STMATH 493 – Problem-set 1

Name: Samuel L. Peoples - 1527650 **Collaborator(s):** First M. Last

(1) [10 points] Let $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a function. Prove that T is linear if and only if

$$T(cx + dy) = cT(x) + dT(y) \tag{1}$$

for every pair $x, y \in \mathbb{R}^m$ and for every $c, d \in \mathbb{R}$.

Proof. Let $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, where $x, y \in \mathbb{R}^m, c, d \in \mathbb{R}$. Notice $T(cx + dy) = T(cx) + T(dy)$ because $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, and $x, y \in \mathbb{R}^m$. Because $c, d \in \mathbb{R}$, being scalar, $T(cx) + T(dy) = cT(x) + dT(y)$. Therefore, T is linear. □

(2) [5 points] If $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear transformation, prove that $T(0) = 0$. Hint: recall that $0x = 0$ for every $x \in \mathbb{R}^m$.

Proof. Let $x \in \mathbb{R}^m$. Then $T(0) = T(0x)$, and from the previous problem, we can conclude $T(0x) = 0T(x) = 0$. □

(3) [10 points] Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \ \forall n \in \mathbb{N}. \tag{2}$$

Proof. Suppose $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \ \forall n \in \mathbb{N}$.
For the base case, let $n = 1$, so $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2\cdot1+1)}{6} = \frac{6}{6} = 1^2$
For the inductive step, we assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
For $k + 1$, notice $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2$
$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$
$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$
$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$
Therefore $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \ \forall n \in \mathbb{N}$. □

(4) [5 points] Using only a for-loop, write MATLAB that computes $\sum_{i=1}^{n} i^2$ (do not use the formula in Problem (3)).

[*Solution*] The required MATLAB code is giving by

```
%Calculating the sum of squares from 1->n%

n = 5 %Set upper bound%
sol=0; %Initial value of 0^2%
for k = 1:n
  sol = sol + k.^2 %calculate and print solution%
end
--
```

```
Output:
n = 5
sol = 1         %1^2 = 1%
sol = 5         %1 + 2^2 = 5%
sol = 14        %5 + 3^2 = 14%
sol = 30        %14+ 4^2 = 30$
sol = 55        %30+ 5^2 = 55$
etc...
```

STMATH 493 – Problem-set 2

Name: Samuel L. Peoples - 1527650 **Collaborator(s):** First M. Last

(1) [10 points] If $G \in M_n(\mathbb{R})$ is triangular, prove that

$$\det G = \prod_{i=1}^{n} g_{ii}.$$

Proof. Because G is triangular, suppose

$$G = \begin{bmatrix} g_{11} & 0 & 0 & ... & 0 \\ g_{21} & g_{22} & 0 & ... & 0 \\ g_{31} & g_{32} & g_{33} & ... & 0 \\ & . & . & . & \\ g_{n1} & g_{n2} & g_{n3} & ... & g_{nn} \end{bmatrix}$$

Where G is lower triangular, without loss of generality.
Then det(G)= g_{11}det(M_{11}) $-g_{12}$det(M_{12}) $+ ... + g_{1n}$det(M_{1n})= $g_{11}M_{11} - 0 + ... + 0$,
Where M_{ij} is the $(n-1) \times (n-1)$ ij minor of G.
It follows that det(G)= g_{11}det(M_{11}), where

$$M_{11} = \begin{bmatrix} g_{22} & 0 & 0 & ... & 0 \\ g_{32} & g_{33} & 0 & ... & 0 \\ g_{42} & g_{43} & g_{44} & ... & 0 \\ & . & . & . & \\ g_{n2} & g_{n3} & g_{n4} & ... & g_{nn} \end{bmatrix}$$

So det(M_{11}) $= g_{22}$det(M_{22}), so det(G)= $g_{11}g_{22}$ det(M_{22}).
Continuing this process displays that det(G)= $g_{11}g_{22}g_{33}g_{44}...g_{nn}$
Therefore

$$\det G = \prod_{i=1}^{n} g_{ii}.$$

□

(2) [5 points] If $A \in M_n(\mathbb{R})$ and I denotes the identity matrix, prove that $AI = IA = A$.

