§8.1 直角坐标下的二重积分

一、填空题（一）

1．根据二重积分的几何意义，计算 $\iint\limits_{x^2+y^2\le a^2} \mathrm{d}\sigma =$ $\underline{\pi a^2}$；
圆面积，

$\iint\limits_{x^2+y^2\le a^2} \sqrt{a^2-x^2-y^2}\,\mathrm{d}\sigma =$ $\underline{\frac{2}{3}\pi a^3}$．上半球体积，

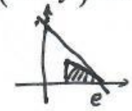
2. 已知 $I_1=\iint\limits_{x^2+y^2\le 1}|xy|\mathrm{d}x\mathrm{d}y, I_2=\iint\limits_{|x|+|y|\le 1}|xy|\mathrm{d}x\mathrm{d}y, I_3=\iint\limits_{\substack{|x|\le 1\\|y|\le 1}}|xy|\mathrm{d}x\mathrm{d}y$，则 I_1,I_2,I_3 的大小为

$\underline{I_2 < I_1 < I_3}$．

3. 设 D 是三角形闭区域，三顶点分别为 $(1,0),(1,1),(\mathrm{e},0)$，比较 $I_1=\iint\limits_D \ln(x+y)\mathrm{d}\sigma$ 与

$I_2=\iint\limits_D (\ln(x+y))^2\mathrm{d}\sigma$ 的大小关系为 $\underline{I_2 < I_1}$．

$1 < x+y < e$



4．改换二次积分 $\int_1^{\mathrm{e}}\mathrm{d}x\int_0^{\ln x} f(x,y)\mathrm{d}y$ 的积分次序为 $\underline{\int_0^1 dy \int_{e^y}^{e} f(x,y)\,dx}$．

5．改换二次积分 $\int_0^1\mathrm{d}y\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\mathrm{d}x$ 的积分次序为 $\underline{\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x,y)\,dy}$．

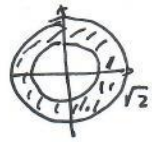
6．设 $D=\{(x,y)\mid -3\le x\le 2, 0\le y\le 1\}$，计算二重积分 $I=\iint\limits_D xy^2\mathrm{d}\sigma=$ $\underline{-\frac{5}{8}}$．

二、试估计二重积分 $I=\iint\limits_D \ln(1+x^2+y^2)\mathrm{d}\sigma$ 的值，其中 $D=\{(x,y)\mid 1\le x^2+y^2\le 2\}$．

$$\ln 2 \le \ln(1+x^2+y^2) \le \ln 3$$

$$\therefore \pi\ln 2 = \iint\limits_D \ln 2\,d\sigma \le \iint\limits_D \ln(1+x^2+y^2)\,d\sigma \le \iint\limits_D \ln 3\,d\sigma = \pi\cdot\ln 3$$

$$\therefore \quad \pi\ln 2 \le I \le \pi\ln 3$$



三、计算下列二重积分

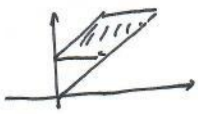
1．$I=\iint\limits_D xy\mathrm{d}\sigma$，其中 D 由 $y=x, x=1$ 及 x 轴所围成．

$$D=\{(x,y)\mid 0\le y\le x,\ 0\le x\le 1\}$$

$$I=\int_0^1 dx\int_0^x xy\,dy = \int_0^1 x\cdot\frac{y^2}{2}\Big|_0^x dx$$

$$=\frac{1}{2}\int_0^1 x^3 dx = \frac{x^4}{8}\Big|_0^1 = \frac{1}{8}$$

2．$I=\iint\limits_D\left(x^2+y^2\right)\mathrm{d}\sigma$，其中 D 由 $y=x, y=x+1, y=1, y=2$ 围成.



$D=\{(x,y)\mid y-1\le x\le y,\ 1\le y\le 2\}$

$$I=\int_1^2 dy\int_{y-1}^{y}(x^2+y^2)\,dx$$

$$=\int_1^2\left(\frac{x^3}{3}+y^2\cdot x\right)\Big|_{y-1}^{y}dy=\int_1^2\left\{\frac{1}{3}\left[y^3-(y-1)^3\right]+y^2\right\}dy$$

$$=\int_1^2\left(2y^2-y+\frac{1}{3}\right)dy=\left(\frac{2}{3}y^3-\frac{y^2}{2}+\frac{y}{3}\right)\Big|_1^2=\frac{7}{2}$$

3．$I=\iint\limits_D\frac{\sin x}{x}\mathrm{d}x\mathrm{d}y$，其中 D 是直线 $y=x$ 及曲线 $y=x^2$ 所围成.

$D=\{(x,y)\mid x^2\le y\le x,\ 0\le x\le 1\}$

$$I=\int_0^1 dx\int_{x^2}^{x}\frac{\sin x}{x}\,dy$$

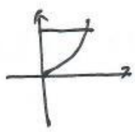
$$=\int_0^1\frac{\sin x}{x}\cdot y\Big|_{x^2}^{x}dx=\int_0^1\frac{\sin x}{x}(x-x^2)\,dx$$

$$=\int_0^1\sin x\,dx-\int_0^1 x\sin x\,dx=-\cos x\Big|_0^1+\int_0^1 x\cdot(\cos x)'\,dx$$

$$=-\cos 1+1+x\cos x\Big|_0^1-\sin x\Big|_0^1=1-\sin 1$$

4．$I=\iint\limits_D x^2\sin y^2\mathrm{d}\sigma$，其中 D 是曲线 $y=x^3$ 和直线 $y=1, x=0$ 所围的位于第一象限的闭区域.

