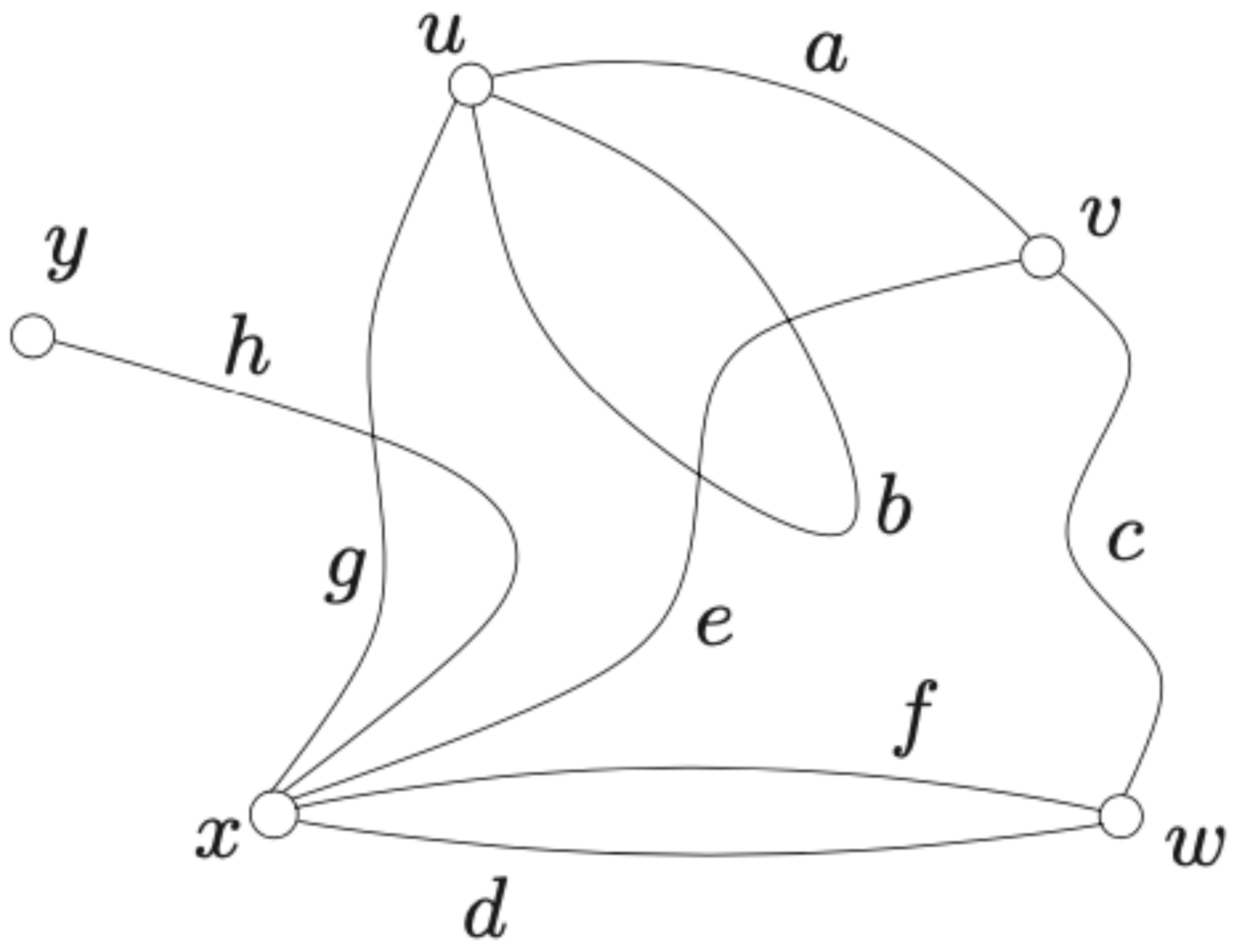
Algebraic Methods in Graph Theory

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$.
Recall its adjacent matrix $A = A(G) = \left(a_{ij}\right)_{n\times n}$, where a_{ij} is
the number of edges joining vertices v_i and v_j, each loop counted as 2 edges.



	u	v	w	x	y
u	2	1	0	1	0
v	1	0	1	1	0
w	0	1	0	2	0
x	1	1	2	0	1
y	0	0	0	1	0

A *walk* in a graph G is a sequence $W = v_0 e_1 v_1 \cdots v_{\ell-1} e_\ell v_\ell$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that v_{i-1} and v_i are the ends of e_i, $1 \leq i \leq \ell$. The length of W is ℓ.

Theorem 46. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix $A = \left(a_{ij}\right)_{n\times n}$. Then the (i, j)-entry of A^k is the number of $\left(v_i, v_j\right)$-walks of length k in G.

Proof. Clearly, the result holds for $k = 1$.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \quad & v_j & \quad & v_n \end{array} \\ \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \end{array}$$

Assume that the result holds for $k-1$. Then the (i,j)-entry of

$$A^k = \underbrace{\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & & v_n \end{array} \\ \begin{array}{c} v_1 \\ \\ v_i \\ \\ v_n \end{array} & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \end{array}}_{A^{k-1}} \begin{array}{cc} \begin{array}{ccccc} v_1 & & v_j & & v_n \end{array} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} & \begin{array}{c} v_1 \\ v_2 \\ \\ v_n \end{array} \end{array}$$

