Turán Problem

Given a graph H, a graph is called H-free if it contains no H as a subgraph.

For a simple graph H, let $ex(n, H)$ denote
the maximum number of edges a simple H-free graph on n vertices can have.
The number $ex(n, H)$ is called Turán number of H.

Let $T_{k,n}$ denote a complete k-partite graph on n vertices, and all parts are as equal in size as possible. The graph $T_{k,n}$ is called a Turán graph.

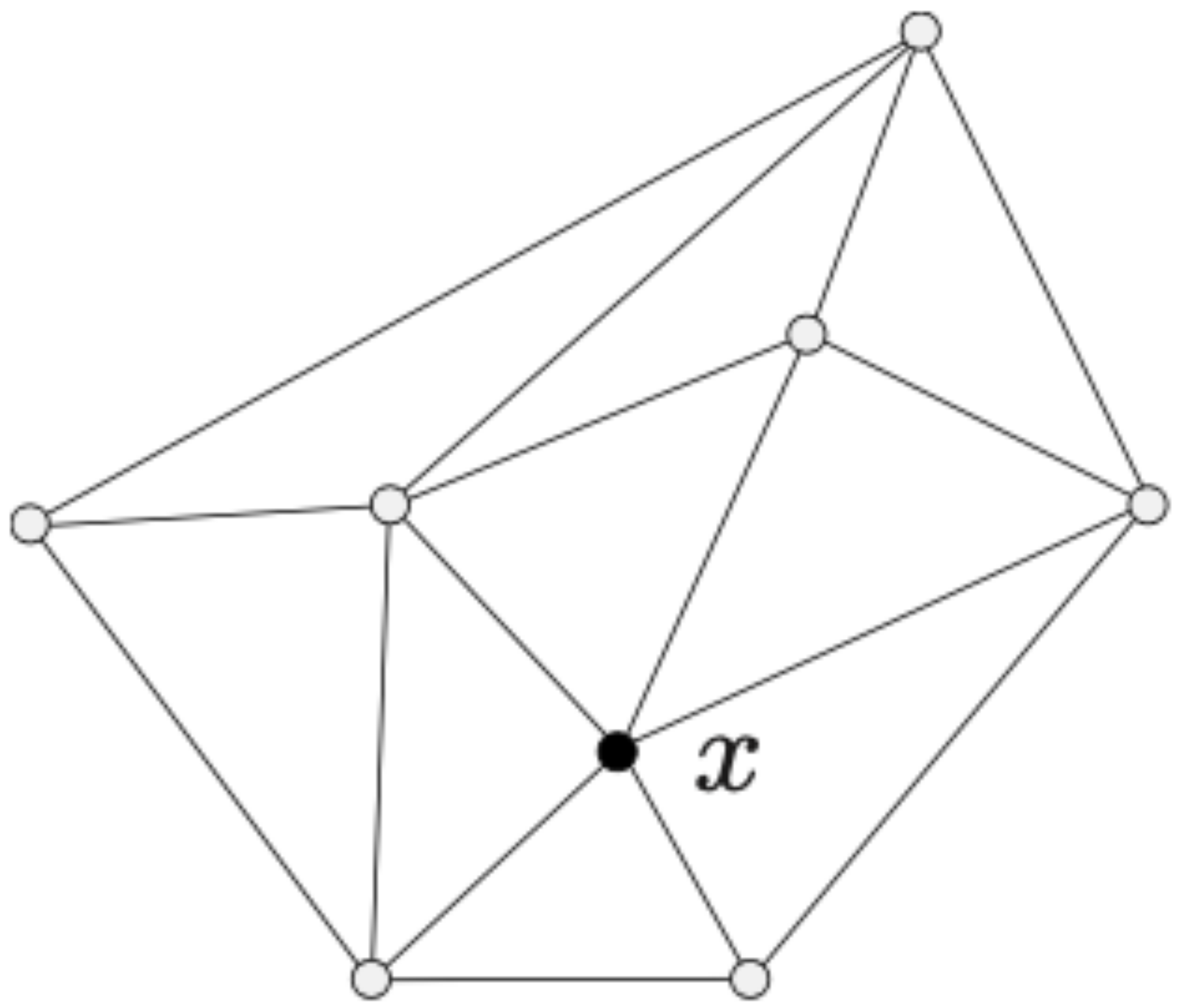
Theorem 42 (Turán). Let G be a simple graph which contains no K_k, where $k \geq 2$. Then $e(G) \leq e\left(T_{k-1,n}\right)$, with equality if and only if $G \cong T_{k-1,n}$.

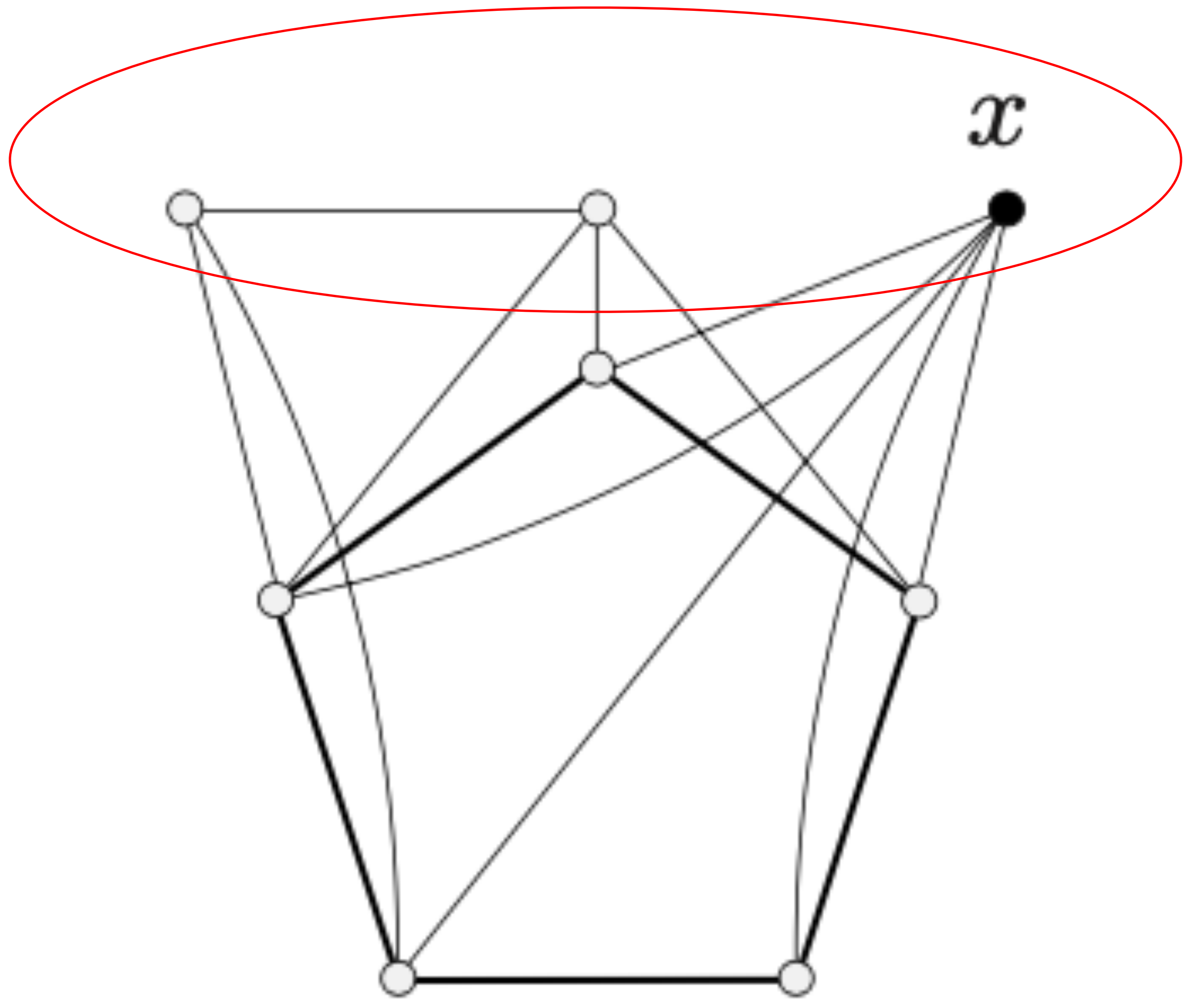
Proof. By induction on k.

