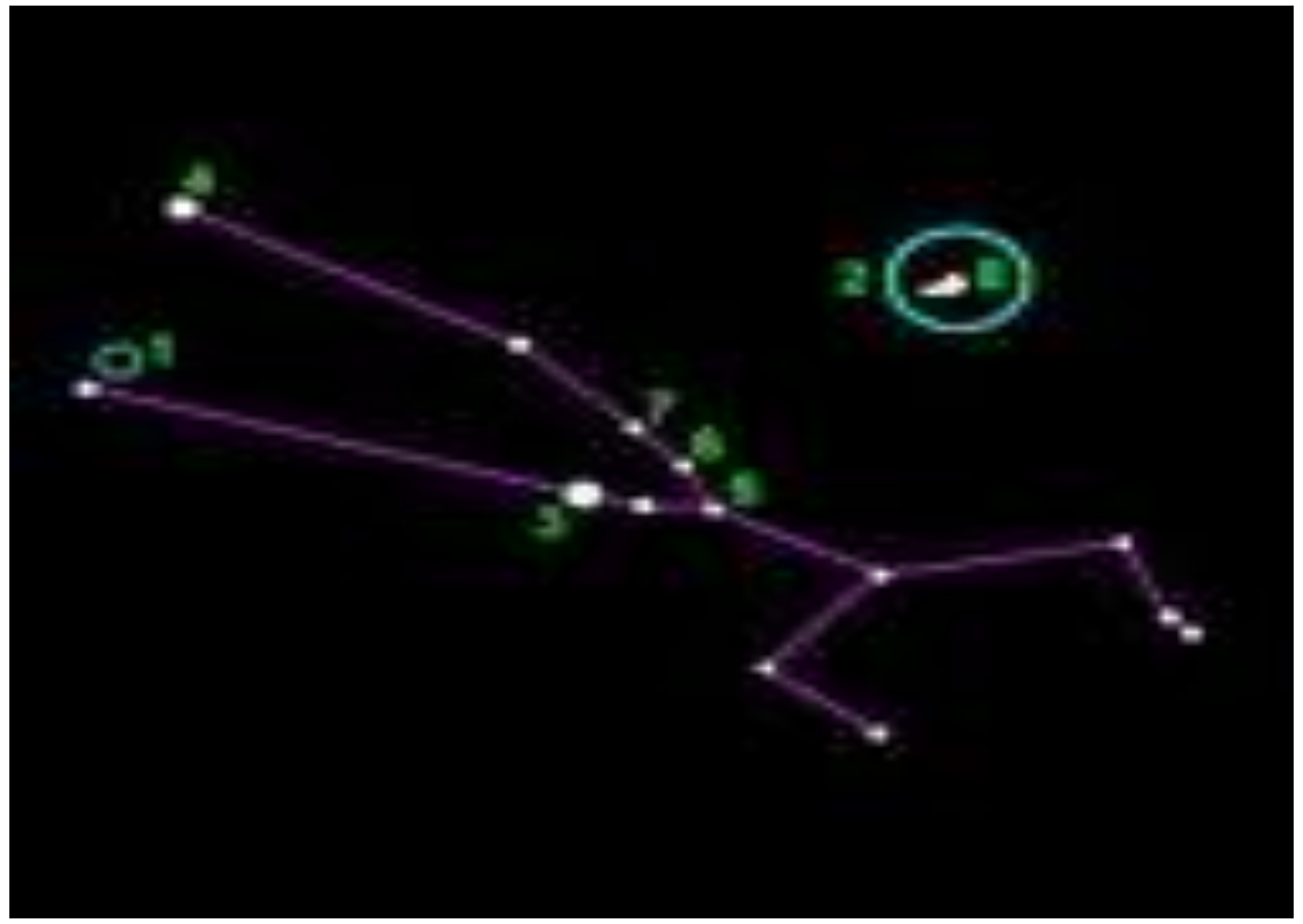
What is Ramsey Theory?

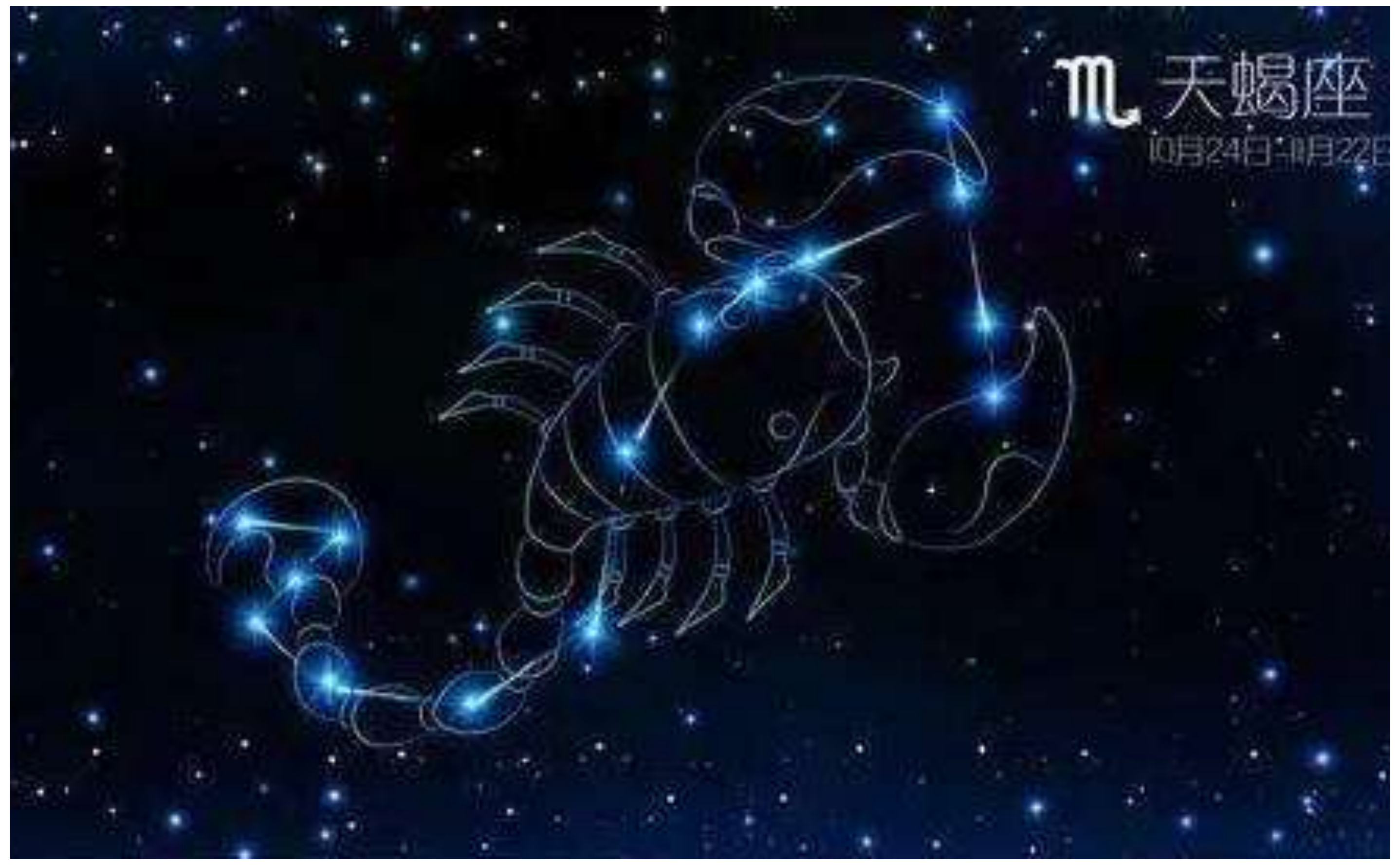
According to a 3500-year-old cuneiform text,
an ancient Sumerian scholar
once looked to the stars in the heavens and saw

A LION

A BULL



A SCORPION



Today, most stargazers would agree that
the night sky appears
to be filled with constellations in the shape of
such as straight lines, rectangles and pentagons, and so on.

Could it be that such geometric patterns
arise from some unknown forces in the cosmos?

Mathematics provides a much more plausible explanation.

In 1928, an English mathematician, philosopher and economist

F.P. Ramsey

proved that such patterns

are actually implicit in any large structure,

no matter it is a group of stars or an array of pebbles!

F.P. Ramsey, On a problem of formal logic,

Proc. London Math. Soc., 30(1930), 361-376.

1. Ramsey theorem

2. Schur theorem

3. Van der Waerden theorem

4. Erdős-Szekeres theorem

1. Ramsey Theorem

PARTY PUZZLE

How many people does it take to form
a group that always contains either n mutual acquaintances
or n mutual strangers?

For $n = 3$, it is known that at least 6 people is needed.

How to prove it?

One method is to list all conceivable combinations,
check each one for an acquainted or unacquainted group of three.

Since we would have to check

$$2^{\binom{6}{2}} = 2^{15} = 32768$$

combinations, this method is neither practical nor insightful.

