Vertex Colorings of Graphs

Let G be a graph. A k-coloring of G is a mapping

$$c : V(G) \mapsto \{1,2,...,k\} .$$

A coloring c is *proper* if no two adjacent vertices are assigned the same color.

A graph G is *k-colorable* (k-可染的) if G has a proper k-coloring.

The minimum integer k for which a graph G is k-colorable is called the *chromatic number* (色数) of G, and denoted $\chi(G)$.

If $\chi(G) = k$, the graph G is said to be *k-chromatic* (k-色的) .

Example 1. Examination Scheduling

The students at a university have annual examinations in all the courses they take. Naturally, examinations in different courses cannot be held concurrently if the courses have students in common.

How can all the examinations be organized in as few parallel sessions as possible?

To find such a schedule, consider the graph G whose vertex set is the set of all courses, two courses being joined by an edge if they give rise to a conflict. Clearly, independent sets of G correspond to conflict-free groups of courses. Thus, the required minimum number of parallel sessions is the chromatic number of G.

Example 2. Chemical Storage

A company manufactures n chemicals $A_1, A_2, \ldots, A_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals A_i and A_j are incompatible.

It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

Let G be a graph.

How to find a proper coloring of G using as few colors as possible?

The Greedy Coloring Heuristic Algorithm

1. Arrange the vertices of G in a linear order:

$$v_1, v_2, \ldots, v_n .$$

2. Color the vertices one by one in this order, assigning to v_i the smallest positive integer not yet assigned to one of its already-colored neighbors.

Let $\Delta(G) = \Delta$. One can check the number of colors used by the greedy heuristic is no more than $\Delta + 1$, regardless of the order in which the vertices are presented.

Properties of 2-connected Graphs

Theorem 17. A graph G with at least three vertices is 2-connected if and only if any two vertices are connected by at least two internally disjoint paths.

Proof. If any two vertices are connected by at least two internally disjoint paths, then G is connected and has no cut vertex. Hence G is 2-connected.

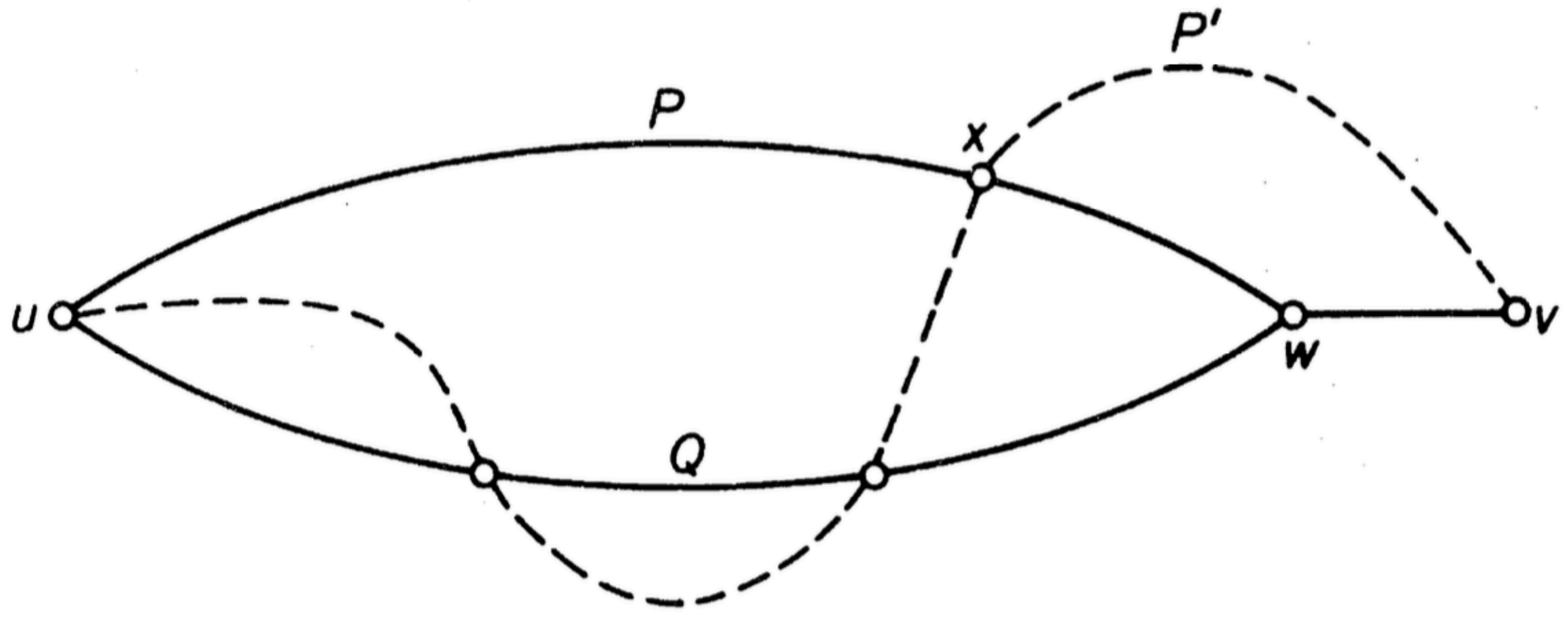
Let G be a 2-connected graph.
We shall prove, by induction on the distance $d(u, v)$ between u and v, that any two vertices u and v are connected by at least two internally disjoint paths.

If $d(u, v) = 1$, then since G is 2-connected, the edge uv is not a cut edge.
Thus, $G - uv$ is connected and there is a path in $G - uv$ connecting u and v.
It follows that u and v are connected by two internally disjoint paths in G.

Assume that the result holds for any two vertices at distance less than k, and let $d(u,v) = k \geq 2$.

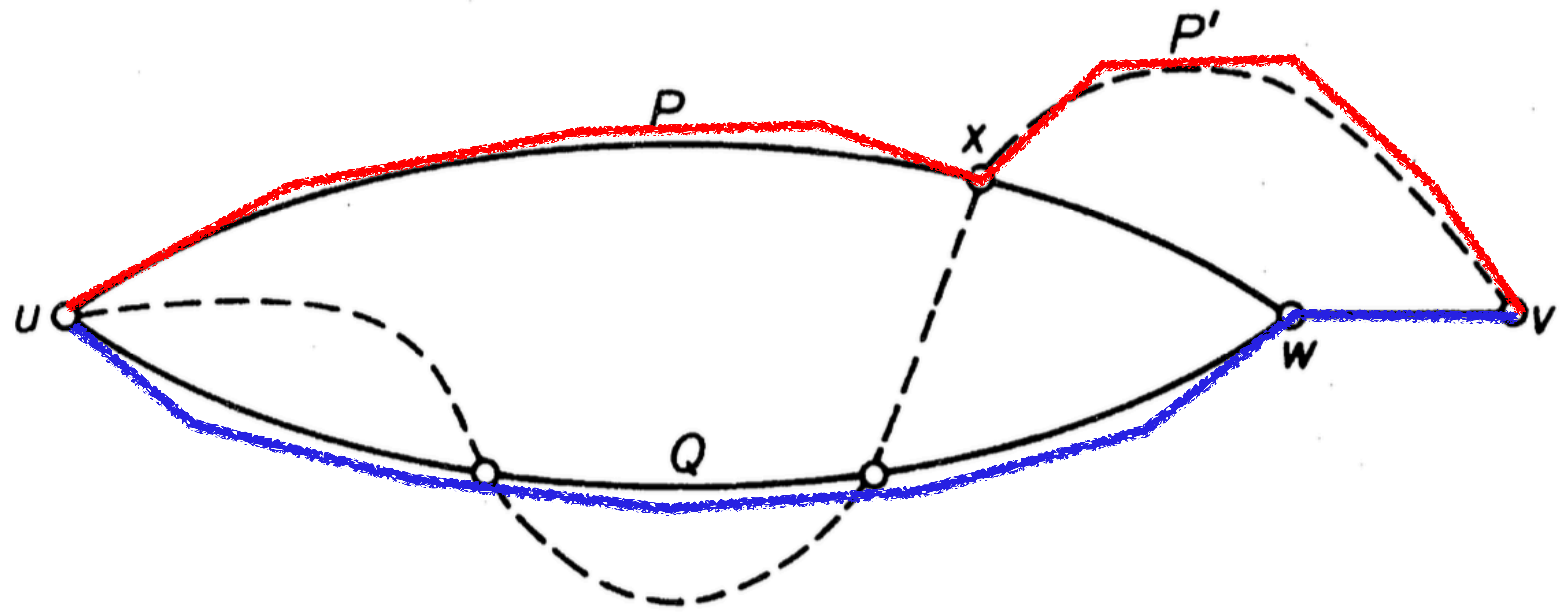
Consider a (u,v)-path of length k, and let w be the vertex that precedes v on this path. Since $d(u,w) = k-1$, by the induction hypothesis, there are two internally disjoint (u,w)-paths P and Q in G.

