Chapter 1: Introduction to Modelling & Estimation in Linear Dynamic Systems

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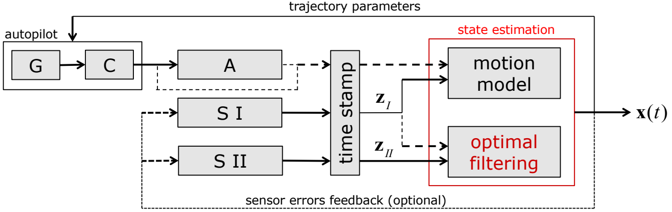
Agenda

- Stochastic and dynamic modelling – where is it useful?
- State space notation
- Modelling examples
- Estimation
- 3-D examples

Where is useful stochastic and dynamic modelling?

Autonomous platform - principle



- Legend
  - G - guidance, C - control, A - actuators
  - S_I - autonomous sensors (IMU, pressure, etc.)
  - S_{II} - non-autonomous sensors (GNSS, vision, ultrasound, etc.)
- Motion model
  - Kinematic (sensor based, i.e. observing forces, rates, ...)
  - Dynamic (model based, i.e. specifying forces, rates, ...)

Where is useful stochastic and dynamic modelling?

Moving platform - state estimation

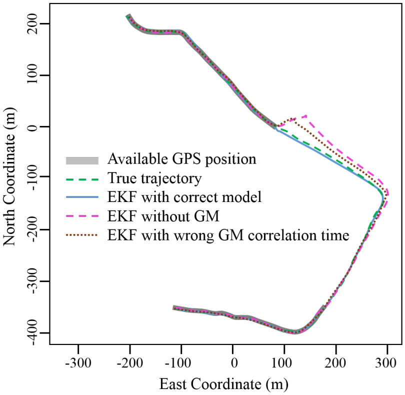
- Why?
  - Platform needs to continuously maintain believe about many parameters (states) $\mathbf{x}(t)$
  - Not all needed parameters (states)[a] are directly observable
  - Errors exists in sensors (& models) $\rightarrow$ filtering needed
- How?
  - Construct **a model** to maintain the state believe in time
  - **Update** the believe according to **observations**
- Models
  - sensor observations $\rightarrow$ **sensor** models (needs stoch. param.)
  - executed action $\rightarrow$ **motion** model & sensor models(s)

[a] related to trajectory, sensors or sensor assembly

Impact of sensor model on trajectory quality

Auto-motive example

- Different models are used to describe sensor (random) errors in *accelerometers* and *gyroscopes* within an inertial system (INS) that is integrated with satellite positioning.
- The realization of *time correlated random errors* in the sensors is estimated by a navigation filter and subtracted via a feedback.
- In the absence of satellite signals the trajectory is entirely based on INS, which performance depends partly on sensor models.



After Clausen et al. 2018.

Where is useful stochastic and dynamic modelling?

Inflation - state estimation

- Why?
  - The rate of price inflation is an important macroeconomic variable
  - Economists often assume that inflation has unobserved (latent) component(s) which are not directly observable
- How?
  - Construct a model to identify the latent components
  - Update the believe according to observations
- Models
  - Various models have been proposed. On of the most common one decomposes inflation into permanent (X_t) and transitory (V_t) component:

$$\begin{aligned} X_t &= X_{t-1} + U_t, \qquad & U_t &\sim \mathcal{N}(0, \sigma_u^2) \\ Z_t &= X_t + V_t, \qquad & V_t &\sim \mathcal{N}(0, \sigma_v^2) \end{aligned}$$

  - This model can be extended in various way, including with time varying parameters (see e.g. Stock and Watson 2007)

Where is useful stochastic and dynamic modelling?

Natural phenomena - state estimation

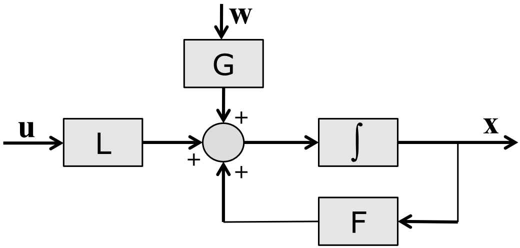
- Why?
  - Natural phenomena have often random yet time-correlated character(s) that is possibly multi-dimensional with an unobserved (latent) component.
- How?
  - Identification of such character can be achieved via analysis of (time, spatial) series.
  - New (and possibly indirect) observations are used to estimate its actual realisation.
- Models
  - Models vary according to phenomena and can be used for forecasting (weather, hydrology, biology, etc.)