$D=\{(x,y)\mid 0\le x\le\sqrt[3]{y},\ 0\le y\le 1\}$



$$I=\int_0^1 dy\int_0^{\sqrt[3]{y}}x^2\sin y^2\,dx$$

$$=\int_0^1\sin y^2\cdot\frac{x^3}{3}\Big|_0^{\sqrt[3]{y}}dy=\frac{1}{3}\int_0^1\sin y^2\cdot y\,dy$$

$$=\frac{1}{6}\int_0^1\sin y^2\,dy^2=-\frac{1}{6}\cos y^2\Big|_0^1=\frac{1}{6}(1-\cos 1)$$

六.计算

1.平面$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$被三坐标面所割出的有限部分的面积.

$z = c - \frac{c}{a}x - \frac{by}{a}$， $z_x = -\frac{c}{a}$， $z_y = -\frac{c}{b}$． $D_{xy} = \{(x,y) \mid \frac{x}{a}+\frac{y}{b} \le 1,\ x \ge 0,\ y \ge 0\}$

(先不妨设 $a>0$, $b>0$)

$$\therefore S = \iint_D \sqrt{1+z_x^2+z_y^2}\, dx\, dy$$

$$= \iint_D \sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}\, dx\, dy = \frac{ab}{2}\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}$$

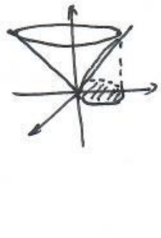
$$= \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

2.曲面$x^2+y^2-z+1=0$在点$P(1,1,1)$处的切平面被柱面$x^2+y^2=1$所截下的面积.

切平面为 $z-1 = 2(x-1)+2(y-1)$ 即 $z = 2x+2y-3$

$$\therefore S = \iint_{x^2+y^2\le 1} \sqrt{1+2^2+2^2}\, dx\, dy = 3\pi$$

七、求圆锥面$z=\sqrt{x^2+y^2}$被柱面$x^2+y^2=x$所割下部分的曲面面积.



$z = \sqrt{x^2+y^2}$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$， $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$．

投影区域 $D = \{(x,y) \mid (x-\frac{1}{2})^2+y^2 \le \frac{1}{4}\}$

$$\therefore S = \iint_D \sqrt{2}\, dx\, dy = \sqrt{2}\cdot\pi\cdot(\frac{1}{2})^2 = \frac{\sqrt{2}}{4}\pi$$

四、计算下列二次积分

1．$I=\int_0^2 \mathrm{d}x\int_x^2 \mathrm{e}^{-y^2}\mathrm{d}y$.

必须改为先 x 再 y 的积分

$D=\{(x,y)\mid 0\le x\le y,\ 0\le y\le 2\}$

$$I=\int_0^2 dy\int_0^y e^{-y^2}dx=\int_0^2 e^{-y^2}\cdot x\Big|_0^y dy$$

$$=\int_0^2 y\,e^{-y^2}dy=-\frac{1}{2}\int_0^2 e^{-y^2}d(-y^2)=-\frac{1}{2}e^{-y^2}\Big|_0^2$$

$$=\frac{1}{2}(1-e^{-4})$$

2．$I=\int_0^1 \mathrm{d}x\int_x^1 x\sin y^3\mathrm{d}y$.

$D_y=\{(x,y)\mid 0\le x\le y,\ 0\le y\le 1\}$

$$I=\int_0^1 dy\int_0^y x\sin y^3\,dx$$

$$=\int_0^1 \sin y^3\cdot\frac{x^2}{2}\Big|_0^y dy=\frac{1}{2}\int_0^1 y^2\cdot\sin y^3\,dy$$

$$=\frac{1}{6}\int_0^1 \sin y^3\,dy^3=-\frac{1}{6}\cos y^3\Big|_0^1=\frac{1}{6}(1-\cos 1)$$

五．若 $f(x,y)$ 在两坐标轴与直线 $x+y=1$ 所围区域上连续，且

$x\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y=f(x,y)-y$，求 $\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y$.

令 $A=\iint\limits_D f(x,y)dxdy$， 则 $Ax=f(x,y)-y$

两边在 D 上积分得 $A\iint\limits_D x\,dxdy=\iint\limits_D f(x,y)dxdy-\iint\limits_D y\,dxdy$

即 $A\iint\limits_D x\,dxdy=A-\iint\limits_D y\,dxdy$

其中 $\iint\limits_D x\,dxdy=\iint\limits_D y\,dxdy=\int_0^1 y\,dy\int_0^{1-y}dx=\int_0^1 y(1-y)dy$

$$=\frac{1}{6}$$

$\therefore\ \frac{A}{6}=A-\frac{1}{6}$，$A=\frac{1}{5}$.

