第一章 图的基本概念

Definition 1. A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v, and the vertices u and v are called the ends of e.

We denote the numbers of vertices and edges in G by $v(G)$ and $e(G)$.

These two basic parameters are called the *order* and *size* of G, respectively.

Example 1.

$$G = (V(G), E(G)),$$

where $\quad V(G) = \{u, v, w, x, y\}, \quad E(G) = \{a, b, c, d, e, f, g, h\},$

and ψ_G is defined by

$$\psi_G(a) = uv,\ \psi_G(b) = uu,\ \psi_G(c) = vw,\ \psi_G(d) = wx,$$
$$\psi_G(e) = vx,\ \psi_G(f) = wx,\ \psi_G(g) = ux,\ \psi_G(h) = xy\,.$$

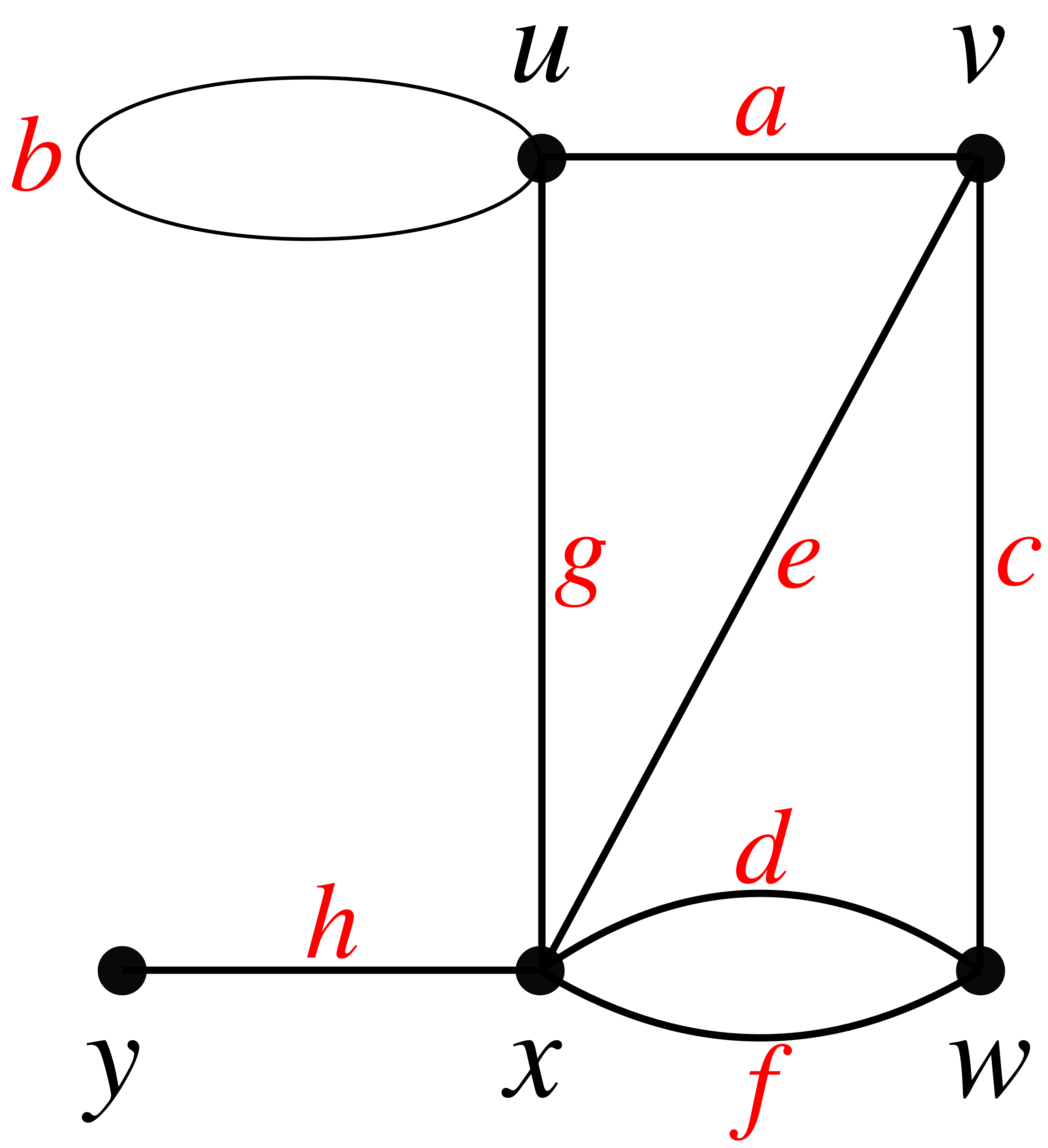
Example 2.

$$H = (V(H), E(H)),$$

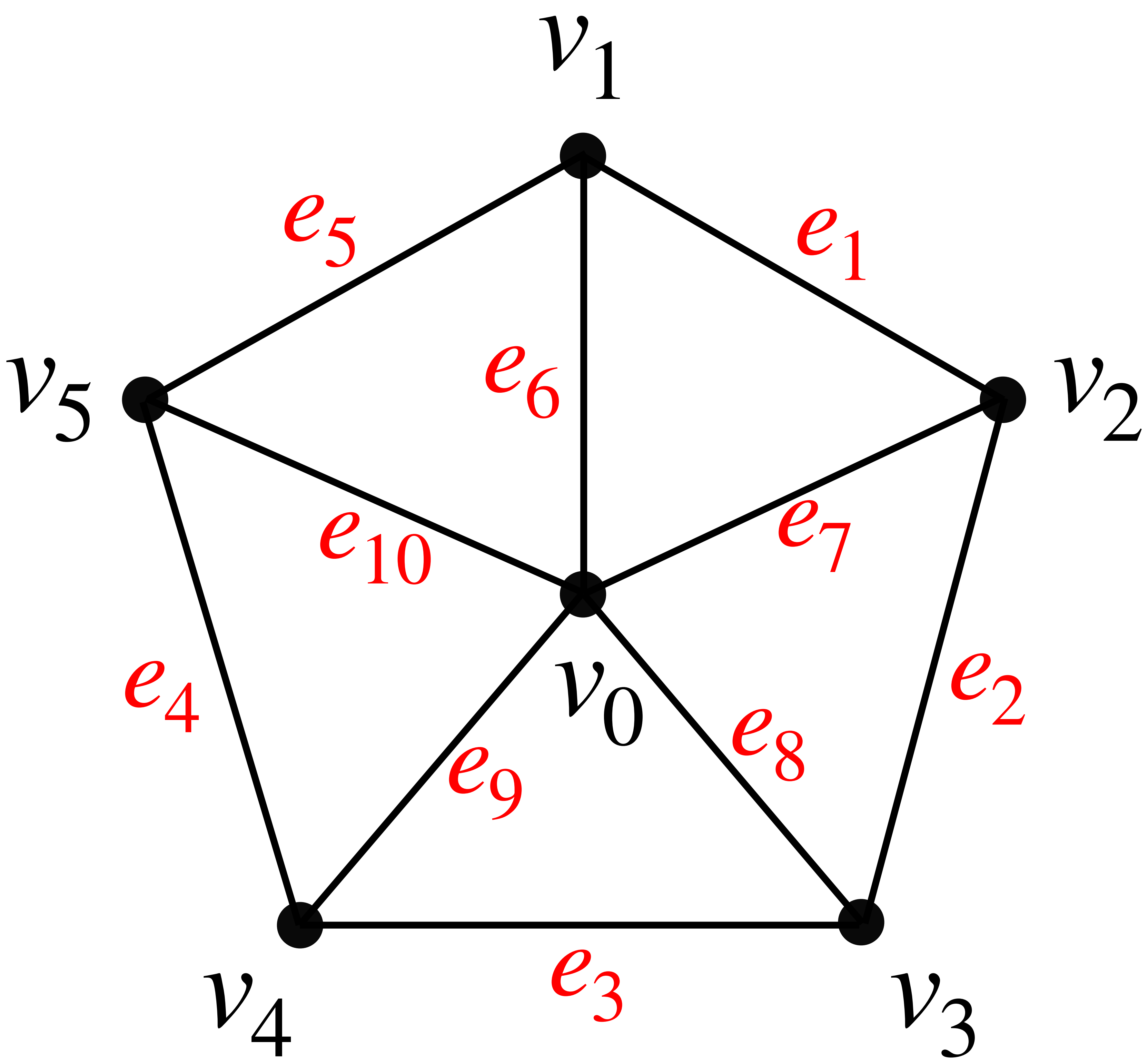
where $V(H) = \{v_i \mid 0 \le i \le 5\}, \ E(H) = \{e_i \mid 1 \le i \le 10\},$

and ψ_H is defined by

$$\psi_H(e_1) = v_1v_2,\ \psi_H(e_2) = v_2v_3,\ \psi_H(e_3) = v_3v_4,\ \psi_H(e_4) = v_4v_5, \psi_H(e_5) = v_5v_1,$$
$$\psi_H(e_6) = v_0v_1,\ \psi_H(e_7) = v_0v_2,\ \psi_H(e_8) = v_0v_3,\ \psi_H(e_9) = v_0v_4, \psi_H(e_{10}) = v_0v_5\,.$$