Representation in time domain

The dynamics of linear (parameter) systems can be represented by a first-order “vector-matrix” differential equation

Continuous form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) + \mathbf{L}(t)\mathbf{u}(t) \tag{1.1}$$



$\dot{\mathbf{x}}(t)$ - system *state vector*
$\mathbf{w}(t)$ - *random forcing function*
$\mathbf{u}(t)$ - *deterministic input*

Choice on $\mathbf{x}$ – any set of quantities *sufficient* to describe the motion at t

State space with higher order derivatives

N^{th} order linear differential equation

$$\left[\frac{\partial^n}{\partial t} + a_{n-1}(t)\frac{\partial^{n-1}}{\partial t} + \cdots a_1(t)\frac{\partial}{\partial t} + a_0(t)\right] y(t) = w(t) \tag{1.2}$$

From vector to a matrix form

- Defining

$$x_1(t) := y(t), \quad x_2(t) := \dot{x}_1(t), \quad \ldots \quad x_n(t) := \dot{x}_{n-1}(t)$$

- Rewriting Eq. (1.2)

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ &\vdots \\ \dot{x}_n(t) &= -a_0(t)x_1(t) - a_1(t)x_2(t) - \ldots - a_{n-1}(t)x_n(t) + w(t) \end{aligned}$$

- Provides the desired matrix-vector form ...

State space in matrix-vector form

N^{th} order linear differential equation

Companion form (single variable)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w(t) \end{bmatrix}$$

Note: if $w = u$, i.e., is the control input that this is the *controllable* canonical form

Generally

- random ($\mathbf{w}$) and forcing ($\mathbf{u}$) inputs are multi-variable
- $\mathbf{F}$, & ($\mathbf{G}$, $\mathbf{L}$) matrices have non-zero (& non-trivial) elements outside the main diagonal

State evolution in time (= transition)

Homogeneous form (deterministic input $\mathbf{u} = 0$ and random forcing inputs of zero mean $\mathbb{E}\{\mathbf{u}\} = 0$):

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\,\mathbf{x}(t) \tag{1.3}$$

with $\mathbf{F}$ being the *dynamic* matrix.

Solution

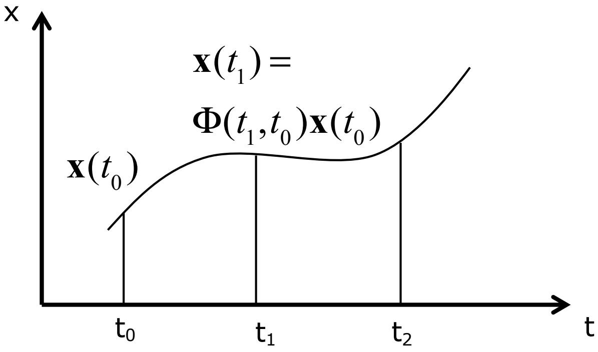
$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}(t_0) \tag{1.4}$$

with $\Phi(t, t_0)$ being the *transition* matrix that is related to F as:

$$\Phi(t - t_0) = e^{\mathbf{F}(t - t_0)} \tag{1.5}$$

State evolution in time (= transition)

Transition matrix allows calculation of the state vector at some time t, given the knowledge of state vector at t_0 ... in the absence of random forcing functions.



State transition - adding probability

Reality:

- All models are partially wrong or incomplete.
- Sensor observations are noisy and/or partial.
- Usually some prior knowledge on initial state at t_0 exist.

State incertitude

- Is generally expressed via a probability density function (PDF).
- Therefore *propagation* of samples $\rightarrow$ *propagation* of PDF.

Special case

- Propagation of *multivariate* Gaussian can be realized by propagation of its 1^{st} and 2^{nd} moments (i.e., the mean and covariance).
- Note #1 - Gaussian remains Gaussiant under a linear transformation.
- Note #2 - Intersection of Gaussians $\rightarrow$ remains Gaussian.

