Solutions Logic Course

Data Analytics Lab

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Exercise 0

Give an example where "$A \Rightarrow B$" and its reciprocal "$A \Leftarrow B$" are not equivalent.

1. "$ab = 0 \Rightarrow a = 0$" is not equivalent to "$ab = 0 \Leftarrow a = 0$".
2. "It rains $\Rightarrow$ I go out" is not equivalent to "It rains $\Leftarrow$ I go out".

Exercise 1

Give the negation and the contraposition of the following assertions.

Let us recall the following useful theorems

Theorem 2.5 (Negation of $\|, \&$ and $\Rightarrow$)

Let A and B be two assertions, we have

1. $n(A \| B) \Leftrightarrow nA \,\&\, nB$
2. $n(A \,\&\, B) \Leftrightarrow nA \| nB$
3. $n(A \Rightarrow B) \Leftrightarrow A \,\&\, nB$

Theorem 2.6 (Negation of assertion with quantifiers)

Let E be a set and $A(x)$ an assertion indexed by $x \in E$, we have

1. $n(\exists x \in E,\, A(x)) \Leftrightarrow \forall x \in E,\, nA(x)$
2. $n(\forall x \in E,\, A(x)) \Leftrightarrow \exists x \in E,\, nA(x)$

We also recall that the contraposition of "$A \Rightarrow B$" is "$nB \Rightarrow nA$". Let us denote it by "$c\,[A \Rightarrow B]$".

- Negation

1. We do this one in details:

$$n\,\{\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [x < y \Rightarrow f(x) < f(y)]\} \Leftrightarrow \exists x \in \mathbb{R}, n\,\{\forall y \in \mathbb{R}, [x < y \Rightarrow f(x) < f(y)]\}$$
$$\Leftrightarrow \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, n\,[x < y \Rightarrow f(x) < f(y)] \Leftrightarrow \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, [x < y \,\&\, f(x) \geq f(y)]$$

2. $n\,\{\forall \varepsilon > 0, \exists N \in \mathbb{N}, [n \geq N \Rightarrow |u_n - u| < \varepsilon]\} \Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, [n \geq N \,\&\, |u_n - u| \geq \varepsilon]$

3. $n\,\{\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}, [|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon]\} \iff$
   $\exists x \in \mathbb{R}, \exists \varepsilon > 0, \forall \delta > 0, \exists y \in \mathbb{R}, [|x-y| < \delta \;\&\; |f(x) - f(y)| \geq \varepsilon]$
4. $n\,\{\forall E \subset \mathbb{N}, [E \neq \emptyset \implies (\exists n \in E, \forall m \in E, m \geq n)]\} \iff \exists E \subset \mathbb{N}, [E \neq \emptyset \;\&\; (\forall n \in E, \exists m \in E, m < n)]$

- Contraposition

1. $c\,\{\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [x < y \implies f(x) < f(y)]\} \iff \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [f(x) \geq f(y) \implies x \geq y]$
2. $c\,\{\forall \varepsilon > 0, \exists N \in \mathbb{N}, [n \geq N \implies |u_n - u| < \varepsilon]\} \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, [|u_n - u| \geq \varepsilon \implies n < N]$
3. $c\,\{\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}, [|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon]\} \iff$
   $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}, [|f(x) - f(y)| \geq \varepsilon \implies |x-y| \geq \delta]$
4. $c\,\{\forall E \subset \mathbb{N}, [E \neq \emptyset \implies (\exists n \in E, \forall m \in E, m \geq n)]\} \iff \forall E \subset \mathbb{N}, [(\forall n \in E, \exists m \in E, m < n) \implies E = \emptyset]$

Exercise 2

Let E and F be two sets and $A(x, y)$ an assertion indexed by $x \in E$ and $y \in F$.
Find an example where "$\exists x \in E, \forall y \in F, A(x, y)$" is not equivalent to "$\forall y \in F, \exists x \in E, A(x, y)$".

1. Set $E = F = \mathbb{N}$ and $A(x, y) =$ "$x \geq y$". In that case, the assertions are not equivalent since "$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \geq y$" is false whereas "$\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, x \geq y$" is true.
2. If we set $A(x, y) =$ "$x \leq y$" instead, the assertions are both true but still not equivalent (why?).

Exercise 3

If f is a function from $\mathbb{R}$ to itself, what do we need to show to prove that f is not continuous in $x_0 \in \mathbb{R}$? Similarly, what do we need to show to prove that an estimator $\hat{\theta}$ of θ_0 is not consistent?