Now, let **points**

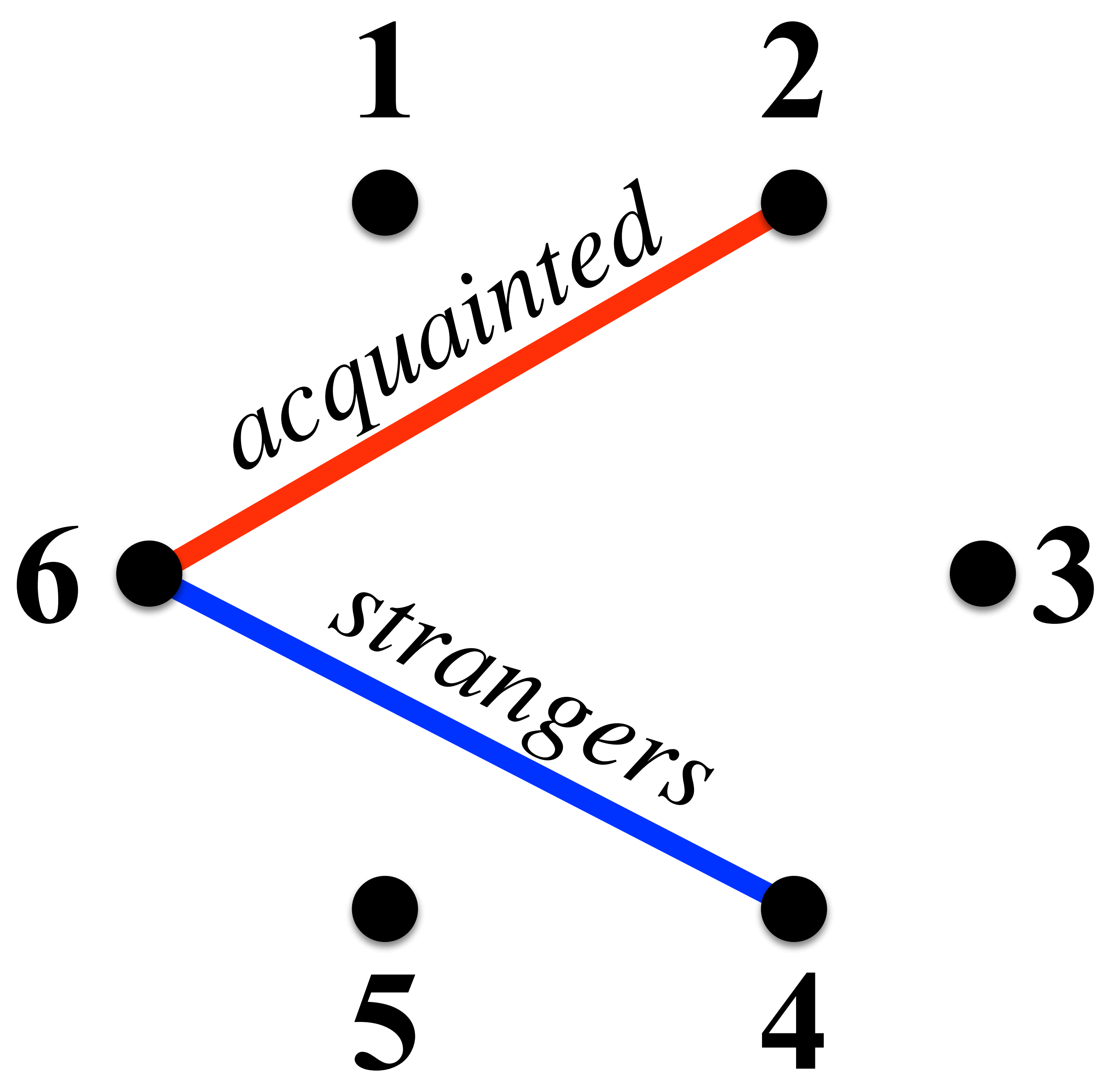
represent people in the party,

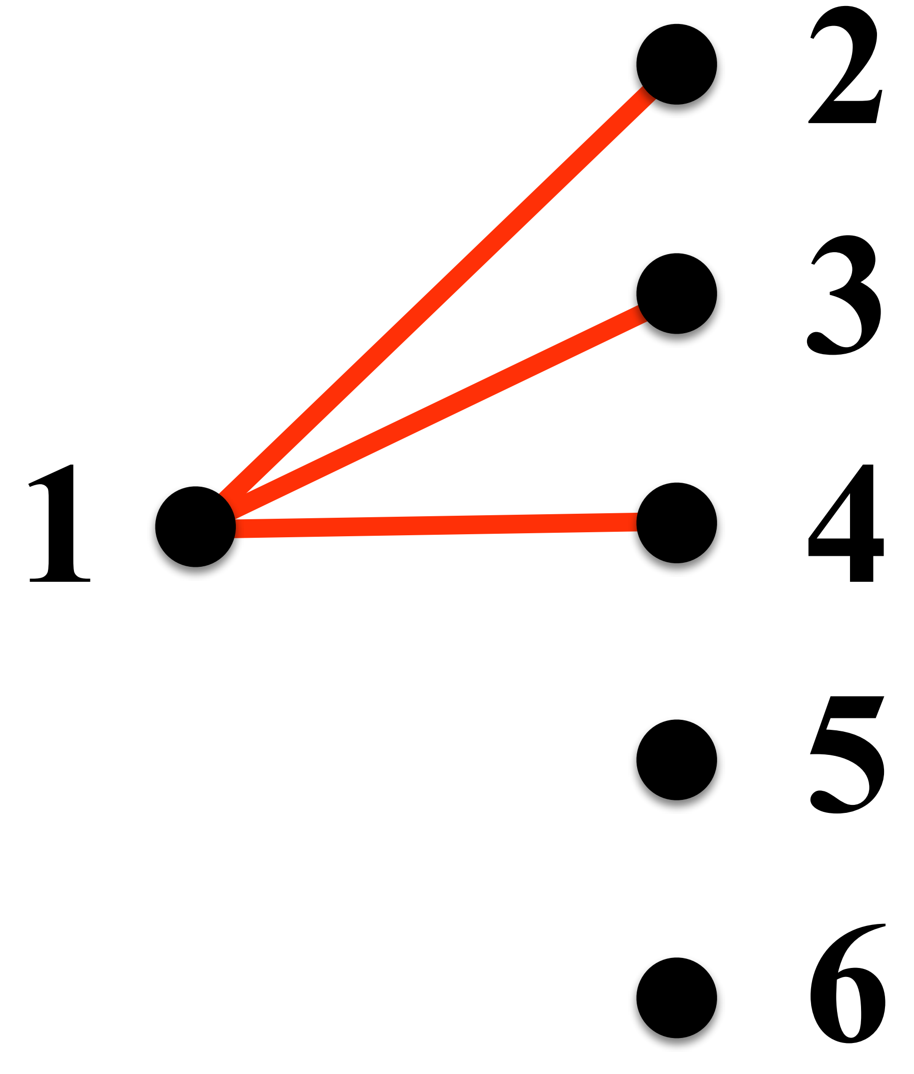
a red edge

joins people who are mutual acquainted,

a blue edge

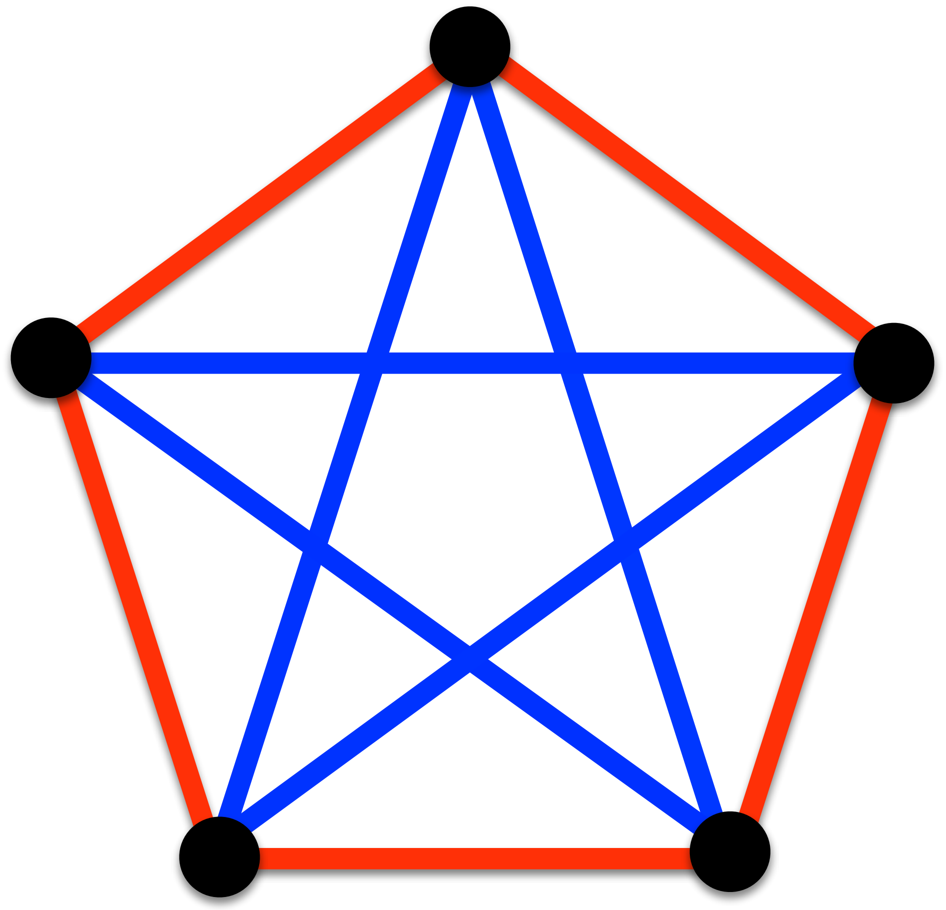
joins people who are mutual strangers.





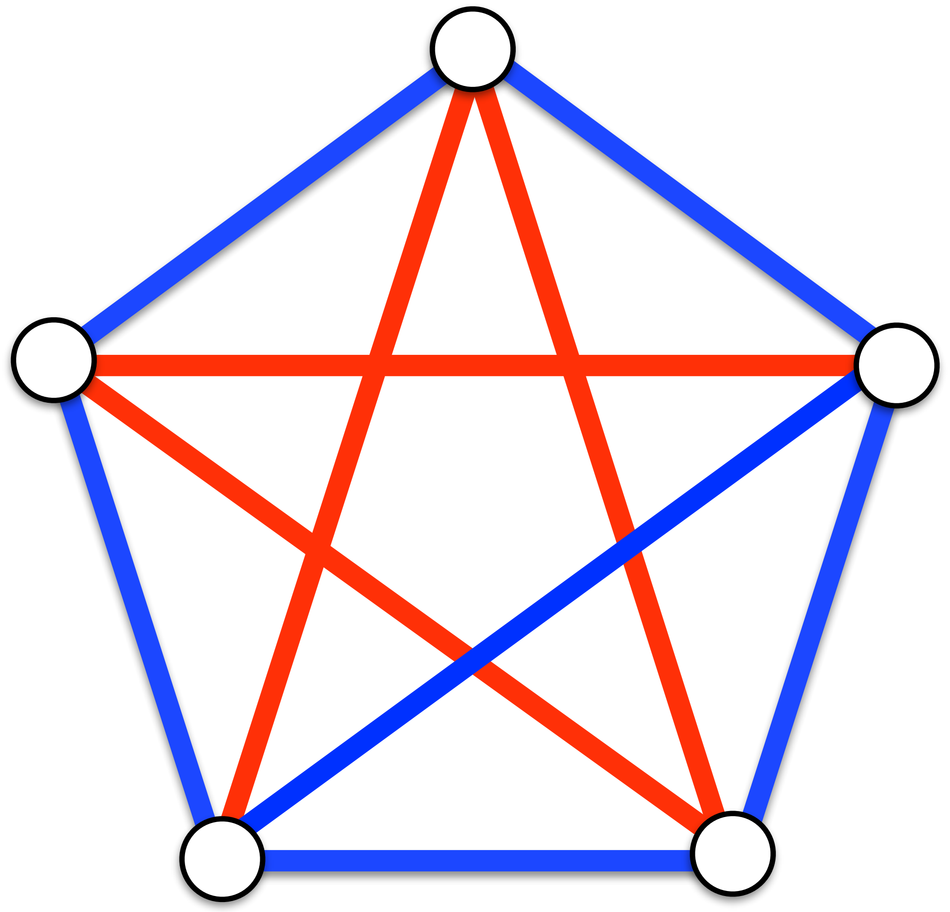
If {2,3,4} are mutual strangers,
then the conclusion holds.

