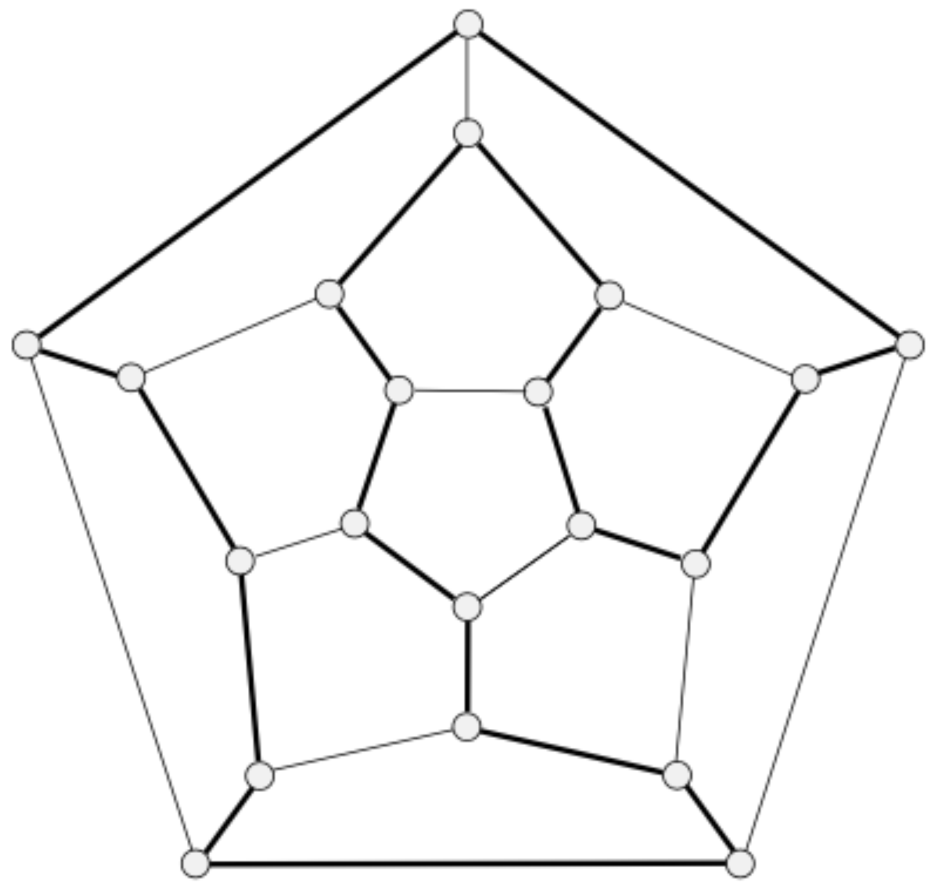
Hamilton Problems

A path or cycle in a graph which contains every vertex of the graph is called a *Hamilton path* or *Hamilton cycle* of the graph.

Such paths and cycles are named after Sir William Rowan Hamilton, who described, in a letter to his friend Graves in 1856, a mathematical game on the dodecahedron in which one person sticks pins in any five consecutive vertices and the other is required to complete the path so formed to a spanning cycle.



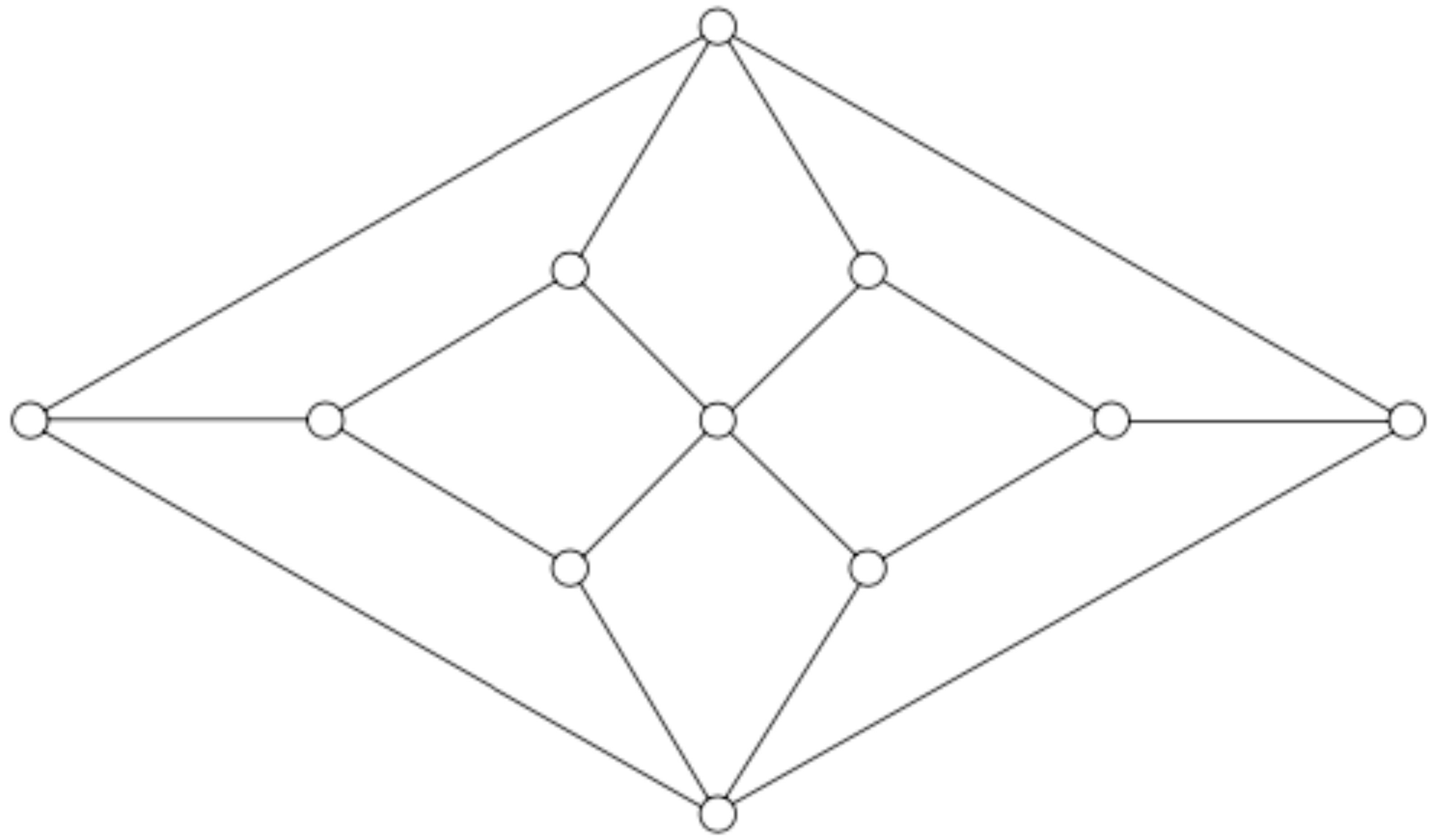
Let G be a graph and $S \subset V(G)$.

