

§7.1 多元函数的极限与连续 §7.2 偏导数和全微分

一、填空题

1. 函数 $z = \arcsin 2x + \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$ 的定义域为 $\{(x,y) \mid y^2 \leq 4x, x^2+y^2 < 1, 0 < x \leq \frac{1}{2}\}$

2. 设三角形区域 D 由直线 $y=1, y=x, y=-x$ 所围, 则 D 可用 X 型和 Y 型区域形式分别表示为 $D = \{(x,y) \mid -y \leq x \leq y, 0 \leq y \leq 1\} \cup \{(x,y) \mid -y \leq x \leq y, 1 \leq y \leq \sqrt{2}\}$ $D = \{(x,y) \mid -y \leq x \leq y, 0 \leq y \leq 1\}$



3. 函数 $z = \frac{1}{\sin x \cdot \sin y}$ 在 $x=k_1\pi, y=k_2\pi, k_1, k_2 \in \mathbb{Z}$ 处是间断的.

4. $\lim_{(x,y) \rightarrow (1,0)} \frac{\ln(x+e^y)}{\sqrt{x^2+y^2}} = \ln 2$;

5. $\lim_{(x,y) \rightarrow (0,0)} \frac{2-\sqrt{xy+4}}{xy} = -\frac{1}{4}$;

6. $\lim_{(x,y) \rightarrow (2,0)} \frac{\sin xy}{y} = 2$;

7. $\lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2+y^2} \sin \frac{1}{x^2+y^2} = 0$;

8. $\lim_{(x,y) \rightarrow (0,1)} \frac{1-x+xy}{x^2+y^2} = 1$;

9. $\lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)e^{x^2y^2}} = 0$;

10. $\lim_{\substack{x \rightarrow \infty \\ y \rightarrow a}} (1 + \frac{1}{xy})^{\frac{x^2}{x+y}} (a \neq 0) = e^{\frac{1}{a}}$. $\left[\left(1 + \frac{1}{xy}\right)^{xy} \right]^{\frac{x}{y(x+y)}} \xrightarrow{\frac{1}{a}} e$

二、讨论函数 $f(x,y) = \begin{cases} (x^2+y^2) \ln(x^2+y^2), & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0 \end{cases}$ 在 $(0,0)$ 点的连续性.

令 $x^2+y^2=t$, 则 $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{t \rightarrow 0} t \ln t = \lim_{t \rightarrow 0} \frac{\ln t}{\frac{1}{t}} = \lim_{t \rightarrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = 0 = f(0,0)$

故 $f(x,y)$ 在 $(0,0)$ 点连续.

三、选择题

1. 二元函数 $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0) \end{cases}$ 在点 $(0, 0)$ 处 (C)

A. 连续, 偏导数存在

B. 连续, 偏导数不存在

C. 不连续, 偏导数存在

D. 不连续, 偏导数不存在

2. 已知函数 $z = x^2 e^y + (x-1) \arctan \frac{y}{x}$, 则 $z_x(1, 0) = (C)$

A. 0

B. 1

C. 2

D. 不存在

$$f_x(x, 0) = x^2$$

$$\therefore f_x = 2x$$

$$f_x(1, 0) = 2$$

四、求下列函数的偏导数:

1. $z = x^2 y - xy^3$;

$$\frac{\partial z}{\partial x} = 2xy - y^3$$

$$\frac{\partial z}{\partial y} = x^2 - 3xy^2$$

2. $z = \ln \cos(2x + y)$;

$$\frac{\partial z}{\partial x} = -2 \tan(2x + y)$$

$$\frac{\partial z}{\partial y} = -\tan(2x + y)$$

3. $u = \left(\frac{x}{y}\right)^2$

$$\frac{\partial u}{\partial x} = 2 \cdot \left(\frac{x}{y}\right)^{2-1} \cdot \frac{1}{y} = \frac{2x}{y^3}$$

$$\frac{\partial u}{\partial y} = 2 \cdot \left(\frac{x}{y}\right)^{2-1} \cdot \left(-\frac{x}{y^2}\right) = -\frac{2x^2}{y^3}$$

$$\frac{\partial u}{\partial y} = \left(\frac{x}{y}\right)^2 \cdot \ln\left(\frac{x}{y}\right)$$

4. $u = \int_x^{x^2} e^t dt$

$$\frac{\partial u}{\partial x} = -e^{x^2} \cdot 2x$$

$$\frac{\partial u}{\partial y} = e^{y^2} \cdot y$$

$$\frac{\partial u}{\partial y} = y e^{y^2} - x e^{x^2}$$

五、求旋转曲面 $z = \sqrt{1+x^2+y^2}$ 与平面 $x=1$ 的交线在点 $(1,1,\sqrt{3})$ 处的切线与 y 轴正向之间的夹角。

曲线 $\begin{cases} z = \sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ 在点 $(1,1,\sqrt{3})$ 处切线对 y 轴的斜率即为 $\frac{\partial z}{\partial y} \Big|_{(1,1)}$

$$\text{即 } \tan \beta = \frac{dz}{dy} \Big|_{y=1} = \frac{y}{\sqrt{2+y^2}} \Big|_{y=1} = \frac{1}{\sqrt{3}}, \quad \therefore \beta = \frac{\pi}{6}$$

六、求下列函数的 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$:

1. $z = x^4 + y^4 - 4x^2y^2$;

$$\frac{\partial z}{\partial x} = 4x^3 - 8xy^2, \quad \frac{\partial z}{\partial y} = 4y^3 - 8x^2y$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 8y^2, \quad \frac{\partial^2 z}{\partial y^2} = 12y^2 - 8x^2, \quad \frac{\partial^2 z}{\partial x \partial y} = -16xy$$

2. $z = x \arcsin \sqrt{y}$;

$$\frac{\partial z}{\partial x} = \arcsin \sqrt{y}, \quad \frac{\partial z}{\partial y} = \frac{x}{\sqrt{1-y} \cdot 2\sqrt{y}}$$

$$\frac{\partial^2 z}{\partial x^2} = 0, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{x(1-2y)}{4\sqrt{y^3(1-y)^3}}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2\sqrt{y} \cdot \sqrt{1-y}}$$

3. $z = e^{xy^2}$.

$$\frac{\partial z}{\partial x} = y^2 e^{xy^2}, \quad \frac{\partial z}{\partial y} = 2xy e^{xy^2}$$

$$\frac{\partial^2 z}{\partial x^2} = y^4 e^{xy^2}, \quad \frac{\partial^2 z}{\partial y^2} = (2x + 4x^2y^2) e^{xy^2},$$

$$\frac{\partial^2 z}{\partial x \partial y} = (2y + 2xy^3) e^{xy^2}$$

七、求函数 $z = 5x^2 + y^2$ 当 $x = 1, y = 2, \Delta x = 0.005, \Delta y = 0.1$ 时的全增量和全微分。