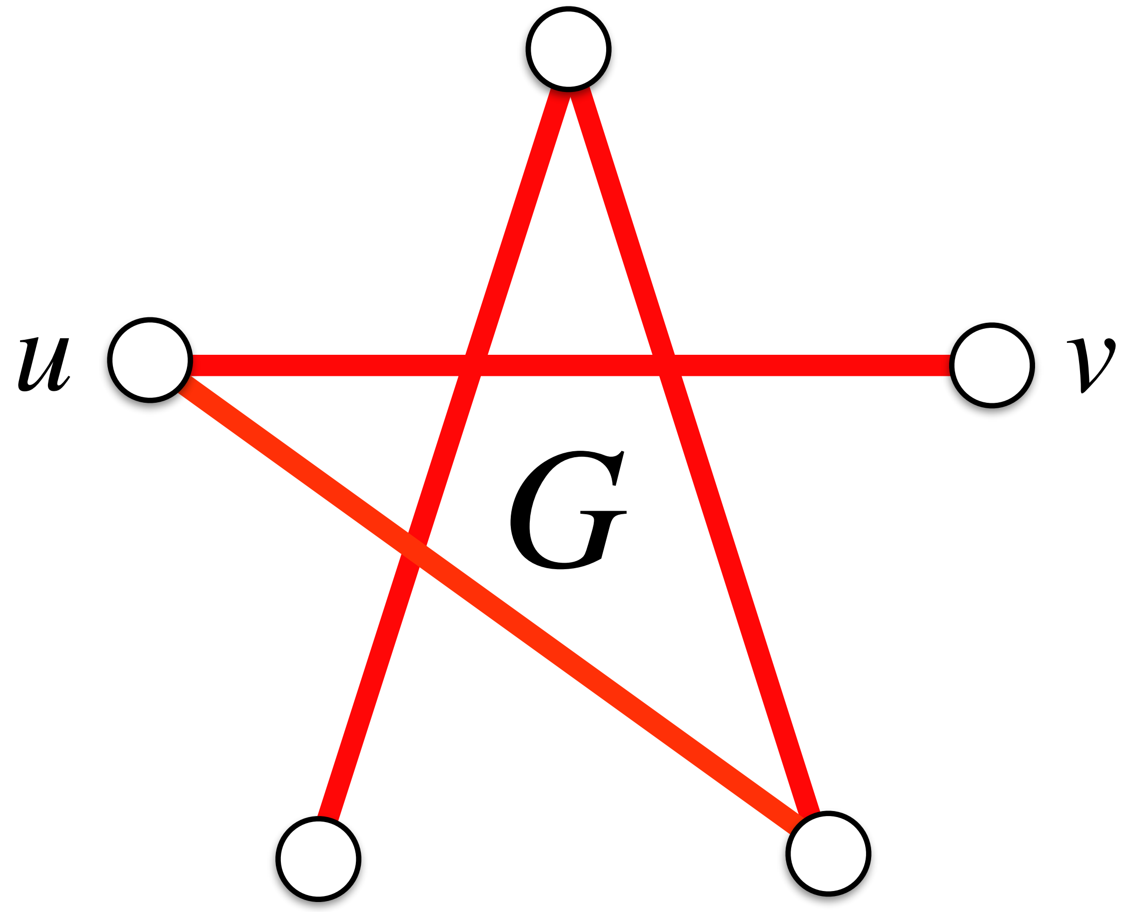
If some two people of {2,3,4} who are mutual acquainted, say 2 and 3 are mutual acquainted, then {1,2,3} are acquainted.



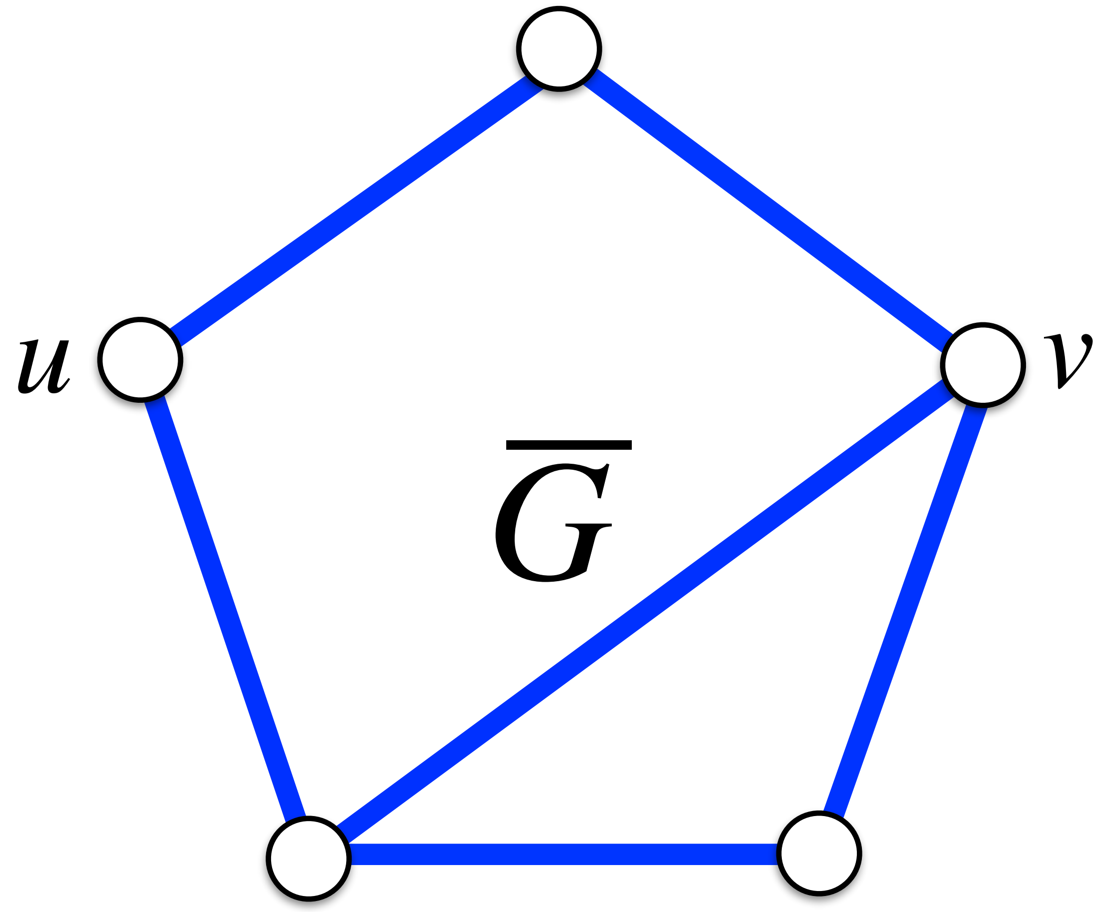
On the other hand,
5 people with relations illustrated in the left shows that 5 is not sufficient.

A complete graph K_N:
a graph on N vertices, any two vertices are connected by an edge.





$$uv \in E(G) \Longleftrightarrow uv \notin E(\overline{G})$$



General Form of PARTY PUZZLE:

Given positive integers p and q, $p, q \geq 2$.

How large $N = N(p, q)$ can guarantee that any 2-edge colorings of a complete graph K_N with red and blue, either K_N contains a red K_p, or a blue K_q ?

Ramsey proved that such an integer N does exist!

Let $R(p,q)$ be the least integer N such that any 2-edge colorings of a complete graph K_N with red and blue, either K_N contains a red K_p, or a blue K_q.

It is easy to see that

$$R(p,2) = p, R(2,q) = q \text{ and } R(p,q) = R(q,p).$$

Theorem 29. For any two integers $p, q \geq 2$, $R(p, q)$ is finite, and

$$R(p, q) \leq R(p-1, q) + R(p, q-1)$$

for $p, q \geq 3$.