§8.2 二重积分的计算（续）

一.填空

1．化下列二次积分为极坐标形式的二次积分 $\int_0^4 \mathrm{d}x \int_0^{\sqrt{4x-x^2}} f\left(x^2+y^2\right)\mathrm{d}y =$ $\int_0^{\frac{\pi}{2}} d\theta \int_0^{4\cos\theta} f(\rho^2)\rho\, d\rho$；

$\int_0^2 \mathrm{d}x \int_0^{\sqrt{3}x} f\left(\arctan\dfrac{y}{x}\right)\mathrm{d}y =$ $\int_0^{\frac{\pi}{3}} d\theta \int_0^{\frac{2}{\cos\theta}} f(\theta)\rho\, d\rho$； $\int_0^1 \mathrm{d}x \int_0^{x^2} f(x,y)\mathrm{d}y =$ $\int_0^{\frac{\pi}{4}} d\theta \int_{\frac{\sin\theta}{\cos^2\theta}}^{\sec\theta} f(\rho\cos\theta, \rho\sin\theta)\rho\, d\rho$.

计算 2．$\iint\limits_{x^2+y^2\le 1} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y =$ $\int_0^{2\pi} d\theta \int_0^1 \rho^2 d\rho$. $= \frac{2}{3}\pi$

计算 3．$\iint\limits_{x^2+y^2\le 1} \mathrm{e}^{-x^2-y^2}\mathrm{d}x\mathrm{d}y =$ $\int_0^{2\pi} d\theta \int_0^1 e^{-\rho^2}\rho\, d\rho$. $= \pi(1-e^{-1})$

4.设 $D=\left\{(x,y)\middle|\,0\le y\le\sqrt{1-x^2},-1\le x\le 1\right\}$，则 $\iint\limits_D xy\mathrm{d}x\mathrm{d}y =$ 0；

5．设 D 由圆周 $x^2+y^2=R^2$ 所围成的闭区域，则

$\iint\limits_D \left(x^2+y^2\right)\mathrm{d}\sigma =$ $\int_0^{2\pi} d\theta \int_0^R \rho^3 d\rho = 2\pi\cdot\frac{R^4}{4} = \frac{\pi}{2}R^4$

$\iint\limits_D x^2\mathrm{d}\sigma =$ $\frac{\pi}{4}R^4$； （由对称性，$\iint\limits_D x^2 dxdy = \iint\limits_D y^2 dxdy$

$\iint\limits_D \left(y^2+2x-6y+9\right)\mathrm{d}\sigma =$ $\frac{\pi}{4}R^4 + 2\times 0 - 4\times 0 + 9\times\pi R^2 = \frac{\pi}{4}R^4 + 9\pi R^2$；

$\iint\limits_D \left(\dfrac{x^2}{9}+\dfrac{y^2}{4}\right)\mathrm{d}\sigma =$ $(\frac{1}{9}+\frac{1}{4})\times\frac{\pi}{4}R^4 = \frac{13}{144}\pi R^4$.

二、利用极坐标计算下列各题

1．$I=\iint\limits_D \sin\sqrt{x^2+y^2}\,\mathrm{d}\sigma$,其中 $D=\left\{(x,y)\mid \pi^2\le x^2+y^2\le 4\pi^2\right\}$.

$D = \{(\rho,\theta) \mid \pi \le \rho \le 2\pi,\ \pi \le \theta \le 2\pi\}$

$I = \int_0^{2\pi} d\theta \int_\pi^{2\pi} \sin\rho\cdot\rho\, d\rho$

$= -2\pi \int_\pi^{2\pi} \rho\cdot(\cos\rho)'\, d\rho$

$= -2\pi\left[\rho\cos\rho\Big|_\pi^{2\pi} - \sin\rho\Big|_\pi^{2\pi}\right] = -2\pi\times 3\pi + 0$

$= -6\pi^2$

2． $I=\iint\limits_D \ln\left(1+x^2+y^2\right)\mathrm{d}\sigma$,其中是 D 由圆周 $x^2+y^2=1$ 与坐标轴所围成在第一象限内的闭区域.

$D=\{(\rho,\theta)\mid 0\le\rho\le 1,\ 0\le\theta\le\frac{\pi}{2}\}$

$$I=\int_0^{\frac{\pi}{2}}d\theta\int_0^1 \ln(1+\rho^2)\cdot\rho\, d\rho$$

$$=\frac{\pi}{4}\int_0^1 \ln(1+\rho^2)\, d\rho^2$$

$$\overset{\rho^2=t}{=}\frac{\pi}{4}\int_0^1 \ln(1+t)\, dt \overset{1+t=u}{=} \frac{\pi}{4}\int_1^2 \ln u\, du$$

$$=\frac{\pi}{4}(u\ln u-u)\Big|_1^2=\frac{\pi}{4}(2\ln 2-1)$$

三、计算二次积分 $I=\int_0^1 \mathrm{d}x\int_{x^2}^{x}\frac{1}{\sqrt{x^2+y^2}}\mathrm{d}y$.

