Matchings in Graphs

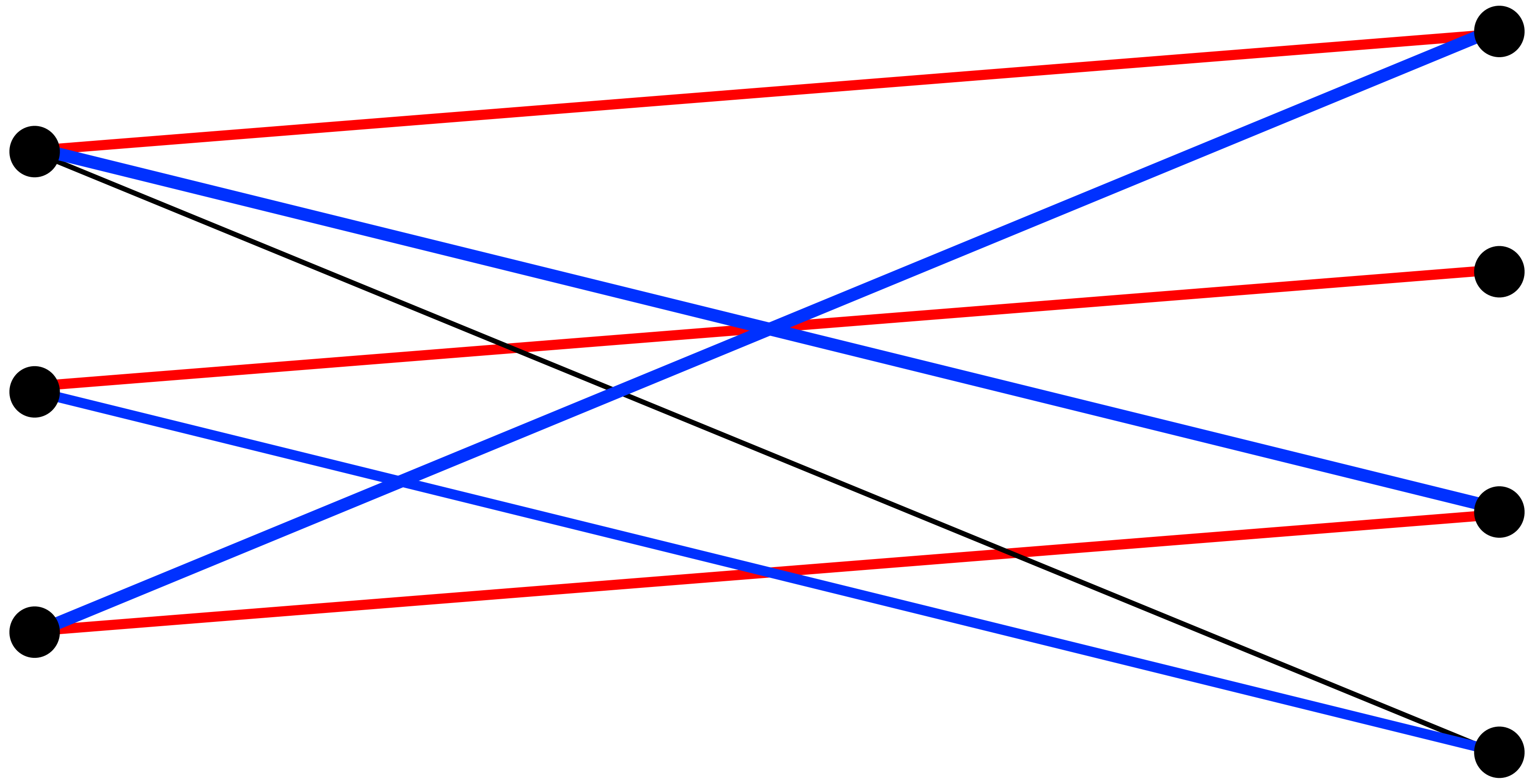
A matching (匹配) in a graph is a set of pairwise nonadjacent edges.

If M is a matching,
the two ends of each edge of M are said to be matched under M,
and each vertex incident with an edge of M is said to be covered by M.

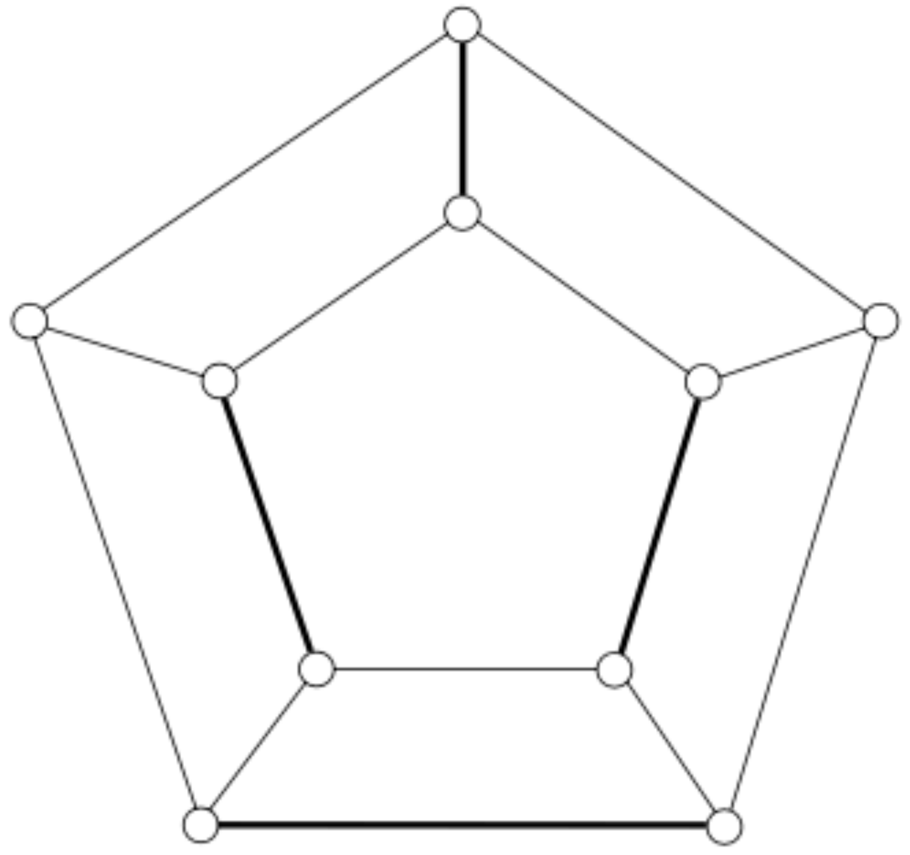
A perfect matching(完美匹配) is one which covers every vertex of the graph,
and a maximum matching is one which covers as many vertices as possible.

A maximal matching is one which cannot be extended to a larger matching.