Also, since G is 2-connected, $G - w$ is connected and so contains a (u,v)-path P'. Let x be the last vertex of P' that is also in $P \cup Q$.



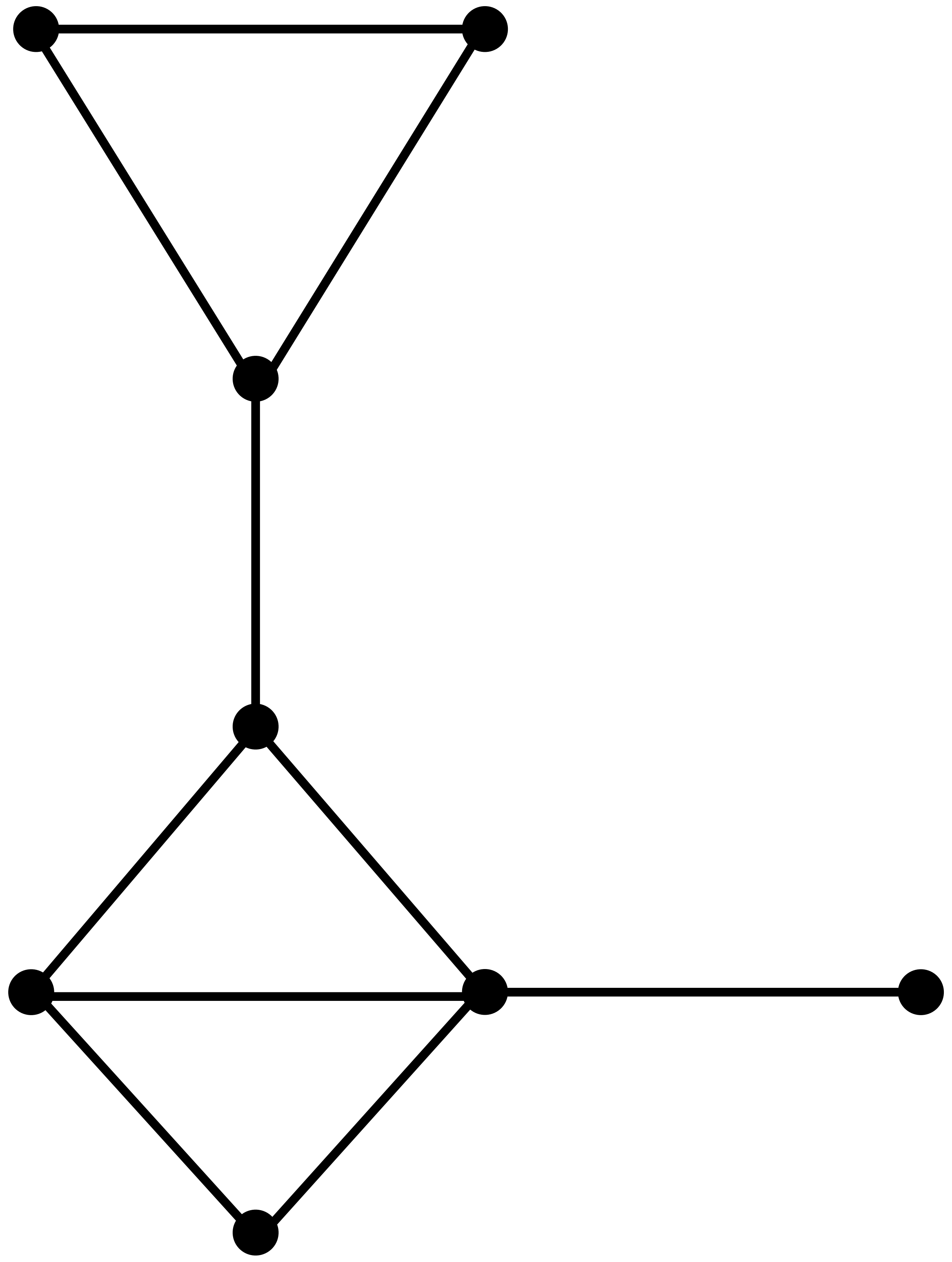
Since u is in $P \cup Q$, there is such an x (possibly $x = u$).

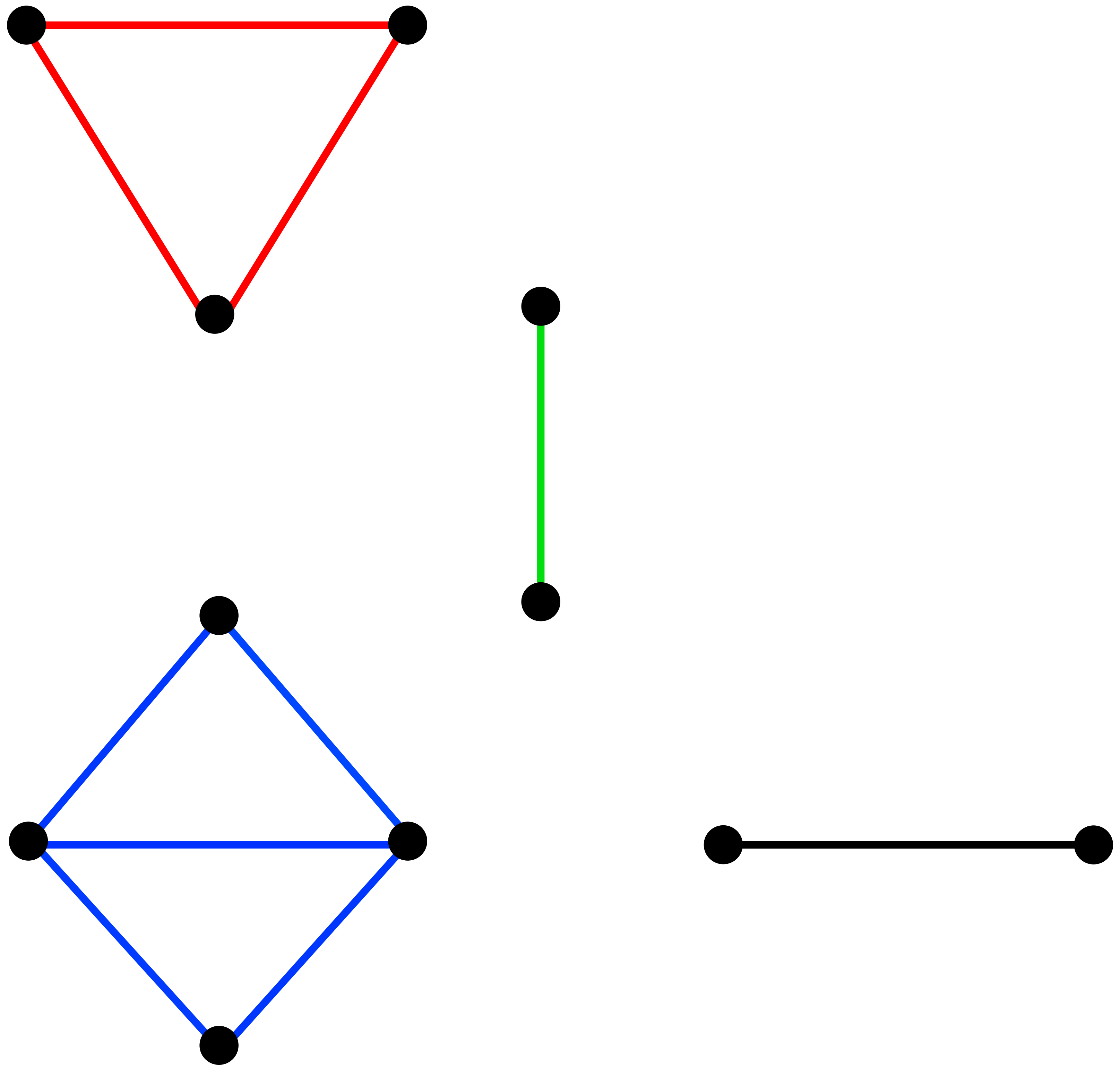
Assume without loss of generality that x is in P. Then $u\overrightarrow{P}x\overrightarrow{P'}v$ and $u\overrightarrow{Q}wv$ are two internally disjoint (u, v)-paths in G.



