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# Direct Formulae for the $\mathcal{H}_\infty$ Sub-Optimal Central Controller [1]

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**Abstract**

Alternative formulae, directly based on the original data of the given interconnected system, are presented for the $\mathcal{H}_\infty$ sub-optimal central controller.



**Key Words**: $\mathcal{H}_\infty$ sub-optimal central controller, Numeric formulae, SLICOT

# 1 Introduction

In the robust and optimal control of finite dimension, linear, time-invariant systems $\mathcal{H}_\infty$ optimization approaches have emerged in the last two decades as a theoretically sound and practically powerful method. The theory has been beautifully developed and the formulae for construction of an $\mathcal{H}_\infty$ (sub-)optimal controller are readily available in the literature [1, 2]. The presentation of the results is usually based on the underlying system being in a normalized form, i.e. $D_{12}$ and $D_{21}$ are orthonormal (see §2 below). That assumption is very useful for the purpose of clear exposition of the theory. The results are also without any loss of generality, since suitable transformations may be applied to change a general form system into that special form and the controller recovered accordingly.

Along with the theoretic development of $\mathcal{H}_\infty$ optimization, industrial applications of the theory have begun to be reported, e.g. [3, 4, 5]. As always the case when an elegant theory is applied in industry, many practical issues, especially the numeric properties of the computations required, will inevitably arise. It is not rare that a solution may become totally unreliable if too many, unnecessary intermediate computations are performed or if an unstable computation step is applied at an early stage which causes some numeric error to propagate through the whole procedure upto an unacceptable degree. Bearing in mind the above considerations, this report presents a computational procedure for the $\mathcal{H}_\infty$ central sub-optimal controller. The procedure, based on the original system data, employs numerically stable algorithms and minimizes any numeric error propagation as much as possible. The procedure comprises one $\mathcal{H}_\infty$ (sub-)optimal solution routine in the SLICOT package. More numerical aspects concerning the implementation of the procedure and other $\mathcal{H}_\infty$ optimization routines in SLICOT will be discussed in another report.

§2 summarises the solution formulae for a normalized system as considered in most literature. §3 describes the equivalence between a general system and a normalized one via suitable transformations. §4 reveals the relations between the major computation ingredients with regard to the above two sets of system data. The complete procedure of computing a sub-optimal $\mathcal{H}_\infty$ central controller for a general system, but with $D_{22} = 0$, is presented in §5. The case of $D_{22} \neq 0$ is considered in §6. §7 concludes the report.

# 2 Formulae for normalized systems

Consider a generalized (interconnected) plant, $\bar{P}(s)$, described by

$$\begin{aligned} \dot{\bar{x}}(t) &= A\bar{x}(t) + \bar{B}_1\bar{w}(t) + \bar{B}_2\bar{u}(t) && (2.1)\\ \bar{z}(t) &= \bar{C}_1\bar{x}(t) + \bar{D}_{11}\bar{w}(t) + \bar{D}_{12}\bar{u}(t) && (2.2)\\ \bar{y}(t) &= \bar{C}_2\bar{x}(t) + \bar{D}_{21}\bar{w}(t) && (2.3) \end{aligned}$$

where $\bar{x}(t) \in R^n$ is the state vector, $\bar{w}(t) \in R^{m_1}$ the exogeneous input vector, $\bar{u}(t) \in R^{m_2}$ the control input vector, $\bar{z}(t) \in R^{p_1}$ the error vector, and $\bar{y}(t) \in R^{p_2}$ the measurement vector, with $p_1 \geq m_2$ and $p_2 \leq m_1$. $\bar{P}(s)$ may be further denoted as

$$\bar{P}(s) = \begin{bmatrix} \bar{P}_{11}(s) & \bar{P}_{12}(s) \\ \bar{P}_{21}(s) & \bar{P}_{22}(s) \end{bmatrix} \quad (2.4)$$

$$= \left[\begin{array}{c|cc} A & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array}\right] \tag{2.5}$$

$$=: \left[\begin{array}{c|c} A & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \tag{2.6}$$

Note that in the above definition it is assumed that there is no direct link between the control input and the measurement output, i.e. $\bar{D}_{22} = 0$. This assumption is reasonable because most industrial control systems are strictly proper and the corresponding $\bar{P}(s)$ would have a zero $\bar{D}_{22}$ in a sensible design configuration. The case of non-zero direct term between $\bar{u}(t)$ and $\bar{y}(t)$ will be however considered in §6 for the sake of completeness.

The '$\mathcal{H}_\infty$ sub-optimal control problem' is to find an internally stabilizing controller $\bar{K}(s)$ such that, for a prespecified positive value $\gamma$,

$$||F_l(\bar{P}, \bar{K})||_\infty < \gamma \tag{2.7}$$

where $F_l(\bar{P}, \bar{K})$ is a lower linear fractional transformation (LFT) on $\bar{K}(s)$, with the coefficient matrix $\bar{P}(s)$, as defined below, and is the closed-loop transfer function $T_{\bar{z}\bar{w}}(s)$ from $\bar{w}$ to $\bar{z}$,

$$F_l(\bar{P}, \bar{K}) = \bar{P}_{11} + \bar{P}_{12}\bar{K}(I - \bar{P}_{22}\bar{K})^{-1}\bar{P}_{21} \quad .$$

The $\mathcal{H}_\infty$ sub-optimal control problem has been fully investigated and state-space formulae for sub-optimal solutions are now available. The formulae use the following two algebraic Riccati equation solutions.