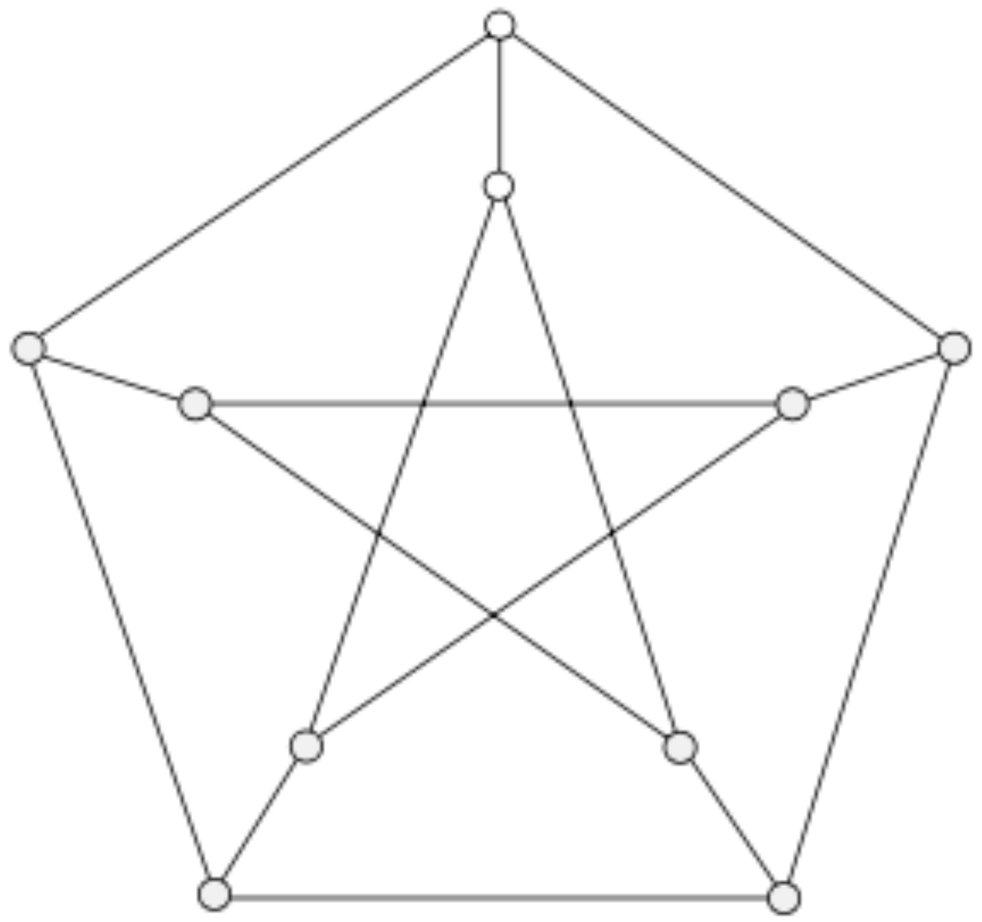
We use $\omega(G - S)$ to denote the number of components of $G - S$.

Theorem 12. Let G be a graph with a Hamilton cycle. Then for any $S \subset V(G)$,

$$\omega(G - S) \leq |S|.$$

Theorem 12 is a necessary condition for a graph to be hamiltonian.

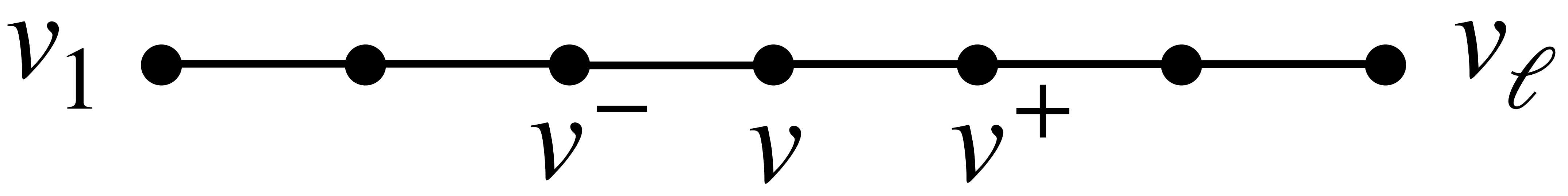




Theorem 13(Dirac). Let G be a simple graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

Proof. It is not difficult to show that G is connected (in fact, 2-connected).