G



H

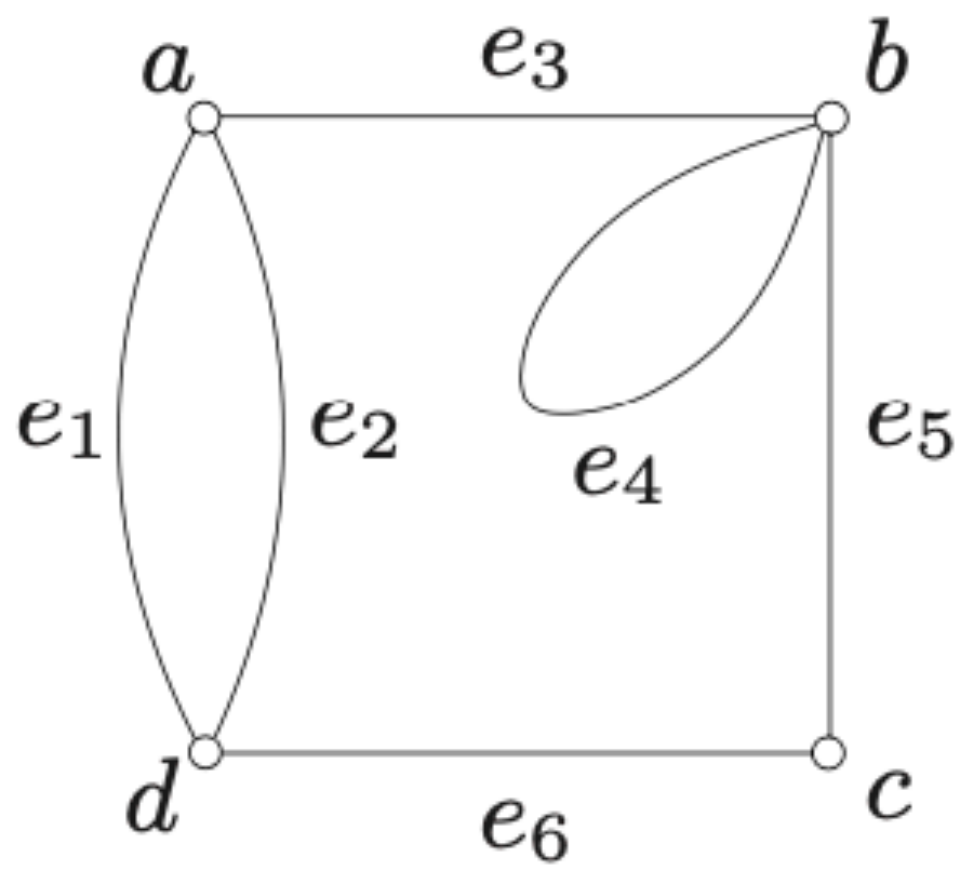
Isomorphism: Given two graphs G and H,

if there are bijections $\theta: V(G) \mapsto V(H)$ and $\phi: E(G) \mapsto E(H)$ such that

$$\psi_G(e) = uv \iff \psi_H(\phi(e)) = \theta(u)\theta(v),$$

then we say G is isomorphic to H, written as $G \cong H$.

Such a pair of mappings is called an *isomorphism* between G and H.



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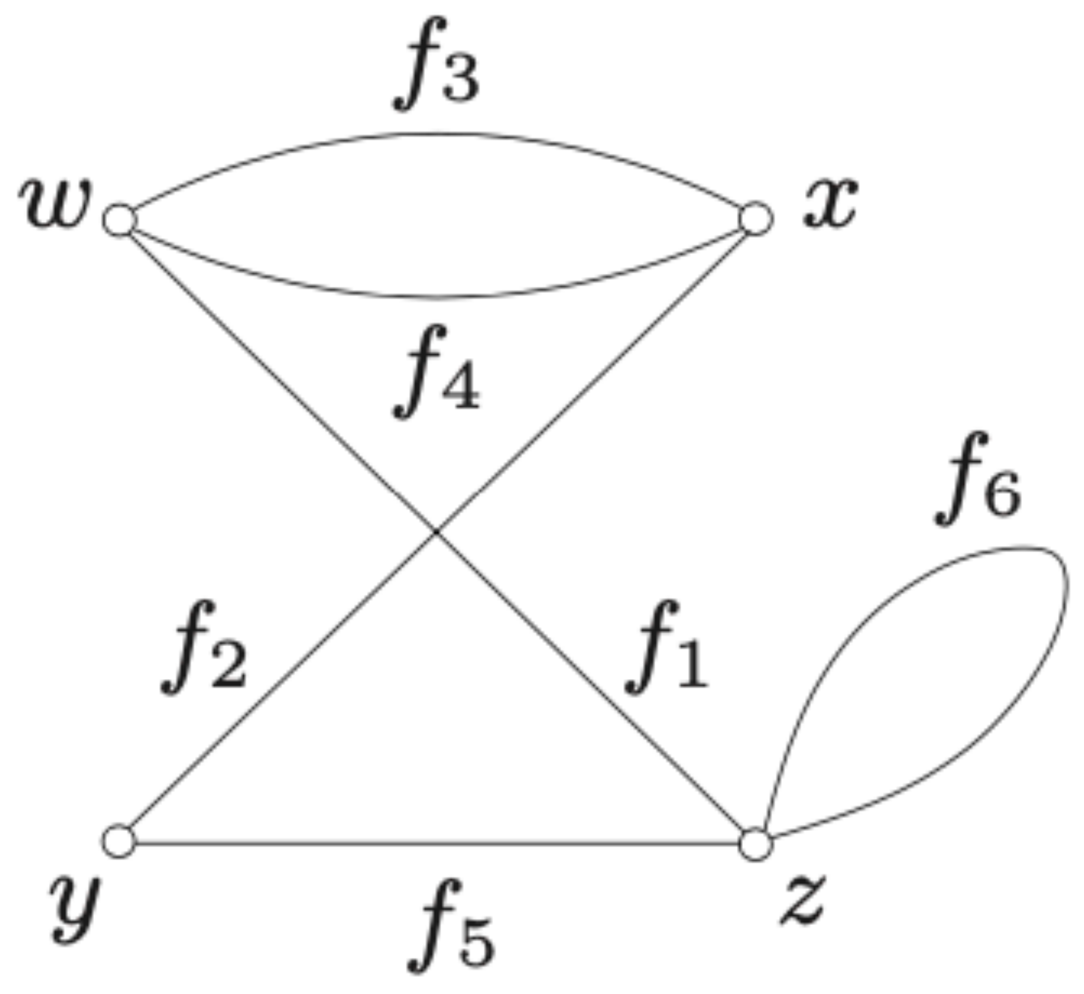

