Numerical problems in robust and H_∞ optimal control *

Anton A. Stoorvogel
Dept. of Mathematics and Comp. Science
Eindhoven Univ. of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
Fax. +31-40-2442489
E-mail. A.A.Stoorvogel@tue.nl

September 1999

*This paper presents research results of the European Community BRITE-EURAM III Thematic Networks Programme NICONET (contract number BRRT-CT97-5040) and is distributed by the Working Group on Software WGS. *W*GS secretariat: Mrs. Ida Tassens, ESAT - Katholieke Universiteit Leuven, K. Mercierlaan 94, 3001-Leuven-Heverlee, BELGIUM. This report is also available by anonymous ftp from *wgs.esat.kuleuven.ac.be* in the directory /pub/WGS/REPORTS/ as file name nic1999-13.ps.Z

1 Introduction

In the last 20 years a lot of research effort has gone into the area of robust and H_∞ optimal control. There are currently quite a few textbooks available in this area (see for instance [16, 20, 24, 26, 31, 38]). For the H_∞ optimal control problem the basic algorithm as originally reported in [9] has been extended in many directions and several other approaches have been developed: a polynomial approach, [21]; an approach based on J-spectral factorization, [15]; an approach based on chain-scattering, [19, 20]; an approach based on linear matrix inequalities, [11, 18].

In robust control control many new ideas and techniques have been developed. As a prime example, we would like to mention μ-analysis and synthesis, [8]. Also a lot of variations on the small gain theorem, [37], have been presented. Finally a lot of mixed-objective problems have recently been approached via linear matrix inequalities, [4]. However from a computational point of view most of these methods still rely on the fundamental underlying H_∞ control problem. We might have to choose weighthing functions appropriately. This is mostly done by iteratively solving a number of H_∞ control problems: this either without guaranteed convergence or with guaranteed convergence by exploiting the structure and using convexity. Linear matrix inequalities transform everything to a large convex optimization problem with a nice structure but the underlying properties can be learned from the H_∞ control problem.

As argued above it is reasonable when studying the numerical properties to concentrate on the H_∞ control problem. In H_∞ control, after the results had been obtained, a lot of effort has been put into exploiting the connections between the different solvability methods:

- The J-spectral factorization is directly related to solvability of the algebraic Riccati equation.
- The polynomial method can be connected to J-spectral factorization but using polynomial factors instead of rational factors.
- The method based on chain-scattering has been connected to J-spectral factorization.

Therefore, it is fair to say that there are currently two main methods to solve H_∞ optimal control which have intrinsically different properties: one based on algebraic Riccati equations and one based on linear matrix inequalities. At this moment there are a few comments to be made on the differences between these two methods:

- Solving algebraic Riccati equations is faster than solving linear matrix inequalities.
- Extending the results to multi-objective problems is intrinsically easier with linear matrix inequalities.
- It is easier to formulate the most general results in terms of linear matrix inequalities but their are drawbacks from a numerical point of view as explained later.

In section 2 we will formulate the H_∞ control problem for linear, time-invariant and finite-dimensional systems. We will then consider the difficulties in computation of the optimal performance in section3. In section 4 we will consider the problem of computing controllers.

2 The H_∞ optimal control problem

We consider the following system:

$$\Sigma : \begin{cases} \sigma x = \ \ Ax + \ \ B_1 w + \ \ B_2 u, \\ \ z \ \ = C_1 x + D_{11} w + D_{12} u, \\ \ y \ \ = C_2 x + D_{21} w + D_{22} u. \end{cases} \tag{2.1}$$

where σ denotes differentiation, $\sigma x = \dot{x}$, or the time-shift, $(\sigma x)(k) = x(k+1)$, for continuous-time systems and discrete-time systems respectively. For all t we have that $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^\ell$ is the unknown disturbance, $y(t) \in \mathbb{R}^p$ is the measured output and $z(t) \in \mathbb{R}^q$ is the unknown output to be controlled.

