Fortran 77 Routines for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Design of Discrete-Time Linear Control Systems [1]

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Abstract

We present Fortran 77 subroutines intended for state-space design of $\mathcal{H}_\infty$ (sub)optimal controllers and $\mathcal{H}_2$ optimal controllers for linear discrete-time control systems. The subroutines make use of LAPACK and BLAS libraries and produce estimates of the condition numbers of the matrices which are to be inverted and estimates of the condition numbers of the matrix algebraic Riccati equations which are to be solved in the computation of the controllers. The subroutines will be included in the SLICOT library.

1 Introduction

In this report we present a set of Fortran 77 subroutines intended for state-space design of $\mathcal{H}_\infty$ (sub)optimal controllers and $\mathcal{H}_2$ optimal controllers for linear discrete-time control systems. The subroutines make use of LAPACK [1] and BLAS [7, 5, 4] libraries and involve solution of the corresponding discrete-time matrix algebraic Riccati equations with condition and error estimation.

The report is organised as follows. In Section 2 we give the formulae for $\mathcal{H}_\infty$ suboptimal design, based on the method proposed in [6], and present the formulae for $\mathcal{H}_2$ optimal design of discrete-time systems derived in [9]. In Section 3 we give detailed description of the algorithms and subroutines intended for design of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers for discrete-time systems . In Section 4 a sixth-order example is given to illustrate the implementation of the routines and to demonstrate their features.

The subroutines will be included in a future release of the SLICOT library [3].

The following notations are used in the report. $\mathcal{R}$ denotes the field of real numbers; $\mathcal{R}^{m\times n}$ – the space of $m \times n$ matrices $A = [a_{ij}]$ over $\mathcal{R}$; A^T – the transposed matrix A; $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_k$ – the singular values of A, $k = \min\{m, n\}$; $\|G(z)\|_2$ and $\|G(z)\|_\infty$ – the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a stable discrete-time transfer function matrix $G(z)$, respectively, and I_n – the unit $n \times n$ matrix.

2 Formulae for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ design of discrete-time systems

Consider a generalised linear discrete-time system, described by the equations

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k \\ y_k &= C_2 x_k + D_{21} w_k + D_{22} u_k \end{aligned} \tag{2.1}$$

where $x_k \in \mathcal{R}^n$ is the state vector, $w_k \in \mathcal{R}^{m_1}$ is the exogenous input vector (the disturbance), $u_k \in \mathcal{R}^{m_2}$ is the control input vector, $z_k \in \mathcal{R}^{p_1}$ is the error vector, and $y_k \in \mathcal{R}^{p_2}$ is the measurement vector, with $p_1 \geq m_2$ and $p_2 \leq m_1$. The transfer function matrix of the system will be denoted by

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.2}$$

$$=: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

2.1 $\mathcal{H}_\infty$ sub-optimal controller

The '$\mathcal{H}_\infty$ sub-optimal discrete-time control problem' is to find an internally stabilising controller $K(z)$ such that, for a pre-specified positive value γ,

$$\|F_\ell(P,K)\|_\infty < \gamma \tag{2.3}$$

where $F_\ell(P,K)$ is the lower linear fractional transformation (LFT) on $K(z)$, equal to the closed-loop transfer function $T_{zw}(z)$ from w to z,

$$T_{zw}(z) := F_\ell(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Further on we shall make use of the method for design of sub-optimal $\mathcal{H}_\infty$ controllers, presented in [6]. This method is derived under the following assumptions.

A1 (A, B_2) is stabilisable and (C_2, A) is detectable;

A2 $\begin{bmatrix} A - e^{j\Theta}I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\Theta \in [0, 2\pi)$;

A3 $\begin{bmatrix} A - e^{j\Theta}I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\Theta \in [0, 2\pi)$.

We shall assume also that a loop-shifting transformation that enables to set $D_{22} = 0$ has been carried out. We shall return to the general case ($D_{22} \neq 0$) later on.

Note that the method under consideration does not involve reduction of the matrices D_{12} and D_{21} to some special form, as it is usually required in the design of continuous-time $\mathcal{H}_\infty$ controllers.

Let

$$\bar{C} = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} D_{11} & D_{12} \\ I_{m_1} & 0 \end{bmatrix}$$

and define

$$J = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -\gamma^2 I_{m_1} \end{bmatrix}, \quad \hat{J} = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -\gamma^2 I_{m_2} \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -\gamma^2 I_{p_1} \end{bmatrix}.$$

Let X_∞ be the solution to the discrete-time Riccati equation

$$X_\infty = \bar{C}^T J \bar{C} + A^T X_\infty A - L^T R^{-1} L \tag{2.4}$$

where

$$R = \bar{D}^T J \bar{D} + B^T X_\infty B =: \begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix}$$

$$L = \bar{D}^T J \bar{C} + B^T X_\infty A =: \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Assume that there exists an $m_2 \times m_2$ matrix V_{12} such that

$$V_{12}^T V_{12} = R_3$$

and an $m_1 \times m_1$ matrix V_{21} such that

$$V_{21}^T V_{21} = -\gamma^{-2}\nabla, \quad \nabla = R_1 - R_2^T R_3^{-1} R_2 < 0.$$