$D=\{(\rho,\theta)\mid 0\le\rho\le\frac{\sin\theta}{\cos^2\theta},\ 0\le\theta\le\frac{\pi}{4}\}$

$$I=\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{\sin\theta}{\cos^2\theta}}\frac{1}{\rho}\cdot\rho\, d\rho=\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos^2\theta}d\theta$$

$$=\frac{1}{\cos\theta}\Big|_0^{\frac{\pi}{4}}=\sqrt{2}-1$$

四、求由曲面 $z=2x^2+4y^2$ 及 $z=6-4x^2-2y^2$ 所围成的立体的体积.

$D: x^2+y^2\le 1$

$$V=\iint\limits_{x^2+y^2\le 1}\left[(6-4x^2-2y^2)-(2x^2+4y^2)\right]dx\,dy=\iint\limits_{x^2+y^2\le 1}6(1-x^2-y^2)\,dx\,dy$$

$$=\int_0^{2\pi}d\theta\int_0^1 6(1-\rho^2)\rho\, d\rho$$

$$=2\pi\times 6\times\left(\frac{1}{2}-\frac{1}{4}\right)=\cancel{3\pi}\ \ 3\pi$$

五.用适当的方法计算二重积分

1. $\iint\limits_{D}(x+y)\mathrm{d}x\mathrm{d}y$ ，其中 $D=\{(x,y)\mid x^2+y^2\le 2x\}$.

$$I=\iint\limits_{D} x\,dx\,dy+\iint\limits_{D} y\,dx\,dy$$

形心坐标

D的面积

$$=\iint\limits_{D} x\,dx\,dy+0\left(=\bar{x}\cdot A(D)=1\times\pi=\pi\right)$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}(\rho\cos\theta)\cdot\rho\,d\rho=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\cdot\left.\frac{\rho^3}{3}\right|_0^{2\cos\theta}d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8}{3}\cos^4\theta\,d\theta=\frac{8}{3}\cdot\cancel{(\cos\theta)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{8}{3}\times2\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}=\pi$$

2.计算二重积分 $\iint\limits_{D}\sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\mathrm{d}\sigma$，其中 D 为 $x^2+y^2\le1$ 在第一象限部分.

$$I=\int_0^{2\pi}d\theta\int_0^1\sqrt{\frac{1-\rho^2}{1+\rho^2}}\rho\,d\rho=\pi\int_0^1\sqrt{\frac{1-\rho^2}{1+\rho^2}}\,d\rho^2$$

$$\overset{\rho^2=t}{=\!=}\pi\int_0^1\sqrt{\frac{1-t}{1+t}}\,dt=\pi\int_0^1\frac{1-t}{\sqrt{1-t^2}}\,dt$$

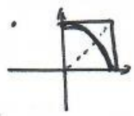
$$=\pi\left(\arcsin t+\sqrt{1-t^2}\right)\Big|_0^1=\pi\left(\frac{\pi}{2}-1\right)$$

3. $\iint\limits_{x^2+y^2\le1}(|x|+|y|)\mathrm{d}x\mathrm{d}y$ $=4\iint\limits_{\substack{x^2+y^2\le1\\x\ge0\\y\ge0}}(x+y)\,dx\,dy=8\iint\limits_{\substack{x^2+y^2\le1\\x\ge0\\y\ge0}}x\,dx\,dy$

$$=8\int_0^{\frac{\pi}{2}}d\theta\int_0^1\rho\cos\theta\cdot\rho\,d\rho=8\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\int_0^1\rho^2d\rho$$

$$=8\times1\times\frac{1}{3}=\frac{8}{3}$$

4. $\iint_D |x^2+y^2-1| \mathrm{d}\sigma$，其中 $D=\{(x,y)|0\le x\le 1,0\le y\le 1\}$.



$$= \iint_{\substack{x^2+y^2\le 1\\ x\ge 0\\ y\ge 0}} (1-x^2-y^2)\,dx\,dy + \iint_{\substack{D\\ x^2+y^2\ge 1\\ 0\le x\le 1\\ 0\le y\le 1}} (x^2+y^2-1)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 (1-\rho^2)\rho\, d\rho + 2\iint_{\substack{x^2+y^2\ge 1\\ 0\le y\le x\le 1}} (x^2+y^2-1)\,dx\,dy$$

$$= \frac{\pi}{2}\times\frac{1}{4} + 2\int_0^{\frac{\pi}{4}} d\theta \int_1^{\frac{1}{\cos\theta}} (\rho^2-1)\rho\, d\rho$$

$$= \frac{\pi}{8} + 2\int_0^{\frac{\pi}{4}} \left(\frac{\rho^4}{4}-\frac{\rho^2}{2}\right)\Big|_1^{\frac{1}{\cos\theta}} d\theta = \frac{\pi}{8} + 2\int_0^{\frac{\pi}{4}} \left(\frac{1}{4\cos^4\theta} - \frac{1}{2\cos^2\theta} + \frac{1}{4}\right) d\theta$$

$$= \frac{\pi}{8} + \frac{1}{2}\int_0^{\frac{\pi}{4}} (\tan^2\theta+1)\, d\tan\theta - \tan\theta\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\times\frac{\pi}{4} = \frac{\pi}{8} + \frac{2}{3} - 1 + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{3}$$

