STMATH 409 – Problem-set 1
Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [10 points] Let z be a nonzero complex number. Use mathematical induction to prove that

$$(z^{-1})^n = (z^n)^{-1}, \tag{1}$$

for every $n \in \mathbb{N}$.

Proof. For the base case, suppose $n = 1$; observe $(z^{-1})^1 = z^{-1} = (z^1)^{-1}$.
For the inductive step, suppose $(z^{-1})^k = (z^k)^{-1}$, so
$(z^{-1})^{k+1} = (z^{-1})^k z^{-1} = (z^k)^{-1} z^{-1} = (z^k z)^{-1} = (z^{k+1})^{-1}$
Therefore $(z^{-1})^n = (z^n)^{-1} \forall n \in \mathbb{N}$, by induction. □

(2) [10 points] Let

$$p(t) = \sum_{k=0}^{n} a_k t^k,$$

and suppose that $a_k \in \mathbb{R}$, $0 \leq k \leq n$. If z is a complex number such that $p(z) = 0$, prove that $p(\overline{z}) = 0$.

Proof. Let $p(z) = \displaystyle\sum_{k=0}^{n} a_k z^k = 0$; and notice $p(\overline{z}) = \sum_{k=0}^{n} a_k \overline{z}^k$
$= \displaystyle\sum_{k=0}^{n} a_k \overline{z^k} = \sum_{k=0}^{n} \overline{a_k z^k} = \overline{\sum_{k=0}^{n} a_k z^k} = \overline{0} = 0$
Therefore, $p(z) = 0 \Rightarrow p(\overline{z}) = 0$ □

(3) [10 points] Let W be a nonempty subset of a vector space V. Prove that W satisfies

$$u + v \in W, \ \forall u, v \in W \tag{2}$$

and

$$cu \in W, \ \forall c \in \mathbb{F}, \ \forall u \in W \tag{3}$$

if and only if W satisfies

$$cu + dv \in W, \ \forall u, v \in W, c, d \in \mathbb{F}. \tag{4}$$

Proof. ($\Rightarrow$) Suppose $u + v \in W$, and $cu \in W, \forall u, v \in W, \forall c \in \mathbb{F}$.
So $cu + dv \in W, \forall c, d \in \mathbb{F}$
($\Leftarrow$) Suppose $cu + dv \in W, \forall u, v \in W, c, d \in \mathbb{F}$, and $u + v \notin W$.
Notice for $1 = c = d \in \mathbb{F}, cu + dv = (1)u + 1(v) = u + v \in W\#$, by contradiction.
Suppose $cu \notin W$, and $d = 0 \in \mathbb{F}$.
Notice $cu + dv = cu + 0v = cu \in W\#$, by contradiction.
Therefore, $u + v \in W$, and $cu \in W, \forall u, v \in W, c \in \mathbb{F}$ iff $cu + dv \in W \forall u, v \in W$, and $c, d \in \mathbb{F}$. □

STMATH 409 – Problem-set 3

Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [10 points] If $p(t) = \sum_{k=0}^{n} a_k t^k$ and $q(t) = \sum_{k=0}^{n} b_k t^k$, prove that $p = q$ if and only if $a_k = b_k$, for every $k \in \{0, 1, \ldots, n\}$.

Proof. Suppose $p = q$, thus $\displaystyle\sum_{k=0}^{n} a_k t^k = \sum_{k=0}^{n} b_k t^k \Rightarrow \sum_{k=0}^{n} a_k t^k - \sum_{k=0}^{n} b_k t^k = \sum_{k=0}^{n} (a_k - b_k) t^k = 0.$
Notice $(a_0 - b_0) + (a_1 - b_1)t + ... + (a_n - b_n)t^n = 0$, and $a_k - b_k = 0 \Rightarrow a_k = b_k, \forall k \in \{0, 1, ..., n\}$.
For sufficiency, assume $a_k = b_k$, so $p(t) = \displaystyle\sum_{k=0}^{n} a_k t^k = \sum_{k=0}^{n} b_k t^k = q(t)$.
Therefore $p = q$ iff $a_k = b_k$ $\square$

(2) [6 points] If $S = \{u\} \subset V$, prove that S is linearly dependent if and only if $u = 0$.