Let $P = v_1 v_2 \cdots v_\ell$ be a longest path in G. For any $v \in V(P)$, let v^-, v^+ be the predecessor and successor of v, respectively.

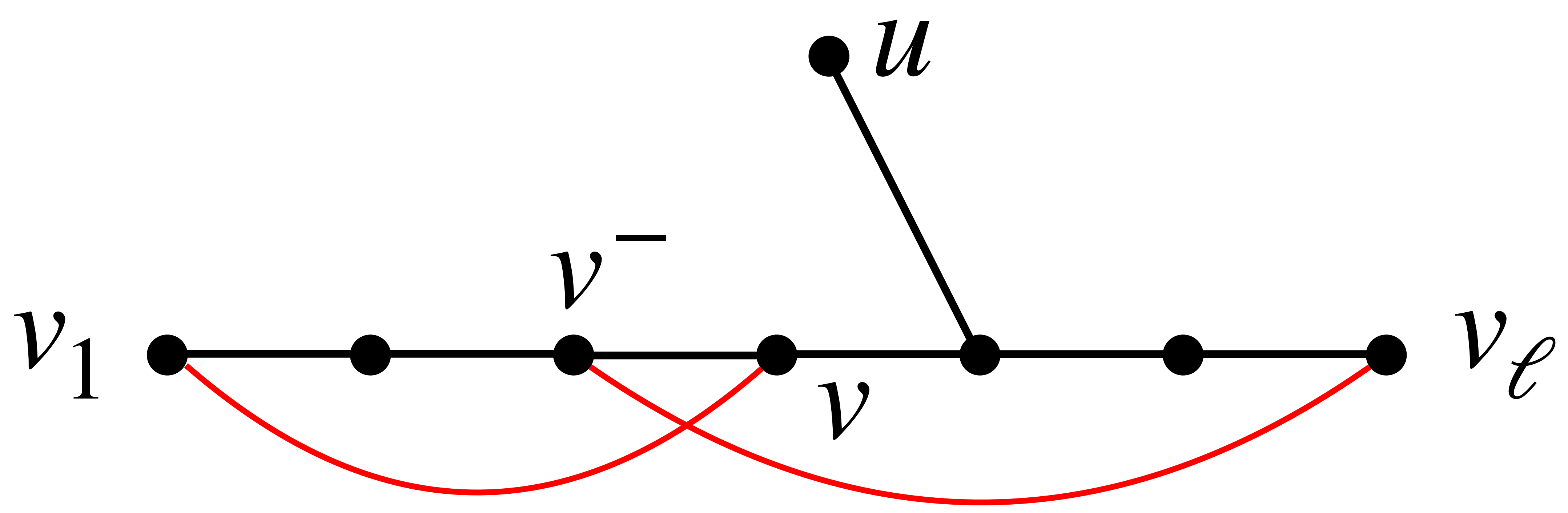


Let

$$N_P^+(v_\ell) = \left\{ v^+ : v \in N_P(v_\ell) \right\}.$$

It is not difficult to see that $N_P^+(v_\ell) \subseteq V(P)$.

If $N_P(v_1) \cap N_P^+(v_\ell) \neq \varnothing$, say $v \in N_P(v_1) \cap N_P^+(v_\ell)$, then $C = v_1 \overrightarrow{P} v^- v_\ell \overleftarrow{P} v v_1$ must be a Hamiltonian cycle, for otherwise,



If $N_P(v_1) \cap N_P^+(v_\ell) = \varnothing$, then since $\delta(G) \geq n/2$, we have

$$|P| \geq |N_P(v_1)| + |N_P^+(v_\ell)| + 1 \geq n/2 + n/2 + 1 = n + 1,$$

which is impossible.

Let G be a graph and define

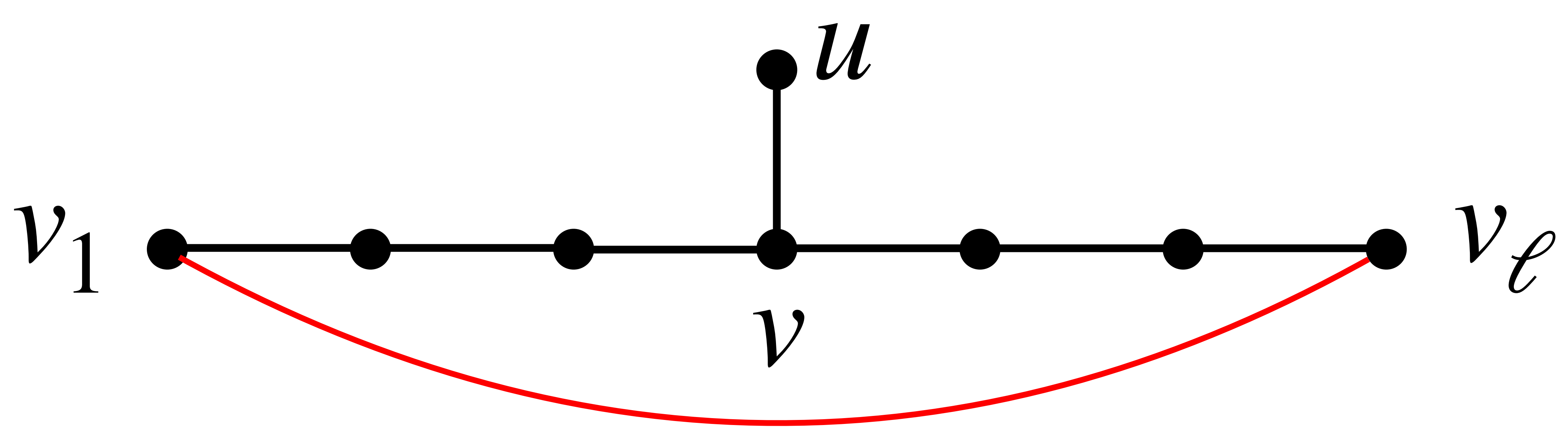
$$\sigma_2(G) = \min \left\{ d(u) + d(v) : uv \notin E(G) \right\}.$$

Theorem 14(Ore). Let G be a simple graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is Hamiltonian.

