Derivation of Neumann Boundary Condition for Linear Elasticity

Introduction

This document explains the derivation of the Neumann boundary condition (traction boundary condition) for a 2D linear elasticity problem with:

- Bottom boundary fixed (Dirichlet condition).
- Traction applied on the top boundary.
- Using the stress-strain relation and strain definitions.

Step 1: Write down the traction boundary condition

On the top boundary, the traction vector is given by:

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

with normal vector:

$$\mathbf{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The traction components are:

$$t_x = \sigma_{xy}, \quad t_y = \sigma_{yy}$$

Step 2: Express stress tensor $\boldsymbol{\sigma}$ in terms of displacement $\mathbf{u} = (u_x, u_y)$

The stress-strain relation is:

$$\boldsymbol{\sigma} = \lambda \, \mathrm{tr}(\boldsymbol{\varepsilon}) \, \mathbf{I} + 2\mu \, \boldsymbol{\varepsilon}$$

where strain tensor $\boldsymbol{\varepsilon}$ is the symmetric part of the displacement gradient:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Component-wise:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

The trace is:

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{xx} + \varepsilon_{yy} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$$

Step 3: Write σ_{yy} and σ_{xy}

$$\sigma_{yy} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}) + 2\mu\varepsilon_{yy} = \lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + 2\mu \frac{\partial u_y}{\partial y}$$

$$\sigma_{xy} = 2\mu\varepsilon_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

Step 4: Neumann boundary condition on top boundary $(y = L)$

The traction vector on the top boundary $y = L$ is:

$$\mathbf{t}(x, L) = \begin{bmatrix} \sigma_{xy} \\ \sigma_{yy} \end{bmatrix}$$

Given:

$$t_y(x, L) = (\lambda + 2\mu)Q \sin(\pi x)$$

Assuming no horizontal traction:

$$t_x(x, L) = 0$$

Summary of Neumann Boundary Conditions

The boundary conditions on the top boundary $y = L$ are:

$$\sigma_{xy}(x, L) = 0$$

$$\sigma_{yy}(x, L) = (\lambda + 2\mu)Q \sin(\pi x)$$

Or equivalently, in terms of displacement derivatives:

$$\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \bigg|_{y=L} = 0$$

$$\left[\lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + 2\mu \frac{\partial u_y}{\partial y} \right]_{y=L} = (\lambda + 2\mu)Q \sin(\pi x)$$