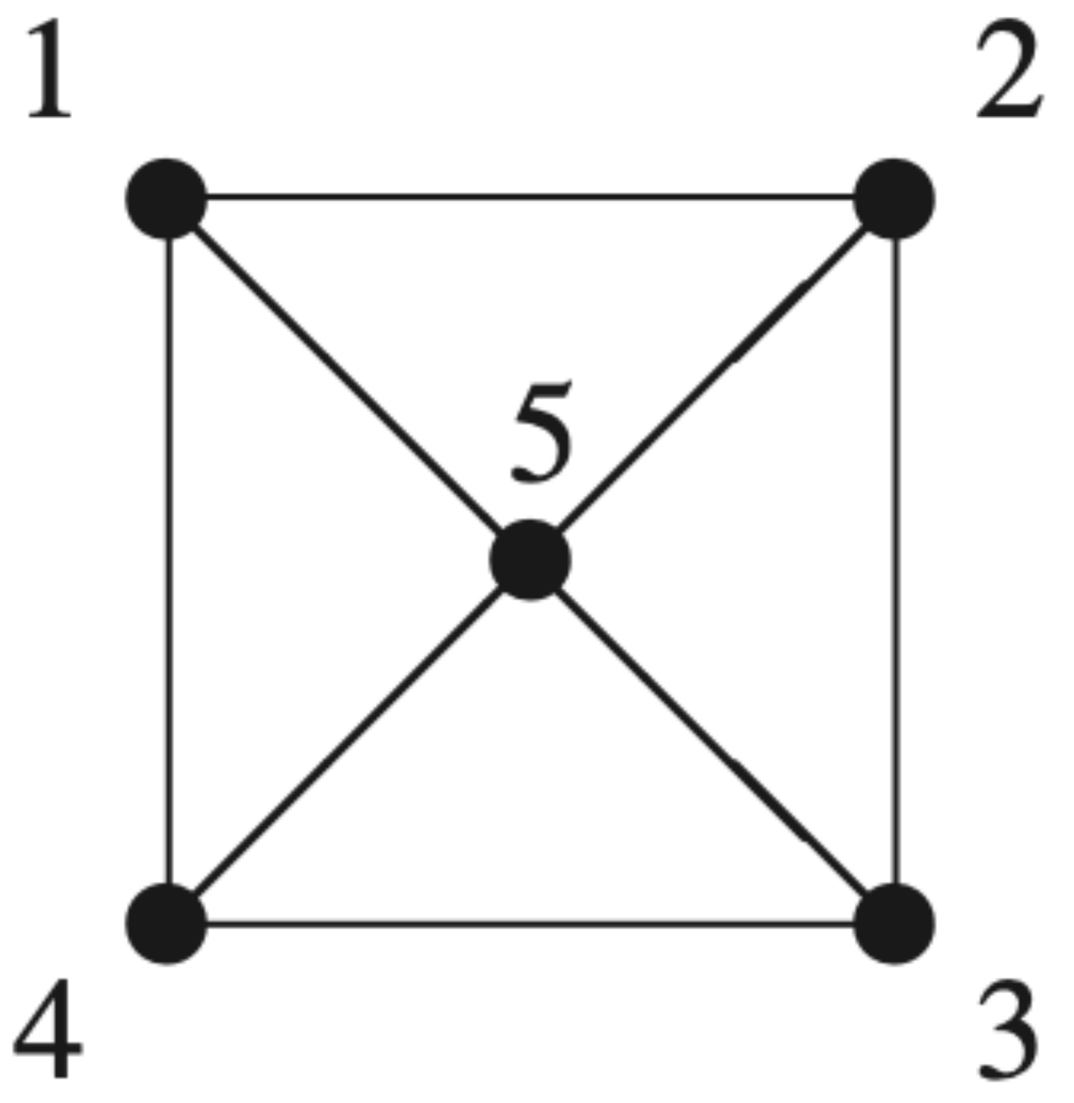
is

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

and so the result follows.

If G is a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, then its adjacent matrix is $A = A(G) = \left(a_{ij}\right)_{n\times n}$, where

$$a_{ij} = 1 \text{ if } v_i v_j \in E(G), \text{ and } a_{ij} = 0 \text{ otherwise.}$$



$$A = \begin{array}{c c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

Thus $A = A(G) = \left(a_{ij}\right)_{n\times n}$ is a symmetric matrix with zero diagonal.
The eigenvalues of A are the n roots of the characteristic polynomial

$$\left|\lambda I - A\right| .$$

They are independent of the labelling of the vertices of G,
because similar matrices have the same characteristic polynomial:
if the labels are permuted we obtain a (0,1)-adjacency matrix $A' = P^{-1}AP$,
where P is a permutation matrix.
Accordingly we speak of the *characteristic polynomial of* G, denoted by
$P_G(x)$, and the *spectrum of* G, which consists of the n *eigenvalues of* G:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n .$$

The eigenvalues of A are the real numbers λ satisfying

$$Ax = \lambda x$$

for some non-zero vector $x \in R^n$. Each such vector x is called an *eigenvector* of the matrix A corresponding to the eigenvalue λ.

The relation $Ax = \lambda x$ can be interpreted in the following way: if $x = \left(x_1, x_2, \ldots, x_n\right)^T$, then

$$A \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix},$$

$$\lambda x_i = \sum_{v_i v_j \in E(G)} x_j \ \left(i = 1,2,\ldots,n\right).$$

Proposition 12. If the graph G has maximum degree $\Delta(G) = \Delta$, then

$$|\lambda| \leq \Delta$$

for every eigenvalue λ of G.

Proof. Let λ be any eigenvalue of $A(G) = A$, and $x = \left(x_1, x_2, \ldots, x_n\right)^T$ be an *eigenvector* of A corresponding to the eigenvalue λ.

Suppose that $|x_i| = \max\left\{|x_j| : 1 \leq j \leq n\right\}$. It is clear that $x_i \neq 0$.

By the arguments before, we have

$$|\lambda||x_i| = \left|\sum_{v_iv_j \in E(G)} x_j\right| \leq \sum_{v_iv_j \in E(G)} |x_j| \leq \Delta |x_i|.$$

Example. For the graph W_4, as shown before, we have

$$\left|\lambda E - A\right| = \begin{vmatrix} \lambda & -1 & 0 & -1 & -1 \\ -1 & \lambda & -1 & 0 & -1 \\ 0 & -1 & \lambda & -1 & -1 \\ -1 & 0 & -1 & \lambda & -1 \\ -1 & -1 & -1 & -1 & \lambda \end{vmatrix} = x^2(x+2)(x^2-2x-4)\,.$$

$\lambda_1 = 1+\sqrt{5},$ $\quad x_1 = \left(-1, -1, -1, -1, -1+\sqrt{5}\right)^T;$

$\lambda_2 = \lambda_3 = 0,$ $\quad x_2 = \left(0,1,0,-1,0\right)^T,\quad x_3 = \left(1,0,-1,0,0\right)^T;$

$\lambda_4 = 1-\sqrt{5},$ $\quad x_4 = \left(-1, -1, -1, -1, -1-\sqrt{5}\right)^T;$

$\lambda_4 = -2,$ $\quad x_5 = \left(1,-1,1,-1,0\right)^T;$

Proposition 13. A graph G is regular (of degree k) if and only if all-1 vector is an eigenvector of G (with corresponding eigenvalue k).