Define the matrices

$$\begin{bmatrix} A_t & \tilde{B}_t \\ C_t & \tilde{D}_t \end{bmatrix} =: \begin{bmatrix} A_t & | & \tilde{B}_{t_1} \tilde{B}_{t_2} \\ -- & | & ---- \\ C_{t_1} & | & \tilde{D}_{t_{11}} \tilde{D}_{t_{12}} \\ C_{t_2} & | & \tilde{D}_{t_{21}} \tilde{D}_{t_{22}} \end{bmatrix} =$$

$$\begin{bmatrix} A - B_1 \nabla^{-1} L_\nabla & | & B_1 V_{21}^{-1} & 0 \\ ------------ & - & -------- & -- \\ V_{12} R_3^{-1}(L_2 - R_2 \nabla^{-1} L_\nabla) & | & V_{12} R_3^{-1} R_2 V_{21}^{-1} & I \\ C_2 - D_{21} \nabla^{-1} L_\nabla & | & D_{21} V_{21}^{-1} & 0 \end{bmatrix}$$

where

$$L_\nabla = L_1 - R_2^T R_3^{-1} L_2.$$

Let Z_∞ be the solution to the discrete-time Riccati equation

$$Z_\infty = \tilde{B}_t \hat{J} \tilde{B}_t^T + A_t Z_\infty A_t^T - M_t S_t^{-1} M_t^T \tag{2.5}$$

in which

$$S_t = \tilde{D}_t \hat{J} \tilde{D}_t^T + C_t Z_\infty C_t^T =: \begin{bmatrix} S_{t_1} & S_{t_2} \\ S_{t_2}^T & S_{t_3} \end{bmatrix}$$

$$M_t = \tilde{B}_t \hat{J} \tilde{D}_t^T + A_t Z_\infty C_t^T =: [M_{t_1} M_{t_2}].$$

Further on we shall refer to equations (2.4) and (2.5) as to *X-Riccati equation* and *Z-Riccati equation*, respectively.

As it is proved in [6], a stabilizing controller that satisfies

$$\|F_\ell(P,K)\|_\infty < \gamma$$

exists, if and only if

1. There exists a solution to the Riccati equation (2.4) satisfying

$$X_\infty \geq 0$$

$$\nabla < 0$$

such that $A - BR^{-1}L$ is asymptotically stable.

2.There exists a solution to the Riccati equation (2.5) such that

$$Z_\infty \geq 0$$

$$S_{t_1} - S_{t_2} S_{t_3}^{-1} S_{t_2}^T < 0$$

with $A_t - M_t S_t^{-1} C_t$ asymptotically stable.

In this case a controller that achieves the objective is

$$\hat{x}_{k+1} = A_t \hat{x}_k + B_2 u_k + M_{t_2} S_{t_3}^{-1} (y_k - C_{t_2} \hat{x}_k)$$

$$V_{12} u_k = -C_{t_1} \hat{x}_k - S_{t_2} S_{t_3}^{-1} (y_k - C_{t_2} \hat{x}_k)$$

which yields

$$K_0 = \left[\begin{array}{c|c} A_t - B_2 V_{12}^{-1} (C_{t_1} - S_{t_2} S_{t_3}^{-1} C_{t_2}) - M_{t_2} S_{t_3}^{-1} C_{t_2} & -B_2 V_{12}^{-1} S_{t_2} S_{t_3}^{-1} + M_{t_2} S_{t_3}^{-1} \\ \hline -V_{12}^{-1} (C_{t_1} - S_{t_2} S_{t_3}^{-1} C_{t_2}) & -V_{12}^{-1} S_{t_2} S_{t_3}^{-1} \end{array} \right].$$

This is the so called *central controller* which is widely used in practice. Later on we consider only the computation of the central sub-optimal controller.

Consider now the general case when $D_{22} \neq 0$. Suppose

$$\hat{K} = \left[\begin{array}{c|c} \hat{A}_k & \hat{B}_k \\ \hline \hat{C}_k & \hat{D}_k \end{array}\right]$$

is a stabilising controller for D_{22} set to zero, and satisfies

$$\|F_\ell\left(P - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}, \hat{K}\right)\|_\infty < \gamma.$$

Then [2]

$$\begin{aligned} F_\ell(P, \hat{K}(I + D_{22}\hat{K})^{-1}) &= P_{11} + P_{12}\hat{K}(I + D_{22}\hat{K} - P_{22}\hat{K})^{-1}P_{21} \\ &= F_\ell\left(P - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}, \hat{K}\right). \end{aligned}$$

In this way a controller $\hat{K}$ for

$$P - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}$$

yields a controller $K = \hat{K}(I + D_{22}\hat{K})^{-1}$ for P. It may be shown that

$$K = \left[\begin{array}{c|c} \hat{A}_k - \hat{B}_k D_{22}(I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{C}_k & \hat{B}_k - \hat{B}_k D_{22}(I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{D}_k \\ \hline (I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{C}_k & (I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{D}_k \end{array}\right].$$

In order to find K from $\hat{K}$ we must exclude the possibility of the feedback system becoming ill-posed, i.e. $\det(I + \hat{K}(\infty)D_{22}) = 0$.

2.2 $\mathcal{H}_2$ optimal controller

The '$\mathcal{H}_2$ optimal discrete-time control problem' is to find an internally stabilising controller $K(z)$ which minimises $\|F_\ell(P, K)\|_2$. The solution of this problem is found under the following assumptions:

A1 (A, B_2) is stabilisable and (C_2, A) is detectable;

A2 $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$;

A3 $\begin{bmatrix} A - e^{j\Theta}I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\Theta \in [0, 2\pi)$;

A4 $\begin{bmatrix} A - e^{j\Theta} I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\Theta \in [0, 2\pi)$.

Whenever the assumption **A2** is fulfilled, we shall say that the system is in *normalised form.*

As in the case of $\mathcal{H}_\infty$ optimisation we shall assume first that $D_{22} = 0$.