Proof. Recall Linear Dependence implies $\exists c \neq 0$ such that $cu = 0$, thus, $u = 0$.
For sufficiency, observe solutions to $S = \{0\}$ are of the form $c \cdot 0 = 0$, so S is LD. $\square$

(3) [6 points] If $S = \{u, v\} \subset V$, prove that S is linearly dependent if and only if $u = cv$, $c \in \mathbb{F}$.

Proof. Assume $\{u, v\}$ are LD, so $\exists a, b \neq 0 \in \mathbb{F}$ such that $au + bv = 0$
Thus $au = bv$, and for $\dfrac{b}{a} = c, u = cv$. For sufficiency, $u = cv, c \in \mathbb{F}$, $(-1)u + cv = 0$,
so for some value c, S is LD. $\square$

(4) [10 points] Recall from the lecture notes that if $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for a vector space V and $x \in V$, then there are unique scalars $a_1, \ldots, a_n$ such that $x = \sum_{k=1}^{n} a_k b_k$. If $f : V \longrightarrow \mathbb{F}^n$, where

$$x = \sum_{k=1}^{n} a_k b_k \longmapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n, \qquad \text{(Note: the symbol `}\longmapsto\text{' reads as ``maps to'')}$$

prove that f is a bijection.

Proof. Suppose $f(a) = f(a'); a, a' \in V$. So $f(a) - f(a') = \displaystyle\sum_{k=1}^{n} (a - a')_k b_k = 0$, and $a = a'$, from (1); thus f is injective. For Surjectivity, notice $x = \displaystyle\sum_{k=1}^{n} x_k b_k$ where $x_1 b_1 + x_2 b_2 + ... + x_n b_n$ forms a basis vector for V, which spans $\mathbb{F}$, so f is bijective. $\square$

STMATH 409 – Problem-set 6
Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [6 points] Let $A \in M_n$, where $a_i \in M_{1\times n}$ denotes the i^{th}-row of A. Use the definition of determinant to prove that

$$\begin{vmatrix} cx+y \\ a_2 \\ \vdots \\ a_n \end{vmatrix} = c\begin{vmatrix} x \\ a_2 \\ \vdots \\ a_n \end{vmatrix} + \begin{vmatrix} y \\ a_2 \\ \vdots \\ a_n \end{vmatrix},$$

for every $x, y \in \mathbb{F}^n$ and for every $c \in \mathbb{F}$.

Proof. Recall the definition of the determinant provides

$$|A| = \sum_{j=1}^{n}(-1)^{i+j}(cx+y)_{1j}\begin{vmatrix} e_j \\ a_2 \\ \vdots \\ a_n \end{vmatrix} = \sum_{j=1}^{n}(-1)^{i+j}\left[c(x)_{1j}\begin{vmatrix} e_j \\ a_2 \\ \vdots \\ a_n \end{vmatrix} + (y)_{1j}\begin{vmatrix} e_j \\ a_2 \\ \vdots \\ a_n \end{vmatrix}\right]$$

$$= c\cdot\sum_{j=1}^{n}(-1)^{i+j}(x)_{1j}\begin{vmatrix} e_j \\ a_2 \\ \vdots \\ a_n \end{vmatrix} + \sum_{j=1}^{n}(-1)^{i+j}(y)_{1j}\begin{vmatrix} e_j \\ a_2 \\ \vdots \\ a_n \end{vmatrix} = c\begin{vmatrix} x \\ a_2 \\ \cdots \\ a_n \end{vmatrix} + \begin{vmatrix} y \\ a_2 \\ \cdots \\ a_n \end{vmatrix}.$$

Therefore the determinant of a matrix $A \in M_n$ is a linear function of each row when the remaining rows are held fixed, where $\begin{vmatrix} cx+y \\ a_2 \\ \vdots \\ a_n \end{vmatrix} = c\begin{vmatrix} x \\ a_2 \\ \vdots \\ a_n \end{vmatrix} + \begin{vmatrix} y \\ a_2 \\ \vdots \\ a_n \end{vmatrix}, \forall x, y \in \mathbb{F}^n, c \in \mathbb{F}$. ☐

(2) Let $A \in M_{m,n}(\mathbb{F})$. Denote by A_{ij} the $(m-1)$-by-$(n-1)$ matrix obtained by deleting the ith-row and j-th column of A. For ease of notation, denote by $a_{ij}^{\top}$ the (i,j)-entry of $A^{\top}$.