State confidence - covariance matrix

Forcing function

- If forcing function is based on (linearly transformed) white noise $\rightarrow$ has a zero mean (influence on state)
- Hence, if the state $\mathbf{x}$ is unbiased at time (t), it will remain unbiased
- In a discrete case

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi_k x_k + \Gamma_k \mathbf{w}_k \\ \mathbb{E}\{\mathbf{x}_{k+1}\} &= \mathbb{E}\{\Phi_k \mathbf{x}_k + \Gamma_k \mathbf{w}_k\} \\ &= \Phi_k \mathbb{E}\{\mathbf{x}_k\} + \Gamma_k \mathbb{E}\{\mathbf{w}_k\}\end{aligned}$$

Randomness of state

- is described in terms of *covariance matrix* $\mathbf{P} := \mathbb{E}\{\check{\mathbf{x}}\check{\mathbf{x}}\}$
- where $\check{\mathbf{x}} := \hat{\mathbf{x}} - \mathbf{x}$ is the error in estimate (estimated - true)
- in a system with two variables $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathbf{T}}$:

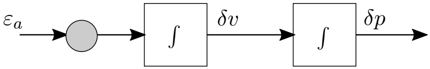
$$\mathbf{P} := \mathbb{E}\left\{\begin{bmatrix} \check{x}_1^2 & \check{x}_1\check{x}_2 \\ \check{x}_1\check{x}_2 & \check{x}_2^2 \end{bmatrix}\right\} = \begin{bmatrix} \mathbb{E}\left\{\check{x}_1^2\right\} & \mathbb{E}\left\{\check{x}_1\check{x}_2\right\} \\ \mathbb{E}\left\{\check{x}_1\check{x}_2\right\} & \mathbb{E}\left\{\check{x}_2^2\right\} \end{bmatrix}$$

Review point

Consider a dynamic system with zero deterministic input that is represented by a state $\mathbf{x}$. Given the knowledge of this state at time $\mathbf{x}(t)$ what is the *needed* and *sufficient* element to predict $\mathbf{x}(t + \Delta t)$?

1-D Accelerometer in space

Relation between the accelerometer error ε_a, velocity error δv and position error δp.



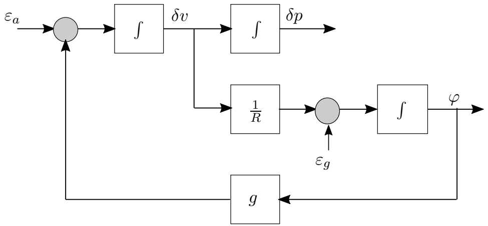
System equation

$$\begin{bmatrix} \delta \dot{p} \\ \delta \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_a \end{bmatrix}$$

The forcing input is the random noise from a realisation i.e., $\varepsilon_a \sim \mathcal{N}\left(0, \sigma_a^2\right)$.

1-D Accelerometer on Earth (1-axis INS)

Relation between the accelerometer error ε_a, velocity error δv and position error δp with respect to platform tilt φ. The accelerometer error is coupled with platform tilt φ via gravity g, while the tilt φ is related to δv via Earth radius R and possibly gyro error ε_g.



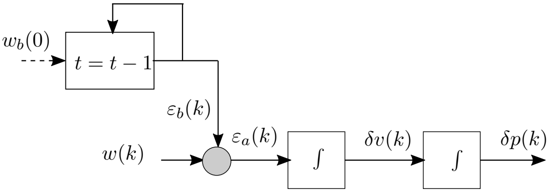
System equation

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & g \\ 0 & \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \varphi \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_a \\ \varepsilon_g \end{bmatrix}$$

With forcing inputs $\varepsilon_a \sim \mathcal{N}\left(0, \sigma_a^2\right)$ and possibly $\varepsilon_g \sim \mathcal{N}\left(0, \sigma_g^2\right)$

1-D Accelerometer in space with a random bias

Discrete case. Relation between the accelerometer error $\varepsilon_a(k)$, velocity error $\delta v(k)$ and position error $\delta p(k)$. The accelerometer error is composed of a realisation of white noise $w(k)$ and a random bias $\varepsilon_b(k)$.



