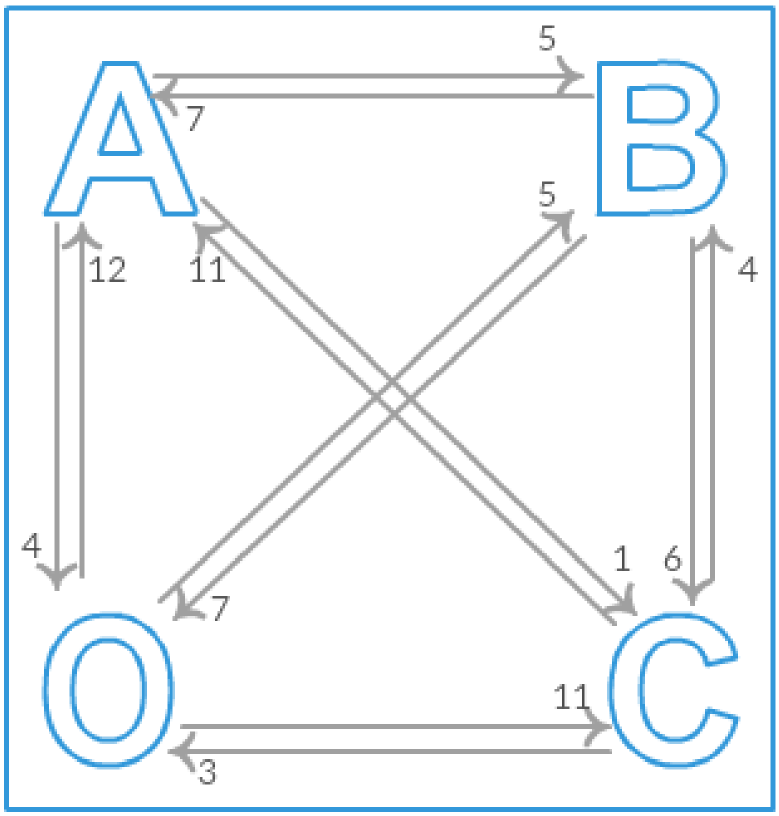
Discrete Mathematics

Homework 3

Samuel L. Peoples, CST, MA-C

July 3, 2016

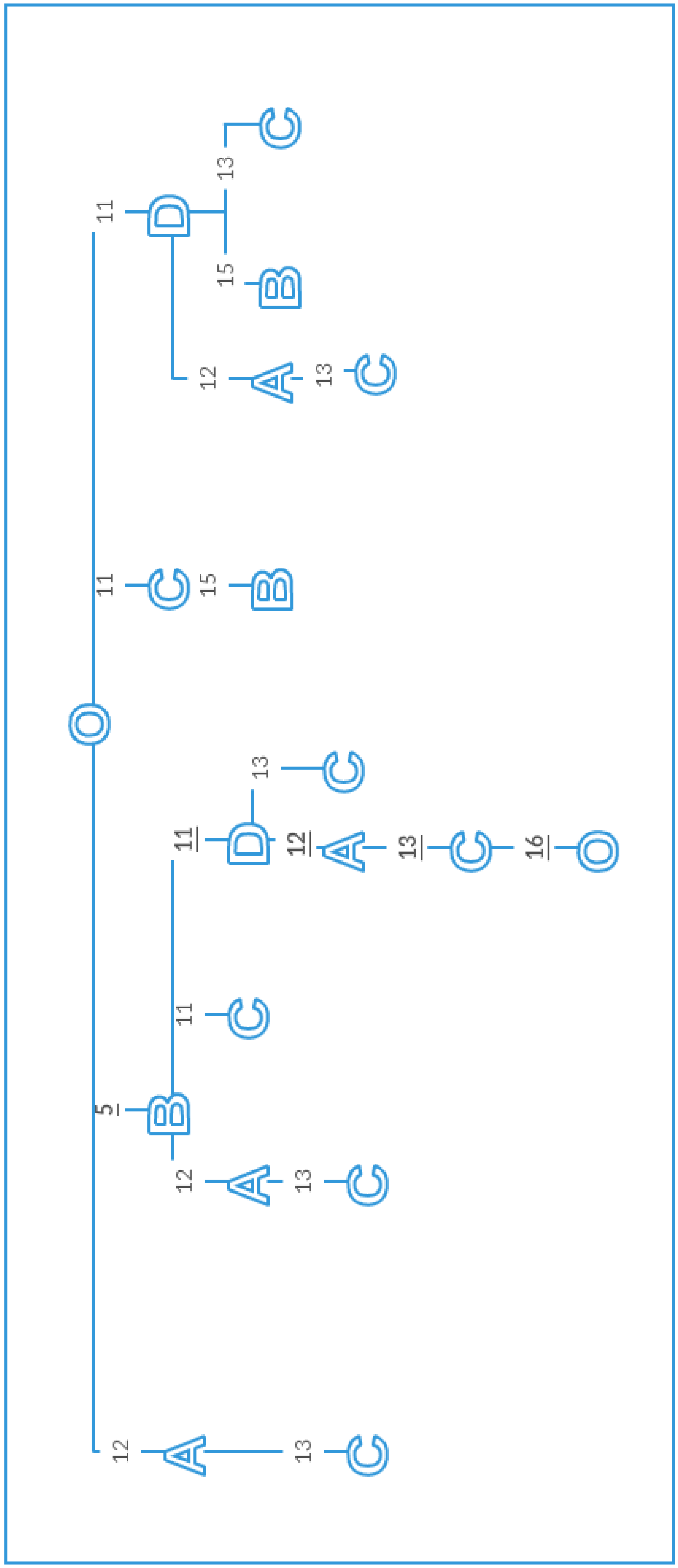
1. a. Draw the complete directed graph with vertices corresponding to O, A, B, and C (ignore D for now) and two edges between each pair of vertices, one in each direction. Label each with the distance (number of blocks) one must travel to get from one location to the other



Ignoring the vertex D, the most simple method for drawing the complete directed graph is above. Without considering the vertices passed through, the graph above merely describes the simple cost from one vertex to another.