We would like to minimize the effect of the disturbance w on the output z by finding an appropriate control input u. We seek a stabilizing controller of the form:

$$\Sigma_F : \begin{cases} \dot{p} \ = \ Kp \ + \ Ly, \\ u = Mp + Ny. \end{cases} \tag{2.2}$$

such that after applying the feedback Σ_F to the system (2.1), the resulting closed-loop system is internally stable and has minimal H_∞ norm. Note that necessary and sufficient conditions are known under which we can find an internally stabilizing controller which makes the H_∞ norm strictly less than some a priori given bound γ. This obviously enables us to obtain the infimum over all stabilizing controllers of the closed loop H_∞ norm as discussed in the next section. Optimal controllers and suboptimal controllers are discussed in section 4. In the literature we encounter the following standard assumptions:

(*i*) $D_{11} = 0$,

(*ii*) $D_{22} = 0$,

(*iii*) $D_{12}^{\mathrm{T}} C_1 = 0$,

(*iv*) $B_1 D_{21}^{\mathrm{T}} = 0$,

(*v*) $D_{12}^{\mathrm{T}} D_{12} = I$,

(*vi*) $D_{21} D_{21}^{\mathrm{T}} = I$,

(*vii*) D_{12} injective,

(*viii*) D_{21} injective,

(*ix*) The subsystem from u to z with realization (A, B_2, C_1, D_{12}) has no invariant zeros on the imaginary axis (continuous time) or on the unit circle (discrete time),

(*x*) The subsystem from w to y with realization (A, B_1, C_2, D_{21}) has no invariant zeros on the imaginary axis (continuous time) or on the unit circle (discrete time), All the above assumptions were made in the original solution to the H_∞ control problem as presented in [9]. But two of the authors immediately indicated in a related paper [14] that assumptions (i),(ii),(iii),(iv),(v),(vi) were not essential at all and just simplified notation. Assumptions (vii),(viii),(ix),(x) are however crucial assumptions in continuous time as we will see shortly. In discrete time assumtions (vii) and (viii) are also not needed but do effect our numerical solution method. The latter is however not always that obvious from the literature.

We will discuss the above assumptions in more detail:

(*i*) The standard approach to handle the general case is to first apply a preliminary static feedback to make the norm of the resulting feedthrough matrix $\tilde{D}_{11}$ less than γ. Then the system is transformed to obtain a new system with the property that the direct feedthrough matrix from w to z is zero and a controller stabilizes this new system and achieves a closed loop H_∞ norm less than γ if and only if this controller stabilizes the original system and achieves a closed loop H_∞ norm less than γ. This basically reduces the original problem to the case $D_{11} = 0$. For details we refer to for instance [31].

This transformation has two drawbacks:

- If we want to minimize the H_∞ norm by iteratively using the condition whether we can make the H_∞ norm less than some γ then we have to apply the above transformation for each value of γ since this transformation to the case $D_{11} = 0$ is intrinsically γ-dependent.
- If we look at the minimum over all static controllers of the resulting direct feedthrough matrix from w to z:

$$\inf_K \|D_{11} + D_{12}K(I - D_{22}K)^{-1}D_{21}\|$$

  and this minimum is close to γ then this transformation becomes numerically sensitive.

(*ii*) First of all the matrix D_{22} does not effect the infimum of the closed-loop H_∞ norm over all stabilizing controllers. If we are only interested in the achievable performance we can therefore assume $D_{22} = 0$ without any consequence. However, the controller does depend on this direct feedthrough matrix! If we design a controller of the form (2.2) after setting $D_{11} = 0$ then in order to construct a controller for the original system we need to invert the matrix $I + D_{11}N$. In general, this matrix need not be invertible. However, a small perturbation of N will make this matrix invertible while preserving stability and the perturbation of the achieved H_∞ norm can be kept arbitrarily small. Obviously, this can create numerical difficulties but will rarely happen.

(*iii*) This condition can always be achieved by a preliminary state feedback. This is obviously not possible in the measurement feedback case. It is in general advisable to use the general formulas for controller design and computation of achievable performance. The formulas might look more difficult but are not intrinsically more difficult while a preliminary feedback can cause numerical problems.

(*iv*) The same comment as above applies but this time this condition can be achieved via a preliminary output injection.

(v) This condition can always be achieved by scaling of the input space as long as condition (vii) is satisfied. Obviously this creates difficulties if some of the singular values of D_{12} are close to zero.

