begin{bmatrix} A - j\omega I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

A4 $\begin{bmatrix} A - j\omega I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

We shall assume also that $D_{22} = 0$ (this will be removed later) and we shall partition D_{11} as

$$D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix},$$

where D_{1122} has m_2 rows and p_2 columns.

The solution of the $\mathcal{H}_\infty$ optimisation problem is closely related to the solution of matrix algebraic Riccati equations. Let A, Q, R be real $n \times n$ matrices with Q and R symmetric. Define the $2n \times 2n$ Hamiltonian matrix

$$H = \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix}.$$

If H has no eigenvalues on the imaginary axis (which means that it will have exactly n eigenvalues with negative real parts) and if the two subspaces

$$\mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \mathrm{Im}\begin{bmatrix} 0 \\ I_n \end{bmatrix}$$

are complementary, where $X_1, X_2 \in R^n$ and the columns of

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

span the stable invariant subspace of H, then the matrix $X = X_2X_1^{-1}$ is uniquely determined by H and is denoted as $X = \mathrm{Ric}(H)$. The domain of the function Ric is taken to consist of Hamiltonian matrices with the two properties described above.

Suppose $H \in \mathrm{dom(Ric)}$ and $X = \mathrm{Ric}(H)$. Then [10],

(a) X is symmetric;

(b) X satisfies the algebraic Riccati equation

$$A^TX + XA + Q - XRX = 0;$$

(c) $A - RX$ is stable.

If in addition

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

with (A, B) stabilisable and (C, A) detectable, then $H \in \mathrm{dom}(\mathrm{Ric})$ and $X = \mathrm{Ric}(H) \geq 0$. Note that if (C, A) is observable, then $\mathrm{Ric}(H) > 0$.

The $\mathcal{H}_\infty$ solution involves two Hamiltonian matrices , $\mathcal{H}_\infty$ and J_∞ which are defined as follows.

$$R := D_{1\bullet}{}^T D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.8}$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}{}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.9}$$

where

$$D_{1\bullet} := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \text{ and } D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \tag{2.10}$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^T C_1 & B^T \end{bmatrix} \tag{2.11}$$

$$J_\infty := \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1 D_{\bullet 1}^T \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^T & C \end{bmatrix} \tag{2.12}$$

$$X_\infty = \mathrm{Ric}(H_\infty) \tag{2.13}$$

$$Y_\infty = \mathrm{Ric}(J_\infty) \tag{2.14}$$

Also define the "state feedback" and "output injection" matrices as

$$F := -R^{-1} \left(D_{1\bullet}{}^T C_1 + B^T X_\infty \right) := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := \begin{bmatrix} F_{11} \\ F_{12} \\ F_2 \end{bmatrix} \tag{2.15}$$

$$H := -\left(B_1 D_{\bullet 1}{}^T + Y_\infty C^T \right) \tilde{R}^{-1} := \begin{bmatrix} H_1 & H_2 \end{bmatrix} := \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} \tag{2.16}$$

where F_1, F_2, F_{11} and F_{12} are of m_1, m_2, $m_1 - p_2$ and p_2 rows, respectively, and H_1, H_2, H_{11} and H_{12} are of p_1, p_2, $p_1 - m_2$ and m_2 columns, respectively.

Now the Glover's-Doyle's result is stated in the following theorem.

Theorem [11] *Suppose $P(s)$ satisfies the assumptions* **A1 – A4**.
(a) There exists an internally stabilising controller $K(s)$ such that $||F_\ell(P, K)||_\infty < \gamma$ if and only if
(i) $\gamma > max(\sigma_{\max}[D_{1111}, D_{1112}], \sigma_{\max}[D_{1111}^T, D_{1121}^T])$
(ii) $H_\infty \in dom(Ric)$ *with* $X_\infty = Ric(H_\infty) \geq 0$
(iii) $J_\infty \in dom(Ric)$ *with* $Y_\infty = Ric(J_\infty) \geq 0$
(iv) $\rho(X_\infty Y_\infty) < \gamma^2$

where $\rho(\bullet)$ denotes the spectral radius.

(b) Given that the conditions of part (a) are satisfied, then all rational, internally stabilising controllers $K(s)$, satisfying $||F_\ell(P,K)||_\infty < \gamma$, are given by

$$K(s) = F_\ell(M, \Phi) \tag{2.17}$$

for any rational $\Phi(s) \in \mathcal{H}_\infty$ such that $||\Phi(s)||_\infty < \gamma$, where $M(s)$ has the realisation

$$M(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right] \tag{2.18}$$

and

$$\hat{D}_{11} = -D_{1121}D_{1111}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122}. \tag{2.19}$$

$\hat{D}_{12} \in R^{m_2 \times m_2}$ and $\hat{D}_{21} \in R^{p_2 \times p_2}$ are any matrices (e.g. Cholesky factors) satisfying

$$\hat{D}_{12}\hat{D}_{12}^T = I_{m_2} - D_{1121}(\gamma^2 I - D_{1111}^T D_{1111})^{-1}D_{1121}^T \tag{2.20}$$

$$\hat{D}_{21}^T\hat{D}_{21} = I_{p_2} - D_{1112}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} \tag{2.21}$$

and

$$\hat{B}_2 = Z(B_2 + H_{12})\hat{D}_{12} \tag{2.22}$$

$$\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12}) \tag{2.23}$$

$$\begin{aligned} \hat{B}_1 &= -ZH_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \\ &= -ZH_2 + Z(B_2 + H_{12})\hat{D}_{11} \end{aligned} \tag{2.24}$$

$$\begin{aligned} \hat{C}_1 &= F_2 + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \\ &= F_2 - \hat{D}_{11}(C_2 + F_{12}) \end{aligned} \tag{2.25}$$

$$\begin{aligned} \hat{A} &= A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \\ &= A + BF - \hat{B}_1(C_2 + F_{12}) \end{aligned} \tag{2.26}$$

where

$$Z = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}. \tag{2.27}$$

■

When $\Phi(s) = 0$ is chosen, the corresponding sub-optimal controller is called the *central* controller which is widely used in the $\mathcal{H}_\infty$ optimal design and has the state-space form

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right] \tag{2.28}$$

In the rest of the report, only the central sub-optimal controller will be considered.