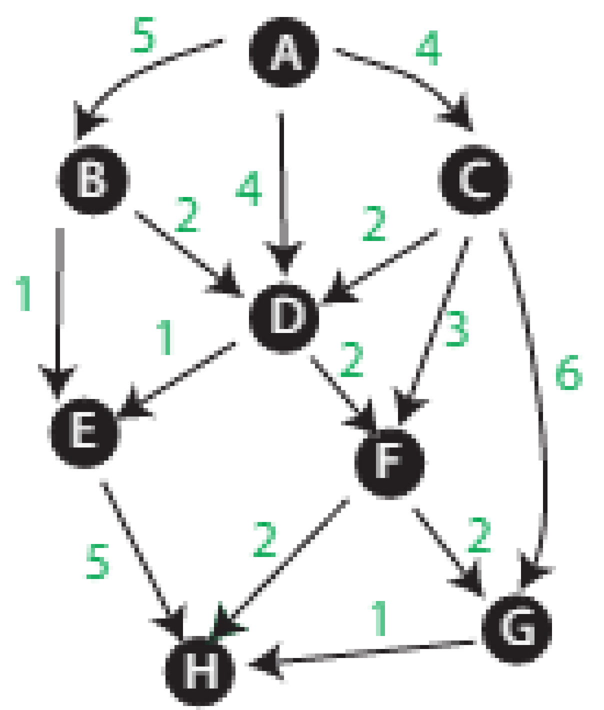
b. Suppose a delivery van leaving from O must visit A, B, C, and D and return to O. Use the branch and bound algorithm to find the best route. Show the portion of the tree that you constructed in finding this route, indicating the partial distances computed and indicating what parts of the tree you were able to prune off. (It is pretty obvious what the best route is in this case; I constructed it to make the branch and bound algorithm fairly simple to apply. You might want to experiment with similar problems where the answer isnt so obvious.)

Contstructing a greedy path, following the cheapest immediate path, yields a value of 16. This value is also the value of the most optimal path, as shown below. Each path terminates before it grows larger than the greedy path. Therefore, the cheapest path is OBDACO, with a value of 16.



11
D
13
C
15
B
12
A
13
C
11
C
15
B
O
13
C
11
D
12
A
13
C
16
O
11
C
5
B
12
A
13
C
12
A
13
C

2. Use Dijkstras algorithm to find the shortest path from A to H in the graph below. Show all work.

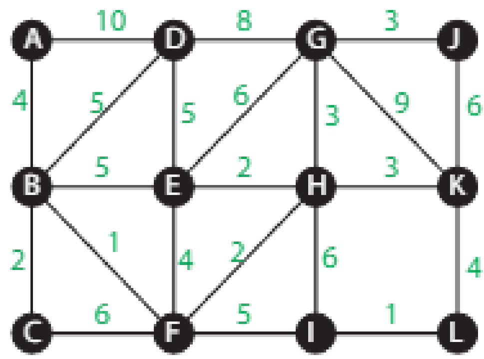


Using Dijkstra's algorithm, beginning with vertex A, there are three possible edges to follow. These edges lead to B, C, and D, respectively. In the table below, it can be seen that the shortest path is ADFH, with the weight of 8.

Node	Tentative Distance		Visted?	Predecessor
A	0			N/A
B	∞	5		A
C	∞	4		A
D	∞	4	x	A
E	∞	5		D
F	∞	6	x	D
G	∞	10		C
H	∞	8	x	F

Notice, the path to E is shorter from D than from B by 1 unit, so E has the predecessor of D. The same can be said for F, where the path through D has the value of 6, and through C has the value of 7. The path through G is greater than any other path with a value of 10, so it can be omitted. The shortest path to H from A is through D and F. The solution is a value of 8 through ADFH.

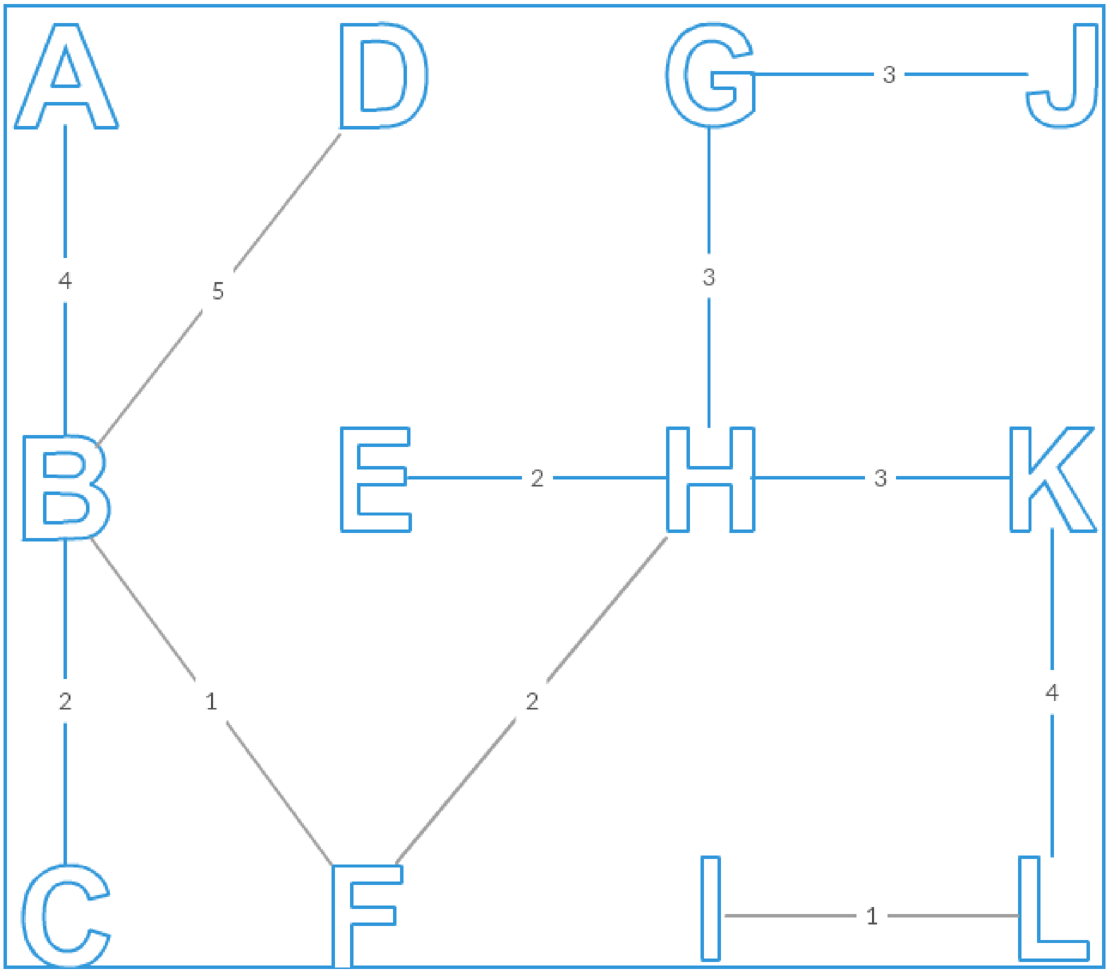
3. Use the table below to complete parts *a* and *b*.



a. Use Kruskals algorithm to find a minimal spanning tree in the graph above.

Using Kruskal's algorithm, one can list the edges in acending order and follow the cheapest paths until all the edges are complete. The table below maps the logic flow, as well as a diagram which highlights the completed edges.