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y) = 5(x+\Delta x)^2 + (y+\Delta y)^2 - (5x^2 + y^2)$$

$$= 10x\Delta x + 2y\Delta y + 5(\Delta x)^2 + (\Delta y)^2$$

$$dz = 10x\Delta x + 2y\Delta y$$

$$\therefore \Delta z(1, 2) \Big|_{\substack{\Delta x = 0.005 \\ \Delta y = 0.1}} = 10 \times 1 \times 0.005 + 2 \times 2 \times 0.1 + 5 \times (0.005)^2 + (0.1)^2$$

$$= 0.460125$$

$$dz(1, 2) \Big|_{\substack{\Delta x = 0.005 \\ \Delta y = 0.1}} = 10 \times 1 \times 0.005 + 2 \times 2 \times 0.1 = 0.45$$

八、设二元函数 $z = xe^{x+y} + (x+1)\ln(1+y)$, 求 dz 和 $dz|_{(1,0)}$ 。

$$\frac{\partial z}{\partial x} = e^{x+y} + xe^{x+y} + \ln(1+y)$$

$$\frac{\partial z}{\partial y} = xe^{x+y} + \frac{x+1}{1+y}$$

$$\therefore dz = [(x+1)e^{x+y} + \ln(1+y)]dx + (xe^{x+y} + \frac{x+1}{1+y})dy$$

$$dz|_{(1,0)} = 2e dx + (e+2)dy$$

九、设二元函数 $f(x, y) = \begin{cases} (x^2 + y^2) \cos \frac{1}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0 \end{cases}$

(1) 求 $f_x(0,0), f_y(0,0)$;

(2) 讨论 $f(x, y)$ 在点 $(0,0)$ 是否可微。

$$(1) \quad f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{x^2 \cos \frac{1}{|x|}}{x} = 0, \quad \text{同理 } f_y(0,0) = 0$$

$$(2) \quad \lim_{\rho \rightarrow 0} \frac{\Delta f - [f_x \Delta x + f_y \Delta y]}{\rho} = \lim_{\rho \rightarrow 0} \frac{f(x,y) - 0 - [0+0]}{\rho} = \lim_{\rho \rightarrow 0} \frac{(x^2+y^2) \cos \frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{\rho \rightarrow 0} \rho \cos \frac{1}{\rho} = 0$$

故 $f(x, y)$ 在 $(0,0)$ 处可微。

§7.3 复合函数和隐函数的偏导数

一、用链法则求下列函数的导数或偏导数:

1. $z = u^v, u = x + 2y, v = x - y$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$;

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \\ &= v \cdot u^{v-1} \cdot 1 + u^v \cdot \ln u \cdot 1 = (x-y)(x+2y)^{x-y-1} + (x+2y)^{x-y} \ln(x+2y) \\ &= (x+2y)^{x-y} \left[\frac{x-y}{x+2y} + \ln(x+2y) \right] \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = v \cdot u^{v-1} \cdot 2 + u^v \ln u \cdot (-1) = (x+2y)^{x-y} \left[\frac{2(x-y)}{x+2y} - \ln(x+2y) \right]\end{aligned}$$

2. $z = \frac{y}{x}, x = e^t, y = 1 - e^{2t}$, 求 $\frac{dz}{dt}$.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \left(-\frac{y}{x^2}\right) \cdot e^t + \frac{1}{x} \cdot (-2e^{2t}) \\ &= -\frac{1-e^{2t}}{e^{2t}} \cdot e^t + \frac{1}{e^t} \cdot (-2e^{2t}) = -e^t - e^{-t}\end{aligned}$$

二、求下列复合函数的一阶偏导数:

1. $u = f(x, xy, xyz)$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$;

$$\frac{\partial u}{\partial x} = f'_1 + y \cdot f'_2 + yz f'_3$$

$$\frac{\partial u}{\partial y} = x f'_2 + xz f'_3$$

$$\frac{\partial u}{\partial z} = xy f'_3$$

2. $z = f(xy, \frac{x}{y}) + \varphi(\frac{y}{x^2})$, 其中 f, φ 均可微, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = y f'_1 + \frac{1}{y} f'_2 + \varphi' \cdot \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = x f'_1 - \frac{x}{y^2} f'_2 + \frac{1}{x} \varphi'$$

三、设函数 $z = f(x, y)$ 在点 $(1, 1)$ 处可微, 且

$$f(1, 1) = 1, \frac{\partial f}{\partial x} \Big|_{(1,1)} = 2, \frac{\partial f}{\partial y} \Big|_{(1,1)} = 3, \varphi(x) = f(x, f(x, x)), \text{ 求 } \frac{d}{dx} \varphi^3(x) \Big|_{x=1}.$$

$$\varphi(1) = f(1, f(1, 1)) = f(1, 1) = 1$$

$$\frac{d}{dx} \varphi^3(x) = 3\varphi^2(x) \cdot [f'_1(x, f(x, x)) + f'_2(x, f(x, x)) (f'_1(x, x) + f'_2(x, x))]$$

$$\text{当 } x=1 \text{ 时, 代入得}$$

$$\frac{d}{dx} \varphi^3(x) \Big|_{x=1} = 3 \cdot 1 \cdot [2 + 3(2+3)] = 51$$

四、设 $z = f(x^2 + y^2)$, 其中 f 具有二阶导数, 求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$.

$$\frac{\partial z}{\partial x} = f'(u) \cdot 2x, \quad \frac{\partial z}{\partial y} = f'(u) \cdot 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2f' + 2x \cdot f'' \cdot 2x = 2f' + 4x^2 f''$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x f'' \cdot 2y = 4xy f''$$

$$\frac{\partial^2 z}{\partial y^2} = 2f' + 2y f'' \cdot 2y = 2f' + 4y^2 f''$$

五、设 $z = yf(e^x, xy)$, 其中 f 具有二阶连续偏导数, 求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$.

$$\frac{\partial z}{\partial x} = y[f'_1 \cdot e^x + y f'_2] = ye^x f'_1 + y^2 f'_2$$

$$\frac{\partial z}{\partial y} = f + xy f'_2$$

$$\frac{\partial^2 z}{\partial x^2} = e^x y f'_1 + y e^{2x} f''_{11} + 2y^2 e^x f''_{12} + y^3 f''_{22}$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x f'_1 + 2y f'_2 + xy e^x f''_{11} + xy^2 f''_{21}$$

$$\frac{\partial^2 z}{\partial y^2} = 2x f'_2 + x^2 y f''_{22}$$

六、求下列方程所确定的隐函数 $y = f(x)$ 的一阶导数:

(1) $x^2 + xy - e^y = 0;$

(2) $x^y = y^x.$

证: 两边对 x 求导,

$$2x + y \cdot xy' - e^y \cdot y' = 0 \Rightarrow y' = \frac{2x+y}{e^y - x}$$

证: 令 $F(x, y) = x^2 + xy - e^y$, 则

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x-e^y}$$

令 $F(x, y) = x^y - y^x$, 则

$$\begin{aligned} \frac{dy}{dx} &= -\frac{F_x}{F_y} = -\frac{y x^{y-1} - y^x \ln y}{x^y \ln x - x^{y-1}} \\ &= \frac{xy \ln y - y^2}{xy \ln x - x^2} = \frac{y \ln y^x - y^2}{x \ln x^y - x^2} = \frac{y \ln x^y - y^2}{x \ln y^x - x^2} \end{aligned}$$

七、求方程 $z = e^{2x-3y} + 2y$ 所确定的隐函数 $z = f(x, y)$ 的一阶偏导数:

$$= \frac{y^2(\ln x - 1)}{x^2(\ln y - 1)}$$

两边对 x 求偏导, $\frac{\partial z}{\partial x} = e^{2x-3y} (2 - 3 \frac{\partial z}{\partial x})$

$$\therefore \frac{\partial z}{\partial x} = \frac{2e^{2x-3y}}{1+3e^{2x-3y}}$$

两边对 y 求偏导, $\frac{\partial z}{\partial y} = e^{2x-3y} (-3 \frac{\partial z}{\partial y}) + 2$

$$\therefore \frac{\partial z}{\partial y} = \frac{2}{1+3e^{2x-3y}}$$

八、已知 $x^2 + y^2 + z^2 = 4z$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$.

令 $F(x, y, z) = x^2 + y^2 + z^2 - 4z$

$$2) \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-4} = \frac{x}{2-z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y}{2z-4} = \frac{y}{2-z}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{x}{2-z}\right)'_x = \frac{2-z+x \cdot \frac{\partial z}{\partial x}}{(2-z)^2} = \frac{2-z+\frac{x^2}{2-z}}{(2-z)^2} = \frac{(2-z)^2+x^2}{(2-z)^3}$$

九、已知 $z + \ln z - \int_y^x e^{-t^2} dt = 0$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

令 $F(x, y, z) = z + \ln z - \int_y^x e^{-t^2} dt$

$$2) \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{-x^2}}{1+\frac{1}{z}} = \frac{z}{z+1} e^{-x^2}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{-y^2}}{1+\frac{1}{z}} = -\frac{z}{z+1} e^{-y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{z}{z+1} e^{-x^2} \right) = \frac{(z+1) \frac{\partial z}{\partial y} - z \frac{\partial z}{\partial y}}{(z+1)^2} e^{-x^2} = \frac{-\frac{z}{z+1} e^{-y^2}}{(z+1)^2} e^{-x^2} = \frac{z}{(z+1)^3} e^{-x^2-y^2}$$

十、设 $u = f(x, y, z)$ 有连续的偏导数, $y = y(x)$ 和 $z = z(x)$ 分别由方程 $e^{xy} - y = 0$ 和 $e^z - xz = 0$ 所确定, 求 $\frac{du}{dx}$.

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dx}$$

$$\text{由 } e^{xy} - y = 0 \text{ 得 } e^{xy} \left(y + x \frac{dy}{dx} \right) - \frac{dy}{dx} = 0, \quad \therefore \frac{dy}{dx} = \frac{ye^{xy}}{1 - xe^{xy}} = \frac{y^2}{1 - xy}$$

$$\text{由 } e^z - xz = 0 \text{ 得 } e^z \frac{dz}{dx} - z - x \frac{dz}{dx} = 0 \quad \therefore \frac{dz}{dx} = \frac{z}{e^z - x} = \frac{z}{xz - x}$$

$$\text{于是 } \frac{du}{dx} = f_x + \frac{y^2}{1 - xy} \cdot f_y + \frac{z}{xz - x} \cdot f_z$$

十一、求由下列方程组所确定的隐函数的导数或偏导数:

$$1. \begin{cases} x + y + z = 0, \\ xyz = 1, \end{cases} \text{ 求 } \frac{dz}{dx}, \frac{dy}{dx}.$$

$$\text{方程组两边对 } x \text{ 求导得 } \begin{cases} 1 + \frac{dy}{dx} + \frac{dz}{dx} = 0 \\ yz + xy \frac{dy}{dx} + xz \frac{dz}{dx} = 0 \end{cases} \Rightarrow \begin{cases} \frac{dy}{dx} + \frac{dz}{dx} = -1 \\ xy \frac{dy}{dx} + xz \frac{dz}{dx} = -yz \end{cases}$$

$$\text{解得 } \frac{dy}{dx} = \frac{y(z-x)}{x(y-z)}, \quad \frac{dz}{dx} = \frac{z(x-y)}{x(y-z)}$$

$$2. \begin{cases} u = f(ux, v+y), \\ v = g(u-x, v^2y), \end{cases} \text{ 其中 } f, g \text{ 具有一阶连续偏导数, 求 } \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}.$$

对 x 求偏导得

$$\begin{cases} \frac{\partial u}{\partial x} = f'_1 \cdot (u + x \frac{\partial u}{\partial x}) + f'_2 \cdot \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} = g'_1 \cdot (\frac{\partial u}{\partial x} - 1) + 2g'_2 \cdot yv \cdot \frac{\partial v}{\partial x} \end{cases}$$

$$\text{解方程组得 } \frac{\partial u}{\partial x} = \frac{-uf'_1 \cdot (2yv g'_2 - 1) - f'_2 \cdot g'_1}{(x f'_1 - 1)(2yv g'_2 - 1) - f'_2 \cdot g'_1}$$

$$\frac{\partial v}{\partial x} = \frac{g'_1 \cdot (x f'_1 + u f'_1 - 1)}{(x f'_1 - 1)(2yv g'_2 - 1) - f'_2 \cdot g'_1}$$

§7.4 可微函数的几何性质

一、填空

1. 曲线 $\begin{cases} z = \frac{1}{4}(x^2 + y^2), \\ y = 4 \end{cases}$ 在点 $(2, 4, 5)$ 处的切线与 x 轴的夹角为 $\frac{\pi}{2}$. $\delta x = \frac{\pi}{2} = 1$

2. 若曲面 $\Sigma: F(x, y, z) = 0$ 上 Q 点的法线经过曲面外一点 $p(a, b, c)$, 则 $Q(x, y, z)$ 点必须满足 $\frac{a-x}{F_x(x,y,z)} = \frac{b-y}{F_y(x,y,z)} = \frac{c-z}{F_z(x,y,z)}$

二、求曲线 $\Gamma: \begin{cases} x^2 + y^2 + z^2 = \frac{9}{4}, \\ 3x^2 + (y-1)^2 + z^2 = \frac{17}{4} \end{cases}$ 在点 $M(1, \frac{1}{2}, 1)$ 处的切线与法平面.