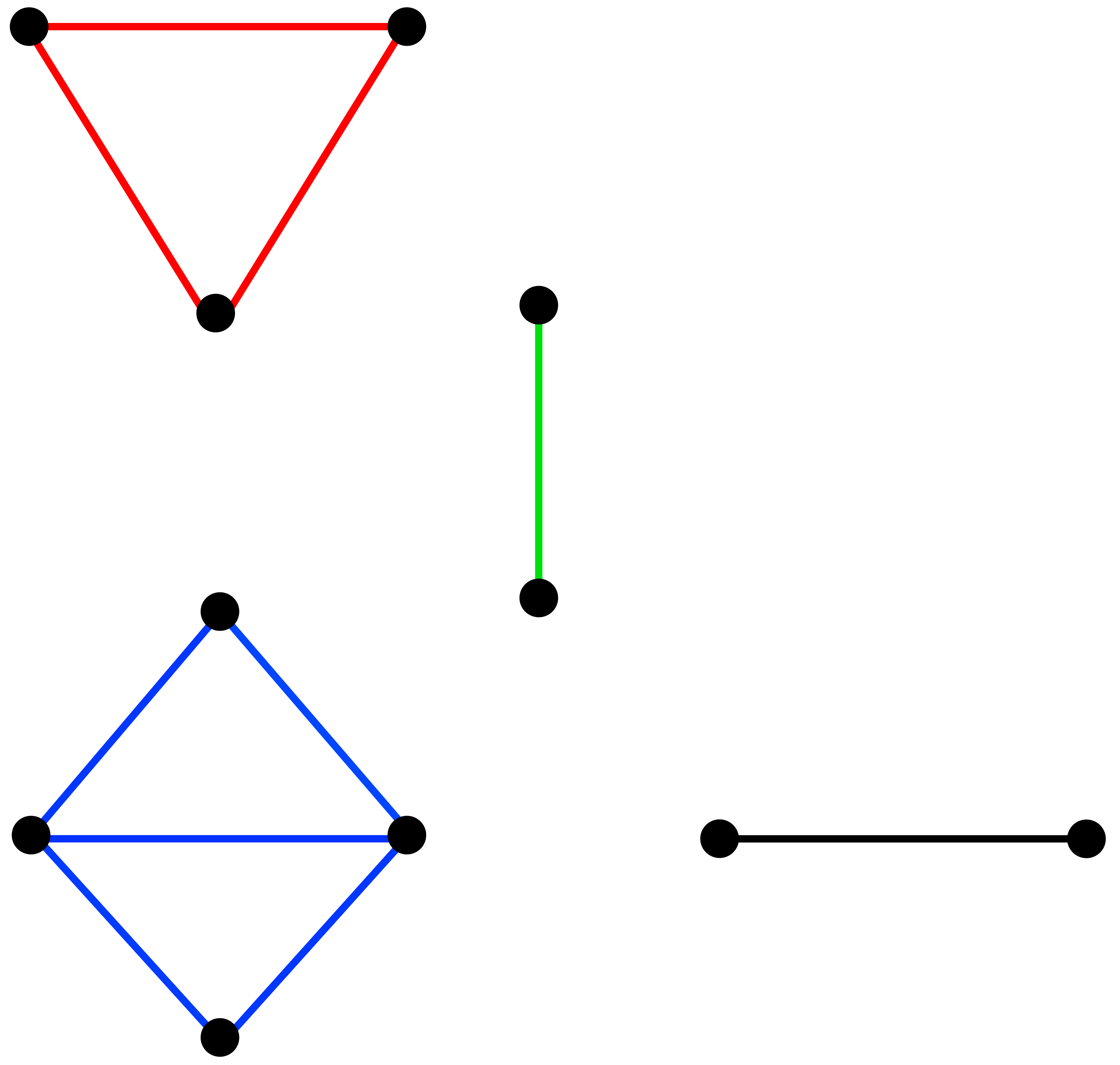
Corollary 1. If G is 2-connected, then any two vertices of G lie on a common cycle.

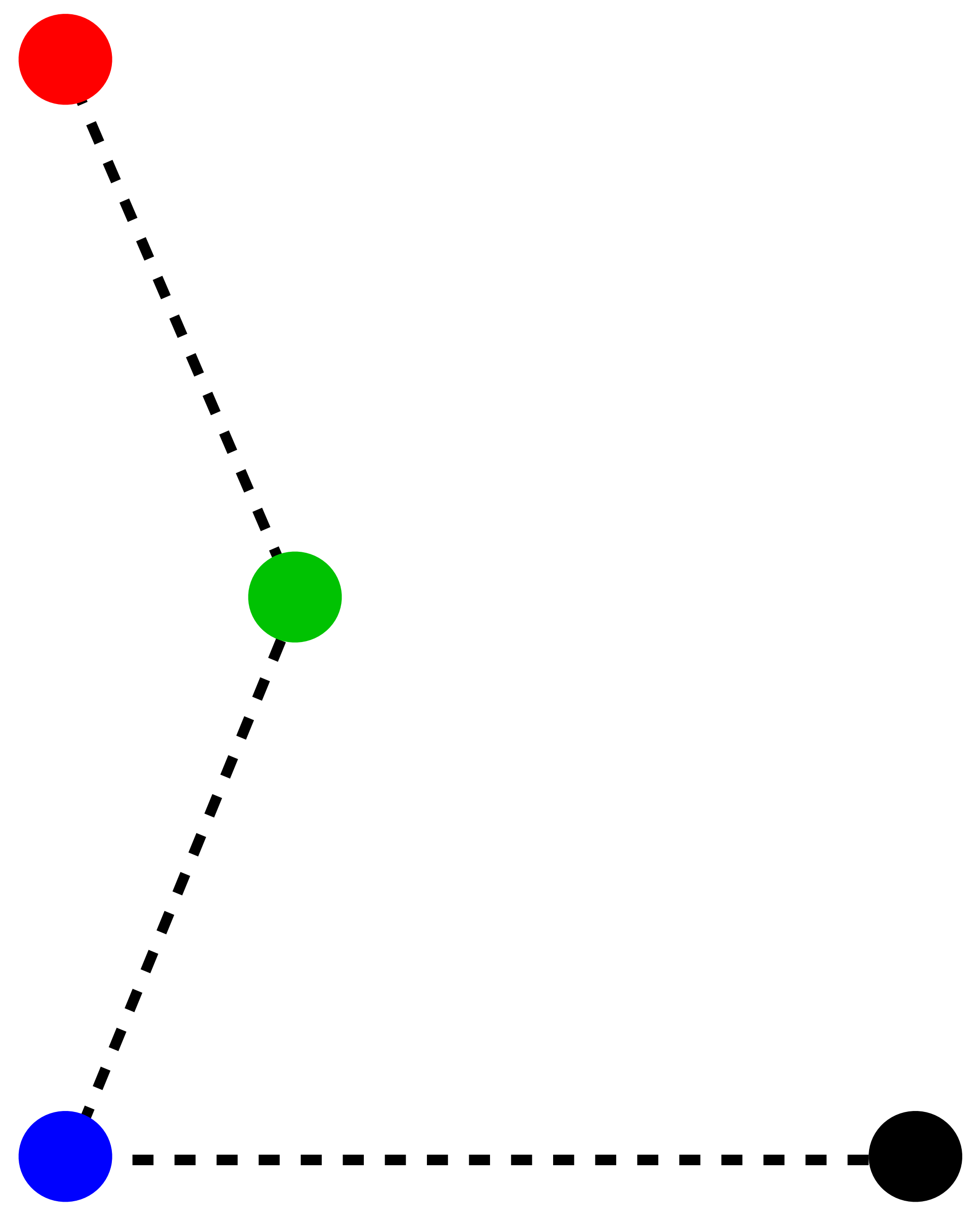
A connected graph that has no cut vertices is called a block. Every block with at least three vertices is 2-connected. A block of a graph is a subgraph that is a block and is maximal with respect to this property.