An algebraic Riccati equation (ARE)

$$E^T X + XE - XWX + Q = 0$$

where $W = W^T$ and $Q = Q^T$, uniquely corresponds to a Hamiltonian matrix $\begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix}$. The stabilizing solution $X$ is a symmetric matrix which solves the ARE and is such that $E - WX$ is a stable matrix. The stabilizing solution may be denoted as

$$X := \mathbf{Ric} \begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix}.$$

Define

$$R_n := \bar{D}_{1*}^T \bar{D}_{1*} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.8}$$

and

$$\tilde{R}_n := \bar{D}_{*1} \bar{D}_{*1}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.9}$$

where

$$\bar{D}_{1*} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \end{bmatrix} \quad \text{and} \quad \bar{D}_{*1} = \begin{bmatrix} \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix}. \tag{2.10}$$

Define

$$\mathbf{H} := \begin{bmatrix} A & 0 \\ -\bar{C}_1^T\bar{C}_1 & -A^T \end{bmatrix} - \begin{bmatrix} \bar{B} \\ -\bar{C}_1^T\bar{D}_{1*} \end{bmatrix} R_n^{-1} \begin{bmatrix} \bar{D}_{1*}^T\bar{C}_1 & \bar{B}^T \end{bmatrix} \tag{2.11}$$

$$\mathbf{J} := \begin{bmatrix} A^T & 0 \\ -\bar{B}_1\bar{B}_1^T & -A \end{bmatrix} - \begin{bmatrix} \bar{C}^T \\ -\bar{B}_1\bar{D}_{*1}^T \end{bmatrix} \tilde{R}_n^{-1} \begin{bmatrix} \bar{D}_{*1}\bar{B}_1^T & \bar{C} \end{bmatrix} \tag{2.12}$$

Under certain assumptions such as those given below, the existence of stabilizing solutions to the AREs corresponding to the above Hamiltonians is guaranteed. Let

$$X := \mathbf{Ric}\ (\mathbf{H}) \tag{2.13}$$

$$Y := \mathbf{Ric}\ (\mathbf{J}). \tag{2.14}$$

Based on $X$ and $Y$, a state feedback matrix $\bar{F}$ and an observer gain matrix $\bar{L}$ can be constructed, which will be used in the solution formulae,

$$\bar{F} := -R_n^{-1}(\bar{D}_{1*}^T\bar{C}_1 + \bar{B}^TX) =: \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} =: \begin{bmatrix} \bar{F}_{11} \\ \bar{F}_{12} \\ \bar{F}_2 \end{bmatrix} \tag{2.15}$$

$$\bar{L} := -(\bar{B}_1\bar{D}_{*1}^T + Y\bar{C}^T)\tilde{R}_n^{-1} =: \begin{bmatrix} \bar{L}_1 & \bar{L}_2 \end{bmatrix} =: \begin{bmatrix} \bar{L}_{11} & \bar{L}_{12} & \bar{L}_2 \end{bmatrix} \tag{2.16}$$

where $\bar{F}_1, \bar{F}_2, \bar{F}_{11}$ and $\bar{F}_{12}$ are of $m_1, m_2, m_1 - p_2$ and $p_2$ rows, respectively, and $\bar{L}_1, \bar{L}_2, \bar{L}_{11}$ and $\bar{L}_{12}$ of $p_1, p_2, p_1 - m_2$ and $m_2$ columns, respectively.

Glover and Doyle [1] derived necessary and sufficient conditions for the existence of an $\mathcal{H}_\infty$ sub-optimal solution and further parametrized all such controllers. The results are obtained under the following assumptions.

**$\bar{\mathbf{A}}\mathbf{1}$** $(A, \bar{B}_2)$ is stabilizable and $(\bar{C}_2, A)$ detectable;

**$\bar{\mathbf{A}}\mathbf{2}$** $\bar{D}_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $\bar{D}_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$;

**$\bar{\mathbf{A}}\mathbf{3}$** $\begin{bmatrix} A - j\omega I & \bar{B}_2 \\ \bar{C}_1 & \bar{D}_{12} \end{bmatrix}$ has full column rank for all $\omega$;

**$\bar{\mathbf{A}}\mathbf{4}$** $\begin{bmatrix} A - j\omega I & \bar{B}_1 \\ \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$ has full row rank for all $\omega$.

Together with appropriate partition of $\bar{D}_{11} = \begin{bmatrix} \bar{D}_{1111} & \bar{D}_{1112} \\ \bar{D}_{1121} & \bar{D}_{1122} \end{bmatrix}$, where $\bar{D}_{1122}$ has $m_2$ rows and $p_2$ columns, the solution formulae are given in the following theorem.