(vi) The same comment as above applies but this time we can achieve this via scaling of the measurement space as long as condition (viii) is satisfied.

(vii) This property is very important from several points of view for continuous time systems. First of all this condition is intrinsic and the standard solutions to the H_∞ control problem cannnot be used if this property is not satisfied. Secondly note that this condition in general implies the existence of invariant zeros at infinity of the subsystem from w to y. Optimal controllers want to cancel this zero which would place a pole at infinity which is however in contradiction with our stability requirement.

Finally if this property is not satisfied then in many case the optimal achievable performance in continuous time does not depend continuously on the system parameters. It is however always upper semicontinuous. After all a small perturbation of the system can move this zero at infinity into a stable zero where pole-zero cancellation is allowed.

The last comment also indicates the main difference between continuous and discrete time. In discrete time the point infinity is not on the boundary between the stable and unstable region and therefore this condition does not induce for instance discontinuous dependence on the system parameters. There is however another issue that is relevant in this respect. Solutions to the H_∞ control problem are related to Riccati equations. In discrete time these are related to symplectic pencils which have eigenvalues which are symmetric with respect to the unit circle. If condition (vii) is not satisfied then the symmetric pencil will have an eigenvalue at zero and infinity and will therefore be a singular pencil. In that case we need to be careful how we are going to solve this Riccati equation (see [1, 17, 34].

($viii$) This is basically a ual condition to condition (vii) and the same comments apply.

(ix) Here we have a zero on the boundary of the stability domain. Again by a small perturbation we can make this zero stable (in which case pole-zero cancellation is possible) while we can at this moment only achieve an approximate pole-zero cancellation. This induces again a discontinuous dependence on the system parameters. This also is a property which is not satisfied in many practical problems and requires a different solution method because the standard solution based on Riccati equations is not applicable.

(x) This is basically a dual condition to condition (x) and the same comments apply.

The above brief comments will be explained in more detail in the following sections.

3 Computation of the optimal performance

Let us first recall the continuous-time solution for the case that only assumptions (vi), (vii), (ix), (x) are satisfied. The result can be obtained more or less directly from any standard textbook on H_∞ control.

Theorem 3.1 *Consider the system* (2.1)*. Assume assumptions* (vi), (vii), (ix), (x) *are satisfied. Then the following two statements are equivalent for any a priori given constant* $\gamma > 0$*.*

(*i*) *For the system* (2.1) *a controller of the form* (2.2) *exists such that the resulting closed-loop system, with transfer matrix* G_F*, is internally stable and has* H_∞ *norm less than* γ*, i.e.* $\|G_F\|_\infty < \gamma$*.*

(*ii*) γ *is such that the matrix*

$$\tilde{D} = \begin{pmatrix} D_{11}^T D_{11} - \gamma^2 I & D_{12}^T D_{11} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{pmatrix}$$

has m *positive eigenvalues and* ℓ *negative eigenvalues and the matrix*

$$\hat{D} = \begin{pmatrix} D_{11} D_{11}^T - \gamma^2 I & D_{21} D_{11}^T \\ D_{21} D_{11}^T & D_{21} D_{21}^T \end{pmatrix}$$

has p *positive eigenvalues and* q *negative eigenvalues. Moreover, there exists a positive semi-definite solution* P *of the algebraic Riccati equation:*

$$A^T P + PA - \begin{pmatrix} B_1^T P + D_{11}^T C_1 \\ B_2^T P + D_{12}^T C_2 \end{pmatrix}^T \tilde{D}^{-1} \begin{pmatrix} B_1^T P + D_{11}^T C_1 \\ B_2^T P + D_{12}^T C_2 \end{pmatrix} + C_1^T C_1 = 0$$

such that

$$A - \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix}^T \tilde{D}^{-1} \begin{pmatrix} B_1^T P + D_{11}^T C_1 \\ B_2^T P + D_{12}^T C_2 \end{pmatrix}$$

has all its eigenvalues in the open left-half plane. Next there exists a positive semi-definite solution Q *of the algebraic Riccati equation:*

$$QA^T + AQ - \begin{pmatrix} C_1 Q + D_{11} B_1^T \\ C_2 Q + D_{21} B_2^T \end{pmatrix}^T \tilde{D}^{-1} \begin{pmatrix} C_1 Q + D_{11} B_1^T \\ C_2 Q + D_{21} B_2^T \end{pmatrix} + B_1^T B_1 = 0$$

such that

$$A - \begin{pmatrix} C_1 Q + D_{11} B_1^T \\ C_2 Q + D_{21} B_2^T \end{pmatrix}^T \hat{D}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

has all its eigenvalues in the open left-half plane. Finally P *and* Q *satisfy* $\rho(PQ) < \gamma^2$