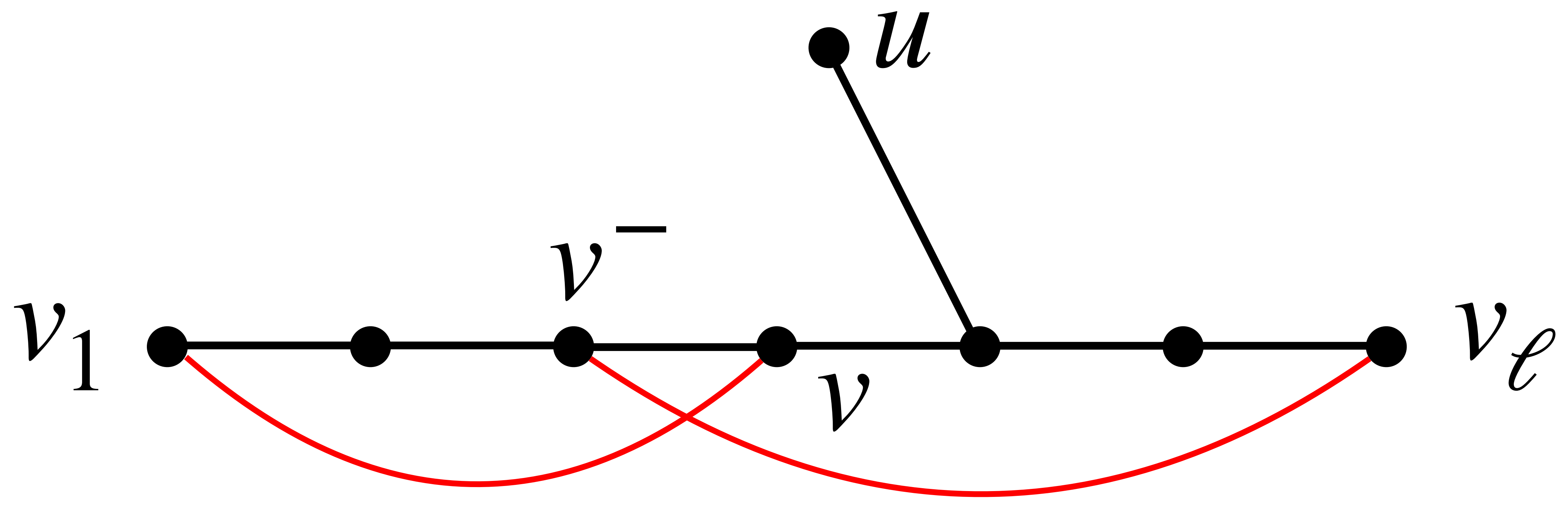
Proof. It is not difficult to show that G is connected (in fact, 2-connected).

Let $P = v_1 v_2 \cdots v_\ell$ be a longest path in G. We still use the notations in Theorem 13.

If $v_1 v_\ell \in E(G)$, then $C = v_1 v_2 \cdots v_\ell v_1$ must be a Hamiltonian cycle, otherwise,



Suppose $v_1 v_\ell \notin E(G)$. If $N_P(v_1) \cap N_P^+(v_\ell) \neq \emptyset$, say $v \in N_P(v_1) \cap N_P^+(v_\ell)$, then $C = v_1 \overrightarrow{P} v^- v_\ell \overleftarrow{P} v v_1$ must be a Hamiltonian cycle, for otherwise,



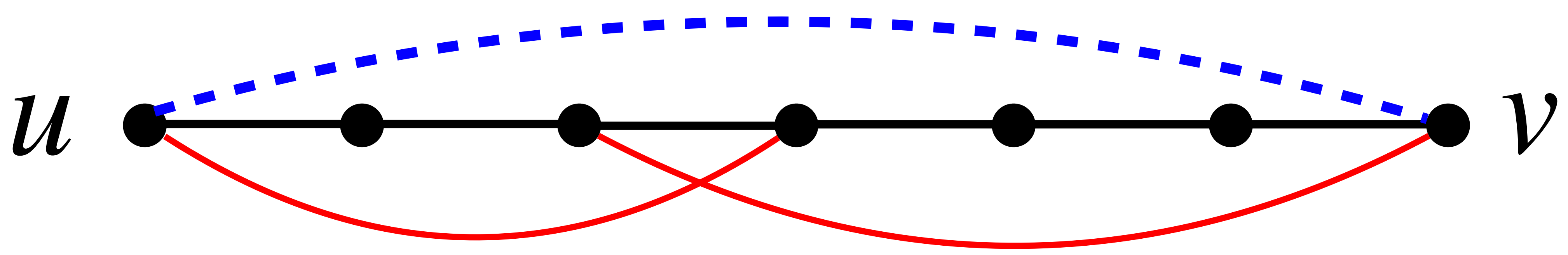
If $N_P(v_1) \cap N_P^+(v_\ell) = \emptyset$, then since $d(v_1) + d(v_\ell) \geq \sigma_2(G) \geq n$, we have

$$|P| \geq |N_P(v_1)| + |N_P^+(v_\ell)| + 1 \geq n + 1,$$

a contradiction.