The theorem holds trivially for $k = 2$.

Assume that it holds for all integers less than k, and let G be a simple graph which contains no K_k.

Choose a vertex x of degree Δ in G, and set $X = N(x)$ and $Y = V(G)\backslash X$.





Then

$$e(G) = e(X) + e(X, Y) + e(Y)\,.$$

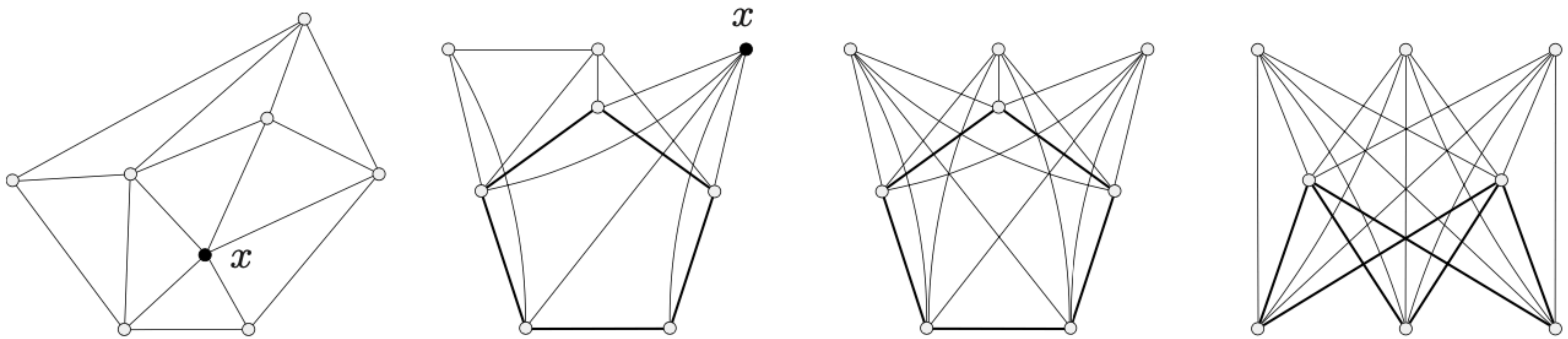
Since G contains no K_k, $G[X]$ contains no K_{k-1}. Therefore, by the induction hypothesis,

$$e(X) \leq e\left(T_{k-2,\Delta}\right),$$

with quality if and only if $G[X] \cong T_{k-2,\Delta}$. In addition, because each edge of G incident with a vertex of Y belongs to either $E[X, Y]$ or $E[Y]$,

$$e(X, Y) + e(Y) \leq \Delta(n - \Delta),$$

with equality if and only if Y is an independent set and all members of which have degree Δ.



Therefore, $e(G) \leq e(H)$,

where H is the graph obtained from a copy of $T_{k-2,\Delta}$,

by adding an independent $n - \Delta$ vertices and joining each vertex of this set to each vertex of $T_{k-2,\Delta}$.

Observe that H is a complete $(k-1)$-partite graph on n vertices.
It is not difficult to show that $e(H) \leq e\left(T_{k-1,n}\right)$,
with equality if and only if $H = T_{k-1,n}$.
It follows that $e(G) \leq e\left(T_{k-1,n}\right)$, with equality if and only if $G = T_{k-1,n}$.

Turán theorem is regarded as the origin of extremal graph theory,
which has applications to diverse areas of mathematics,
including combinatorial number theory and combinatorial geometry.
We present an application of Turán's theorem to combinatorial geometry.

The diameter of a set of points in the plane is the maximum distance between two points of the set. It should be noted that this is a purely geometric notion and is unrelated to the graph-theoretic concepts of diameter and distance.

We discuss sets of diameter one.

A set of n points determines $\binom{n}{2}$ distances between pairs of these points.

It is clear that if n is 'large', some of these distances must be 'small'.

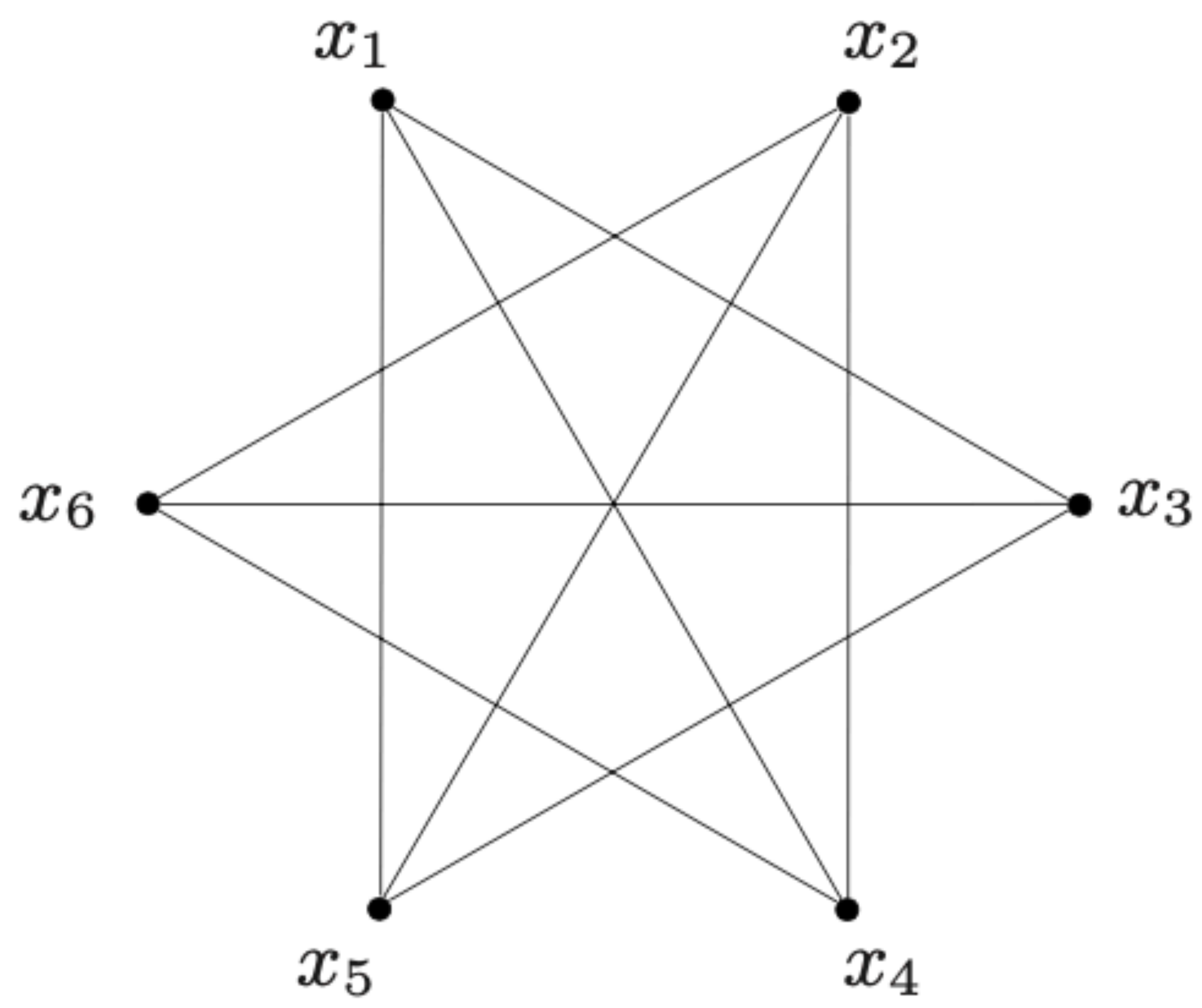
Therefore, for any d between 0 and 1,

it makes sense to ask how many pairs of points in a set $\{x_1, x_2, \ldots, x_n\}$ of diameter one can be at distance greater than d.