5. $\iint_D x\mathrm{d}x\mathrm{d}y$. 其中 $D=\left\{(\rho,\theta)\,|\,2\le\rho\le 2(1+\cos\theta), -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$

由对称性， $\iint_D x\,dx\,dy = 2\iint_{D_上} x\,dx\,dy = 2\int_0^{\frac{\pi}{2}} d\theta \int_2^{2(1+\cos\theta)} \rho^2\cos\theta\, d\rho$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} \left[(1+\cos\theta)^3 - 1\right]\cos\theta\, d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$$

$$= \frac{16}{3}\left(3\times\frac{\pi}{4} + 3\times\frac{2}{3} + \frac{3\times 1}{4\times 2}\times\frac{\pi}{2}\right) = \frac{16}{3}\left(2 + \frac{15}{16}\pi\right)$$

$$= \frac{32}{3} + 5\pi$$

§8.3 三重积分

一、填空

1．设$\Omega: x^2+y^2+z^2 \le R^2$，则$\iiint\limits_{\Omega}\left[\left(x^2+y^2\right)z+3\right]\mathrm{d}v =$ $0+\iiint\limits_{\Omega} 3\,\mathrm{d}v = 4\pi R^3$

2．设Ω为$a \le x \le b, c \le y \le d, l \le z \le m$，则$\iiint\limits_{\Omega} xy^2z^3\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$ $\frac{1}{24}(b^2-a^2)(d^3-c^3)(m^4-l^4)$

3．设Ω由曲面$z=2x^2+3y^2$及$z=3-x^2$所围成的闭区域，化三重积分

$I=\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$为三次积分是 $\int_{-1}^{1}\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y\int_{2x^2+3y^2}^{3-x^2} f(x,y,z)\mathrm{d}z$

二、设Ω是由平面$x+y+z=1$及三坐标面所围成的区域，计算

1．$I=\iiint\limits_{\Omega} z\mathrm{d}v$；

法一. 投影法 $\Omega=\{(x,y,z) \mid 0\le z\le 1-x-y,\ 0\le y\le 1-x,\ 0\le x\le 1\}$

$\therefore I=\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y} z\,\mathrm{d}z=\int_0^1 \mathrm{d}x\int_0^{1-x}\frac{z^2}{2}\Big|_0^{1-x-y}\mathrm{d}y$

$=\frac{1}{2}\int_0^1 \mathrm{d}x\int_0^{1-x}(1-x-y)^2\mathrm{d}y=-\frac{1}{2}\int_0^1\mathrm{d}x\int_0^{1-x}(1-x-y)^2\mathrm{d}(1-x-y)$

$=-\frac{1}{6}\int_0^1(1-x-y)^3\Big|_0^{1-x}\mathrm{d}x=\frac{1}{6}\int_0^1(1-x)^3\mathrm{d}x=\frac{1}{24}$

法二. 截面法

$I=\int_0^1 z\,\mathrm{d}z\iint\limits_{D_z}\mathrm{d}x\mathrm{d}y$

$=\int_0^1 z\cdot\frac{1}{2}(1-z)^2\mathrm{d}z$

$=\frac{1}{2}\int_0^1(1-z)^2\mathrm{d}z-\frac{1}{2}\int_0^1(1-z$

$=\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

2．$I=\iiint\limits_{\Omega}(x+2y+3z)\mathrm{d}v$．

Ω关于x,y,z具有轮换对称性，故

$\iiint\limits_{\Omega} x\,\mathrm{d}v=\iiint\limits_{\Omega} y\,\mathrm{d}v=\iiint\limits_{\Omega} z\,\mathrm{d}v$

$\therefore I=(1+2+3)\iiint\limits_{\Omega} z\,\mathrm{d}v=6\times\frac{1}{24}=\frac{1}{4}$

法三. $I=\bar{z}\cdot V(\Omega)$

$=\frac{0+0+0+1}{4}\times\frac{1}{6}=\frac{}{}$

形心坐标 体积

五 ~~三~~、计算 $I=\iiint\limits_{\Omega} z\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω 是由 $z=\sqrt{x^2+y^2}$ 与平面 $z=1$ 所围成的形体.

$D=\{(\rho,\theta)\mid 0\le\rho\le1,\ 0\le\theta\le2\pi\}$，$\rho\le z\le1$

$$\therefore I=\iiint\limits_{\Omega} z\cdot\rho\cdot\rho\,d\rho\,d\theta\,dz$$

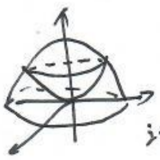
$$=\int_0^{2\pi}d\theta\int_0^1 d\rho\int_\rho^1 z\cdot\rho^2\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^1 \rho^2\cdot\frac{z^2}{2}\Big|_\rho^1 d\rho=2\pi\cdot\frac{1}{2}\int_0^1(\rho^2-\rho^4)d\rho$$

$$=\pi\cdot\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2}{15}\pi$$

六 ~~四~~、计算 $I=\iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω 是由上半球面 $z=\sqrt{4-x^2-y^2}$ 及抛物面 $x^2+y^2=3z$ 所围成的形体.