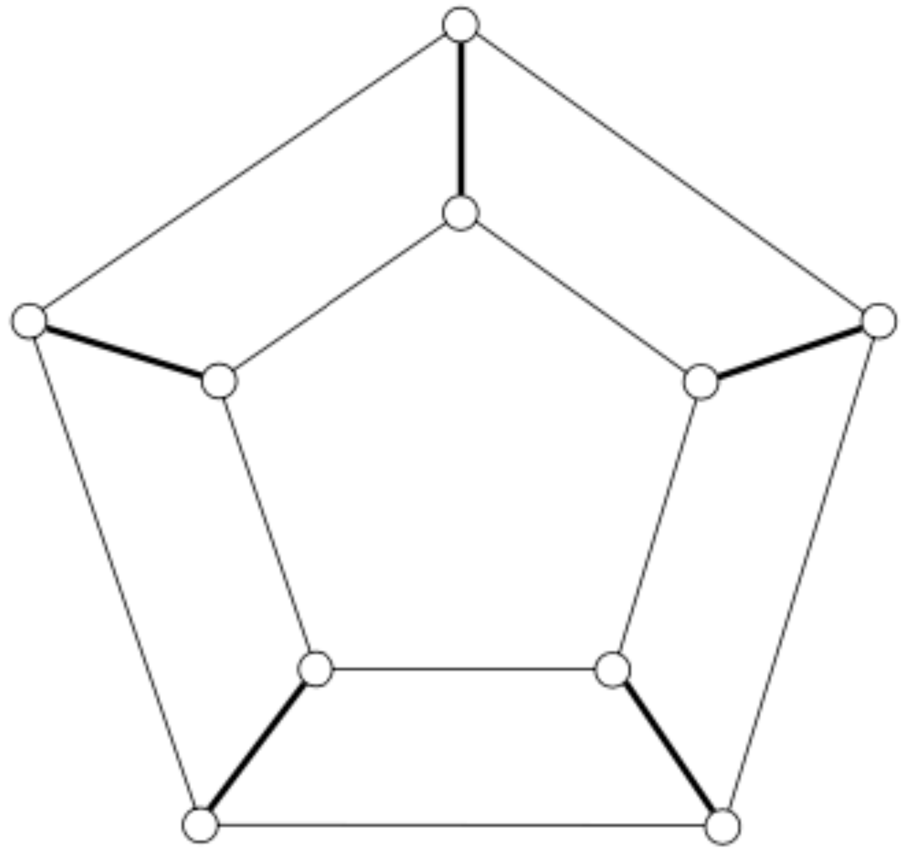
The number of edges in a maximum matching of a graph G
is called the matching number (匹配数) of G, and denoted $\alpha'(G)$.



Maximum matchings



A maximal matching



A perfect matching

Example 3. Jobs Assignment

A certain number of jobs are available to be filled.

Given a group of applicants for these jobs, fill as many of them as possible, assigning applicants only to jobs for which they are qualified.

This situation can be represented by means of a bipartite graph $G[X, Y]$ in which X represents the set of applicants, Y the set of jobs, and an edge xy with $x \in X$ and $y \in Y$ signifies that applicant x is qualified for job y.

An assignment of applicants to jobs, one person per job, corresponds to a matching in G, and the problem of filling as many vacancies as possible amounts to finding a maximum matching in G.

How to find a maximum matching in a graph G?

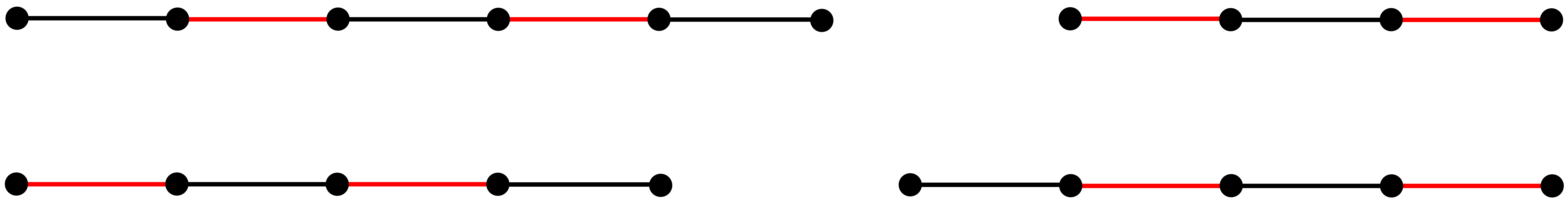
Augmenting Paths (增广路)

Let M be a matching of a graph G.