Edge	Cost	Y/N
BF	1	Y
IL	1	Y
BC	2	Y
EH	2	Y
FH	2	Y
HK	3	Y
GH	3	Y
GJ	3	Y
AB	4	Y
EF	4	N
KL	4	Y
BD	5	Y - Done
BE	5	N/A
DE	5	N/A
FI	5	N/A
CF	6	N/A
EG	6	N/A
HI	6	N/A
JK	6	N/A
DG	8	N/A
GK	9	N/A
AD	10	N/A
COST	30	



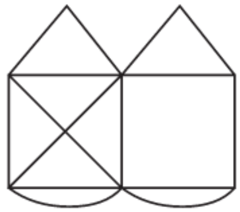
Notice that there are no circuits created as the paths are made, and that the total cost is 30.

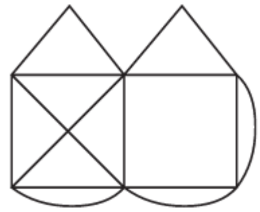
b. Use Prims algorithm to find a minimal spanning tree in the graph above.

Using Prim's algorithm, a random vertex is chosen, being A, naturally. The edge of minimal cost that has not been traversed is B, with a value of 4. The same can be highlighted through the whole path in the table below. The diagram from 3a. displays the same path followed from Kruskal's algorithm.

Starting Vertex	Minimal Cost Edge	Net Cost	Total Cost
A	AB	4	4
B	BF	1	5
F	FH	2	7
H	HE	2	9
B	BC	2	11
H	HK	3	14
K	KL	4	18
L	LI	1	19
H	HG	3	22
G	GJ	3	25
B	BD	5	30
TOTAL COST			30

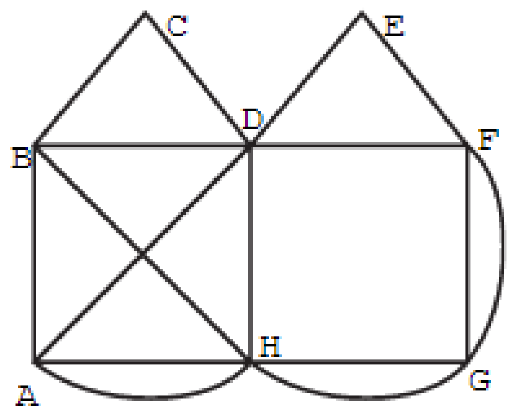
4. Of the graphs below, decide which contain an Eulerian closed chain and use the algorithm discussed in class to find the Eulerian closed chain.





As discussed in class, for a graph to contain an Eulerian closed chain, each vertex must contain an even number of edges. Notice that the graph on the left, at its most bottom right vertex and the vertex above and adjecent contain only three edges. Therefore, the graph on the left does not contain an Eularian closed chain.

After labeling the vertices in the diagram below, beginning with edge AB, the graph can be followed with the corresponding table. Although there are many solutions, this is one which is immediately obvious. Therefore, a possible Eularian closed chain beginning at A would consist of: ABCDEFGHGFDHADBHA.



Edge	Possible Paths				Chosen Path
AB	BC	BD	BH		BC
BC	CD				CD
CD	DE	DF	DH		DE
DE	EF				EF
EF	FG(Curved)	FG(Straight)	FD		FG(Curved)
FG(Curved)	GH(Curved)	GH(Straight)			GH(Curved)
GH(Curved)	HG	HD			HG
HG	GF				GF
GF	FD				FD
FD	DH	DA	DB		DH
DH	HA(Curved)	HA(Straight)	HB		HA(Curved)
HA(Curved)	AH	AD			AD
AD	DB				DB
DB	BH				BH
BH	HA				HA
HA	Complete				

5. Here is a puzzle that tormented me in high school. Referring to the figure below, the object of the game is to draw a line a continuous line that passes through each wall of the house (think blueprint) exactly once. Convert this puzzle to a graph theory problem in the following way: let the rooms of the house (including outside as one of the rooms) be represented by vertices of a graph and let the edges of this graph represent the walls. Two vertices of this graph are connected by an edge if the corresponding rooms share a wall. Decide if this puzzle has a solution, and if so, find it.

Viewing this problem as an Eularian closed chain, it is immediately obvious that there are multiple vertices which contain an odd number of edges. In order to traverse each edge once and only once, there must be an even number of edges along each vertex. This problem is known as Houdini's Last Trick, and has no solution. By traversing a possible vertex with an odd number of edges, the user is "backed into a corner" and immediately must abandon their attempt.

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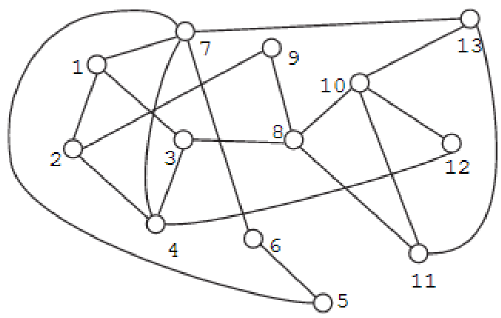
Discrete Mathematics

Homework 4
Samuel L. Peoples, CST, MA-C
July 17, 2016

1. Use the graph below to answer the following questions.

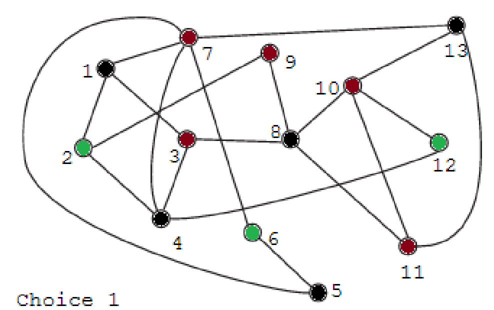


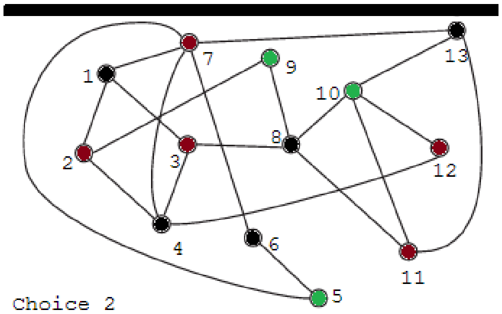
a. Based on the valences of the vertices alone (Wallis Theorem 7.2), what is an upper bound on the number of colors needed for a proper coloring of this graph.

   Based on the referenced theorem, notice that the number of colors required is less than or equal to the valence + 1. Vertex 7 has a valence of 5, an upper bound for the number of colors is 6.

b. Use the greedy algorithm two times (there are lots of ways you could order your choice of vertices as you go) to color this graph. For both ways, document the order in which you did it and what colors you used (you will want to label the nodes of the graph for this).

   From the following table, nodes were assigned colors in an increasing order, allowing for a chromatic number of three.