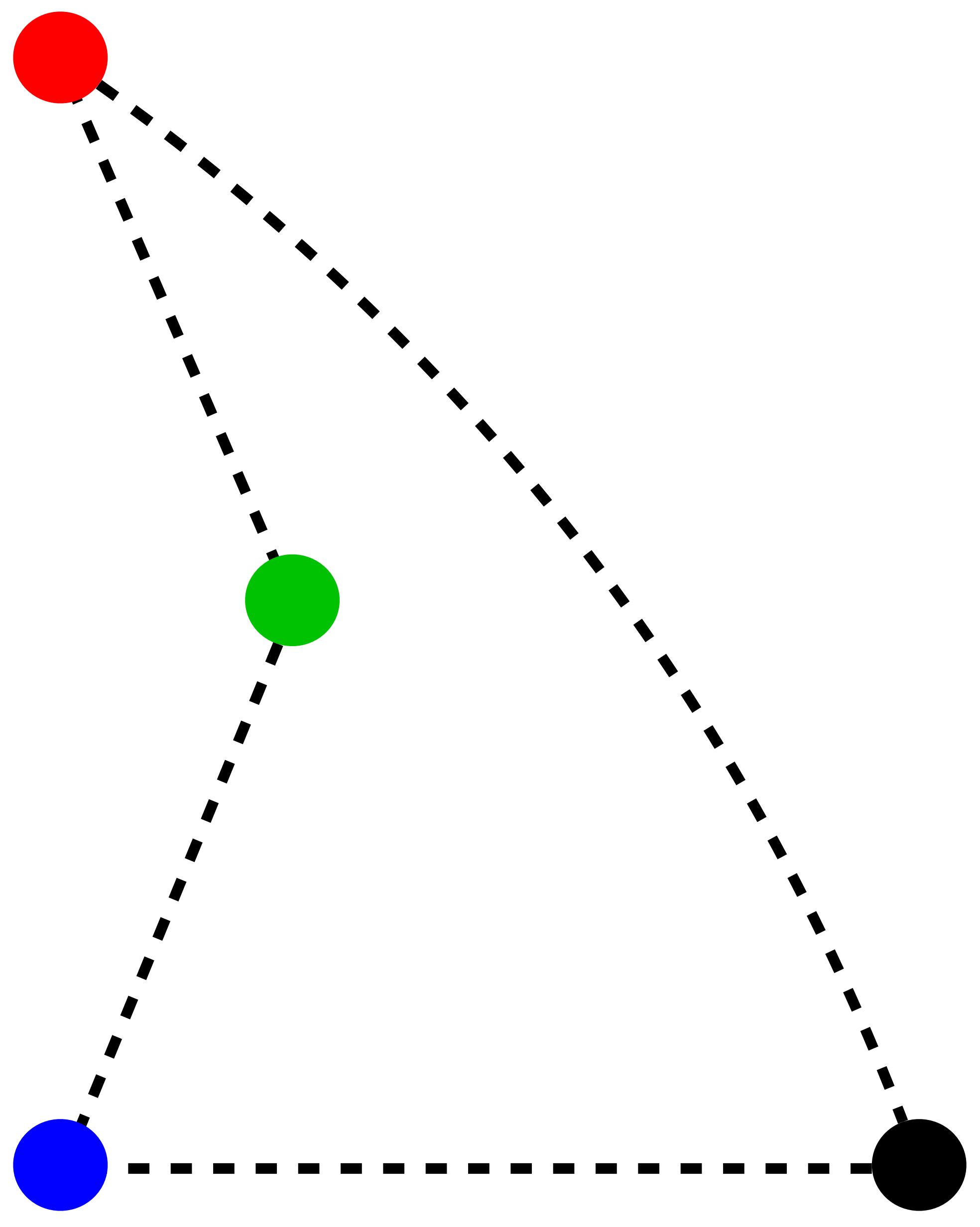


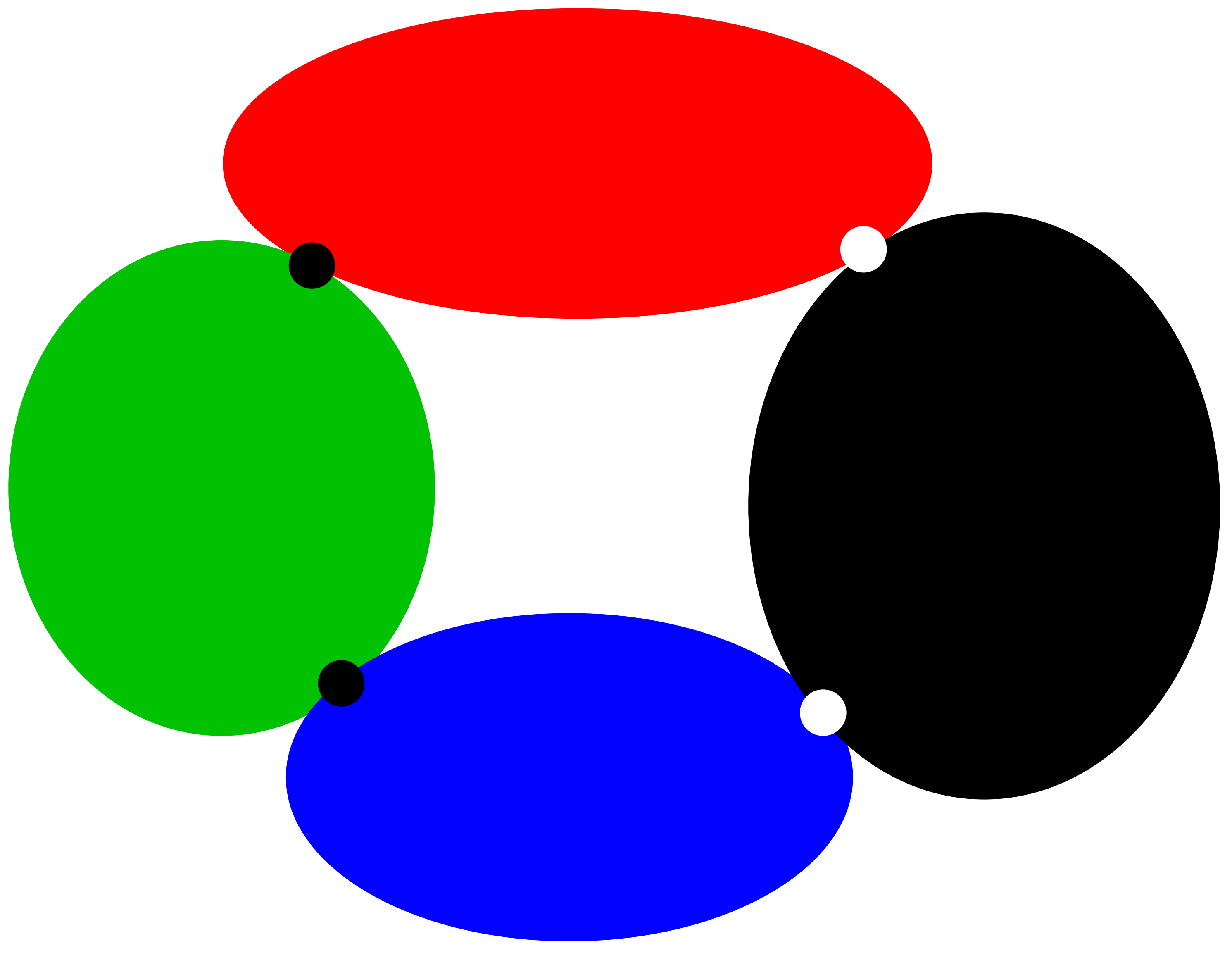
Let G be a connected graph containing cut vertices, and $H_1, H_2, \ldots, H_k$ be all blocks of G. Now, take $H_1, H_2, \ldots, H_k$ as vertices, H_i and H_j is adjacent if and only if H_i and H_j sharing a common cut vertex. Such a graph $B(G)$ is called block-cut vertex graph of G.