The main computational task in the solution procedure of the $\mathcal{H}_\infty$ optimisation problem is to solve the two matrix algebraic Riccati equations

$$A_X^T X_\infty + X_\infty A_X + C_X - X_\infty D_X X_\infty = 0 \tag{2.29}$$

$$A_Y Y_\infty + Y_\infty A_Y^T + C_Y - Y_\infty D_Y Y_\infty = 0 \tag{2.30}$$

for X_∞, Y_∞, respectively, where

$$A_X = A - BR^{-1}D_{1\bullet}^T C_1,$$

$$C_X = C_1^T C_1 - C_1^T D_{1\bullet} R^{-1} D_{1\bullet}^T C_1,$$

$$D_X = BR^{-1}B^T$$

and

$$A_Y = A - B_1 D_{\bullet 1}^T \tilde{R}^{-1} C,$$

$$C_Y = B_1 B_1^T - B_1 D_{\bullet 1}^T \tilde{R}^{-1} D_{\bullet 1} B_1^T,$$

$$D_Y = C^T \tilde{R}^{-1} C.$$

We shall refer to equations (2.29) and (2.30) as to *X-Riccati equation* and *Y-Riccati equation*, respectively.

In what follows we will show that the inversion of the $m \times m$ matrix R, $m = m_1 + m_2$, and correspondingly the $p \times p$ matrix $\tilde{R}$, $p = p_1 + p_2$, in the solution of the Riccati equations may be reduced to the inversion of $m_1 \times m_1$ matrix and $p_1 \times p_1$ matrix, respectively, taking into account the structure of the given matrices. According to (2.8) we have that

$$R := \begin{bmatrix} R_{11} & R_{21}^T \\ R_{21} & I_{m_2} \end{bmatrix} \tag{2.31}$$

where

$$R_{11} = D_{11}^T D_{11} - \gamma^2 I_{m_1},$$

$$R_{21} = [D_{1121} \; D_{1122}].$$

Now it is straightforward to show that

$$R^{-1} := \begin{bmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{bmatrix} \tag{2.32}$$

where

$$Z_{11} = (R_{11} - R_{21}^T R_{21})^{-1} = \left[\begin{bmatrix} D_{1111}^T \\ D_{1112}^T \end{bmatrix} \begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix} - \gamma^2 I_{m_1} \right]^{-1},$$

$$Z_{21} = - \begin{bmatrix} D_{1121} & D_{1122} \end{bmatrix} Z_{11},$$

$$Z_{22} = \begin{bmatrix} D_{1121} & D_{1122} \end{bmatrix} Z_{11} \begin{bmatrix} D_{1121}^T \\ D_{1122}^T \end{bmatrix} + I_{m_2}.$$

The benefit of using the formula (2.32) is twofold. First, the dimension of the matrix which has to be inverted is decreased and second, the condition number of the matrix $R_{11} - R_{21}^T R_{21}$ can be much smaller than the condition number of R. In fact, for large values of γ the condition number of $R_{11} - R_{21}^T R_{21}$ is of order one, while the condition number of R is of order γ^2.

In a similar way, the matrix $\tilde{R}^{-1}$ is represented as

$$\tilde{R}^{-1} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{12}^T & \tilde{Z}_{22} \end{bmatrix} \tag{2.33}$$

where

$$\tilde{Z}_{11} = (\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{12}^T)^{-1} = \left[\begin{bmatrix} D_{1111} \\ D_{1121} \end{bmatrix} \begin{bmatrix} D_{1111}^T & D_{1121}^T \end{bmatrix} - \gamma^2 I_{p_1} \right]^{-1},$$

$$\tilde{Z}_{12} = -\tilde{Z}_{11} \begin{bmatrix} D_{1112} \\ D_{1122} \end{bmatrix},$$

$$\tilde{Z}_{22} = \begin{bmatrix} D_{1112}^T & D_{1122}^T \end{bmatrix} \tilde{Z}_{11} \begin{bmatrix} D_{1112} \\ D_{1122} \end{bmatrix} + I_{p_2}.$$

Consider now how to normalise a given system in order to fulfil the assumption **A2**. Suppose we are given a system with transfer function matrix

$$P_p(s) \;=\; \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array} \right] \tag{2.34}$$

for which $\mathrm{rank}(D_{p12}) = m_2$ and $\mathrm{rank}(D_{p21}) = p_2$. By using the input and output transformations

$$u_p = T_u u, \;\; y_p = T_y^{-1} y, \;\; z_p = Q_{12} z, \;\; w_p = Q_{21}^T w$$

where the matrices Q_{12} and Q_{21} are orthogonal and the matrices T_u and T_y are nonsingular, the system (2.34) will be reduced to a system with the transfer function matrix

$$P(s) \;=\; \begin{bmatrix} Q_{12}^T & 0 \\ 0 & T_y \end{bmatrix} P_p(s) \begin{bmatrix} Q_{21}^T & 0 \\ 0 & T_u \end{bmatrix} \tag{2.35}$$

$$\;=\; \left[\begin{array}{c|cc} A_p & B_{p1}Q_{21}^T & B_{p2}T_u \\ \hline Q_{12}^T C_{p1} & Q_{12}^T D_{p11} Q_{21}^T & Q_{12}^T D_{p12} T_u \\ T_y C_{p2} & T_y D_{p21} Q_{21}^T & T_y D_{p22} T_u \end{array} \right] \tag{2.36}$$

If the matrices Q_{12}, Q_{21}, T_u, T_y are chosen such that

$$D_{12} \;=\; Q_{12}^T D_{p12} T_u = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}, \tag{2.37}$$

$$D_{21} \;=\; T_y D_{p21} Q_{21}^T = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix} \tag{2.38}$$

then the matrices of the new system $P(s)$ will satisfy the assumption **A2**. After determining the controller $K(s)$ for the transformed system $P(s)$, the controller $K_p(s)$ for $P_p(s)$ is found as $K_p(s) = T_u K(s) T_y$, which preserves the closed-loop system $\mathcal{H}_\infty$ norm,

$$\|F_\ell(P_p, K_p)\|_\infty = \|F_\ell(P, K)\|_\infty.$$

The transformations (2.37), (2.38) can be performed by using QR or Singular Value Decomposition (SVD). Here we shall use SVD which allows to determine reliably the numerical rank of the matrices D_{p12}, D_{p21} and to find easily the condition numbers of the transformation matrices T_u, T_y.