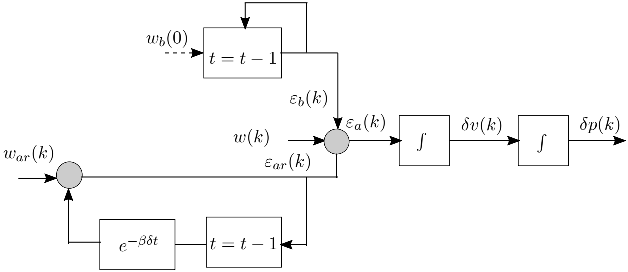
System equation

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \dot{\varepsilon}_b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \varepsilon_b \end{bmatrix} + \begin{bmatrix} 0 \\ w \\ 0 \end{bmatrix}$$

The system is augmented by the one-time realisation of $\varepsilon_b(0) \sim \mathcal{N}\left(0, {w_b}^2\right)$

1-D Accelerometer in space - bias & latent process

Relation between the accelerometer error $\varepsilon_a(k)$, velocity error $\delta v(k)$ and position error $\delta p(k)$ in a discrete case.



Accelerometer error components

- white noise process $w(k)$
- random bias $\varepsilon_b(k)$
- correlated random noise ε_{ar} modelled by a 1-st order auto-regressive process, in a differential form: $\dot{\varepsilon}_{ar} = -\beta\varepsilon_{ar} + w_{ar}$

Review point

What is the system equation in the last example?

1-D Accelerometer in space - bias & latent process

System equation

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \dot{\varepsilon}_b \\ \dot{\varepsilon}_{ar} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \varepsilon_b \\ \varepsilon_{ar} \end{bmatrix} + \begin{bmatrix} 0 \\ w \\ 0 \\ w_{ar} \end{bmatrix}$$

The system is augmented time-correlated random errors:

- the one-time realisation of $\varepsilon_b(0) \sim \mathcal{N}\left(0, {\sigma_b}^2\right)$
- the auto-regressive process: $\varepsilon_{ar}(k) = e^{-\beta(t_k - t_{k-1})}\varepsilon_{ar}(k-1) + w_{ar}(k)$ with $w_{ar} \sim \mathcal{N}\left(0, \sigma_{ar}^2\right)$

Estimation of time correlated dependencies

Situation

- Large family of processes (including sensor noise) can be modeled by putting white noise through a linear system.
- Such process/error models can be added to system model as long as the parameters (e.g., σ, β) are known.
- The realisation of such process (including time correlated noise) can be estimated by observing transformed quantities [a]. *Bayesian* estimation techniques are often used and the most popular filtering method will be summarised later.

[a] e.g., position / velocity errors in the preceding example

Dependency on process (error) parameters

- The effectiveness of process estimation (error filtering) depends, among others, on the correctness of the parameters describing the process/error models.
- The determination of such parameters by time-series analysis is the main subject of this course.

General non-linear form

Ordinary differential equations (ODE) - components

$$\dot{\mathbf{x}}(t) = \mathbf{f}_1 \left\{ \left[\mathbf{x}_1(t), \mathbf{x}_2(t) \right], t \right\} \tag{1.6}$$

where $\mathbf{x_1}$ are system states, $\mathbf{x_2}$ are the augmented states related to the random forcing input $\mathbf{u}$.

Linearization for estimation

- True state is approximated: $\widehat{\mathbf{x}}_1(t) = \mathbf{x}_1(t) - \delta\mathbf{x}_1(t)$
- Eq. (1.6) takes the form: $\widehat{\dot{\mathbf{x}}}(t) = \mathbf{f}_1 \left\{ \left[\mathbf{x}_1(t) - \delta\mathbf{x}_1(t), \mathbf{x}_2(t) \right], t \right\}$
- Taylor expansion: $\delta\dot{\mathbf{x}}_1(t) = \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} \delta\mathbf{x}_1(t) = \mathbf{F}_1(t)\delta\mathbf{x}_1(t)$, where $\delta\dot{\mathbf{x}}_1(t) = \dot{\mathbf{x}}_1(t) - \widehat{\dot{\mathbf{x}}}_1(t)$
- Augmented states are modelled: $\delta\dot{\mathbf{x}}_2(t) = \mathbf{F}_2(t)\delta\mathbf{x}_2(t) + \mathbf{G}_2(t)\mathbf{w}(t)$
- Both parts are put together in the general linearized form

General (compound) linearized form

The general linearized form $\delta\,\dot{\mathbf{x}}(t) = \mathbf{F}(t) + \mathbf{G}(t)$ is found as

$$\begin{bmatrix} \delta\mathbf{x}_1(t) \\ \delta\mathbf{x}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1(t) & \mathbf{F}_{12}(t) \\ \mathbf{0} & \mathbf{F}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_2(t) \end{bmatrix} \tag{1.7}$$