Proof. If G is k-regular, then each row of $A = A(G)$ has exactly k 1's, so

$$A\begin{pmatrix}1\\1\\\cdots\\1\end{pmatrix} = k\begin{pmatrix}1\\1\\\cdots\\1\end{pmatrix}.$$

On the other hand, if

$$A\begin{pmatrix}1\\1\\\cdots\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\\cdots\\1\end{pmatrix},$$

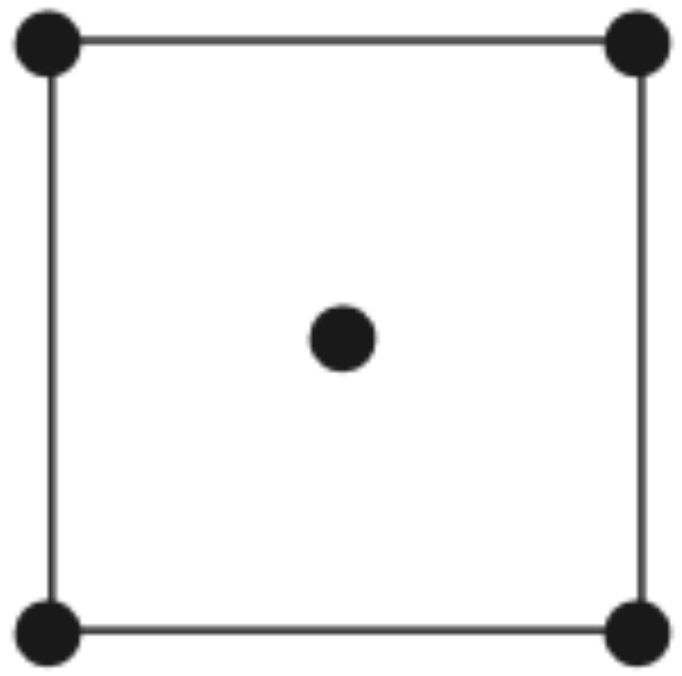
then each row of $A = A(G)$ has exactly λ 1's, so G is regular of degree λ.

Example. The spectrum of Petersen graph is

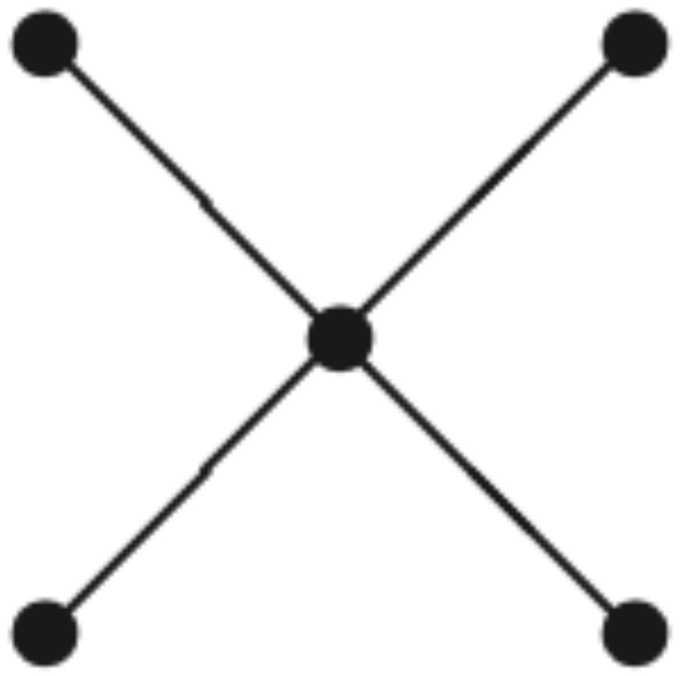
$$3,1,1,1,1,1,-2,-2,-2,-2.$$

We say that two graphs are *cospectral* if they have the same spectrum.

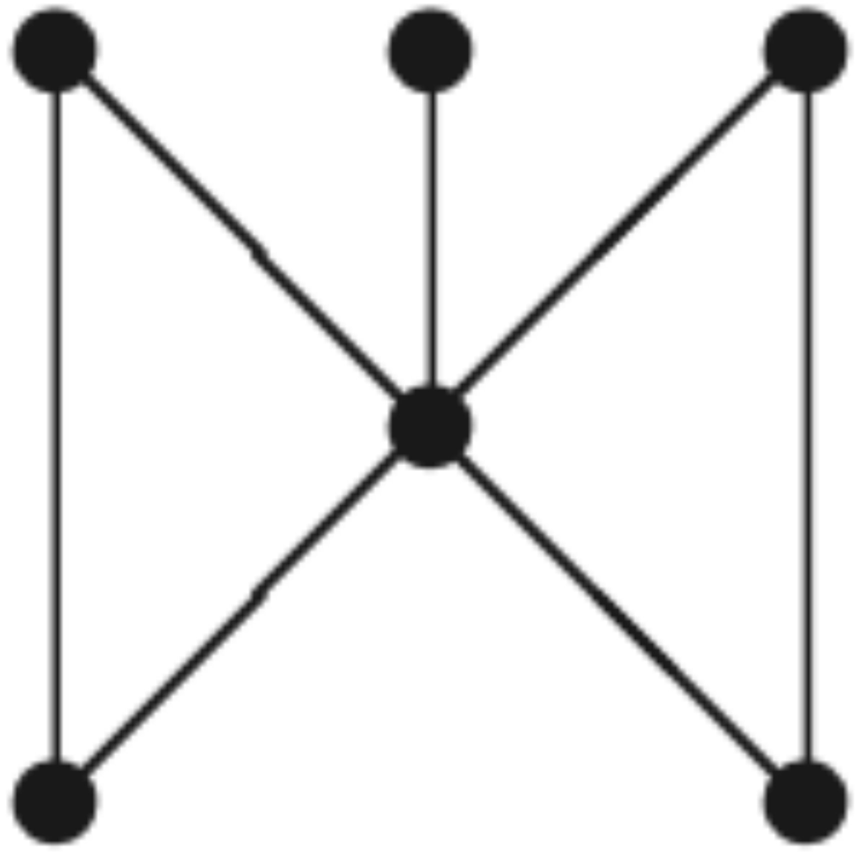
Clearly, isomorphic graphs are cospectral (in other words, the spectrum is a graph invariant). However, cospectral graphs are not necessarily isomorphic:

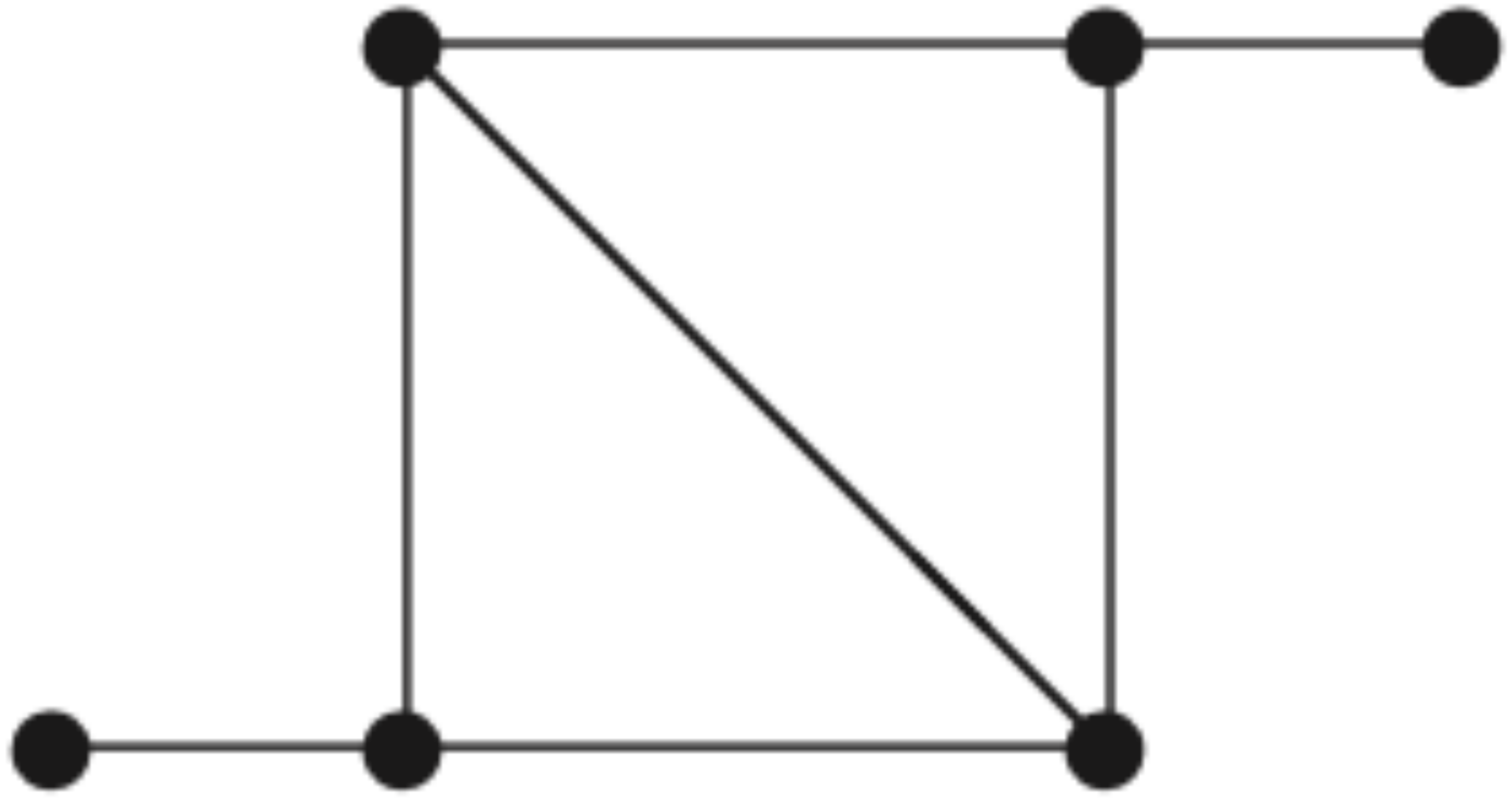


$$2,0,0,0,-2$$



Example. non-isomorphic cospectral *connected* graphs with fewest vertices.





$$(x-1)(x+1)^2(x^3-x^2-5x+1)$$

Theorem 47. Let $\mathscr{F} = \{F_1, \ldots, F_k\}$ be a decomposition of K_n into complete bipartite graphs. Then $k \geq n - 1$.

Proof. Let $V = V(K_n)$ and F_i have bipartition (X_i, Y_i), $1 \leq i \leq k$.

Associated each $v \in V$ a variable x_v. Consider the following linear equations:

$$\sum_{v \in V} x_v = 0, \qquad \sum_{v \in X_i} x_v = 0, \quad 1 \leq i \leq k.$$

If $k < n - 1$, then this linear equations has a solution $x_v = c_v$, $v \in V$, with $c_v \neq 0$ for at least one $v \in V$. Thus

$$\sum_{v \in V} c_v = 0, \qquad \sum_{v \in X_i} c_v = 0, \quad 1 \leq i \leq k.$$