**Theorem 1** [2] *Suppose $\bar{P}(s)$ satisfies the assumptions* $\bar{\mathbf{A}}\mathbf{1} - \bar{\mathbf{A}}\mathbf{4}$.
*(a) There exists an internally stabilizing controller $\bar{K}(s)$ such that $||F_l(\bar{P}, \bar{K})||_\infty < \gamma$ if and only*

*if*
*(i)*

$$\gamma > max(\bar{\sigma}[\bar{D}_{1111}, \bar{D}_{1112}], \bar{\sigma}[\bar{D}_{1111}^T, \bar{D}_{1121}^T]) \tag{2.17}$$

*and*
*(ii) there exist solutions $X \geq 0$ and $Y \geq 0$ satisfying the two AREs respectively and such that*

$$\rho(XY) < \gamma^2 \tag{2.18}$$

*where $\rho(\bullet)$ denotes the spectral radius.*

*(b) Given that the conditions of part (a) are satisfied, then all rational, internally stabilizing controllers, $\bar{K}(s)$, satisfying $||F_l(\bar{P}, \bar{K})||_\infty < \gamma$ are given by*

$$\bar{K}(s) = F_l(M, \Phi) \tag{2.19}$$

*for any rational $\Phi(s) \in \mathcal{H}_\infty$ such that $||\Phi(s)||_\infty < \gamma$, where $M(s)$ has the realization*

$$M(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right] \tag{2.20}$$

*and*

$$\hat{D}_{11} = -\bar{D}_{1121}\bar{D}_{1111}^T(\gamma^2 I - \bar{D}_{1111}\bar{D}_{1111}^T)^{-1}\bar{D}_{1112} - \bar{D}_{1122}. \tag{2.21}$$

$\hat{D}_{12} \in R^{m_2 \times m_2}$ *and* $\hat{D}_{21} \in R^{p_2 \times p_2}$ *are any matrices (e.g. Cholesky factors) satisfying*

$$\hat{D}_{12}\hat{D}_{12}^T = I - \bar{D}_{1121}(\gamma^2 I - \bar{D}_{1111}^T\bar{D}_{1111})^{-1}\bar{D}_{1121}^T \tag{2.22}$$

$$\hat{D}_{21}^T\hat{D}_{21} = I - \bar{D}_{1112}^T(\gamma^2 I - \bar{D}_{1111}\bar{D}_{1111}^T)^{-1}\bar{D}_{1112} \tag{2.23}$$

*and*

$$\hat{B}_2 = Z(\bar{B}_2 + \bar{L}_{12})\hat{D}_{12} \tag{2.24}$$

$$\hat{C}_2 = -\hat{D}_{21}(\bar{C}_2 + \bar{F}_{12}) \tag{2.25}$$

$$\begin{aligned} \hat{B}_1 &= -Z\bar{L}_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \\ &= -Z\bar{L}_2 + Z(\bar{B}_2 + \bar{L}_{12})\hat{D}_{11} \end{aligned} \tag{2.26}$$

$$\begin{aligned} \hat{C}_1 &= \bar{F}_2 + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \\ &= \bar{F}_2 - \hat{D}_{11}(\bar{C}_2 + \bar{F}_{12}) \end{aligned} \tag{2.27}$$

$$\begin{aligned} \hat{A} &= A + \bar{B}\bar{F} + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \\ &= A + \bar{B}\bar{F} - \hat{B}_1(\bar{C}_2 + \bar{F}_{12}) \end{aligned} \tag{2.28}$$

*where*

$$Z = (I - \gamma^{-2} Y X)^{-1}. \tag{2.29}$$

■

When $\Phi(s) = 0$ is chosen, the corresponding sub-optimal controller is called the *central* controller which is widely used in the $\mathcal{H}_\infty$ optimal design and has the state-space form

$$\bar{K}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right] \tag{2.30}$$

In the rest of the report, only the central sub-optimal controller will be considered.

# 3 Normalization transformations

In the assumptions made in §2, $\bar{\mathbf{A}}\mathbf{1}$ is necessary for the existence of stabilizing controllers for $\bar{P}(s)$. $\bar{\mathbf{A}}\mathbf{3}$ and $\bar{\mathbf{A}}\mathbf{4}$, together with $\bar{\mathbf{A}}\mathbf{1}$, guarantee the existence of the two stabilizing ARE solutions and, hence, are necessary for the method. The assumption $\bar{\mathbf{A}}\mathbf{2}$ assumes that the matrices $\bar{D}_{12}$ and $\bar{D}_{21}$ are in normalized forms and the system $\bar{P}(s)$ is thus so called a normalized system. The solution still exists if $\bar{\mathbf{A}}\mathbf{2}$ is replaced by the assumption that those two matrices are full rank but not necessarily in the normalized forms. The transformations between a general system and a normalized one will be discussed in this section.