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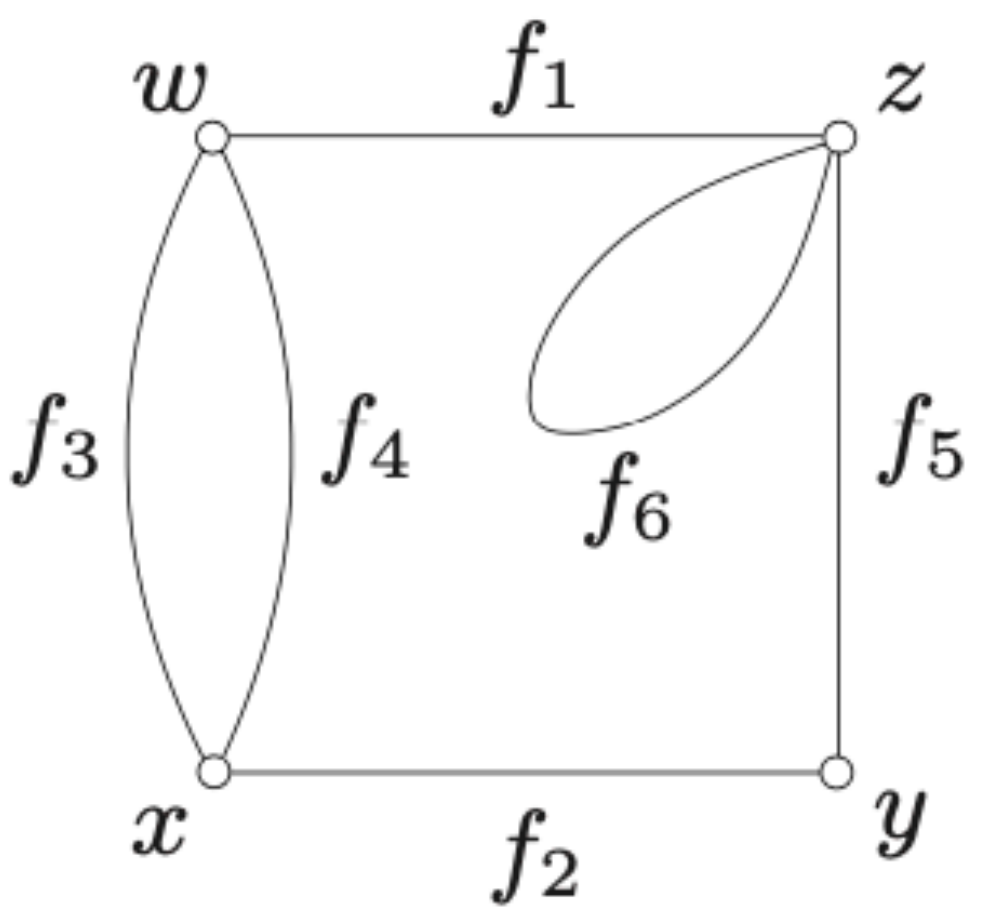
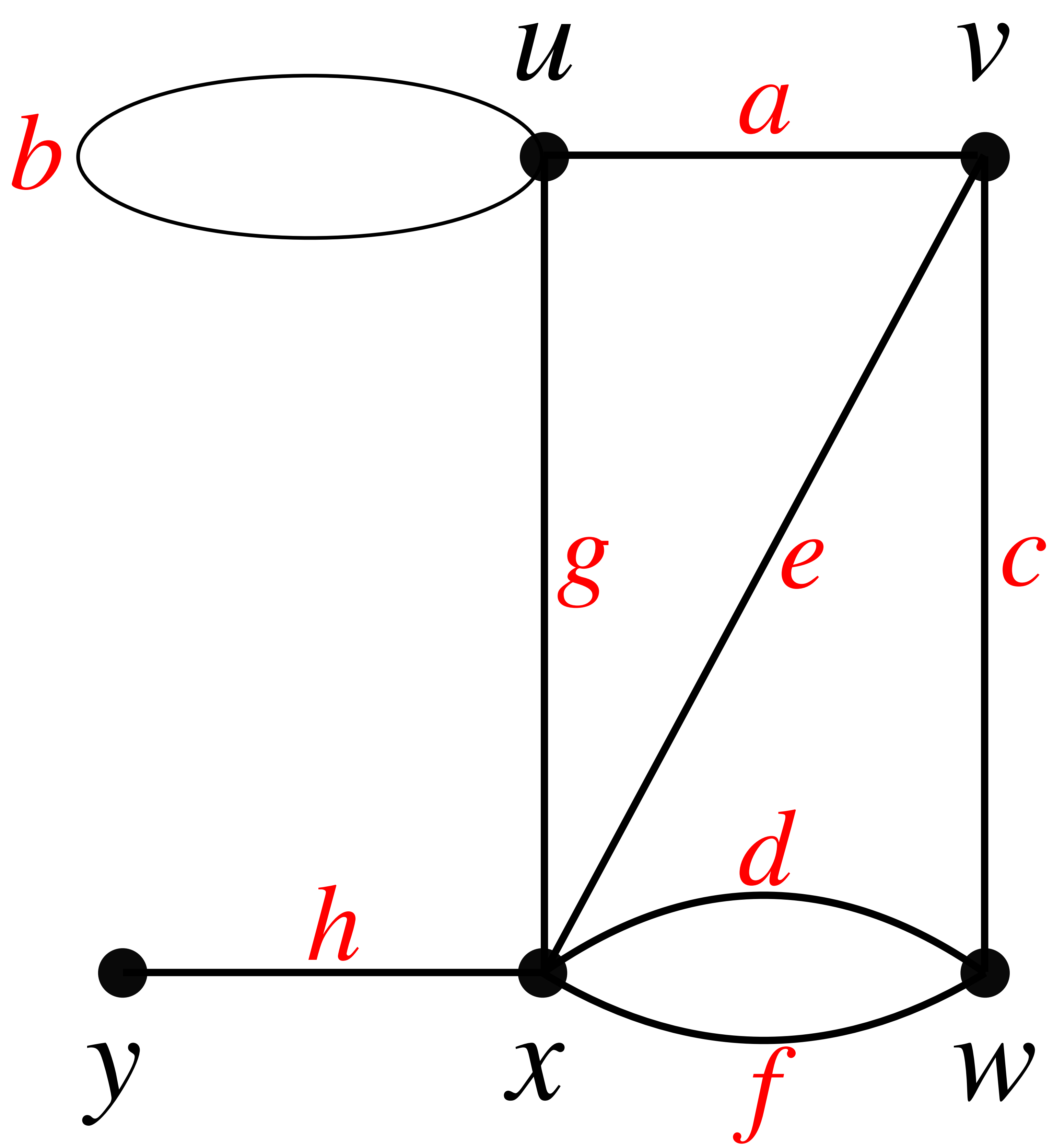


H

Adjacency Matrix and Incidence Matrix of a Graph



	a	b	c	d	e	f	g	h
u	1	2	0	0	0	0	1	0
v	1	0	1	0	1	0	0	0
w	0	0	1	1	0	1	0	0
x	0	0	0	1	1	1	1	1
y	0	0	0	0	0	0	0	1

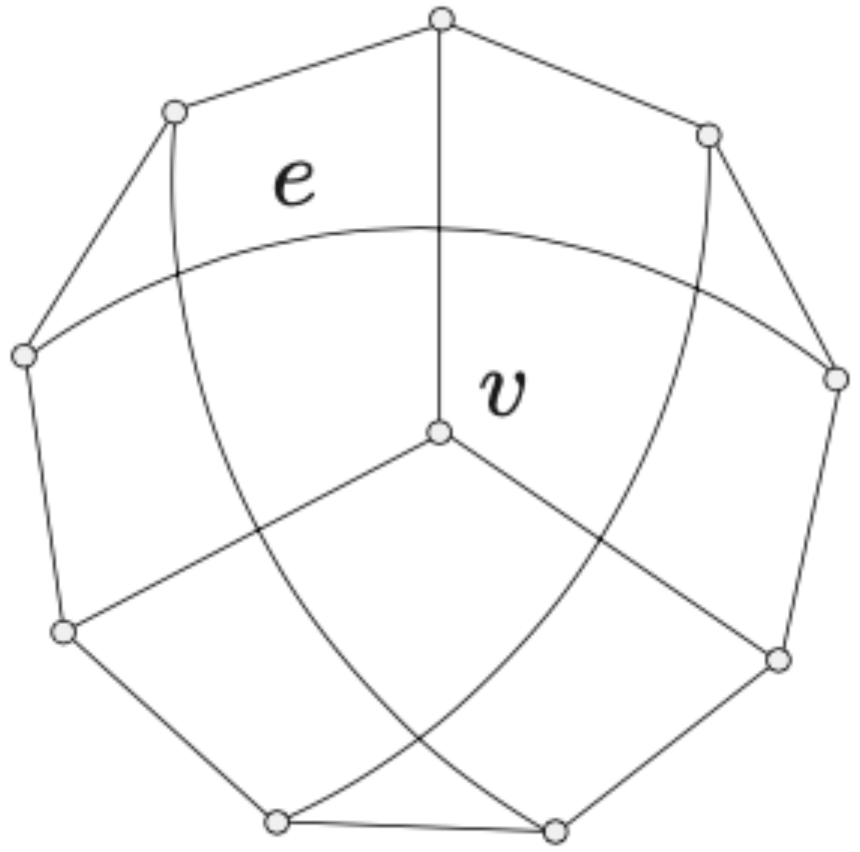
Incidence Matrix

	u	v	w	x	y
u	2	1	0	1	0
v	1	0	1	1	0
w	0	1	0	2	0
x	1	1	2	0	1
y	0	0	0	1	0

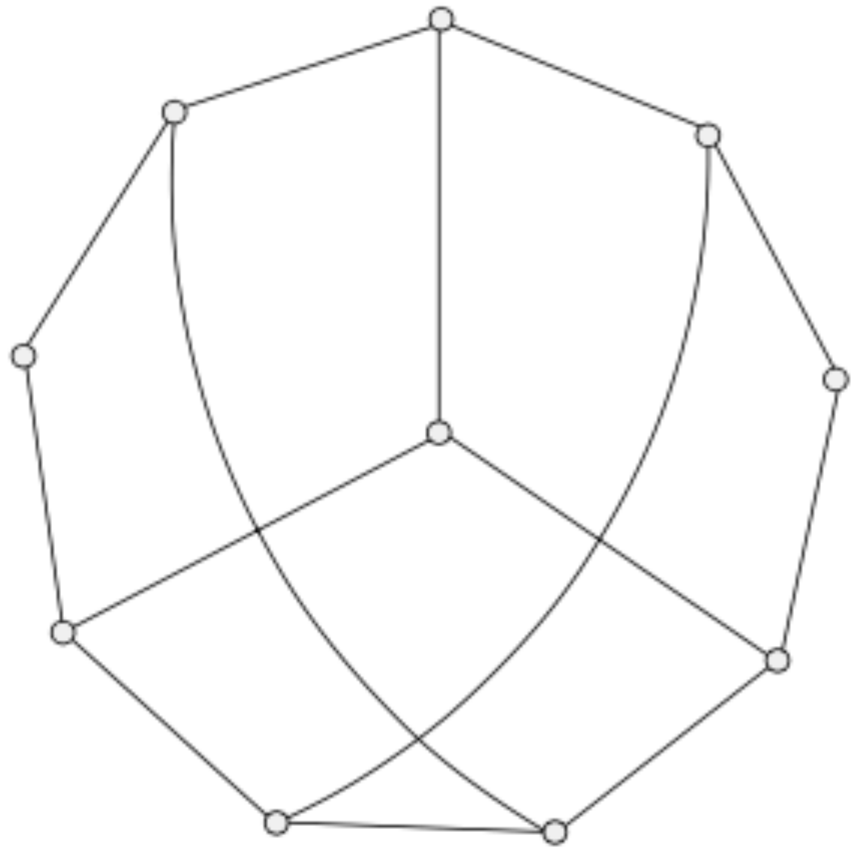
Adjacency Matrix

If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G.
Moreover, if $V(H) = V(G)$, then H is a *spanning subgraph* of G.