Moreover, if $R(p-1, q)$ and $R(p, q-1)$ are both even, then

$$R(p, q) < R(p-1, q) + R(p, q-1) .$$

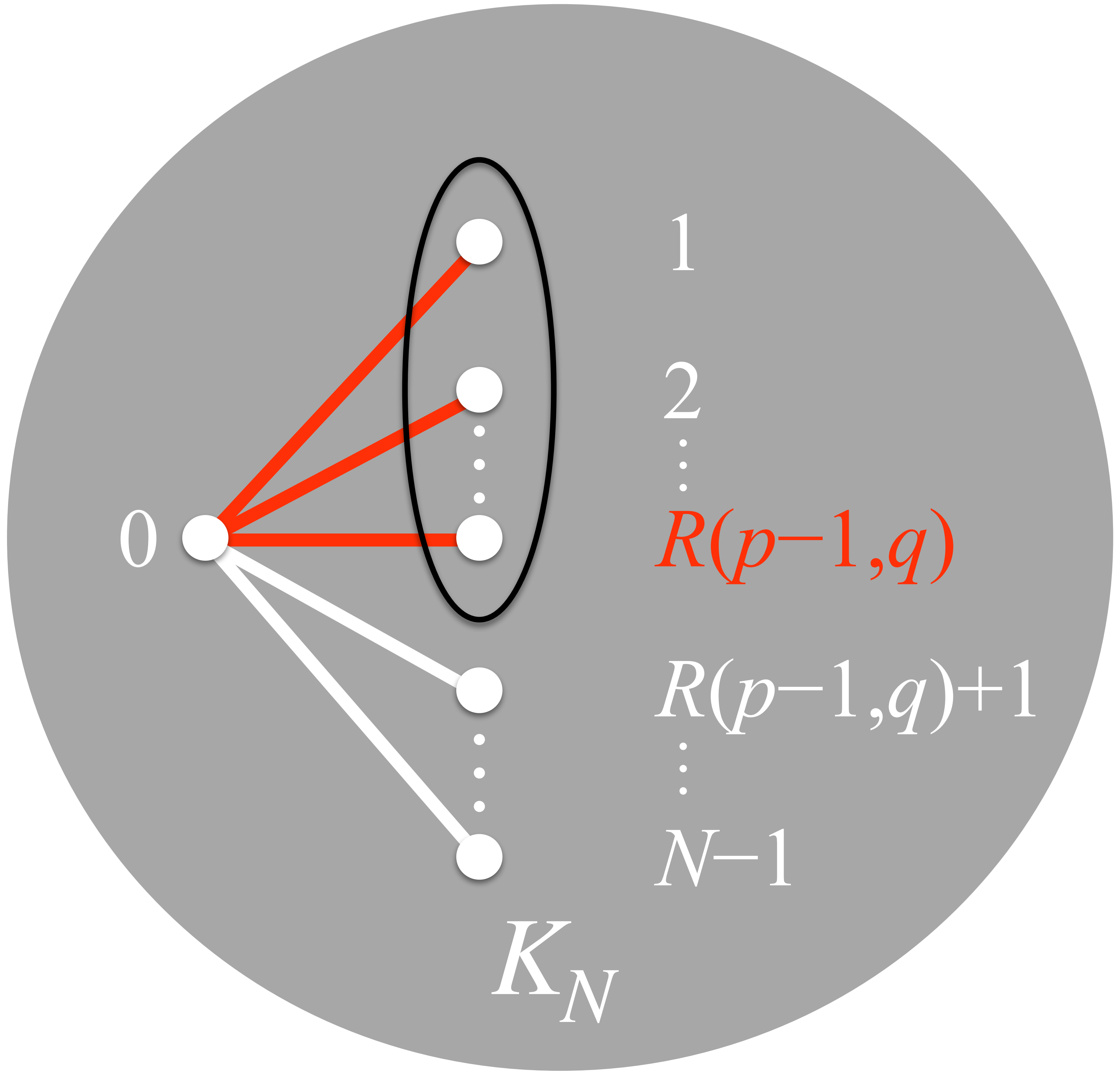
Because

$$R(p, 2) = p \text{ and } R(2, q) = q,$$

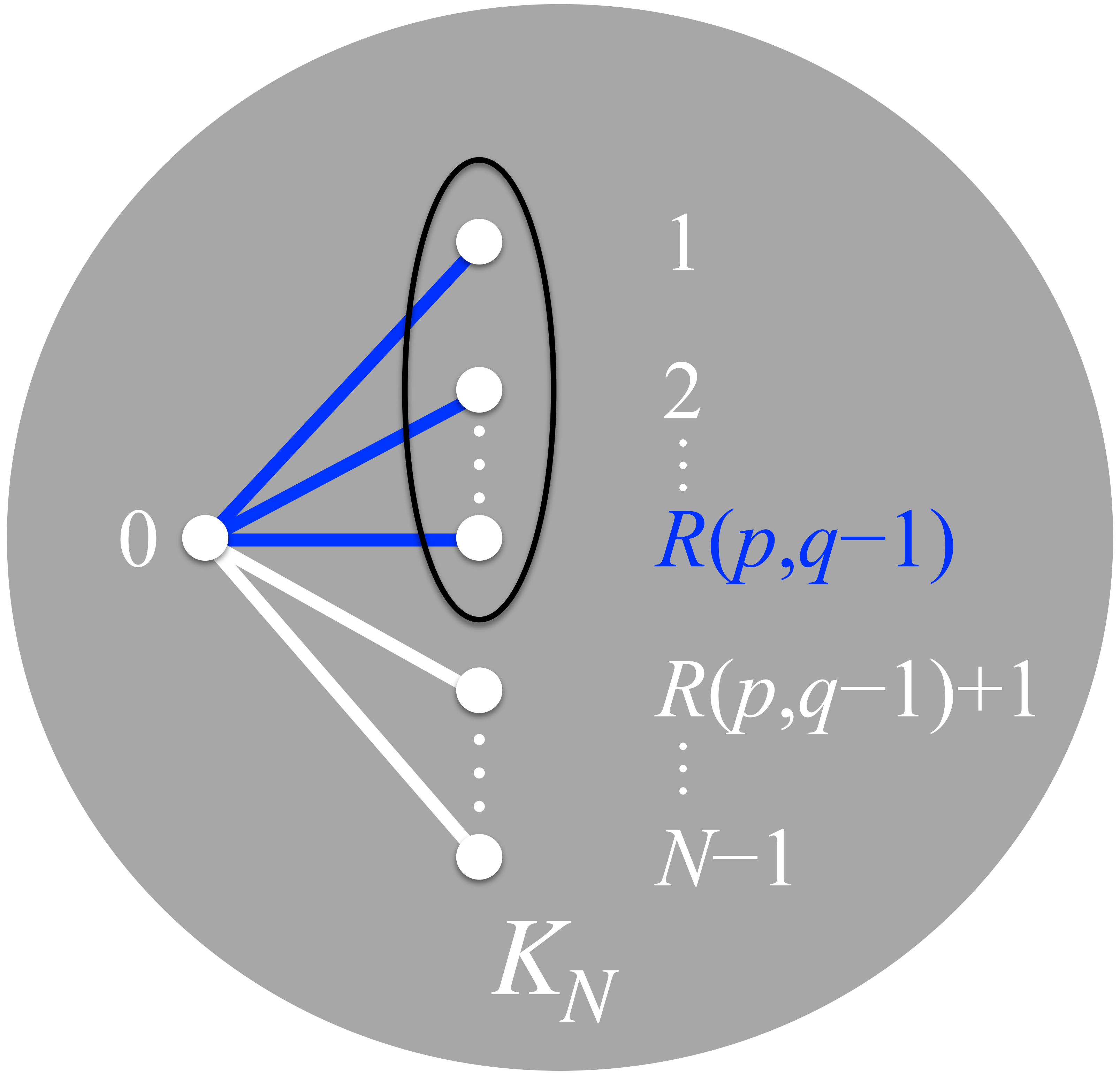
the inequality above implies $R(p, q)$ is finite.

Let $N = R(p-1,q) + R(p,q-1)$.

A vertex incident at least $R(p-1,q)$ red edges



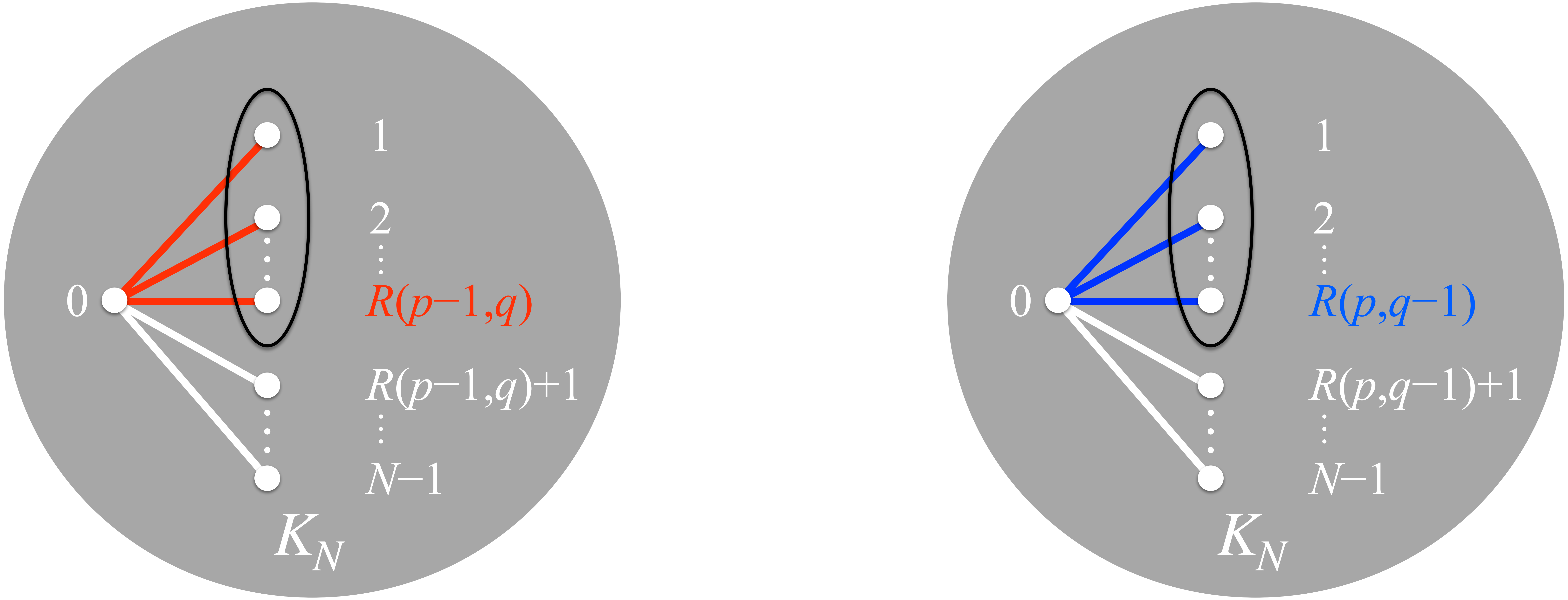
A vertex incident at least $R(p,q-1)$ blue edges



Let $N = R(p-1,q) + R(p,q-1) - 1$.

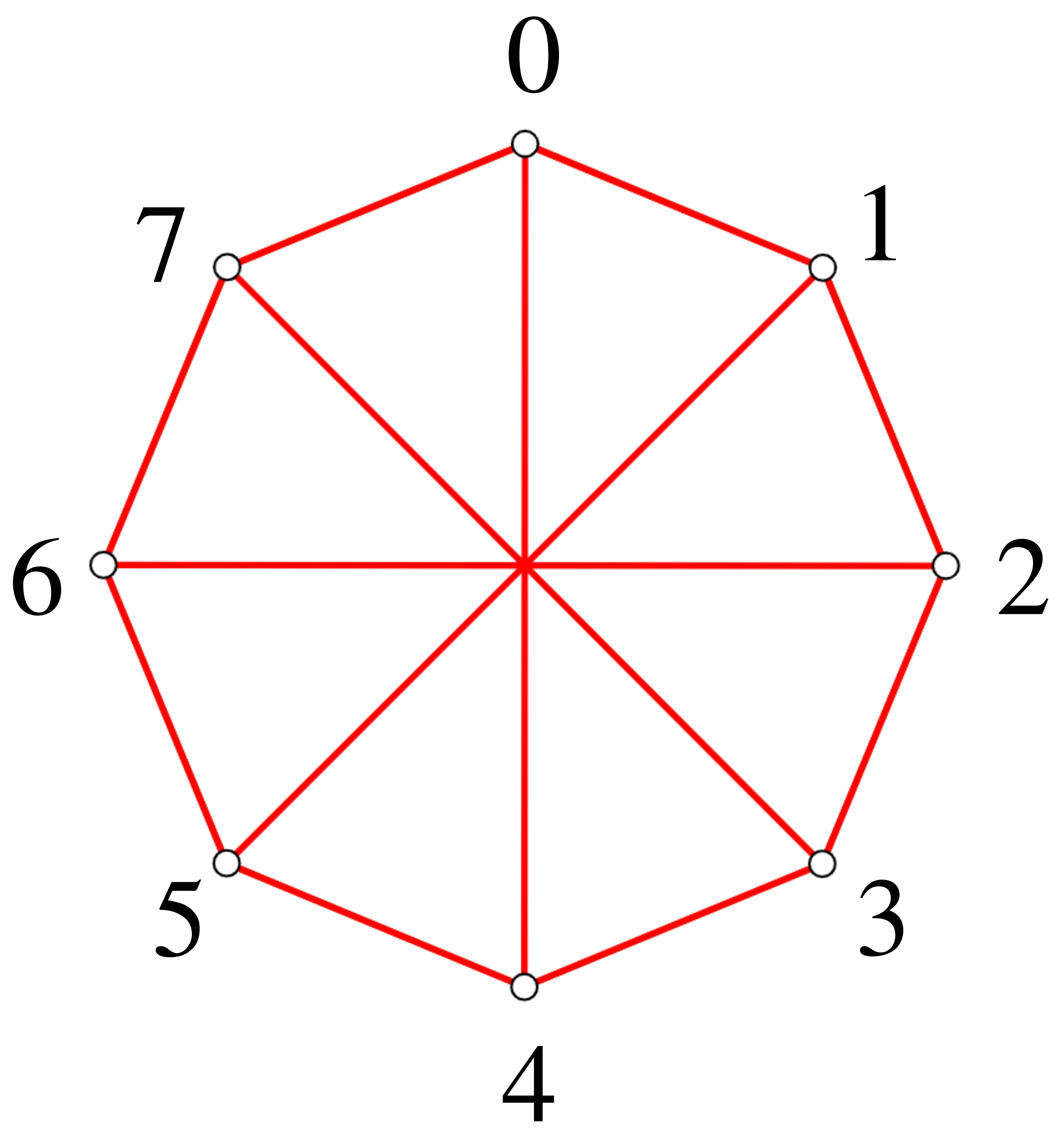
Since N is odd, there is a vertex v in red subgraph $d(v)$ is even.