Consider now a general, interconnected system $P(s)$

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{3.1}$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{3.2}$$
$$y(t) = C_2 x(t) + D_{21} w(t), \tag{3.3}$$

with the dimensions same as the corresponding ones in (2.1) – (2.3). Alternatively, $P(s)$ can be denoted as

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \tag{3.4}$$

$$= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3.5}$$

$$=: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3.6}$$

The closed-loop system of $P(s)$ and a controller $K(s)$ is depicted in Figure 1.

Assume that $D_{12}$ is of full column rank and $D_{21}$ of full row rank, respectively. There exist orthonormal matrices $U_{12}$, $V_{12}$, $U_{21}$ and $V_{21}$, using SVD or otherwise, such that

$$U_{12} D_{12} V_{12}^T = \begin{bmatrix} 0 \\ \Sigma_{12} \end{bmatrix} \tag{3.7}$$

$$U_{21} D_{21} V_{21}^T = \begin{bmatrix} 0 & \Sigma_{21} \end{bmatrix} \tag{3.8}$$

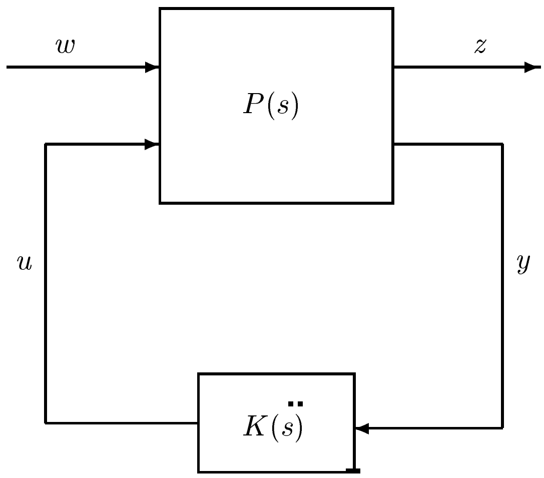


Figure 1: Closed-loop configuration of P(s) and K(s)

where $\Sigma_{12} : m_2 \times m_2$ and $\Sigma_{21} : p_2 \times p_2$ are nonsingular. Furthermore, we may have

$$U_{12} D_{12} V_{12}^T \Sigma_{12}^{-1} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3.9}$$

$$\Sigma_{21}^{-1} U_{21} D_{21} V_{21}^T = \begin{bmatrix} 0 & I \end{bmatrix} \tag{3.10}$$

The right hand sides of the above equations are now in the normalized forms.

When $p_1 > m_2$ and $p_2 < m_1$, the matrices $U_{12}$ and $V_{21}$ can be partitioned as

$$U_{12} = \begin{bmatrix} U_{121} \\ U_{122} \end{bmatrix} \tag{3.11}$$

$$V_{21} = \begin{bmatrix} V_{211} \\ V_{212} \end{bmatrix} \tag{3.12}$$

with $U_{121} : (p_1 - m_2) \times p_1, U_{122} : m_2 \times p_1, V_{211} : (m_1 - p_2) \times m_1$ and $V_{212} : p_2 \times m_1$.

The normalization of $P(s)$ into $\bar{P}(s)$ is based on the above transformations and clearly shown in Figure 2.

Given $P(s)$, the state-space form of $\bar{P}(s)$ in (2.5) is obtained as the following

$$\bar{B}_1 = B_1 V_{21}^T \tag{3.13}$$

$$\bar{B}_2 = B_2 V_{12}^T \Sigma_{12}^{-1} \tag{3.14}$$

$$\bar{C}_1 = U_{12} C_1 \tag{3.15}$$

$$\bar{C}_2 = \Sigma_{21}^{-1} U_{21} C_2 \tag{3.16}$$

$$\bar{D}_{11} = U_{12} D_{11} V_{21}^T = \begin{bmatrix} U_{121} D_{11} V_{211}^T & U_{121} D_{11} V_{212}^T \\ U_{122} D_{11} V_{211}^T & U_{122} D_{11} V_{212}^T \end{bmatrix} \tag{3.17}$$