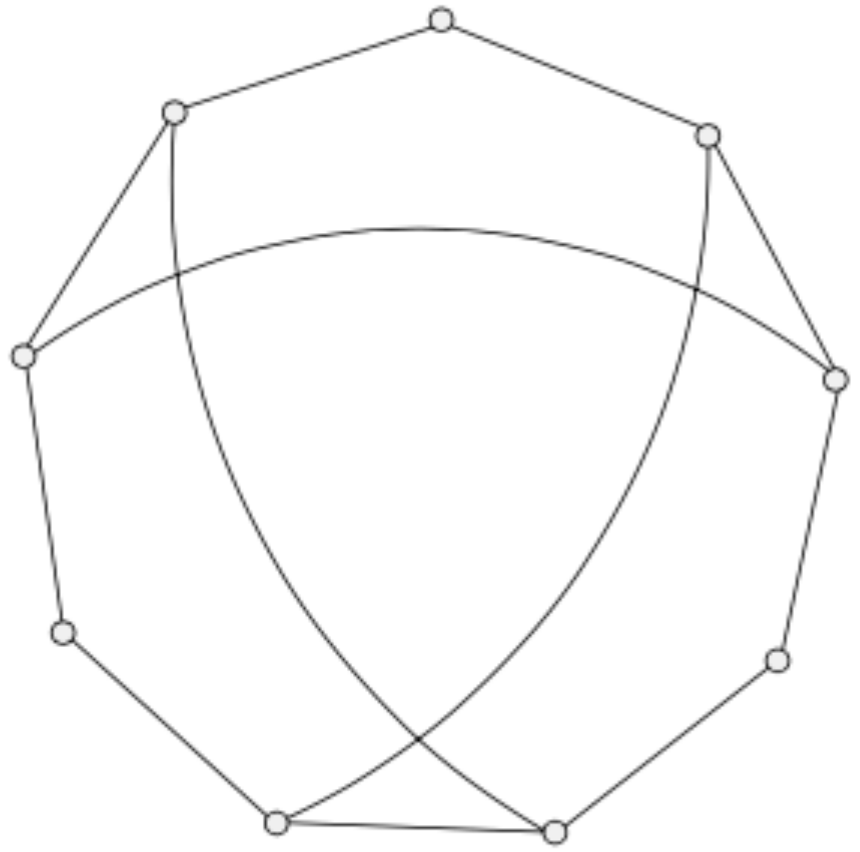
Edge-deleted subgraph and vertex-deleted subgraph:



G



$G \setminus e$



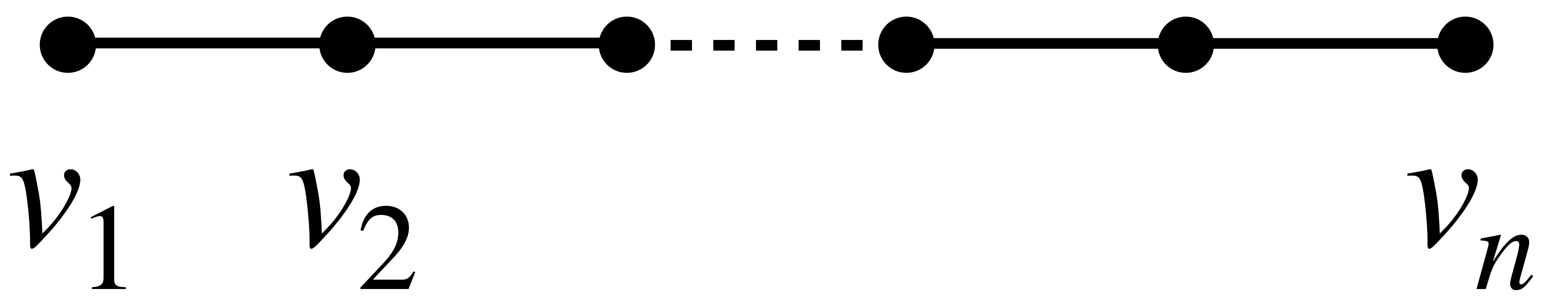
$G - v$

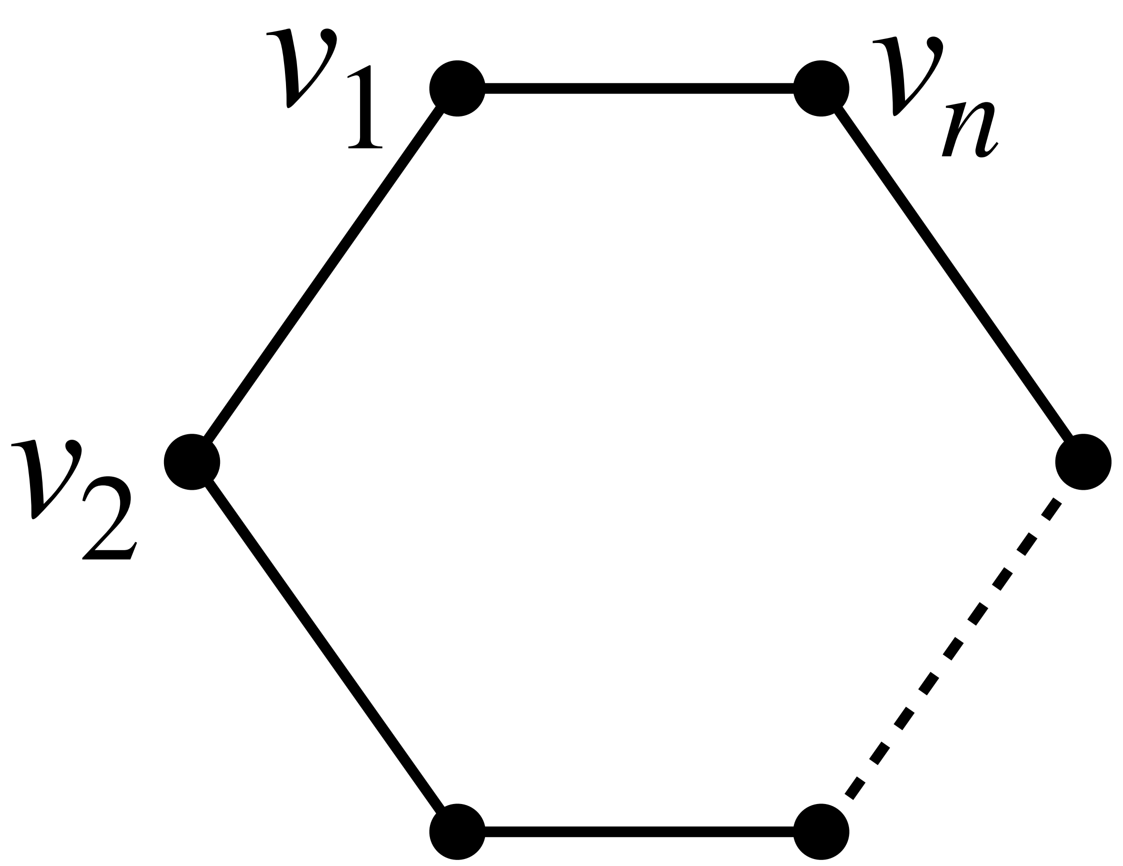
A graph is *simple* if it has no loops or parallel edges.

If $e = uv$ is an edge in a graph G, then u, v are *incident* with e.
Two distinct vertices which are incident a common edge are *adjacent*.
Two distinct adjacent vertices are *neighbors*.
The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$.