(a) [6 points] Prove that $(A_{ij})^{\top} = (A^{\top})_{ji}$.

Proof. Observe that A can be represented as the $m - by - n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{m(j-1)} & a_{mj} & a_{m(j+1)} & \cdots & a_{mn} \end{bmatrix}.$$

Notice the matrix obtained by deleting the $i-th$ row and $j-th$ column can be represented as the $(m-1)-by-(n-1)$ matrix

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m(j-1)} & a_{m(j+1)} & \dots & a_{mn} \end{bmatrix},$$

where the definition of transpose provides that $\forall a_{ij} \in A \in M_{n,m}(\mathbb{F}), a_{ij} = a_{ji} \in A^T \in M_{m,n}(\mathbb{F})$, where

$$(A_{ij})^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{(j-1)1} & a_{(j+1)1} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{(j-1)2} & a_{(j+1)2} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1(i-1)} & a_{2(i-1)} & \dots & a_{(j-1)(i-1)} & a_{(j+1)(i-1)} & \dots & a_{n(i-1)} \\ a_{1(i+1)} & a_{2(i+1)} & \dots & a_{(j-1)(i+1)} & a_{(j+1)(i+1)} & \dots & a_{n(i+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{(j-1)m} & a_{(j+1)m} & \dots & a_{nm} \end{bmatrix}.$$

Similarly, observe that the transpose of A can be represented by the $n-by-m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{(j-1)1} & a_{j1} & a_{(j+1)1} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{(j-1)2} & a_{j2} & a_{(j+1)2} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1(i-1)} & a_{2(i-1)} & \dots & a_{(j-1)(i-1)} & a_{j(i-1)} & a_{(j+1)(i-1)} & \dots & a_{n(i-1)} \\ a_{1i} & a_{2i} & \dots & a_{(j-1)i} & a_{ji} & a_{(j+1)i} & \dots & a_{ni} \\ a_{1(i+1)} & a_{2(i+1)} & \dots & a_{(j-1)(i+1)} & a_{j(i+1)} & a_{(j+1)(i+1)} & \dots & a_{n(i+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{(j-1)m} & a_{jm} & a_{(j+1)m} & \dots & a_{nm} \end{bmatrix},$$

where the matrix obtained by removing $j-th$ row and $i-th$ column of A^T provides

$$(A^T)_{ji} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{(j-1)1} & a_{(j+1)1} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{(j-1)2} & a_{(j+1)2} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1(i-1)} & a_{2(i-1)} & \dots & a_{(j-1)(i-1)} & a_{(j+1)(i-1)} & \dots & a_{n(i-1)} \\ a_{1(i+1)} & a_{2(i+1)} & \dots & a_{(j-1)(i+1)} & a_{(j+1)(i+1)} & \dots & a_{n(i+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{(j-1)m} & a_{(j+1)m} & \dots & a_{nm} \end{bmatrix} = (A_{ij})^T.$$

□

(b) [12 points] Use part (a) and induction to prove that $\det(A) = \det(A^\top)$ for every n-by-n matrix A.

Proof. Suppose $|A| = 0$. Recall A is not invertible, and thus A^T is not invertible, so $0 = |A| = |A^T| = 0$.

If $|A| \neq 0$, suppose for the base case $n = 1$, observe $A = A^T$, and $|A| = |A^T|$. For the inductive step, suppose $A \in M_k$ where

$$|A| = \sum_{j=1}^{k} (-1)^{i+j} a_{1j} |\tilde{A}_{1j}|$$

$$= a_{11}|\tilde{A_{11}}| - a_{12}|\tilde{A_{12}}| + \ldots + (-1)^{1+k} a_{1k}|\tilde{A_{1k}}|,$$

and $\tilde{A_{ij}}$ denotes the $(k-1)-by-(k-1)$ matrix obtained by removing the $i-th$ row and $j-th$ column. Similarly, we can expand $|A^T|$ as

$$|A^T| = a_{11}|(\tilde{A}^T)_{11}| - a_{12}|(\tilde{A}^T)_{21}| + \ldots + (-1)^{1+k} a_{1k}|(\tilde{A}^T)_{k1}|$$

$$= a_{11}|(\tilde{A}_{11})| - a_{12}|(\tilde{A}_{12})| + \ldots + (-1)^{1+k} a_{1k}|(\tilde{A}_{1k})| = |A|.$$

