Conditional Probability Models.

* Recall:

$P[A|B] \to$ Prob. of an event, given the fact that we know event B has occurred.

Except when the events are indep, the knowledge of B occurring will change $P[A]$.

* $P[A|B] \to$ New probability as a result of additional knowledge.

$P[A] \to$ Prior prob

$P[A|B] \to$ Posterior probability

Ex: There are 2 coins $\to$ ① Fair coin
② Biased Coin.

Q: Choose the coin & toss it 4 times. What is the prob that we get 2 or more heads?

Soln: $P[2 \text{ or More heads}]$ Dep. on the Coin we choose

Because one of the coins is biased.

$\therefore$ we define conditional PMF.

$$p_X[k \mid \text{fair coin was chosen}]$$

$$p_X[k \mid \text{Biased coin was chosen}]$$

$\rightarrow$ $p_X[k \geq 2]$.

Once we know which coin is chosen, we can easily specify the PMF.

$\rightarrow$ Once the conditional PMF is known, we can get the num of heads.

$p_X[k] \rightarrow$ Prob. of observing k-heads

$$p_X[k] = p_X\left[k \mid \begin{matrix}\text{Fair coin}\\ \text{Chosen}\end{matrix}\right] p[fc]$$

$$+ \; p_X\left[k \mid \begin{matrix}\text{Biased coin}\\ \text{Chosen}\end{matrix}\right] p[BC].$$

$\Rightarrow$ Required PMF $\rightarrow$ Directly dep. on conditional probabilities.

* When is Conditional prob models appropriate?

Useful when experiment consists of Sequence of Subexpts where the Second part of the expt depends on the first part.

* Conditional PMF

Let's look the previous ex of Choosing a coin and tossing the chosen coin 4 times.

Observation: 2 or more heads.

1. Step 1: Choose the coin either F or B

Coin F:

$p(H) = p_1$

$p(T) = 1 - p_1$

Coin B

$p(H) = p_2$

$p(T) = 1 - p_2$

2. Let X be the DRV describing the outcome of the coin choice expt

$$x = \begin{cases} 1 & \text{if } F \text{ is Chosen} \\ 2 & \text{if } B \text{ is Chosen} \end{cases}$$

$$\therefore \quad S_x = \{1, 2\}.$$

$$p_x[i] = \begin{cases} \alpha & \text{if } i = 1 \\ 1-\alpha & \text{if } i = 2 \end{cases}$$

$$0 < \alpha < 1.$$

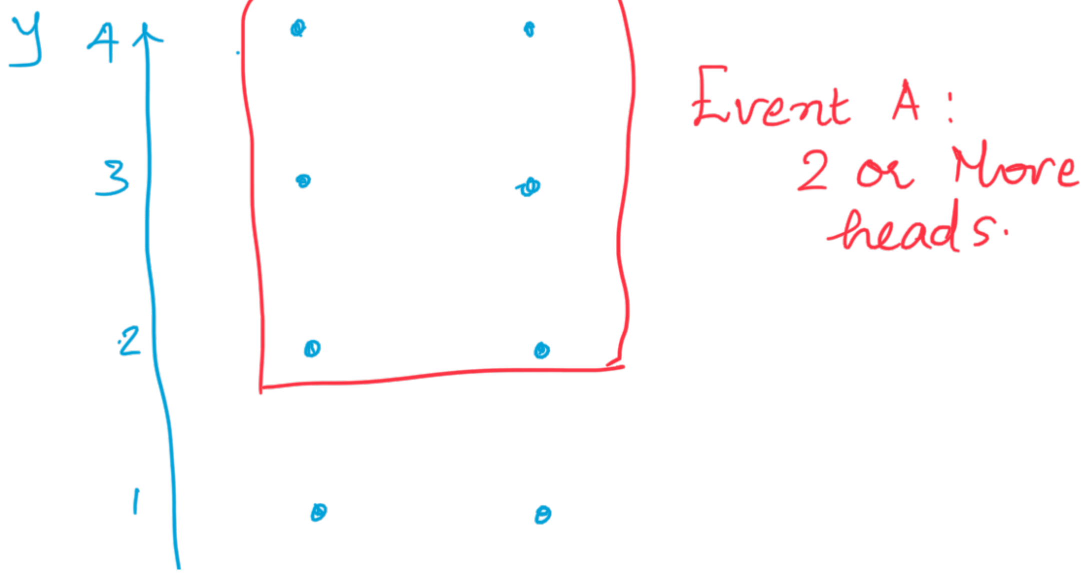
* Second part of the expt is tossing the Chosen coin 4 times in Succession.