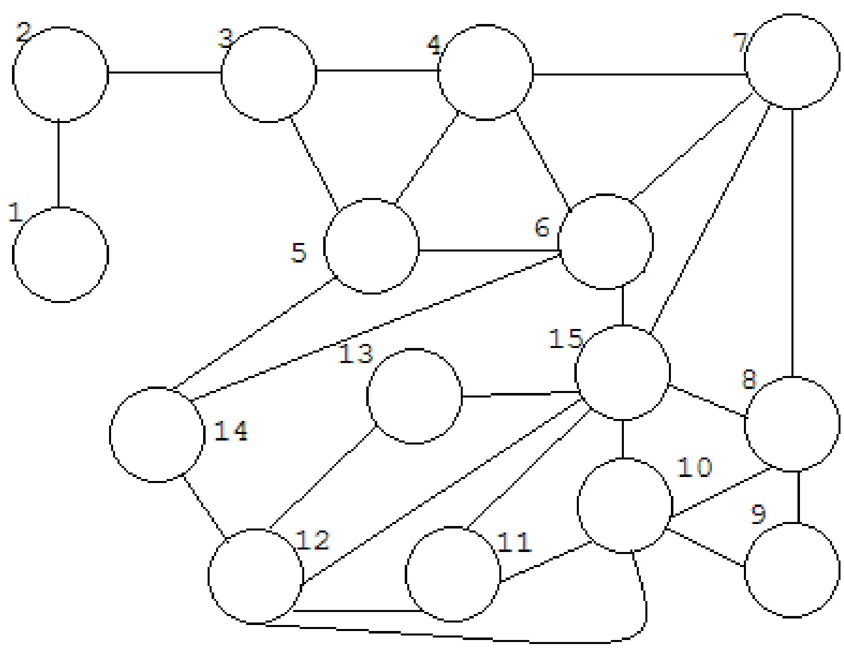




Ch 1	Color	Ch 2
13	Black	1
7	Red	2
10	Red	3
11	Red	7
1	Black	4
4	Black	6
5	Black	8
8	Black	13
3	Red	11
9	Red	12
2	Green	5
6	Green	9
12	Green	10

2.

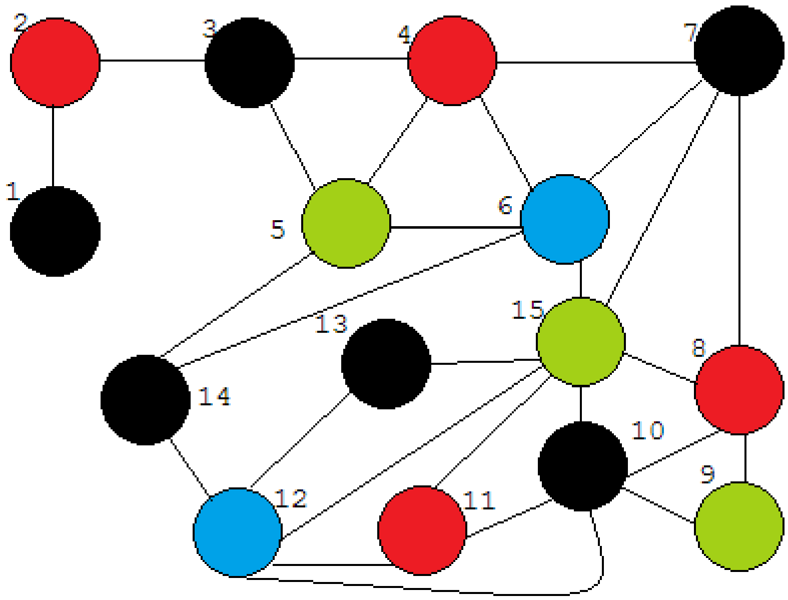
a. Create the planar graph corresponding to the blue shaded region of northern Mexico in the image above. In this graph, states are represented by vertices and two vertices are connected by an edge if they share a border.



b. Calculate the Euler characteristic of your graph. Did you get the number you expected to get. Explain.

By using the equation $V - E + F = 2$, one can observe that there exists 15 vertices, 27 edges, and 14 faces. $15 - 27 + 14 = 2$. This is expected under the fact that this is a planar finite graph.

c. Use the greedy algorithm to find a “good” coloring of the map of Mexico. See if you can find a proper 4-coloring (4 is the theoretical maximum number of colors required; can you do better than that?).



Ch	Clr
1	Blk
2	Red
3	Blk
4	Red
5	Grn
7	Blk
8	Red
6	Blu
15	Grn
10	Blk
9	Grn
11	Red
12	Blu
13	Blk
14	Blk

The graph and associated chart above displays a “good” coloring of the map of Mexico. The chromatic number is 4. Observe that it can be no less than that because there exist vertices which require at least four colors to to satisfy the problem.

3. a. Read the book chapter on Critical Path Analysis Then do #4 on page 195 from that chapter.

An extension is to be built to a sports hall. Details of the activities are given below.

	activity	time
A	lay foundations	7
B	build walls	10
C	lay drains and floor	15
D	install fittings	8
E	make and fit door frames	2
F	erect roof	5
G	plaster ceiling	2
H	fit and paint doors and windows	8
I	fit gutters and pipes	2
J	paint outside	3

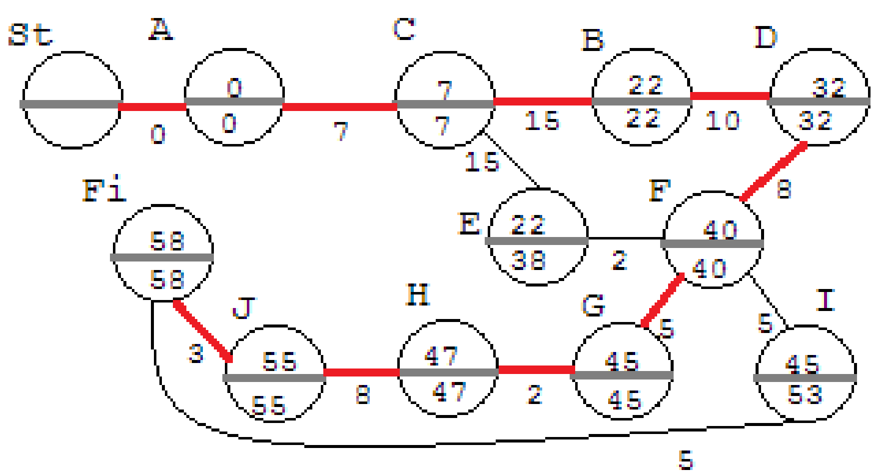
Some of these activities cannot be started until others have been completed:

B must be after C
C must be after A
D must be after B
E must be after C
F must be after D and E
G must be after F
H must be after G
I must be after F
J must be after H

Complete an activity network for this problem.

b. Find the critical path for the activity network you found in part *a*. What is the signifcance of this critical path?