Notice if $A \in M_{k+1}$

$$|A| = \sum_{j=1}^{k+1} (-1)^{i+j} a_{1j} |\tilde{A}_{1j}|$$

$$= a_{11}|\tilde{A_{11}}| - a_{12}|\tilde{A_{12}}| + \ldots + (-1)^{1+k} a_{1k}|\tilde{A}_{1k}| + (-1)^{1+(k+1)} a_{1(k+1)}|\tilde{A_{1(k+1)}}|,$$

and $\tilde{A_{ij}}$ denotes the $(k)-by-(k)$ matrix obtained by removing the $i-th$ row and $j-th$ column. Similarly, we can expand $|A^T|$ as

$$|A^T| = a_{11}|(\tilde{A}^T)_{11}| - a_{12}|(\tilde{A}^T)_{21}| + \ldots + (-1)^{1+k} a_{1k}|(\tilde{A}^T)_{k1}| + (-1)^{1+(k+1)} a_{1(k+1)}|(\tilde{A}^T)_{(k+1)1}|$$

$$= a_{11}|(\tilde{A}_{11})| - a_{12}|(\tilde{A}_{12})| + \ldots + (-1)^{1+k} a_{1k}|(\tilde{A}_{1k})| + (-1)^{1+(k+1)} a_{1(k+1)}|(\tilde{A}_{1(k+1)})| = |A|.$$

Therefore, $|A| = |A^T|, \forall A \in M_n$. □

(3) [6 points] If $A \in M_n$ and $c \in \mathbb{F}$, prove that $|cA| = c^n|A|$

Proof. For the base case, suppose $n = 1$, where $|cA| = ca = c|A|$. For the inductive step, suppose $A \in M_k$, where $\tilde{A}_{ij} \in M_{k-1}$; thus

$$|cA| = \sum_{j=1}^{k} (-1)^{i+j} (cA_{1j}) |c\tilde{A}_{1j}| = \sum_{j=1}^{k} (-1)^{i+j} c(A_{1j}) c^{k-1} |\tilde{A}_{1j}| = c^k \sum_{j=1}^{k} (-1)^{i+j} (A_{1j}) |\tilde{A}_{1j}| = c^k |A|.$$

For $A \in M_{k+1}$, notice

$$|cA| = \sum_{j=1}^{k+1} (-1)^{i+j} (cA_{1j}) |c\tilde{A}_{1j}| = \sum_{j=1}^{k+1} (-1)^{i+j} c(A_{1j}) c^{(k+1)-1} |\tilde{A}_{1j}| = c^{k+1} \sum_{j=1}^{k+1} (-1)^{i+j} (A_{1j}) |\tilde{A}_{1j}| = c^{k+1} |A|.$$

Therefore $|cA| = c^n|A|, \forall A \in M_n, c \in \mathbb{F}$. □

STMATH 409 – Problem-set 7
Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [8 points] Let $A \in \mathsf{M}_n$ and suppose that $\lambda \in \mathbb{F}$. Prove that $E_\lambda = \mathsf{N}(\lambda I_n - A)$.

Proof. Because λ is an eigenvalue where $E_\lambda \neq 0$, and $\exists v \in E_\lambda$ such that

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0, v \neq 0 \Leftrightarrow (A - \lambda I_n)v = 0, v \neq 0 \Leftrightarrow v \in N(A - \lambda I_n), v \neq 0.$$

□

(2) [8 points] Recall that if $A, B \in \mathsf{M}_n$, then A *is similar to* B, denoted $A \sim B$, if there is an invertible matrix S such that $A = SBS^{-1}$. Prove that similarity is an equivalence relation on M_n.

Proof. Recall the definition of similar is $A, B \in M_n, A \sim B \rightarrow A = SBS^{-1}$, where S is an invertible matrix.
Observe that $I_n A I_n^{-1} = A$, so similarities are reflexive.
If $A \sim B$, then $A = SBS^{-1} \Leftrightarrow S^{-1}AS = S^{-1}SBS^{-1}S = B \Leftrightarrow B \sim A$, so similarities are symmetric.
If $A \sim B, B \sim C$, observe $A = SBS^{-1}, B = TCT^{-1}$, where $T \in M_n$ is an invertible matrix.
Observe $A = STCT^{-1}S^{-1} = (ST)C(T^{-1}S^{-1}) = (ST)C(ST)^{-1} \Leftrightarrow A \sim C$, and similarities are transitive.
Therefore, similarities are an equivalence relation on M_n □

(3) [8 points] If $S = \{v_1, \ldots, v_k\} \subset V$ $(k \in \mathbb{N})$ is linearly independent and $T \subseteq S$, prove that T is linearly independent.