To obtain the minimally achievable H_∞ norm γ^* we need to interatively check the above solvability conditions for different values of γ. One can do a straightforward bilinear search or use more fancy tools which guarantee quadratic convergence. This was originally derived for the computation of the H_∞ norm in [3, 5] and later for the H_∞ control problem in [28].

As we can see from the above theorem we do not need a transformation to achieve $D_{11} = 0$ or any of the other conditions (iii),(iv),(v),(vi). However, if either one of the conditions (vii),(viii),(ix),(x) then we can have discontinuous dependence of the minimally achieveable H_∞ norm, γ^* on the system parameters as shown in [13]. This is problematic because there are many practical applications where these conditions are not satisfied.

When these conditions are not satisfied then there are many alternative approaches to solve the so-called singulat H_∞ control problem:

- Quadratic matrix inequalities [31, 32]. These can be reduced to lower-order Riccati equations and this method is closely related to the work in [29, 30].
- Generalized eigenvalues and zero compensation [7].
- Linear matrix inequalities [11, 18]
- Extensions to J-spectral factorization [2]

Note that all these methods are intrinsically numerically sensitive because we have a discontinuous dependence on the system parameters. The only paper which studies this case in considerable detail is [12]. The method in [12] is in spirit very close to the work in [7, 29–32]. In [12] that paper they also make the correct observation that since the achievable H_∞ norm is upper-semicontinuous and since one wants some form of guaranteed or robust performance, it is natural to perturb the system which are really close to being singular, and make it singular. This will potentially increase the achieved norm but only so much as needed to guarantee this performance in a ball around the nominal model. Because of this argument singular problems are not that bad. The above suggests that in the case of rank evaluations you should always choose to make the problem more singular to get more reliable answers!!!

If one wants to solve singular problems via regularization then one has to be careful due to the discontinuous dependence on the plant parameters. However, there exists perturbations which are guaranteed to yield continuous dependence on the perturbation and which regularize the problem. This was studied originally for H_2 Riccati equations in [36] and later also for H_∞ Riccati equations in [33].

Finally it is good to know that if D_{12} is not injective or D_{21} is not surjective but (A, B_2, C_1, D_{12}) is left-invertible and (A, B_1, C_2, D_{21}) is right-invertible then although the problem is singular the problems of discontinuity do not arise because the systems do not have zeros at infinity . This case was studied in [25]. The above result is then still valid as soon as we replace all inverses by generalized inverses.

For the discrete-time case we obviously get a different type of Riccati equation. The following theorem, gives the discrete-time equivalent of theorem 3.1:

Theorem 3.2 *Consider the system* (2.1). *Assume assumptions* (ix) *and* (x) *are satisfied. The following statements are equivalent:*

(*i*) *There exists a dynamic compensator Σ_F of the form* (2.2) *such that the resulting closed loop system is internally stable and the closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.*

(*ii*) *There exist symmetric matrices $P \geqslant 0$ and $Q \geqslant 0$ such that* (*a*) *We have* $R > 0$ *where*

$$\begin{aligned} V &:= B_2^T P B_2 + D_{12}^T D_{12}, \\ R &:= \gamma^2 I - D_{11} D_{11}^T - B_1^T P B_1 + (B_1^T P B_2 + D_{11}^T D_{12}) V^{\dagger} (B_2^T P B_1 + D_{12}^T D_{11}). \end{aligned}$$