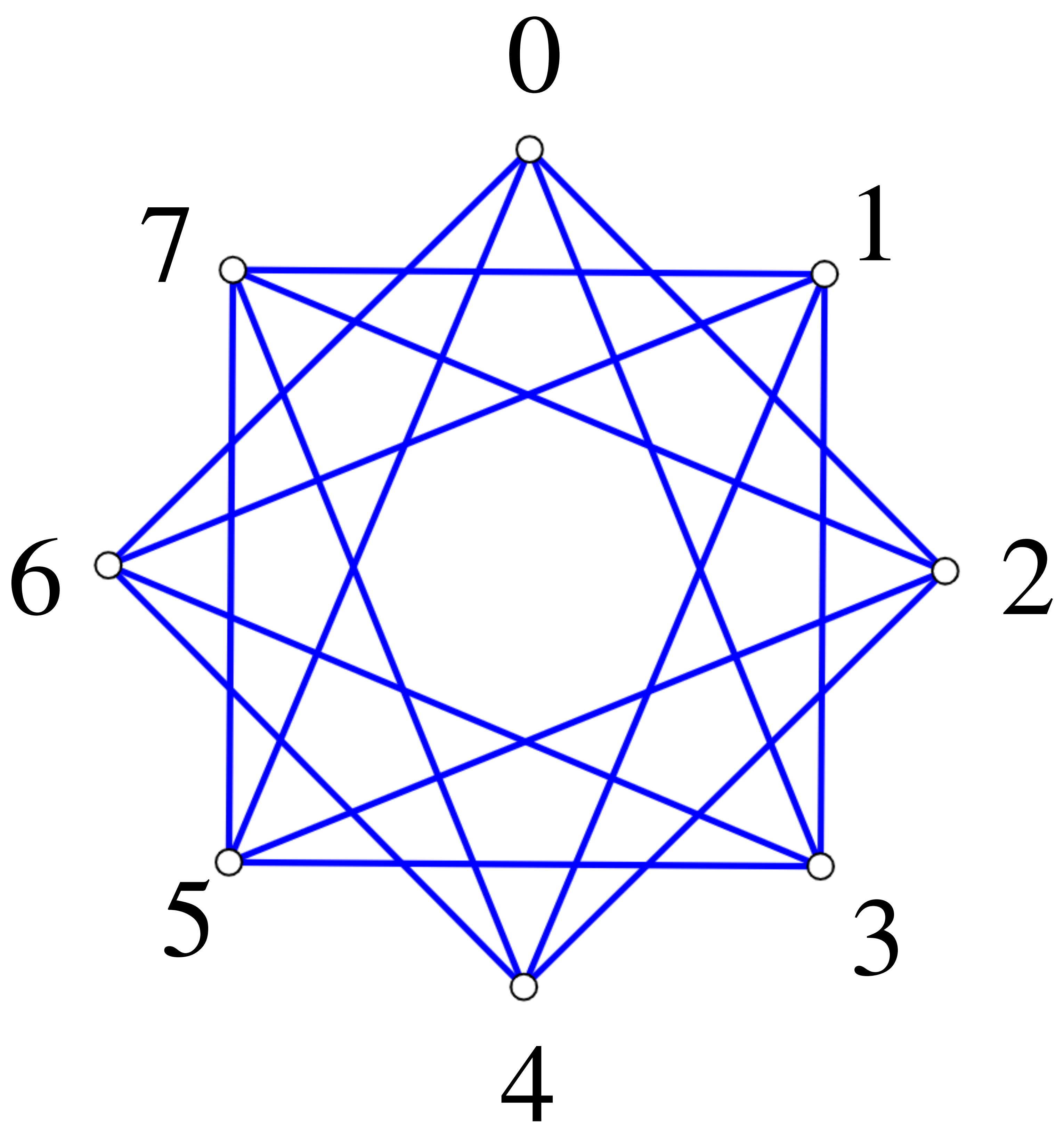
That is, $d(v) \geq R(p-1,q)$ or $d(v) \leq R(p-1,q) - 2$.



$$R(3,4) = 9.$$

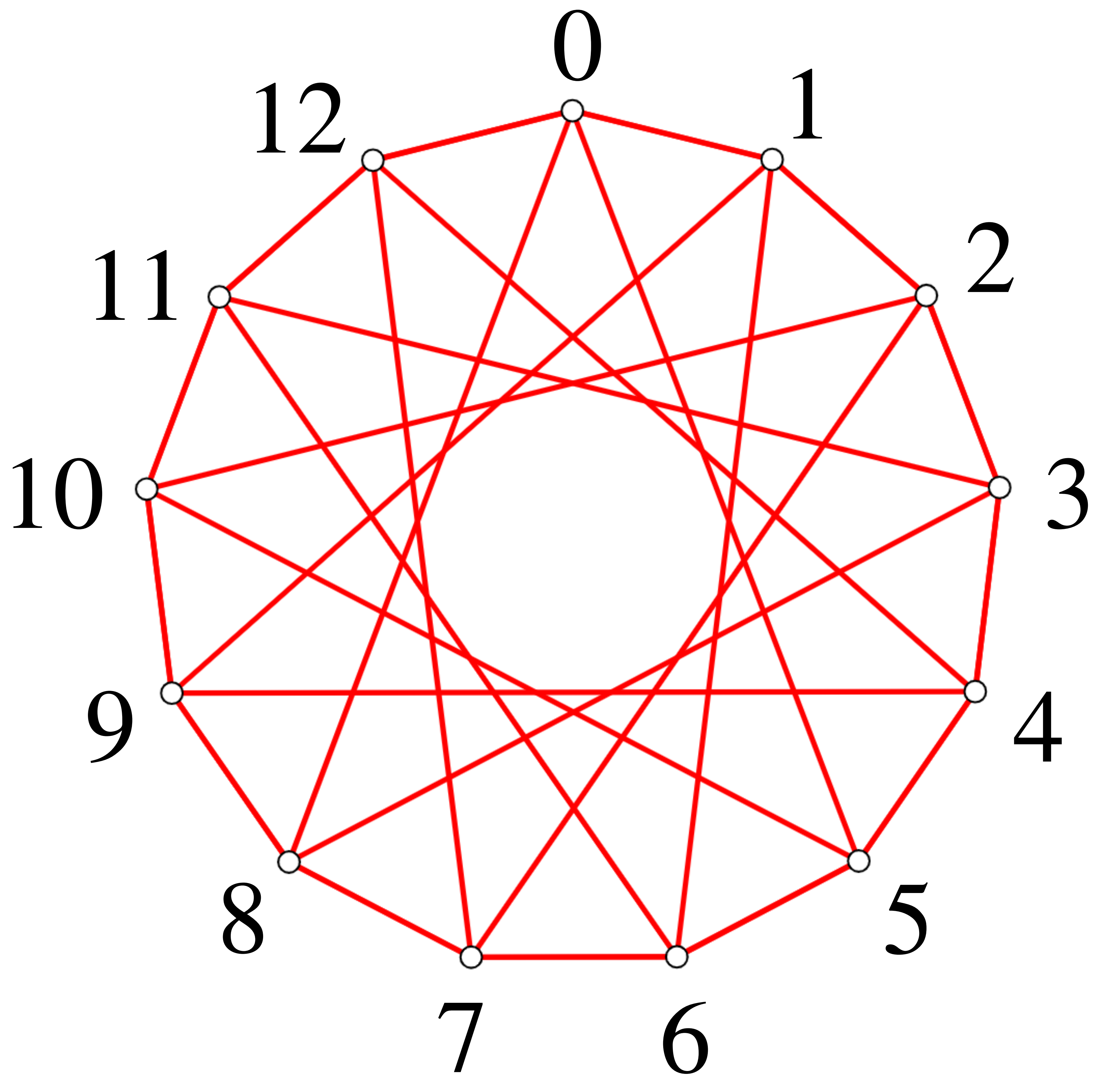
$$R(3,4) \le R(3,3) + R(2,4) - 1 = 9.$$

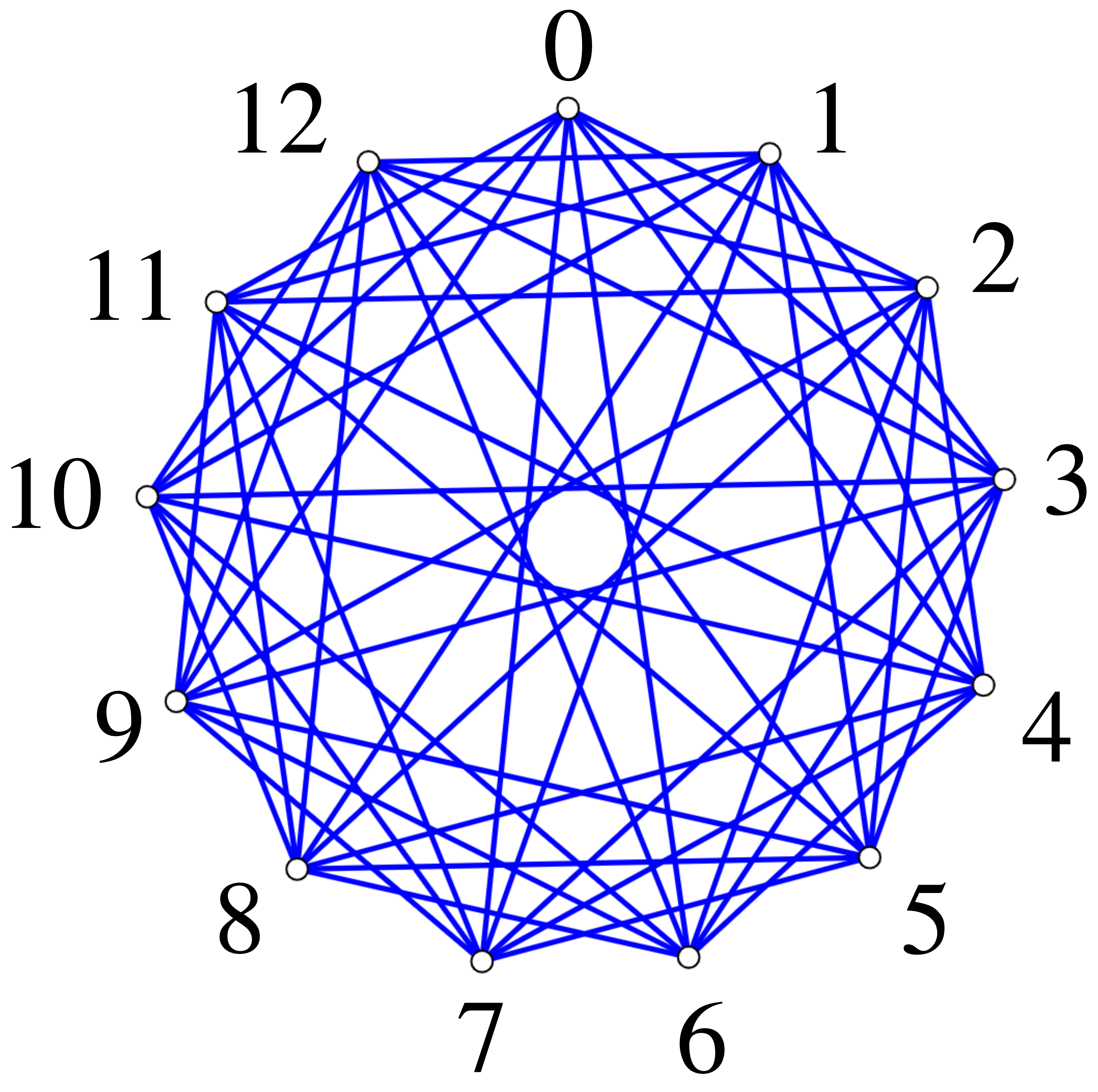




$R(3,5) = 14.$

$R\left(3,5\right) \leq R(3,4) + R(2,5) = 14.$





How about $R(3,6)$?

$$R\left(3,6\right) \leq R(3,5) + R(2,6) - 1 = 14 + 6 - 1 = 19.$$

Is it true that $R(3,6) = 19$?