Because $\mathscr{F}$ is a decomposition of K_n,

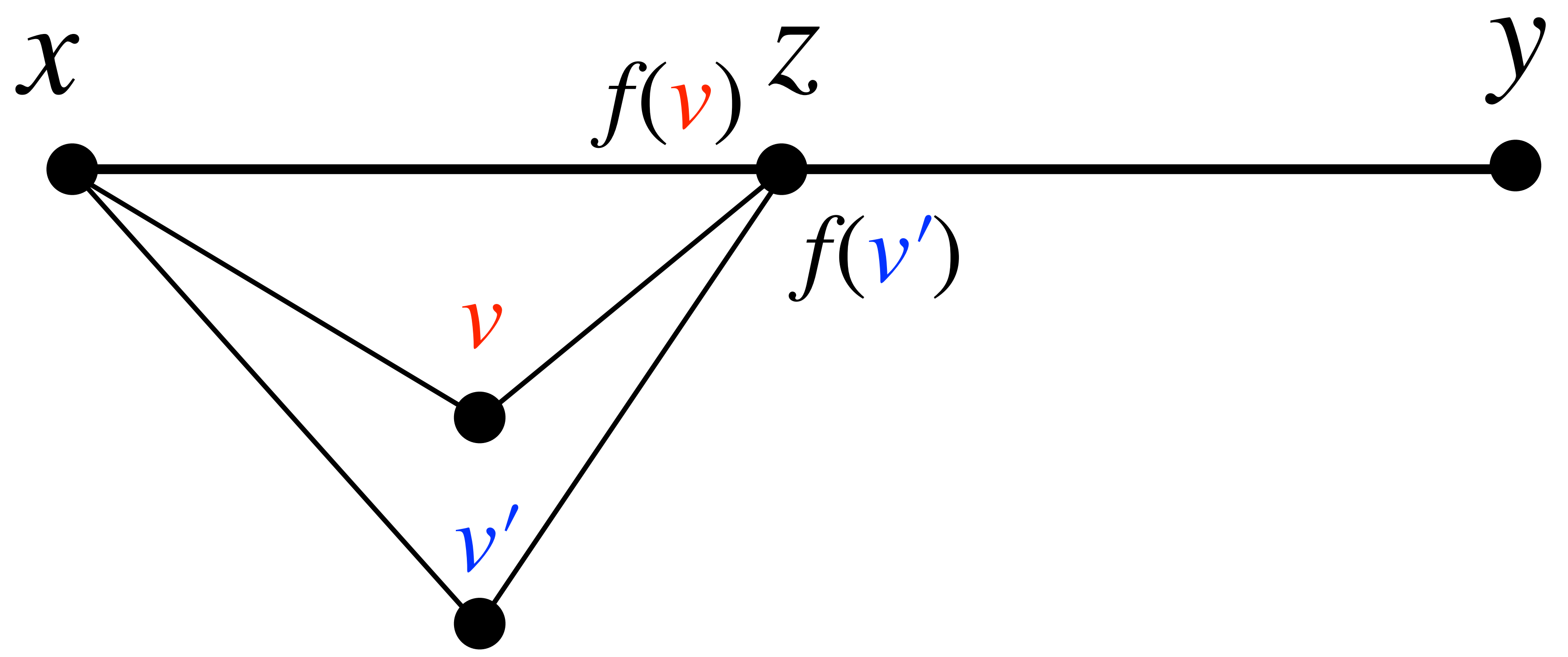
$$\sum_{uv\in E} c_u c_v = \sum_{i=1}^{k} \left(\sum_{u\in X_i} c_u \right) \left(\sum_{v\in Y_i} c_v \right).$$

Therefore,

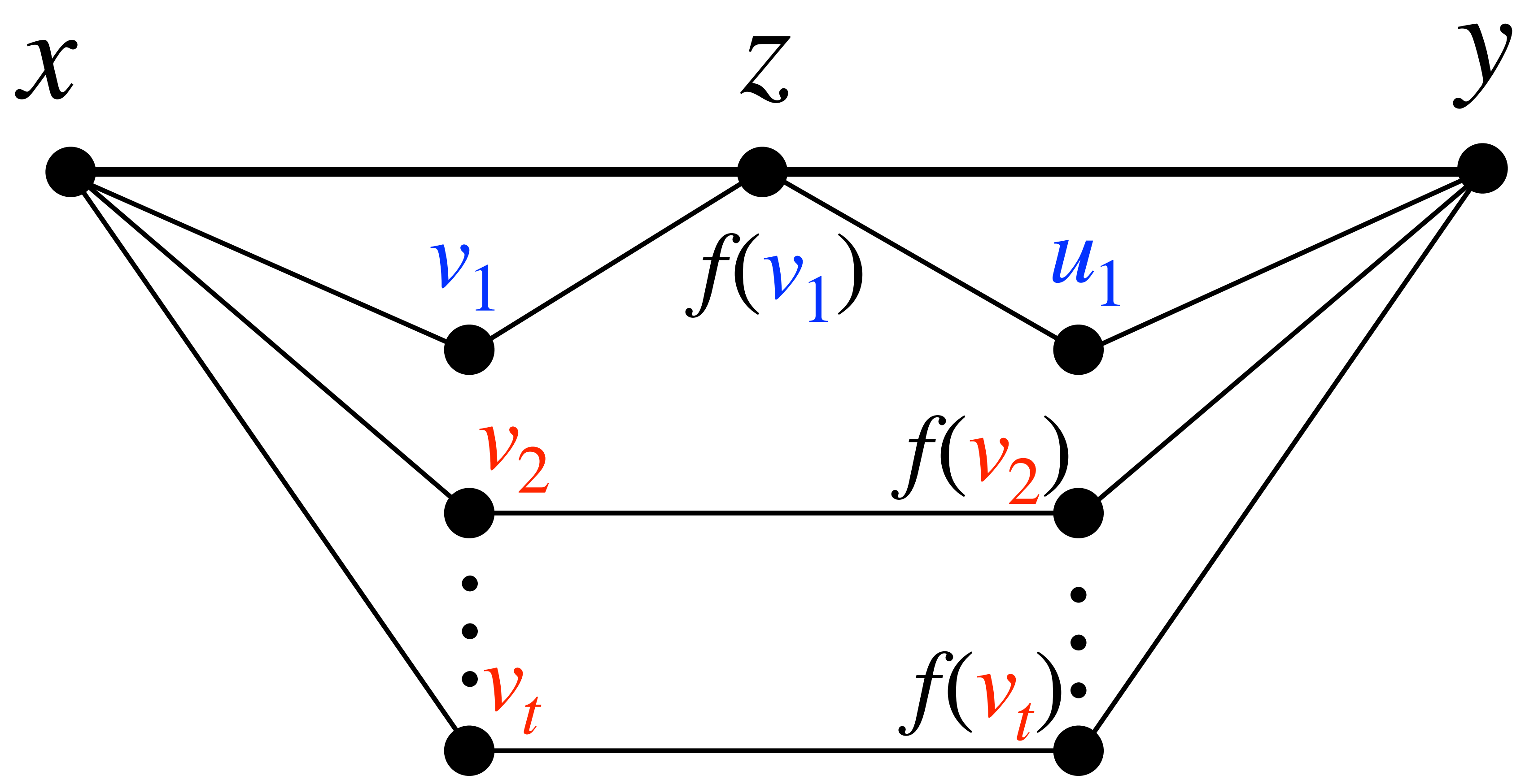
$$0 = \left(\sum_{u\in V} c_u \right)^2 = \sum_{u\in V} c_u^2 + 2 \sum_{i=1}^{k} \left(\sum_{u\in X_i} c_u \right) \left(\sum_{v\in Y_i} c_v \right) = \sum_{u\in V} c_u^2 > 0.$$

Theorem 48 (*The Friendship Theorem*). Let G be a simple graph of order n in which any two vertices have exactly one common neighbor. Then G has a vertex of degree $n-1$.