Let

$$D_{p12} = U_{12} \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T$$

be the SVD of D_{p12} where the matrices $U_{12},\ V_{12}$ are orthogonal and Σ_{12} is a diagonal matrix containing the m_2 singular values of D_{p12} on the diagonal. (If D_{p12} is of full column rank, then the singular values are positive and Σ_{12} is non-singular.) Partitioning the matrix U_{12} as

$$U_{12} = [U_{121}\ U_{122}]$$

where U_{121} has m_2 columns and U_{122} has $p_1 - m_2$ columns and defining

$$Q_{12} = [U_{122}\ U_{121}]\,,$$

$$T_u = V_{12}\Sigma_{12}^{-1}\,,$$

we obtain the necessary representation of D_{p12}. In addition, the condition number of T_u in respect to inversion is $\sigma_{\max}(\Sigma_{12})/\sigma_{\min}(\Sigma_{12})$. The determination of this condition number allows to monitor the conditioning of T_u during the reduction and to obtain an estimation about the precision of the computations.

In a similar way, let

$$D_{p21} = U_{21}\left[\begin{array}{cc} \Sigma_{21} & 0 \end{array}\right] V_{21}^T$$

be the SVD of D_{p12} where $U_{21},\ V_{21}$ are orthogonal and Σ_{21} is a diagonal matrix. If D_{p12} is of full row rank, then the p_2 singular values are positive and Σ_{21} is non-singular. Partition the matrix V_{21}^T as

$$V_{21}^T = \left[\begin{array}{c} V_{211}^T \\ V_{212}^T \end{array}\right]$$

where V_{211}^T has p_2 rows and V_{212}^T has $m_1 - p_2$ rows and taking

$$Q_{21} = \left[\begin{array}{c} V_{212}^T \\ V_{211}^T \end{array}\right],$$

$$T_y = \Sigma_{21}^{-1}U_{21}^T$$

we obtain the necessary reduction of D_{p21}. The condition number of T_y in respect to inversion is $\sigma_{\max}(\Sigma_{21})/\sigma_{\min}(\Sigma_{21})$.

Consider finally the general case when $D_{22} \neq 0$. Suppose K is a stabilising controller for D_{22} set to zero, and satisfies

$$\|F_\ell\left(P - \left[\begin{array}{cc} 0 & 0 \\ 0 & D_{22} \end{array}\right], K\right)\|_\infty < \gamma.$$

Then [2]

$$\begin{array}{rcl} F_\ell(P, K(I + D_{22}K)^{-1}) & = & P_{11} + P_{12}K(I + D_{22}K - P_{22}K)^{-1}P_{21} \\ & = & F_\ell\left(P - \left[\begin{array}{cc} 0 & 0 \\ 0 & D_{22} \end{array}\right], K\right). \end{array}$$

In this way a controller K for

$$P - \left[\begin{array}{cc} 0 & 0 \\ 0 & D_{22} \end{array}\right]$$

yields a controller $\tilde{K} = K(I + D_{22}K)^{-1}$ for P. In order to find $\tilde{K}$ from K we must exclude the possibility of the feedback system becoming ill-posed, i.e. $\det(I + D_{22}\tilde{K}(\infty)) = 0$.

Alternative formulae for $\mathcal{H}_\infty$ design are proposed in [7]

Consider now a strictly proper system with transfer function

$$
\begin{aligned}
P(s) &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] && (2.39)\\
&:= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] && (2.40)
\end{aligned}
$$

The '$\mathcal{H}_2$ optimal control problem' is to find an internally stabilising controller $K(s)$ which minimises $\|F_\ell(P,K)\|_2$. Under the assumption that (A, B_2) is stabilisable and (C_2, A) is detectable, the optimal controller exists and is unique.

Suppose that the system is transformed to normalised form, just as in the case of H_∞ design, such that

$$D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \text{ and } D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}.$$

The $\mathcal{H}_2$ optimal state feedback and output injection matrices are given by [11]

$$F = -D_{12}^T C_1 - B_2^T X_2, \quad (2.41)$$
$$H = -B_1 D_{21}^T - Y_2 C_2^T, \quad (2.42)$$

respectively, where X_2, Y_2 are the positive semi-definite solutions of the corresponding X- and Y-Riccati equations

$$
\begin{aligned}
A_X^T X_2 + X_2 A_X + C_X - X_2 D_X X_2 &= 0, && (2.43)\\
A_Y Y_2 + Y_2 A_Y^T + C_Y - Y_2 D_Y Y_2 &= 0 && (2.44)
\end{aligned}
$$

and

$$
\begin{aligned}
A_X &= A - B_2 D_{12}^T C_1, \\
C_X &= C_1^T C_1 - C_1^T D_{12} D_{12}^T C_1, \\
D_X &= B_2 B_2^T, \\
A_Y &= A - B_1 D_{21}^T C_2, \\
C_Y &= B_1 B_1^T + B_1 D_{21}^T D_{21} B_1^T, \\
D_Y &= C_2^T C_2.
\end{aligned}
$$

The matrices in the state-space description of the optimal $\mathcal{H}_2$ controller $K(s)$ for the normalised system are given by

$$
\begin{aligned}
A_k &= A + HC_2 + B_2 F + HD_{22} F, && (2.45)\\
B_k &= -H, && (2.46)\\
C_k &= F, && (2.47)\\
D_k &= 0_{m_2 \times p_2} && (2.48)
\end{aligned}
$$

and the controller for the original system is found as $K_p(s) = T_u K(s) T_y$.

3 Numerical algorithms and software

The synthesis of $\mathcal{H}_\infty$ (sub)optimal controller is accomplished by the algorithms *HINFT*, *HINFN* and *HINFC*.

For given matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21} and D_{22} and for a specified value of γ, algorithm *HINFT* checks the rank conditions against an user supplied threshold *tol* and computes the transformation matrices Q_{12}, Q_{21}, T_u and T_y which reduce the generalised system into the normalised form. Note that the block D_{22} is not transformed, since it is required in its original form by the algorithm *HINFC* in the computation of the controller matrices. The algorithm produces also the condition numbers of the nonsingular transformation matrices T_u and T_y.