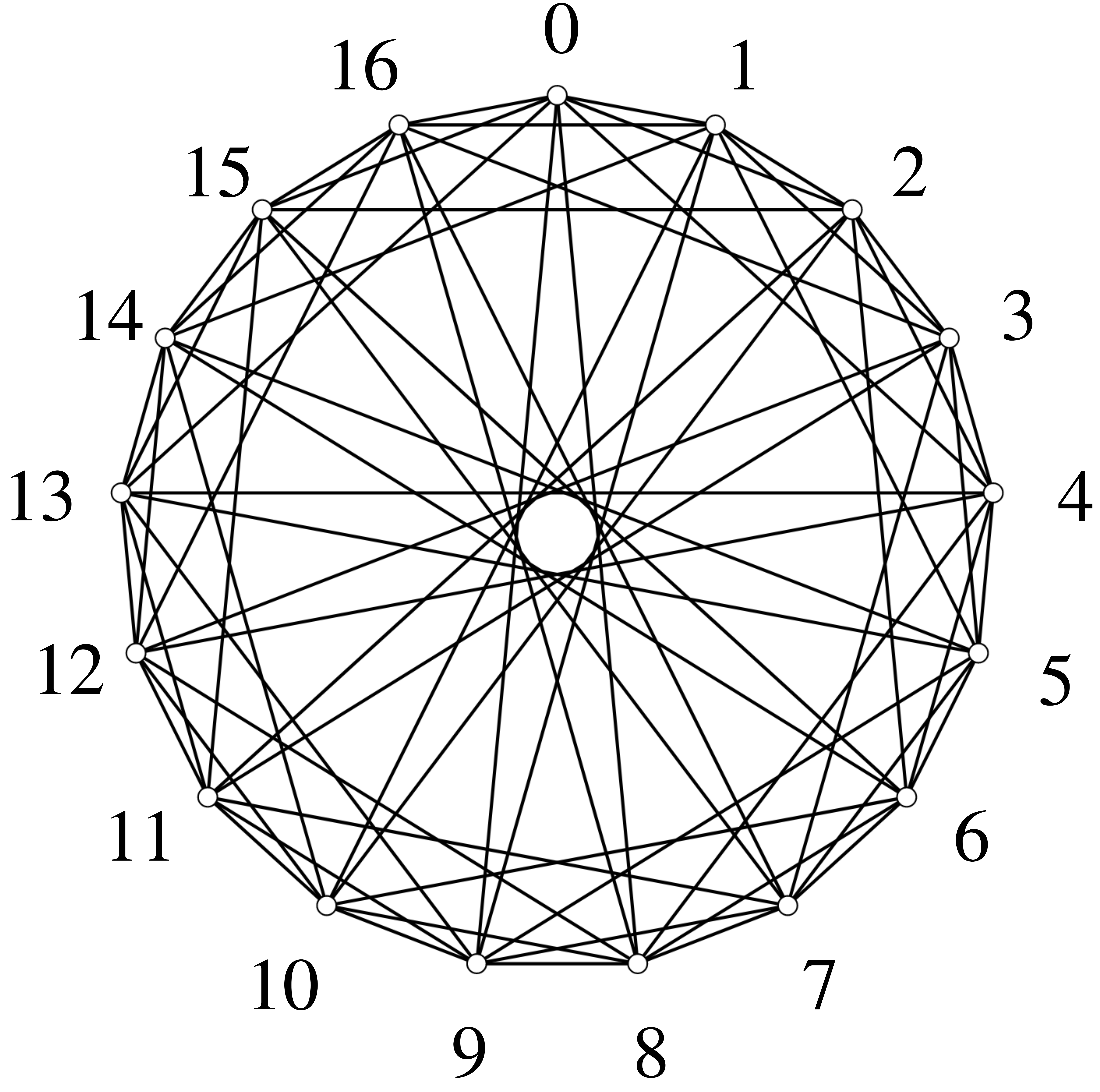
$$R(3,6) = 18!$$

$$R(3,7) \leq R(3,6) + R(2,7) = 18 + 7 = 25.$$

$$R(3,7) = 23!$$

$$R(4,4) \leq R(3,4) + R(4,3) = 9 + 9 = 18.$$

$$R(4,4) = 18.$$


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

Up to know, all known values of classical Ramsey numbers $R(p,q)$ are as follows:

p	3	3	3	3	3	3	3	4	4
q	3	4	5	6	7	8	9	4	5
$R(p,q)$	6	9	14	18	23	28	36	18	25

It is very difficult to evaluate classical Ramsey numbers, both exactly and asymptotically. Most of the values in the table are obtained with help of computers.

A challenging problem is to calculate $R(5,5)$.

It is known that $43 \leq R(5,5) \leq 48$.

Bounds for Classical Ramsey numbers

Theorem 30(Erdős, 1947):

$$R(p,p) > 2^{\frac{p}{2}} \text{ for } p \geq 3.$$

Proof. The number of all red-blue edge colorings of K_n is

$$2^{\binom{n}{2}}.$$

The number of colorings with monochromatic K_p is at most

$$\binom{n}{p} \cdot 2 \cdot 2^{\binom{n}{2} - \binom{p}{2}}.$$

If

$$\binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1} < 2^{\binom{n}{2}},$$

then there exists

a red-blue colorings such that K_n has no monochromatic K_p.

By the definition of $R(p,p)$, we have

$$R(p,p) > n\,.$$

It is not difficult to show that if $n \le 2^{\frac{p}{2}}$, then

$$\binom{n}{p} < \frac{n^p}{2^{p-1}} \le 2^{\frac{p^2}{2}-p+1}$$

$$= 2^{\frac{1}{2}p(p-1)-1} \cdot 2^{-\frac{p}{2}+2} \le 2^{\binom{p}{2}-1}.$$

Thus,

$$\binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1} < 2^{\binom{n}{2}}.$$

By the argument above, we have

$$R(p,p) > 2^{\frac{p}{2}}.$$

Theorem 31(Erdős and Szekeres, 1935)

$$R(p+1, q+1) \leq \binom{p+q}{p}.$$

Proof. If $p+q \leq 3$, then it is easy to see the result holds.
Since $R(p+1, q+1) \leq R(p, q+1) + R(p+1, q)$, we have

$$R(p+1, q+1) \leq \binom{p+q-1}{p-1} + \binom{p+q-1}{p}$$
$$= \binom{p+q}{p}.$$

It is believed that this upper bound is far more being satisfied!

Theorem 32(Graham and Rödl, 1987)

$$R(p+1,q+1) \le \frac{\binom{p+q}{p}}{\log\log(p+q)}.$$

$R(p,q)$ is called *diagonal Ramsey number* if $p=q$, and *off-diagonal Ramsey number* otherwise.

For $R(p,p)$, more advances are obtained recently.

Theorem 33 (Conlon, 2009)

$$R(p+1,p+1) \leq p^{-c\log p/\log\log p}\binom{2p}{p}.$$

where c is a constant.

Theorem 34 (Sah, 2023)

$$R(p+1,p+1) \leq e^{-c(\log p)^2}\binom{2p}{p}.$$

where c is a constant.

Theorem 35 (Campos, Griffiths, Morris, Sahasrabudhe, 2023)

$$R(p,p) \le (4-\varepsilon)^p$$

for some constant ε (p is sufficiently large).

It is known that

$$R(p,p) \le \binom{2p-2}{p-1} = O\left(\frac{4^p}{\sqrt{p}}\right).$$

Problem 2. Does $\lim_{p\to\infty} \sqrt[p]{R(p,p)}$ exist?

If true, then $\sqrt{2} \le \lim_{p\to\infty} \sqrt[p]{R(p,p)} < 4.$

For off-diagonal Ramsey number $R(p,q)$,

if p is fixed and $q \to \infty$, then

Erdős and Szekeres' upper bound implies

$$R(p,q) \leq q^{p-1} .$$

Ajtai, Komlós and Szemerédi improved the bound

$$R(p,q) \leq c_p \frac{q^{p-1}}{(\log q)^{p-2}} .$$

where c_p is a constant dependent on p .

When $p = 3$, it says there is a constant c such that

$$R(3,q) \leq c\frac{q^2}{\log q}.$$

Kim's lower bound for $R(3,q)$ (1995):

$$R(3,q) \geq c\frac{q^2}{\log q},$$

where c is a constant. Thus, the asymptotical order of $R(3,q)$ is:

$$R(3,q) = \Theta\left(q^2/\log q\right).$$

Exercise 10.

1. Show that $R(3,3) = 6$.

2. Show that $R(n, n) > 2^{\frac{n}{2}}$ for $n \geq 3$.

Definition of multiple colors Ramsey number

Let G_i be a simple graph of order n_i, $1 \le i \le k$.

The *Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the minimum integer N with the following property:

If the edges of K_N are colored by k colors, then there is some i with $1 \le i \le k$ such that K_N has a subgraph in color i, which is isomorphic to G_i.

If each G_i is a complete graph for $1 \le i \le k$, then we call $R(G_1, G_2, \ldots, G_k)$ *classical Ramsey number*, and write $R(G_1, G_2, \ldots, G_k) = R(n_1, n_2, \ldots, n_k)$.

General Form of Ramsey Theorem

Theorem 36. For any two positive integers r, k, and $q_1, q_2, \ldots, q_k \geq r$, there is $N = N(q_1, q_2, \ldots, q_k)$, such that for any $n \geq N$, and any k-colorings of $[n]^{(r)}$, there exists some i with $1 \leq i \leq k$ and a q_i-set $S_i \subseteq [n]$ such that $S_i^{(r)}$ are in color i.

$[n] = \{1,2,\ldots,n\}$; $S_i^{(r)}$ denotes all r-subsets of S_i.

The smallest integer N satisfies Theorem 2 is called Ramsey number, write as $R^{(r)}(n_1, n_2, \ldots, n_k)$.