Here, we present a solution, by Erdős,

of one special case of this problem, namely when $d = 1/\sqrt{2}$.

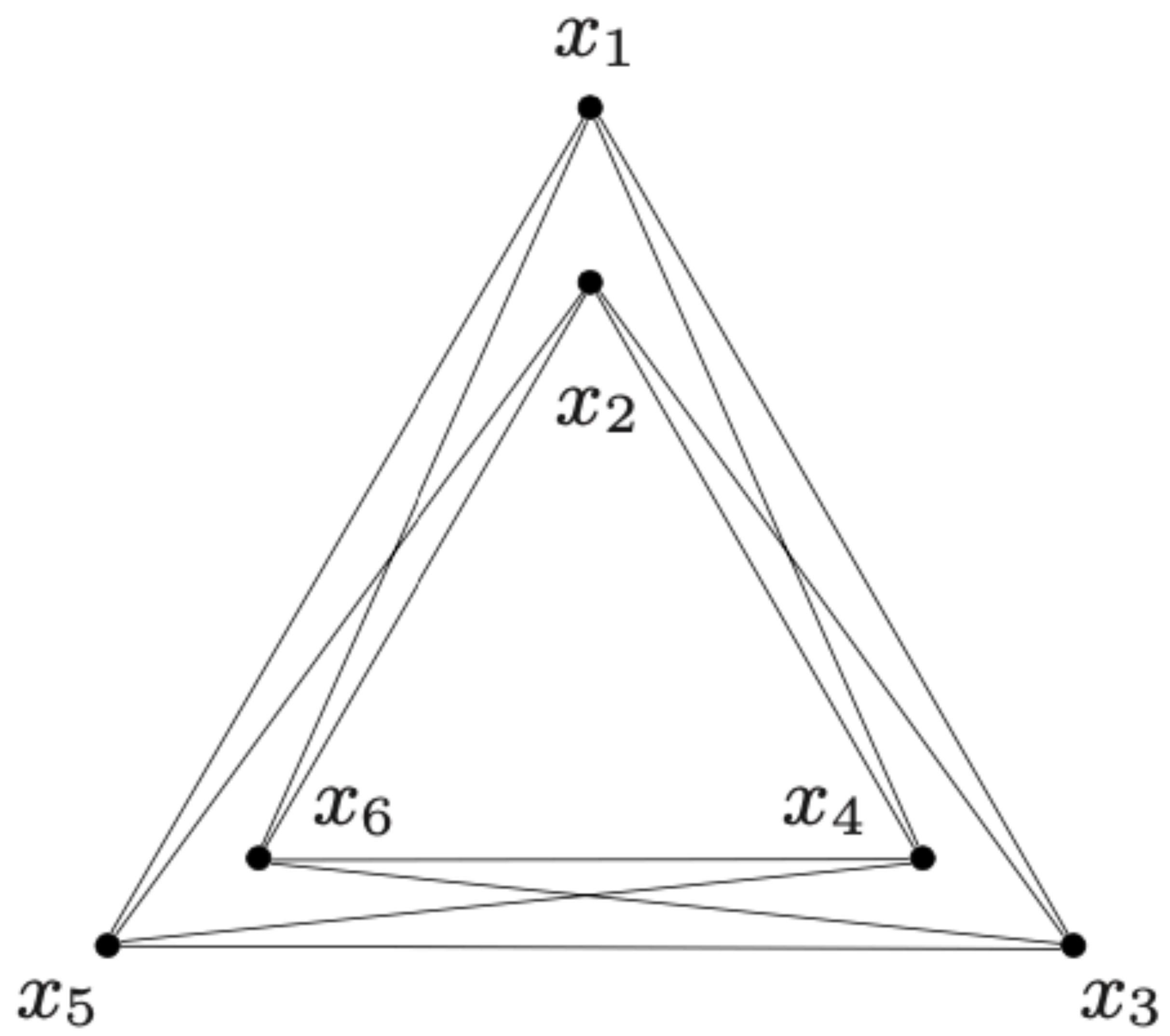
Example. $n = 6$, the set is $\{x_1, x_2, \ldots, x_6\}$.



1 $(x_1, x_4), (x_2, x_5), (x_3, x_6)$

$1/2$ $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_6), (x_6, x_1)$

$\sqrt{3}/2$ $(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_5, x_1), (x_6, x_2)$



All pairs except $(x_1, x_2), (x_3, x_4), (x_5, x_6)$

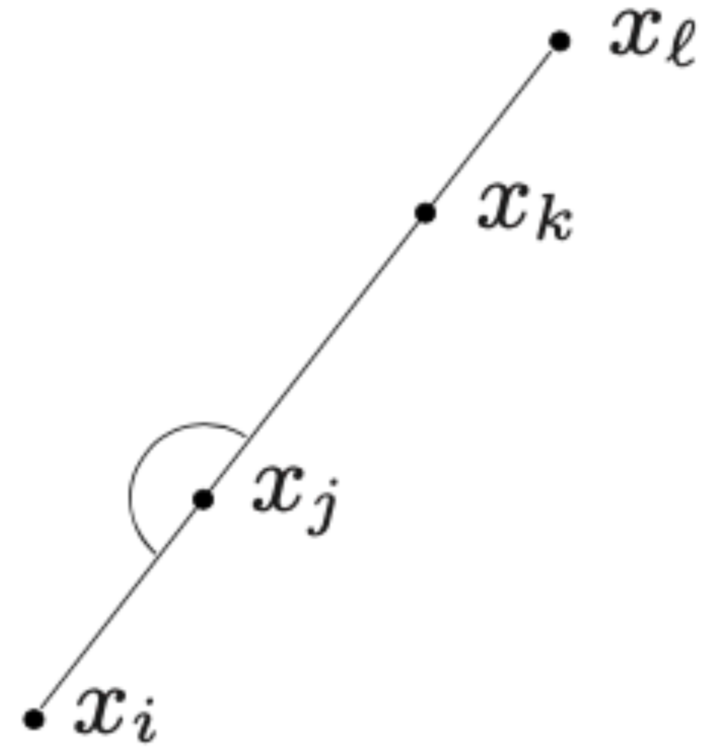
are at distance greater than $1/\sqrt{2}$.