Proposition 5. Let G be a connected graph. Then $B(G)$ is a tree.





If a connected graph contains cut vertices, then it contains at least two blocks. The block contains exactly one cut vertex is called an end-block.

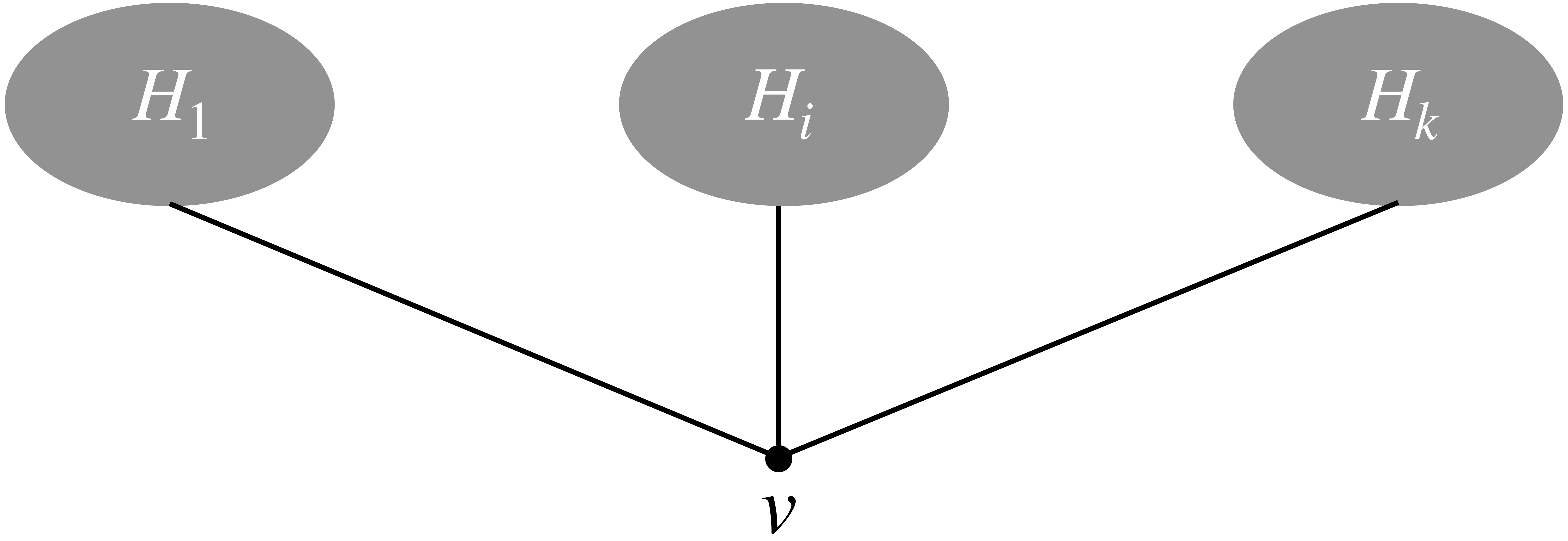
By Proposition 5, if a connected graph contains cut vertices, then it contains at least two end-blocks.

Theorem 18(Brooks). Let G be a connected graph with $\Delta(G) = \Delta$. If G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.

Proof. If G is not Δ-regular, say $d(x_n) < \Delta$, then since G is connected, G has a spanning tree T rooted at x_n. This implies the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that each v_i $(1 \leq i \leq n-1)$ has at least one neighbor in $\{v_{i+1}, \ldots, v_n\}$. By the greedy coloring heuristic algorithm, $\chi(G) \leq \Delta$.

Assume that G is Δ-regular and $\Delta \geq 3$.

If G contains a cut vertex v, let $H_1, \ldots, H_\ell$ be all components of $G - v$, and let G_i be the subgraph of G induced by the $V(H_i) \cup \{v\}$ for $1 \leq i \leq k$.
It is obvious that $d_{G_i}(v) < \Delta$ for $1 \leq i \leq k$, which implies that each G_i is not Δ-regular, and so $\chi(G_i) \leq \Delta$. Thus, $\chi(G) \leq \Delta$.

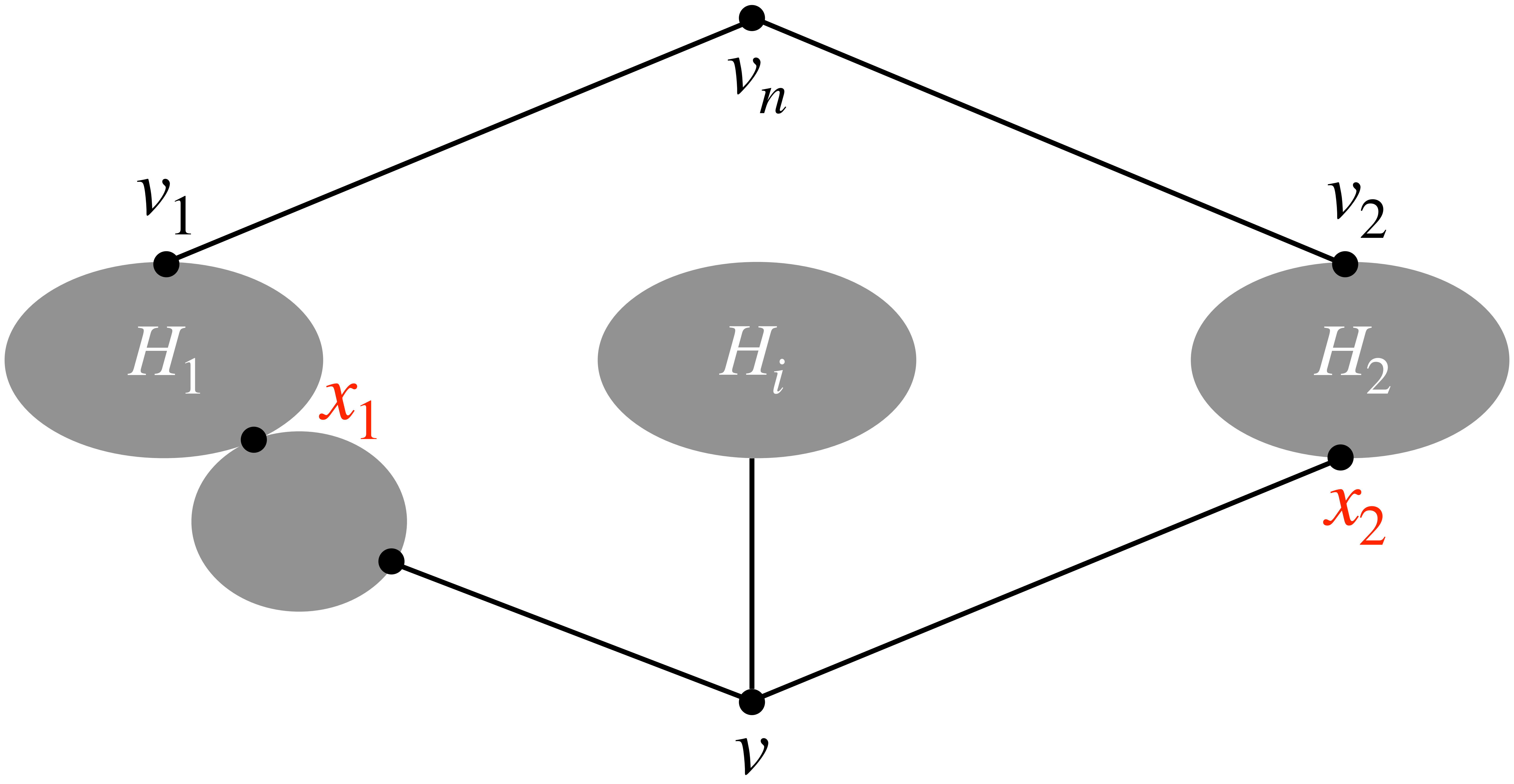


So we may assume that G is 2-connected.