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ a_{31} & a_{32} & a_{33} & ... & a_{3n} \\ & . & . & . & \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nn} \end{bmatrix} \quad I = \begin{bmatrix} 1_{11} & 0 & 0 & ... & 0 \\ 0 & 1_{22} & 0 & ... & 0 \\ 0 & 0 & 1_{33} & ... & 0 \\ & . & . & . & \\ 0 & 0 & 0 & ... & 1_{nn} \end{bmatrix}$$

So

$$AI = \begin{bmatrix} a_{11}1_{11} + a_{12}0 + ... + 0 & a_{11}0 + a_{12}1_{22} + a_{13}0 + ... + 0 & ... & a_{11}0 + ...a_{1n}1_{nn} \\ a_{21}1_{11} + a_{22}0 + ... + 0 & a_{22}1_{22} + 0 & ... & a_{2n}1_{nn} + 0 \\ a_{31}1_{11} + 0 & a_{32}1_{22} + 0 & ... & a_{3n}1_{nn} + 0 \\ ... & ... & ... & ... \\ a_{n1}1_{11} + 0 & a_{n2}1_{22} + 0 & ... & a_{nn}1_{nn} + 0 \end{bmatrix}$$

$$
= \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ a_{31} & a_{32} & a_{33} & ... & a_{3n} \\ & . & . & . & \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nn} \end{bmatrix} = A
$$

And

$$
IA = \begin{bmatrix} 1_{11}a11 + 0a_{21} + ...0 & 1_{11}a_{12} + 0a_{22} + ...0 & ... & 1_{11}a_{1n} + 0a_{2n} + ...0 \\ 0a_{11} + 1_{22}a_{21} + 0a_{31} + ...0 & 0a_{11} + 1_{22}a_{22} + 0a_{32} + ...0 & ... & 0a_{1n} + 1_{22}a_{2n} + 0a_{3n} + ...0 \\ ... & & & \\ & ... & & \\ & & ... & 0a_{1n} + 0a_{2n} + ... + 1_{nn}a_{nn} \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ a_{31} & a_{32} & a_{33} & ... & a_{3n} \\ & . & . & . & \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nn} \end{bmatrix} = A
$$

Therefore $AI = IA = A$ □

(3) [3 points] Write MATLAB code that generates a random upper-triangular matrix U of size n.

Solution. The required MATLAB code is given by

```
function triangle(n) %Generate function for size n triangular matrix%
U = zeros(n); %Generate a matrix of size n zeros%
for i = 1:n
    for j= 1:n
        if i<=j %When i is less than j, assign a random integer (1:9) to the matrix%
            U(i,j) = U(i,j) = randi(9);
        end
    end
end
U %Print U$
--
>> triangle(5)

U =

     5     1     4     2     8
     0     3     5     2     6
     0     0     3     6     7
     0     0     0     7     5
     0     0     0     0     1
```

□

(4) [6 points] Recall from the lecture-notes that if $Ux = y$, and $U \in M_n(\mathbb{R})$ is upper-triangular, then

$$
x_i = u_{ii}^{-1} \left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right), \; i \in \{1, \dots, n\}, \tag{1}
$$

with the convention that the sum is zero when the lower limit of summation exceeds the upper limit. Using (1), write MATLAB code in the spirit of the pseudo-code (1.3.5) on page 26 for *row-oriented back substitution.*

Solution. The required MATLAB code is given by

```
function triangley(n)
U = zeros(n);
b = randi(9,[n,1]);
x = zeros(n,1);
for i = 1:n
    for j= 1:n
        if i<=j
            U(i,j) = randi(9);
        end
    end
end
b
for i = 1:n-1
    for j = 1:n-1
        b(i)= b(i) - U(i,j) * b(j);
    end
    if U(i,i)==0
        error('Pivot(%d,%d)=0',i,i);
    end
end
b(n) = b(n) / U(n,n);
%I had trouble with this one!%
```

□

STMATH 493 – Problem-set 3
Name: Samuel L Peoples, 1527650 **Collaborator(s):** First M. Last

(1) [10 points] Recall that a matrix B is *row-equivalent* to A, denoted as $B \sim A$, if there are finitely-many elementary matrices $E_1, \ldots, E_r$ $(r \geq 1)$ such that $B = \prod_{i=1}^{r} E_{r-i+1} A$. Prove that $\sim$ is an *equivalence relation*.

Proof. Recall that an equivalence relation requires that it is symmetric, reflexive, and trasitive. Notice $B = \prod_{i=1}^{r} E_{r-i+1} A = E_1 ... E_r A$, So $A = E_r^{-1} ... E_1^{-1} B$. Because $E_i^{-1}, i \in [1, r]$ is an elementary matrix, if A can be row reduced to B, the converse is true as well. Thus symmetry holds.
Because every matrix is row equivalent to itself, there exists $E_1 E_1^{-1} ... E_r E_r^{-1} A = A$. So reflexivity holds.
Suppose $B \sim C$ such that $E'_1 ... E'_r B = C$. Then $E_1 ... E_r A = E_r'^{-1} ... E_1^{-1} C$
$\Rightarrow$E'$_1 ... E'_r E_1 ... E_r A = C$. Thus if $A \sim B, B \sim C$ then $A \sim C$ and transitivity holds.
Therefore, $\sim$ is an equivalence relation. □

(2) [10 points] Prove that the product of two lower triangular matrices is lower triangular.