Lemma 1. Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq n$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

Proof. ($\Rightarrow$) If G is hamiltonian, so too is $G + uv$.
($\Leftarrow$) Suppose that $G + uv$ has a Hamilton cycle C.
If C does not contain the edge uv, then C is a Hamilton cycle of G.
If C does contain the edge uv, then $P = C - uv$ is a Hamilton path of G starting at u ending at v. Obviously, P is a longest path of G such that u and v are nonadjacent satisfying $d(u) + d(v) \geq n$.

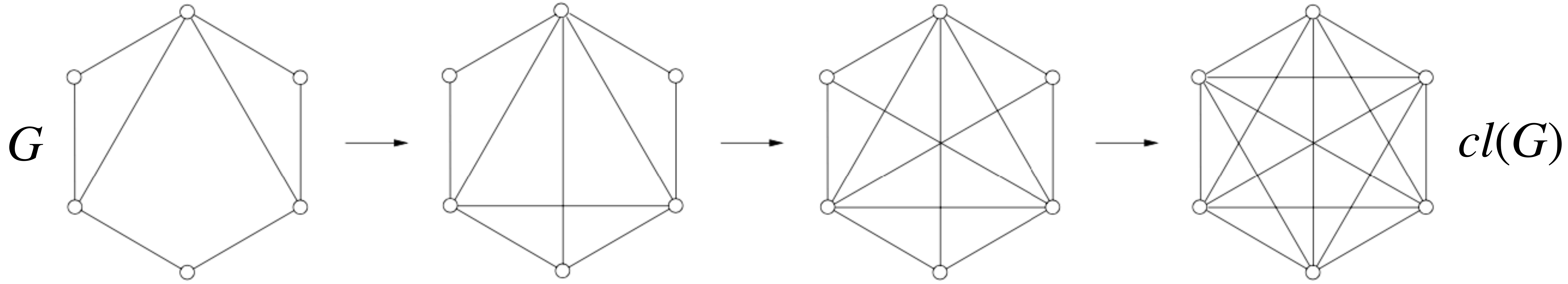


Then, as in the proof of Theorem 14, we can show that G is Hamiltonian.

Let G be a graph of order n. The *closure* of G, write as $cl(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices, whose degree sum is at least n, until no such pair remains.

Lemma 2. The closure of a graph is well defined.

Theorem 15. A simple graph G is Hamiltonian if and only if its closure $cl(G)$ is Hamiltonian.

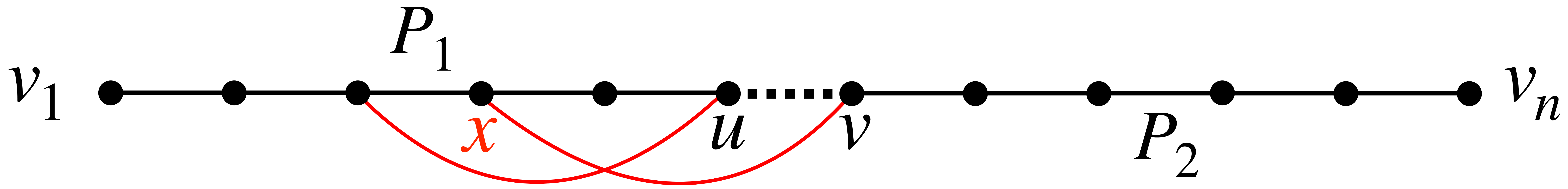


Lemma 3. Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq n - 1$. Then G has hamiltonian path if and only if $G + uv$ has a hamiltonian path.

Proof. $(\Rightarrow)$ If G has a hamiltonian path, so too is $G + uv$.
$(\Leftarrow)$ Suppose that $G + uv$ has a Hamiltonian path $P = v_1 v_2 \cdots v_n$.
If P does not contain the edge uv, then P is a Hamiltonian path of G.
If P contains uv, let $u = v_i$ and $v = v_{i+1}$, $P_1 = v_1 \cdots v_i$ and $P_2 = v_{i+1} \cdots v_n$.

If $N_{P_1}(v) \cap N^+_{P_1}(u) \neq \emptyset$, say $x \in N_{P_1}(v) \cap N^+_{P_1}(u)$,

then $P' = v_1 \overrightarrow{P} x^- u \overleftarrow{P} x v \overrightarrow{P} v_n$ is a Hamiltonian path of G, so the result follows.

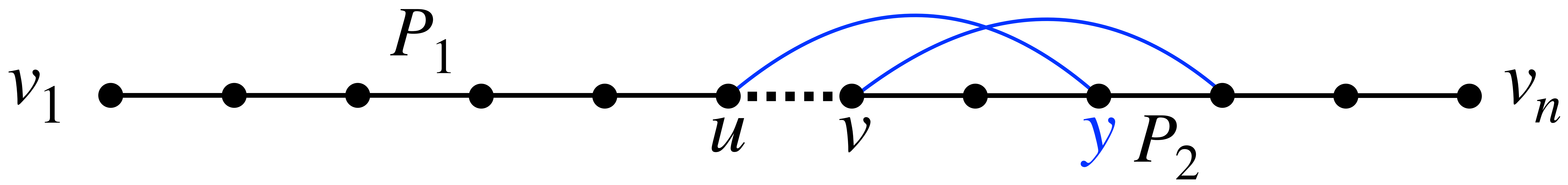


If $N_{P_1}(v) \cap N^+_{P_1}(u) = \varnothing$, then note that $v_1 \notin N_{P_1}(v) \cup N^+_{P_1}(u)$, we have

$$d_{P_1}(u) + d_{P_1}(v) = |N_{P_1}(v)| + |N^+_{P_1}(u)| \le |P_1| - 1.$$

If $N_{P_2}(u) \cap N^-_{P_2}(v) \neq \varnothing$, say $y \in N_{P_2}(u) \cap N^-_{P_2}(v)$,

then $P'' = v_1 \overrightarrow{P} u y \overleftarrow{P} v y^+ \overrightarrow{P} v_n$ is a Hamiltonian path of G, so the result follows.