Denote

$$A_X = A - B_2 D_{12}^T C_1, \;\; A_Y = A - B_1 D_{21}^T C_2$$

$$C_X = C_1^T C_1 - C_1^T D_{12} D_{12}^T C_1, \;\; C_Y = B_1 B_1^T + B_1 D_{21}^T D_{21} B_1^T$$

$$D_X = B_2 B_2^T, \;\; D_Y = C_2^T C_2.$$

Let $X_2 \geq 0$ and $Y_2 \geq 0$ be the stabilising solutions to the following discrete-time Riccati equations

$$X_2 = C_X + A_X^T (I_n + X_2 D_X)^{-1} X_2 A_X$$

and

$$Y_2 = C_Y + A_Y (I_n + Y_2 D_Y)^{-1} Y_2 A_Y^T.$$

Define

$$\begin{aligned} F_2 &:= -(I_n + B_2^T X_2 B_2)^{-1}(B_2^T X_2 A + D_{12}^T C_1) \\ F_0 &:= -(I_n + B_2^T X_2 B_2)^{-1}(B_2^T X_2 B_1 + D_{12}^T D_{11}) \\ L_2 &:= -(A Y_2 C_2^T + B_1 D_{21}^T)(I_n + C_2 Y_2 C_2^T)^{-1} \\ L_0 &:= (F_2 Y_2 C_2^T + F_0 D_{21}^T)(I_n + C_2 Y_2 C_2^T)^{-1} \end{aligned}$$

Then the unique $\mathcal{H}_2$ optimal controller is given by [9]

$$K_{opt}(z) = \left[\begin{array}{c|c} A + B_2 F_2 + L_2 C_2 - B_2 L_0 C_2 & -(L_2 - B_2 L_0) \\ \hline F_2 - L_0 C_2 & L_0 \end{array}\right].$$

Consider finally how to normalise a given system in order to fulfil the assumption **A2**. Suppose we are given a system with transfer function matrix

$$P_p(z) = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right] \tag{2.6}$$

for which $\mathrm{rank}(D_{p12}) = m_2$ and $\mathrm{rank}(D_{p21}) = p_2$. By using the input and output transformations

$$u_p = T_u u, \;\; y_p = T_y^{-1} y, \;\; z_p = Q_{12} z, \;\; w_p = Q_{21}^T w$$

where the matrices Q_{12} and Q_{21} are orthogonal and the matrices T_u and T_y are nonsingular, the system (2.6) will be reduced to a system with the transfer function matrix

$$\begin{aligned} P(z) &= \begin{bmatrix} Q_{12}^T & 0 \\ 0 & T_y \end{bmatrix} P_p(z) \begin{bmatrix} Q_{21}^T & 0 \\ 0 & T_u \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_p & B_{p1}Q_{21}^T & B_{p2}T_u \\ \hline Q_{12}^T C_{p1} & Q_{12}^T D_{p11} Q_{21}^T & Q_{12}^T D_{p12} T_u \\ T_y C_{p2} & T_y D_{p21} Q_{21}^T & T_y D_{p22} T_u \end{array}\right] \end{aligned} \tag{2.7}$$

If the matrices $Q_{12},\ Q_{21},\ T_u,\ T_y$ are chosen such that

$$D_{12} = Q_{12}^T D_{p12} T_u = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}, \tag{2.8}$$

$$D_{21} = T_y D_{p21} Q_{21}^T = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix} \tag{2.9}$$

then the matrices of the new system $P(z)$ will satisfy the assumption **A2**. After determining the controller $K(z)$ for the transformed system $P(z)$, the controller $K_p(z)$ for $P_p(z)$ is found as $K_p(z) = T_u K(z) T_y$, which preserves the closed-loop system $\mathcal{H}_\infty$ norm,

$$\|F_\ell(P_p, K_p)\|_\infty = \|F_\ell(P, K)\|_\infty.$$

The transformations (2.8), (2.9) can be performed by using QR or Singular Value Decomposition (SVD). Here we shall use SVD which allows to determine reliably the numerical rank of the matrices $D_{p12},\ D_{p21}$ and to find easily the condition numbers of the transformation matrices $T_u,\ T_y$.

Let

$$D_{p12} = U_{12} \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T$$

be the SVD of D_{p12} where the matrices $U_{12},\ V_{12}$ are orthogonal and Σ_{12} is a diagonal matrix containing the m_2 singular values of D_{p12} on the diagonal. (If D_{p12} is of full column rank, then the singular values are positive and Σ_{12} is non-singular.) Partitioning the matrix U_{12} as

$$U_{12} = [U_{121} \; U_{122}]$$

where U_{121} has m_2 columns and U_{122} has $p_1 - m_2$ columns and defining

$$Q_{12} = [U_{122}\ U_{121}]$$

$$T_u = V_{12}\Sigma_{12}^{-1}$$

we obtain the necessary representation of D_{p12}. In addition, the condition number of T_u in respect to inversion is $\sigma_{\max}(\Sigma_{12})/\sigma_{\min}(\Sigma_{12})$. The determination of this condition number allows to monitor the conditioning of T_u during the reduction and to obtain an estimation about the precision of the computations.