The activity network from part *a* and its associated critical path are located below. The path marked in red is the critical path; this is the essential tasks that must be completed on time in order to meet the deadline.



c. There are two main methods for creating activity networks: Activity-On-Node and Activity-On-Edge. I mostly used the activity-on-node method in class. Read This Discussion Regarding Two Methods of Doing Activity Networks and give a brief summary.

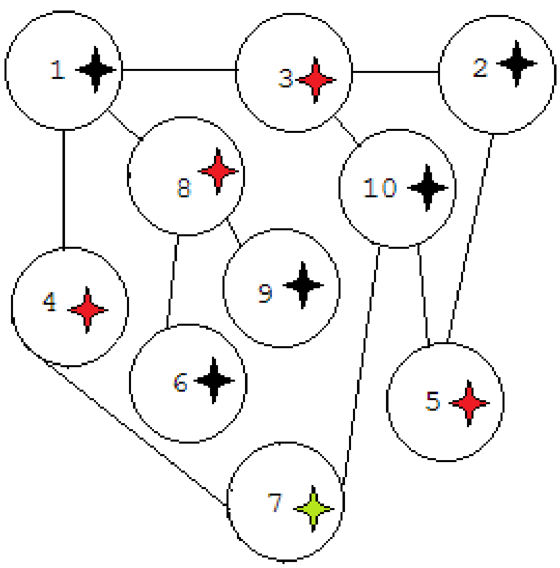
In the short article, the author relates the AoN and AoE methods by defining them as follows:

AoN – activities are represented by nodes and precedence relations by arcs

AoE – activities are represented by edges and the precedence relations are implicitly embedded in the network nodes

The main distinction between the two is the defining factors about the preceding element and succeeding element in the graph. For AoE activity netoworks, the edges are marked with sets of activities, rather than the preceding element. The article emphasized that the AoE method has more restrictions than AoN. The other main distinguishing feature of AoE when compared to AoN is the use of 'dummy arcs' which stand in the place of a true path.

4. Make up and solve a final exam scheduling problem (like the one presented in class) using 10 courses and conflicts of your own choosing (don't make it trivial, though).



	1	2	3	4	5	6	7	8	9	10
1			X	X				X		
2			X		X					
3	X	X								X
4	X						X			
5		X								X
6								X		
7				X						X
8	X					X			X	
9								X		
10			X		X		X			

Ch	1	3	8	4	2	10	9	6	5	7
Co										

Observe that, for the tables and figure above, there are multiple conflicts in the final examination schedule. Approaching the scenario as a graph theory and graph coloring problem allows a possible solution to become evident. With a chromatic number of 3, the final examinations can be administered during at least three distinct times.

For example, allow the classes colored with black to be administered on Monday, Red on Wednesday, and Green on Friday. This allows the students to never take more than one exam on a single day.

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Discrete Mathematics

Homework 5
Samuel L. Peoples, CST, MA-C
July 25, 2016

1. Using the defintion of eigenvector, verify that $\vec{x} = (1, -1, 2)^T$ is a (right) eigenvector of the matrix $A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{pmatrix}$ Without actually computing the eigenvalue coresponding to x, what must that eigenvalue be?

   Based on the definition of eigenvector, $Ax = \lambda x$ for some λ. Multiplying the vector $\vec{x}$ yields $\begin{pmatrix} -6 & -6 & -3 \end{pmatrix}$, which implies that the eigenvalue is $\lambda = 3$.

2. By hand, find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

   Based on the definition of eigenvalue, $|B - \lambda \cdot I| = 0$
   $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -\lambda \\ -\lambda & 2-\lambda \end{pmatrix}$
   $|\begin{pmatrix} 2-\lambda & -\lambda \\ -\lambda & 2-\lambda \end{pmatrix}| = \lambda^2 - 4\lambda + 3$
   Thus, $\lambda = \{1, 3\}$
   Observe $B - \lambda_1 \cdot \vec{v_1} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \cdot \vec{v_1} = 0$
   Thus, $\vec{v_1} = (-1, 1)$
   Observe $B - \lambda_2 \cdot \vec{v_2} = \begin{pmatrix} -1 & -4 \\ -4 & -1 \end{pmatrix} \cdot \vec{v_2} = 0$
   Thus, $\vec{v_2} = (1, 1)$

3. For the matrix in #1, use Mathematica to compute the following.

   (a) $A\vec{y}$ where $\vec{y}$ is the column vector $\vec{y} = (.25, .5, .25)^T$

   ```
   A = {{-2, -1, 0}, {0, 1, 1}, {-2, -2, -1}};
   y = {{.25, .5, .25}};
   A.Transpose[y]

   {{-1.}, {0.75}, {-1.75}}
   ```

   (b) A^5

   ```
   A = {{-2, -1, 0}, {0, 1, 1}, {-2, -2, -1}};
   MatrixPower[A, 5]

   {{-10, -1, 4}, {8, 1, -3}, {-18, -2, 7}}
   ```

   (c) $|A|$

   ```
   Det[A]

   0
   ```

   (d) Find the eigenvalues of A.

   ```
   Eigenvalues[A]

   {-1, -1, 0}
   ```

   (e) Find the eigenvectors of A.

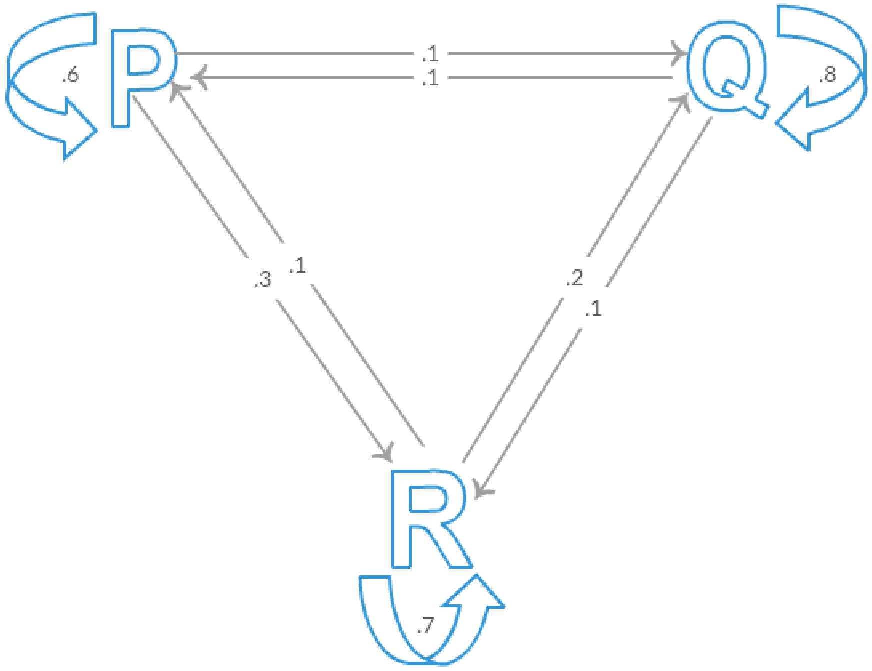
   ```
   Eigenvectors[A]

   {{1, -1, 2}, {0, 0, 0}, {1, -2, 2}}
   ```

4. An automobile rental company has three locations which we will call P, Q, and R. When an automobile is rented at one of these locations, it may be returned to any of these three locations. The movement of cars from location to location each week, depending on the location from which cars are rented, is captured in the following table.