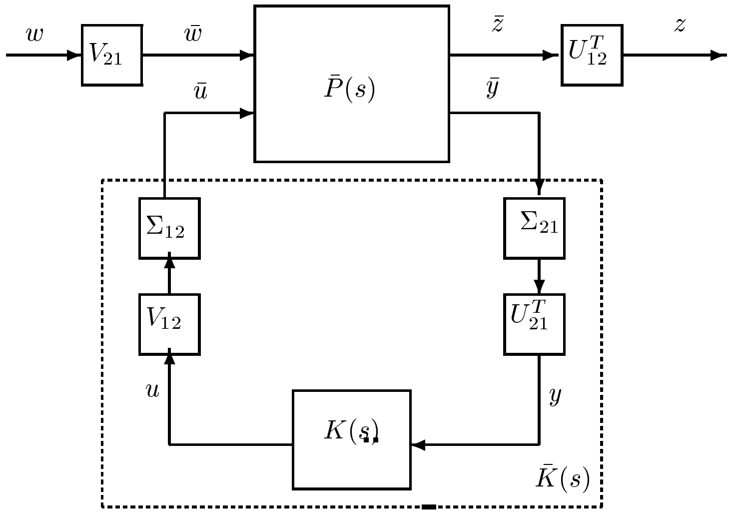


Figure 2: Normalization configuration

$$
\begin{aligned}
\bar{D}_{12} &= \begin{bmatrix} 0 \\ I \end{bmatrix} = U_{12} D_{12} V_{12}^T \Sigma_{12}^{-1} && (3.18) \\
\bar{D}_{21} &= \begin{bmatrix} 0 & I \end{bmatrix} = \Sigma_{21}^{-1} U_{21} D_{21} V_{21}^T && (3.19)
\end{aligned}
$$

Since $V_{21}$ and $U_{12}$ are orthonormal, $||T_{zw}||_\infty = ||T_{\bar{z}\bar{w}}||_\infty$, with obviously $K(s) = V_{12}^T \Sigma_{12}^{-1} \bar{K}(s) \Sigma_{21}^{-1} U_{21}$.

# 4 Two algebraic Riccati equations

With the formulae derived at the end of §3, we are now ready to discuss the two algebraic Riccati equations which are essential in the computations of $\mathcal{H}_\infty$ sub-optimal controllers. The conclusion is a bit surprising: the two AREs remain the same with matrices replaced by corresponding matrices in the state-space form of $P(s)$. This is obtained by the following routine matrix manipulations,

$$
\begin{aligned}
\bar{C}_1^T \bar{C}_1 &= C_1^T U_{12}^T U_{12} C_1 = C_1^T C_1 \\
\begin{bmatrix} \bar{B} \\ -\bar{C}_1^T \bar{D}_{1*} \end{bmatrix} &= \begin{bmatrix} B_1 V_{21}^T & B_2 V_{12}^T \Sigma_{12}^{-1} \\ -C_1^T U_{12}^T U_{12} D_{11} V_{21}^T & -C_1^T U_{12}^T U_{12} D_{12} V_{12}^T \Sigma_{12}^{-1} \end{bmatrix} = \begin{bmatrix} B \\ -C_1^T D_{1*} \end{bmatrix} \begin{bmatrix} V_{21}^T & 0 \\ 0 & V_{12}^T \Sigma_{12}^{-1} \end{bmatrix} \\
\bar{D}_{1*}^T \bar{D}_{1*} &= \begin{bmatrix} V_{21} D_{11}^T U_{12}^T \\ \Sigma_{12}^{-T} V_{12} D_{12}^T U_{12}^T \end{bmatrix} \begin{bmatrix} U_{12} D_{11} V_{21}^T & U_{12} D_{12} V_{12}^T \Sigma_{12}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} V_{21} & 0 \\ 0 & \Sigma_{12}^{-T} V_{12} \end{bmatrix} D_{1*}{}^T D_{1*} \begin{bmatrix} V_{21}^T & 0 \\ 0 & V_{12}^T \Sigma_{12}^{-1} \end{bmatrix}
\end{aligned}
$$

where

$$
D_{1*} := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}. \tag{4.1}
$$

Thus,

$$R_n \;=\; \begin{bmatrix} V_{21} & 0 \\ 0 & \Sigma_{12}^{-T}V_{12} \end{bmatrix} R \begin{bmatrix} V_{21}^T & 0 \\ 0 & V_{12}^T\Sigma_{12}^{-1} \end{bmatrix}$$

with

$$R \;:=\; {D_{1*}}^T D_{1*} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.2}$$

Consequently, the Hamiltonian $\mathbf{H}$ defined in (2.11) is also given by

$$\mathbf{H} = \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_{1*} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1*}^T C_1 & B^T \end{bmatrix} \tag{4.3}$$

Similarly, the Hamiltonian $\mathbf{J}$ is given by

$$\mathbf{J} = \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1 D_{*1}^T \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{*1} B_1^T & C \end{bmatrix} \tag{4.4}$$

where

$$D_{*1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \tag{4.5}$$

and

$$\tilde{R} := D_{*1}{D_{*1}}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.6}$$

with

$$\tilde{R}_n \;=\; \begin{bmatrix} U_{12} & 0 \\ 0 & \Sigma_{21}^{-1}U_{21} \end{bmatrix} \tilde{R} \begin{bmatrix} U_{12}^T & 0 \\ 0 & U_{21}^T\Sigma_{21}^{-T} \end{bmatrix}$$

¿From now on, the matrices $X$ and $Y$ are referred to the stabilizing solutions to the AREs corresponding to $\mathbf{H}$ and $\mathbf{J}$ defined in (4.3) and (4.4), respectively.