解法一: Γ 上 M 点处切向量为 $(2x, 2y, 2z) \times (6x, 2y-1, 2z)|_M = 4(1, 2, -2)$

\therefore 切线 $\frac{x-1}{1} = \frac{y-\frac{1}{2}}{2} = \frac{z-1}{-2}$, 法平面: $x-1+2(y-\frac{1}{2})+(-2)(z-1)=0$
即 $x+2y-2z=0$

解法二: 方程组两边对 x 求导: $\begin{cases} 2x + 2y \cdot y' + 2z \cdot z' = 0 \\ 3x + (y-1) \cdot y' + z \cdot z' = 0 \end{cases}$ $M(1, \frac{1}{2}, 1)$ 代入 $\begin{cases} y'=2 \\ z'=-2 \end{cases}$

\therefore 切向量为 $(1, 2, -2)$,

三、证明: 曲线 $\Gamma: \begin{cases} x^2 - z = 0, \\ 3x + 2y + 1 = 0 \end{cases}$ 上点 $(1, -2, 1)$ 处的法平面与直线 $\begin{cases} 9x - 7y - 21z = 0, \\ x - y - z = 0 \end{cases}$ 平行.

取 x 为参数, 则 $\begin{cases} 2x - \frac{dz}{dx} = 0 \\ 3 + 2 \frac{dy}{dx} = 0 \end{cases} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}, \frac{dz}{dx} = 2x$

曲线在 $M(1, -2, 1)$ 处切向量为 $\vec{t} = (1, -\frac{3}{2}, 2x)|_{(1, -2, 1)} = (1, -\frac{3}{2}, 2)$

而直线方向为 $\vec{s} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 9 & -7 & -21 \\ 1 & -1 & -1 \end{vmatrix} = (-14, -12, -2)$

$\therefore \vec{t} \cdot \vec{s} = 0$, 得证.

四、求曲面 $z - e^z + 2xy = 3$ 在点 $(1, 2, 0)$ 处的切平面和法线方程.

$$\text{令 } F(x, y, z) = z - e^z + 2xy - 3, \text{ 则}$$

$$F_x|_{(1,2,0)} = 2y|_{(1,2,0)} = 4$$

$$F_y|_{(1,2,0)} = 2x|_{(1,2,0)} = 2$$

$$F_z|_{(1,2,0)} = (1 - e^z)|_{(1,2,0)} = 0$$

$$\therefore \text{切平面方程: } 4(x-1) + 2(y-2) = 0, \text{ 即 } 2x + y - 4 = 0$$

$$\text{法线: } \frac{x-1}{4} = \frac{y-2}{2} = \frac{z-0}{0}$$

五、求曲面 $2x^2 + 3y^2 + z^2 = 9$ 的切平面, 使之平行于平面 $2x - 3y + 2z = 1$.

$$\text{令 } F = 2x^2 + 3y^2 + z^2 - 9$$

$$\text{则 } F_x = 4x, F_y = 6y, F_z = 2z$$

$$\text{曲面上点 } M_0(x_0, y_0, z_0) \text{ 处法向量 } \vec{n} = (4x_0, 6y_0, 2z_0)$$

$$\text{于是 } \begin{cases} 2x_0^2 + 3y_0^2 + z_0^2 = 9 \\ \frac{4x_0}{2} = \frac{6y_0}{-3} = \frac{2z_0}{2} \end{cases} \Rightarrow x_0 = \pm 1, y_0 = \mp 1, z_0 = \pm 2$$

$$\therefore \text{切平面为 } 2(x-x_0) - 3(y-y_0) + 2(z-z_0) = 0, \text{ 即}$$

$$2x - 3y + 2z = \pm 9$$

六、求由曲线 $\begin{cases} 3x^2 + 2y^2 = 12 \\ z = 0 \end{cases}$ 绕 y 轴旋转一周所得的旋转曲面在点 $(0, \sqrt{3}, \sqrt{2})$ 处的指向

外侧的单位法向量.

$$\text{旋转曲面方程为 } 3(x^2 + z^2) + 2y^2 = 12$$

$$\text{令 } F = 3x^2 + 2y^2 + 3z^2 - 12, \text{ 则 } \vec{n} = (F_x, F_y, F_z) = (6x, 4y, 6z),$$

$$\text{在点 } (0, \sqrt{3}, \sqrt{2}) \text{ 处 } \vec{n} = (0, 4\sqrt{3}, 6\sqrt{2})$$

由于指向外侧的单位法向量 $\cos \theta > 0$

$$\text{故取 } \vec{e}_n = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{0^2 + (4\sqrt{3})^2 + (6\sqrt{2})^2}} (0, 4\sqrt{3}, 6\sqrt{2}) = (0, \frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5})$$

七、设直线 $l: \begin{cases} x+y+b=0, \\ x+ay-z-3=0 \end{cases}$ 在平面 π 上, 而平面 π 与曲面 $z=x^2+y^2$ 相切于点

$(1, -2, 5)$, 求 a, b 的值.

方法一. 先求出曲面 $z=x^2+y^2$ 在点 $M_0(1, -2, 5)$ 处的切平面, 因法向量为 $\vec{n}=(2, -4, -1)$

故切平面为 $2(x-1)-4(y+2)-(z-5)=0$, 即 $2x-4y-z-5=0$,

将 l 的方程代入切平面中, 即 $\begin{cases} x+y+b=0 \\ x+ay-z-3=0 \end{cases} \Rightarrow \begin{cases} y=-x-b \\ z=x-3+a(-x-b) \end{cases}$

$\therefore 2x+4(x+b)-(x-3-ax-ab)-5=0$, 此方程有无穷解 $\Rightarrow \begin{cases} 5+a=0 \\ 4b+ab-2=0 \end{cases} \Rightarrow a=-5, b=-2$

方法二. 过 l 的平面来与切平面重合, $(x+y+b)+\lambda(x+ay-z-3)=0 \Rightarrow (1+\lambda)x+(1+a\lambda)y-\lambda z+b-3\lambda=0$

令 $\frac{1+\lambda}{2} = \frac{1+a\lambda}{-4} = \frac{-\lambda}{-1} = \frac{b-3\lambda}{-5} \Rightarrow \lambda=1, a=-5, b=-2$

八、求下列函数在指定点 M_0 处沿指定方向 l 的方向导数:

1. $z=xe^{xy}, M_0(-3, 0), l$ 为从点 $(-3, 0)$ 到点 $(-1, 3)$ 的方向;

$$\frac{\partial z}{\partial x} \Big|_{M_0} = (e^{xy} + xy e^{xy}) \Big|_{(-3, 0)} = 1$$

$$\frac{\partial z}{\partial y} \Big|_{M_0} = x^2 e^{xy} \Big|_{(-3, 0)} = 9$$

$$\vec{l} = (2, 3) \text{ 的方向余弦 } \cos \alpha = \frac{2}{\sqrt{13}}, \cos \beta = \frac{3}{\sqrt{13}}$$

$$\therefore \frac{\partial z}{\partial l} \Big|_{M_0} = \frac{\partial z}{\partial x} \Big|_{M_0} \cos \alpha + \frac{\partial z}{\partial y} \Big|_{M_0} \cos \beta = 1 \cdot \frac{2}{\sqrt{13}} + 9 \cdot \frac{3}{\sqrt{13}} = \frac{29}{\sqrt{13}}$$

2. $u = x \arctan \frac{y}{z}, M_0(1, 2, -2), l = (1, 1, -1)$.

$$\frac{\partial u}{\partial x} \Big|_{M_0} = \arctan \frac{y}{z} \Big|_{(1, 2, -2)} = \arctan(-1) = -\frac{\pi}{4}$$

$$\frac{\partial u}{\partial y} \Big|_{M_0} = \frac{xz}{z^2+y^2} \Big|_{(1, 2, -2)} = -\frac{1}{2}$$

$$\frac{\partial u}{\partial z} \Big|_{M_0} = \frac{-xy}{z^2+y^2} \Big|_{(1, 2, -2)} = -\frac{1}{2}$$

$$\vec{l} \text{ 的方向余弦 } \cos \alpha = \frac{1}{\sqrt{3}}, \cos \beta = \frac{1}{\sqrt{3}}, \cos \gamma = -\frac{1}{\sqrt{3}}$$

$$\begin{aligned} \therefore \frac{\partial u}{\partial l} \Big|_{M_0} &= \frac{\partial u}{\partial x} \Big|_{M_0} \cos \alpha + \frac{\partial u}{\partial y} \Big|_{M_0} \cos \beta + \frac{\partial u}{\partial z} \Big|_{M_0} \cos \gamma \\ &= -\frac{\pi}{4} \cdot \frac{1}{\sqrt{3}} - \frac{1}{2} \cdot \frac{1}{\sqrt{3}} - \frac{1}{2} \cdot \left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{4\sqrt{3}} \end{aligned}$$

九、求函数 $z = x^2 - y^2$ 在点 $M(1,1)$ 沿与 x 轴正向组成角 $\alpha = 60^\circ$ 的方向 l 上的方向导数.

$$\left. \frac{\partial z}{\partial x} \right|_{(1,1)} = 2x \Big|_{(1,1)} = 2, \quad \left. \frac{\partial z}{\partial y} \right|_{(1,1)} = (-2y) \Big|_{(1,1)} = -2$$

$$\cos \varphi = \cos 60^\circ = \frac{1}{2}, \quad \sin \varphi = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} \therefore \left. \frac{\partial z}{\partial l} \right|_{(1,1)} &= \left. \frac{\partial z}{\partial x} \right|_{(1,1)} \cos \varphi + \left. \frac{\partial z}{\partial y} \right|_{(1,1)} \sin \varphi = 2 \times \frac{1}{2} + (-2) \times \frac{\sqrt{3}}{2} \\ &= 1 - \sqrt{3} \end{aligned}$$

十、设 \mathbf{n} 是曲面 $2x^2 + 3y^2 + z^2 = 6$ 在点 $P(1,1,1)$ 处的指向外侧的法向量, 求函数

$$u = \frac{\sqrt{6x^2 + 8y^2}}{z} \text{ 在点 } P \text{ 处沿方向 } \mathbf{n} \text{ 的方向导数.}$$

$$\text{令 } F(x, y, z) = 2x^2 + 3y^2 + z^2 - 6, \quad F_x = 4x, \quad F_y = 6y, \quad F_z = 2z$$

$$\therefore \text{在 } P(1,1,1) \text{ 处曲线外侧向量 } \vec{n} = (4, 6, 2)$$

$$\cos \alpha = \frac{2}{\sqrt{16}} = \frac{1}{2}, \quad \cos \beta = \frac{6}{\sqrt{36}} = \frac{1}{2}, \quad \cos \gamma = \frac{2}{\sqrt{4}} = \frac{1}{2}$$

$$\left. \frac{\partial u}{\partial x} \right|_P = \frac{6x}{z\sqrt{6x^2 + 8y^2}} \Big|_P = \frac{6}{\sqrt{14}}, \quad \left. \frac{\partial u}{\partial y} \right|_P = \frac{8y}{z\sqrt{6x^2 + 8y^2}} \Big|_P = \frac{8}{\sqrt{14}}, \quad \left. \frac{\partial u}{\partial z} \right|_P = -\frac{\sqrt{6x^2 + 8y^2}}{z^2} \Big|_P = -\sqrt{14}$$

$$\therefore \left. \frac{\partial u}{\partial l} \right|_P = \left. \frac{\partial u}{\partial x} \right|_P \cos \alpha + \left. \frac{\partial u}{\partial y} \right|_P \cos \beta + \left. \frac{\partial u}{\partial z} \right|_P \cos \gamma = \frac{6}{\sqrt{14}} \times \frac{1}{2} + \frac{8}{\sqrt{14}} \times \frac{1}{2} - \sqrt{14} \times \frac{1}{2} = \frac{11}{7}$$

十一、二元函数 $u = x^2 - xy + y^2$ 在点 $(-1,1)$ 沿哪个方向变化得最快? 沿哪个方向 u 的值不变?

$$\text{grad } u(-1,1) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \Big|_{(-1,1)} = (2x - y, 2y - x) \Big|_{(-1,1)} = (-3, 3)$$

方向导数取得最大值方向为梯度方向, 故变化最快的方向为

$\pm \frac{1}{\sqrt{2}}(-1, 1)$. (增加最快的方向为 $\frac{1}{\sqrt{2}}(-1, 1)$, 减少最快的方向为 $-\frac{1}{\sqrt{2}}(-1, 1)$)

与梯度方向垂直的方向上变化率为 0, 即 $\pm \frac{1}{\sqrt{2}}(1, 1)$ 方向上, 它与

x 轴正向夹角为 $\frac{\pi}{4}$ 或 $\frac{5\pi}{4}$.