Algorithm *HINFT*: *Reduction of the system to normalised form and checking the rank conditions for $\mathcal{H}_\infty$ design*

If $\sigma_{n+m_2}\left(\begin{bmatrix} A - j\omega I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}\right) < tol$, stop

If $\sigma_{n+p_2}\left(\begin{bmatrix} A - j\omega I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}\right) < tol$, stop

Decompose $D_{12} = U_{12} \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T$

If $\sigma_{m_2}(\Sigma_{12}) < tol$, stop

Decompose $D_{21} = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \end{bmatrix} V_{21}^T$

If $\sigma_{p_2}(\Sigma_{21}) < tol$, stop

Compute $Q_{12} = U_{12} \begin{bmatrix} 0 & I_{m_2} \\ I_{p_1-m_2} & 0 \end{bmatrix}$

Compute $T_u = V_{12}\Sigma_{12}^{-1}$

Compute $\mathrm{cond}(T_u) = \sigma_1(\Sigma_{12})/\sigma_{m_2}(\Sigma_{12})$

Compute $Q_{21} = \begin{bmatrix} 0 & I_{m_1-p_2} \\ I_{p_2} & 0 \end{bmatrix} V_{21}^T$

Compute $T_y = \Sigma_{21}^{-1}U_{21}^T$

Compute $\mathrm{cond}(T_y) = \sigma_1(\Sigma_{21})/\sigma_{p_2}(\Sigma_{21})$

$B_1 \leftarrow B_1 Q_{21}^T$

$B_2 \leftarrow B_2 T_u$

$C_1 \leftarrow Q_{12}^T C_1$

$C_2 \leftarrow T_y C_2$

$D_{11} \leftarrow Q_{12}^T D_{11} Q_{21}^T$

Set $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$

Set $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$

The computation of the $\mathcal{H}_\infty$ (sub)optimal state feedback F and output injection H matrices for the normalised system is done by the algorithm *HINFN*. The algorithm makes use of the Schur algorithm for solving the matrix Riccati equation with condition and accuracy estimates, described in [9].

Algorithm *HINFN*: ***Computation of the state feedback and output injection matrices for the (sub)optimal $\mathcal{H}_\infty$ controller***

Compute $Z_{11} = \left[\begin{bmatrix} D_{1111}^T \\ D_{1112}^T \end{bmatrix} \begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix} - \gamma^2 I_{m_1} \right]^{-1}$

Compute $Z_{21} = -\begin{bmatrix} D_{1121} & D_{1122} \end{bmatrix} Z_{11}$

Compute $Z_{22} = -Z_{21} \begin{bmatrix} D_{1121}^T \\ D_{1122}^T \end{bmatrix} + I_{m_2}$

Form $R^{-1} = \begin{bmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{bmatrix}$

Compute $D_{1\bullet}^T C_1 = \begin{bmatrix} D_{11}^T C_1 & [0 \;\; I_{m_2}] C_1 \end{bmatrix}$

Compute $A_X = A - BR^{-1} D_{1\bullet}^T C_1$

Compute $C_X = C_1^T C_1 - C_1^T D_{1\bullet} R^{-1} D_{1\bullet}^T C_1$

Compute $D_X = BR^{-1}B^T$

Solve $A_X^T X_\infty + X_\infty A_X + C_X - X_\infty D_X X_\infty = 0$

Compute $F = -R^{-1}\left(D_{1\bullet}^T C_1 + B^T X_\infty\right)$

Compute $\tilde{Z}_{11} = \left[\begin{bmatrix} D_{1111} \\ D_{1121} \end{bmatrix} \begin{bmatrix} D_{1111}^T & D_{1121}^T \end{bmatrix} - \gamma^2 I_{p_1} \right]^{-1}$

Compute $\tilde{Z}_{12} = -\tilde{Z}_{11} \begin{bmatrix} D_{1112} \\ D_{1122} \end{bmatrix}$

Compute $\tilde{Z}_{22} = -\begin{bmatrix} D_{1112}^T & D_{1122}^T \end{bmatrix} \tilde{Z}_{12} + I_{p_2}$

Form $\tilde{R}^{-1} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{12}^T & \tilde{Z}_{22} \end{bmatrix}$

Compute $B_1 D_{\bullet 1}^T = \begin{bmatrix} B_1 D_{11}^T & B_{11} \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix} \end{bmatrix}$

Compute $A_Y = A - B_1 D_{\bullet 1}^T \tilde{R}^{-1} C$

Compute $C_Y = B_1 B_1^T - B_1 D_{\bullet 1}^T \tilde{R}^{-1} D_{\bullet 1} B_1^T$

Compute $D_Y = C^T \tilde{R}^{-1} C$

Solve $A_Y Y_\infty + Y_\infty A_Y^T + C_Y - Y_\infty D_Y Y_\infty = 0$

Compute $H = -\left(B_1 D_{\bullet 1}^T + Y_\infty C^T\right) \tilde{R}^{-1}$

The computation of the central controller matrices A_k, B_k, C_k and D_k for the original (non-transformed) system from the state feedback and output injection matrices is done by the algorithm *HINFC*.

Algorithm *HINFNC: Computation of the $\mathcal{H}_\infty$ controller matrices*

Set $F_{11} = F(1:(m_1 - p_2), 1:n)$

Set $F_{12} = F((m_1 - p_2 + 1):m_1, 1:n)$

Set $F_2 = F((m_1+1):(m_1+m_2), 1:n)$

Set $H_{11} = H(1:n, 1:(p_1 - m_2))$

Set $H_{12} = H(1:n, (p_1 - m_2 + 1):p_1)$

Set $H_2 = H(1:n, (p_1+1):(p_1+p_2))$

Set $\hat{D}_{11} = -D_{1122}$

Set $\hat{D}_{12}\hat{D}_{12}^T = I_{m_2}$

Set $\hat{D}_{21}^T\hat{D}_{21} = I_{p_2}$

If $p_1 > m_2$, then
  If $m_1 = p_2$, then

    Compute $\hat{D}_{21}^T\hat{D}_{21} = I_{p_2} - D_{1112}^T D_{1112}/\gamma^2$

    Else

      If $m_1 > p_2$, then

        Compute $\hat{D}_{11} = -D_{1121}D_{1111}^T(\gamma^2 I_{p_1-m_2} - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122}$