If $N_{P_2}(u) \cap N^-_{P_2}(v) = \varnothing$, then note that $v_n \notin N_{P_2}(u) \cup N^-_{P_2}(v)$, we have

$$d_{P_2}(u) + d_{P_2}(v) = |N_{P_2}(u)| + |N^-_{P_2}(v)| \le |P_2| - 1.$$

Thus, we have $d(u) + d(v) \le |P_1| + |P_2| - 2 \le n - 2$, a contradiction.

Let G be a graph. If $S \subset V(G)$ and $vv' \notin E(G)$ for any two vertices $v, v' \in S$, then S is called an *independent set*. Define

$$\alpha(G) = \max\{\,|S| : S \text{ is an independent set of } G\},$$

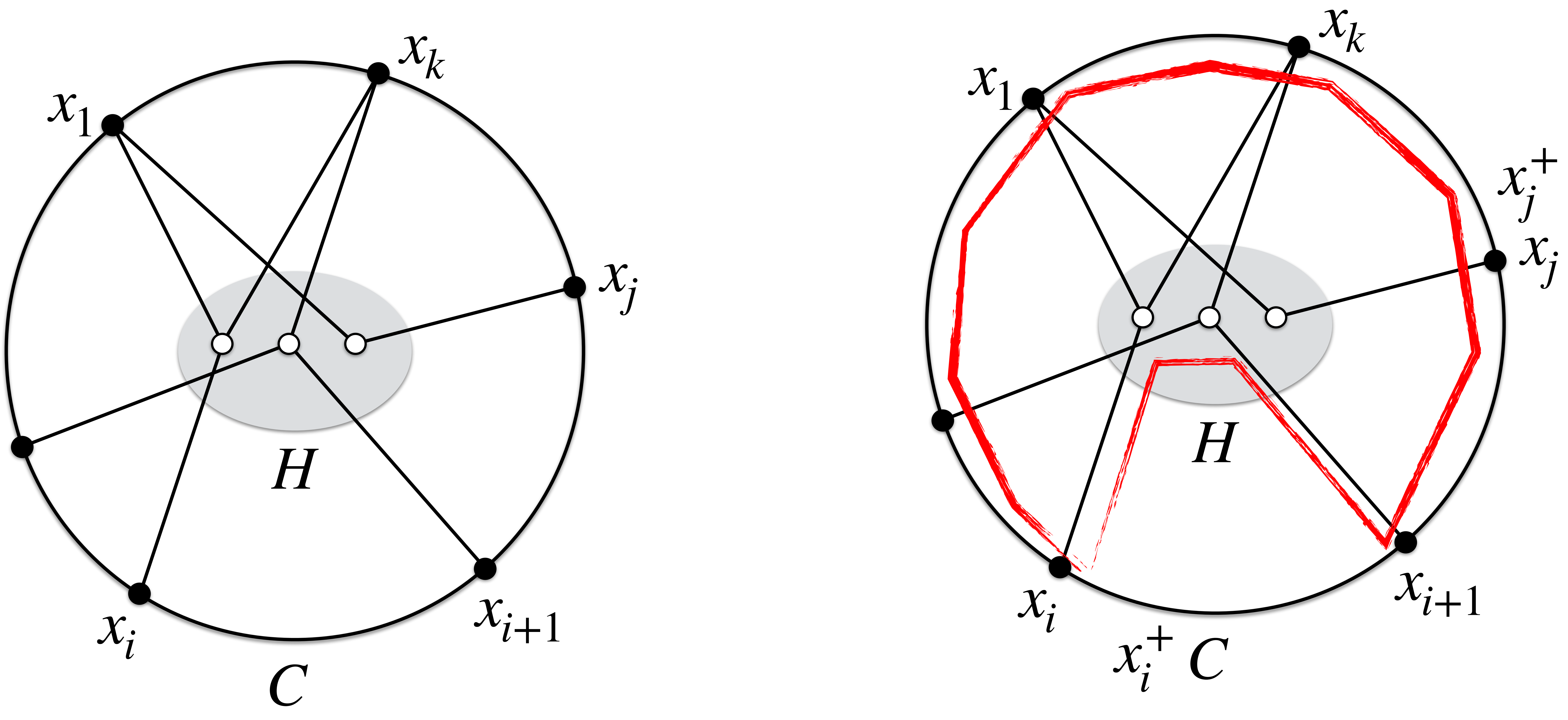
called *independence number* (独立数) of G.

Theorem 16(Chvátal and Erdős). Let G be a simple graph of order at least 3. If $\alpha(G) \le \kappa(G)$, then G is Hamiltonian.