Estimation

Bayesian approach

- General *probabilistic approach* utilizing (any) PDF (probability density function)
- Prior probability of the system state $P(x)$
- Stream of observations z and actions (process) u: $(u_1, z_1, \ldots, u_t, z_t)$
- *Action (process)* model $P(\tilde{x}|x, u)$
- *Sensor* model $P(z|x)$

Markov chain assumptions for recursive estimation

- Current state depends only on previous state & current action.
- Observations depends only on a current state.
- Implementations: Hidden Markov models, Particle filter, *Kalman filter*, ..

Kalman Filter

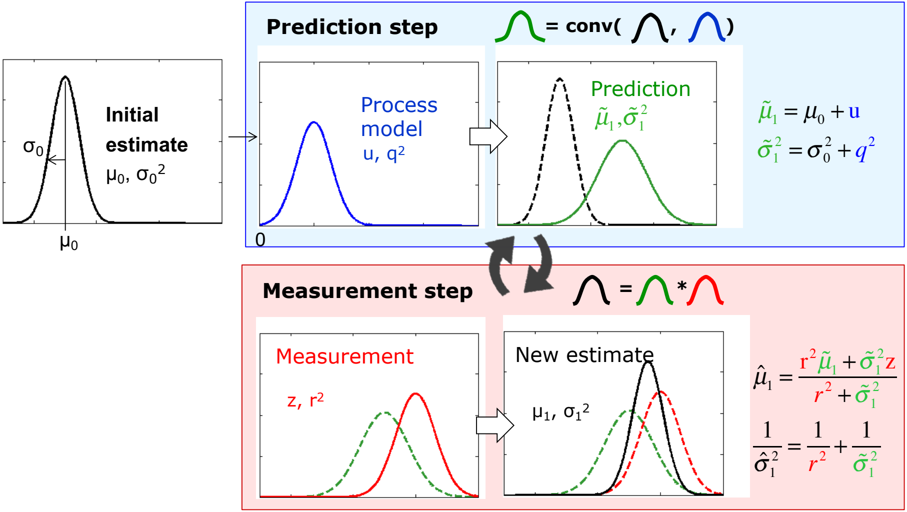
Properties

- Bayes filter with continuous (or discrete) states
- State represented with normal distribution - mean $\mathbf{x}$, covariance $\mathbf{P}$
- Very efficient (per state dim. n and obs. dim. k): $\mathcal{O}\left(k^{2.376} + n^2\right)$
- Most relevant filter in practice since 1950'...
- Optimal for linear Gaussian systems
- Most systems are non-linear $\rightarrow$ linearization of process & observation is needed!

Drawbacks

- *Spatial conditions* between states are difficult to handle.
- *Only one* dynamic model is possible for the same phenomena.
- Alternatives: method(s) allowing expressing different or several dynamic, temporal and spatial constrains / models within one framework (e.g., dynamic network).

Kalman filter - 1D example



Discrete Kalman Filter

Relations

- State propagation $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1})_{k-1}$
- Covariance propagation $(-)$=$(\sim)$ $\mathbf{P}_k^- = \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}^+\boldsymbol{\Phi}_{k-1}^T + \boldsymbol{\Gamma}_{k-1}\mathbf{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^T$
- Measurement $\mathbf{z}$ with covariance $\mathbf{R}$ $\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k$
- Gain computation $\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}$
- Covariance update $(+)$=("hat") $\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{P}_k^-$
- State update $\mathbf{x}_k^+ = \mathbf{x}_k^- + \mathbf{K}_k\left[\mathbf{z}_k - \mathbf{h}\left(\mathbf{x}_k^-\right)\right]$

Linearization

- Observations $\mathbf{H}_k = \frac{\partial \mathbf{h}\left(\mathbf{x}_k^*\right)}{\partial \mathbf{x}}$
- Process $\mathbf{F}_{k-1} = \frac{\partial \mathbf{f}\left(\mathbf{x}_{k-1}^*, \mathbf{u}_{k-1}\right)}{\partial \mathbf{x}}$