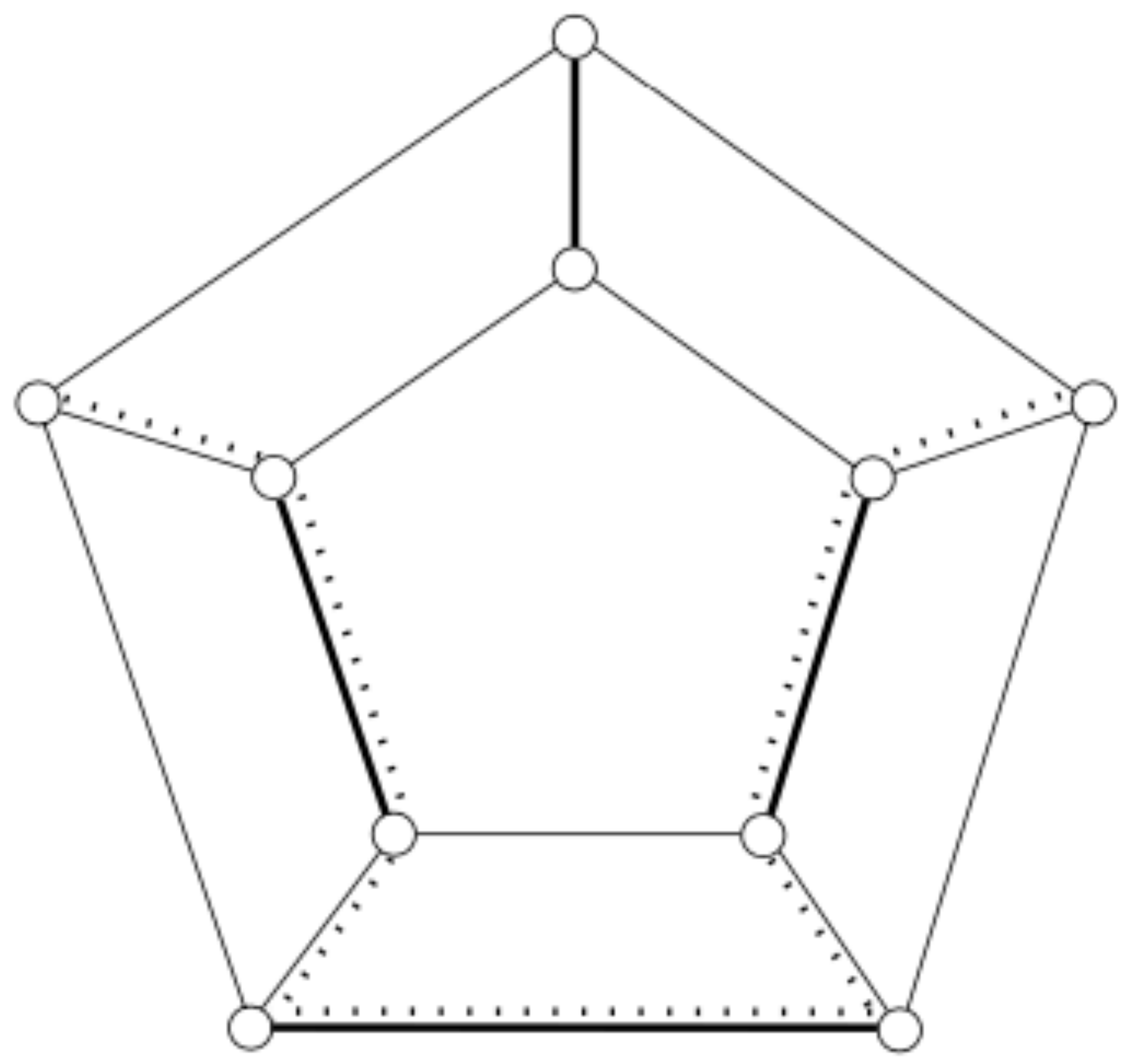
An M-alternating path or cycle in G

is a path or cycle whose edges are alternately in M and $E \backslash M$.

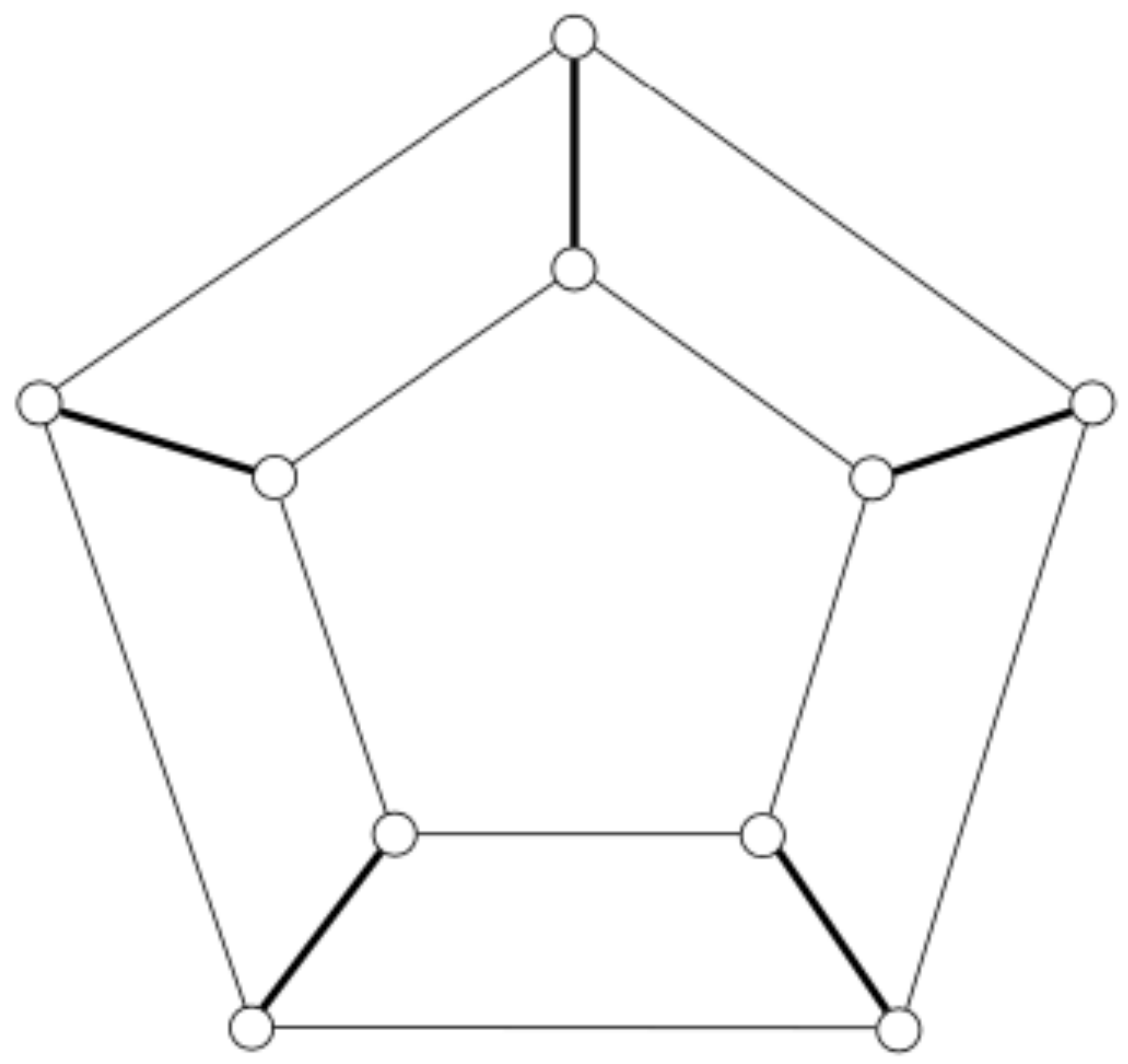
An M-alternating path might or might not start or end with edges of M.



For an M-alternating path, if neither its origin nor its terminus is covered by M, then the path is called an M-augmenting path.