§7.5 多元函数的极值

一、选择题

1. 点 (x_0, y_0) 使 $f'_x(x, y) = 0$ 且 $f'_y(x, y) = 0$ 成立, 则 (D)

- A. (x_0, y_0) 是 $f(x, y)$ 的极值点 B. (x_0, y_0) 是 $f(x, y)$ 的最小值点
C. (x_0, y_0) 是 $f(x, y)$ 的最大值点 D. (x_0, y_0) 可能是 $f(x, y)$ 的极值点

2. 函数 $z = xy(1-x-y)$ 的极值点是 (C) $\frac{1}{3}, \frac{1}{3}$ $A = -\frac{2}{3}, B = -\frac{1}{3}, C = -\frac{1}{3}$ $\Delta C - B^2 > 0, A < 0$

A. $(\frac{1}{3}, -\frac{1}{3})$ B. $(-\frac{1}{3}, \frac{1}{3})$ C. $(\frac{1}{3}, \frac{1}{3})$ D. $(-\frac{1}{3}, -\frac{1}{3})$

3. 已知函数 $f(x, y)$ 在点 $(0, 0)$ 的某个邻域内连续, 且 $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y) - xy}{(x^2 + y^2)^2} = 1$, 则 (A)

- (A) 点 $(0, 0)$ 不是 $f(x, y)$ 的极值点 (B) 点 $(0, 0)$ 是 $f(x, y)$ 的极大值点 $f(x, y) = xy + o(r^4)$
(C) 点 $(0, 0)$ 是 $f(x, y)$ 的极小值点 (D) 根据所给条件无法判断点 $(0, 0)$ 是否为 $f(x, y)$ 的极值点 $f(x, -x) = -x^2$
(x 充分小时)

二、求下列函数的极值

1. $z = x^3 + y^3 - 3x^2 - 3y^2$

$$\begin{cases} f'_x = 3x^2 - 6x = 0 \\ f'_y = 3y^2 - 6y = 0 \end{cases} \Rightarrow \begin{matrix} x = 0, 2 \\ y = 0, 2 \end{matrix} \Rightarrow \text{驻点 } (0, 0), (0, 2), (2, 0), (2, 2)$$

$$A = 6(x-1), B = 0, C = 6(y-1)$$

$$\Delta C - B^2 = 36(x-1)(y-1)$$

$\therefore \Delta C - B^2|_{(0,0)} = 36 > 0, A|_{(0,0)} = -6 < 0$, 有极大值 $f(0,0) = 0$
 $\Delta C - B^2|_{(0,2)} = -36 < 0, A|_{(0,2)} = 0$, $\therefore (0,2)$ 和 $(2,0)$ 不是极值点
 $\Delta C - B^2|_{(2,2)} = 36 > 0, A|_{(2,2)} = 6 > 0$, 有极小值 $f(2,2) = -8$.

2. $f(x, y) = e^{2x}(x + y^2 + 2y)$

$$\begin{cases} f'_x = e^{2x}(2x + 2y^2 + 4y + 1) = 0 \\ f'_y = e^{2x}(2y + 2) = 0 \end{cases} \Rightarrow \text{驻点 } (\frac{1}{2}, -1)$$

$$A = f''_{xx}|_{(\frac{1}{2}, -1)} = e^{2x}(4x + 4y^2 + 8y + 4) = 2e > 0$$

$$B = f''_{xy}|_{(\frac{1}{2}, -1)} = e^{2x}(4y + 4)|_{(\frac{1}{2}, -1)} = 0$$

$$C = f''_{yy}|_{(\frac{1}{2}, -1)} = 2e^{2x}|_{(\frac{1}{2}, -1)} = 2e, \therefore \Delta C - B^2 = 4e^2 > 0$$

$$\therefore \text{极小值 } f(\frac{1}{2}, -1) = -\frac{e}{2}$$

三. 求函数 $f(x, y) = x^2 + 2y^2$ 在闭域 $x^2 + y^2 \leq 1$ 内的最大值和最小值, 并对上述计算结论作出几何解释.

$f(x, y)$ 的驻点为 $(0, 0)$, 在闭域内边界为 $x^2 + y^2 = 1$,

在边界上 $f(x, y) = 2 - x^2$ 或 $1 + y^2$, 由于 $0 \leq x^2 \leq 1$, $0 \leq y^2 \leq 1$,

就有 $1 \leq f(x, y) \leq 2$, 与 $f(0, 0)$ 比较得最小值为 $f(0, 0) = 0$
最大值为 $f(0, 1) = 2$



几何上, 意指: 椭圆抛物面被圆柱面 $x^2 + y^2 \leq 1$ 所截得部分
的最低点在坐标原点, 最高点在边界上的
 $A_1(0, 1, 2)$, $A_2(0, -1, 2)$ 处.

四. 设 $\{(a_i, b_i)\}_{i=1}^n$ 是不在同一直线上的 n 个点,

(1) 求出函数 $f(x, y) = \sum_{i=1}^n [b_i - (xa_i + y)]^2$ 的驻点;

(2) 用二阶导数的方法证明这个驻点是函数的极小值点;

(3) 说明这个极小值点也是这个函数的最小值点.

$$(1) \text{ 以 } \begin{cases} f_x = -2 \sum_{i=1}^n (b_i - xa_i - y)a_i = 0 \\ f_y = -2 \sum_{i=1}^n (b_i - xa_i - y) = 0 \end{cases} \Leftrightarrow \begin{cases} x \sum_{i=1}^n a_i^2 + y \sum_{i=1}^n b_i = \sum_{i=1}^n a_i b_i \\ x \sum_{i=1}^n a_i + ny = \sum_{i=1}^n b_i \end{cases}$$

$$\therefore \text{驻点为 } x = \frac{n \sum_{i=1}^n a_i b_i - n \sum_{i=1}^n a_i \sum_{i=1}^n b_i}{n \sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2}, \quad y = \frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i - n \sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i}{n \sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2}$$

$$(2) f_{xx} = 2 \sum_{i=1}^n a_i^2, \quad f_{xy} = 2 \sum_{i=1}^n a_i, \quad f_{yy} = 2n$$

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 4 \left(n \sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2 \right) > 0, \quad \text{且 } f_{xx} > 0, \therefore \text{该驻点为极小点}$$

(3) 因为 $f(x, y) \geq 0$, \therefore 最小值一定存在, 而驻点唯一,
故该点必为最小值点. (为椭圆抛物面)

五、设有曲线 $L: \begin{cases} z = x^2 + 3y^2, \\ z = 4 - 3x^2 - y^2, \end{cases}$ 求 L 在 xOy 平面上的投影, 并求 L 上的 z 坐标的最大值

和最小值.

L 方程消去 z 得 $x^2 + 3y^2 = 4 - 3x^2 - y^2 \Rightarrow x^2 + y^2 = 1, \therefore$ 投影为圆周 $\begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$

最值的求法一, 消去 x 得

$$z = 1 + 2y^2 \quad (-1 \leq y \leq 1). \quad \therefore z_{\max} = 3, \quad z_{\min} = 1.$$

求法二. 即求 $z = x^2 + 3y^2$ (或 $z = 4 - 3x^2 - y^2$) 在条件 $x^2 + y^2 = 1$ 下极值,

$$\text{令 } L = x^2 + 3y^2 + \lambda(x^2 + y^2 - 1)$$

$$\text{求法三. } \text{令 } L = z + \lambda(z - x^2 - 3y^2) + \mu(3x^2 + y^2 + z - 4)$$

六、在 $x^2 + 4y^2 = 4$ 上求一点, 使其到直线 $2x + 3y - 6 = 0$ 的距离最短.