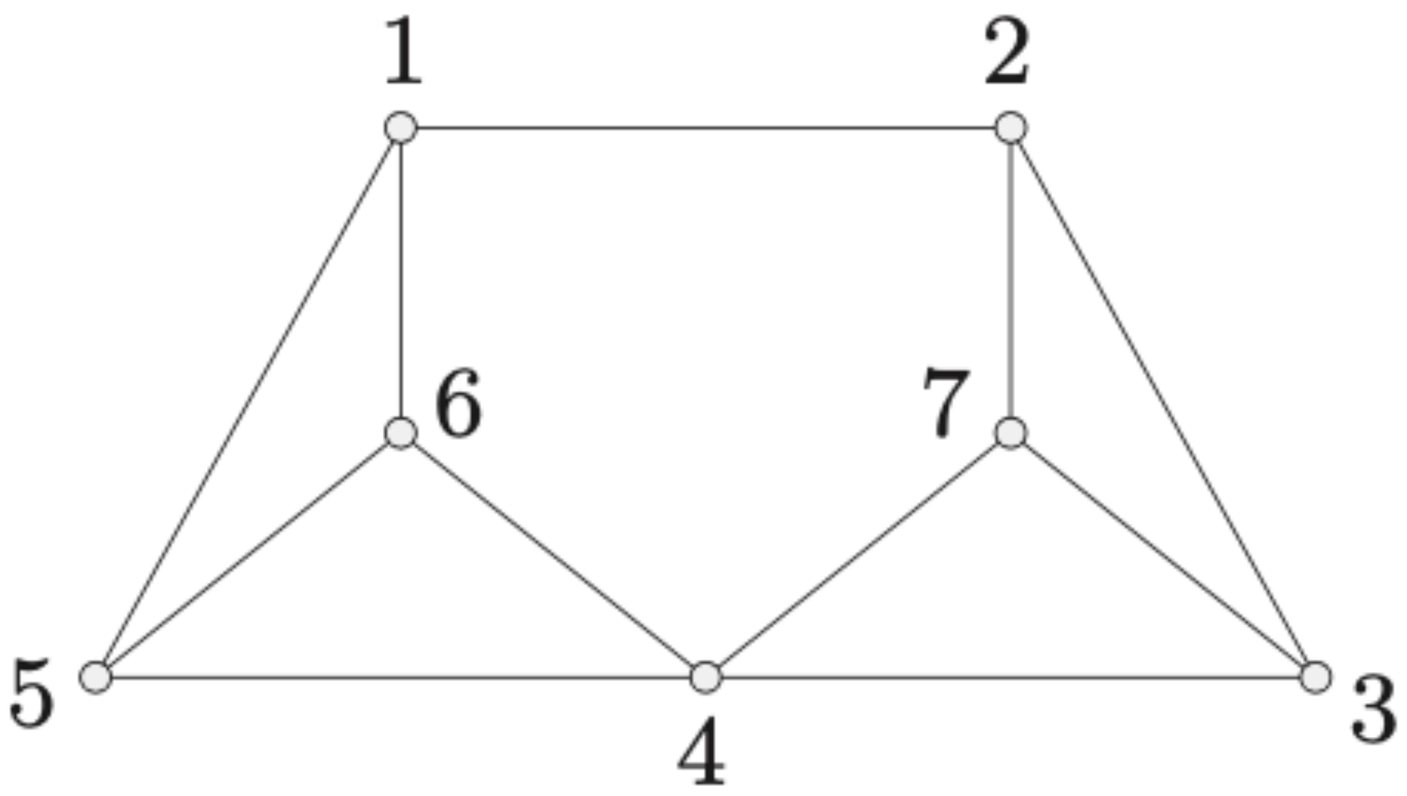
A *path* is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.

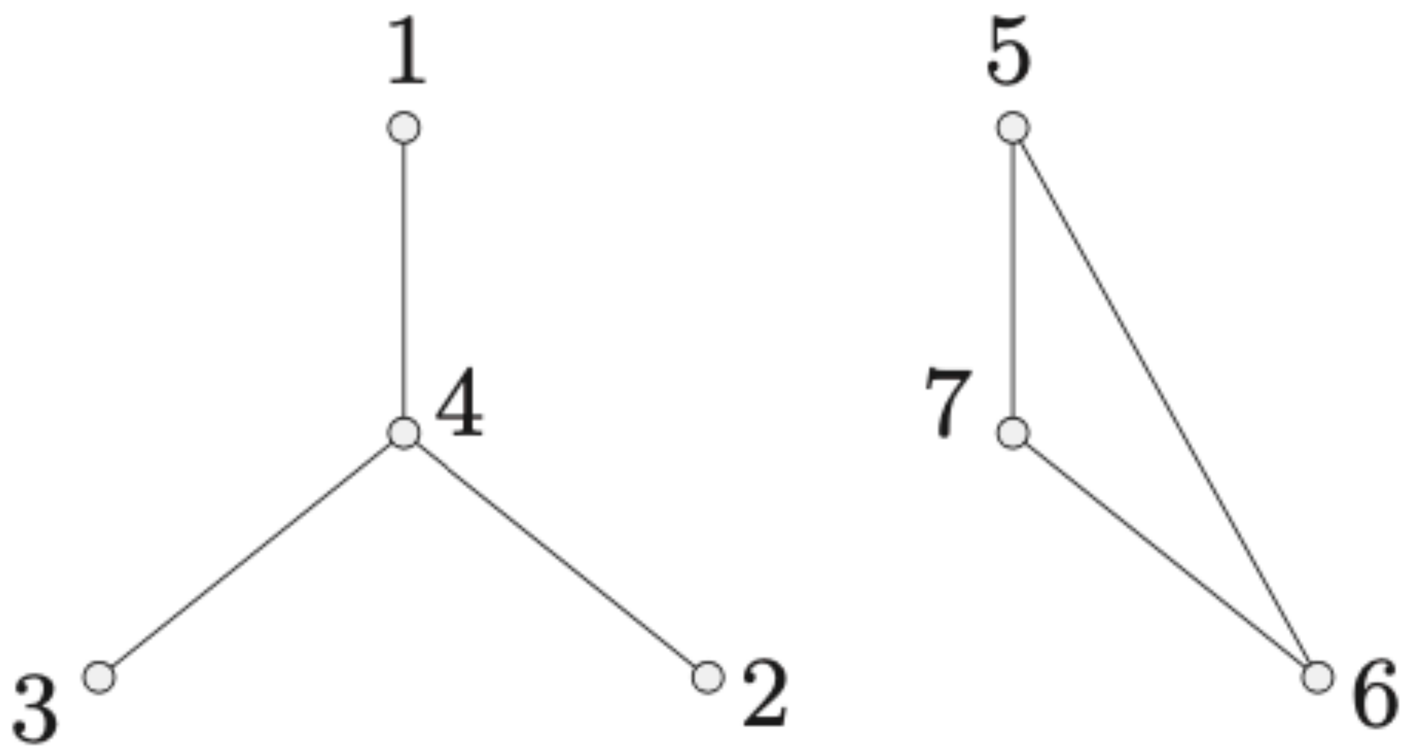
Likewise, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.





A graph is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and one end in Y; otherwise the graph is *disconnected*.