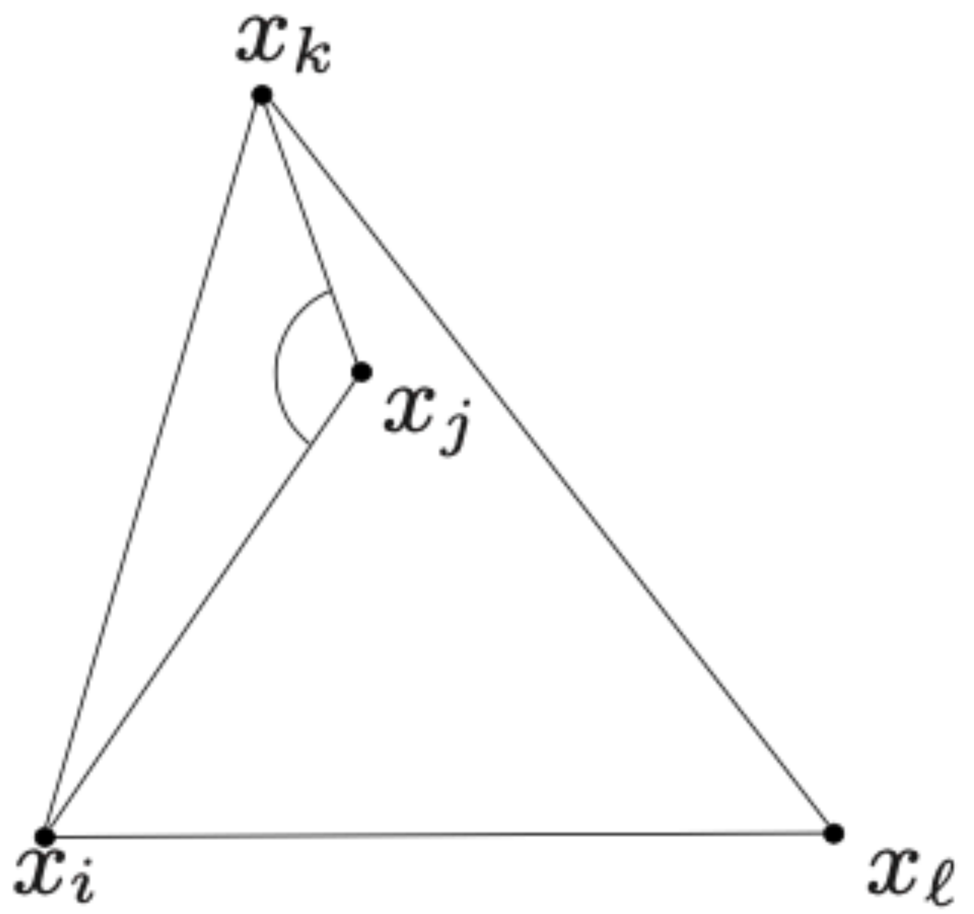
This is best possible!

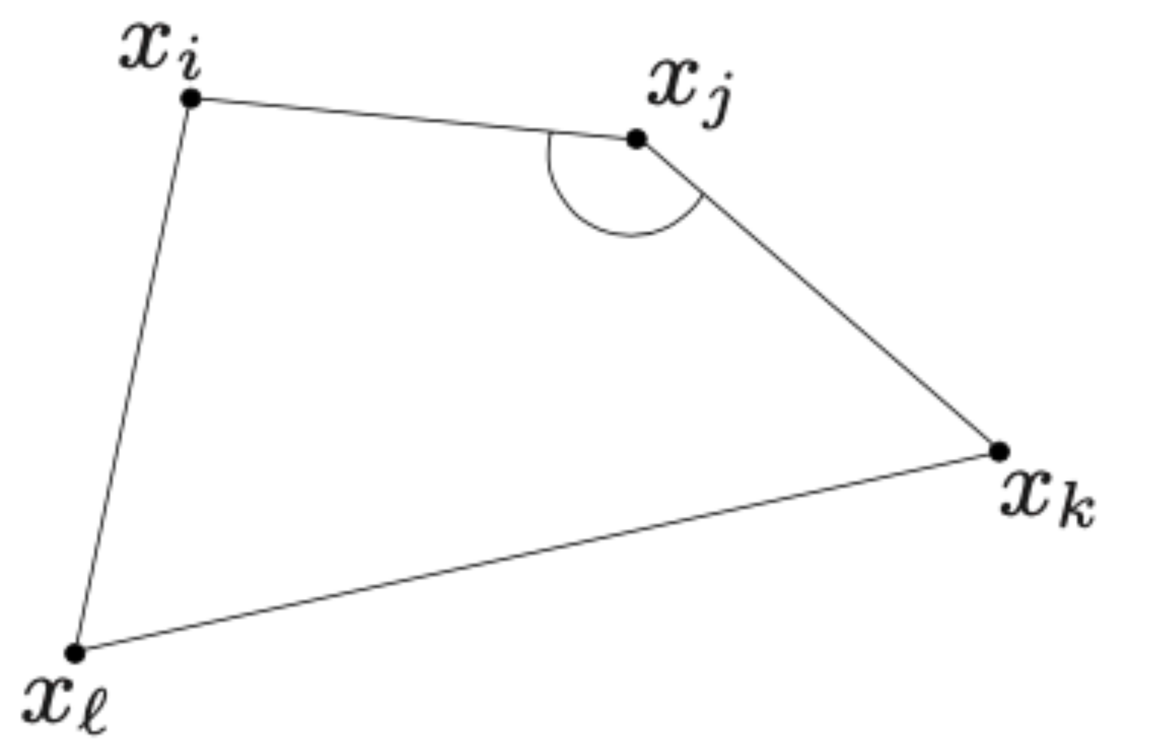
Theorem 43 (Erdős). Let S be a set of diameter one in the plane. Then the number of pairs of points of S whose distance is greater than $1/\sqrt{2}$ is at most $\lfloor n^2/3 \rfloor$, where $n = |S|$. Moreover, for each $n \geq 2$, there is a set of n points of diameter one in which exactly $\lfloor n^2/3 \rfloor$ pairs of points are at distance greater than $1/\sqrt{2}$.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$. Consider the graph G with vertex set S and edge set $\{x_i x_j \mid d(x_i, x_j) > 1/\sqrt{2}\}$, where $d(x_i, x_j)$ denotes the euclidean distance between x_i and x_j. We show that G cannot contain a copy of K_4.

Note that any 4 points in the plane determine an angle of at least 90 degrees between three of them: the convex hull of the points is a line, a triangle, or a quadrilateral, and in each case there is an angle $\angle x_i x_j x_k$ at least 90 degrees.