Proof. Observe for $T = \{S\}, T$ is vacuously linearly independent. Because S is linearly independent, $\sum_{i=1}^{k} a_i v_i = 0, a_i \in \mathbb{F}$, not all zero. Suppose T is a nonempty subest of S, where without loss of generality, $t_1 = v_1, t_2 = v_2, ..., t_l = v_l, l < k$, and $t_k = t_{k-1} = ... = t_{l+1} = 0$. Notice $\sum_{i=1}^{l} b_i t_i = \sum_{i=1}^{k} a_i v_i = 0, a_i, b_i \in \mathbb{F}$, not all zero. So every subset of a linearly independent set is linearly independent. □

(4) [8 points] If

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathsf{M}_n,$$

prove that $\dim(\mathsf{N}(J - \lambda I_n)) = 1$

Proof. Suppose $\Delta = J - \lambda I_n$, where

$$\Delta = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix},$$

Where $\text{rank}(N(\Delta)) = n - 1$, $N(\Delta) = \{x \in N(\Delta) \mid \Delta x = 0\} = \{0_n\}$, and $\dim(N(\Delta)) = 1$. □

(5) [8 points] If

$$A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix} \in \mathsf{M}_n,$$

where $B \in \mathsf{M}_k$ and $D \in \mathsf{M}_{n-k}$, prove that $p_A = p_B p_D$. *Hint*: proceed by induction on n and expand the determinant across the first row.

Proof. Notice for $n = 1$, the result is vacuously true. For the inductive hypothesis, suppose the result holds for $n = m$, where $p_A =$

$$\sum_{j=1}^{m}(-1)^{i+j}a_{1j} \mid \tilde{A}_{1k} \mid = \sum_{j=1}^{k}(-1)^{i+j}a_{1j} \mid \tilde{B}_{1j} \mid\mid D \mid = \sum_{j=1}^{k}(-1)^{i+j}a_{1j} \mid \tilde{B}_{1j} \mid \sum_{j=n}^{k+m}(-1)^{i+j}d_{1j} \mid \tilde{D}_{1j} \mid = p_B p_D$$

For $A \in M_{m+1}$, notice $p_A =$

$$\sum_{j=1}^{m+1}(-1)^{i+j}a_{1j} \mid \tilde{A}_{1k} \mid = \sum_{j=1}^{k}(-1)^{i+j}a_{1j} \mid \tilde{B}_{1j} \mid\mid D \mid = \sum_{j=1}^{k}(-1)^{i+j}a_{1j} \mid \tilde{B}_{1j} \mid \sum_{j=n}^{k+m+1}(-1)^{i+j}d_{1j} \mid \tilde{D}_{1j} \mid = p_B p_D$$

$\square$

STMATH 409 – Extra-credit Problem-set
Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [15 points] Let $P = [p_{ij}] \in \mathsf{M}_n(\{0,1\})$ (i.e., $p_{ij} = 0$ or $p_{ij} = 1$) and suppose that

$$\sum_{i=1}^{n} p_{ij} = 1, \; i = 1, \ldots, n$$

and

$$\sum_{j=1}^{n} p_{ij} = 1, \; j = 1, \ldots, n.$$

Prove that $P^\top P = I_n$.

Proof. Recall the definition of Transpose provides $P_{ij} = P^T_{ji}$. Observe

$$(P^T P)_{ij} = \sum_{k=1}^{n} P^T_{ik} P_{kj} = \sum_{k=1}^{n} P_{ki} P_{kj}$$

Notice $P_{ki}P_{kj} = 0, \forall i \neq j$, because each column only contains one "1", and is zero elsewhere; so $P_{ki}P_{kj} = P^T_{ik}P_{kj} = 1, \forall i = j$, where

$$(P^T P)_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow P^T P = I_n, \text{ by definition.}$$

$\square$