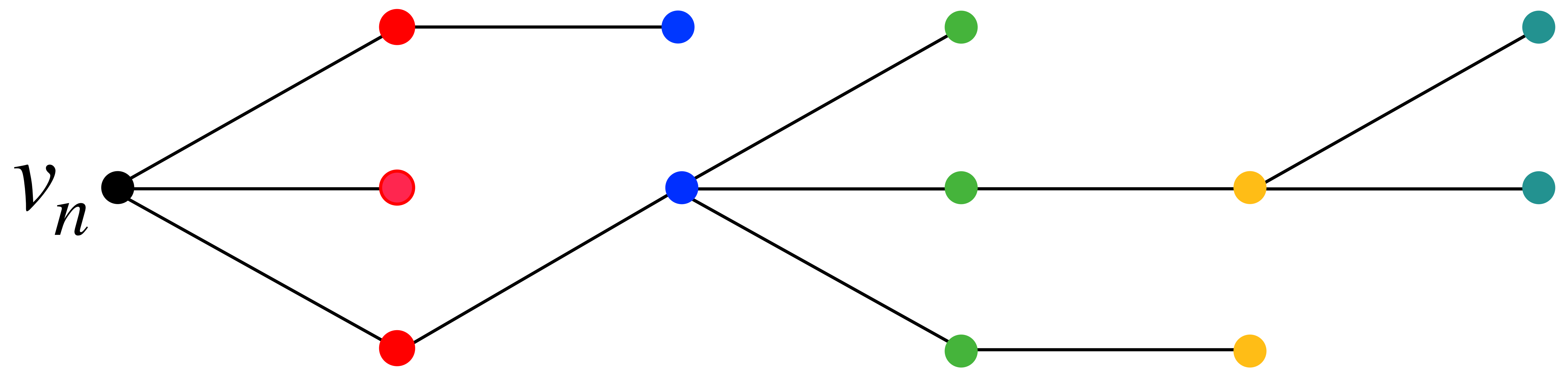
If G is 3-connected, then since G is not a complete graph, v_n must have two nonadjacent neighbors v_1, v_2 such that $G - \{v_1, v_2\}$ is connected.

If $\kappa(G) = 2$, then we assume v_n is contained in some minimum cut set.

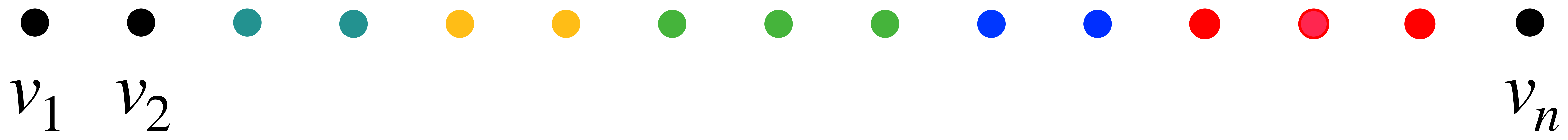
Thus, $G - v_n$ is connected and contains cut vertices.
Assume that H_1, H_2 are two end-blocks of $G - v_n$, x_i is the unique cut vertex in H_i, $i = 1{,}2$. Since G is 2-connected, v_n has a neighbor v_i in $H_i - x_i$, $i = 1{,}2$.



It is clear that v_1, v_2 are nonadjacent and $G - \{v_1, v_2\}$ is connected.
Let T be a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n:



Order the vertices of G as the following:



In this order, by the greedy coloring heuristic algorithm, $\chi(G) \leq \Delta$.

Proposition 6. Let G be a connected graph of order n. Then

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Critical Graphs

A graph G is called color-critical, abbreviate to critical, if $\chi(H) < \chi(G)$ for every proper subgraph H of G.

Such graphs were first investigated by Dirac in 1951.

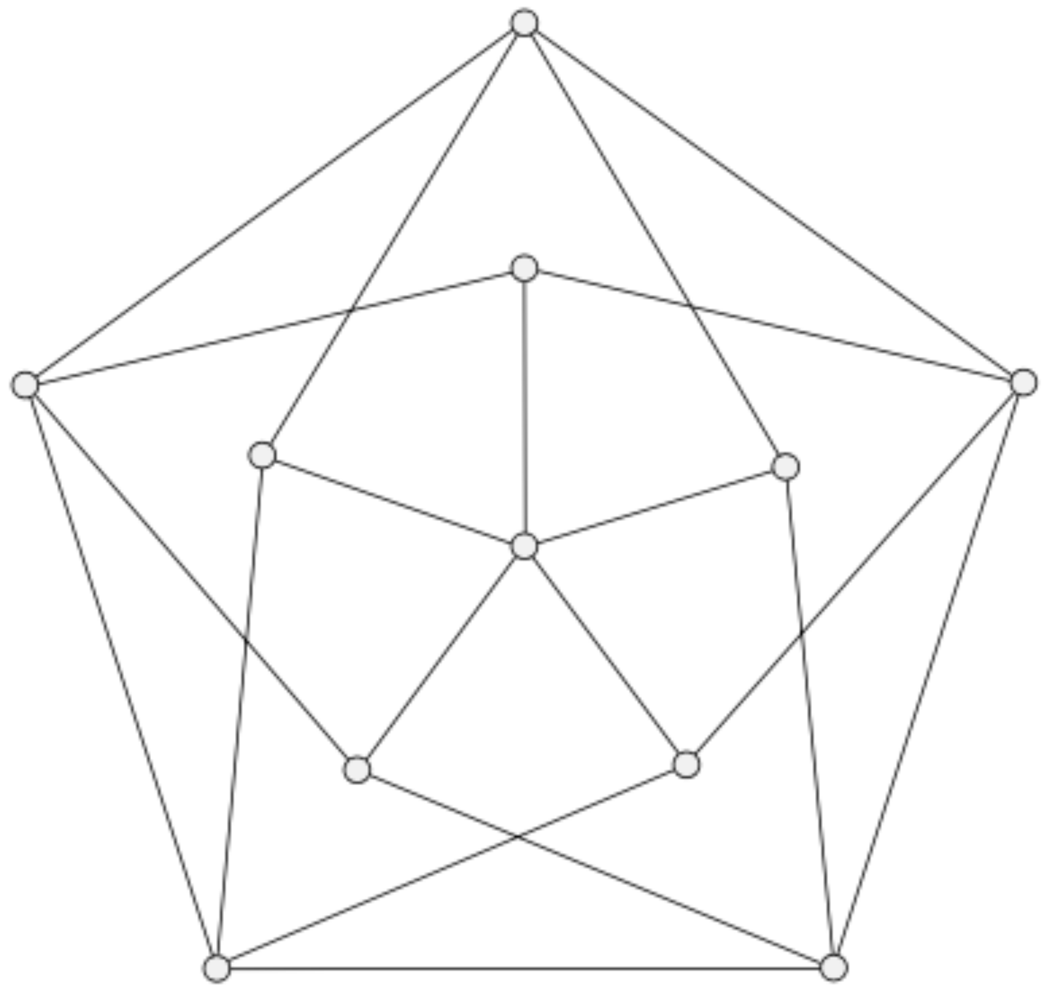
A k-critical graph is one that is k-chromatic and critical.