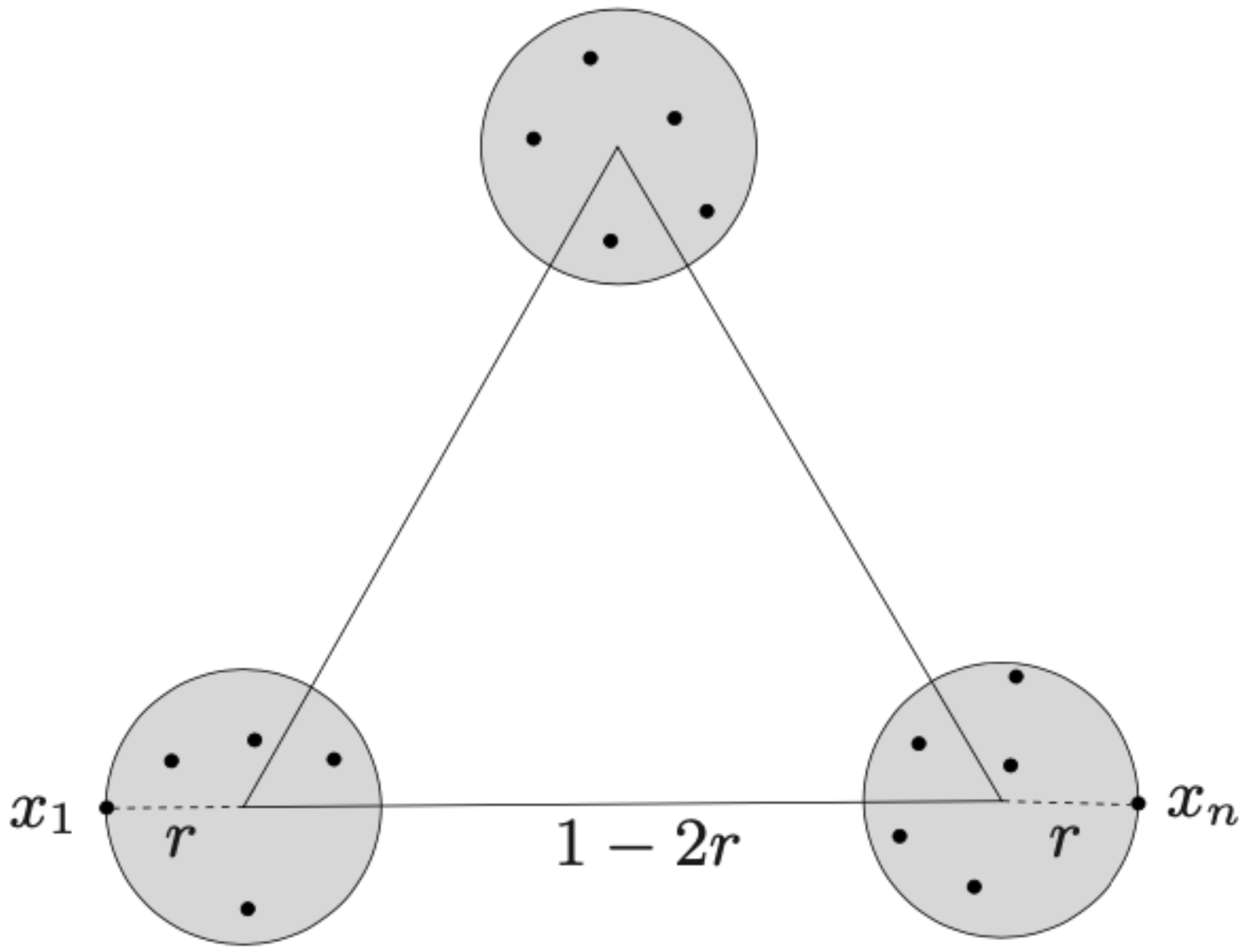
Now look at the three points x_i, x_j, x_k which determine this angle. Not all the distances $d(x_i, x_j), d(x_i, x_k), d(x_j, x_k)$ can be greater than $1/\sqrt{2}$ and less than or equal to 1. For, if $d(x_i, x_j) > 1/\sqrt{2}$ and $d(x_j, x_k) > 1/\sqrt{2}$, then $d(x_i, x_k) > 1$. The set $\{x_1, x_2, \ldots, x_n\}$ is assumed to have diameter one. It follows that, of any 4 points in G, at least one pair cannot be joined by an edge, and hence that G cannot contain a copy of K_4.

By Turán theorem,

$$e(G) \leq e\left(T_{3,n}\right) = \left\lfloor \frac{n^2}{3} \right\rfloor .$$

One can construct a set $\left\{x_1, x_2, \ldots, x_n\right\}$ of diameter one in which exactly $\lfloor n^2/3 \rfloor$ pairs of points are at distance greater than $1/\sqrt{2}$ as follows.

$$0 < r < \frac{1}{4}\left(1 - \frac{1}{\sqrt{2}}\right)$$



Place n points into 3 circles of radius r, s.t. the numbers of points in any 2 circles differ at most 1.

For a nonbipartite graph H,
the value of $ex(n, H)$ was determined asymptotically.

Theorem 43 (Erdős-Stone-Simonovits). Let H be a graph with chromatic number $\chi(H) = \chi \geq 3$, then

$$ex(n, H) = \left(\frac{\chi - 2}{\chi - 1} + o(1)\right)\binom{n}{2}.$$

For a bipartite graph H,
the value of $ex(n, H)$ is far more from being known.

Theorem 44 (Reiman). For all $n \geq 4$,

$$ex\left(n, C_4\right) < \frac{1}{4}n\left(1 + \sqrt{4n - 3}\right).$$

Theorem 45(Füredi)

$$ex\left(q^2+q+1,C_4\right)\le \frac{1}{2}q(q+1)^2 \text{ for } q>13,$$

and the equality holds for all prime power q.

The extremal graph for Theorem 45 is *polarity graph* G_q, which is defined by Brown, and by Erdős, Rényi and Sós, independently.

Let $F_q=GF(q)$ be the Galois field with q elements.
Define an equivalence relation on $F_q^3\backslash\{(0,0,0)\}$ by letting $(a,b,c)=\left(a',b',c'\right)$ if there is a nonzero $k\in F_q$ such that $(ka,kb,kc)=\left(a',b',c'\right)$.
Let V be the set of all equivalence classes.