(*b*) P *satisfies the discrete algebraic Riccati equation:*

$$P = A^T P A + C_1^T C_1 - \begin{pmatrix} B_2^T P A + D_{12}^T C_1 \\ B_1^T P A + D_{11}^T C_1 \end{pmatrix}^T G(P)^{\dagger} \begin{pmatrix} B_2^T P A + D_{12}^T C_1 \\ B_1^T P A + D_{11}^T C_1 \end{pmatrix}, \tag{3.1}$$

where

$$G(P) := \begin{pmatrix} D_{12}^T D_{12} & D_{12}^T D_{11} \\ D_{11}^T D_{12} & D_{11}^T D_{11} - \gamma^2 I \end{pmatrix} + \begin{pmatrix} B_2^T \\ B_1^T \end{pmatrix} P \begin{pmatrix} B_2 & B_1 \end{pmatrix}. \tag{3.2}$$

(*c*) *All the zeros of the matrix pencil*

$$\begin{pmatrix} zI - A & -B_2 & -B_1 \\ B_2^T P A + D_{12}^T C_1 & B_2^T P B_2 + D_{12}^T D_{12} & B_2^T P B_1 + D_{12}^T D_{11} \\ B_1^T P A + D_{11}^T C_1 & B_1^T P B_2 + D_{11}^T D_{12} & B_1^T P B_1 + D_{11}^T D_{11} - I \end{pmatrix}, \tag{3.3}$$

are inside the unit circle.

(*d*) *We have* $S > 0$ *where*

$$\begin{aligned} W &:= D_{21} D_{21}^T + C_2 Q C_2^T, \\ S &:= \gamma^2 I - D_{11} D_{11}^T - C_1 Q C_1^T + (C_1 Q C_2^T + D_{11} D_{21}^T) W^{\dagger} (C_2 Q C_1^T + D_{21} D_{11}^T). \end{aligned}$$

(*e*) Q *satisfies the following discrete algebraic Riccati equation:*

$$Q = A Q A^T + B_1 B_1^T - \begin{pmatrix} C_2 Q A^T + D_{21} B_1^T \\ C_1 Q A^T + D_{11} B_1^T \end{pmatrix}^T H(Q)^{\dagger} \begin{pmatrix} C_2 Q A^T + D_{21} B_1^T \\ C_1 Q A^T + D_{11} B_1^T \end{pmatrix}. \tag{3.4}$$

where

$$H(Q) := \begin{pmatrix} D_{21} D_{21}^T & D_{21} D_{11}^T \\ D_{11} D_{21}^T & D_{11} D_{11}^T - \gamma^2 I \end{pmatrix} + \begin{pmatrix} C_2 \\ C_1 \end{pmatrix} Q \begin{pmatrix} C_2^T & C_1^T \end{pmatrix}. \tag{3.5}$$

(*f*) *All the zeros of the matrix pencil*

$$\begin{pmatrix} zI - A & A Q C_2^T + B_1 D_{21}^T & A Q C_1^T + B_1 D_{11}^T \\ -C_2 & C_2 Q C_2^T + D_{21} D_{21}^T & C_2 Q C_1^T + D_{21} D_{11}^T \\ -C_1 & C_1 Q C_2^T + D_{11} D_{21}^T & C_1 Q C_1^T + D_{11} D_{11}^T - I \end{pmatrix} \tag{3.6}$$

are inside the unit circle.

(*g*) $\rho(PQ) < \gamma^2$.

In many Matlab toolboxes the discrete time H_∞ result is obtained via the bilinear transformation. This is obviously possible but numerically not very reliable. All the comments made in the continuous time about zeros on the imaginary axis apply here as well but then for zeros on the unit circle. General results allowing for zeros on the unit circle are available from for instance [32].

The only intrinsic difference with respect to continuous time, is with respect to zeros at infinity. First of all if, (A, B_2, C_1, D_{12}) is left-invertible and (A, B_1, C_2, D_{21}) is right-invertible then in the above result we do not need the generalized inverse. Note that in general the discrete time Riccati equation is connected to symplectic pencils [1, 17, 34]. Some of the difficulties in obtaining solutions of the Riccati equation in the general case have been clarified in [34]. An alternative approach to solve these Riccati equations is connected to exploiting the relationship with continuous-time Riccati equations and is presented in [6]. In general this method appears to work well.