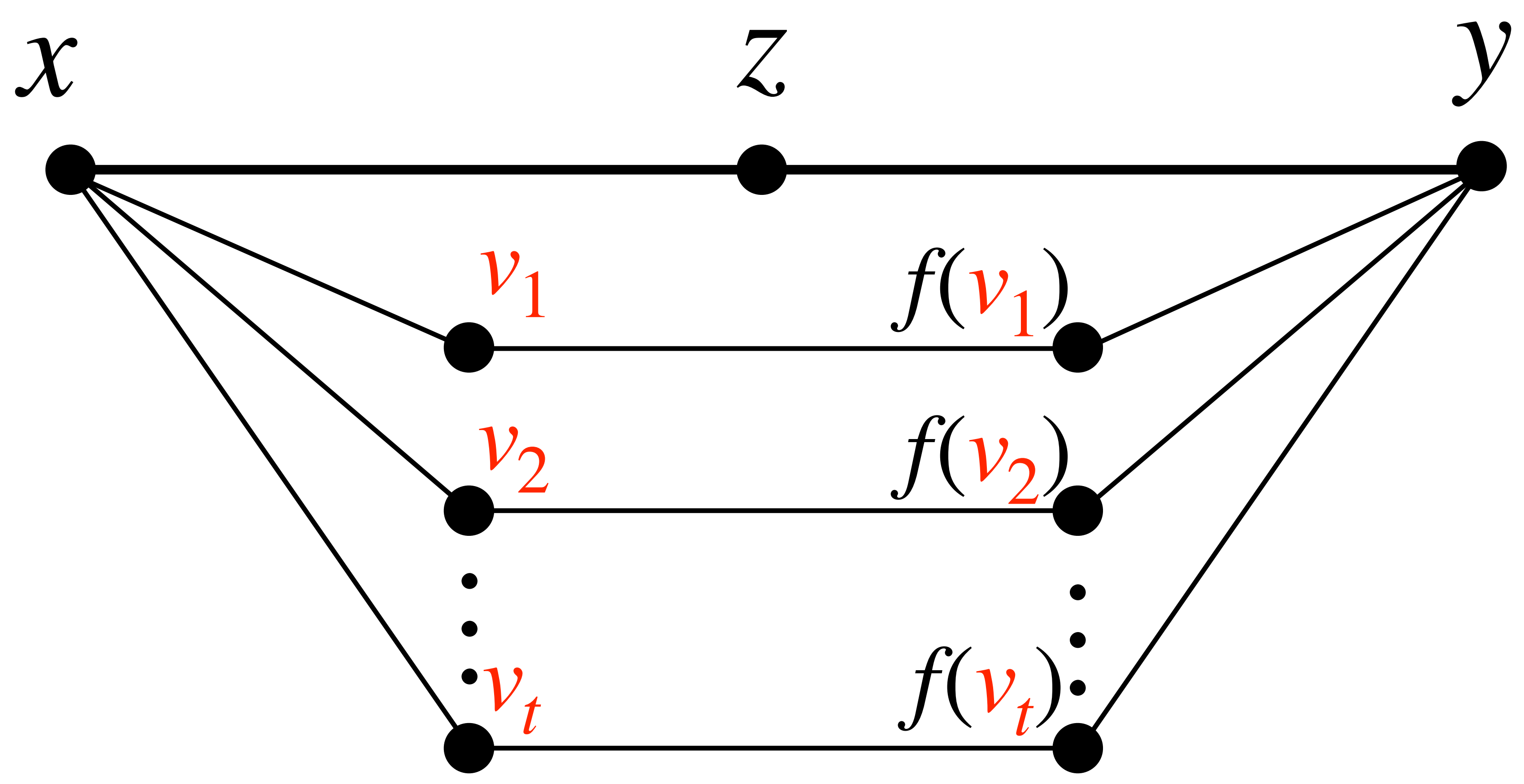
Proof. Assume to the contrary that $\Delta(G) < n-1$.
Consider two nonadjacent vertices x and y.
Assume $d(x) \geq d(y)$ and z the unique common neighbor of x and y.
For any $v \in N_G(x)\backslash\{z\}$, let $f(v)$ be the unique common neighbor of v and y.
There are at most one $v \in N_G(x)\backslash\{z\}$ such that $f(v) = z$, for otherwise,



There is one $v \in N_G(x)\backslash\{z\}$, say v_1, such that $f(v_1) = z$.



There is no $v \in N_G(x)\backslash\{z\}$ such that $f(v) = z$.



By the arguments above, $d(x) = d(y)$ for any nonadjacent vertices x and y.
We now show that G is regular.
It suffices to show that $\overline{G}$ is connected. Because $\delta(\overline{G}) = n - 1 - \Delta(G) > 0$, each component of $\overline{G}$ has at least 2 vertices.
If $\overline{G}$ is disconnected, then G has a quadrilateral C_4, a contradiction.
So G is k-regular for some integer k.

Count the number of paths of length 2 in G in two ways, we have

$$n\binom{k}{2} = \binom{n}{2},$$

which implies

$$n = k^2 - k + 1.$$

Let A be the adjacency matrix of G. Then

$$A^2 = \begin{pmatrix} k & 1 & \cdots & 1 \\ 1 & k & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & k \end{pmatrix} = J + (k-1)I\,.$$

The eigenvalues of J are: $0 \quad (n-1)$, and $n \quad (1)$,

A^2 are: $k-1 \quad (n-1)$, and $k^2 \quad (1)$,

A are: $\pm\sqrt{k-1} \quad (n-1)$, and $k \quad (1)$.

Because $tr(A) = 0$, we have $t\sqrt{k-1} = k$ for some integer t, which implies

$$k = 2, \quad n = 3.$$

Exercise 13.

1. Show that the eigenvalues of a cycle C_n are

$$2\cos\frac{2\pi i}{n},\ i = 0,1,\ldots,n-1.$$

Moore Graphs

A *Moore graph* is a graph with diameter d and girth $2d + 1$, for some $d > 1$. The 5-cycle and the Petersen graph are two known examples with $d = 2$.

Lemma 1. A Moore graph is regular.

Proof. Let G be a Moore graph with diameter d.