1. By definition, f is continuous in $x_0 \in \mathbb{R}$ if and only if

$$\text{"}\forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}, [|x_0 - y| < \delta \implies |f(x_0) - f(y)| < \varepsilon]\text{"}.$$

   To show that f is not continuous in $x_0 \in \mathbb{R}$ we need to show that the negation of the last assertion is true, that is

$$\text{"}\exists \varepsilon > 0, \forall \delta > 0, \exists y \in \mathbb{R}, [|x_0 - y| < \delta \;\&\; |f(x_0) - f(y)| \geq \varepsilon]\text{"}.$$

2. By definition, an estimator $\hat{\theta}$ of θ_0 is consistent if and only if

$$\text{"}\forall \varepsilon > 0, \forall \delta > 0, \exists N \geq 0, \forall n \geq N,\ \mathbb{P}\left(\|\hat{\theta} - \theta_0\| \geq \delta\right) \leq \varepsilon\text{"}.$$

   To show that $\hat{\theta}$ is not consistent we need to show that the negation of the last assertion is true, that is,

$$\text{"}\exists \varepsilon > 0, \exists \delta > 0, \forall N \geq 0, \exists n \geq N,\ \mathbb{P}\left(\|\hat{\theta} - \theta_0\| \geq \delta\right) > \varepsilon\text{"}.$$

Exercise 4

What do we need to show if we want to prove

$$\text{"}p \text{ divide } ab\text{"} \Rightarrow \text{"}p \text{ divide } a \parallel p \text{ divide } b\text{"},$$

by contraposition? by contradiction?

- By contraposition we need to show that $c\,[p$ divide $ab \Rightarrow p$ divide $a \parallel p$ divide $b]$ is true, that is

$$p \text{ doesn't divide } a \;\&\; p \text{ doesn't divide } b \Rightarrow p \text{ doesn't divide } ab.$$

- By contradiction we need to suppose that

$$\text{"}p \text{ divide } ab\text{"} \;\&\; \text{"}p \text{ doesn't divide } a \parallel p \text{ doesn't divide } b\text{"},$$

  is true and show that it leads to a contradiction.

Exercise 5

1. *Prove by contradicton that "$\nexists a \in, a \times 0 = 1$".*
   Suppose that "$\exists a \in, a \times 0 = 1$" is true. Therefore we have

$$2 = 1 + 1 = a \times 0 + a \times 0 = a \times (0 + 0) = a \times 0 = 1,$$

   which is a contradiction.

2. *Prove by induction that "$\forall n \in \mathbb{N}, \ \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$"*
   First, we define for all $n \in \mathbb{N}$ the assertion $A_n :=$ "$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$".

   (a) A_0 is true since: $\sum_{k=0}^{0} k^2 = 0 = \frac{0(0+1)(2\times 0+1)}{6}$

   (b) Suppose that A_n is true for some $n \geq 0$ and let's show that it implies that A_{n+1} is true, that is,

$$\sum_{k=0}^{n+1} k^2 = \frac{(n+1)(n+2)(2\times(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

   We compute

$$\sum_{k=0}^{n+1} k^2 = (n+1)^2 + \sum_{k=0}^{n} k^2 \overset{(IH)}{=} (n+1)^2 + \frac{n(n+1)(2n+1)}{6} = \cdots = \frac{(n+1)(n+2)(2n+3)}{6},$$

   which ends the proof.

3. *Prove by induction that "$\forall n \in \mathbb{N}, \ 3^{2n} - 2^n$ is a multiple of 7."*
   First, we define for all $n \in \mathbb{N}$ the assertion $A_n :=$ "$3^{2n} - 2^n$ is a multiple of 7".

   (a) A_0 is true since: $3^{2\times 0} - 2^0 = 0 = 0 \times 7$.

(b) Suppose that A_n is true for some $n \geq 0$ and let's show that it implies that A_{n+1} is true, that is,

$$3^{2(n+1)} - 2^{n+1} \text{ is a multiple of } 7.$$

By the induction hypothesis, there exists $m \in \mathbb{Z}$ such that $3^{2n} = 7m + 2^n$. We compute

$$3^{2(n+1)} - 2^{n+1} = 3^{2n}3^2 - 2^n 2 \overset{(IH)}{=} (7m + 2^n)3^2 - 2^n 2 = \cdots = 7(9m + 2^n),$$

which ends the proof.

