Exercise sheet No. 2

Quantum Simulators

Initial state and Time-evolution

Prof. Karem Rodríguez

1. What potentials whold be required to prepare each of the following (unnormalized) state functions as ground state? (a) $\psi(r) = \exp(-\alpha r^2)$; (b) $\psi(r) = \exp(-\alpha r)$; (c) $\psi(x) = 1/\cosh(x)$. Cases (a) and (b) are three-dimensional. Case (c) is one-dimensional. What restrictions, if any, must be imposed on the assumed ground state energy E, in order that the potentials be physically reasonable?

2. Consider the Hamiltonian,

$$\hat{H} = \hat{\sigma}_z + \Gamma\hat{\sigma}_x := \begin{pmatrix} 1 & \Gamma \\ \Gamma & -1 \end{pmatrix}.$$

If we start the dynamics from the up-spin state

$$\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the spin precesses around the axis of the magnetic field $\vec{B} = (\Gamma, 0, 1)$ with the period $T = \pi/\sqrt{1+\Gamma^2}$.

(a) Calculate the dynamics exactly. Diagonalize the Hamiltonian and use its eigenbasis to solve the problem.

(b) Use the Suzuki-Trotter approximant for the time-evolution operator,

$$\hat{U}(t+\Delta t; t) \approx e^{-i\Delta t\hat{\sigma}_z} e^{-i\Delta t\Gamma\hat{\sigma}_x}.$$

(c) Finally, use the perturbational approximant,

$$\hat{U}(t+\Delta t; t) \approx \hat{I} - i\Delta t\hat{H} = \hat{I} - i\Delta t(\hat{\sigma}_z + \Gamma\hat{\sigma}_x).$$

The exact dynamics should conserve the energy expectation $\langle\hat{H}\rangle$. Show three graphs of the energy as a function of time for the three different solutions. Discuss about the necessity of having unitary operators in Physics, and how this fact is reflected in your calculations?

3. Consider the chaotic dynamics of the system

$$K(\vec{p}) = \frac{1}{2}\left(p_1^2 + p_2^2\right) \quad \text{and} \quad V(\vec{q}) = \frac{1}{2}q_1^2 q_2^2.$$

Show in coordinate space (q_1, q_2) that the system is confined in the area surrounded by four hyperbolas given by $|q_1 q_2|$ =constant (consider the constant to be 2). Plot the movement of the system in the (q_1, q_2) space with the initial conditions: $p_1 = p_2 = 0, q_1 = 2$ and $q_2 = 1$ with the energy $E = 2$. Calculate the energy as a function of time using the Trotter approximation and the perturbational approximant and make the respective plots. Guide yourself on the paper we had been working in the Lecture: Finding Exponential Product Formulas of Higher Orders, Naomichi Hatano and Masuo Suzuki.