In a similar way, let

$$D_{p21} = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \end{bmatrix} V_{21}^T$$

be the SVD of D_{p12} where U_{21}, V_{21} are orthogonal and Σ_{21} is a diagonal matrix. If D_{p12} is of full row rank, then the p_2 singular values are positive and Σ_{21} is non-singular. Partition the matrix V_{21}^T as

$$V_{21}^T = \begin{bmatrix} V_{211}^T \\ V_{212}^T \end{bmatrix}$$

where V_{211}^T has p_2 rows and V_{212}^T has $m_1 - p_2$ rows and taking

$$Q_{21} = \begin{bmatrix} V_{212}^T \\ V_{211}^T \end{bmatrix}$$

$$T_y = \Sigma_{21}^{-1}U_{21}^T$$

we obtain the necessary reduction of D_{p21}. The condition number of T_y in respect to inversion is $\sigma_{\max}(\Sigma_{21})/\sigma_{\min}(\Sigma_{21})$.

3 Numerical algorithms and software

The synthesis of an $\mathcal{H}_\infty$ (sub)optimal controller for a discrete-time system is accomplished by the algorithm *DHINF*.

For given matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21} and D_{22} and for a specified value of γ, algorithm *DHINF* checks the rank conditions against a threshold *tol* and computes first the controller for the case $D_{22} = 0$ after that it finds the controller for the original system.

Algorithm *DHINF: Computation of an $\mathcal{H}_\infty$ suboptimal controller for a discrete-time system*

If $\sigma_{n+m_2}\left(\begin{bmatrix} A - e^{j\Theta}I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}\right) < tol$, stop

If $\sigma_{n+p_2}\left(\begin{bmatrix} A - e^{j\Theta}I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}\right) < tol$, stop

If $\sigma_{m_2}(D_{12}) < tol$, stop

If $\sigma_{p_2}(D_{21}) < tol$, stop

Compute $R_0 = \begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}$

Compute $C_X = C_1^T C_1$

Compute $S_X = C_1^T [D_{11} D_{12}]$

Solve $X_\infty = C_X + A^T X_\infty A - (S_X + A^T X_\infty B)(R_0 + B^T X_\infty B)^{-1}(S_X^T + B^T X_\infty A)$

Compute $R_1 = D_{11}^T D_{11} + B_1^T X_\infty B_1 - \gamma^2 I_{m_1}$

Compute $R_2 = D_{12}^T D_{11} + B_2^T X_\infty B_1$

Compute $R_3 = D_{12}^T D_{12} + B_2^T X_\infty B_2$

If $\mathrm{cond}(R_3) > 1/tol$, stop

Compute the Cholesky factorization of R_3, $V_{12}^T V_{12} = R_3$

Compute $\nabla = R_1 - R_2^T R_3^{-1} R_2$

If $\mathrm{cond}(\nabla) > 1/tol$, stop

Compute the Cholesky factorization of $-\nabla/\gamma^2$, $V_{21}^T V_{21} = -\nabla/\gamma^2$

Compute $L_1 = D_{11}^T C_1 + B_1^T X_\infty A$

Compute $L_2 = D_{12}^T C_1 + B_2^T X_\infty A$

Compute $L_\nabla = L_1 - R_2^T R_3^{-1} L_2$

Compute $A_t = A - B_1 \nabla^{-1} L_\nabla$

Compute $\tilde{B}_{t_1} = B_1 V_{21}^{-1}$

Compute $C_{t_1} = V_{12}R_3^{-1}(L_2 - R_2\nabla^{-1}L_\nabla)$

Compute $C_{t_2} = C_2 - D_{21}\nabla^{-1}L_\nabla$

Compute $\tilde{D}_{t_{11}} = V_{12}R_3^{-1}R_2V_{21}^{-1}$

Compute $\tilde{D}_{t_{21}} = D_{21}V_{21}^{-1}$

Compute $\tilde{R}_0 = \begin{bmatrix} \tilde{D}_{t_{11}} \\ \tilde{D}_{t_{21}} \end{bmatrix} \begin{bmatrix} \tilde{D}_{t_{11}}^T & \tilde{D}_{t_{21}}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_2} & 0 \\ 0 & 0 \end{bmatrix}$

Compute $C_Z = \tilde{B}_{t_{11}}\tilde{B}_{t_{11}}^T$

Compute $S_Z = \tilde{B}_{t_1}[\tilde{D}_{t_{11}}^T \tilde{D}_{t_{21}}^T]$

Solve $Z_\infty = C_Z + A_tZ_\infty A_t^T - (S_Z + A_tZ_\infty C_t^T)(\tilde{R}_0 + C_tZ_\infty C_t^T)^{-1}(S_Z^T + C_tZ_\infty A_t^T)$

Compute $S_{t_1} = \tilde{D}_{t_{11}}\tilde{D}_{t_{11}}^T + C_{t_1}Z_\infty C_{t_1}^T - \gamma^2 I_{m_2}$

Compute $S_{t_2} = \tilde{D}_{t_{11}}\tilde{D}_{t_{21}}^T + C_{t_1}Z_\infty C_{t_2}^T$

Compute $S_{t_3} = \tilde{D}_{t_{21}}\tilde{D}_{t_{21}}^T + C_{t_2}Z_\infty C_{t_2}^T$

If $\mathrm{cond}(S_{t_3}) > 1/tol$, stop

If $S_{t_1} - S_{t_2}S_{t_3}^{-1}S_{t_2}^T$ is not negative definite, stop

Compute $M_{t_1} = \tilde{B}_{t_1}\tilde{D}_{t_{11}}^T + A_tZ_\infty C_{t_1}^T$

Compute $M_{t_2} = \tilde{B}_{t_1}\tilde{D}_{t_{21}}^T + A_tZ_\infty C_{t_2}^T$

Compute $\hat{A}_k = A_t - B_2V_{12}^{-1}(C_{t_1} - S_{t_2}S_{t_3}^{-1}C_{t_2}) - M_{t_2}S_{t_3}^{-1}C_{t_2}$