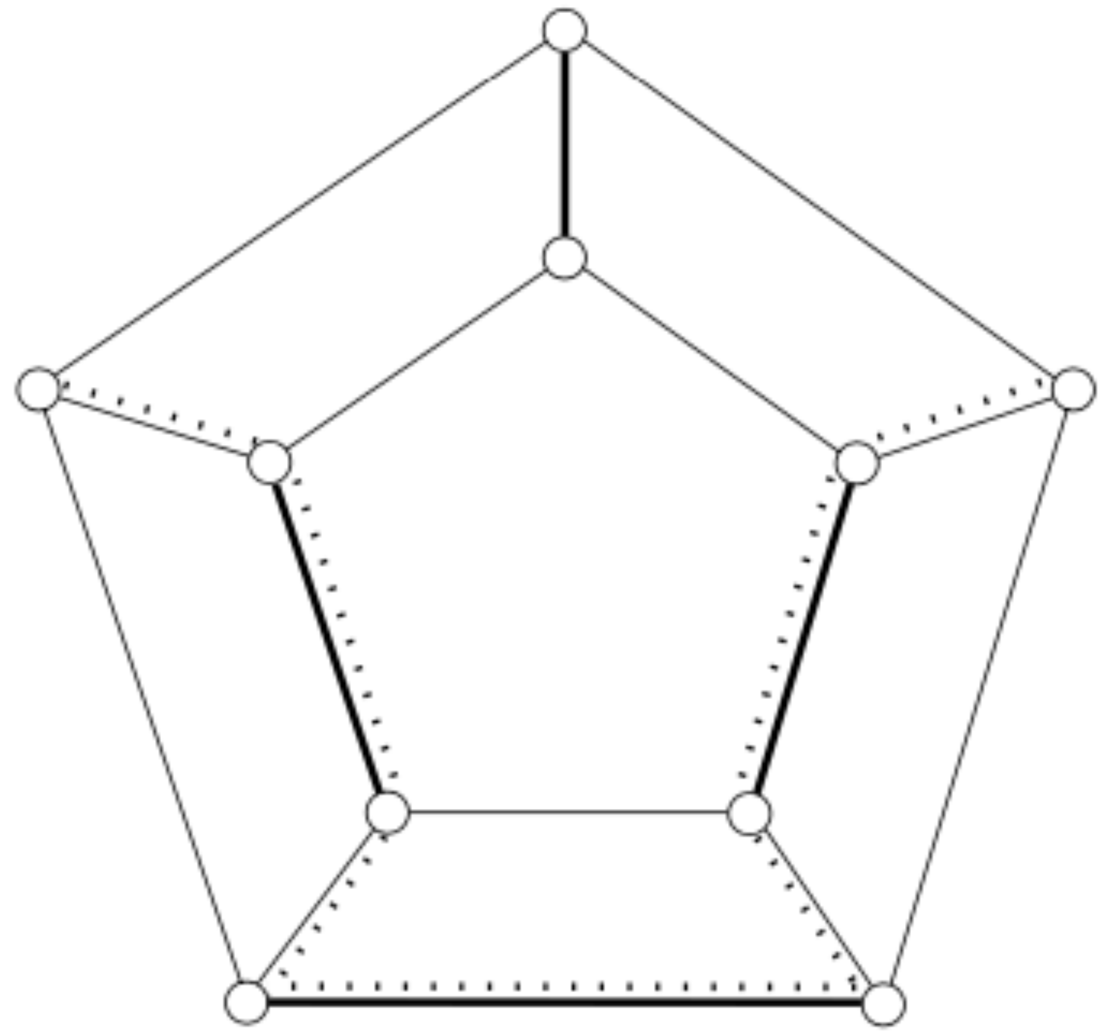
An M-augmenting path P

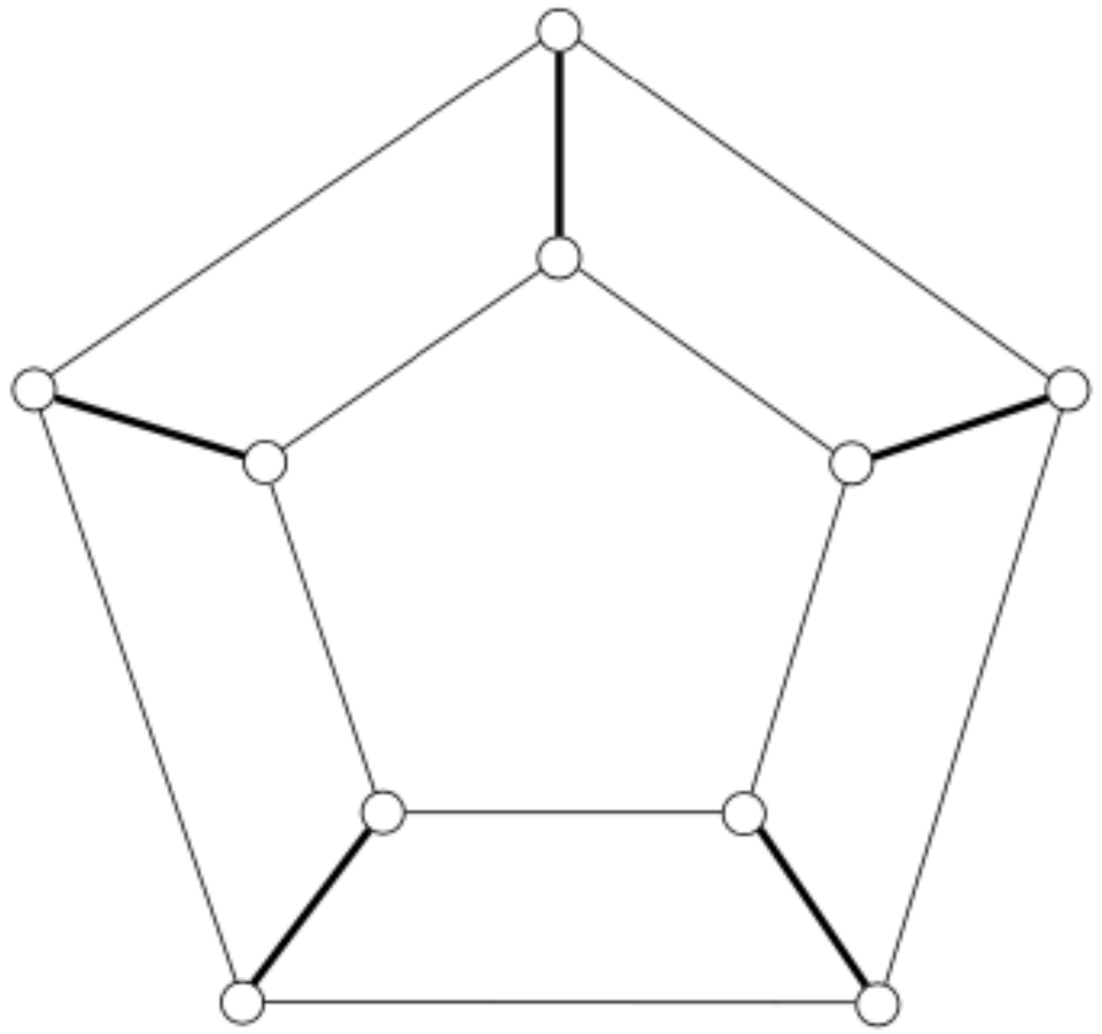


The matching $M \,\Delta\, E(P)$

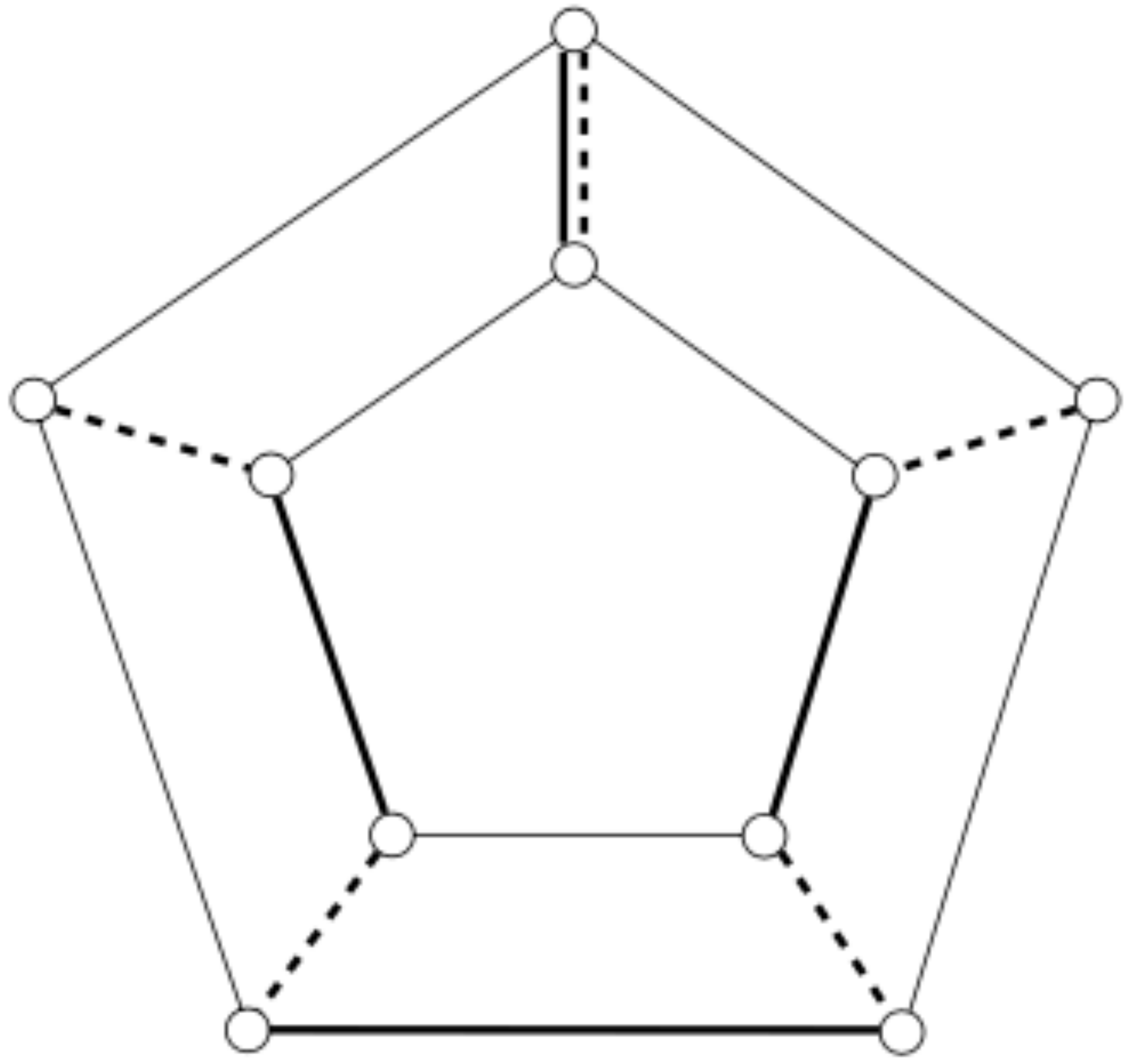
Theorem 21(Berge). A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

Proof. Let M be a matching in G. Suppose that G contains an M-augmenting path P. Then $M' = M \,\Delta\, E(P)$ is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.

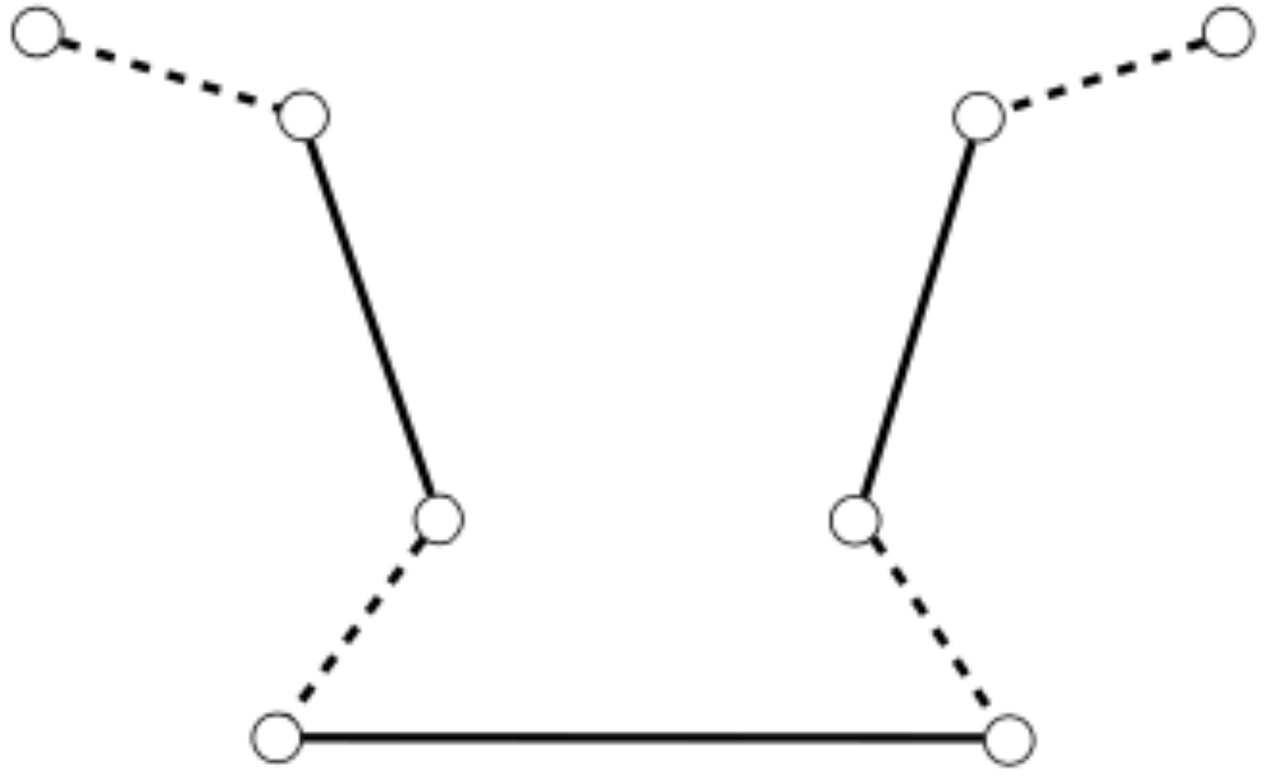




Suppose that M is not a maximum matching, and let M^* be a maximum matching in G, so that $|M^*| > |M|$. Set $H = G[M \,\Delta\, M^*]$.

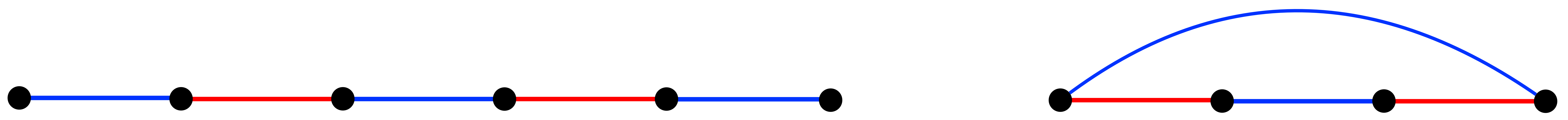


Matchings M (heavy) and M^* (broken)



The subgraph $H = G[M \,\Delta\, M^*]$

Clearly, each vertex of H has degree one or two in H,
for it can be incident with at most one edge of M and one edge of M^*.
Consequently, each component of H is either an even cycle with edges alternately in M and M^*, or else a path with edges alternately in M and M^*.