Proof. Suppose A, B are lower triangular matrices, where if $i < j$, $A_{ij} = B_{ij} = 0$.
Then $[AB]_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$. When $i < k$, $A_{ik} = 0$, so $[AB]_{ij} = 0$.
The same can be said when $k < j$, where $B_{kj} = 0$ so $[AB]_{ij} = 0$.
Because $i < k < j$, and $[AB]_{ij} = 0$ when $i < j$, AB is lower triangular. □

(3) [15 points] Write a MATLAB function, called `gewop`, that solves the linear system $Ax = b$ (with $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$) via *Gaussian elimination without pivoting.* Your code should compute the LU decomposition of A, where the matrices L and U are stored over A. Furthermore, your code should solve the system by solving the lower-triangular system $Ly = b$ (via row-oriented forward substitution [this code was generated in class]) and then solving the upper-triangular system $Ux = y$ (via row-oriented back substitution [Problem (4) from PS2]). Please write your code below and please submit a MATLAB m-file.

Solution. The required MATLAB code is given by

```
function x = gewop(n)

A = randn(n);
b = randn(n,1);
L = eye(n); %Assigns 1s to the diagonal
U = zeros(n);


%LU Decomposition of A%
for k = 1 : n
    % Create L from its zero matrix
    L(k + 1 : n, k) = A(k + 1 : n, k) / A(k, k);
```

```
    % Create U from A
    for l = k + 1 : n
        A(l, :) = A(l, :) - L(l, k) * A(k, :);
    end
end
U = A; %Set U to A%

%ROFS
for i=1:n
    for j=1:i-1
        b(i)=b(i)-L(i,j)*b(j);
    end
    b(i)=b(i)/L(i,i);
end
y = b;

%ROBS
for i=1:n
    y(i)=y(i)/U(i,i);
    for j=1:i-1
        y(i)=y(i)-U(i,j)*y(j)/U(i,i);
    end
end
x = y;
```

□

STMath 493 – Problem-Set 6

Name: Samuel L. Peoples **Collaborator(s):** First M. Last

(1) [10 points] Let $Q \in M_n(\mathbb{R})$ and suppose that $\langle Qx, Qy\rangle = \langle x, y\rangle$ for every x, $y \in \mathbb{R}^n$. Prove that Q is orthogonal.

Proof. We know that $Q^TQ = I$ if Q is orthogonal, and that $\langle x, y\rangle = y^Tx$.
So $\langle x, y\rangle = \langle Qx, Qy\rangle = (Qy)^T(Qx) = y^TQ^TQx = y^Tx$.
Thus, $Q^TQ = I$, and Q is orthogonal. □

(2) [5 points] If Q is an orthogonal matrix, prove that $\|Q\|_2 = \|Q^{-1}\|_2 = 1$.

Proof. We know that $Q^T = Q^{-1}$ and $\|Qx\|_2 = \|x\|_2 = \|Q\|_2\|x\|_2$
So $\|Q\|_2 = 1$. Because Q^T is orthogonal, $\|Q^Tx\|_2 = \|x\|_2$ and $\|Q^T\|_2 = \|Q^{-1}\|_2 = 1$. □

(3) Let $u \in \mathbb{R}^n$ and suppose that $\|u\|_2 = 1$. If $P = uu^\top$ and $Q = I - 2uu^\top$, prove that

(i) [3 points] $P^2 = P$.

Proof. Notice $P^2 = (uu^T)(uu^T) = (uu^Tu)u^T = uu^T = P$ □

(ii) [3 points] $P^T = P$.

Proof. Notice $P^T = (uu^T)^T = (u^T)^T(u^T) = uu^T = P$ □

(iii) [5 points] $Q^T = Q$.

Proof. Notice $Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T = I - 2(uu^T)^T = I - 2uu^T = Q$ □

(iv) [4 points] $Q^2 = I$.

Proof. Notice $Q^2 = (I - 2uu^T)^2 = (I - 2uu^T)(I - 2uu^T) = I^2 - 4uu^TI + 4uu^Tuu^T$
$= I - 4P + 4P = I$ □

(4) [10 points] Let $u \in \mathbb{R}^n$ and suppose that $\|u\|_2 = 1$. If $\mathcal{H}(u) := \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\}$, prove that

$$\alpha v + \beta w \in \mathcal{H}(u)$$

for every v, $w \in \mathcal{H}(u)$ and α, $\beta \in \mathbb{R}$.

Proof. Recall when $\langle u, v\rangle = 0$, u and v are orthogonal.
We also know that $\langle u, \alpha v + \beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$.
Because $v, w \in \mathcal{H}(u)$, we know $\alpha\langle u, v\rangle + \beta\langle u, w\rangle = \alpha \cdot 0 + \beta \cdot 0 = 0$
Thus, $\forall \alpha, \beta \in \mathbb{R}$, and $v, w \in \mathcal{H}(u)$, $\alpha v + \beta w \in \mathcal{H}(u)$ □