Compute $\hat{B}_k = -B_2V_{12}^{-1}S_{t_2}S_{t_3}^{-1} + M_{t_2}S_{t_3}^{-1}$

Compute $\hat{C}_k = -V_{12}^{-1}(C_{t_1} - S_{t_2}S_{t_3}^{-1}C_{t_2})$

Compute $\hat{D}_k = -V_{12}^{-1}S_{t_2}S_{t_3}^{-1}$

If $\mathrm{cond}(I_{m_2} + \hat{D}_kD_{22}) > 1/tol$, stop

Compute $A_k = \hat{A}_k - \hat{B}_kD_{22}(I_{m_2} + \hat{D}_kD_{22})^{-1}\hat{C}_k$

Compute $B_k = \hat{B}_k - \hat{B}_kD_{22}(I_{m_2} + \hat{D}_kD_{22})^{-1}\hat{D}_k$

Compute $C_k = (I_{m_2} + \hat{D}_kD_{22})^{-1}\hat{C}_k$

Compute $D_k = (I_{m_2} + \hat{D}_kD_{22})^{-1}\hat{D}_k$

The algorithm *DHINF* is implemented by the double precision Fortran 77 subroutine SB10DD. The subroutine produces the suboptimal controller matrices along with estimates of the condition numbers of the matrices $R_3, \nabla, V_{21}, S_{t_3}, V_{12}$ and $I_{m_2} + \hat{D}_k D_{22}$, which are to be inverted, and estimates of the condition numbers of the X- and Z-Riccati equations which are to be solved. These condition numbers gives an information about the accuracy of the computations in obtaining the controller. Almost all computations in this routine are done by using BLAS Level-3 routines [4] and LAPACK routines [1]. The matrix Riccati equations are solved by the subroutine SB02OD from SLICOT [3] and the condition numbers of these equations are estimated by the subroutine SB02SD from the same library.

The synthesis of the $\mathcal{H}_2$ optimal controller is accomplished by the algorithms *DH2T, DH2N* and *DH2C*.

For given matrices $A,\ B_1,\ B_2,\ C_1,\ C_2,\ D_{11},\ D_{12},\ D_{21}$ and D_{22} algorithm *DH2T* checks the rank conditions against the threshold *tol* and computes the transformation matrices Q_{12}, Q_{21}, T_u and T_y which reduce the generalised system into the normalised form. Note that the block D_{22} is not transformed, since it is required in its original form by the algorithm *DH2C* in the computation of the controller matrices. The algorithm produces also the condition numbers of the nonsingular transformation matrices T_u and T_y.

Algorithm *DH2T:* *Reduction of the system to normalised form and checking the rank conditions for optimal $\mathcal{H}_2$ design of a discrete-time system*

If $\sigma_{n+m_2}\left(\begin{bmatrix} A - e^{j\Theta} I_n & B_2 \\ C_1 & D_{12} \end{bmatrix}\right) < tol$, stop

If $\sigma_{n+p_2}\left(\begin{bmatrix} A - e^{j\Theta} I_n & B_1 \\ C_2 & D_{21} \end{bmatrix}\right) < tol$, stop

Decompose $D_{12} = U_{12} \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T$

If $\sigma_{m_2}(\Sigma_{12}) < tol$, stop

Decompose $D_{21} = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \end{bmatrix} V_{21}^T$

If $\sigma_{p_2}(\Sigma_{21}) < tol$, stop

Compute $Q_{12} = U_{12} \begin{bmatrix} 0 & I_{m_2} \\ I_{p_1-m_2} & 0 \end{bmatrix}$

Compute $T_u = V_{12}\Sigma_{12}^{-1}$

Compute $\text{cond}(T_u) = \sigma_1(\Sigma_{12})/\sigma_{m_2}(\Sigma_{12})$

Compute $Q_{21} = \begin{bmatrix} 0 & I_{m_1-p_2} \\ I_{p_2} & 0 \end{bmatrix} V_{21}^T$

Compute $T_y = \Sigma_{21}^{-1}U_{21}^T$

Compute $\text{cond}(T_y) = \sigma_1(\Sigma_{21})/\sigma_{p_2}(\Sigma_{21})$

$B_1 \leftarrow B_1Q_{21}^T$

$B_2 \leftarrow B_2T_u$

$C_1 \leftarrow Q_{12}^TC_1$

$C_2 \leftarrow T_yC_2$

$D_{11} \leftarrow Q_{12}^TD_{11}Q_{21}^T$

Set $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$

Set $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$

Note that the algorithm *DH2T* is practically the same as the algorithm *HINFT* presented in [8].

The computation of the matrices $\hat{A}_k,\ \hat{B}_k,\ \hat{C}_k,\ \hat{D}_k$ of the $\mathcal{H}_2$ optimal controller for the normalised system is done by the algorithm *DH2N*.