Because $|M^*| > |M|$, the subgraph H contains more edges of M^* than of M, and therefore there exists some component of H which is a path P starting and ending with edges of M^*.
The origin and terminus of P, being covered by M^*, are not covered by M.
The path P is thus an M-augmenting path in G.

Matchings in Bipartite Graphs

Let $S \subset V(G)$. Define $N(S) = \cup_{v \in S} N(v)$.

Theorem 22(Hall). A bipartite graph $G = G[X, Y]$ has a matching which covers every vertex in X if and only if

$$|N(S)| \geq |S| \text{ for all } S \subseteq X.$$

Proof. Let $G = G[X, Y]$ be a bipartite graph which has a matching M covering every vertex in X.

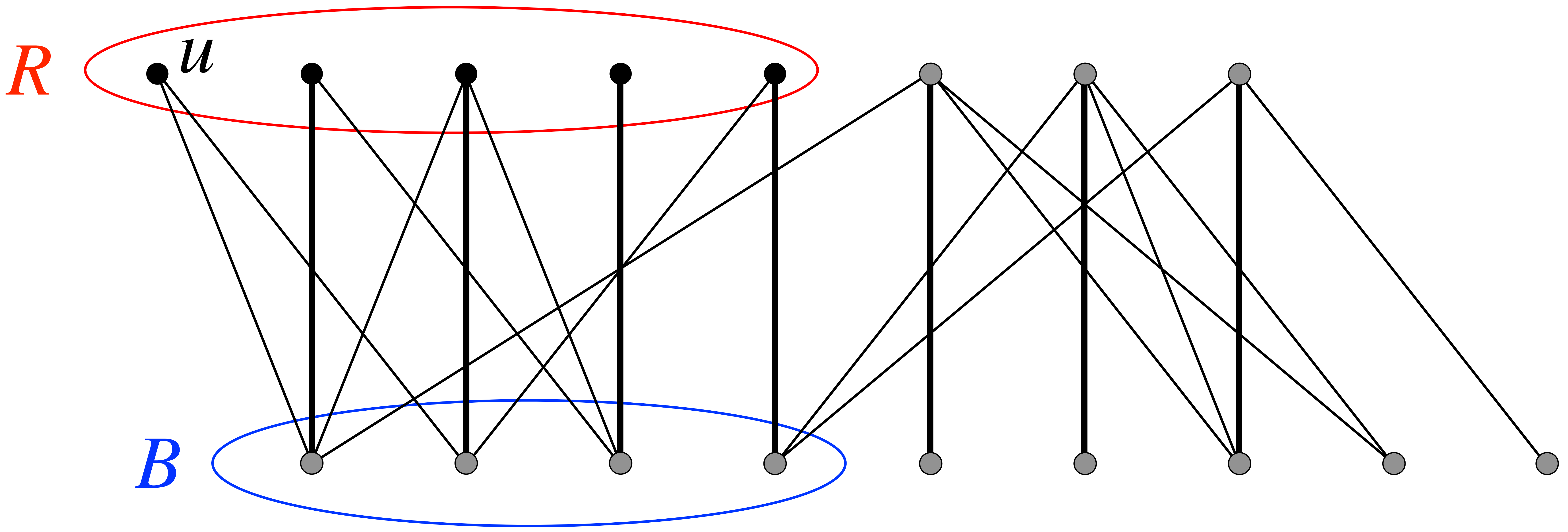
Consider any subset S of X.

The vertices in S are matched under M with distinct vertices in $N(S)$.

Therefore, $|N(S)| \geq |S|$.

Conversely, let $G = G[X, Y]$ be a bipartite graph which has no matching covering every vertex in X.

Let M^* be a maximum matching in G and u a vertex in X not covered by M^*. Denote by Z the set of all vertices reachable from u by M^*-alternating paths. Because M^* is a maximum matching, it follows from Theorem 21 that u is the only vertex in Z not covered by M^*. Set $R = X \cap Z$ and $B = Y \cap Z$.



Clearly, the vertices of $R \backslash \{u\}$ are matched under M^* with the vertices of B. Therefore, $|B| = |R| - 1$ and $B \subseteq N(R)$.
In fact, we have $N(R) = B$, because every vertex in $N(R)$ is connected to u by an M^*-alternating path.
These two equations imply that $|N(R)| = |B| = |R| - 1$, which contradicts the assumption.

Corollary 3. A bipartite graph $G[X, Y]$ has a perfect matching if and only if $|X| = |Y|$ and $|N(S)| \geq |S|$ for all $S \subseteq X$.

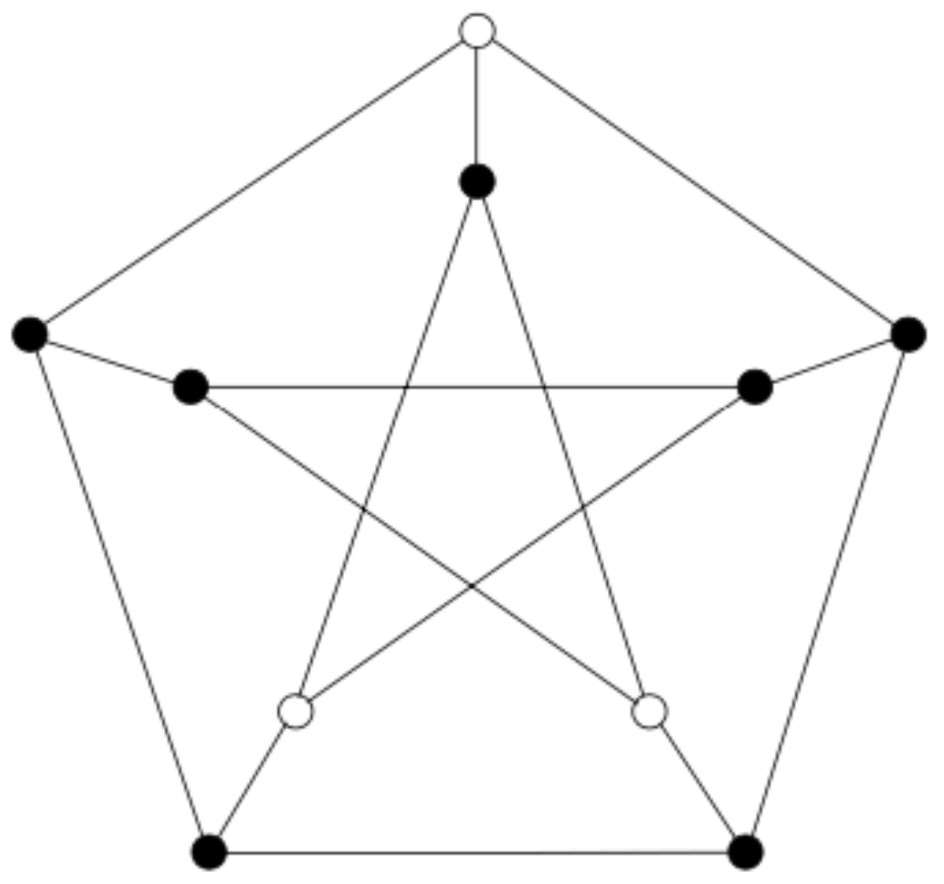
Corollary 4. Every nonempty regular bipartite graph has a perfect matching.