解法一. 点 $M(x, y)$ 到直线 $ax + by + c = 0$ 的距离为 $d = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|2x + 3y - 6|}{\sqrt{13}}$

$$\text{令 } F(x, y, \lambda) = (2x + 3y - 6)^2 + \lambda(x^2 + 4y^2 - 4)$$

$$\text{由 } \begin{cases} F_x = 4(2x + 3y - 6) + 2\lambda x = 0 & (1) \\ F_y = 6(2x + 3y - 6) + 8\lambda y = 0 & (2) \\ x^2 + 4y^2 - 4 = 0 & (3) \end{cases}$$

$$\text{由 (1), (2) 得 } x = \frac{8}{3}y \text{ 代入 (3) 得 } (\frac{8}{5}, \frac{3}{5}), (-\frac{8}{5}, -\frac{3}{5}).$$

由于 d_{\max} 和 d_{\min} 一定存在, 因 $d(\frac{8}{5}, \frac{3}{5}) < d(-\frac{8}{5}, -\frac{3}{5})$, 故在点 $(\frac{8}{5}, \frac{3}{5})$ 处距离最短.

解法二. 所求问题为椭圆 $x^2 + 4y^2 = 4$ 到直线 $2x + 3y - 6 = 0$ 的距离最短问题,

只需作椭圆的平行于已知直线的切线, 设切点为 (x_0, y_0) , 则

$$\text{切线方程为 } x_0 x + 4y_0 y = 4. \quad \therefore \begin{cases} \frac{x_0}{2} = \frac{4y_0}{3} \\ x_0^2 + 4y_0^2 = 4 \end{cases} \Rightarrow (x_0, y_0) = \pm(\frac{8}{5}, \frac{3}{5})$$

根据直线所在位置知, 所求点为 $(\frac{8}{5}, \frac{3}{5})$.

七、在已给的椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 内一切内接的长方体（各棱分别平行坐标轴）中，求其体积最大者。

设 x, y, z 为长方体在第一卦限中的顶点坐标，则
 $V = 8xyz$ ，满足条件 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\text{令 } L = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

$$\begin{cases} L_x = 8yz + \frac{2x}{a^2} \lambda = 0 \\ L_y = 8xz + \frac{2y}{b^2} \lambda = 0 \\ L_z = 8xy + \frac{2z}{c^2} \lambda = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases} \Rightarrow \begin{cases} \frac{x^2}{a^2} \lambda = -4xyz \\ \frac{y^2}{b^2} \lambda = -4xyz \\ \frac{z^2}{c^2} \lambda = -4xyz \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

这是唯一驻点，故 $V_{\max} = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8}{3\sqrt{3}} abc$
 （而最大值一定存在）

八、欲造一无盖的长方体容器，已知底部造价为每平方米 3 元，侧面造价均为每平方米 1 元，现想用 36 元造一个容积最大的容器，求它的尺寸。

设容器的长、宽、高分别为 x, y, z (米)，($x > 0, y > 0, z > 0$)

问题归结为求 $V = xyz$ 在条件 $3xy + 2yz + 2xz = 36$ 时的最大值。

$$\text{令 } L(x, y, z, \lambda) = xyz + \lambda (3xy + 2yz + 2xz - 36)$$

$$\begin{cases} L_x = yz + 3\lambda y + 2\lambda z = 0 \\ L_y = xz + 3\lambda x + 2\lambda z = 0 \\ L_z = xy + 2\lambda y + 2\lambda x = 0 \\ 3xy + 2yz + 2xz = 36 \end{cases} \Rightarrow \begin{cases} xyz + \lambda(3xy + 2xz) = 0 \\ xyz + \lambda(3xy + 2yz) = 0 \\ xyz + \lambda(2yz + 2xz) = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ 3xy = 2xz \\ \Rightarrow z = \frac{3}{2}x \end{cases}$$

将 $y = x, z = \frac{3}{2}x$ 代入 $3xy + 2yz + 2xz = 36$ 得 $x = y = 2, z = 3$ ， $(2, 2, 3)$ 是唯一驻点。

根据实际问题，故所求最大容器长 2 米，宽 2 米，高 3 米。
 （最大值一定存在）

自测题二 (多元函数的微分学)

一、选择题 (每题 3 分, 共 15 分)

1. $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{3xy}{\sqrt{xy+1}-1} = (\quad B \quad)$

A、3, B、6 C、不存在但不是无穷大, D、 ∞

2. 若 $\frac{\partial f}{\partial x}|_{(x_0, y_0)} = 0$, $\frac{\partial f}{\partial y}|_{(x_0, y_0)} = 0$, 则 $f(x, y)$ 在 (x_0, y_0) ($\quad D \quad$)

A、连续且可微, B、连续但不一定可微 C、可微但不一定连续
D、不一定可微也不一定连续。 ()

3. $z = f(x, y)$ 在 (x_0, y_0) 处可微且 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, 则 $z = f(x, y)$ 在 (x_0, y_0)

处 ($\quad B \quad$) $\text{如 } f(x, y) = \begin{cases} \frac{\ln(1+x^2+y^2)}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$ B. $f_x(0, 1) = (\quad B \quad)$

A、无极值 B、可能有极值, 也可能没有 C、有极大值 D、有极小值 () A. 0, B. 1, C. 2, D. 不存在

4. 二元函数 $f(x, y)$ 在 $(0, 0)$ 可微的一个必要条件是 ($\quad A \quad$) $\lim_{x \rightarrow 0} \frac{f(x, 1) - f(0, 1)}{x} = \lim_{x \rightarrow 0} \frac{\ln(1+x^2) - 0}{x^2} = 1$

A、 $\lim_{(x, y) \rightarrow (0, 0)} (f(x, y) - f(0, 0)) = 0$ B、 $\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = 0$

C、 $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0)}{\sqrt{x^2 + y^2}} = 0$ D、 $f(x, y)$ 在 $(0, 0)$ 偏导数存在且连续

5. 曲线 $\begin{cases} y = 1 - 2x, \\ z = \frac{1}{2} - \frac{5}{2}x^2 \end{cases}$ 在点 $(1, -1, -2)$ 处 切线与直线 $\begin{cases} 5x - 3y + 3z - 9 = 0, \\ 3x - 2y + z - 1 = 0 \end{cases}$ 的夹角 $\varphi = (\quad D \quad)$ $(1, 4, -1)$

A. 0 B. $\frac{\pi}{4}$ C. $\frac{\pi}{3}$ D. $\frac{\pi}{2}$

二、填空题 (每题 3 分, 共 15 分)