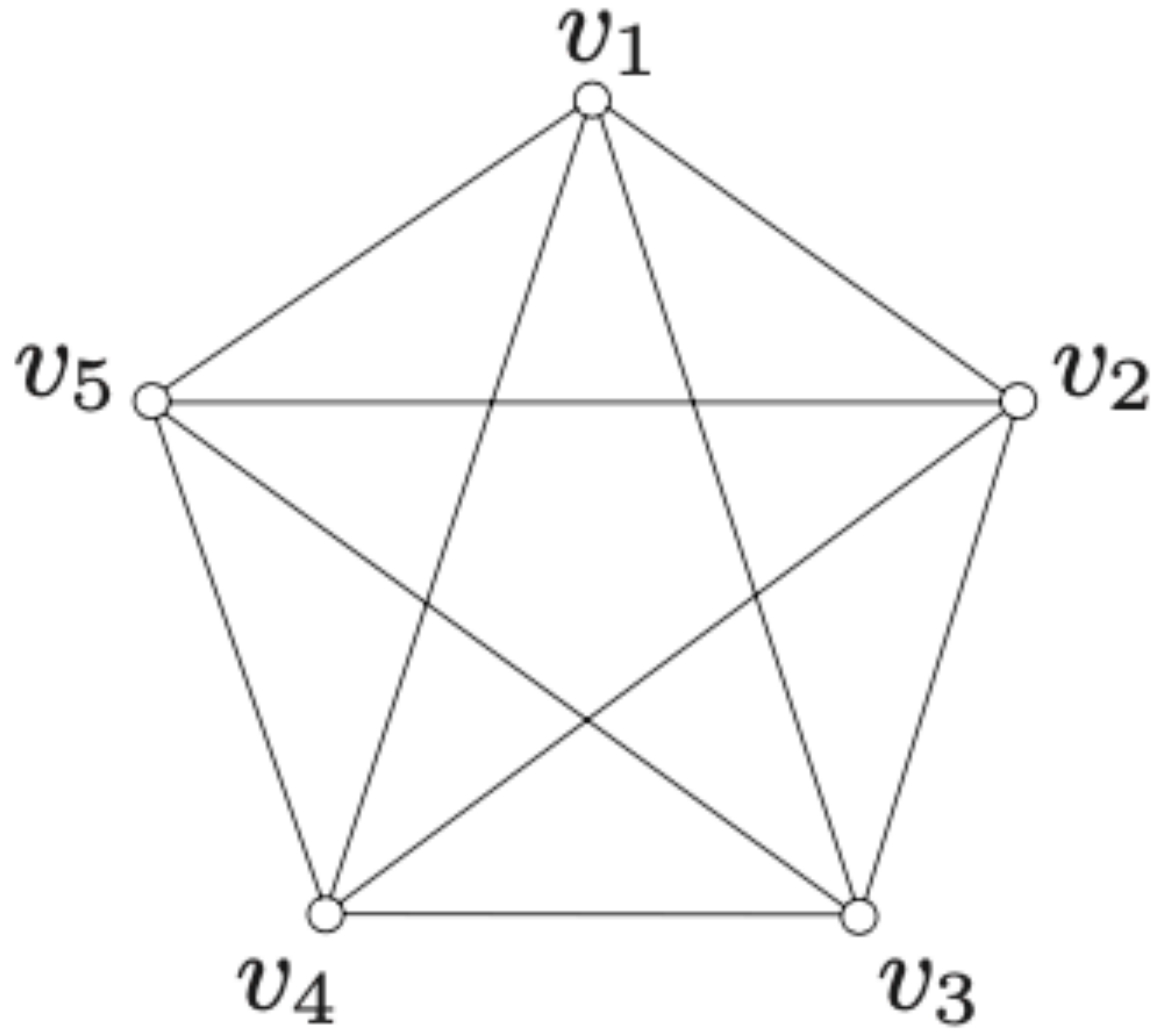
Some Family of Graphs

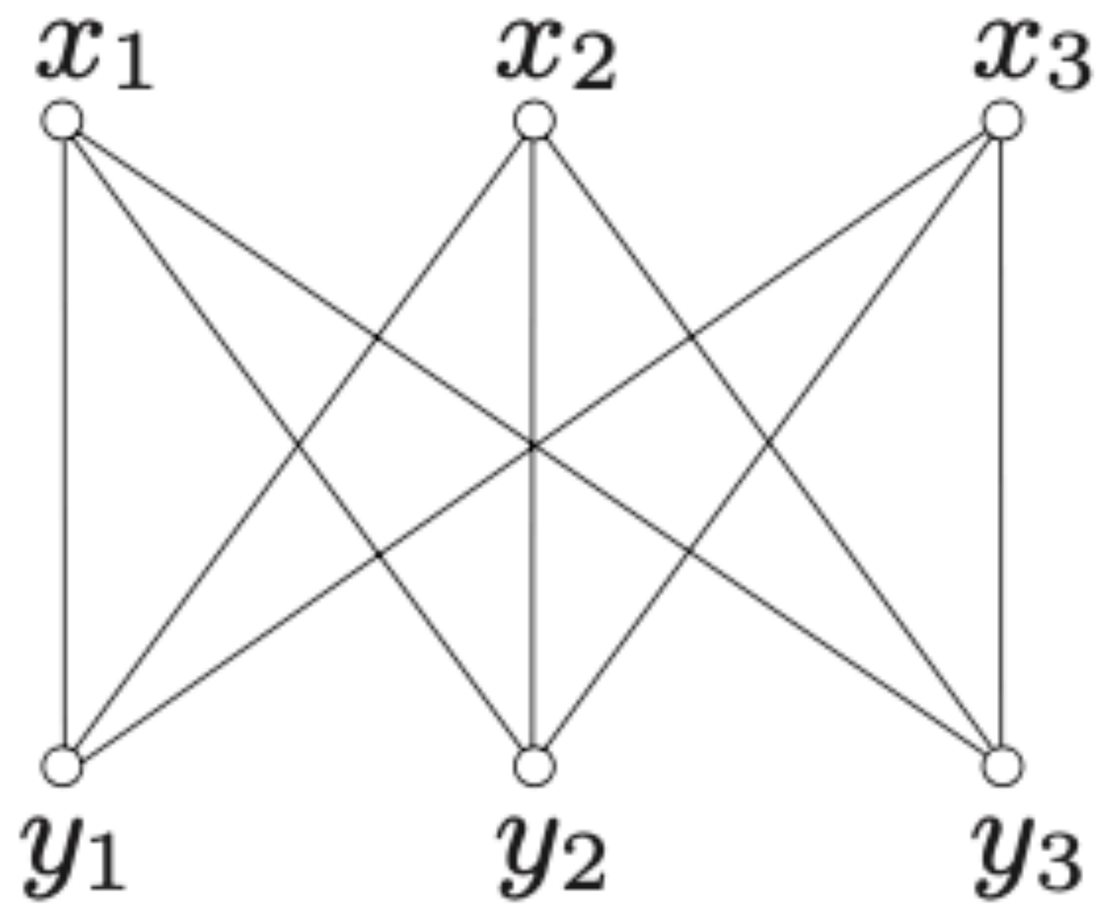
A *complete graph* is a simple graph in which any two vertices are adjacent, and an *empty graph* one in which no two vertices are adjacent.

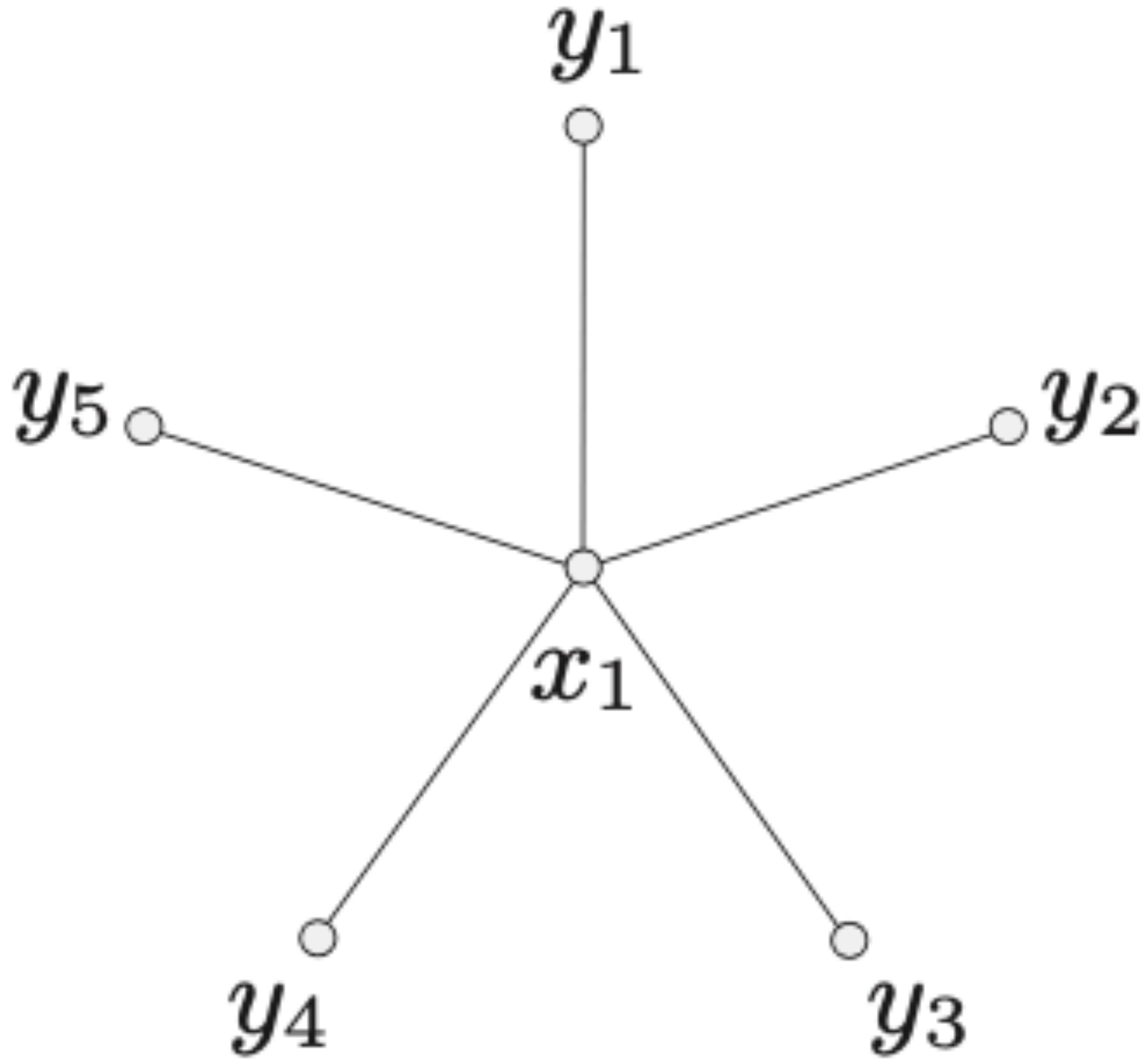
A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. Such a partition (X, Y) is called a bipartition of the graph, and denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then G is called a *complete bipartite graph*. A star is a complete bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = 1$.

A *k-partite graph* is one whose vertex set can be partitioned into k subsets, or parts, in such a way that no edge has both ends in the same part.

Example 3.







K_5 $\qquad$ $K_{3,3}$ $\qquad$ $K_{1,5}$

A complete graph $\qquad$ **A complete bipartite graph** $\qquad$ **A star**

Vertex Degrees

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v, each loop counting as two edges. In particular, if G is a simple graph, $d_G(v)$ is the number of neighbors of v in G. A vertex of degree zero is called an *isolated vertex*.

Let $\delta(G)$ and $\Delta(G)$ be the *minimum* and *maximum degrees* of the vertices of G, respectively.