Note that a minimal k-chromatic subgraph of a k-chromatic graph is -critical, so every k-chromatic graph has a k-critical subgraph.

Theorem 19. If G is a k-critical, then $\delta(G) \geq k - 1$.

Theorem 20. If G is a k-critical, then G has no clique cut.

Corollary 2. If G is a k-critical, then G has no cut vertices.



The Grötzsch graph: a 4-critical graph

Proposition 7. Let G be a connected graph with clique number k, that is, the number of vertices of a maximum complete graph in G. Then

$$\chi(G) \geq k.$$

Mycielski's Construction

Theorem 21. For any positive integer k, there is a triangle-free k-chromatic graph.

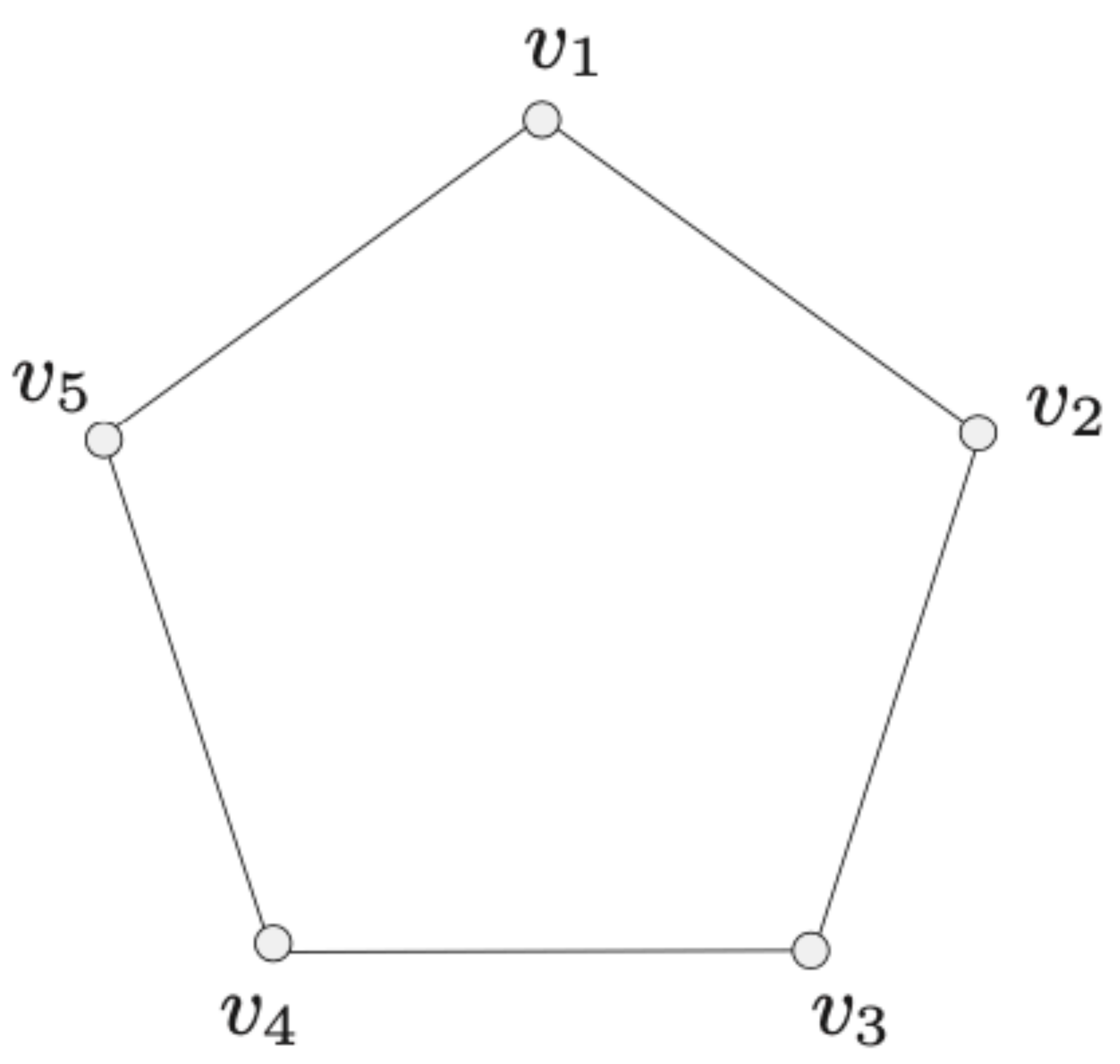
Proof. For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We proceed by induction on k.

Suppose that we have constructed a triangle-free graph G_k with chromatic number $k \geq 2$. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$.

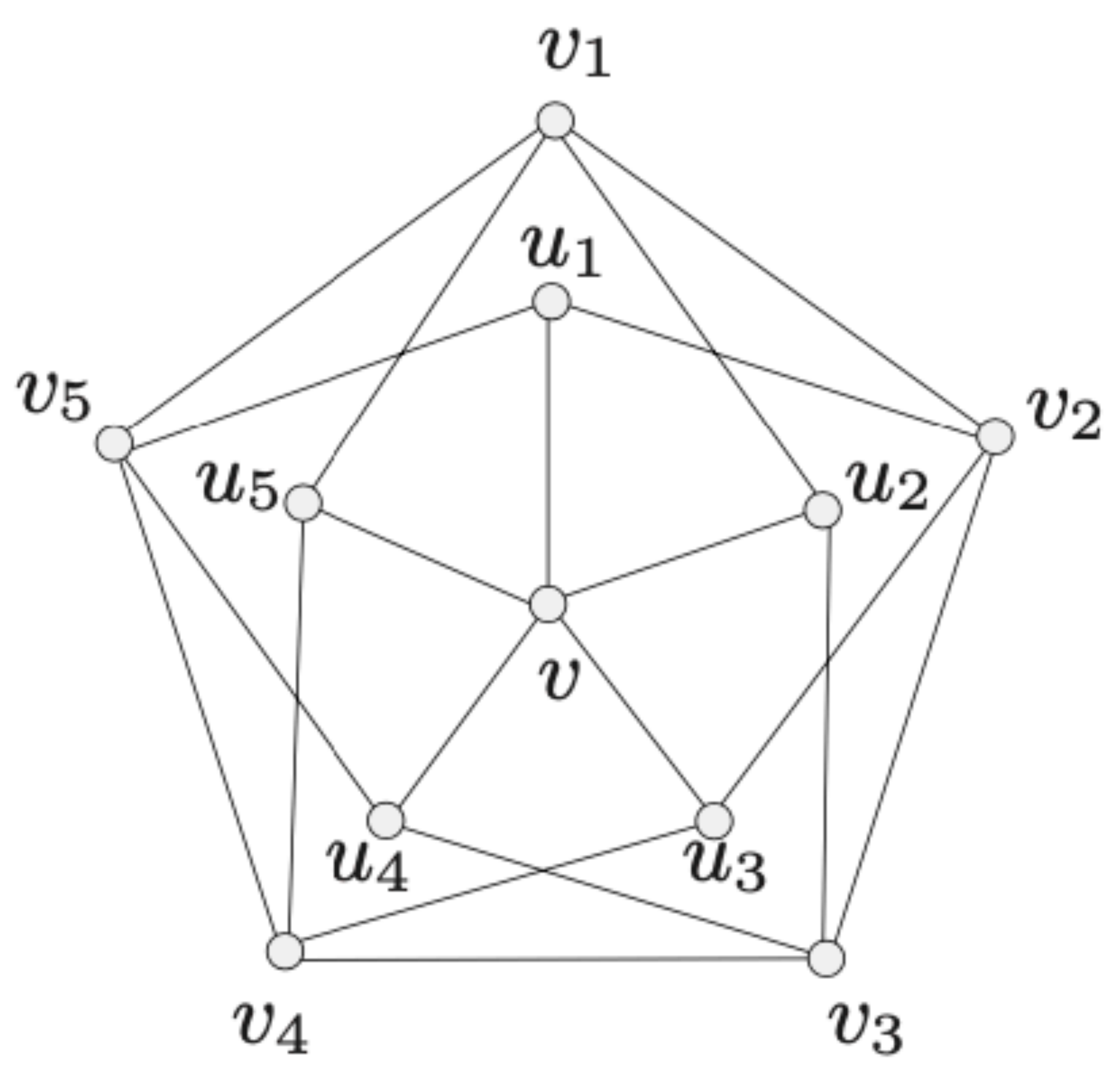
Form the graph G_{k+1} from G_k as follows:

Add $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$ and then, for $1 \leq i \leq n$, join u_i to the neighbours of v_i in G_k, and also to v.