Location P	60% stay at P	10% go to Q	30% go to R
Location Q	10% go to P	80% stay at Q	10% go to R
Location R	10% go to P	20% go to Q	70% stay at R

(a) Make a transition diagram that describes this Markov chain.



(b) Make a transition matrix P for this Markov chain.

$$P = \begin{pmatrix} .6 & .1 & .1 \\ .1 & .8 & .2 \\ .3 & .1 & .7 \end{pmatrix}$$

(c) Suppose that initially there are 100 cars at location P, 150 cars at location Q, and 200 cars at location R. After one week, how many cars will be at each location? Explain your solution in terms of matrix operations. You may use the computer to make your computations.

Using Mathematica, the vector $\vec{c} = (100, 150, 200)$ is multiplied by the matrix P, which is raised to the seventh power, to pass time for one week.

```
P = Transpose[{{.6, .1, .3}, {.1, .8, .1}, {.1, .2, .7}}];
c = {{100, 150, 200}};
MatrixForm[MatrixPower[P, 7].Transpose[c]]
```

$$\begin{pmatrix} 90.0781 \\ 200.829 \\ 159.093 \end{pmatrix}$$

(d) Suppose that, initially, a randomly selected car is equally likely to be at any of the three locations.

i. What is the initial probability vector $\vec{p}^{(0)}$?

Because there is equal probability, $\vec{p}^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

ii. What are the probabilities that this randomly selected car will be at each of the three locations after one week? How about after two weeks? Three weeks? As the number of weeks becomes very large? Again, you may use the computer.

```
P = Transpose[{{.6, .1, .3}, {.1, .8, .1}, {.1, .2, .7}}];
d = {{1/3, 1/3, 1/3}};
Print["One Week: " MatrixForm[MatrixPower[P, 7].Transpose[d]]]
Print["Two Weeks: " MatrixForm[MatrixPower[P, 14].Transpose[d]]]
Print["Three Weeks: " MatrixForm[MatrixPower[P, 21].Transpose[d]]]
Print["Many Many Weeks: " MatrixForm[MatrixPower[P, 1000].Transpose[d]]]
```

One Week: $\begin{pmatrix} 0.201042 \\ 0.444043 \\ 0.354915 \end{pmatrix}$

Two Weeks: $\begin{pmatrix} 0.200008 \\ 0.449812 \\ 0.35018 \end{pmatrix}$

Three Weeks: $\begin{pmatrix} 0.2 \\ 0.449995 \\ 0.350005 \end{pmatrix}$

Many Many Weeks: $\begin{pmatrix} 0.2 \\ 0.45 \\ 0.35 \end{pmatrix}$

(e) Repeat part (d), but this time assume that the initial randomly selected car has probability 0.25 of being in location P, probability 0.5 of being in location Q, and probability 0.25 of being in location R. What did you discover?

```
P = Transpose[{{.6, .1, .3}, {.1, .8, .1}, {.1, .2, .7}}];
d = {{.25, .5, .25}};
Print["One Week: " MatrixForm[MatrixPower[P, 7].Transpose[d]]]
Print["Two Weeks: " MatrixForm[MatrixPower[P, 14].Transpose[d]]]
Print["Three Weeks: " MatrixForm[MatrixPower[P, 21].Transpose[d]]]
Print["Many Many Weeks: " MatrixForm[MatrixPower[P, 1000].Transpose[d]]]
```

One Week: $\begin{pmatrix} 0.200391 \\ 0.450391 \\ 0.349219 \end{pmatrix}$

Two Weeks: $\begin{pmatrix} 0.200003 \\ 0.450003 \\ 0.349994 \end{pmatrix}$

Three Weeks: $\begin{pmatrix} 0.2 \\ 0.45 \\ 0.35 \end{pmatrix}$

Many Many Weeks: $\begin{pmatrix} 0.2 \\ 0.45 \\ 0.35 \end{pmatrix}$

In the figure above, it is evident that the final probability approaches $(.2, .45, .35)$ as time moves forward to infinity. The steady state vector is independent of the initial probability vector. The only influence it has on the steady state is the amount of time it takes to reach steady state.

(f) By experimenting on the computer, notice that P^n converges (as n gets large) to a matrix whose columns are all identical stochastic vectors. This common column vector is called a *steady state vector*. Explain how this happens for this particular Markov chain in terms of eigenvalues and eigenvectors.

Because one of the eigenvalues is 1, the corresponding eigenvector to this dominant eigenvalue is remarkably close to the steady state vector. In this instance, the eigenvector $(-0.331042, -0.744845, -0.579324)$ is equivalent to the steady state vector, divided by $-.604$, a scalar multiple.

(g) What does the steady state vector say about the long term behavior of the Markov chain when started with any initial probability vector $\vec{p}^{(0)}$?

The transient behavior will be affected by the initial probability vector, but the steady state vector will not be changed. Moreover, given any initial probability vector, the first few 'days' of the Markov Chain will be slightly different, but will eventually approach the same steady state vector.

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Discrete Mathematics

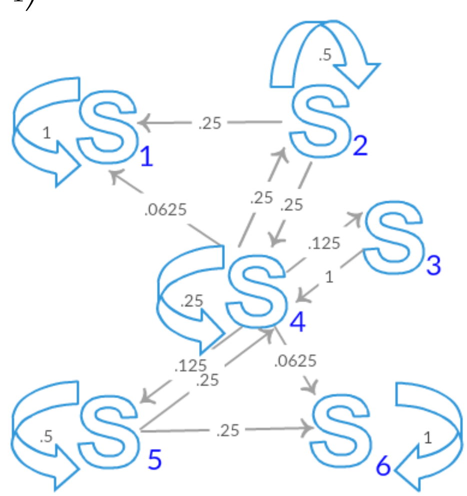
Homework 6
Samuel L. Peoples, CST, MA-C
October 4, 2016

1. We start with two animals of opposite sex, mate them, select two of their offspring of opposite sex , and mate those, and so forth. To simplify the example, we will assume that the trait under consideration is independent of sex. Here a state is determined by a pair of animals. Hence, the states of our process will be: $s_1 = (GG; GG), s_2 = (GG; Gg), s_3 = (GG; gg), s_4 = (Gg; Gg), s_5 = (Gg; gg)$, and $s_6 = (gg; gg)$. We illustrate the calculation of transition probabilities in terms of the state s_4. When the process is in this state, both parents have Gg genes. Each parent contributes one if its two genes to create the offspring. So, for example, a $(Gg; Gg)$ mating could yield outcomes GG, Gg, and gg with probabilities .25, .5, and .25, respectively. Transitioning to the next state happens by a selection of two offspring of opposite sexes. Thus, the probability of transition from state s_4 to state s_1 is (.25)(.25), to state s_2 is 2(.25)(.5), to state s_3 is 2(.25)(.25), to state s_4 is (.5)(.5), to state s_5 is 2(.5)(.25), and to state s_6 is (.25)(.25).