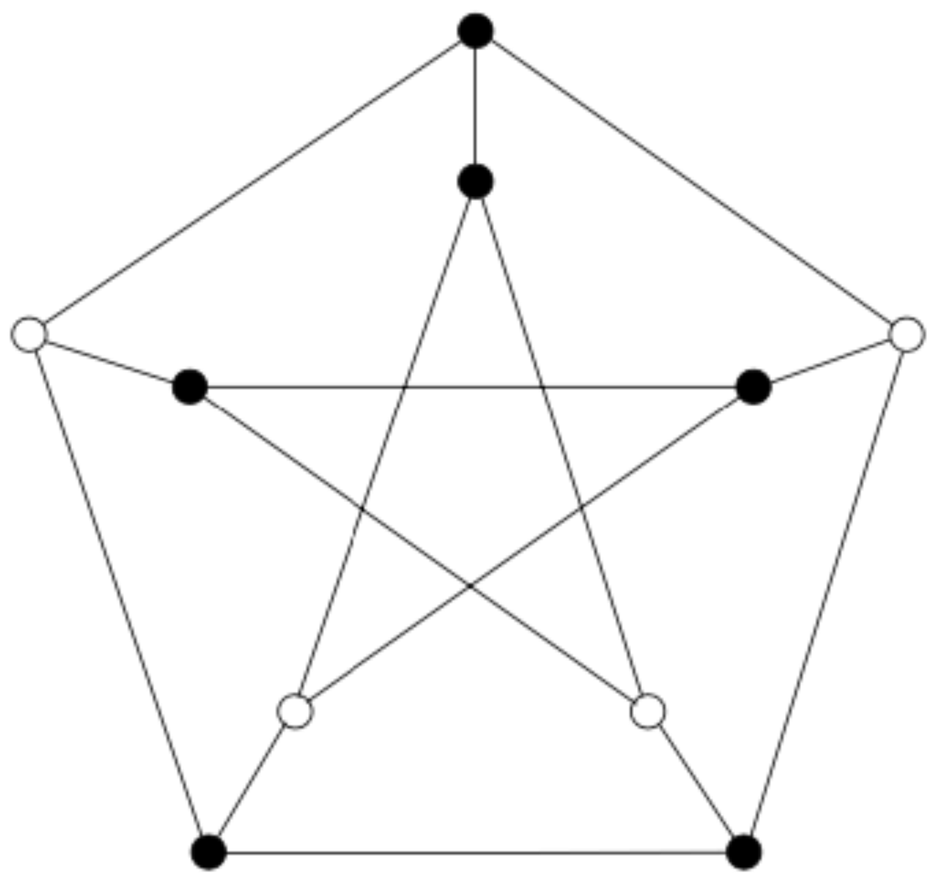
Matchings and Coverings

A covering (覆盖) of a graph G is a subset K of $V(G)$ such that every edge of G has at least one end in K.

A covering K^* is a minimum covering if G has no covering K such that $|K| < |K^*|$. The number of vertices in a minimum covering of G is called the covering number of G, and is denoted by $\beta(G)$.

A covering is minimal if none of its proper subsets is itself a covering.





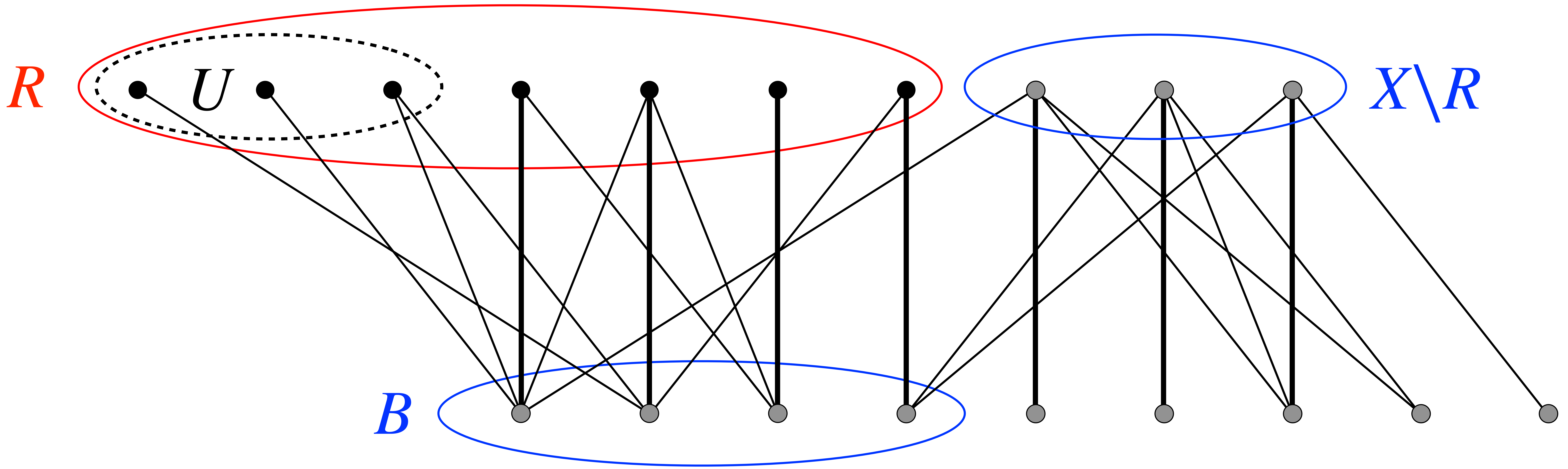
If M is a matching of a graph G, and K is a covering of G, then at least one end of each edge of M belongs to K. Because all these ends are distinct, one can deduce that $|M| \leq |K|$.

Proposition 8. Let M be a matching and K a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

Theorem 23. For a bipartite graph $G = G[X, Y]$, $\alpha'(G) = \beta(G)$.

Proof. Let M^* be a maximum matching in G and U be the set of vertices in X not covered by M^*. Denote by Z the set of all vertices reachable from U by M^*-alternating paths. Set $R = X \cap Z$ and $B = Y \cap Z$. Then $K^* = (X \backslash R) \cup B$ is a covering of G with $|K^*| = |M^*|$.

By Proposition 8, K^* is a minimum covering of G.