For example, if $G_2 = K_2$, then G_3 is the 5-cycle and G_4 the Grötzsch graph:



G_3



G_4

The graph G_{k+1} certainly has no triangles.

For, because $u_1, u_2, \ldots, u_n$ is an independent set in G_{k+1}, no triangle can contain more than one u_i; and if $u_i v_j v_k u_i$ were a triangle in G_{k+1}, then $v_i v_j v_k v_i$ would be a triangle in G_k, contrary to our assumption.

We now show that G_{k+1} is $(k+1)$-chromatic.

Note, first, that G_{k+1} is $(k+1)$-colorable, because any k-coloring of G_k can be extended to a $(k+1)$-coloring of G_{k+1} by assigning the color of v_i to u_i, $1 \leq i \leq n$, and then assigning a new color to v.

Therefore, it remains to show that G_{k+1} is not k-colorable.

Suppose that G_{k+1} has a k-coloring.

This k-coloring, when restricted to $\{v_1, v_2, \ldots, v_n\}$, is a k-coloring of the k-chromatic graph G_k.

It is not difficult to show that for each color j, there exists a vertex v_i of color j which is adjacent in G_k to vertices of every other color.

Because u_i has precisely the same neighbors in G_k as v_i, the vertex u_i must also have color j.

Therefore, each of the k colors appears on at least one of the vertices u_i.

But no color is now available for the vertex v, a contradiction.

We conclude that G_{k+1} is indeed $(k+1)$-chromatic, and the theorem follows.

List Colorings of Graphs

Let G be a graph and let L be a function which assigns to each vertex v of G a set $L(v)$ of positive integers, called the list of v. A coloring

$$c : V \mapsto N$$

such that $c(v) \in L(v)$ for all $v \in V$, is called a list coloring of G with respect to L, or an L-coloring.

We say that G is L-colorable if $c(u) \neq c(v)$ whenever u is adjacent to v. Observe that if $L(v) = \{1,2,...,k\}$ for all $v \in V$,

an L-coloring is simply a k-coloring.

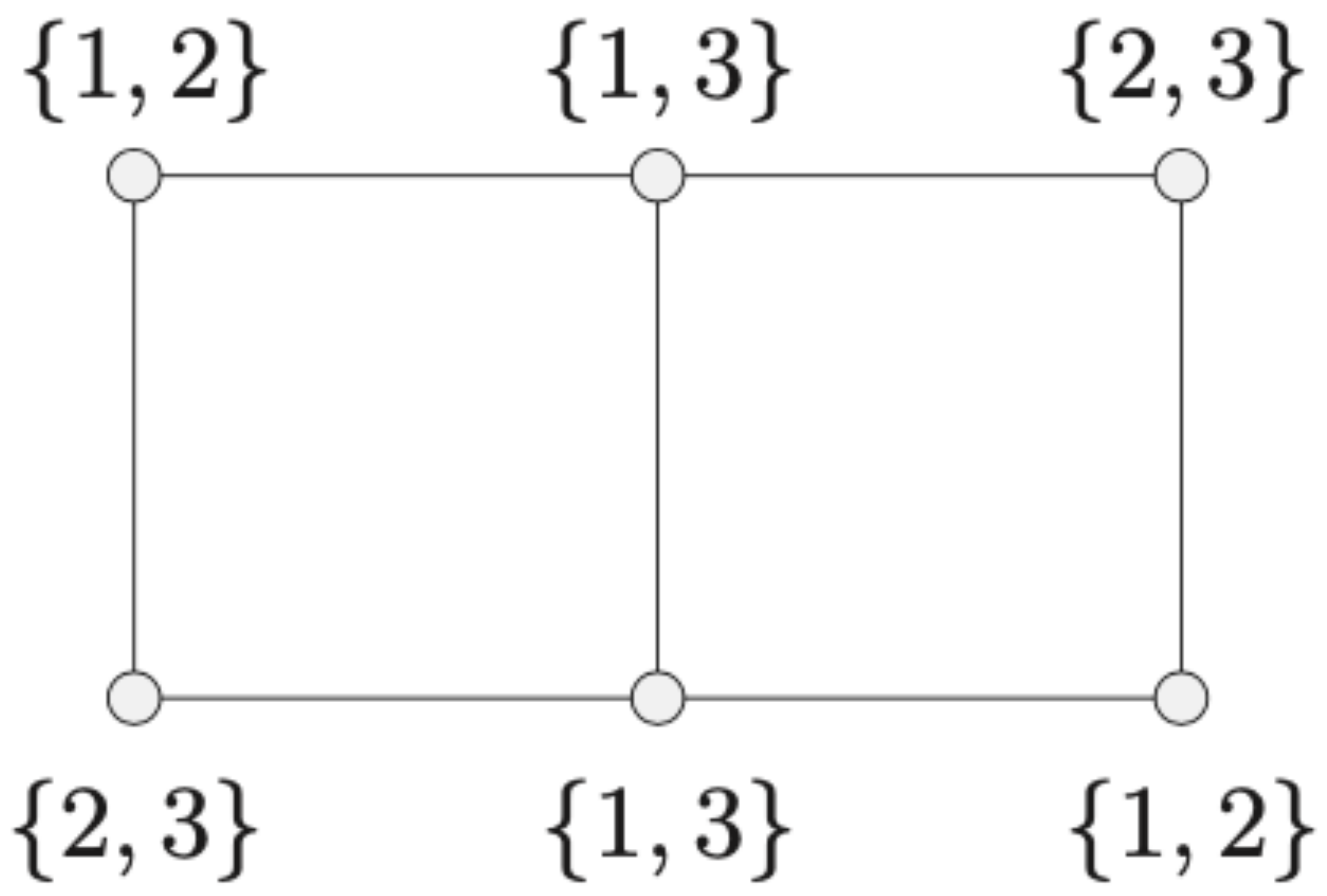
For instance, if G is a bipartite graph and $L(v) = \{1,2\}$ for all $v \in V$, then G has the L-coloring which assigns color 1 to all vertices in one part and color 2 to all vertices in the other part.

A graph G is said to be k-list-colorable if it is L-colorable when ever all the lists have length k.

Clearly, every graph G on n vertices is n-list-colorable.

The smallest value of k for which G is k-list-colorable is called the list chromatic number of G, denoted by $\chi_L(G)$.

For example, the list chromatic number of the following graph is 3.



For any graph G, we always have

$$\chi(G) \leq \chi_L(G) \leq \Delta(G) + 1.$$

In fact, there exist 2-chromatic graphs whose list chromatic number can be arbitrarily large. For example, $\chi_L\left(K_{n,n^n}\right) = n + 1$.

Exercise 5.

1. Let G be a non-regular graph of order n. Show that $\chi(G) \leq \Delta$ by induction on n.

2. Show that a graph G is 3-critical if and only if G is an odd cycle.

3. Show that $\chi_L\left(K_{n,n^n}\right) = n + 1$.