   (a) After you understand how the transition probabilities for s_4 were computed above, compute the other transition probabilities and create a transition diagram and a transition matrix for the Markov chain.

   In the following matrix, the probabilities in each column correspond to the probabilities of each pair transitioning to each other state. For example, s_1 transitions to itself with a probability of 1, while s_2 transitions to s_1 with a probability of .25, s_2 with a probability of .5, and s_4 with a probability of .25.

$$\begin{pmatrix} 1 & .25 & 0 & .0625 & 0 & 0 \\ 0 & .5 & 0 & .25 & 0 & 0 \\ 0 & 0 & 0 & .125 & 0 & 0 \\ 0 & .25 & 1 & .25 & .25 & 0 \\ 0 & 0 & 0 & .25 & .5 & 0 \\ 0 & 0 & 0 & .0625 & .25 & 1 \end{pmatrix}$$



   (b) Identify the transient states and the absorbing states and write the canonical form of the transition matrix from (a).

   The transient states are: s_2, s_3, s_4,and s_5, while the absorbing states are s_1, and s_6. Thus, the layout of the matrix below is $s_2, s_3, s_4, s_5, s_1, s_6$.

$$\begin{pmatrix} .5 & 0 & .25 & 0 & 0 & 0 \\ 0 & 0 & .125 & 0 & 0 & 0 \\ .25 & 1 & .25 & .25 & 0 & 0 \\ 0 & 0 & .25 & .5 & 0 & 0 \\ .25 & 0 & .0625 & 0 & 1 & 0 \\ 0 & 0 & .0625 & .25 & 0 & 1 \end{pmatrix}$$

(c) What is the probability, regardless of the starting state, that the process will eventually be absorbed?

Based on the canonical graph above, it can be observed that, after much time has passed, the transient states will eventually absorb into s_1 or s_6. The probability of absorbtion is 1.

```
A = {{.5, 0, .25, 0, 0, 0}, {0, 0, .125, 0, 0, 0}, {.25, 1, .25, .25, 0, 0},
   {0, 0, .25, .5, 0, 0}, {.25, 0, .0625, 0, 1, 0}, {0, 0, .0625, .25, 0, 1}};
MatrixForm[MatrixPower[A, 999 999 999 999]]
```

$$\begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.75 & 0.5 & 0.5 & 0.25 & 1. & 0. \\ 0.25 & 0.5 & 0.5 & 0.75 & 0. & 1. \end{pmatrix}$$

(d) If the starting state is s_3, how many times should we expect the process to be in state s_4 before absorbing?

In the following Mathematica code, it can be observed that, based on the canoncial matrix, Q is defined as such. Observe $N = (I - Q)^{-1}$. Thus, the process will be in state s_4 between two and three times before being absorbed. Observe that, in this instance, $J = I$.

```
A = {{.5, 0, .25, 0, 0, 0}, {0, 0, .125, 0, 0, 0}, {.25, 1, .25, .25, 0, 0}, {0, 0, .25, .5, 0, 0},
   {.25, 0, .0625, 0, 1, 0}, {0, 0, .0625, .25, 0, 1}};
J = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Q = {{.5, 0, .25, 0}, {0, 0, .125, 0}, {.25, 1, .25, .25}, {0, 0, .25, .5}};
s3 = Transpose[{{0, 0, 0, 1}}];
MatrixForm[Inverse[J - Q]]
```

$$\begin{pmatrix} 2.66667 & 1.33333 & 1.33333 & 0.666667 \\ 0.166667 & 1.33333 & 0.333333 & 0.166667 \\ 1.33333 & 2.66667 & 2.66667 & 1.33333 \\ 0.666667 & 1.33333 & 1.33333 & 2.66667 \end{pmatrix}$$

(e) If the starting state is equally likely to be any one of the four transient states, how many times should we expect the chain to be in each of transient states before absorbing? Explain.

Based on the probability of .25 for each state, the dot product of N, here denoted as Nm yields the number of times each of the transient states are visited before being absorbed. s_2 is visited 1.5 times, s_3 is visited .5 times, s_4 is visited twice, and s_5 is visited 1.5 times.

```
Nm = Inverse[J - Q];
p = Transpose[{{.25, .25, .25, .25}}];
MatrixForm[Nm.p]
```

$$\begin{pmatrix} 1.5 \\ 0.5 \\ 2. \\ 1.5 \end{pmatrix}$$

(f) For each transient state, how many steps should we expect it to take before the process is absorbed?

Based on N, the sum of the columns yields the following for s_2, s_3, s_4, and s_5

```
Nm = Inverse[J - Q];
MatrixForm[Total[Nm, {1}]]
```

$$\begin{pmatrix} 4.83333 \\ 6.66667 \\ 5.66667 \\ 4.83333 \end{pmatrix}$$

(g) For each of the transient states s_i, if s_i is the starting state, what is the probability of absorbing into each of the absorbing states?

Because absorbtion probabilities are defined as $B = RN$, the following mathematica code yields a probability of .75 for s_2 to absorb into s_1 and .25 to absorb into s_6; a probability of .5 for s_3 to absorb into s_1 or s_6; a probability of .5 for s_4 to absorb into s_1 or s_6; and a probability of .25 for s_5 to absorb into s_1 and .75 to absorb into s_6;

```
R = {{.25, 0, .0625, 0}, {0, 0, .0625, .25}};
Nm = Inverse[J - Q];
MatrixForm[R.Nm]
```

$$\begin{pmatrix} 0.75 & 0.5 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.5 & 0.75 \end{pmatrix}$$

(h) If the starting state is equally likely to be any one of the four transient states, what is the probability of absorbing into each of the absorbing states?

Based on the mathematica code below, the product of the vector $\vec{p}$ and B, denoted as $R.Nm$, yields an equal probability of .5 of absorbing into s_1 or s_6 from the transient states.