        Compute $\hat{D}_{12}\hat{D}_{12}^T = I_{m_2} - D_{1121}(\gamma^2 I_{m_1-p_2} - D_{1111}^T D_{1111})^{-1}D_{1121}^T$

        Compute $\hat{D}_{21}^T\hat{D}_{21} = I_{p_2} - D_{1112}^T(\gamma^2 I_{p_1-m_2} - D_{1111}D_{1111}^T)^{-1}D_{1112}$

      End if

  End if

  Else

    If $m_1 > p_2$, then

      Compute $\hat{D}_{12}\hat{D}_{12}^T = I_{m_2} - D_{1121}D_{1121}^T/\gamma^2$

    End if

End if

Compute $\hat{D}_{12}$ as the Cholesky factor of $\hat{D}_{12}\hat{D}_{12}^T$

Compute $\hat{D}_{21}$ as the Cholesky factor of $\hat{D}_{21}^T\hat{D}_{21}$

Compute $Z = (I_n - Y_\infty X_\infty/\gamma^2)^{-1}$

Compute $\hat{B}_2 = (B_2 + H_{12})\hat{D}_{12}$

Compute $\hat{B}_1 = -H_2 + (B_2 + H_{12})\hat{D}_{11}$

Compute $\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12})Z$

Compute $\hat{C}_1 = F_2 Z - \hat{D}_{11}(C_2 + F_{12})Z$

Compute $\hat{A} = A + HC + (B_2 + H_{12})\hat{C}_1$

Compute $\hat{B} = \begin{bmatrix} \hat{B}_1 T_y & \hat{B}_2 \end{bmatrix}$

Compute $\hat{C} = \begin{bmatrix} T_u \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}$

Compute $\hat{D} = \begin{bmatrix} T_u \hat{D}_{11} T_y & T_u \hat{D}_{12} \\ \hat{D}_{21} T_y & 0 \end{bmatrix}$

Form the controller matrices for the case when $D_{22} \neq 0$

If $\mathrm{cond}(I_{m_2} + \hat{D}_{11} D_{22}) > 1/\varepsilon$, stop

Compute $M = (I_{m_2+p_2} + \begin{bmatrix} D_{22} & 0 \\ 0 & 0 \end{bmatrix} \hat{D})^{-1}$

Compute $\hat{M} = (I_{m_2+p_2} + \hat{D} \begin{bmatrix} D_{22} & 0 \\ 0 & 0 \end{bmatrix})^{-1}$

$\hat{A} \leftarrow \hat{A} - \hat{B}(I_{m_2+p_2} - M)\hat{D}^{-1}\hat{C}$

$\hat{B} \leftarrow \hat{B}M$

$\hat{C} \leftarrow \hat{M}C$

$\hat{D} \leftarrow DM$

Set $A_k = \hat{A}$

Set $B_k = \hat{B}(1:n, 1:p_2)$

Set $C_k = \hat{C}(1:m_2, 1:n)$

Set $D_k = \hat{D}(1:m_2, 1:p_2)$

The algorithms *HINFT, HINFN* and *HINFC* are implemented by the double precision Fortran 77 subroutines DHINFT, DHINFN and DHINFC, respectively. Almost all computations in these routines are done by using BLAS Level-3 routines [4] and LAPACK routines [1]. The matrix Riccati equations are solved by the subroutine DGRSVX, presented in [9]. The driver subroutine DHFSYN calls all routines necessary to compute the $\mathcal{H}_\infty$ (sub)optimal controller. Listings of the subroutines DHFSYN, DHINFT, DHINFN, and DHINFC will shortly be included in the SLICOT library.

The synthesis of the $\mathcal{H}_2$ optimal controller is accomplished by the algorithms *H2T, H2N* and *H2C*, which are analogical to the corresponding algorithms for $\mathcal{H}_\infty$ synthesis.

Algorithm *H2T*: *Reduction of the system to normalised form*

Decompose $D_{12} = U_{12} \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T$

If $\sigma_{m_2}(\Sigma_{12}) < tol$, stop

Decompose $D_{21} = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \end{bmatrix} V_{21}^T$

If $\sigma_{p_2}(\Sigma_{21}) < tol$, stop

Compute $Q_{12} = U_{12} \begin{bmatrix} 0 & I_{m_2} \\ I_{p_1-m_2} & 0 \end{bmatrix}$

Compute $T_u = V_{12}\Sigma_{12}^{-1}$

Compute $\text{cond}(T_u) = \sigma_1(\Sigma_{12})/\sigma_{m_2}(\Sigma_{12})$

Compute $Q_{21} = \begin{bmatrix} 0 & I_{m_1-p_2} \\ I_{p_2} & 0 \end{bmatrix} V_{21}^T$

Compute $T_y = \Sigma_{21}^{-1}U_{21}^T$

Compute $\text{cond}(T_y) = \sigma_1(\Sigma_{21})/\sigma_{p_2}(\Sigma_{21})$

$B_1 \leftarrow B_1 Q_{21}^T$

$B_2 \leftarrow B_2 T_u$

$C_1 \leftarrow Q_{12}^T C_1$

$C_2 \leftarrow T_y C_2$

Set $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$

Set $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$

Algorithm *H2N*: *Computation of the $\mathcal{H}_2$ optimal state feedback and output injection matrices*

Compute $A_X = A - B_2 D_{12}^T C_1$

Compute $C_X = C_1^T C_1 - C_1^T D_{12} D_{12}^T C_1$

Compute $D_X = B_2 B_2^T$

Solve $A_X^T X_2 + X_2 A_X + C_X - X_2 D_X X_2 = 0$

Compute $F = -D_{12}^T C_1 - B_2^T X_2$

Compute $A_Y = A - B_1 D_{21}^T C_2$

Compute $C_Y = B_1 B_1^T + B_1 D_{21}^T D_{21} B_1^T$

Compute $D_Y = C_2^T C_2$

Solve $A_Y Y_2 + Y_2 A_Y^T + C_Y - Y_2 D_Y Y_2 = 0$

Compute $H = -B_1 D_{21}^T - Y_2 C_2^T$

Algorithm *H2C:* *Computation of the $\mathcal{H}_2$ optimal controller matrices*

Compute $A_k = A + HC_2 + B_2 F + HD_{22} F$

Compute $B_k = -HT_y$

Compute $C_k = T_u F$

Set $D_k = 0_{m_2 \times p_2}$

The algorithms *H2T, H2N* and *H2C* are implemented by the double precision Fortran 77 routines DH2T, DH2N and DH2C, respectively. The driver subroutine DH2SYN calls all routines necessary to compute the $\mathcal{H}_2$ optimal controller. Listings of the subroutines DH2SYN, DH2T, DH2N and DH2C again will shortly appear in the SLICOT library.