Algorithm *DH2N: Computation of the $\mathcal{H}_2$ optimal controller for a normalised discrete-time system*

Compute $A_X = A - B_2 D_{12}^T C_1$

Compute $C_X = C_1^T C_1 - C_1^T D_{12} D_{12}^T C_1$

Compute $D_X = B_2 B_2^T$

Solve $X_2 = C_X + A_X^T (I_n + X_2 D_X)^{-1} X_2 A_X$

Compute $F_2 = -(I_n + B_2^T X_2 B_2)^{-1} (B_2^T X_2 A + D_{12}^T C_1)$

Compute $F_0 = -(I_n + B_2^T X_2 B_2)^{-1} (B_2^T X_2 B_1 + D_{12}^T D_{11})$

Compute $A_Y = A - B_1 D_{21}^T C_2$

Compute $C_Y = B_1 B_1^T - B_1 D_{21}^T D_{21} B_1^T$

Compute $D_Y = C_2^T C_2$

Solve $Y_2 = C_Y + A_Y (I_n + Y_2 D_Y)^{-1} Y_2 A_Y^T$

Compute $L_2 = -(A Y_2 C_2^T + B_1 D_{21}^T)(I_n + C_2 Y_2 C_2^T)^{-1}$

Compute $L_0 = (F_2 Y_2 C_2^T + F_0 D_{21}^T)(I_n + C_2 Y_2 C_2^T)^{-1}$

Compute $\hat{A}_k = A + B_2(F_2 - L_0 C_2) + L_2 C_2$

Compute $\hat{B}_k = -L_2 + B_2 L_0$

Compute $\hat{C}_k = F_2 - L_0 C_2$

Set $\hat{D}_k = L_0$

The computation of the central controller matrices $A_k,\ B_k,\ C_k$ and D_k for the original (unnormalised) system from the matrices $\hat{A}_k,\ \hat{B}_k,\ \hat{C}_k,\ \hat{D}_k$ of the optimal controller for the normalised system is done by the algorithm DH2C.

Algorithm *DH2C:* *Computation of the $\mathcal{H}_2$ optimal controller for a discrete-time system from the optimal controller for the normalised system*

$\hat{B}_k \leftarrow \hat{B}_k T_y$

$\hat{C}_k \leftarrow T_u \hat{C}_k$

$\hat{D}_k \leftarrow T_u \hat{D}_k T_y$

If $\text{cond}(I_{m_2} + \hat{D}_k D_{22}) > 1/tol$, stop

Compute $A_k = \hat{A}_k - \hat{B}_k D_{22}(I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{C}_k$

Compute $B_k = \hat{B}_k - \hat{B}_k D_{22}(I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{D}_k$

Compute $C_k = (I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{C}_k$

Compute $D_k = (I_{m_2} + \hat{D}_k D_{22})^{-1}\hat{D}_k$

The algorithms *DH2T*, *DH2N* and *DH2C* are implemented by the double precision Fortran 77 routines SB10PD, SB10SD and SB10TD, respectively. The driver subroutine SB10ED calls all routines necessary to compute the $\mathcal{H}_2$ optimal controller. Apart from the controller matrices this routine produces also estimates of the condition numbers of the matrices T_u, T_y, $I_{m_2} + B_2' X_2 B_2$, $I_{p_2} + C_2 Y_2 C_2^T$ and $I_{m_2} + \hat{D}_k D_{22}$, and estimates of the condition numbers of the X- and Z-Riccati equations.

4 A numerical example

As an illustrative example for discrete-time $\mathcal{H}_\infty$ design, consider the computation of optimal controller for a sixth order system with $m = 5$, $p = 5$ and $m_2 = 2$, $p_2 = 2$ and matrices

$$A = \begin{bmatrix} -0.7 & 0 & 0.3 & 0 & -0.5 & -0.1 \\ -0.6 & 0.2 & -0.4 & -0.3 & 0 & 0 \\ -0.5 & 0.7 & -0.1 & 0 & 0 & -0.8 \\ -0.7 & 0 & 0 & -0.5 & -1.0 & 0 \\ 0 & 0.3 & 0.6 & -0.9 & 0.1 & -0.4 \\ 0.5 & -0.8 & 0 & 0 & 0.2 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & 1 \\ -3 & -4 & 0 & 2 & -2 \\ 1 & -2 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 3 \\ 1 & 0 & 3 & -1 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 2 & -2 & 0 & -3 \\ -3 & 0 & 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -4 & 0 & -2 \\ 1 & -3 & 0 & 0 & 3 & 1 \\ 0 & 1 & -2 & 1 & 0 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

Using the subroutine SB10DD it was found by some trial and error that the optimal value of γ for this system is $\gamma_{opt} = 111.2931936924534$. The controller matrices in this case are (up to four digits)

$$A_k = \begin{bmatrix} -17.9945 & 52.0119 & 26.0705 & -0.4270 & -40.8826 & 18.0767 \\ 18.8109 & -57.5960 & -29.0798 & 0.5870 & 45.3091 & -19.8544 \\ -26.5868 & 77.9314 & 39.0182 & -1.4019 & -60.0839 & 26.6776 \\ -21.4062 & 62.1415 & 30.7357 & -0.9200 & -48.5988 & 21.8244 \\ -0.8905 & 4.2767 & 2.3276 & -0.2424 & -3.0361 & 1.2162 \\ -5.3260 & 16.1877 & 8.4786 & -0.2489 & -12.2288 & 5.1562 \end{bmatrix},$$

$$B_k = \begin{bmatrix} 16.9705 & 14.1579 \\ -18.9123 & -15.6649 \\ 25.1924 & 21.2745 \\ 20.1024 & 16.8240 \\ 1.4098 & 1.2034 \\ 5.3156 & 4.5127 \end{bmatrix},$$

$$C_k = \begin{bmatrix} -9.1896 & 27.5030 & 13.7297 & -0.3638 & -21.5879 & 9.5978 \\ 3.6473 & -10.6143 & -5.2747 & 0.2431 & 8.1069 & -3.6275 \end{bmatrix},$$

$$D_k = \begin{bmatrix} 9.0273 & 7.5311 \\ -3.3990 & -2.8205 \end{bmatrix}.$$

The estimates of the condition numbers of the X- and Z-Riccati equations in this case are $\mathrm{cond}_X = 3.1763 \times 10^3$ and $\mathrm{cond}_Z = 2.6594 \times 10^{13}$, respectively, which shows that the Z-Riccati equation is extremally ill-conditioned.

