Introduction to Logic - Part II

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Assertions

Definition 2.1.

An assertion is a statment that can be either true or false, BUT NEVER both of them. For example, "the weather is nice" is not an assertion and "1+1=3" is an assertion.

In mathematics, there are two types of true assertions:

- Axioms: considered to be true (without justification or proof).
- Theorems: need to be proved (using the axioms and other theorems) to be considered true.

Remark: example and counter-example

An example of an assertion only illustrate it but doesn't validate it.
However a counter-example of an assertion proves that it is false.

In this lesson, we are going to present different ways to prove assertions.

Quantifiers and direct proof

An assertion often (almost) involves the quantifiers $\exists$ (there exists) and $\forall$ (for all) in the following way,

$$"\exists x \in E,\ A(x)" \quad \text{and} \quad "\forall x \in E,\ A(x)", \tag{1}$$

where E is a set and $A(x)$ is an assertion indexed by x. These assertions have to be red as

"there exists x in E such that $A(x)$" and "for all x in E we have $A(x)$".

Definition 2.2 (Direct proof of (1)).

To show that "$\exists x \in E,\ A(x)$", we need to find a certain $x \in E$ such that $A(x)$ is true.
To show that "$\forall x \in E,\ A(x)$", we need to verify that $A(x)$ is true for any $x \in E$.

Formal proof of "$\forall x \in E,\ A(x)$"

A direct proof of "$\forall x \in E,\ A(x)$" always starts by "Let $x \in E$ (taken arbitrarily)" and is followed by the arguments showing that $A(x)$ is true.

Operations on assertions

Given two assertions A and B, we can generate the following 4 new assertions

1. nA (negation of A),
2. $A \; \& \; B$, (A and B)
3. $A \| B$, (A or B)
4. $A \Rightarrow B$ (if A then B / A imply B).

Direct proofs of these assertions go as follows:

- For nA: show that nA is true :)
- For $A \; \& \; B$: show that A and B are true :)
- For $A \| B$: suppose that A is false and show that B is true (or conversly)
- For $A \Rightarrow B$: suppose that A is true and show that B is true.

Equivalence: "$A \Leftrightarrow B$"

Another very common assertion is "$A \Leftrightarrow B$" which stands for "A if and only if B". This assertion can be defined as being ""$A \Rightarrow B$" & "$A \Leftarrow B$"".

A direct proof of "$A \Leftrightarrow B$" shows that "$A \Rightarrow B$" AND "$A \Leftarrow B$" are true.
Note that "$A \Leftarrow B$" is called the reciprocal assertion of "$A \Rightarrow B$".

Proof by contraposition

Most of the theorem are of the form "$A \Rightarrow B$". The following theorem shows that this assertion is equivalent to two other assertions.

Theorem 2.3 (Equivalent assertions of the implication).

The assertions "$A \Rightarrow B$", "$nB \Rightarrow nA$" and "$nA \parallel B$" are equivalent that is, they have the same logical value (true or false) independently of the logical value of A and B.

Thank to this theorem, the assertion "$A \Rightarrow B$" can be shown by showing either "$nB \Rightarrow nA$" or "$nA \parallel B$". The idea behind is that one the three assertion might be easier to show than the other.

Definition 2.4 (Proof by contraposition).

A **proof by contraposition** of "$A \Rightarrow B$" consist in proving that "$nB \Rightarrow nA$" is true. Such a reasoning needs to be announce at the begin of the proof (e.g. "The proof is done by contraposition,...").

Negation of (complex) assertions

Clearly, in order to make a proof by contraposition, one needs to be able to the negation of assertions. What is, for example, "$n(A \parallel B)$" or "$n(A \Rightarrow B)$"???

Theorem 2.5 (Negation of $\parallel$, & and $\Rightarrow$).

Let A and B be two assertions, we have

1. $n(A \parallel B) \;\Leftrightarrow\; nA \,\&\, nB$
2. $n(A \,\&\, B) \;\Leftrightarrow\; nA \parallel nB$
3. $n(A \Rightarrow B) \;\Leftrightarrow\; A \,\&\, nB$

What about the assertion with quantifiers??