Another pair of matrices may be defined from the solutions $X$ and $Y$,

$$F \;:=\; -R^{-1}({D_{1*}}^T C_1 + B^T X) =: \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{4.7}$$

$$L \;:=\; -(B_1 {D_{*1}}^T + Y C^T)\tilde{R}^{-1} =: \begin{bmatrix} L_1 & L_2 \end{bmatrix} \tag{4.8}$$

with $F_1 : m_1 \times n, F_2 : m_2 \times n, L_1 : n \times p_1$ and $L_2 : n \times p_2$. It is easy to check that

$$\bar{F} \;=\; \begin{bmatrix} V_{21} & 0 \\ 0 & \Sigma_{12}V_{12} \end{bmatrix} F$$

and

$$\begin{aligned}
\bar{F}_{11} &= V_{211}F_1 \\
\bar{F}_{12} &= V_{212}F_1 \\
\bar{F}_2 &= \Sigma_{12}V_{12}F_2
\end{aligned}$$

Dually, we have

$$\bar{L} = L\begin{bmatrix} U_{12}^T & 0 \\ 0 & U_{21}^T\Sigma_{21} \end{bmatrix}$$

and

$$\begin{aligned}
\bar{L}_{11} &= L_1U_{121}^T \\
\bar{L}_{12} &= L_1U_{122}^T \\
\bar{L}_2 &= L_2U_{21}^T\Sigma_{21}
\end{aligned}$$

# 5 Solution procedures

As described in §2, the central $\mathcal{H}_\infty$ sub-optimal controller for $\bar{P}(s)$ is $\bar{K}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right]$. It can be seen from the formulae given in **Theorem 1** that as far as just the central controller is concerned there is no need to calculate $\hat{D}_{12}$ and $\hat{D}_{21}$. Let the central $\mathcal{H}_\infty$ sub-optimal controller for $P(s)$ be

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{5.1}$$

¿From $K(s) = V_{12}^T\Sigma_{12}^{-1}\bar{K}(s)\Sigma_{21}^{-1}U_{21}$, we have

$$\begin{aligned}
A_K &= \hat{A} &(5.2)\\
B_K &= \hat{B}_1\Sigma_{21}^{-1}U_{21} &(5.3)\\
C_K &= V_{12}^T\Sigma_{12}^{-1}\hat{C}_1 &(5.4)\\
D_K &= V_{12}^T\Sigma_{12}^{-1}\hat{D}_{11}\Sigma_{21}^{-1}U_{21} &(5.5)
\end{aligned}$$

Using the partition of $\bar{D}_{11}$ given in (3.17), the matrix $\hat{D}_{11}$ defined in (2.21) can be obtained as

$$\begin{aligned}
\hat{D}_{11} &= -\bar{D}_{1121}\bar{D}_{1111}^T(\gamma^2I - \bar{D}_{1111}\bar{D}_{1111}^T)^{-1}\bar{D}_{1112} - \bar{D}_{1122} \\
&= -U_{122}D_{11}V_{211}^TV_{211}D_{11}^TU_{121}^T(\gamma^2I - U_{121}D_{11}V_{211}^TV_{211}D_{11}^TU_{121}^T)^{-1}U_{121}D_{11}V_{212}^T - U_{122}D_{11}V_{212}^T \\
&= -\gamma^2U_{122}(\gamma^2I - D_{11}V_{211}^TV_{211}D_{11}^TU_{121}^TU_{121})^{-1}D_{11}V_{212}^T
\end{aligned}$$

Hence,

$$\begin{aligned}
D_K &= V_{12}^T\Sigma_{12}^{-1}\hat{D}_{11}\Sigma_{21}^{-1}U_{21} \\
&= -\gamma^2(U_{122}D_{12})^{-1}U_{122}(\gamma^2I - D_{11}V_{211}^TV_{211}D_{11}^TU_{121}^TU_{121})^{-1}D_{11}V_{212}^T(D_{21}V_{212}^T)^{-1} &(5.6)
\end{aligned}$$

where the facts that $(U_{122}D_{12})(V_{12}^T\Sigma_{12}^{-1}) = I_{m_2}$ and $(\Sigma_{21}^{-1}U_{21})(D_{21}V_{212}^T) = I_{p_2}$ were applied.

Furthermore, using $D_{12} = U_{122}^T\Sigma_{12}V_{12}$ and $D_{21} = U_{21}^T\Sigma_{21}V_{212}$, the following formulae for $B_K, C_K$ and $A_K$ can be derived,