We show first that $d(u) = d(v)$ for any two vertices u, v of G at distance d.

Let $P(u, v)$ be the unique path of length d from u to v, and let w be any neighbor of v not on $P(u, v)$. Then $d(u, w) = d$ and the path $P(u, w)$ includes a neighbor w' of u not on $P(u, v)$. Different w determine different w', and so $d(v) \leq d(u)$. Similarly, $d(u) \leq d(v)$.

Next, let C be a cycle of length $2d + 1$ in G.

If x, y are adjacent vertices of C, then there exists a vertex z of C such that $d(x, z) = d(y, z) = d$, and so $d(x) = d(y)$. It follows that all vertices of C have the same degree.

Finally, consider a vertex u not on C, and a shortest path of length ℓ say, from u to C. We may add $d - \ell$ consecutive edges of C to this path to reach a vertex u' of C at distance d from u. Then $d(u) = d(u')$, and it follows that all vertices of G have the same degree.

Theorem 45. If G is a k-regular Moore graph of order n diameter 2, then we have $k \in \{2,3,7,57\}$.

Definition 1. A *strongly regular* graph with parameters (n, k, p, q) is a k-regular on n vertices in which any two adjacent vertices have exactly p common neighbors and any two nonadjacent vertices have exactly q common neighbors.

Proposition 14. A k-regular Moore graph G of order n diameter 2 is strongly regular with parameters $(n, k, 0, 1)$.

Proof of Theorem 45. For any two nonadjacent vertices u, v, there exists a unique walk of length 2 between u and v. It follows that the adjacency matrix A of G satisfies

$$A^2 + A - (k-1)I = J.$$

Since J is a polynomial of A, A and J have a common set of eigenvectors. One of these eigenvectors is $\xi = (1,1,...,1)^T$. Thus, we have

$$J\xi = n\xi, \qquad A\xi = k\xi\,.$$

Let η be any other eigenvector corresponding to eigenvalue λ, then

$$J\eta = 0, \qquad A\eta = \lambda\eta,$$

which implies that

$$\lambda^2 + \lambda - (k-1) = 0.$$

Hence A has other two distinct eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(-1 + \sqrt{4k-3}\right),$$

$$\lambda_2 = \frac{1}{2}\left(-1 - \sqrt{4k-3}\right).$$

If k is such that λ_1 and λ_2 are not rational, then each has multiplicity $(n-1)/2$, because A is rational. Thus,

$$tr(A) = k + \frac{n-1}{2}\left(\lambda_1 + \lambda_2\right) = k - \frac{k^2}{2} = 0$$

which means that $k = 0$ and $n = 1$, or $k = 2$ and $n = 5$, that is, $G = C_5$.

If k is such that λ_1 and λ_2 are rational, then since any rational eigenvalues of A are also integral, $\sqrt{4k-3}$ is a square integer, say $\sqrt{4k-3} = t$. Assume the multiplicity of λ_1 is ℓ. Then

$$tr(A) = k + \ell \cdot \frac{t-1}{2} + (n - \ell - 1) \cdot \frac{-t-1}{2} = 0.$$

Noting that $n = k^2 + 1$ and $k = (t^2 + 3)/4$, we have

$$t^5 + t^4 + 6t^3 - 2t^2 + \left(9 - 32\ell\right)t - 15 = 0.$$

Since the equality above requires solutions in integers, the only candidates for t are the factors of 15. The solutions are

$$\begin{array}{llll}
t = 1, & \ell = 0, & k = 1, & n = 2; \\
t = 3, & \ell = 5, & k = 3, & n = 10; \\
t = 5, & \ell = 28, & k = 7, & n = 50; \\
t = 15, & \ell = 1729, & k = 57, & n = 3250;
\end{array}$$

Probabilistic Methods in Graph Theory

A (finite) *probability space* (Ω, P) consists of a finite set Ω, called the *sample space*, and a *probability function* P: $\Omega \mapsto [0,1]$ satisfying

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

We may regard the set $\mathcal{G}_n$ of all labelled graphs on n vertices(or, equivalently, the set of all spanning subgraphs of K_n) as the sample space of a finite probability space $\left(\mathcal{G}_n, P\right)$. The result of selecting an element G of this sample space according to the probability function P is called a *random graph*.

The simplest example of such a probability space arises
when all graphs $G \in \mathcal{G}_n$ have the same probability of being chosen. Because $|\mathcal{G}_n| = 2^N$, where $N = \binom{n}{2}$, the probability function in this case is:

$$P(G) = \frac{1}{2^N} \text{ for all } G \in \mathcal{G}_n .$$

A natural way of viewing this probability space is to imagine the edges of K_n as being considered for inclusion one by one, each edge being chosen with probability one half (for example, by flipping a fair coin), these choices being made independently of one another.
The result of such a procedure is a spanning subgraph G of K_n,
with all $G \in \mathcal{G}_n$ being equiprobable.