法一. 投影法：$I=\int_0^{2\pi}d\theta\int_0^{\sqrt3}\rho d\rho\int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} z\,dz=2\pi\int_0^{\sqrt3}\rho\cdot\frac{z^2}{2}\Big|_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}}d\rho$



$$=\pi\int_0^{\sqrt3}\left(4-\rho^2-\frac{\rho^4}{9}\right)\rho\,d\rho=\pi\left(2\rho^2-\frac{\rho^4}{4}-\frac{\rho^6}{54}\right)\Big|_0^{\sqrt3}=\frac{13}{4}\pi$$

法二. 截面法

$$I=\int_0^1 z\,dz\iint\limits_{D_{z下}}dxdy+\int_1^2 z\,dz\iint\limits_{D_{z上}}dxdy$$

$$=\int_0^1 z\,dz\cdot(3z\cdot\pi)+\int_1^2 z\,dz\cdot(4-z^2)\cdot\pi=\pi\int_0^1 3z^2dz+\pi\int_1^2 z(4-z^2)dz$$

$$=\pi z^3\Big|_0^1+\pi\left(2z^2-\frac{z^4}{4}\right)\Big|_1^2=\pi+\pi\left(6-\frac{15}{4}\right)=\frac{13}{4}\pi$$

七 ~~五~~、设 $\Omega: x^2+y^2+z^2\le1$，计算

1．$I=\iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$；

Ω 关于 xOy 面对称，z 关于 z 是奇函数，$\therefore I=0$

（或 $I=\bar z\cdot V(\Omega)=0\times\frac{4}{3}\pi=0$）

2．$I=\iiint\limits_{\Omega} z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$；

解法一．截面法．

$$I = 2\iiint\limits_{\Omega_上} z^2 dxdydz = 2\int_0^1 z^2 dz \iint\limits_{D_z} dxdy = 2\int_0^1 z^2 dz \cdot \pi(1-z^2)$$

$$= 2\pi\int_0^1 (z^2 - z^4) dz = \frac{4}{15}\pi$$

解法二．球坐标法．

$$I = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 (r\cos\varphi)^2 \cdot r^2 \sin\varphi \, dr$$

$$= 2\pi \int_0^{\pi} \cos^2\varphi \cdot \sin\varphi \, d\varphi \int_0^1 r^4 dr$$

$$= 2\pi \times \frac{2}{3} \times \frac{1}{5} = \frac{4\pi}{15}$$

3．$I=\iiint\limits_{\Omega} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$．

由轮换对称性，$\iiint\limits_{\Omega} x^2 dxdydz = \iiint\limits_{\Omega} y^2 dxdydz = \iiint\limits_{\Omega} z^2 dxdydz$

$$\therefore I = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\cdot\frac{4\pi}{15}$$

三 ~~六~~. 设Ω是平面$z=0, z=y, y=1$以及抛物柱面$y=x^2$所围成的几何体，计算.

1. $\iiint\limits_{\Omega} z\mathrm{d}x\mathrm{d}y\mathrm{d}z$；

$$\Omega = \{(x,y,z) \mid 0 \le z \le y,\ x^2 \le y \le 1,\ -1 \le x \le 1\}$$

$$\therefore \iiint\limits_{\Omega} z\,dxdydz = \int_{-1}^1 dx \int_{x^2}^1 dy \int_0^y z\,dz$$

$$= \int_{-1}^1 dx \int_{x^2}^1 \frac{y^2}{2} dy = \int_{-1}^1 \frac{y^3}{6}\Big|_{x^2}^1 dx = \frac{1}{6}\int_{-1}^1 (1-x^6)dx$$

$$= \frac{1}{6} \times 2 \times \frac{6}{7} = \frac{2}{7}$$

2. $\iiint\limits_{\Omega} xz\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

$$I = \int_{-1}^{1} x \cdot \left(\frac{y^3}{6}\right)\Big|_{x^2}^{1} dx = \frac{1}{6}\int_{-1}^{1} x(1-x^6)\,dx = 0$$

四 ~~七~~ . 设 f 在 $[0,1]$ 上连续，证明：$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \pi\int_{-1}^{1} f(u)(1-u^2)\mathrm{d}u$.

由截面法知，$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)\,dx\,dy\,dz = \int_{-1}^{1} f(z)\,dz \iint\limits_{D_z} dx\,dy$

$$= \int_{-1}^{1} f(z)\cdot \pi(1-z^2)\,dz = \pi\int_{-1}^{1} f(u)(1-u^2)\,du.$$

自测题三（重积分）

一、选择题（每题 3 分，共 15 分）

1、设有空间闭区域$\Omega_1=\{(x,y,z)|x^2+y^2+z^2\le R^2,z\ge 0\}$，

$\Omega_2=\{(x,y,z)|x^2+y^2+z^2\le R^2,x\ge 0,y\ge 0,z\ge 0\}$则有（C）

A、$\iiint_{\Omega_1}xdv=4\iiint_{\Omega_2}xdv$，　　B、$\iiint_{\Omega_1}ydv=4\iiint_{\Omega_2}ydv$，