Theorem 2.6 (Negation of assertion with quantifiers).

Let E be a set and $A(x)$ an assertion indexed by $x \in E$, we have

1. $n(\exists x \in E,\, A(x)) \;\Leftrightarrow\; \forall x \in E,\, nA(x)$
2. $n(\forall x \in E,\, A(x)) \;\Leftrightarrow\; \exists x \in E,\, nA(x)$

Negation of (complex) assertions

These last two theorems are very useful to negate complex assertion such as, for example

$$A := \text{"}\forall x \in E, \forall y \in E, [x < y \Rightarrow \exists z \in E, x < z < y]\text{"}.$$

Indeed,

$$\begin{aligned} n\,[x < y \Rightarrow \exists z \in E, x < z < y] \;&\Leftrightarrow\; x < y \;\&\; n\,[\exists z \in E, x < z < y] \\ x < y \;\&\; [\forall z \in E,\; n(x < z < y)] \;&\Leftrightarrow\; x < y \;\&\; [\forall z \in E,\; (x \geq z \;\|\; z \geq y)] \end{aligned}$$

Therefore we have

$$nA = \text{"}\exists x \in E, \exists y \in E, \{x < y \;\&\; [\forall z \in E,\; (x \geq z \;\|\; z \geq y)]\}\text{"}.$$

Proof of A, when $E := \mathbb{R}$.

Let $x, y \in \mathbb{R}$ and suppose that $x < y$. We need to find $z \in \mathbb{R}$ such that $x < z < y$. Taking $z := \frac{x+y}{2}$ ends the proof. (Why and how?) $\square$

Proof by contradiction

More importantly Theorem 2.3, 2.5 and 2.6 give us the tools to define how to make a proof by contradiction.

Definition 2.7 (Proof by contradiction).

A **proof by contradiction** of "$A \Rightarrow B$" consist in proving that "A & nB" is false. Such a reasoning needs to be announce at the begin of the proof (e.g. "The proof is done by contradiction, etc...").

For example, if we want to prove by contradiction that

$$\text{A:=}"\forall x \in E, \forall y \in E, \{xy = 0 \Rightarrow [x = 0 \,\|\, y = 0]\}",$$

we need to show that nA is false, i.e., by Theorem 2.5 and 2.6,

$$\text{nA=}"\exists x \in E, \exists y \in E, \{xy = 0 \,\&\, [x \neq 0 \,\&\, y \neq 0]\}".$$

Proof by induction

The last technique is called proof by induction and is based on the following theorem.

Theorem 2.8 (Induction principle).

Let A_n be an assertion indexed by $n \in \mathbb{N}$. Then

$$"A_0 \;\&\; [\forall n \in \mathbb{N}, \;\; A_n \;\Rightarrow\; A_{n+1}]" \quad \Leftrightarrow \quad "\forall n \in \mathbb{N}, \;\; A_n"$$

A proof by induction needs to be announced at the begin of the proof by "The proof is by induction, etc...". Formally, a proof by induction goes as follows:

1. Define the assertion A_n, for all $n \in \mathbb{N}$,
2. show that A_0 is true, (initial step)
3. show that $\forall n \in \mathbb{N}, \;\; A_n \;\Rightarrow\; A_{n+1}$ (induction step).

Proof by induction: example

We are going to show by induction that "$\forall n \in \mathbb{N}, \ \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$".

Proof.

This proof is done by induction setting $A_n := \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \ \forall n \in \mathbb{N}$.

1. A_0 is true since: $\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$.
2. Let $n \in \mathbb{N}$ (arbitrary) and suppose that A_n is true, we want to show that A_{n+1} is true, i.e., $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$. This is indeed the case since

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + n + 1 = \frac{n(n+1)}{2} + n + 1 = \left(\frac{n}{2} + 1\right)(n+1) = \frac{(n+1)(n+2)}{2}.$$

$\square$

Exercises

0) Give an example where "$A \Rightarrow B$" and its reciprocal "$A \Leftarrow B$" are not equivalent.