If $r = 1$, then Theorem 2 is drawer principle.

If $r = 2$, then $R^{(2)}(n_1, n_2, \ldots, n_k) = R(n_1, n_2, \ldots, n_k)$, that is, multiple colors Ramsey number.

2. Schur Theorem

The following result, due to Schur (1916), which is viewed as one of the originations of Ramsey theory:

Theorem 37(Schur, 1916). For any given integer k, there exists an N, such that if $n \geq N$, then for any k-colorings of $[n]$, there exist $x, y, z \in [n]$ of the same color such that $x + y = z$.

Let S_k be the least possible value of N in Theorem 3.

S_k is called Schur number. All known Schur numbers up to now:

$$S_1 = 2,\ S_2 = 5,\ S_3 = 14,\ S_4 = 45.$$

The value of S_4 is determined by computer in 1965.

Schur theorem can be proved by Ramsey theorem.

For any given integer k, take

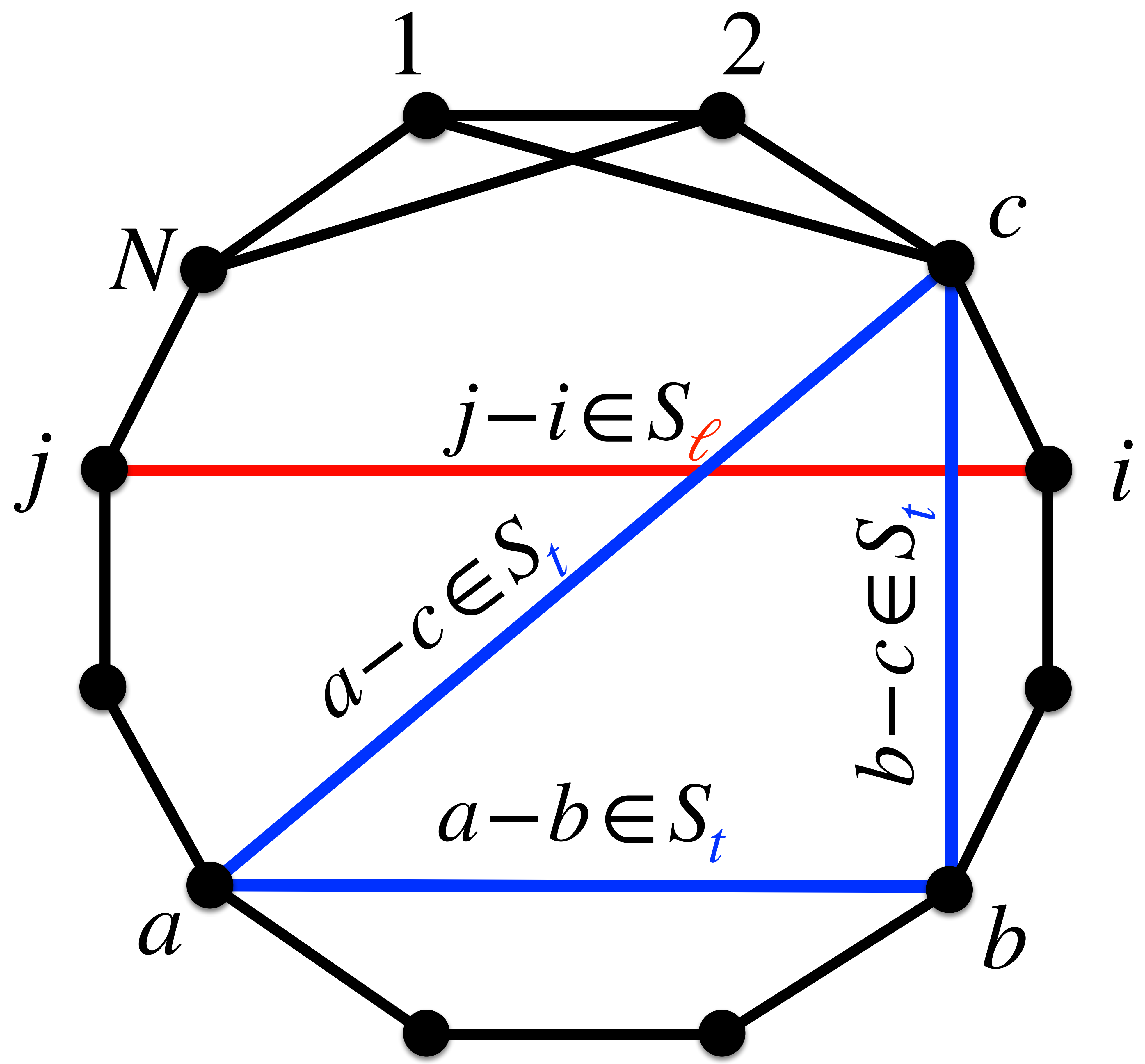
$$N = R^{(2)}\overbrace{(3,3,...,3)}^{k} = R\overbrace{(K_3, K_3, \ldots, K_3)}^{k}.$$

Let $S_1, S_2, \ldots, S_k$ be any partition of $[N]$.
For any 2-subset $\{i,j\}$ of $[N]$, color $\{i,j\}$ with ℓ if $|i-j| \in S_\ell$.
Thus, we get a k-colorings of all edges.

By Ramsey theorem, $[N]$ has a 3-subset $\{a,b,c\}$ such that all of its 2-subsets $\{a,b\}, \{b,c\}, \{a,c\}$ receive the same color.

Assume $a > b > c, x = a-b, y = b-c,$ and $z = a-c$.
Clearly, $x, y, z \in S_\ell$ for some ℓ with $1 \le \ell \le k$, and $x + y = z$.

If $|j-i| \in S_\ell$, color the edge ij with ℓ, $1 \le \ell \le k$.



Let $x = a-b$, $y = b-c$ and $z = a-c$, then $x + y = z$.

3. Van der Waerden Theorem

Van der Waerden (1928) proved the following.

Theorem 38 (Van der Waerden). For any two positive integers $\ell, k,$ there exists a positive integer $W = W(\ell, k)$ such that for any k-colorings of $[W]$, $[W]$ has an arithmetic progressions with ℓ terms of the same color.

The least possible value of $W(\ell, k)$ in Theorem 4, is called Van der Waerden number.

Some known Van der Waerden numbers:

$W(3,2) = 9, W(3,3) = 27, W(3,4) = 76, W(4,2) = 35, W(5,2) = 178.$

Proof of $W(3,2)=9$.

Case 1. 4 and 6 has the same color, say in red.

1	2	3	4	5	6	7	8	9
	2		4	5	6		8	

Case 2. 4 and 6 are in different colors.

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

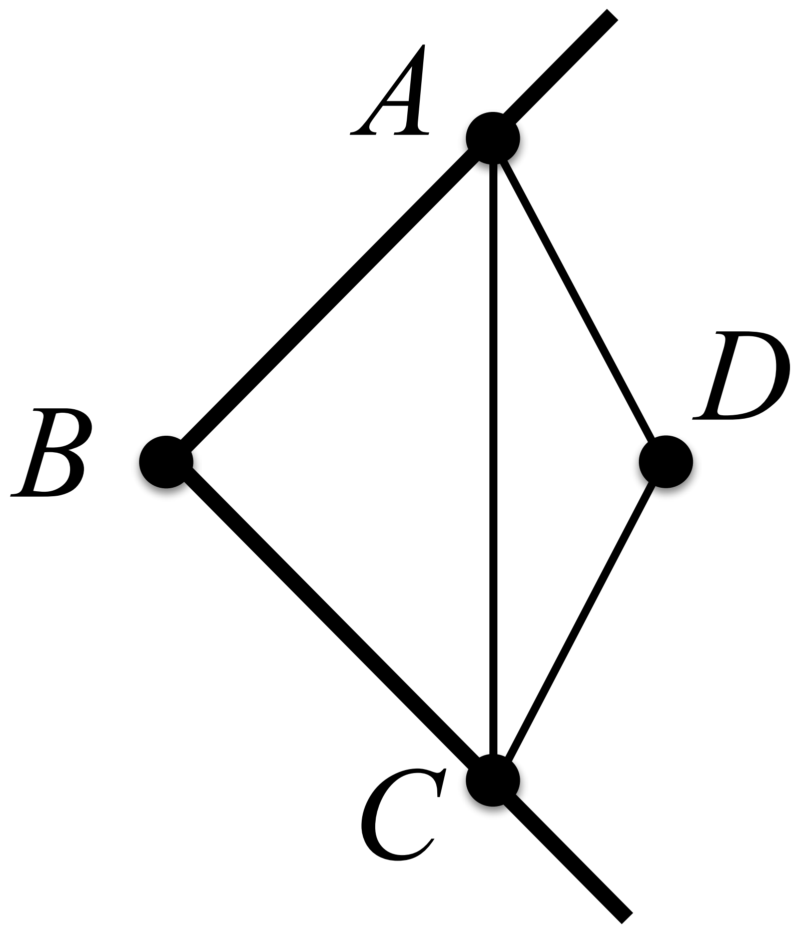
4. Erdős-Szekeres Theorem

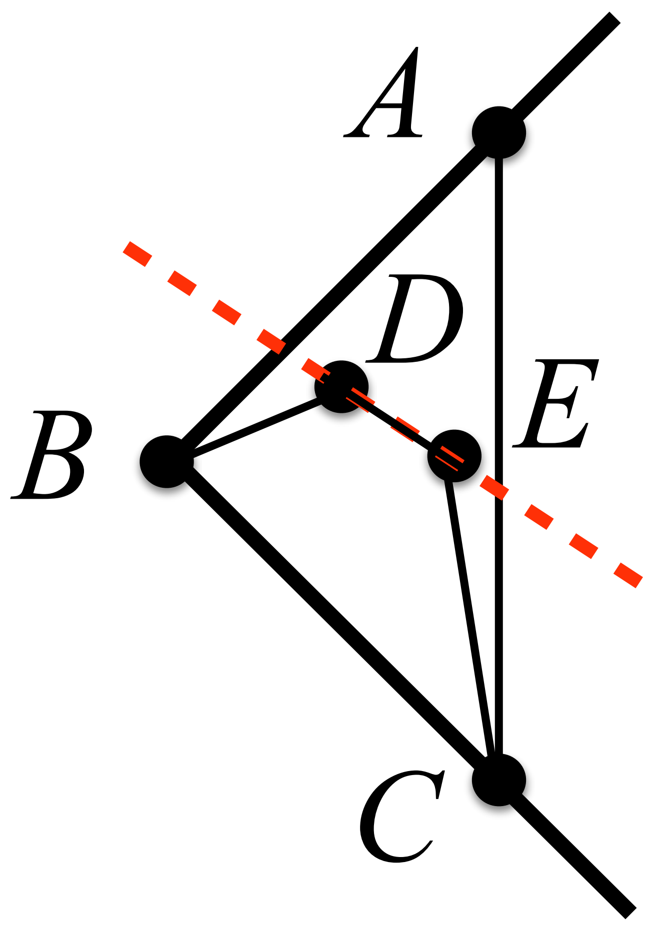
Erdős and Szekeres (1935) published the following:

Theorem 39(Erdős and Szekeres). Let m be a positive integer. Then there is a positive integer N, such that for any N points lie in a plane so that no three points form a straight line, there have m points form a convex m-polygon.

