# Chapter 1: The Foundations – Logic and Proofs

This chapter introduces the fundamental concepts of logic, reasoning, and proof techniques, which are essential for **mathematical problem-solving, programming, and algorithm analysis**.

## 1.1 Propositional Logic

Propositional logic deals with **statements (propositions)** that are either **true (T) or false (F)** but not both.

### Basic Logical Operators:

1. **Negation ($\neg p$)**: "Not p" (opposite truth value).
2. **Conjunction ($p \land q$)**: "p AND q" (true only if both p and q are true).
3. **Disjunction ($p \lor q$)**: "p OR q" (true if at least one of p or q is true).
4. **Implication ($p \rightarrow q$)**: "If p, then q" (false only if p is true and q is false).
5. **Biconditional ($p \leftrightarrow q$)**: "p if and only if q" (true when both have the same truth value).

### Example: Truth Table for $p \rightarrow q$

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

## 1.2 Applications of Propositional Logic

Used in **circuit design, programming conditions, and AI decision-making**.

### Example: Conditional Statements

- "If it rains, then the ground is wet." → **(rains → wet ground)**
- "If a number is even, then it is divisible by 2." → **(even → divisible by 2)**

## 1.3 Propositional Equivalences

Logical expressions can be simplified using **logical identities**.

### Example: De Morgan's Laws

- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $\neg(p \lor q) \equiv \neg p \land \neg q$

## 1.4 Predicates and Quantifiers

Extends logic with **variables** and **quantifiers**.

- **Universal Quantifier ($\forall x\, P(x)$)**: "For all x, P(x) is true."
- **Existential Quantifier ($\exists x\, P(x)$)**: "There exists an x such that P(x) is true."

### Example: Universal vs. Existential

- $\forall x\,(x > 0 \rightarrow x^2 > 0)$ → "For all x, if x is positive, then $x^2$ is positive."
- $\exists x\,(x^2 = 4)$ → "There exists an x such that $x^2 = 4$" ($x = \pm 2$).

## 1.5 Rules of Inference

Used to derive conclusions from given premises.

### Example: Modus Ponens ($p \rightarrow q, p \vdash q$)

1. "If it is raining, then the ground is wet." (**$p \rightarrow q$**)

2. "It is raining." (**p**)
3. ∴ "The ground is wet." (**q**)

## 1.6 Introduction to Proofs

Mathematical proofs verify the correctness of statements.

### Types of Proofs:

- **Direct Proof**: Assume p is true, then show q is true.
- **Proof by Contradiction**: Assume ¬q is true and derive a contradiction.
- **Proof by Induction**: Used for statements about integers.

### Example: Direct Proof

**Theorem**: If n is even, then $n^2$ is even.

**Proof**: Let $n = 2k$ (where k is an integer), then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. ✅

## 1.7 Proof Methods and Strategy

Choosing the right proof method based on the problem.

### Example: Proof by Contradiction

**Theorem**: $\sqrt{2}$ is irrational.

**Proof**: Assume $\sqrt{2}$ is rational, meaning $\sqrt{2} = a/b$ (where a, b are integers with no common factors). Squaring both sides:

$$2b^2 = a^2$$

This means $a^2$ is even, so a is even → $a = 2k$. Substituting, we get b is also even.

This contradicts our assumption that a and b have no common factors. ∴ $\sqrt{2}$ is irrational. ✅

### Conclusion

This chapter builds the foundation for **logical reasoning, proof writing, and mathematical rigor**, which are crucial for **computer science, AI, and algorithm analysis**.

## Mathematical Symbols

Here are the **symbols** used in the explanation along with their meanings:

### Logical Symbols (Propositional Logic)

| Symbol | Meaning | Example |
|---|---|---|
| ¬p | Negation ("NOT p") | ¬(x > 5) → "x is not greater than 5" |
| ∧ | Conjunction ("AND") | (p ∧ q) → "p and q are both true" |
| ∨ | Disjunction ("OR") | (p ∨ q) → "p is true or q is true (or both)" |
| → | Implication ("If p, then q") | (p → q) → "If p is true, then q must be true" |
| ↔ | Biconditional ("p if and only if q") | (p ↔ q) → "p is true if and only if q is true" |

### Quantifier Symbols (Predicate Logic)

| Symbol | Meaning | Example |
|---|---|---|
| ∀ | Universal Quantifier ("For all") | ∀x (x > 0 → $x^2$ > 0) → "For all x, if x is positive, then $x^2$ is positive" |
| ∃ | Existential Quantifier ("There exists") | ∃x ($x^2$ = 4) → "There exists an x such that $x^2$ = 4" |

### Set Theory Symbols

| Symbol | Meaning | Example |
|---|---|---|
| ∈ | Element of | x ∈ A → "x belongs to set A" |
| ∉ | Not an element of | x ∉ A → "x does not belong to set A" |
| ⊆ | Subset | A ⊆ B → "A is a subset of B" |
| ⊂ | Proper Subset | A ⊂ B → "A is a subset of B but not equal to B" |
| ∩ | Intersection ("AND") | A ∩ B → "Elements common in both A and B" |
| ∪ | Union ("OR") | A ∪ B → "All elements in A or B or both" |
| ∅ | Empty Set | A = ∅ → "Set A has no elements" |

### Mathematical Symbols

| Symbol | Meaning | Example |
|---|---|---|
| ≡ | Logical equivalence | ¬(p ∧ q) ≡ (¬p ∨ ¬q) → "De Morgan's Law" |
| ⊢ | Logical derivation (Inference) | p → q, p ⊢ q → "From p → q and p, we infer q" |
| √ | Square Root | √4 = 2 |
| ∴ | Therefore | ∴ x = 5 (conclusion of proof) |

**The Foundation (Mathematics): Logic and Proof**

- The rules of logic specify the meaning of mathematical statement
- These rules help us understand and reason with mathematical statement
- Logic is the basis of all mathematical reasoning and of all automated reasoning
- It has practical applications to
    - The Design of Computer machine
    - The specification of systems
    - Artificial Intelligence
    - Computer Programming
    - Programming Language
    - Other area of Computer Science
    - Many other field of studies

- To understand mathematics, understand what makes up a correct mathematical arguments, that is, a proof
- Once we prove a mathematical statement is true, we called it a theorem
- A collection of theorems on a topics organize what we know about this topics
- To learn a mathematical topics, a person need to actively construct mathematical arguments on this topic
- Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situation

- Proofs are important throughout mathematics and computer science
- In fact, proofs are used to
    - Verify that computer programs produce the correct output for all possible input values
    - Show that algorithms always produce the correct result
    - Establish the security of a system
    - Create artificial intelligence
- Furthermore, automated reasoning systems have been created to allow computers to construct their own proofs

**Introduction**

- The rules of logic give precise meaning of mathematical statements
- These rules are used to distinguish between valid and invalid mathematical arguments

1. Proposition – Example 2
2. Definition 1 – Example 2
3. Definition 2 – Example 1
4. Definition 3 – Example 2
5. Definition 4 – Example 2
6. Definition 5 (Conditional Statement) – Example 2
7. Converse, Contrapositive, and Inverse – Example 1
8. Definition 6 – Example 1
9. Implicit use of Bi-conditional
10. Truth Tables of Compound Propositions – Example 1
11. Precedence of Logical Operator
12. Logic and Bit Operation
13. Definition 7 – Example 2
14. Exercises