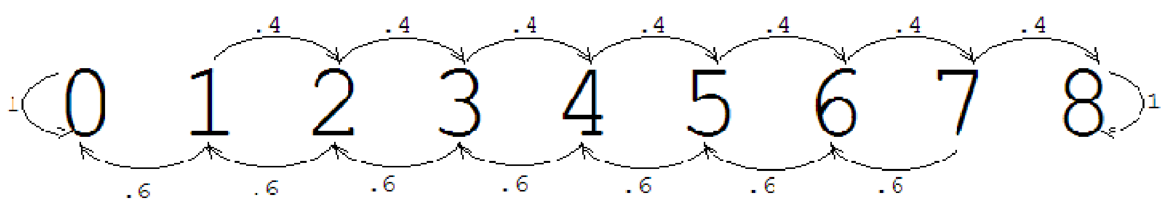
```
R = {{.25, 0, .0625, 0}, {0, 0, .0625, .25}};
Nm = Inverse[J - Q];
p = Transpose[{{.25, .25, .25, .25}}];
MatrixForm[(R.Nm).p]
```

$$\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

2. Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability .4 and loses A dollars with probability .6. Find the probability that he wins 8 dollars before losing all of his money if:

(a) he bets 1 dollar each time (timid strategy).

Based on the information given, the following diagram and matrix can be constructed. Finding the fundamental matrix, N, here denoted as Nm, it can be observed that there is a probability of .0964314 of winning 8 dollars before losing all his money.




```
A = {{1, .6, 0, 0, 0, 0, 0, 0, 0}, {0, 0, .6, 0, 0, 0, 0, 0, 0}, {0, .4, 0, .6, 0, 0, 0, 0, 0},
   {0, 0, .4, 0, .6, 0, 0, 0, 0}, {0, 0, 0, .4, 0, .6, 0, 0, 0}, {0, 0, 0, 0, .4, 0, .6, 0, 0},
   {0, 0, 0, 0, 0, .4, 0, .6, 0}, {0, 0, 0, 0, 0, 0, .4, 0, 0}, {0, 0, 0, 0, 0, 0, 0, .4, 1}};
MatrixForm[A]
```

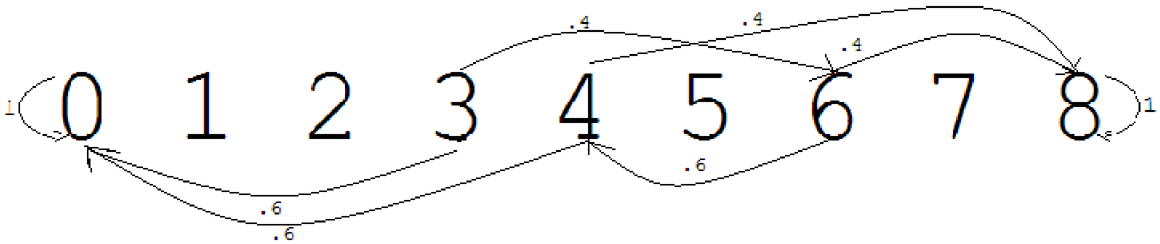
$$\begin{pmatrix} 1 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 1 \end{pmatrix}$$

```
R = {{.6, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, .4}};
Nm = Inverse[J - Q];
MatrixForm[R.Nm]
```

$$\begin{pmatrix} 0.979699 & 0.949247 & 0.903569 & 0.835052 & 0.732276 & 0.578113 & 0.346868 \\ 0.0203013 & 0.0507534 & 0.0964314 & 0.164948 & 0.267724 & 0.421887 & 0.653132 \end{pmatrix}$$

(b) He bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy).

Changing the way the bets are made, by betting the maximum amount possible, the following diagram and matrix can be constructed. Following the same process as above, there is a probability of .256 of winning 8 dollars before losing all his money.



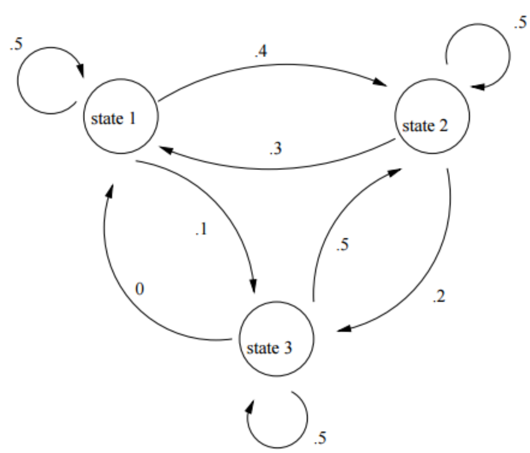

```
A = {{1, 0, 0, .6, .6, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, .6, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, .4, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, .4, 0, .4, 0, 1}};

J = {{1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1}};
Q = {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, .6, 0},
    {0, 0, 0, 0, 0, 0, 0}, {0, 0, .4, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}};
R = {{0, 0, .6, .6, 0, 0, 0}, {0, 0, 0, .4, 0, .4, 0}};
Nm = Inverse[J - Q];
MatrixForm[R.Nm]
```

$$\begin{pmatrix} 0. & 0. & 0.744 & 0.6 & 0. & 0.36 & 0. \\ 0. & 0. & 0.256 & 0.4 & 0. & 0.64 & 0. \end{pmatrix}$$

(c) Which strategy gives Smith the better chance of getting out of jail?

The best strategy for Smith to get out of jail is, by far, the latter. By betting the maximum amount, Smith has nearly a 26% chance of getting out, while betting the minimum will almost certainly leave him with empty pockets.

3. The fire danger during the summer in Mount Baker National Forest is classified into one of three danger levels. These are 1 =low, 2 =moderate, 3 =high. The probability of daily transitions between these states is given by the following flow diagram:



(a) Write the model in matrix form to project the fire danger probability from one day to the next.

$$A = \begin{pmatrix} .5 & .3 & 0 \\ .4 & .5 & .5 \\ .1 & .2 & .5 \end{pmatrix}$$

(b) If we are in State 1 today, what is the probability that we will be in State 2 the day after tomorrow?

Based on the following mathematica code, there is a probability of .45 after two days have passed.

```
A = {{.5, .3, 0}, {.4, .5, .5}, {.1, .2, .5}};
MatrixForm[MatrixPower[A, 2]]
```

$$\begin{pmatrix} 0.37 & 0.3 & 0.15 \\ 0.45 & 0.47 & 0.5 \\ 0.18 & 0.23 & 0.35 \end{pmatrix}$$

(c) If the matrix you found is correct, then it has eigenvalues and eigenvectors given by:

$$\Lambda = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & .0697 & 0 \\ 0 & 0 & .4303 \end{pmatrix}$$

$$\mathbb{R} = \begin{pmatrix} -.4699 & -.5551 & -.7801 \\ -.7832 & .7961 & .1813 \\ -.4072 & -.2410 & .5988 \end{pmatrix}$$

Based on these, what is the equilibrium probability of being in each state?

Based on the dominant eigenvector, the equation $-.4699x - .7832x - .4072x = 1$ can be solved for x, where $x = -.60230078$. Multiplying the vector by x yields $(.283, .472, .245)^T$, where the equilibrium probability of being in state 1 is .283, 2 is .472, and 3 is .245.

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