For $m=4$, $N=5$.

Choose three points, say A,B,C *s.t.* $\angle ABC < 180^0$ and the other two points D, E lie in $\angle ABC$.





Let $N(m)$ be the smallest integer in Theorem 5.
We have known that

$$N(3) = 3,$$
$$N(4) = 5,$$
$$N(5) = 9.$$

No other values of $N(m)$ are known up to now.

Erdős conjectured that

$$N(m) = 1 + 2^{m-2}.$$

The conjecture is true for $m = 3,4,5$.

Generalized Ramsey Numbers

Because the extreme difficulty encountered in the study of classical Ramsey numbers,

Chvátal and Harary in a series of papers:

1. Periodica Math. Hungarica 3(1973) 115-124,
2. Proceedings of AMS 32(1972) 389-394,
3. Pacific J. Math. 41(1972) 335-345,

suggested studying *generalized Ramsey number*, i.e., $R(G, H)$ for G or H is not a complete graph, both for their own sake, and for the light they might shed on the classical Ramsey numbers.

At the early stage of this study,
most people focused on the Ramsey numbers for

a complete graph versus

a complete graph minus one edge,

and hoped this study might shed some light on the classical Ramsey numbers. However, they found shortly that to study this class of Ramsey numbers is still very hard, and it is almost the same difficult as that of classical Ramsey numbers.

By now, the most important result for generalized Ramsey number may be the following one due to Chvátal:

Theorem 40(Chvátal). Let T_m be a tree of order m, and K_n a complete graph of order n, then

$$R(T_m, K_n) = (m-1)(n-1)+1.$$

Proof. Let G be a graph of order $(m-1)(n-1)+1$.
If its complement $\overline{G}$ has no K_n, then $\alpha(G) \leq n-1$, which implies

$$\chi(G) \geq m.$$

Let H be a critical subgraph of G with respect to its chromatic number m,

$$\delta(H) \geq m-1.$$

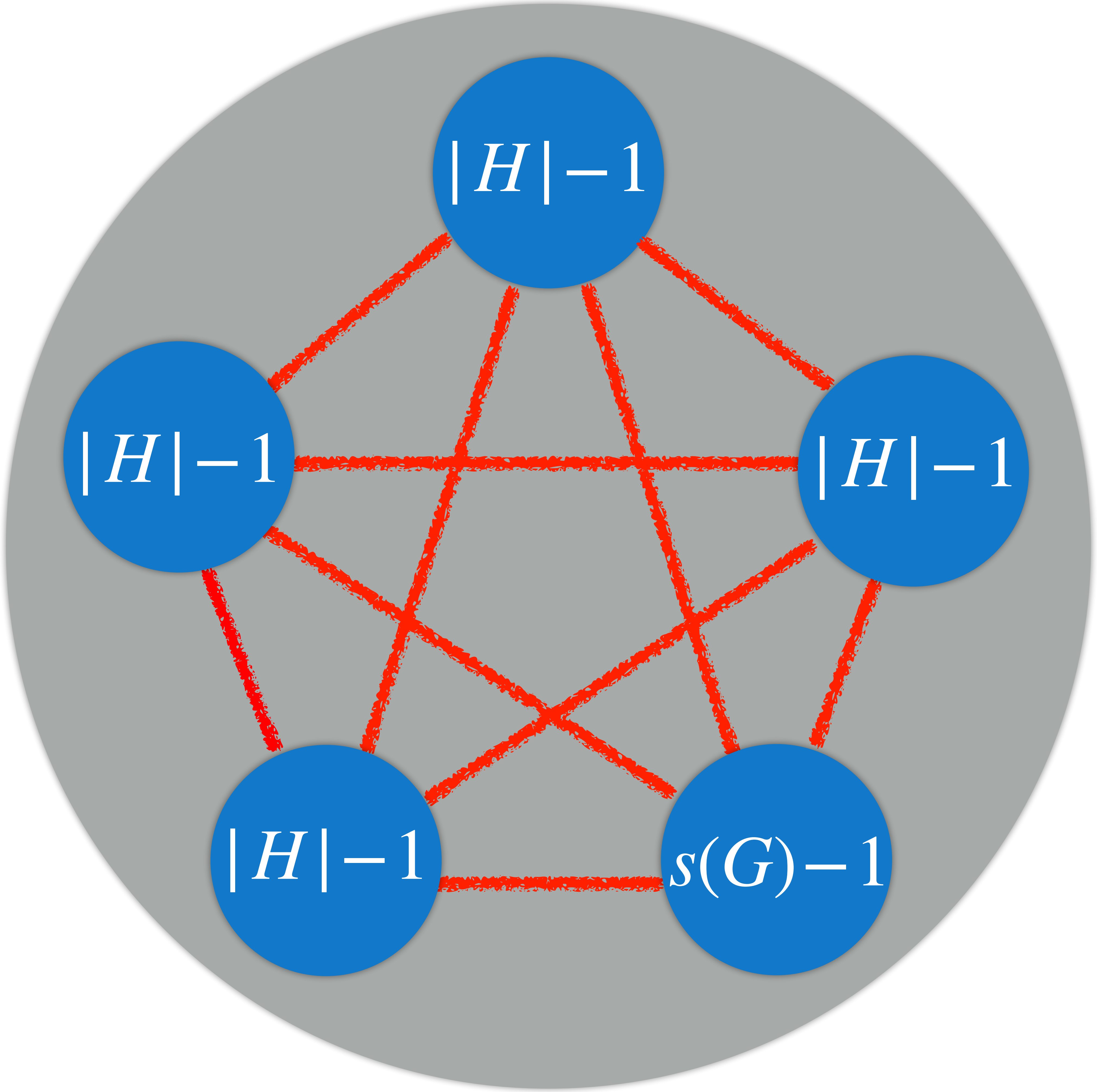
That is to say that H contains all trees T_m.

Let G be a graph with chromatic number k and
$s(G)$ the chromatic surplus of G,
which is defined as the minimum number of vertices
in some color class under all vertex colorings of G with k colors.

Maybe motivated by Theorem 40, Burr proved in the following lower bound for two given graphs:

Theorem 41(Burr). If H is a connected graph, and $|H|$ is at least $s(G)$, then

$$R(G,H) \geq \left(\chi(G)-1\right)\left(|H|-1\right)+s(G).$$



$$K_{(\chi(G)-1)(|H|-1)+s(G)-1}$$

Blue subgraph contains no H.

Let G_r be the red subgraph.
Then

$$\chi(G_r) \le \chi(G).$$

If $\chi(G_r) = \chi(G)$, then

$$s(G_r) \le s(G) - 1.$$

Red subgraph G_r has no G.

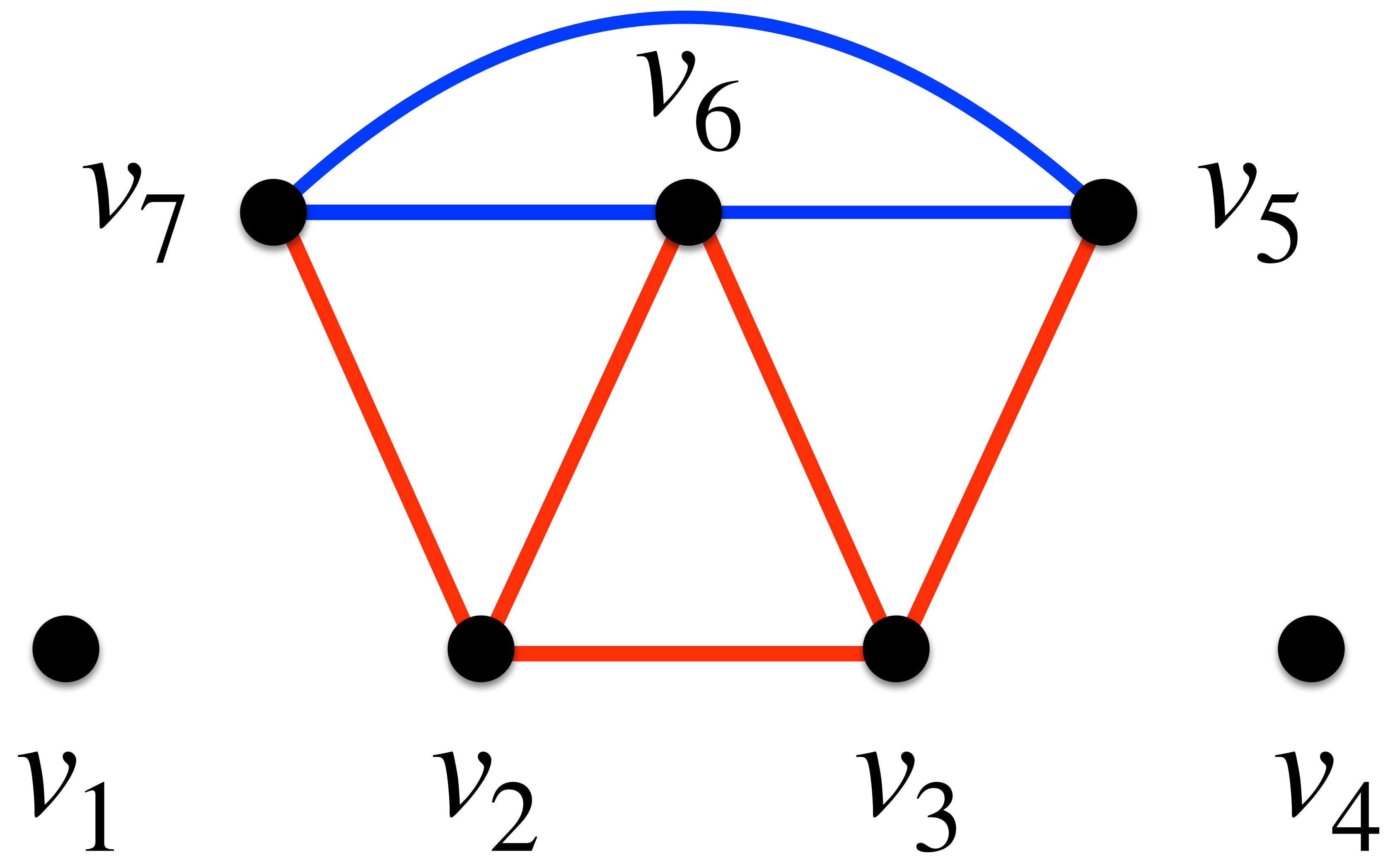
Example. $R(K_3, C_4) = 7$.

By Theorem 41, we have

$$R(K_3, C_4) \geq (\chi(K_3) - 1)(4 - 1) + s(K_3) = 7.$$

To show $R(K_3, C_4) \leq 7$, let K_7 be colored with red and blue.

Assume that K_7 has no red K_3. Since $r(3,3) = 6$, K_7 has a blue K_3.



Exercise 11.

1. Show that $R(K_3, C_4) = 7$.

2. Show that Schur number $S_3 = 14$.