1. $\lim_{(x, y) \rightarrow (0, 0)} (x+y) \sin \frac{1}{x} \sin \frac{1}{y} = \underline{0}$

2. 设 f 有一阶连续偏导数, $z = f(x^2 - y^2, e^{xy})$, 则 $dz = \underline{(2xf'_1 + ye^{xy}f'_2)}dx + \underline{(-2f'_1 + xe^{xy}f'_2)}dy$

3. 设连续函数 $z = f(x, y)$ 满足 $\lim_{(x, y) \rightarrow (0, 1)} \frac{f(x, y) - 2x - 3y}{\sqrt{x^2 + (y-1)^2}} = 0$ 则 $dz|_{(0, 1)} = \underline{2}dx + \underline{3}dy$

4. $F(x, y) = \int_0^{xy} \frac{\sin t}{1+t^2} dt$, 则 $\frac{\partial^2 F}{\partial x^2}|_{(0, 2)} = \underline{4}$ $F_x = y \frac{\sin xy}{1+(xy)^2}$ $F_{xx} = y \frac{\cos(xy) \cdot (1+xy^2) - \sin(xy) \cdot 2xy}{(1+xy^2)^2}$

5. $u = (\frac{x}{y})^{\frac{1}{2}}$ 在 $(1, 1, 1)$ 的梯度为 $\underline{(1, -1, 0)}$

二、解下列各题 (每题 10 分, 共 40 分)

1、设 $z = x^y$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\ln z = y \ln x$$

$$\therefore \frac{1}{z} \cdot \frac{\partial z}{\partial x} = \frac{\partial x^y}{\partial x} \cdot \ln x + \frac{1}{x} \cdot x^y = (1+y \ln x) x^{y-1},$$

$$\therefore \frac{\partial z}{\partial x} = (1+y \ln x) x^{y+y-1}$$

$$\text{类似可得 } \frac{\partial z}{\partial y} = x^{x^y+y} + \ln x$$

2、设 $x^2 + z^2 = y \varphi\left(\frac{z}{y}\right)$, 其中 φ 为可微函数, 求 $\frac{\partial z}{\partial y}$

$$\text{令 } F(x, y, z) = x^2 + z^2 - y \varphi\left(\frac{z}{y}\right)$$

$$F_y = -\varphi + \frac{z}{y} \varphi', \quad F_z = 2z - \varphi'$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{\cancel{-\varphi} + \frac{z}{y} \varphi'}{2z - \varphi'}$$

3、求由方程组 $\begin{cases} u+v=x \\ u^2+v^2=y \end{cases}$ 所确定的函数 $u = u(x, y)$ 的二阶偏导数 $\frac{\partial^2 u}{\partial x \partial y}$.

$$u_x = -\frac{v}{u-v} \quad u_y = \frac{+1}{2(u-v)} \quad v_x = \frac{u}{u-v} \quad v_y = \frac{-1}{2(u-v)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{v}{u-v} \right) = \frac{v_y(v-u) - v(v_y - u_y)}{(u-v)^2} = \frac{u+v}{2(u-v)^3}$$

4、求由方程 $x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$ 所确定的函数 $z = z(x, y)$ 的极值.

$$\text{令 } F = x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$$

$$\text{令 } \frac{\partial F}{\partial x} = -\frac{F_x}{F_z} = -\frac{x-1}{z-2}, \quad \frac{\partial F}{\partial y} = -\frac{F_y}{F_z} = -\frac{y+1}{z-2}$$

$$\text{令 } \begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \end{cases} \quad \text{即 } x=1, y=-1, \text{ 又 } z=1, y=-1 \text{ 时, 原方程为 } z^2 - 4z - 12 = 0$$

故得两点 $(1, -1, -2)$ 及 $(1, -1, 6)$, 计算得

$$A=8, B=0, C=-\frac{1}{z-2}, \text{ 故 } C-B^2 = \frac{1}{(z-2)^2} > 0.$$

三、解下列各题 (每题 10 分, 共 30 分)

1、试证光滑曲面 $F(z-x, y-z) = 0$ 所有切平面都与一固定的非零向量平行.

故 $(1, -1)$ 为极小点, 极值 -2, $(1, -1)$ 为极大点, 极大值 6.

$$\text{令 } G(x, y, z) = F(z-x, y-z).$$

$$\text{法向量 } \vec{n} = (G_x, G_y, G_z) = (-F_1', F_2', F_1' - F_2')$$

$$\text{因此 } \vec{n} \cdot (1, 1, 1) = 0.$$

这说明该光滑曲面的所有法向量都与向量 $(1, 1, 1)$ 垂直,

(即所有切平面都与 $(1, 1, 1)$ 平行). 证毕.

2、设 $u = x + 2y + 2, v = x - y - 1, z = z(x, y)$ 有二阶连续偏导数，变换方程

$$2z_{xx} + z_{xy} - z_{yy} + z_x + z_y = 0.$$

$$\partial_x = \partial_u \cdot u_x + \partial_v \cdot v_x = \partial_u + \partial_v. \quad \partial_y = \partial_u \cdot u_y + \partial_v \cdot v_y = 2\partial_u - \partial_v$$

$$\partial_{xx} = \frac{\partial}{\partial x} \partial_u + \frac{\partial}{\partial x} \partial_v = (\partial_{uu} \cdot u_x + \partial_{uv} \cdot v_x) + (\partial_{vu} \cdot u_x + \partial_{vv} \cdot v_x) = \partial_{uu} + 2\partial_{uv} + \partial_{vv}$$

$$\partial_{xy} = (\partial_{uu} \cdot u_y + \partial_{uv} \cdot v_y) + (\partial_{vu} \cdot u_y + \partial_{vv} \cdot v_y) = 2\partial_{uu} + \partial_{uv} - \partial_{vv}$$

$$\partial_{yy} = 2(\partial_{uu} \cdot u_y + \partial_{uv} \cdot v_y) - (\partial_{vu} \cdot u_y + \partial_{vv} \cdot v_y) = 4\partial_{uu} - 4\partial_{uv} + \partial_{vv}$$

$$2\partial_{xx} + \partial_{xy} - \partial_{yy} + \partial_x + \partial_y = 9\partial_{uv} + 3\partial_u = 0$$

$$\therefore \underline{3\partial_{uv} + \partial_u = 0}$$

3、设 $u = f(x, y, z)$ 有连续偏导数，且 $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$ ，证

明：若 $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$ ，则 u 与 r 无关

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= \frac{\partial u}{\partial x} \cdot \sin \theta \cos \varphi + \frac{\partial u}{\partial y} \cdot \sin \theta \sin \varphi + \frac{\partial u}{\partial z} \cdot \cos \theta$$

$$= \frac{1}{r} (x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}) = \frac{1}{r} \cdot 0 = 0$$

$$\therefore u \text{ 与 } r \text{ 无关.} \quad \#$$