In Figure 1 we show the closed-loop system response to a unit step disturbance w_1 obtained for the optimal controller.

In Figure 2 we show the closed-loop system root loci for a family of (sub)optimal controllers obtained for different values of $\gamma \in [\gamma_{opt}, 1000]$. The circles denote the location of the closed-loop poles for the optimal controller.

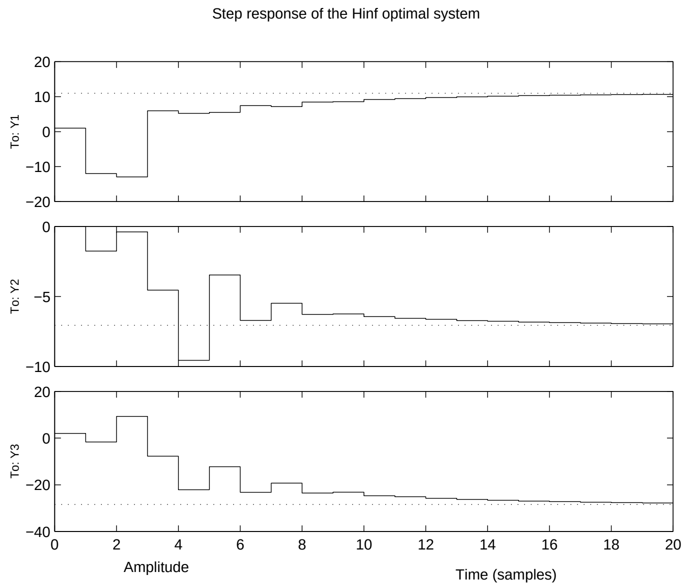

Figure 1: Step response of the H_∞-optimal closed-loop system

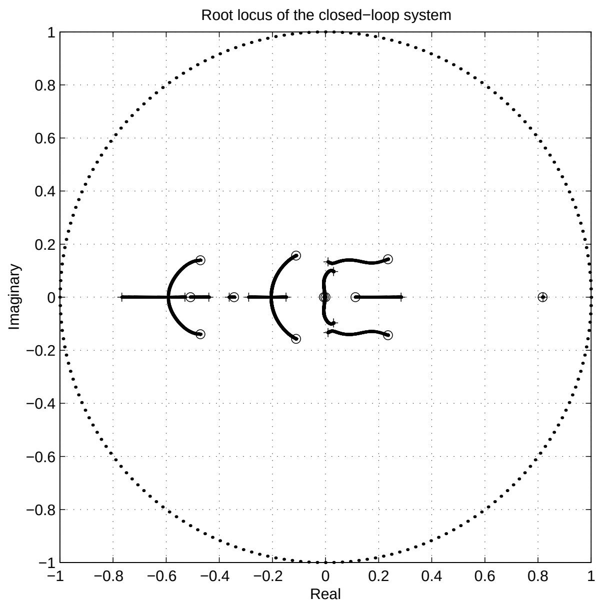

Figure 2: Root loci of the closed-loop system for different values of γ

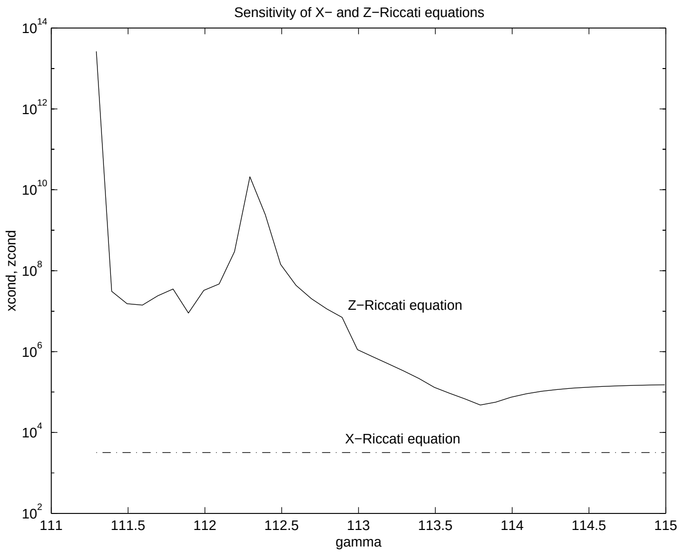

Figure 3: Sensitivity of X- and Z-Riccati equations for different values of γ

The dependence of the condition numbers of X- and Z-Riccati equations on γ is shown in Figure 3. It is seen, that with the decreasing of γ the conditioning of the Z-Riccati equation deteriorates very much.

In Figures 4 we show the sensitivity of the $\mathcal{H}_\infty$ norm of the closed-loop system for $\gamma = \gamma_{opt}$ with respect to variations in the elements A_{k25}, A_{k41} of the controller.