The *polarity graph* $\mathcal{G}_q$ is the graph with vertex set V,

two equivalence classes in V, abc and xyz, being adjacent if and only if

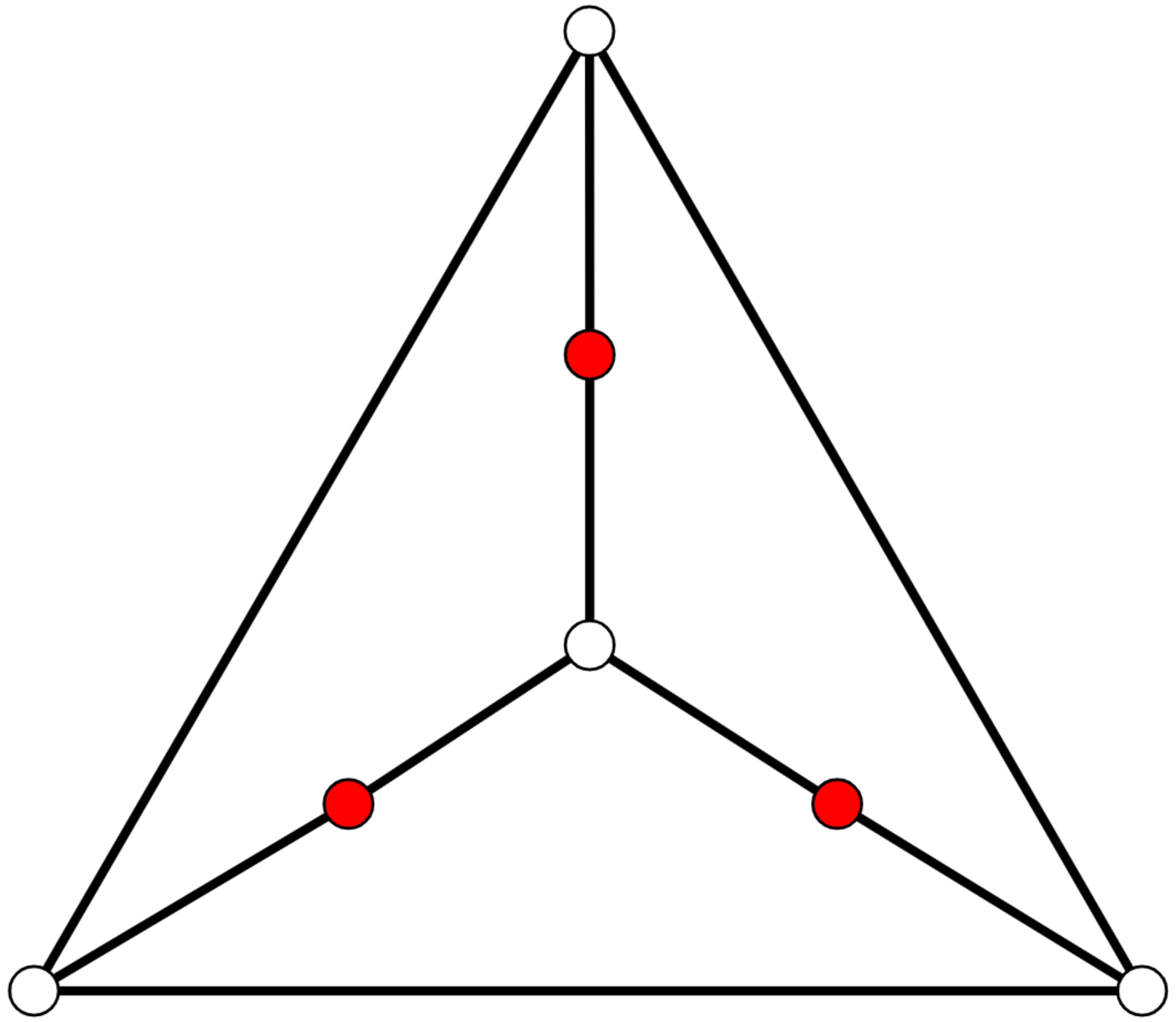
$$ax+by+cz=0.$$

The simple polarity graph G_q is the graph obtained from $\mathcal{G}_q$ by deleting all $q+1$ loops.

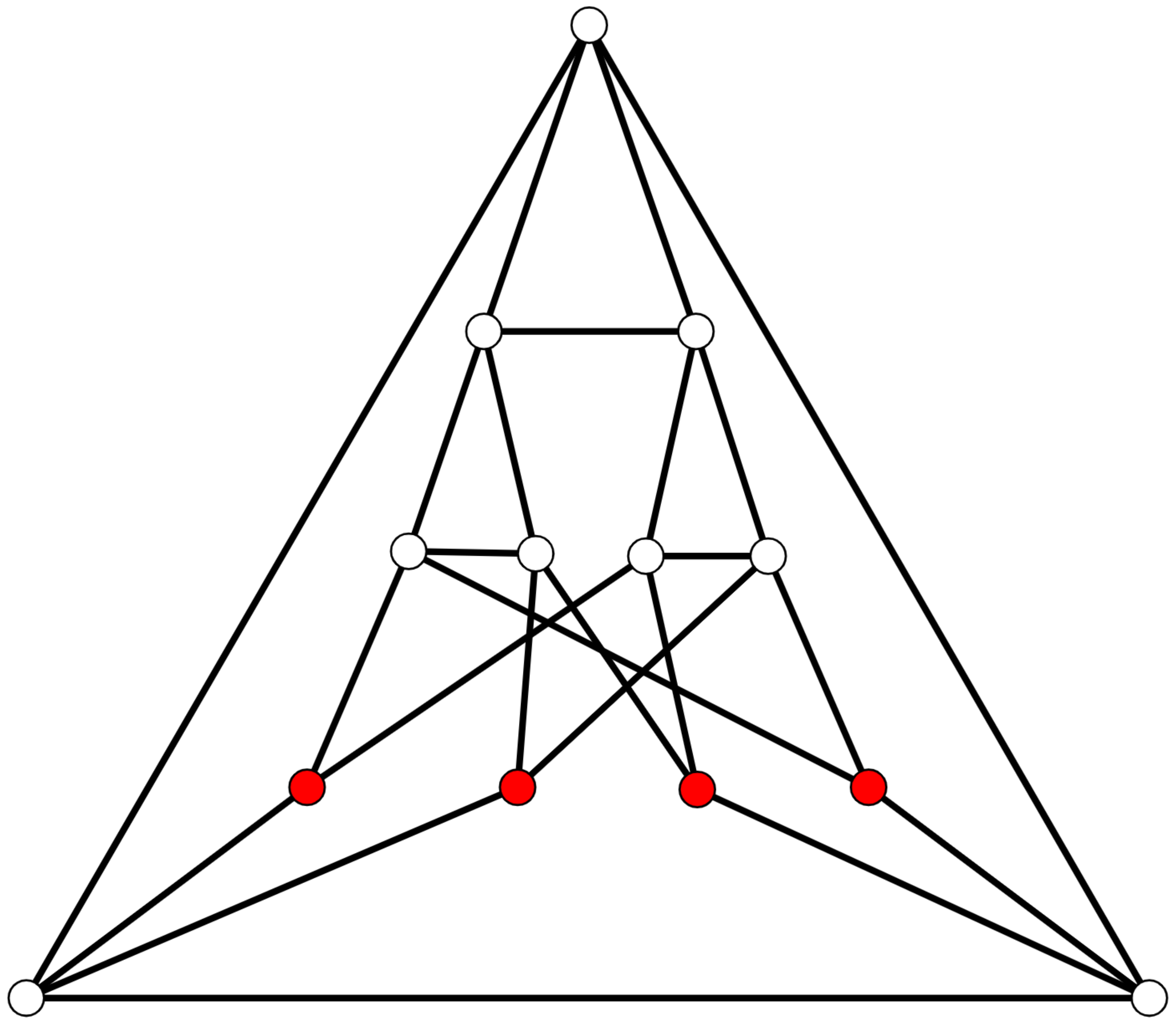
The graph G_q has q^2+q+1 vertices,

$q+1$ vertices are of degree q, which form an independent set of G_q,

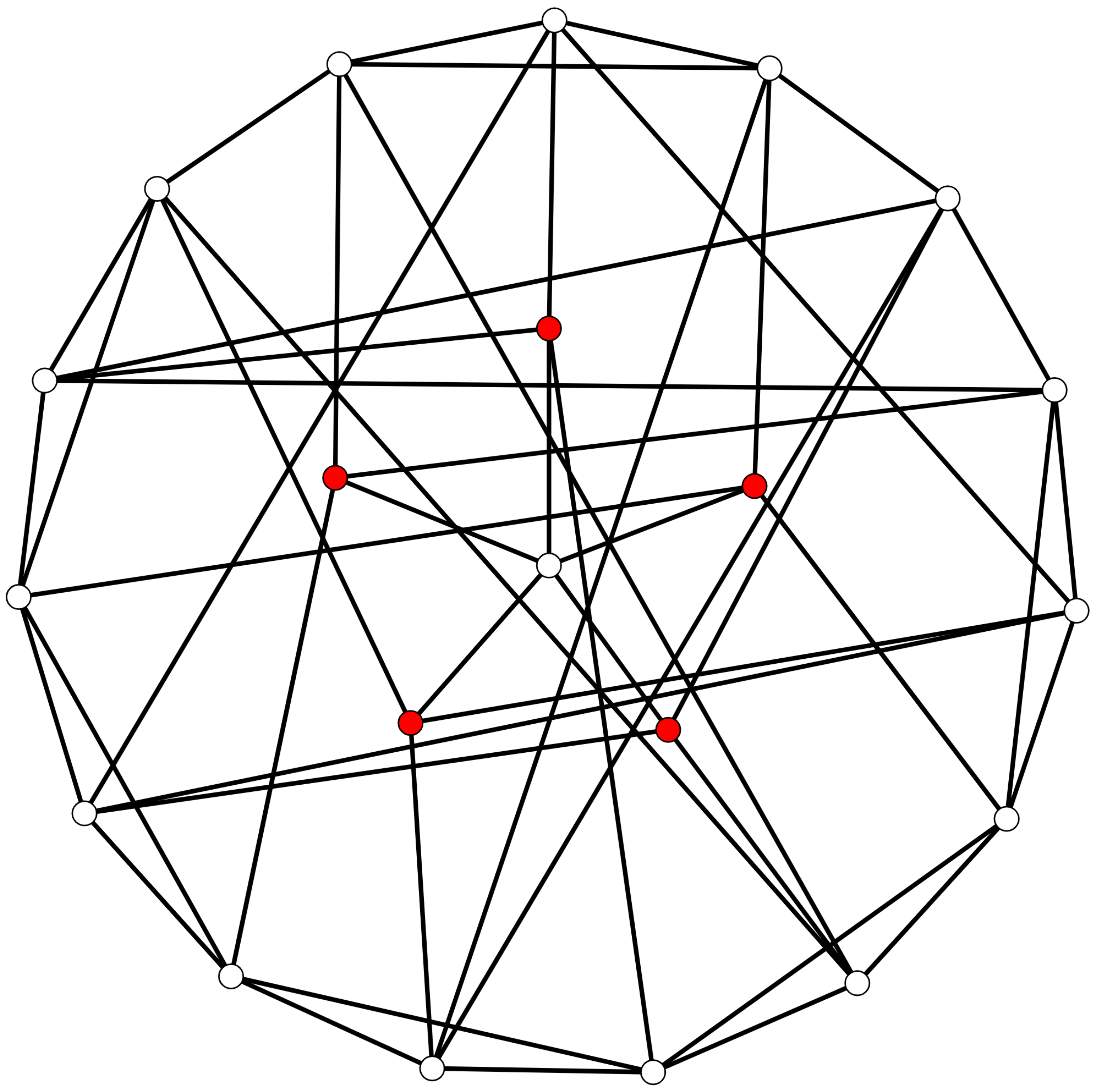
and the other vertices are of degree $q+1$.