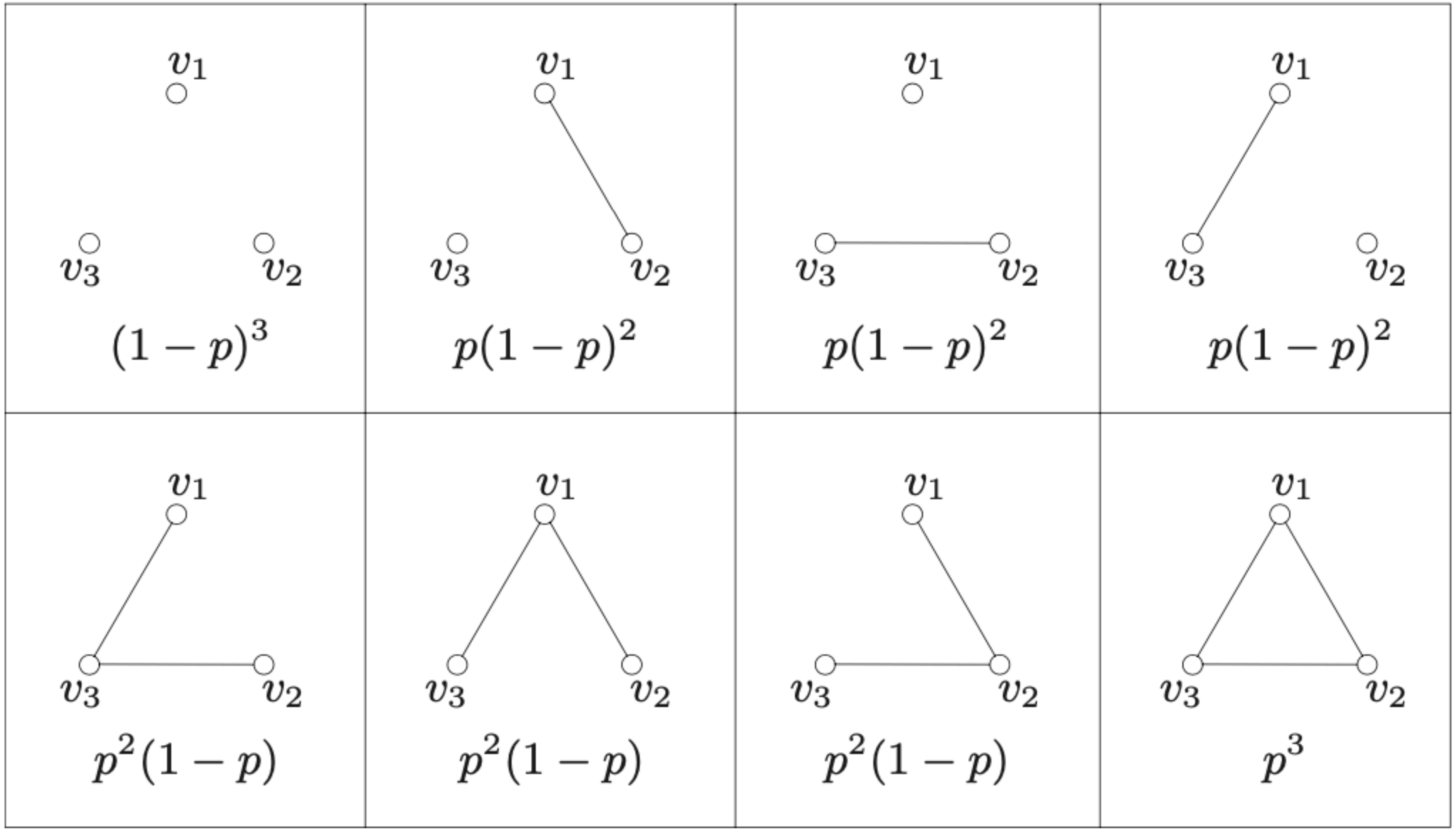
A more refined probability space on the set $\mathcal{G}_n$ may be obtained by fixing a real number p between 0 and 1 and choosing each edge with probability p, these choices again being independent of one another. Because $1 - p$ is the probability that any particular edge is not chosen, the resulting probability function P is given by

$$P(G) = p^m(1-p)^{N-m} \text{ for each } G \in \mathcal{G}_n,$$

where $m = e(G)$. This probability space is denoted by $\mathcal{G}_{n,p}$.

Example.

$\mathcal{G}_{3,p}$ has as sample space the $2^{\binom{3}{2}} = 8$ spanning subgraphs of K_3 shown in the following figure, with the probability function indicated.



The probability space $\mathscr{G}_{3,p}$

Note that the smaller the value of p, the higher the probability of obtaining a sparse graph. We are interested in computing or estimating the probability that a random graph has a particular property.

To each graph property, such as connectedness, there corresponds a subset of $\mathscr{G}_n$, namely those members of $\mathscr{G}_n$ which have the given property.
The probability that a random graph has this particular property is then just the sum of the probabilities of these graphs.

Example. For a random graph G in $\mathscr{G}_{3,p}$,
the probability that it is connected is $3p^2(1-p)+p^3 = p^2(3-2p)$;
the probability that it is bipartite is

$$(1-p)^3 + 3(1-p)^2 p + 3(1-p)p^2 = 1-p^3;$$

and the probability that it is both connected and bipartite is $3p^2(1-p)$.

Recall the lower bound for classical diagonal Ramsey number $R(p,p)$:

Erdős' lower bound (1947):

$$R(p,p) > 2^{\frac{p}{2}} \text{ for } p \geq 3.$$

The number of all red-blue edge colorings of K_n is

$$2^{\binom{n}{2}}.$$

The number of colorings with monochromatic K_p is at most

$$\binom{n}{p} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{p}{2}}.$$

If

$$\binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1} < 2^{\binom{n}{2}},$$

then there exists

a red-blue colorings such that K_n has no monochromatic K_p.

By the definition of $R(p,p)$, we have

$$R(p,p) > n\,.$$

It is not difficult to show that if $n \le 2^{\frac{p}{2}}$, then

$$\binom{n}{p} < \frac{n^p}{2^{p-1}} \le 2^{\frac{p^2}{2}-p+1}$$

$$= 2^{\frac{1}{2}p(p-1)-1} \cdot 2^{-\frac{p}{2}+2} \le 2^{\binom{p}{2}-1}.$$

Thus,

$$\binom{n}{p} \cdot 2^{\binom{n}{2}-\binom{p}{2}+1} < 2^{\binom{n}{2}}.$$

By the argument above, we have

$$R(p,p) > 2^{\frac{p}{2}}.$$

Proof using Probabilistic Method:

Color the edges of a complete graph K_N randomly.

That is, color each edge red with probability 1/2,

and blue with probability 1/2.

Since the probability that a given copy of K_p has all edges red is

$$2^{-\binom{p}{2}},$$

the expected number of red copies of K_p is

$$2^{-\binom{p}{2}}\binom{N}{p}.$$

Similarly, the expected number of blue copies of K_p is

$$2^{-\binom{p}{2}}\binom{N}{p}.$$

Therefore, the expected number of monochromatic copies of K_p is

$$2^{1-\binom{p}{2}}\binom{N}{p}.$$

Since

$$2^{1-\binom{p}{2}}\binom{N}{p} \leq 2^{1-\binom{p}{2}}\left(\frac{eN}{p}\right)^{p},$$

take

$$N = \left(1 - o(1)\right) \frac{p}{\sqrt{2}e} \left(\sqrt{2}\right)^p,$$

we have

$$2^{1-\binom{p}{2}} \binom{N}{p} \leq 2^{1-\binom{p}{2}} \left(\frac{eN}{p}\right)^p < 1.$$

This implies

$$R(p,p) \geq \left(1 - o(1)\right) \frac{p}{\sqrt{2}e} \left(\sqrt{2}\right)^p.$$