For a given plant

$$
\begin{aligned}
G(s) &= \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \\
&:= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]
\end{aligned}
$$

and for a given controller

$$
K(s) := \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]
$$

the algorithm $CLOOP$ computes the matrices A_c, B_c, C_c and D_c of the closed-loop system

$$
\begin{aligned}
G_c(s) &:= F_\ell(G, K) = G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21} \\
&:= \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]
\end{aligned}
$$

Algorithm *CLOOP:* *Computation of the closed-loop system matrices*

Compute $A_c = \begin{bmatrix} A + B_2 D_k (I_{p_2} - D_{22} D_k)^{-1} C_2 & B_2 (I_{m_2} - D_k D_{22})^{-1} C_k \\ B_k (I_{p_2} - D_{22} D_k)^{-1} C_2 & A_k + B_k D_{22} (I_{m_2} - D_k D_{22})^{-1} C_k \end{bmatrix}$

Compute $B_c = \begin{bmatrix} B_1 + B_2 D_k (I_{p_2} - D_{22} D_k)^{-1} D_{21} \\ B_k (I_{p_2} - D_{22} D_k)^{-1} D_{21} \end{bmatrix}$

Compute $C_c = \begin{bmatrix} C_1 + D_{12} D_k (I_{p_2} - D_{22} D_k)^{-1} C_2 & D_{12} (I_{m_2} - D_k D_{22})^{-1} C_k \end{bmatrix}$

Compute $D_c = D_{11} + D_{12} D_k (I_{p_2} - D_{22} D_k)^{-1} D_{21}$

Algorithm $CLOOP$ is implemented by the double precision Fortran 77 subroutine DCLOOP. This subroutine may be used either with the subroutine DHFSYN or with the subroutine DH2SYN.

4 Numerical examples

As an illustrative example for $\mathcal{H}_\infty$ design, consider the computation of suboptimal controllers for a sixth order system with $m = 5, p = 5$ and $m_2 = 2,\ p_2 = 2$ and matrices

$$A = \begin{bmatrix} -1 & 0 & 4 & 5 & -3 & -2 \\ -2 & 4 & -7 & -2 & 0 & 3 \\ -6 & 9 & -5 & 0 & 2 & -1 \\ -8 & 4 & 7 & -1 & -3 & 0 \\ 2 & 5 & 8 & -9 & 1 & -4 \\ 3 & -5 & 8 & 0 & 2 & -6 \end{bmatrix}, B = \begin{bmatrix} -3 & -4 & -2 & 1 & 0 \\ 2 & 0 & 1 & -5 & 2 \\ -5 & -7 & 0 & 7 & -2 \\ 4 & -6 & 1 & 1 & -2 \\ -3 & 9 & -8 & 0 & 5 \\ 1 & -2 & 3 & -6 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 2 & -4 & 0 & -3 \\ -3 & 0 & 5 & -1 & 1 & 1 \\ -7 & 5 & 0 & -8 & 2 & -2 \\ 9 & -3 & 4 & 0 & 3 & 7 \\ 0 & 1 & -2 & 1 & -6 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & -2 & -3 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 \\ 5 & -3 & -4 & 0 & 1 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 7 & 1 \end{bmatrix}.$$

Using the subroutine DHFSYN it was found by some trial and error that the optimal value of γ for this system is $\gamma_{opt} = 10.18425636157899$. The controller matrices in this case are (up to four digits)

$$A_k = 10^9 \begin{bmatrix} -0.6113 & 3.3477 & 2.7572 & 1.3328 & 1.6416 & 1.4269 \\ -0.2162 & 1.1839 & 0.9751 & 0.4713 & 0.5805 & 0.5046 \\ -0.6637 & 3.6348 & 2.9936 & 1.4471 & 1.7823 & 1.5493 \\ -0.2620 & 1.4346 & 1.1815 & 0.5711 & 0.7034 & 0.6115 \\ -0.9298 & 5.0922 & 4.1939 & 2.0273 & 2.4970 & 2.1705 \\ 0.2313 & -1.2669 & -1.0434 & -0.5044 & -0.6212 & -0.5400 \end{bmatrix},$$

$$B_k = \begin{bmatrix} -0.2225 & -0.1085 \\ -0.8518 & -0.6521 \\ 0.8173 & 0.5794 \\ 0.0843 & 0.0100 \\ -0.5632 & -0.2479 \\ 0.0068 & -0.7619 \end{bmatrix},$$

$$C_k = 10^8 \begin{bmatrix} -0.0170 & 0.0928 & 0.0765 & 0.0370 & 0.0455 & 0.0396 \\ -0.3556 & 1.9476 & 1.6041 & 0.7754 & 0.9550 & 0.8302 \end{bmatrix}, D_k = \begin{bmatrix} 0.0552 & 0.1334 \\ -0.3195 & 0.0333 \end{bmatrix}.$$

This controller is characterised by very high gains.

The estimates of the condition numbers of the X- and Y-Riccati equations in this case are $\text{cond}_X = 40.7182$ and $\text{cond}_Y = 7625.7242$, respectively, which shows that the Y-Riccati equation is ill-conditioned.