$q = 2$



$q = 3$



$q = 4$

Perfect difference set

Let Z_{m^2+m+1} be the additive group of integers modulo m^2+m+1 and D an $(m+1)$-subset of Z_{m^2+m+1}.

If each nonzero element of Z_{m^2+m+1} can be expressed uniquely as a difference $d_1 - d_2$ of two elements in D,

then we call D *a perfect difference set*.

For example: $Z_7 = \{0,1,2,3,4,5,6\}$,

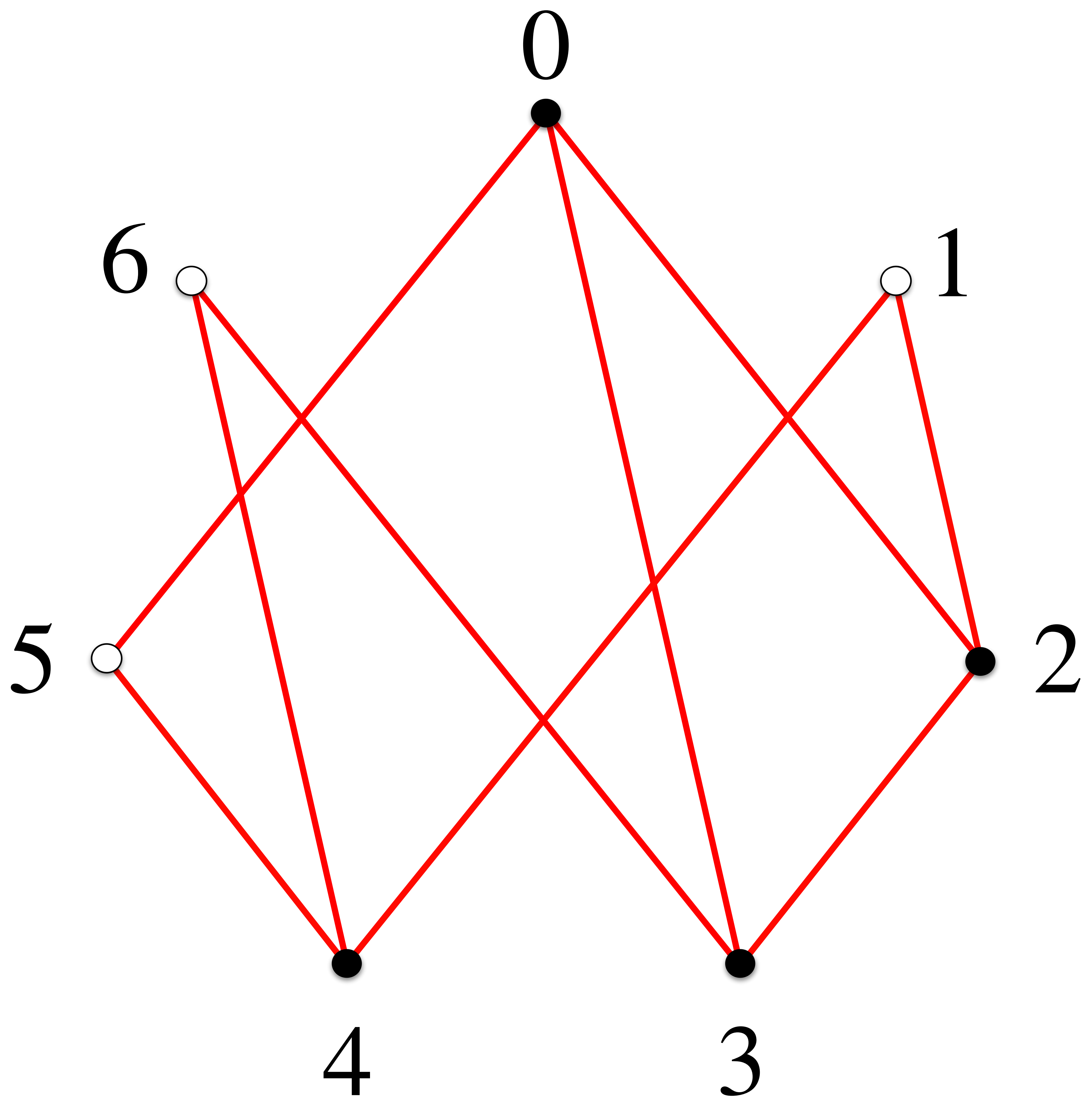
$\{2,3,5\}$ is a perfect difference set of Z_7,

$\{0,4,6\}$ is also a perfect difference set of Z_7.