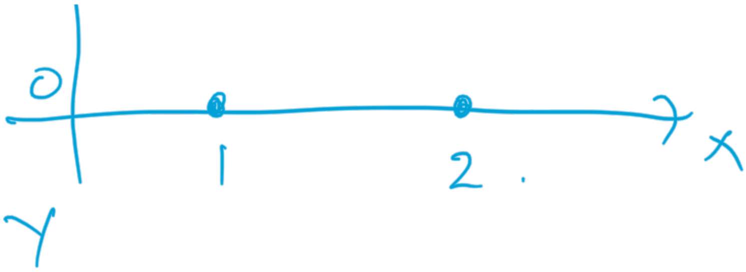
* Let Y denote the # of heads observed

$$S_Y = \{0, 1, 2, 3, 4\}$$

* The overall sample sp. of the Combined expt is

$$S_{XY} = S_X \times S_Y.$$





Overall outcome is (X, Y)

X: Coin chosen

Y: # of heads observed for 4 tosses of the chosen coin.

$$P[2 \text{ or more heads}] = P[A]$$

$$= \sum_{\{(i,j) : (i,j) \in A\}} p_{X,Y}[i,j]$$

$$= \sum_{i=1}^{2} \sum_{j=0}^{4} p_{X,Y}[i,j]$$

$$* \; p_{X,Y}[i,j] = P[X = i, Y = j]$$

$$= p[Y = j \mid X = i] \, p_X[X = i]$$

$$= p[Y = j \mid X = i] \, p[i]$$

Look at the following:

The chosen coin is tossed 4 times.

What can we say about the distribution of no. of heads?

$\Rightarrow$ No. of heads observed is Binomial

$$P[Y=j \mid X=i] = {}^4C_j \, p_i^j (1-p_i)^{4-j}$$

$j = 0, 1, 2, 3, 4.$

$\Downarrow$

Prob depends on the outcome $X=i$ via p_i.

* For a particular value of $X=i$,

(i) $0 \le P[Y=j \mid X=i] \le 1.$

(ii) $\sum_{j=0}^{4} P[Y=j \mid X=i] = 1.$

$\Rightarrow$ Having chosen a coin i, $i \in \{1,2\}$, now we look at the prob mass fn of no. of heads

→ If we sum the prob. values it must sum to 1

Ex: Assume, we have chosen fair coin, which means $X = 1$.

$$\Rightarrow P[Y=j \mid X=1]$$

$$\Rightarrow P[Y=0 \mid X=1],\ P[Y=1 \mid X=1],$$
$$P[Y=2 \mid X=1] \ \ldots\ P[Y=4 \mid X=1]$$

⇒ PMF of # of heads w.r.to fair coin

$$\Rightarrow \sum_j p_{Y|X}(Y=j \mid X=1) = 1$$

Q: What is the prob. of getting 2 or more heads? $p_X[k]$

$$p_{X,Y}[i,j] = p_{Y|X}[j|i]\, p_X[i].$$

$$P[A \cap B] = P[A|B]\, P[B].$$

∴ Joint PMF

$P_{X,Y}[i,j]$ = Conditional PMF of heads having chosen coin 1 + Cond. PMF of heads having chosen Coin-2.

Binomial

$$= P_{Y|X}[j|1]\, P_X[1] + P_{Y|X}[j|2]\, P_X[2]$$

Binomial

$$= \binom{4}{j} p_1^{j}(1-p_1)^{4-j}\alpha + {}^{4}C_j\, p_2^{j}(1-p_2)^{4-j}(1-\alpha).$$

$\Downarrow$ Pr [choosing Fair]

$\downarrow$ Prob of choosing biased coin

$\Downarrow$ Pr [choosing Fair]

$\downarrow$ Prob of choosing biased

coin'

$$+ \sum_{j=2}^{4} {}^4C_j \, p_2^j (1-p_2)^{4-j} (1-\alpha).$$

* Given the joint PMF & the marginal PMFs, how do we get Conditional PMF?

Cond. Prob. $P[A|B] = \dfrac{P[A \cap B]}{P[B]}$

Along the same lines, we look at the conditional PMF?

$$P_{Y|X}[Y=j \mid X=i] = \frac{p_{X,Y}[i,j]}{p_X[i]}$$

$$= \frac{P[X=i \cap Y=j]}{P[X=i]}.$$

$\Rightarrow \; p_X[i] > 0.$

$\Rightarrow$ Marginal PMF $> 0.$

$p_{X,Y}(x_i, y_j) \rightarrow$ joint PMF

$p_X[x_i] \rightarrow$ Marginal PMF $\neq 0$

$p_{Y|X}[y_j | x_i] \rightarrow$ Conditional PMF.

Note: (i) $p_X[x_i] \neq 0$

(ii) The conditional PMF $\rightarrow$ looks like a fn of both x_i & y_j.

But the moment we fix x_i, then

$p[y_j | x_i = x] \rightarrow$ family of PMFs.

Set of PMFs.

$p_{Y|X}[j|1]$ & $p_{Y|X}[j|2]$

$\Rightarrow$ PMF$(Y) \Rightarrow \{ p_{Y|X}(j|1), p_{Y|X}(j|2) \}$.

Valid PMFs

Each member of the set is a valid PMF.

$$* \quad \sum_i p_{Y|X}(y=j \mid x=1) = 1$$

$$\sum_j p_{Y|X}(y=j \mid x=2) = 1.$$

$$* \quad \sum_i p_{Y|X}(y=j \mid x=i) \neq 1.$$

$$= p_{Y|X}(y=j \mid x=1) + p_{Y|X}(y=j \mid x=2)$$