4 Computation of (sub-)optimal controllers

In the literature there are closed form solutions for a suboptimal internally stabilizing controller which makes the H_∞ norm strictly less than γ. This under the assumptions assumptions (vi), (vii), (ix), (x) in continuous time and under assumptions (ix) and (x) in discrete time. See for instance [35] for the discrete time and [14] for continuous time. In the singular cases when these assumptions are not satisfied the situation is quite different.

The linear matrix inequality approach does give formulas for controllers. The approaches based on reduced-order Riccati equations, quadratic matrix inequalities or matrix pencils do not generally give directly formulas for controllers. Here basically finding controllers is achieved via regularization. The latter has to be done in the proper way as described in [33] due to the discontinuous dependence on the system parameters. Another approach is based on the relationship between H_∞ control problems and almost disturbance decoupling as established in [31]. Finding controllers that solve almost disturbance decoupling problems are studied in detail in [22, 23] using a so-called direct method which has been successful in particular for ill-posed problems.

A different story is the construction of controllers which achieve near-optimal performance. Optimal H_∞ controllers are proper while most methods construct strictly proper controllers. The closer one gets to the optimal performance the closer one gets to a pole-zero cancellation at infinity with the associated numerical difficulties. The main problem is the near singularity of $\gamma^2 I - PQ$ near the optimum. The inverse of this matrix appears in nearly all formulas for H_∞ controllers and therefore numerical problems arise. One possible solution is suggested in [10]. Here proper suboptimal controllers are used where the direct feedthrough term of the controller is designed to minimize this specific numerical difficulty. Another way is to use a descriptor description of the controller which does not require inversion of this matrix as described in [27].

References

[1] C. Oară, *Generalized Riccati theory: a Popov function approach*, PhD thesis, Polytechnic University Bucharest, Bucharest, Romania, 1995.

[2] L. Baramov and H. Kimura, “Nonstandard H_∞ control based on (J, J^0)-dissipative factorization”, Submitted for publication, 1999.

[3] S. Boyd, V. Balakrishnan, and P. Kabamba, “A bisection method for computing the H_∞-norm of a transfer matrix and related problems”, Math. Contr. Sign. & Syst., 2, 1989, pp. 207–219.

[4] S.P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*, no. 15 in Studies in Applied Mathematics, SIAM, 1994.

[5] N.A. Bruinsma and M. Steinbuch, “A fast algorithm to compute the H_∞ norm of a transfer matrix”, Syst. & Contr. Letters, 14, 1994, pp. 287–293.

[6] B.M. Chen, A. Saberi, and Y. Shamash, “A non-recursive method for solving the general discrete-time Riccati equation related to the H_∞ control problem”, Int. J. Robust & Nonlinear Control, 4(4), 1994, pp. 503–519.

[7] B.R. Copeland and M.G. Safonov, “A zero compensation approach to singular H^2 and H^∞ problems”, Int. J. Robust & Nonlinear Control, 5, 1995, pp. 71–106.

[8] J.C. Doyle, “Structured uncertainty in control system design”, in Proc. 24th CDC, Ft. Lauderdale, FL, 1985, pp. 260–265.

[9] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, “State space solutions to standard H_2 and H_∞ control problems”, IEEE Trans. Aut. Contr., 34(8), 1989, pp. 831–847.

[10] P. Gahinet, “Reliable computation of H_∞ central controllers near the optimum”, in Proc. ACC, Chicago, IL, 1992, pp. 738–742.

[11] P. Gahinet and P. Apkarian, “A linear matrix inequality approach to H_∞ control”, Int. J. Robust & Nonlinear Control, 4(4), 1994, pp. 421–448.

[12] P. Gahinet and A.J. Laub, “Numerically reliable computation of optimal performance in singular H_∞ optimal control”, SIAM J. Contr. & Opt., 35(5), 1997, pp. 1690–1710.

[13] P. Gahinet and A.A. Stoorvogel, “Continuity properties of the H_∞ optimal gain”, in Proc. European Control Conference, 1993, pp. 1171–1175.

[14] K. Glover and J. Doyle, “State-space formulaes for all stabilizing controllers that satisfy an H_∞ norm bound and relations to risk sensitivity”, Syst. & Contr. Letters, 11, 1988, pp. 167–172.