$$
\begin{aligned}
B_K &= \hat{B}_1\Sigma_{21}^{-1}U_{21} \\
&= (-Z\bar{L}_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11})\Sigma_{21}^{-1}U_{21} \\
&= -Z\left[L_2U_{21}^T\Sigma_{21} - (B_2V_{12}^T\Sigma_{12}^{-1} + L_1U_{122}^T)\hat{D}_{11}\right]\Sigma_{21}^{-1}U_{21} \\
&= -Z\left[L_2 - (B_2 + L_1D_{12})D_K\right] \qquad (5.7) \\
C_K &= V_{12}^T\Sigma_{12}^{-1}\hat{C}_1 \\
&= V_{12}^T\Sigma_{12}^{-1}(\bar{F}_2 + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2) \\
&= V_{12}^T\Sigma_{12}^{-1}\left[\Sigma_{12}V_{12}F_2 - \hat{D}_{11}(\Sigma_{21}^{-1}U_{21}C_2 + V_{212}F_1)\right] \\
&= F_2 - D_K(C_2 + D_{21}F_1) \qquad (5.8) \\
A_K &= \hat{A} \\
&= A + \bar{B}\bar{F} + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \\
&= A + B\begin{bmatrix} V_{21}^T & 0 \\ 0 & V_{12}^T\Sigma_{12}^{-1}\end{bmatrix}\begin{bmatrix} V_{21} & 0 \\ 0 & \Sigma_{12}V_{12}\end{bmatrix}F - \hat{B}_1(\bar{C}_2 + \bar{F}_{12}) \\
&= A + BF + Z\left[L_2U_{21}^T\Sigma_{21} - (B_2V_{12}^T\Sigma_{12}^{-1} + L_1U_{122}^T)\hat{D}_{11}\right]\left[\Sigma_{21}^{-1}U_{21}C_2 + V_{212}F_1\right] \\
&= A + BF + Z\left[L_2 - (B_2 + L_1D_{12})D_K\right](C_2 + D_{21}F_1) \qquad (5.9) \\
&= A + BF - B_K(C_2 + D_{21}F_1) \qquad (5.10)
\end{aligned}
$$

The above is now summarised in the following corollary, with modified assumptions:

**A1** $(A, B_2)$ is stabilizable and $(C_2, A)$ detectable;

**A2** $D_{12}$ is of full column rank and $D_{21}$ of full row rank;

**A3** $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12}\end{bmatrix}$ has full column rank for all $\omega$;

**A4** $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21}\end{bmatrix}$ has full row rank for all $\omega$.

**Corollary 1** *Suppose $P(s)$ satisfies the assumptions* **A1 − A4**.
*(a) There exists an internally stabilizing controller $K(s)$ such that $||F_l(P,K)||_\infty < \gamma$ if and only if*
*(i)*

$$\gamma > max\{\bar{\sigma}(U_{121}D_{11}), \bar{\sigma}(D_{11}V_{211}^T)\} \qquad (5.11)$$

*where $U_{121}$ and $V_{211}$ are as defined in (3.11) and (3.12), and*
*(ii) there exist solutions $X \geq 0$ and $Y \geq 0$ satisfying the two AREs corresponding to (4.3) and (4.4), respectively, and such that*

$$\rho(XY) < \gamma^2 \qquad (5.12)$$

*where $\rho(\bullet)$ denotes the spectral radius.*

*(b) Given that the conditions of part (a) are satisfied, then the central $\mathcal{H}_\infty$ sub-optimal controller for $P(s)$ is given by*

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

*where*

$$\begin{aligned} D_K &= -\gamma^2(U_{122}D_{12})^{-1}U_{122}(\gamma^2 I - D_{11}V_{211}^T V_{211}D_{11}^T U_{121}^T U_{121})^{-1}D_{11}V_{212}^T(D_{21}V_{212}^T)^{-1} && (5.13)\\ B_K &= -Z\left[L_2 - (B_2 + L_1 D_{12})D_K\right] && (5.14)\\ C_K &= F_2 - D_K(C_2 + D_{21}F_1) && (5.15)\\ A_K &= A + BF - B_K(C_2 + D_{21}F_1) && (5.16)\end{aligned}$$

*where*

$$Z = (I - \gamma^{-2}YX)^{-1}, \qquad (5.17)$$

*and $U_{12}, V_{21}, F$ and $L$ are as defined in (3.7) and (3.11), (3.8) and (3.12), (4.7), and (4.8), respectively.* ■

A computation procedure for the central controller can now be given as the following.

**Computation Procedure**

**Step 1:** Apply orthogonal transformations on $D_{12}$ and $D_{21}$ as in (3.7) and (3.8), partition $U_{12}$ and $V_{21}$ as in (3.11) and (3.12);
*Stop:* if $\Sigma_{12}$ or $\Sigma_{21}$ singular;

**Step 2:** Find the ARE solutions $X$ and $Y$ corresponding to the Hamiltonians defined in (4.3) and (4.4), respectively;
*Stop:* if one of the following does not hold

1. $X \geq 0$
2. $Y \geq 0$
3. $\rho(XY) < \gamma^2$

**Step 3:** Define $F$ and $L$ as in (4.7) and (4.8), define $Z$ as in (5.17), construct $D_K, B_K, C_K$ and $A_K$ as in (5.13) – (5.16).

■

**Remark:** The above Corollary and Procedure address the general case of $p_1 > m_2$ and $p_2 < m_1$. There are three special cases of the dimensions in which the solution formulae would be simpler.