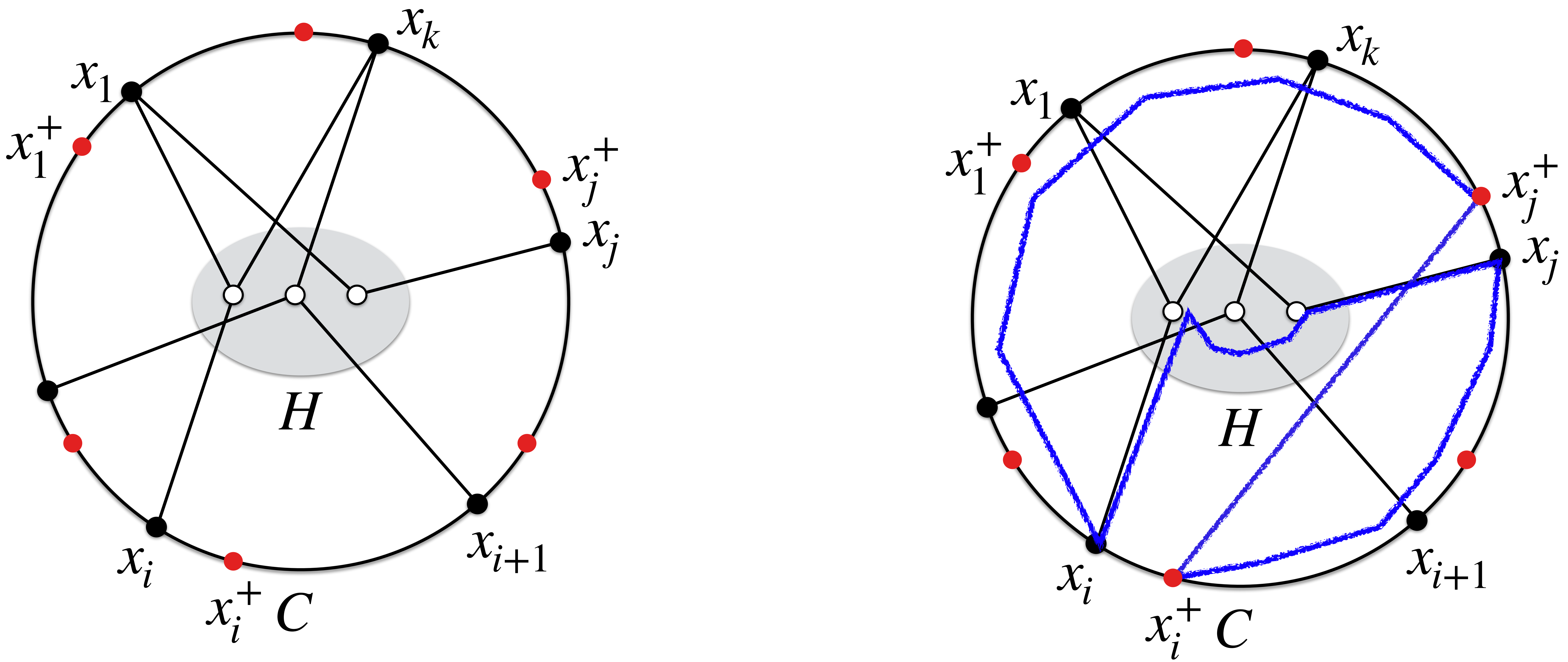
Proof. Let C be a longest cycle of G. Suppose to the contrary that C is not Hamiltonian.

Let H be any component of $G - V(C)$, and all neighbors of H in C are $x_1, x_2, \ldots, x_k$, which occur along a given orientation of C. For any $v \in V(C)$, let v^+ be the successor of v along the given orientation of C.

Let x_iHx_j be a longest (x_i, x_j)-path with internal vertices in H.