For the same system the program SB10ED produced the following optimal $\mathcal{H}_2$ controller,

$$A_k = \begin{bmatrix} -0.0551 & -2.1891 & -0.6607 & -0.2532 & 0.6674 & -1.0044 \\ -1.0379 & 2.3804 & 0.5031 & 0.3960 & -0.6605 & 1.2673 \\ -0.0876 & -2.1320 & -0.4701 & -1.1461 & 1.2927 & -1.5116 \\ -0.1358 & -2.1237 & -0.9560 & -0.7144 & 0.6673 & -0.7957 \\ 0.4900 & 0.0895 & 0.2634 & -0.2354 & 0.1623 & -0.2663 \\ 0.1672 & -0.4163 & 0.2871 & -0.1983 & 0.4944 & -0.6967 \end{bmatrix},$$

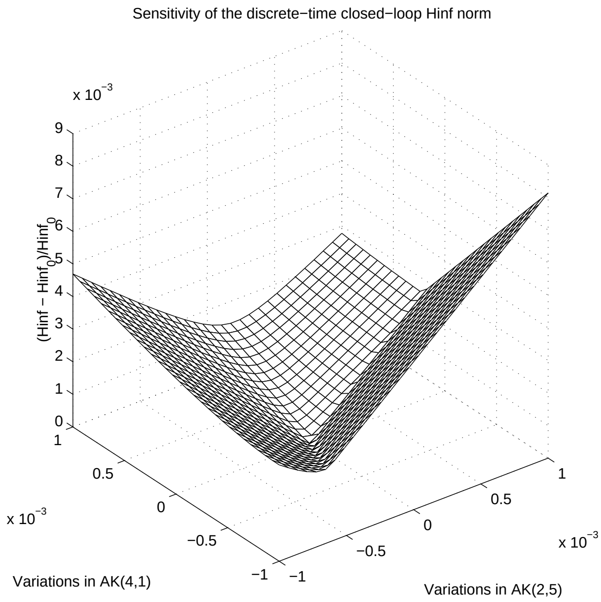

Figure 4: Sensitivity of the closed-loop $\mathcal{H}_\infty$ norm for $\gamma = \gamma_{opt}$

$$B_k = \begin{bmatrix} -0.5985 & -0.5464 \\ 0.5285 & 0.6087 \\ -0.7600 & -0.4472 \\ -0.7288 & -0.6090 \\ 0.0532 & 0.0658 \\ -0.0663 & 0.0059 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 0.2500 & -1.0200 & -0.3371 & -0.2733 & 0.2747 & -0.4444 \\ 0.0654 & 0.2095 & 0.0632 & 0.2089 & -0.1895 & 0.1834 \end{bmatrix},$$

$$D_k = \begin{bmatrix} -0.2181 & -0.2070 \\ 0.1094 & 0.1159 \end{bmatrix}.$$

(Note that the controller is not strictly proper, i.e. $D_k \neq 0$).

In Figure 5 we show the unit step response of the $\mathcal{H}_2$ optimal closed-loop system. In Figure 6 we show the sensitivity of the $\mathcal{H}_2$ norm of the closed loop system for the same variations in the elements A_{k25} and A_{k41} as in the case of $\mathcal{H}_\infty$ optimal design.

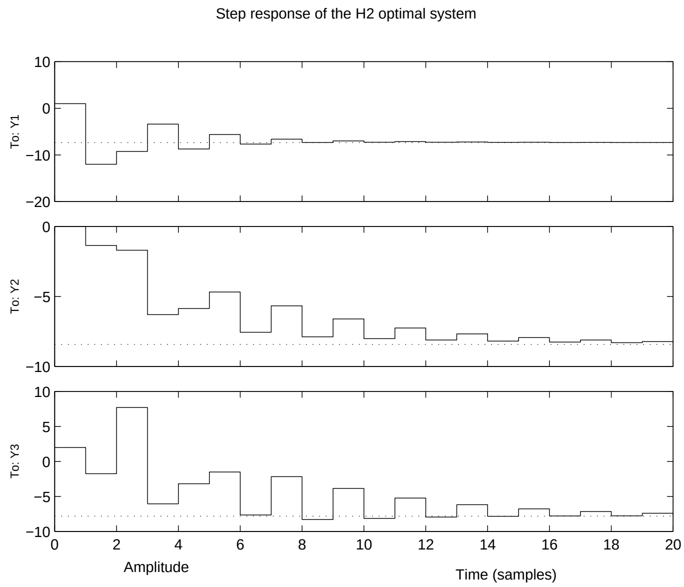

Figure 5: Step response of the H_2-optimal closed-loop system

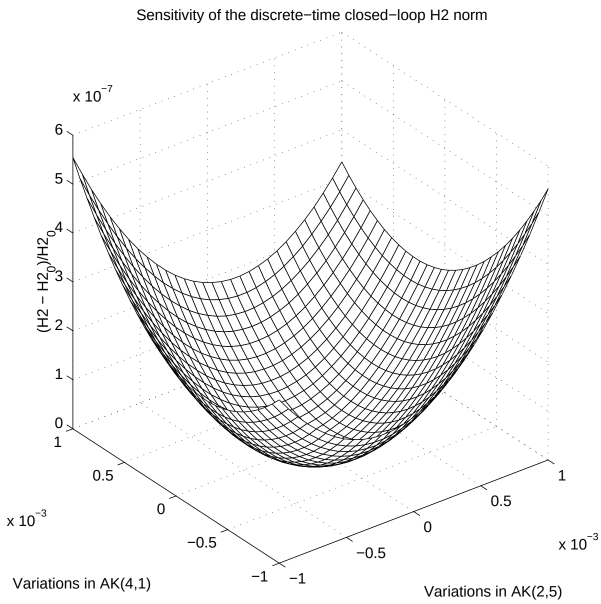

Figure 6: Sensitivity of the closed-loop $\mathcal{H}_2$ norm

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