1) Give the negation and the contraposition of the following assertions

   1. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [x < y \Rightarrow f(x) < f(y)]$
   2. $\forall \varepsilon > 0, \exists N \in \mathbb{N}, [n \geq N \Rightarrow |u_n - \varepsilon| < \varepsilon]$
   3. $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}, [|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon]$
   4. $\forall E \subset \mathbb{N}, [E \neq \varnothing \Rightarrow (\exists n \in E, \forall m \in E, m \geq n)]$

2) Let E and F be two sets and $A(x, y)$ and assertion indexed by $x \in E$ and $y \in F$. Find an example where "$\exists x \in E, \forall y \in F, A(x, y)$" is not equivalent to "$\forall y \in F, \exists x \in E, A(x, y)$".

3) If f is a function from $\mathbb{R}$ to itself, what do we need to show to prove that f is not continuous in $x_0 \in \mathbb{R}$? Similarly, what do we need to show to prove that an estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}_0$ is not consistent?

4) What do we need to show if we want to prove

$$\text{"}p \text{ divide } ab\text{"} \Rightarrow \text{"}p \text{ divide } a \parallel p \text{ divide } b\text{"},$$

by contraposition? by contradiction?

Exercises

5) Prove the following
   1. by contradicton that "$\nexists a \in \mathbb{R}, a \times 0 = 1$",
   2. by induction that "$\forall n \in \mathbb{N}, \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$",
   3. by induction that "$3^{2n} - 2^n$ is a multiple of 7 $\forall n \in \mathbb{N}$" .
6) We consider functions f from $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ to $\mathbb{R}$.
   1. We define $f(n) = \mathcal{O}_{\delta \in \mathbb{N}^*}\left\{n^{-\delta}\right\}$, if and only if
   $$\exists n' > 0, \ \exists M > 0, \ \forall \delta \in \mathbb{N}^*, \ \forall n \geq n', \ |f(n)| \leq M^{\delta} n^{-\delta}.$$
   Suppose that the following assertion is true
   $$\text{"}f(n) = \mathcal{O}_{\delta \in \mathbb{N}^*}\left\{n^{-\delta}\right\} \Rightarrow \exists n^* > 0, \forall n \geq n^*, f(n) = 0.\text{"} \tag{2}$$
   Prove that "$\exp(-n) \neq \mathcal{O}_{\delta \in \mathbb{N}}\left\{n^{-\delta}\right\}$" using (2).
   2. By definition, $f(n) = \mathcal{O}\left\{n^{-\delta}\right\}, \forall \delta \in \mathbb{N}^*$ if and only if
   $$\forall \delta \in \mathbb{N}^*, \ \exists n_\delta > 0, \ \exists M_\delta > 0, \ \forall n \geq n_\delta, \ |f(n)| \leq M_\delta^{\delta} n^{-\delta}.$$
   Taking $f(n) := \exp(-n)$, show that the following assertion is false
   $$\text{"}f(n) = \mathcal{O}\left\{n^{-\delta}\right\}, \delta \in \mathbb{N}^* \Rightarrow \exists n^* > 0, \forall n \geq n^*, f(n) = 0.\text{"}$$

Exercises

7) **Proofs of Theorems 2.3 and 2.5**

The truth table of the operators $n, \|, \&$ and $\Rightarrow$ are defined below,

A	nA
V	F
F	V

A	B	$A \| B$
V	V	V
V	F	V
F	V	V
F	F	F

A	B	$A\&B$
V	V	V
V	F	F
F	V	F
F	F	F

A	B	$A \Rightarrow B$
V	V	V
V	F	F
F	V	V
F	F	V

These tables can be used to prove that two assertions are equivalent, i.e., their logical value (V or F) is the same independently of the logical value of the input assertions A and B. For example, $n(A \| B)$ is equivalent to $nA\&nB$ because

A	B	$A \| B$	$n(A \| B)$	nA	nB	$nA\&nB$
V	V	V	F	F	F	F
V	F	V	F	F	V	F
F	V	V	F	V	F	F
F	F	F	V	V	V	V

Prove Theorem 2.3 and 2.5 using truth tables.