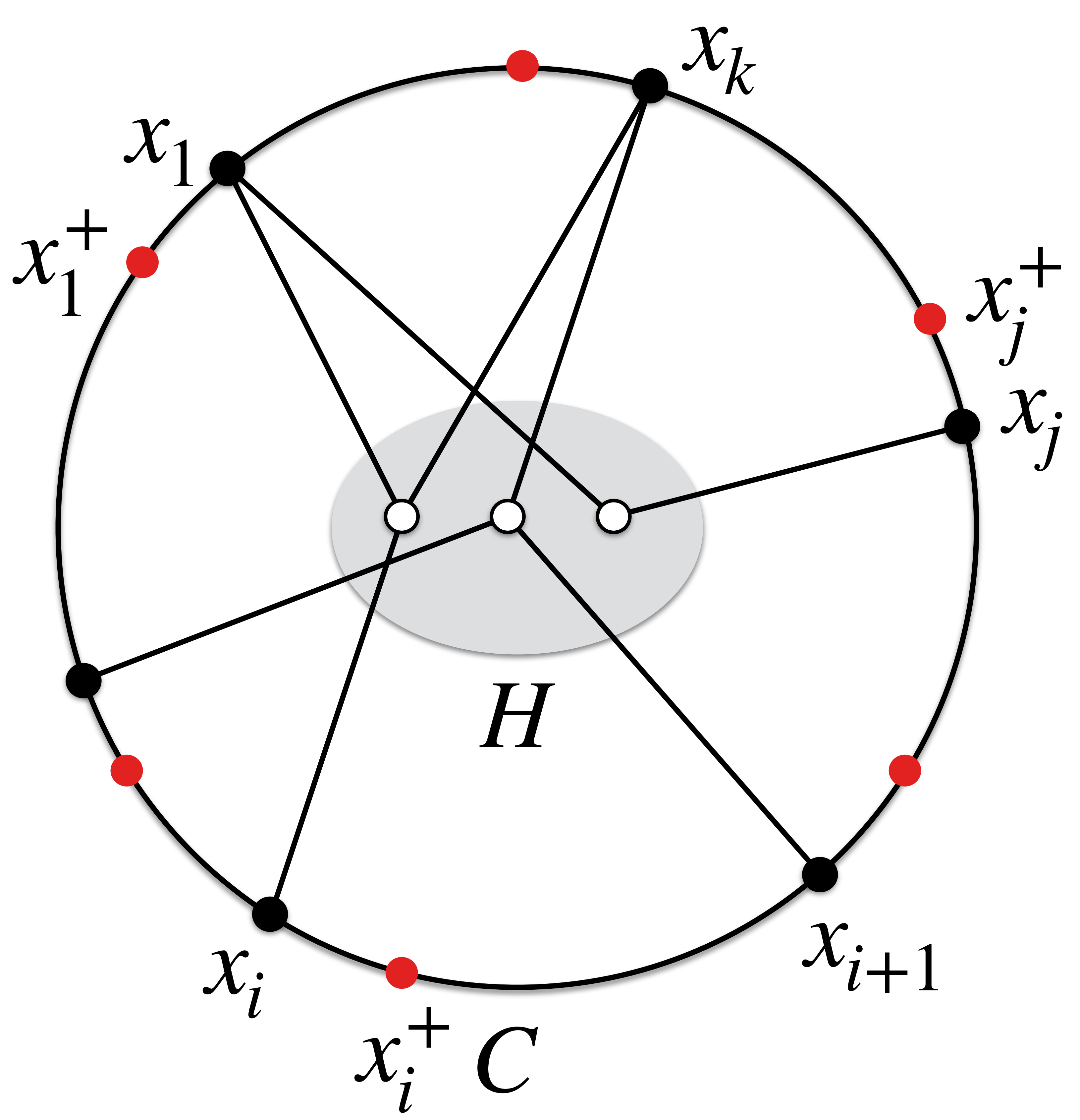


Because C is a longest cycle of G, x_i and x_{i+1} cannot be consecutive vertices in C, for otherwise, $x_iHx_{i+1}\overrightarrow{C}x_i$ is a cycle longer than C.



By the maximality of C, we can see $x_1^+, x_2^+, \ldots, x_k^+$ is an independent set in G.

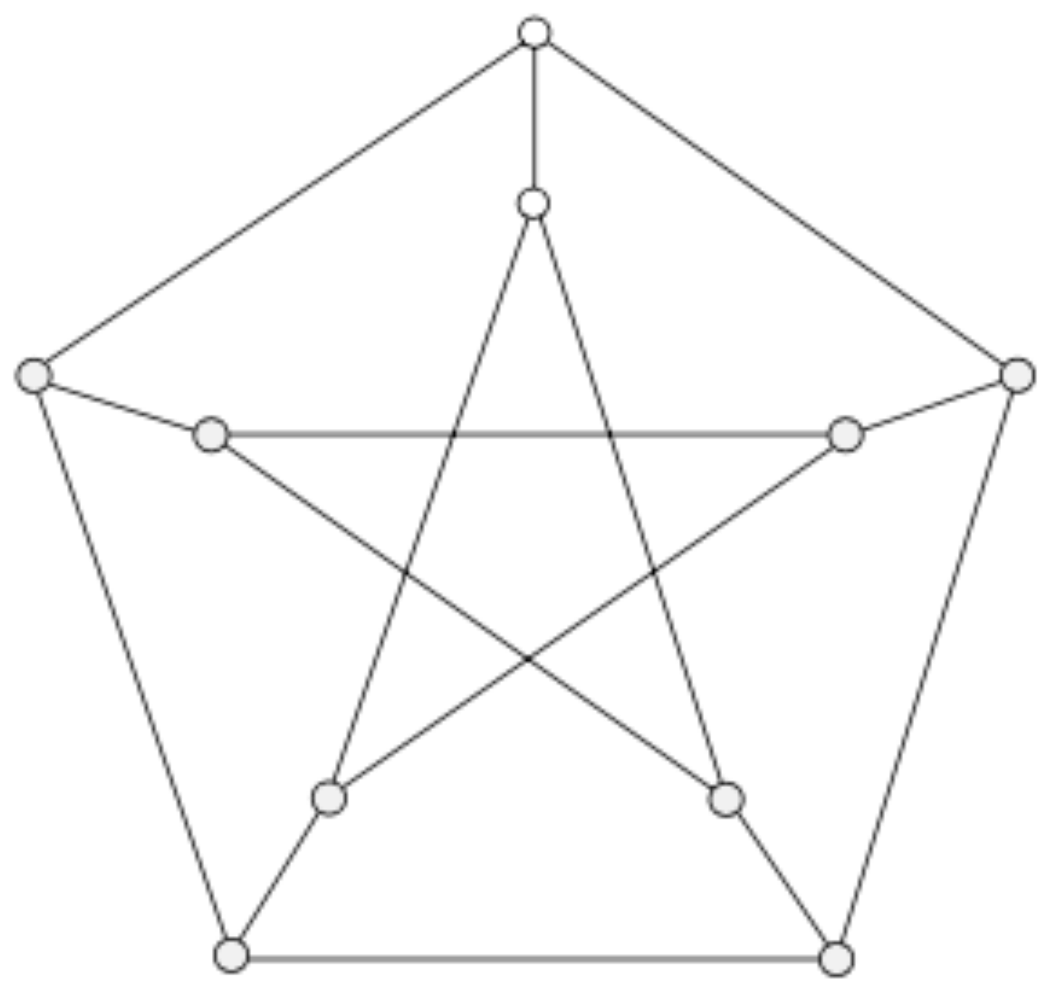
If $x_i^+ x_j^+ \in E(G)$, then $x_i^+ \overrightarrow{C} x_j H x_i \overleftarrow{C} x_j^+ x_i^+$ is a cycle longer than C.



Clearly, $S = \{x_1, x_2, \ldots, x_k\}$ is a cut set of G, and $\{h, x_1^+, x_2^+, \ldots, x_k^+\}$ is an independent set for any $h \in V(H)$. This implies $\kappa(G) \leq k$ and $\alpha(G) \geq k+1$ which contradicts the assumption that $\alpha(G) \leq \kappa(G)$.

Exercise 4.

1. Show that Petersen graph has no Hamilton cycle.



2. Let G be a graph and let H be the graph obtained from G by adding a new vertex and joining it to every vertex of G. Show that H is hamiltonian if and only if G has a Hamiltonian path.