Extended Kalman Filter

Relations

- Approx. state $\mathbf{x}^*$ = estimated state $\mathbf{x}^+$ $\qquad \mathbf{x}^*_{k-1} = \mathbf{f}\left(\mathbf{x}^+_{k-1}, \mathbf{u}_{k-1}\right)$
- Transition matrix $\qquad \Phi = e^{\mathbf{F}\Delta t} = \mathbf{I} + \mathbf{F}\Delta t + \frac{1}{2}\mathbf{F}^2\Delta t^2 + \dots$
- Process noise

$$\mathbf{Q}_k = \int_{k-1}^{k} \Phi\mathbf{G}Q(\tau)\mathbf{G}^T\Phi^T d\tau \approx \Phi_{k-1}\mathbf{G}Q\Delta t\mathbf{G}^T\Phi^T_{k-1} = \Gamma_{k-1}\mathbf{Q}_{k-1}\Gamma^T_{k-1}$$

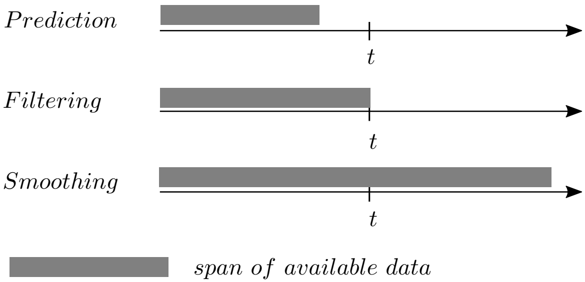
- Initial conditions $\qquad \mathbf{x}_0 = \mathbb{E}\left[\mathbf{x}(0)\right] \qquad \mathbf{P}_0 = \mathbb{E}\left[(\mathbf{x}(0) - \mathbf{x}_0)(\mathbf{x}(0) - \mathbf{x}_0)^T\right]$

Assumptions

- Absence of correlations: process - observation $\qquad \mathbb{E}\left[\mathbf{w}_k\mathbf{v}_j^T\right] = 0,\ \forall j, k \in \mathbb{Z}^+$
- Innovation sequence $\qquad \mathbf{v}_k = \mathbf{z}_k - \mathbf{h}\left(\mathbf{x}^-_k\right) \sim \mathcal{N}\left(0, \sigma_v^2\right)$
- Residual sequence $\qquad \mathbf{r}_k = \mathbf{z}_k - \mathbf{h}\left(\mathbf{x}^+_k\right) \sim \mathcal{N}\left(0, \sigma_r^2\right)$

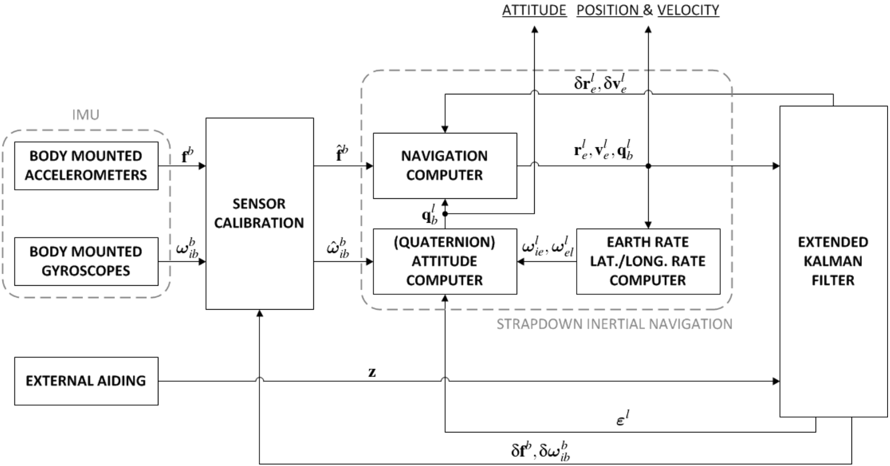
State estimation in time

Depending on the available (or used) span of data with respect to time t, the estimation distinguishes between