C、$\iiint_{\Omega_1}zdv=4\iiint_{\Omega_2}zdv$，　　D、$\iiint_{\Omega_1}xyzdv=4\iiint_{\Omega_2}xyzdv$

2、设有平面闭区域$D=\{(x,y)|-a\le x\le a,x\le y\le a\}$

$D_1=\{(x,y)|0\le x\le a,x\le y\le a\}$，则$\iint\limits_D(xy+\cos x\sin y)dxdy=$（A）

A、$2\iint\limits_{D_1}\cos x\sin ydxdy$，　B、$2\iint\limits_{D_1}xydxdy$　C、$4\iint\limits_{D_1}xydxdy$，　D、0

3 $I=\int_0^1dy\int_0^{\sqrt{1-y}}f(x,y)dx$，则交换积分次序后$I=$（C）

B、$I=\int_0^1dx\int_0^{\sqrt{1-x}}f(x,y)dy$，　B、$I=\int_0^{\sqrt{1-y}}dx\int_0^1f(x,y)dy$　C、$I=\int_0^1dx\int_0^{1-x^2}f(x,y)dy$

E、$I=\int_0^1dx\int_0^{1+x^2}f(x,y)dy$

4、已知$\int_0^1f(x)dx=\int_0^1xf(x)dx$，$D=\{(x,y)|x+y<1,x>0,y>0\}$则$\iint_Df(x)dxdy=$（D）　D

B、2 B、3，　C、1D、0

解 $\int_0^1dx\int_0^{1-x}f(x)dy = \int_0^1(1-x)f(x)dx=0$

5、$I=\iint_{x^2+y^2\le 4}\sqrt[3]{1-x^2-y^2}dxdy$，则必有：（B）

B、$I>0$　B、$I<0$　C、$I=0$　D、$I\ne 0$但无法确定符号

$$\cancel{I=2\pi\int_0^2\sqrt[3]{1-r^2}dr=-\frac{3}{4}\pi[(-3)^{\frac{4}{3}}-1]<0}$$ ×

$1/\sqrt{2}$ → 分成三部分估计.

二、填空题（每题 3 分，共 15 分）

1、设$f(x,y)$连续，$f(0,0)=1$，$D=\left\{(x,y)\middle|x^2+y^2\le t^2\right\}$，则$\lim\limits_{t\to 0^+}\frac{1}{\pi t^2}\iint\limits_D f(x,y)\mathrm{d}\sigma=$ 1

$= f(\xi,\eta)\pi t^2$

2、$\int_0^1dy\int_{\arcsin y}^{\pi-\arcsin y}xdx=$ π　$\int_0^\pi xdx\int_0^{\sin x}dy=\int_0^\pi x\sin xdx=\pi$

3、$\int_0^1dx\int_x^1e^{-y^2}dy=$ $\frac{1}{2}(1-e^{-1})$　$\int_0^1e^{-y^2}dy\int_0^ydx=\frac{1}{2}(1-e^{-1})$

4、$f(x,y)$在矩形区域$D:\{(x,y)|0\le x\le 1,0\le y\le 1\}$上连续，且

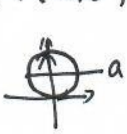
$x\left(\iint\limits_Df(x,y)dxdy\right)^2=f(x,y)-\frac{1}{2}$则$f(x,y)=$ $x+\frac{1}{2}$

$xA^2=f(x,y)-\frac{1}{2}$

$A^2\iint\limits_D xdxdy=A-\frac{1}{2}$

$A^2\cdot\frac{1}{2}=A-\frac{1}{2}\Rightarrow A=1$

5、将二重积分$\int_{-a}^{a}dx\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}}f(x,y)dy$化为极坐标后的二次积分为$\underline{\int_0^{\pi}}d\theta\int_0^{2\sin\theta}f(\rho\cos\theta,\rho\sin\theta)\rho d$



三、解下列各题（每题 10 分，共 40 分）

1、计算三重积分$\iiint\limits_{\Omega}zdV$ 其中Ω由锥面$z=\dfrac{h}{R}\sqrt{x^2+y^2}$与平面$z=h(R>0,h>0)$围成.

$$=\int_0^h z\,dz\iint\limits_{x^2+y^2\le(\frac{R}{h}z)^2}dx\,dy=\int_0^h z\cdot\pi\left(\frac{R}{h}z\right)^2=\frac{\pi R^2}{h^2}\int_0^h z^3\,dz=\frac{\pi}{4}R^2h^2$$

2、Ω是由平面$x+y+z=1$与三个坐标平面所围成的空间区域，计算三重积分$\iiint\limits_{\Omega}(x+y+z)dxdydz$.

$$\iiint\limits_{\Omega}x\,dxdydz=\iiint\limits_{\Omega}y\,dxdydz=\iiint\limits_{\Omega}z\,dxdydz$$

$$\therefore\ \iiint\limits_{\Omega}(x+y+z)\,dxdydz=3\iiint\limits_{\Omega}z\,dxdydz$$

$$=3\int_0^1 z\,dz\cdot\iint\limits_{\substack{x+y\le1-z\\x\ge0\\y\ge0}}dx\,dy$$

$$=3\int_0^1 z\cdot\frac{(1-z)^2}{2}dz=\frac{1}{8}$$

3、计算二重积分 $I=\iint\limits_D r^2\sin\theta\sqrt{1-r^2\cos 2\theta}\,drd\theta$，其中

$D=\{(r,\theta)\mid 0\le r\le\sec\theta, 0\le x\le 1\}$.