In order to obtain controller with lower gains, we repeat the design using $\gamma = 10.2$. In this case one obtains

$$A_k = \begin{bmatrix} -27.9697 & 157.6572 & 121.1820 & 63.0841 & 69.8475 & 62.1293 \\ -2.3387 & 48.4182 & 30.5453 & 19.0161 & 20.3798 & 28.3055 \\ -41.1711 & 176.1441 & 115.5797 & 65.2052 & 79.0261 & 62.4076 \\ -29.5081 & 70.7619 & 51.7856 & 6.5753 & 28.4639 & 19.0043 \\ -9.5356 & 211.6222 & 175.2037 & 116.0559 & 95.5228 & 104.0833 \\ 4.4455 & -53.1745 & -32.9523 & -32.7705 & -24.4632 & -33.7591 \end{bmatrix},$$

$$B_k = \begin{bmatrix} -0.2225 & -0.1065 \\ -0.8571 & -0.6381 \\ 0.8147 & 0.5869 \\ 0.0777 & 0.0309 \\ -0.5658 & -0.2429 \\ 0.0074 & -0.7633 \end{bmatrix},$$

$$C_k = \begin{bmatrix} -0.3296 & 0.1973 & 0.2138 & 0.2886 & 0.6822 & 0.0863 \\ 1.3514 & 8.0244 & 8.1873 & 3.7391 & 5.3173 & 6.1380 \end{bmatrix}, D_k = \begin{bmatrix} 0.0552 & 0.1334 \\ -0.3195 & 0.0333 \end{bmatrix}.$$

This suboptimal controller is relatively easy to implement due to the low gains. In the given case the estimate of the condition numbers of the Riccati equations are $\mathrm{cond}_X = 40.6183$ and $\mathrm{cond}_Y = 3417.03508$, so that the conditioning is slightly improved.

It is interesting to compare the condition numbers of the matrices R and $R_{11} - R_{21}^T R_{21}$ ($\tilde{R}$ and $\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{12}^T$, respectively). In the table we show the condition numbers of these matrices for different values of γ. The results show that for large γ the inversion of $R_{11} - R_{21}^T R_{21}$ ($\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{12}^T$, respectively) will give much better results.

γ	γ_{opt}	10.2	15.0	100
$\mathrm{cond}(R)$	89.52	89.84	210.57	9985.39
$\mathrm{cond}(R_{11} - R_{21}^T R_{21})$	1.16	1.15	1.07	1.00
$\mathrm{cond}(\tilde{R})$	92.39	92.72	213.68	9988.69
$\mathrm{cond}(\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{12}^T)$	1.33	1.33	1.13	1.00

In Figure 1 we show a family of closed-loop step responses of the component $z_3(t)$ for (sub)optimal controllers computed for several values of γ from $\gamma = \gamma_{opt}$ to $\gamma = 14.0$ and for $w_1(t) = -1(t), w_2(t) = 0$. It is seen that with the decreasing of γ the step response becomes faster and the steady-state value of the response decreases.

In Figure 2 we show the closed-loop system root loci for the same family of (sub)optimal controllers. The circles denote the location of the closed-loop poles for the optimal controller.

The dependance of the condition numbers of X- and Y-Riccati equations on γ is shown in Figure 3. It is seen, that with the decreasing of γ the conditioning of both equations deteriorates.

In Figures 4 and 5 we show the sensitivity of the $\mathcal{H}_\infty$ norm of the closed-loop system for two values of γ and relatively large variations in the elements A_{k14} and A_{k24} of the controller.

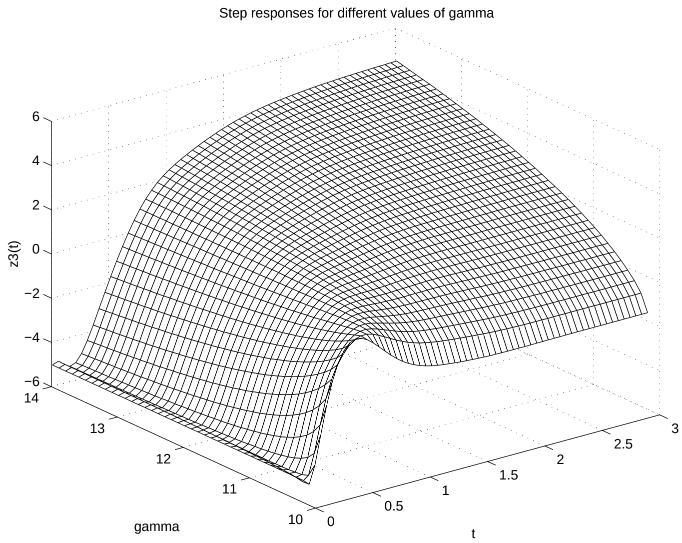

Figure 1: Step responses for different values of γ

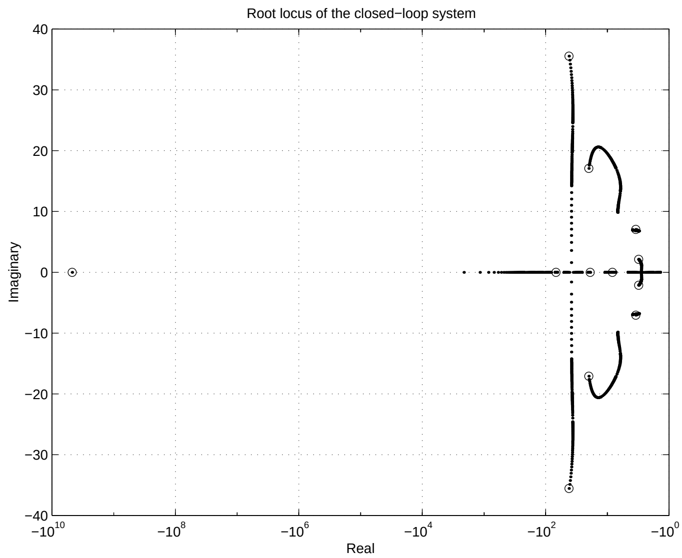

Figure 2: Root loci of the closed-loop system for different values of γ

For the same system, but without the direct link between w and z ($D_{11} = 0$), the program DH2SYN produced the following optimal $\mathcal{H}_2$ controller,

$$A_k = \begin{bmatrix} 95.9378 & -226.9695 & -60.4604 & 108.5332 & -101.4971 & -65.6736 \\ 3.0306 & -37.4541 & 11.7116 & -28.7165 & 3.9497 & -7.3540 \\ 69.8247 & -188.1127 & -62.3360 & 87.8148 & -81.5804 & -61.9814 \\ -10.9869 & -300.2112 & 97.5830 & -156.6650 & 5.7765 & -117.6236 \\ 118.2050 & 376.0538 & -248.4209 & 425.6672 & -124.9145 & 168.6505 \\ -27.2564 & -165.1314 & 94.3864 & -147.5332 & 33.3057 & -76.2206 \end{bmatrix},$$