**Case 1:** $p_1 = m_2$ and $p_2 < m_1$.
In this case, the ortogonal transformation on $D_{12}$ is not needed. The condition $(a)/(i)$ in Corollary 1 is reduced to $\gamma > \bar{\sigma}(D_{11}V_{211}^T)$, and

$$D_K = -D_{12}^{-1}D_{11}V_{212}^T(D_{21}V_{212}^T)^{-1}$$

**Case 2:** $p_1 > m_2$ and $p_2 = m_1$.

In this case, the ortogonal transformation on $D_{21}$ is not needed. The condition $(a)/(i)$ in Corollary 1 is reduced to $\gamma > \bar{\sigma}(U_{121}D_{11})$, and

$$D_K = -(U_{122}D_{12})^{-1}U_{122}D_{11}D_{21}^{-1}$$

**Case 3:** $p_1 = m_2$ and $p_2 = m_1$.

In this case, both ortogonal transformations are not needed. The condition $(a)/(i)$ in Corollary 1 is reduced to any positive $\gamma$, and

$$D_K = -D_{12}^{-1}D_{11}D_{21}^{-1}$$

Another special case is that $D_{11} = 0$, in which the central controller is simply given by

$$K(s) = \left[\begin{array}{c|c} A + BF + ZL_2(C_2 + D_{21}F_1) & -ZL_2 \\ \hline F_2 & 0 \end{array}\right] \tag{5.18}$$

# 6 The case of $D_{22} \neq 0$

When there is a direct link between the control input and the measurement output, the matrix $D_{22}$ will not disappear in (3.3). The central controller formulae for the case $D_{22} \neq 0$ are discussed in this section.

As a matter of fact, the $D_{22}$ term can be easily separated from the rest of the system as depicted in Figure 3. A controller $K(s)$ for the system without $D_{22}$ will be synthesized first, and then the controller $\tilde{K}(s)$ for the original system may be recovered from $K(s)$ and $D_{22}$ by

$$\tilde{K}(s) = K(s)(I + D_{22}K(s))^{-1} \tag{6.1}$$

The state-space model of $\tilde{K}(s)$ is derived as

$$\tilde{K}(s) = \left[\begin{array}{c|c} A_K - B_K D_{22}(I + D_K D_{22})^{-1}C_K & B_K(I + D_{22}D_K)^{-1} \\ \hline (I + D_K D_{22})^{-1}C_K & D_K(I + D_{22}D_K)^{-1} \end{array}\right] \tag{6.2}$$

where we assume that

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

# 7 Conclusions

This report describes a set of direct formulae for the $\mathcal{H}_\infty$ sub-optimal central controller. The formulae are using the original data of the system and avoid forming a normalized system on which usually solutions are based. The form of the direct formulae is not much more complicated than that of the normalized one, but makes the computation procedure shorter and possibly

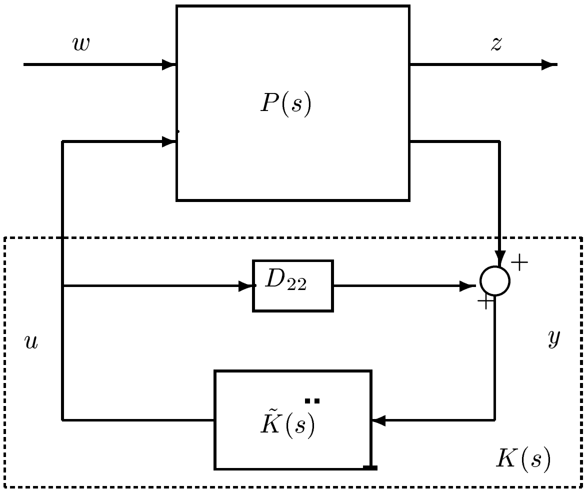


Figure 3: The case of non-zero $D_{22}$

more reliable. The direct formulae will be programmed in SLICOT and provide the user with an alternative routine for the $\mathcal{H}_\infty$ sub-optimal central controller.

The implementation of the formulae via normalization will be discussed in another report, which has a few computational advantages, e.g. the inversion of a small size matrix. It would be interesting to compare the two routines applied in various industrial case studies. We believe the experiences gained would benefit greatly practising control engineers.

# References

[1] K. Glover and J. Doyle. "State-space formulae for all stabilizing controllers that satisfy an $\mathcal{H}_\infty$ norm bound and relations to risk sensitivity" *Systems and Control Letters*, **v-11**, pp. 167 - 172, 1988.

[2] K. Zhou, J. Doyle and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.

[3] G. Murad, I. Postlethwaite and D.W. Gu. "Robust control system design applied to a glass tube shaping process" *Proc. of the 12th IFAC World Congress*, Sydney, **v-4**, pp.385 - 388, 1993.

[4] R. Hyde and K. Glover. "Application of Scheduled $\mathcal{H}_\infty$ Controllers to a VSTOL Aircraft" *IEEE Trans. Auto. Control*, **v-38, No.7**, pp. 1021 - 1039, 1993.

[5] G. Murad, I. Postlethwaite and D.W. Gu. "A Discrete-Time Internal Model-Based $\mathcal{H}^\infty$ Controller & Its Application to a Binary Distillation Column." *Journal of Process Control*, **v-7, No.6**, pp. 451 – 465, 1997.