Theorem 1. Let G be a graph of size m, then

$$\sum_{v \in V} d(v) = 2m .$$

Proof. Consider the incidence matrix of G:

$$
\begin{array}{c|c|c|c}
 & e_1 & e_j & e_m \\
\hline
v_1 & & * & \\
 & & & \\
\hline
v_i & * & * & * \\
\hline
 & & & \\
v_n & & * &
\end{array}
$$

The sum of the entries in the i-th row corresponding to v_i is precisely $d(v_i)$.

The sum of the entries in the j-th column corresponding to e_j is 2 for each j.

Corollary 1. In any graph, the number of vertices of odd degree is even.

Proposition 1. Let $G[X, Y]$ be a bipartite graph without isolated vertices such that $d(x) \geq d(y)$ for all $xy \in E(G)$, where $x \in X$ and $y \in Y$. Then $|X| \leq |Y|$, with equality if and only if $d(x) = d(y)$ for all $xy \in E(G)$.

Proof. Consider the bipartite adjacency matrix of $G[X, Y]$:

	y_1	y_j	y_m
x_1		a_{1j}	
x_i	a_{i1}	a_{ij}	a_{im}
x_n		a_{nj}	

M

	y_1	y_j	y_m
x_1		$\frac{a_{1j}}{d(x_1)}$	
x_i	$\frac{a_{i1}}{d(x_i)}$	$\frac{a_{ij}}{d(x_i)}$	$\frac{a_{im}}{d(x_i)}$
x_n		$\frac{a_{nj}}{d(x_n)}$	

M_1

For the matrix M_1:

The sum of the entries in the i-th row corresponding to x_i is precisely 1.

The sum of the entries in the j-th column corresponding to y_j is

$$\sum_{y_j x_i \in E} \frac{a_{ij}}{d(x_i)} \leq \sum_{y_j x_i \in E} \frac{a_{ij}}{d(y_j)} = \frac{d(y_j)}{d(y_j)} = 1.$$

Therefore,

$$|X| = n = \sum_{i=1}^{n} \underline{\sum_{x_i y_j \in E} \frac{a_{ij}}{d(x_i)}} = \sum_{j=1}^{m} \underline{\sum_{y_j x_i \in E} \frac{a_{ij}}{d(x_i)}} \leq m = |Y|.$$

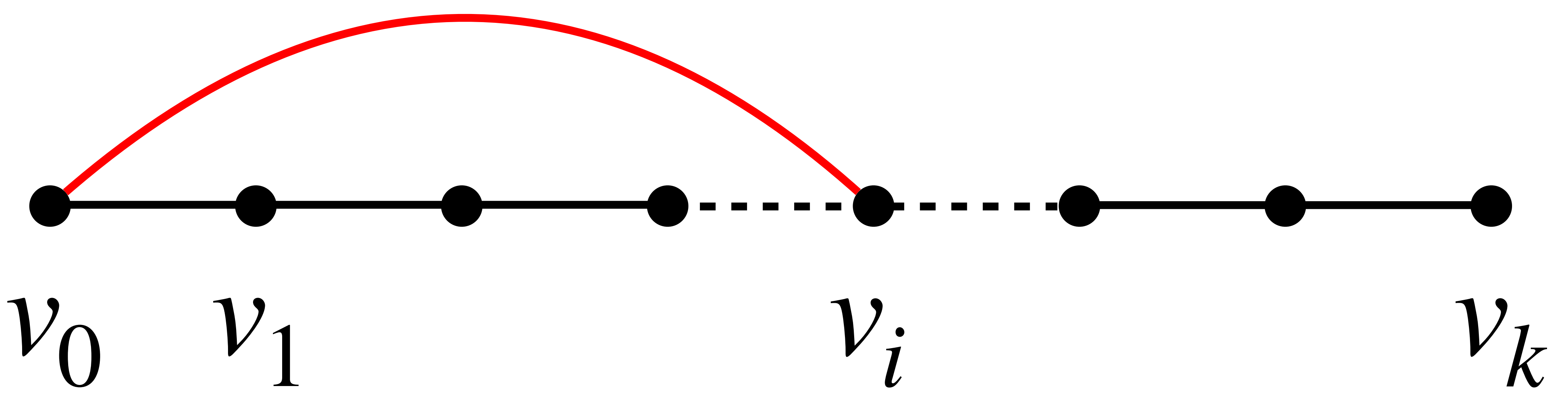
Theorem 2. Let G be a graph with $\delta(G) \geq 2$. Then G contains a cycle.

Proof. If G has a loop, it contains a cycle of length one, and if G has parallel edges, it contains a cycle of length two. So we may assume that G is simple.

Let $P = v_0 v_1 \cdots v_{k-1} v_k$ be a longest path in G.

Because the degree of v_0 is at least two, it has a neighbour v different from v_1. If v is not on P, then the path $v v_0 v_1 \cdots v_{k-1} v_k$ is longer than P, this contradicts the choice of P.

Therefore, $v = v_i$ for some i, $2 \leq i \leq k$, and $v_0 v_1 \cdots v_i v_0$ is a cycle in G.



Theorem 3. Any simple graph G with

$$\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$$

contains a quadrilateral.

Proof. Let p_2 be the number of paths of length two in G, and $p_2(v)$ the number of such paths whose central vertex is v. Clearly,

$$p_2(v) = \binom{d(v)}{2}.$$

As each path of length two has a unique central vertex,

$$p_2 = \sum_{v \in V} p_2(v) = \sum_{v \in V} \binom{d(v)}{2}.$$

On the other hand, since each such path also has a unique pair of ends, the set of all paths of length two can be partitioned into

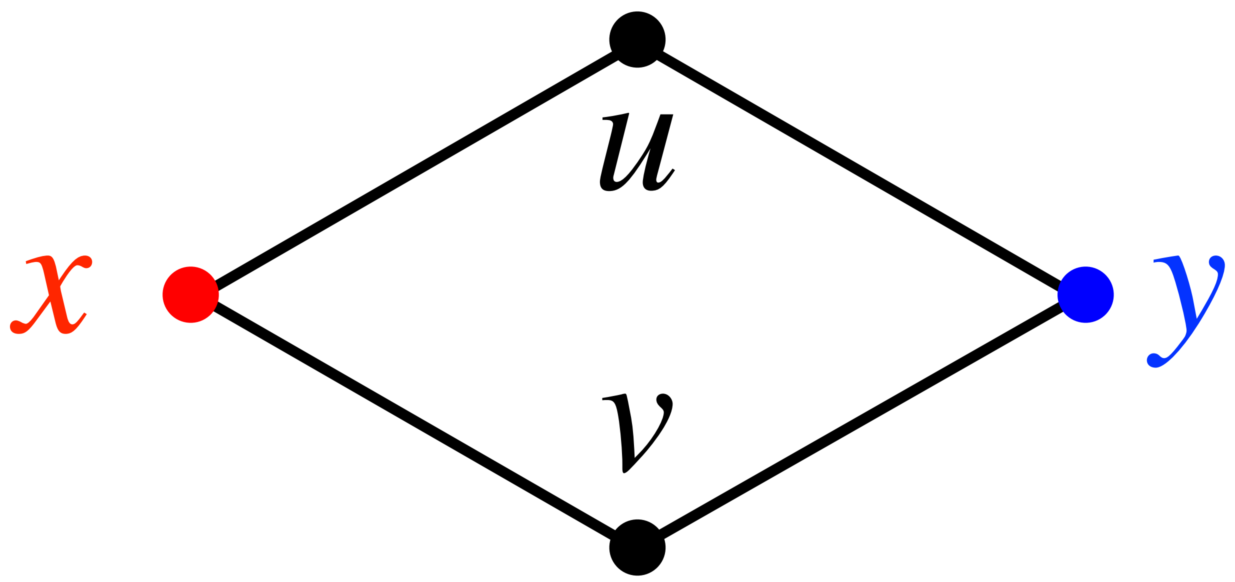
$$\binom{n}{2}$$

subsets according to their ends.

By the hypothesis, one of these subsets contains two or more paths.

That is, there exist two paths of length two with the same pair of ends.

The union of these two paths is a quadrilateral.