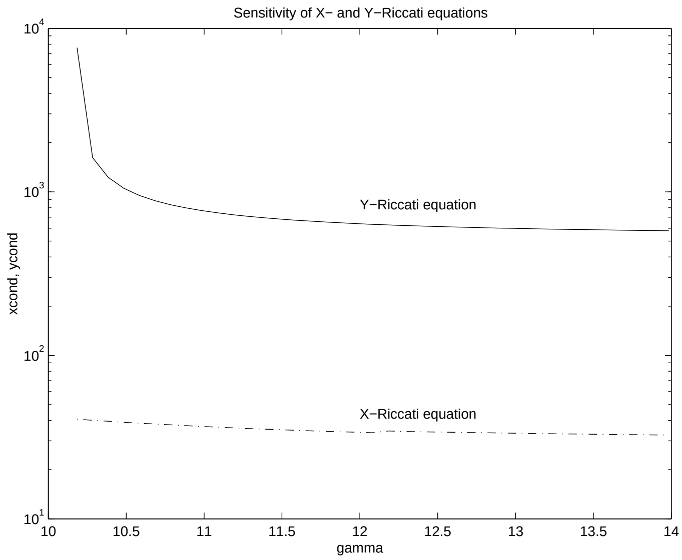

Figure 3: Sensitivity of X- and Y-Riccati equations for different values of γ

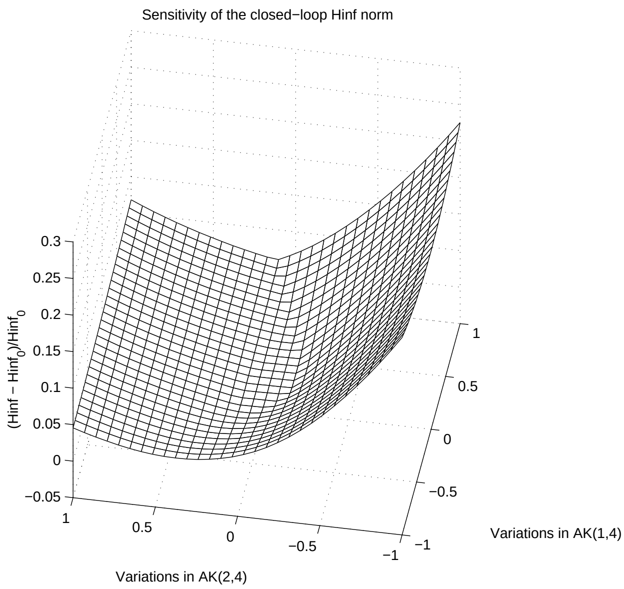

Figure 4: Sensitivity of the closed-loop $\mathcal{H}_\infty$ norm for $\gamma = 10.5$

$$B_k = \begin{bmatrix} 7.0802 & 1.2614 \\ -0.6137 & 0.4321 \\ 7.0772 & 1.8920 \\ 3.8301 & 6.9575 \\ -0.2246 & -12.7117 \\ 0.6129 & 3.9564 \end{bmatrix},$$

$$C_k = \begin{bmatrix} -0.2239 & 5.5887 & -1.9518 & 3.2632 & 0.2160 & 2.0672 \\ 7.8742 & -8.1895 & -3.3916 & 7.6804 & -4.0844 & 0.7101 \end{bmatrix}$$

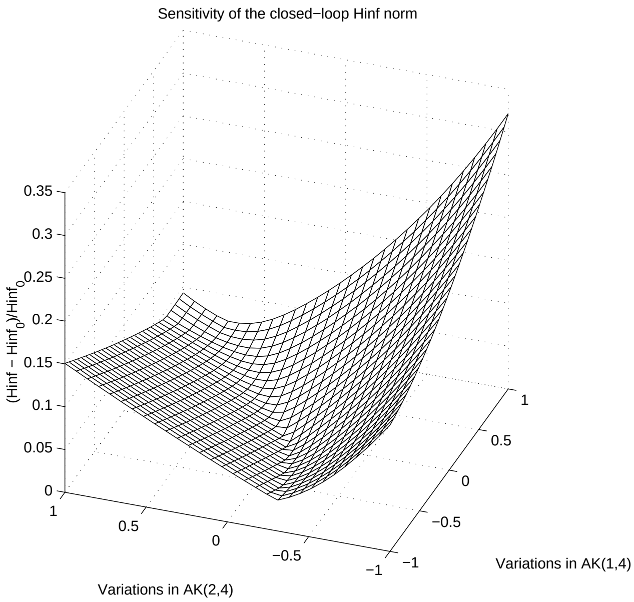

Figure 5: Sensitivity of the closed-loop $\mathcal{H}_\infty$ norm for $\gamma = 10.18426$

(In this case the controller is strictly proper, i.e. $D_k = 0$).

In Figure 6 we show the sensitivity of the $\mathcal{H}_2$ norm of the closed loop system for the same variations in the elements A_{k14} and A_{k24}.

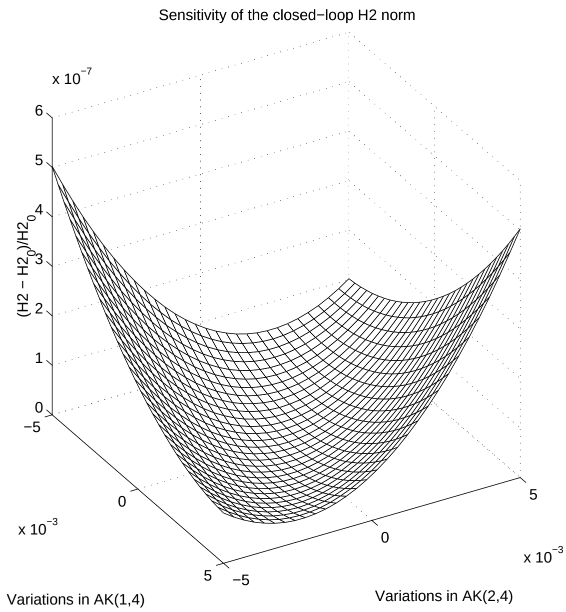

Figure 6: Sensitivity of the closed-loop $\mathcal{H}_2$ norm

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