ρ

$$I=\iint\limits_D y\sqrt{1-x^2+y^2}\,dxdy$$

$$=\frac12\int_0^1 dx\int_0^x\sqrt{1-x^2+y^2}\,d\sqrt{1-x^2+y^2}=\frac12\int_0^1\frac23(1-x^2+y^2)^{\frac32}\Big|_0^x dx$$

$$=\frac13\int_0^1\left[1-(1-x^2)^{\frac32}\right]dx=\frac13-\frac13\int_0^1\sqrt{1-x^2}^{\,3}dx$$

$$\overset{x=\sin t}{=\!=}\frac13-\frac13\int_0^{\frac{\pi}{2}}\cos^4 t\,dt=\frac13-\frac13\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}=\frac13-\frac{\pi}{16}.$$

4、设 Ω 是由半球面 $z=\sqrt{4-x^2-y^2}$ 与旋转抛物面 $3z=x^2+y^2$ 所围空间闭区域，求它的体积.

$$D=\{(x,y)\mid x^2+y^2\le 3\}$$

$$\therefore V=\iint\limits_D\left(\sqrt{4-x^2-y^2}-\frac{x^2+y^2}{3}\right)dxdy$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt3}\left(\sqrt{4-\rho^2}-\frac{\rho^2}{3}\right)\rho\,d\rho$$

$$=2\pi\cdot\left[-\frac13(4-\rho^2)^{\frac32}-\frac{\rho^4}{12}\right]_0^{\sqrt3}=2\pi\cdot\left(\frac73-\frac34\right)=\frac{19\pi}{6}$$

四、解下列各题（每题 10 分，共 30 分）

1、设 $f(x)=g(x)=\begin{cases}a, & 0\le x\le 1\\ 0, & \text{其它}\end{cases}$，$D$ 是全平面。求二重积分

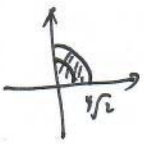
$I=\iint\limits_D f(x)g(y-x)dxdy$.

$g(y-x)$ 不为 0 的区域为 $0\le y-x\le 1$，即 $x\le y\le x+1$，所以

$$I=\int_0^1 dx\int_x^{x+1}a^2\,dy=a^2$$

2、设平面区域 $D=\{(x,y)\mid x^2+y^2\le\sqrt{2},x\ge0,y\ge0\}$，$[x]$表示不超过 x 的最大整数，计算：

$I=\iint\limits_D xy\left[1+x^2+y^2\right]\mathrm{d}x\mathrm{d}y$.



当 $x^2+y^2<1$ 时，$[1+x^2+y^2]=1$

$\therefore \iint\limits_{x^2+y^2<1} xy[1+x^2+y^2]dxdy = \iint\limits_{x^2+y^2\le1} xy\,dxdy = \int_0^{\frac{\pi}{2}}d\theta\int_0^1(\rho\cos\theta)(\rho\sin\theta)\rho\,d\rho$

$=\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^1\rho^3d\rho = \frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$

当 $1\le x^2+y^2<\sqrt{2}$ 时，$[1+x^2+y^2]=2$

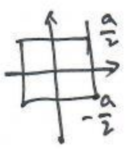
$\therefore \iint\limits_{1\le x^2+y^2<\sqrt{2}} xy[1+x^2+y^2]dxdy = 2\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_1^{\sqrt[4]{2}}\rho^3d\rho = 1\times\frac{2-1}{4}=\frac{1}{4}$

综上，$I=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$

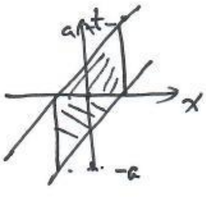
3、设 $f(t)$ 在 R 上连续，常数 $a>0$，区域 $D=\{(x,y)\mid |x|\le a/2,|y|\le a/2\}$

证明：$\iint\limits_D f(x-y)dxdy=\int_{-a}^{a}f(t)(a-|t|)dt$.

$\iint\limits_D f(x-y)dxdy=\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\int_{-\frac{a}{2}}^{\frac{a}{2}}f(x-y)dy \xlongequal{x-y=t}\int_{-\frac{a}{2}}^{\frac{a}{2}}dx\int_{x-\frac{a}{2}}^{x+\frac{a}{2}}f(t)dt$



交换次序 $\int_{-a}^{0}f(t)dt\int_{-\frac{a}{2}}^{t+\frac{a}{2}}dx+\int_0^a f(t)dt\int_{t-\frac{a}{2}}^{\frac{a}{2}}dx$



$=\int_{-a}^{0}f(t)dt\cdot(t+a)+\int_0^a f(t)dt\cdot(a-t)$

$=\int_{-a}^{a}f(t)(a-|t|)dt$.