More on optimal estimation, e.g. Gelb 1988

3D integrated navigation example



Source: *Stebler, 2012*

Experiment Setup

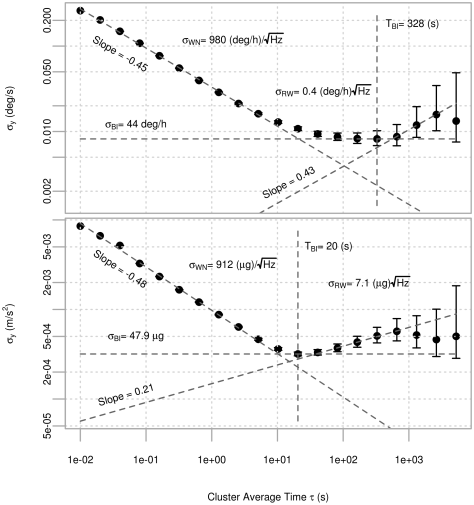
Data:

- Static data collected during 4.5 hours @100Hz
- Constant temperature condition
- MEMS IMU (Xsens MTi-G $\sim$ 2012)

Sensor calibration approaches:

- Allan Variance approach (IEEE standard)
- KF-(Self)-tuning approach
- new approach (GMWM)

Allan Variance Approach



EKF-(self)-tuning approach

Procedure:

- AV parameters used as initial approximation
- Ad-hoc adaptation of model parameters based on:
  - Analysis of KF residuals
  - Analysis of position drift during GNSS artificial outages with cm-level (GNSS-PPK) positioning as a reference

Model:

$Y_t \sim F_\theta$ where F_θ is such that

$$Y_t = Y_{t,WN} + Y_{t,GM}$$

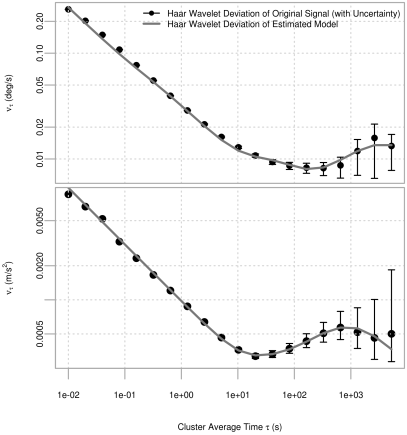
where $Y_{t,WN}$ and $Y_{t,GM}$ denote, respectively, a white noise and a Gauss-Markov process.

GMWM approach

Model:

$Y_t \sim F_\theta$ where F_θ is such that

$$Y_t = Y_{t,WN} + \sum_{k=1}^{3} Y_{t,GM}^{(k)}$$



Validation

Models comparison is non-trival...

- True model F_θ is unknown...
- Calibration on signal acquired in static conditions.

Proposed procedure:

1. Reference solution ($\sim$2-5 cm & $\sim$0.005-0.01° in position /attitude)
2. Emulation of synthetic IMU signals along the reference
3. Addition of real (MEMS-IMU) static noise signal on IMU synthetic signals
4. Introduction of artificial GNSS gaps
5. Processing procedure using closed-loop EKF to implement the models
6. Quality judged by analyzing the navigation error and EKF-predicted accuracy during inertial coasting model

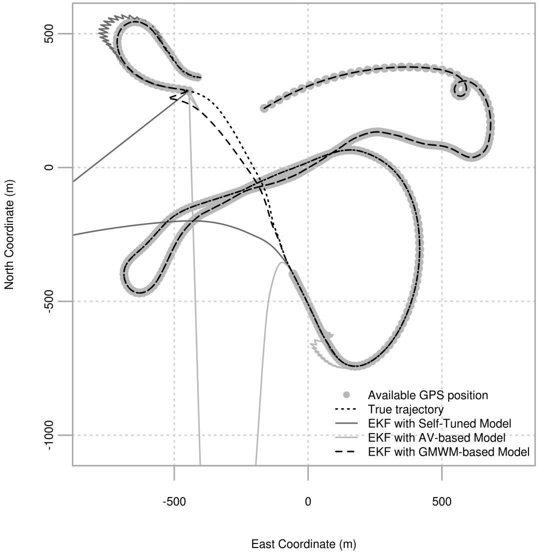
Helicopter experiment - setup

Reference:

- Sensors #1: GNSS receivers Javad Legacy L1/L2 @10Hz in the helicopter & on the ground
- Sensor #2: tactical grad (LN-200) IMU @ 400 Hz
- Helicopter trajectory: post-processed (with optimal filtering / smoothing)

After Stebler et al. 2014

Helicopter experiment - impact



UAV experiment - impact

Interactive demo after Khaghani et al. 2019 in a web browser.

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