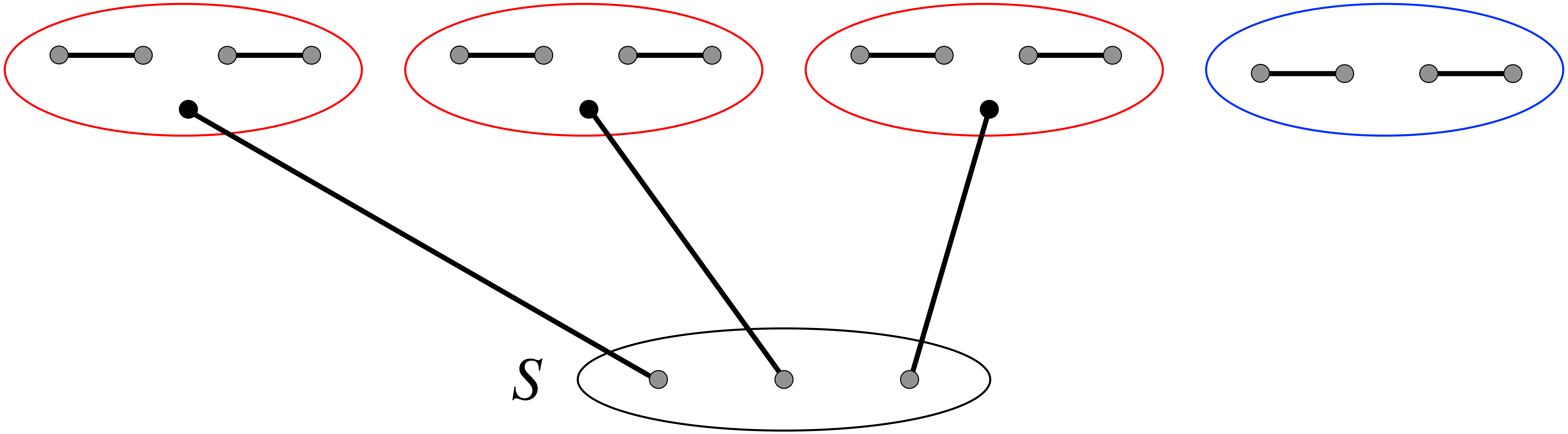
$K^* = (X \backslash R) \cup B$ is a minimum covering of G.

Perfect Matchings in Arbitrary Graphs

Let $S \subset V(G)$ and $o(G - S)$ be the number of odd components of $G - S$.

Theorem 24(Tutte). A graph G has a perfect matching if and only if

$$o(G - S) \leq |S| \text{ for all } S \subseteq V(G).$$



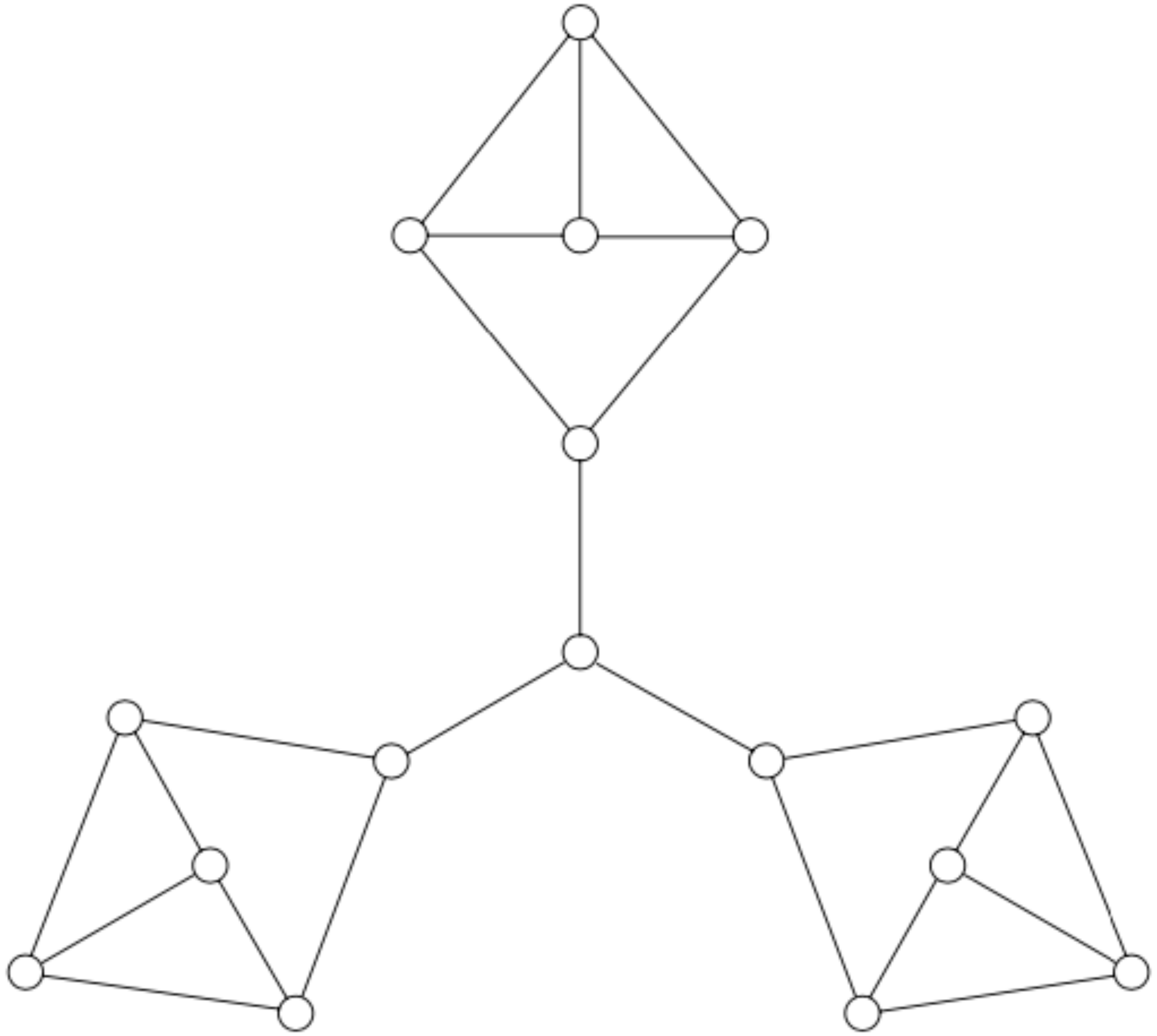
Theorem 25(Petersen). Every 3-regular graph without cut edges has a perfect matching.

Proof. Let G be a 3-regular graph without cut edges. Suppose $S \subset V(G)$ and the odd components of $G - S$ are $S_1, \ldots, S_k$. Since G has no cut edges, there are at two edges between each S_i and S, i.e., $d(S_i) \geq 2$ for $1 \leq i \leq k$. Because $|S_i|$ is odd, one can deduce that $d(S_i) \geq 3$ for $1 \leq i \leq k$.

Note that the edge cuts $\partial(S_i)$ are pairwise disjoint, and $\partial(S_i) \subseteq \partial(S)$, so

$$3k \leq \sum_{i=1}^{k} d(S_i) = d\left(\cup_{i=1}^{k} S_i \right) = d(S) \leq 3|S|.$$

That is, $o(G - S) \leq |S|$. By Theorem 24, the result follows.



A 3-regular graph without perfect matchings

Exercise 6.

1. Show that it is impossible, using 1×2 rectangles, to tile an 8×8 square from which two opposite 1×1 corner squares have been removed.

2. Show that if G is triangle-free on n vertices, then $\alpha'(G) = n - \chi(\overline{G})$, where $\overline{G}$ is the complement graph of G.