Trees and Branchings

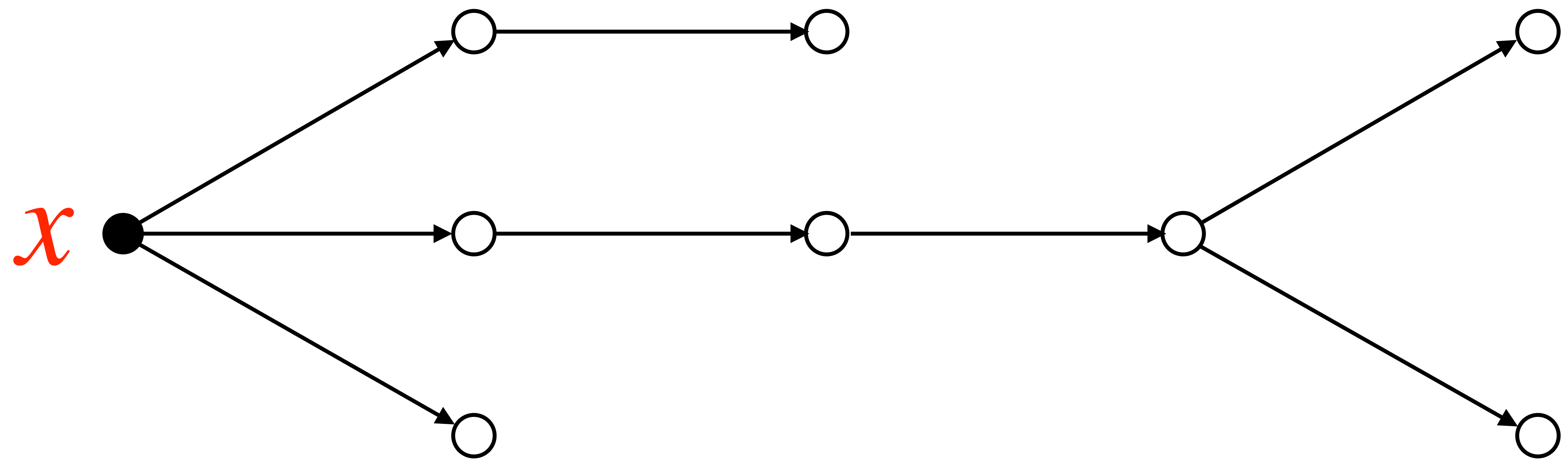
A graph is *acyclic* if it has no cycles.

A *tree* is a connected acyclic graph and a *forest* is an acyclic graph.

A *rooted tree* $T(x)$ is a tree with a specified vertex x, called the *root* of T.

An orientation of a rooted tree in which

every vertex but the root has indegree one is called a *branching*. We refer to a rooted tree or branching with root x as an x-tree or x-branching, respectively.



Proposition 2. In a tree, any two vertices are connected by exactly one path.

Proof. Suppose to the contrary that $u, v \in V(T)$ and

$$P = ux_1x_2\cdots x_sv,$$

$$Q = uy_1y_2\cdots y_tv$$

are two distinct path connecting u and v.

Consider the graph $H = E(P) \cup E(Q) - E(P) \cap E(Q)$.

($E(P) \cup E(Q) - E(P) \cap E(Q)$ is called *symmetric difference* of P and Q)

Let H_1 be the largest component of H, then we must have $\delta(H_1) \geq 2$.

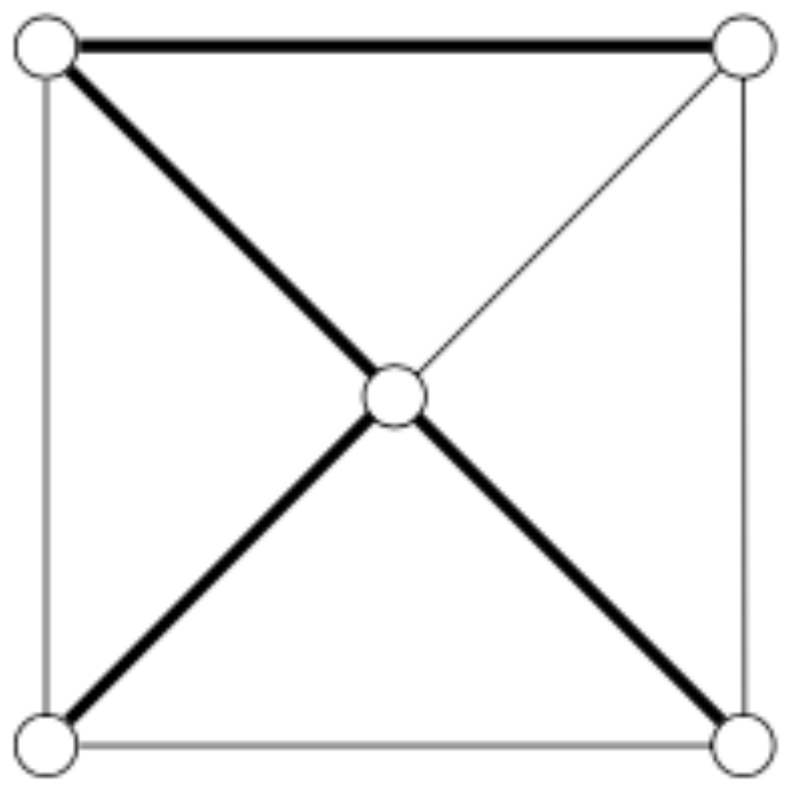
This implies that H_1 contains a cycle, and so is T, a contradiction.

Theorem 4. If T is a tree, then $e(T) = v(T) - 1$.

Spanning Trees

A subtree of a graph is a subgraph which is a tree.

If this tree is a spanning subgraph, it is called a *spanning tree* of the graph.



Theorem 5. A graph is connected if and only if it has a spanning tree.

Theorem 6. A graph G is bipartite if and only if it contains no odd cycle.

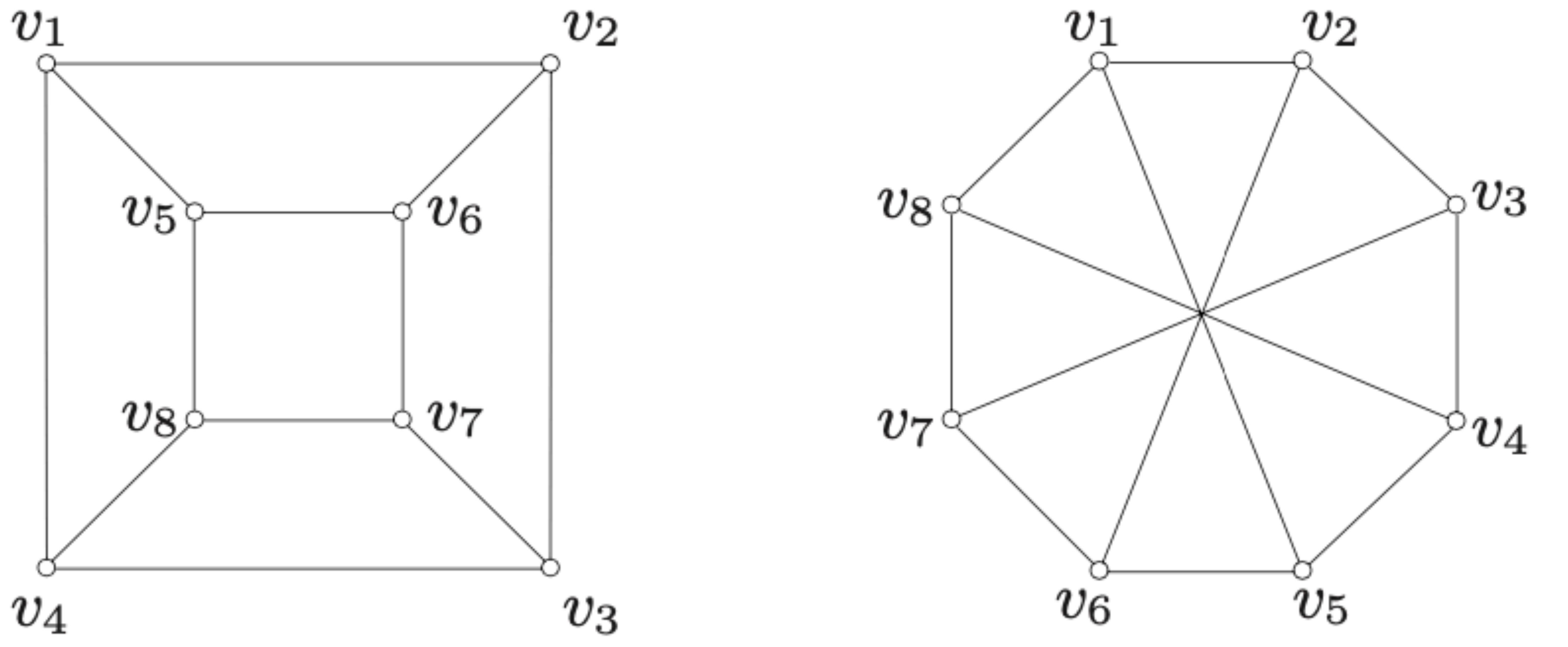
Proof. It suffices to consider connected bipartite graphs.

($\Rightarrow$) Let $G[X, Y]$ be a bipartite graph. It is easy to see that each cycle C of G has the form $x_1 y_1 x_2 y_2 \cdots x_\ell y_\ell x_1$, where $x_i \in X$ and $y_i \in Y$, and so $|C|$ is even.

($\Leftarrow$) By Theorem 3, the graph has a spanning tree T. Let x be any vertex in T. By Proposition 2, for any v, there is unique path in T connecting x and v. Let X be the set of vertices v for which this path is of even length, $Y = V(G) \backslash X$. For any edge $uv \in E(G) \backslash E(T)$, let $P = uTv$ be the unique (u, v)-path in T. Since $P + uv$ is an even cycle, P is of odd length, and so u, v must belong to different parts of X, Y. It follows that (X, Y) is indeed a bipartite of G.

Exercise 1.

1. Show that the following two graphs are not isomorphic, that is, $G \ncong H$.



G H

2. Let G be a simple graph of order n and size m. Show that if $m \geq n$, then G contains a cycle.

3. Show that any simple graph G has a path of order at least $\delta(G) + 1$.