[15] M. Green, K. Glover, D.J.N. Limebeer, and J.C. Doyle, “A J-spectral factorization approach to H_∞ control”, SIAM J. Contr. & Opt., 28(6), 1990, pp. 1350–1371.

[16] M. Green and D.J.N. Limebeer, *Linear robust control*, Information and System Sciences Series, Prentice Hall, 1994.

[17] V. Ionescu and C. Oară, “Generalized discrete-time Riccati theory”, SIAM J. Contr. & Opt., 34(2), 1996, pp. 601–619.

[18] T. Iwasaki and R.E. Skelton, “All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas”, Automatica, 30(8), 1994, pp. 1307–1317.

[19] H. Kimura, “Conjugation, interpolation and model-matching in H_∞”, Int. J. Contr., 49, 1989, pp. 269–307.

[20] ——, *Chain-scattering approach to H_∞ control*, System & Control: Foundations and Appl., Birkhäuser, 1996.

[21] H. KWAKERNAAK, "Robust control and H_∞ optimization – tutorial paper", Automatica, 29(2), 1993, pp. 255–274.

[22] Z. LIN, A. SABERI, P. SANNUTI, AND Y. SHAMASH, "A direct method of constructing H_2 suboptimal controllers – continuous-time systems", J. Optim. Th. & Appl., 99(3), 1998, pp. 585–616.

[23] ——, "A direct method of constructing H_2 suboptimal controllers – discrete-time systems", J. Optim. Th. & Appl., 99(3), 1998, pp. 617–654.

[24] J.M. MACIEJOWSKI, *Multivariable feedback design*, Addison-Wesley, Reading, MA, 1989.

[25] J.B. MATSON, T. MITA, AND B.D.O. ANDERSON, "A complete and simple parameterization of controllers for a nonstandard H_∞ control problem", To appear in Automatica, 1997.

[26] R. SÁNCHEZ-PE NA AND M. SZNAIER, *Robust Systems Theory and Applications*, Wiley, 1998.

[27] M.G. SAFONOV, D.J. LIMEBEER, AND R.Y. CHIANG, "Simplifying the H_∞ theory via loop-shifting, matrix-pencil and descriptor concepts", Int. J. Contr., 50, 1989, pp. 2467–2488.

[28] C. SCHERER, "H_∞ control by state feedback and fast algorithms for the computation of optimal H_∞ norms", IEEE Trans. Aut. Contr., 35, 1990, pp. 1090–1099.

[29] ——, "H_∞ -optimization without assumptions on finite or infinite zeros", SIAM J. Contr. & Opt., 30(1), 1992, pp. 143–166.

[30] ——, "H_∞ control by state feedback for plants with zeros on the imaginary axis", SIAM J. Contr. & Opt., 30(1), 1992, pp. 123–142.

[31] A.A. STOORVOGEL, *The H_∞ control problem: a state space approach*, Prentice-Hall, Englewood Cliffs, 1992.

[32] ——, "The H_∞ control problem with zeros on the boundary of the stability domain", Int. J. Contr., 32(1), 1996, pp. 29–47.

[33] A.A. STOORVOGEL AND A. SABERI, "Continuity properties of solutions to H_2 and H_∞ Riccati equations", Syst. & Contr. Letters, 27(4), 1996, pp. 209–222.

[34] ——, "The discrete algebraic Riccati equation and linear matrix inequality", Lin. Alg. Appl., 274, 1998, pp. 317–365.

[35] A.A. STOORVOGEL, A. SABERI, AND B.M. CHEN, "The discrete-time H_∞ control problem with measurement feedback", Int. J. Robust & Nonlinear Control, 4(4), 1994, pp. 457–480.

[36] H.L. TRENTELMAN, "Families of linear-quadratic problems: continuity properties", IEEE Trans. Aut. Contr., 32, 1987, pp. 323–329.

[37] J.C. WILLEMS, *The analysis of feedback systems*, MIT Press, 1971.

[38] K. ZHOU, J.C. DOYLE, AND K. GLOVER, *Robust and optimal control*, Prentice Hall, Upper Saddle River, NJ, 1996.