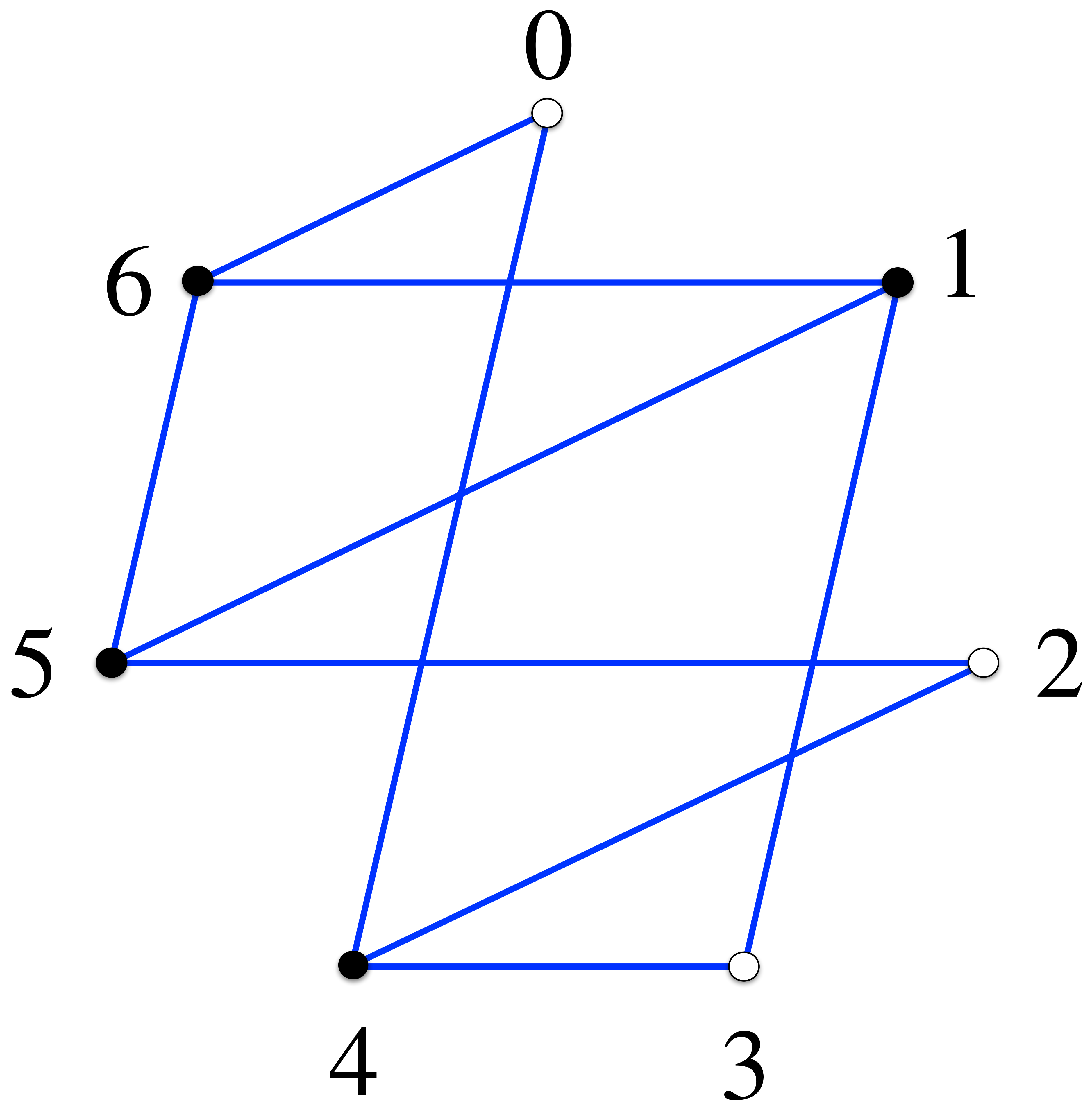
If q is a prime power, then a perfect difference set of order $q+1$ is called a *Singer difference set*.

Theorem 46 (Jungnickel). For any prime power q, there exists two disjoint Singer difference sets of order $q+1$ in Z_{q^2+q+1}.

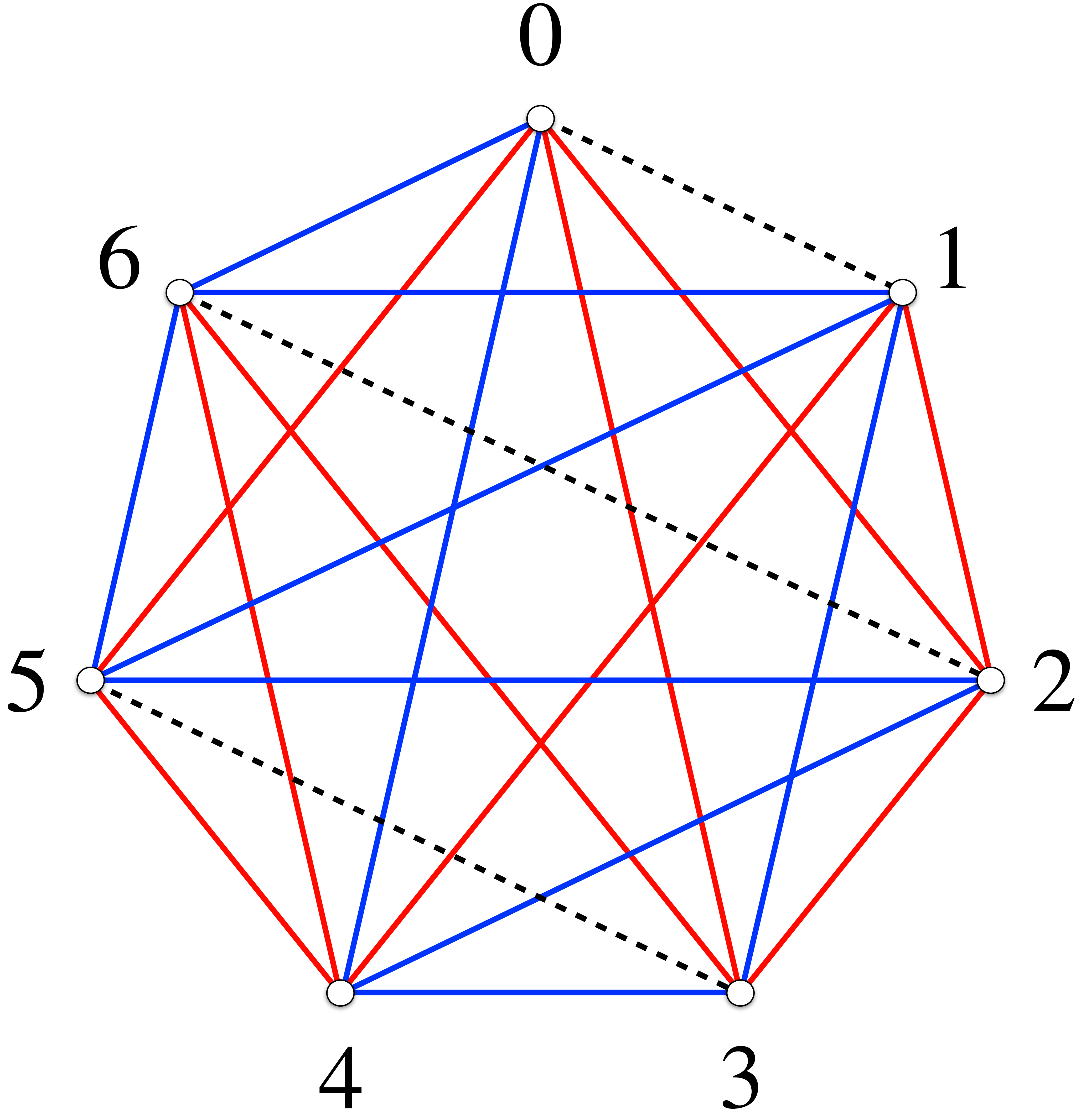
Let q be a prime power,
and D a Singer difference set of order $q+1$ in Z_{q^2+q+1}.
A polarity graph G_D is a simple graph with vertex set Z_{q^2+q+1}, two distinct vertices i and j in Z_{q^2+q+1} being adjacent if and only if $i+j$ is in D.



$\{2,3,5\}$



$\{0,4,6\}$


0
1
2
3
4
5
6

Example.

A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered each other.

At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any one of them, arrives.

Show that at least one more game can now be played.

Exercise 12.

1. A flat circular city of radius six miles is patrolled by eighteen police cars, which communicate with one another by radio. If the range of a radio is nine miles, show that, at any time, there are always at least two cars each of which can communicate with at least five others.

2. Let P_4 be a path of order 4. Determine the Turán number $ex(n, P_4)$.