Exercise 6

We consider functions f from $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ to $\mathbb{R}$. We define $f(n) = O_{\delta \in \mathbb{N}^*}\left\{n^{-\delta}\right\}$, if and only if

$$\exists n' > 0,\ \ \exists M > 0,\ \ \forall \delta \in \mathbb{N}^*,\ \ \forall n \geq n',\ \ |f(n)| \leq M^\delta n^{-\delta}.$$

Suppose that the following assertion is true

$$''f(n) = O_{\delta \in \mathbb{N}^*}\left\{n^{-\delta}\right\} \Rightarrow \exists n^* > 0, \forall n \geq n^*, f(n) = 0.'' \tag{1}$$

(a) *Prove that* "$\exp(-n) \neq O_{\delta \in \mathbb{N}}\left\{n^{-\delta}\right\}$" *using* (1).
We prove this by contraposition. In this case, the assertion of interest is

$$\forall n^* > 0, \exists n \geq n^*, f(n) \neq 0 \Rightarrow f(n) \neq O_{\delta \in \mathbb{N}^*}\left\{n^{-\delta}\right\}.$$

By hypothesis, this assertion is true. Now, since $\exp(-n) \neq 0$ for all $n \in \mathbb{N}^*$, we clearly have

$$\forall n^* > 0, \exists n \geq n^*, \exp(-n) \neq 0,$$

which therefore implies that $\exp(-n) \neq O_{\delta \in \mathbb{N}}\left\{n^{-\delta}\right\}$.

By definition, $f(n) = O\left\{n^{-\delta}\right\}, \forall \delta \in \mathbb{N}^*$ if and only if

$$\forall \delta \in \mathbb{N}^*,\ \ \exists n_\delta > 0,\ \ \exists M_\delta > 0,\ \ \forall n \geq n_\delta,\ \ |f(n)| \leq M_\delta n^{-\delta}.$$

(b) *Taking* $f(n) := \exp(-n)$, *show that the following assertion is false*

$$''f(n) = O\left\{n^{-\delta}\right\},\ \forall \delta \in \mathbb{N}^* \Rightarrow \exists n^* > 0, \forall n \geq n^*, f(n) = 0.''$$

We show that $\exp(-n)$ is a counterexample of the assertion. Indeed, "$\exists n^* > 0, \forall n \geq n^*, \exp(-n) = 0$" is clearly false. Therefore we only need to show that "$\exp(-n) = O\left\{n^{-\delta}\right\},\ \forall \delta \in \mathbb{N}^*$" is true.

Let's consider an arbitrary $\delta \in \mathbb{N}^*$. We need to find $n_\delta > 0$ and $M_\delta > 0$ such that for all $n \geq n_\delta$, we have $|\exp(-n)| \leq M_\delta n^{-\delta}$. Since for all $n \geq 1$ we have

$$\exp(n) = \sum_{k=0}^{\infty} \frac{n^k}{k!} > \frac{n^\delta}{\delta!} \quad \Longrightarrow \quad \exp(-n) < \delta!\, n^{-\delta},$$

we can set $n_\delta := 1$ and $M_\delta := \delta!$ which ends the proof.

Exercise 7

We show Theorems 2.3 and 2.5 using truth tables.

Theorem 2.5 (Equivalent assertion of the implication)

Let A and B be two assertions, we have

$$"A \Rightarrow B" \;\Leftrightarrow\; "nB \Rightarrow nA" \;\Leftrightarrow\; "nA \parallel B"$$

Proof:

A	B	$A \Rightarrow B$	nB	nA	$nB \Rightarrow nA$	$nA \parallel B$
V	V	V	F	F	V	V
V	F	F	V	F	F	F
F	V	V	F	V	V	V
F	F	V	V	V	V	V

Theorem 2.5 (Negation of $\parallel$, & and $\Rightarrow$)

Let A and B be two assertions, we have

1. $n(A \parallel B) \;\Leftrightarrow\; nA \,\&\, nB$
2. $n(A \,\&\, B) \;\Leftrightarrow\; nA \parallel nB$
3. $n(A \Rightarrow B) \;\Leftrightarrow\; A \,\&\, nB$

Proof:

1. $n(A \parallel B) \;\Leftrightarrow\; nA \,\&\, nB$

A	B	$A \parallel B$	$n(A \parallel B)$	nA	nB	$nA \& nB$
V	V	V	F	F	F	F
V	F	V	F	F	V	F
F	V	V	F	V	F	F
F	F	F	V	V	V	V

2. $n(A \,\&\, B) \;\Leftrightarrow\; nA \parallel nB$

A	B	$A \,\&\, B$	$n(A \,\&\, B)$	nA	nB	$nA \parallel nB$
V	V	V	F	F	F	F
V	F	F	V	F	V	V
F	V	F	V	V	F	V
F	F	F	V	V	V	V

3. $n(A \Rightarrow B) \;\Leftrightarrow\; A \,\&\, nB$

A	B	$A \Rightarrow B$	$n(A \Rightarrow B)$	nB	$A \,\&\, nB$
V	V	V	F	F	F
V	F	F	V	V	V
F	V	V	F	F	F
F	F	V	F	V	F