Edge Colorings of Graphs

Let G be a graph. A k-edge-coloring of G is a mapping

$$c : E(G) \mapsto \{1,2,...,k\} .$$

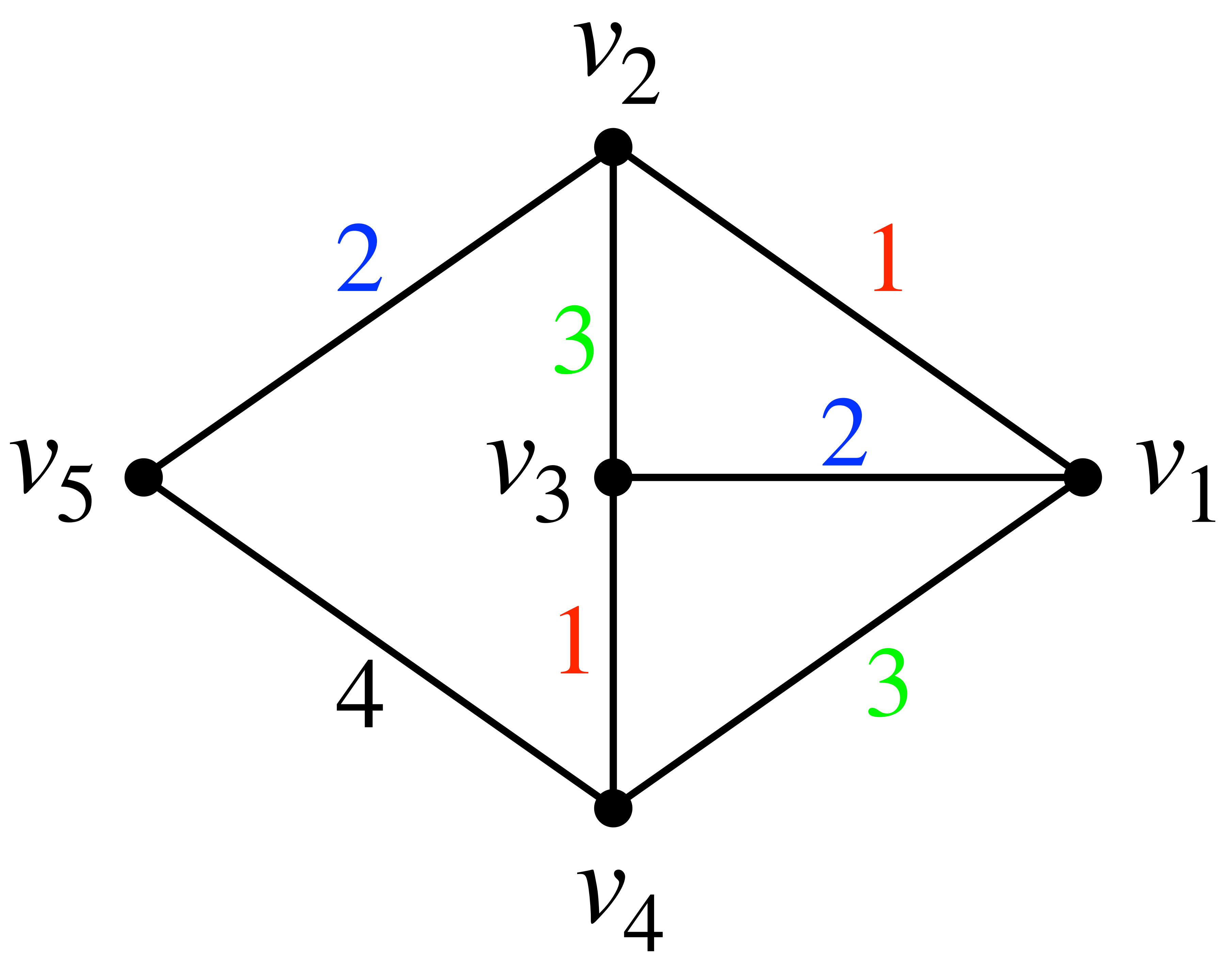
A edge-coloring c is *proper* if no two adjacent edges are assigned the same color. A graph G is *k-edge-colorable* (k-边可染的) if G has a proper k-edge-coloring.

The minimum integer k for which a graph G is k-edge-colorable is called the *edge chromatic number* (边色数) of G, and denoted $\chi'(G)$.

If $\chi'(G) = k$, the graph G is said to be *k-edge-chromatic* (k-边色的) .

Proposition 8. Let G be a connected graph with $\Delta(G) = \Delta$. Then

$$\chi'(G) \geq \Delta\,.$$



A 4-edge-chromatic graph

Edge Colorings of Bipartite Graphs

Theorem 21. If G is a bipartite graph with $\Delta(G) = \Delta$, then $\chi'(G) = \Delta$.

Proof. By induction on the size m. Let $e = uv$ be an edge of G.

We assume that $H = G \backslash e$ has a Δ-edge-coloring $\{M_1, M_2, \ldots, M_\Delta\}$.

If some color is available for e,

that color can be assigned to e to yield a Δ-edge-coloring of G.

So we may assume that each of the Δ colors is represented either at u or at v.

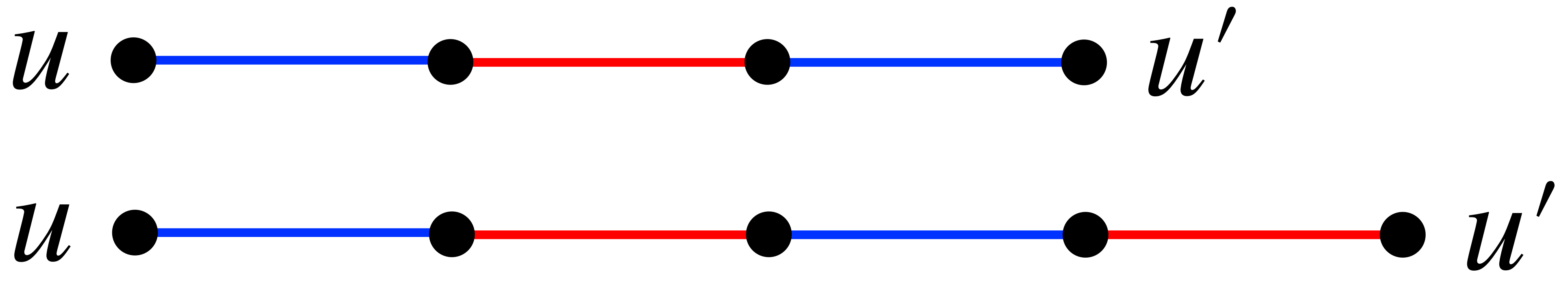
Because the degree of u in $G \backslash e$ is at most $\Delta - 1$,

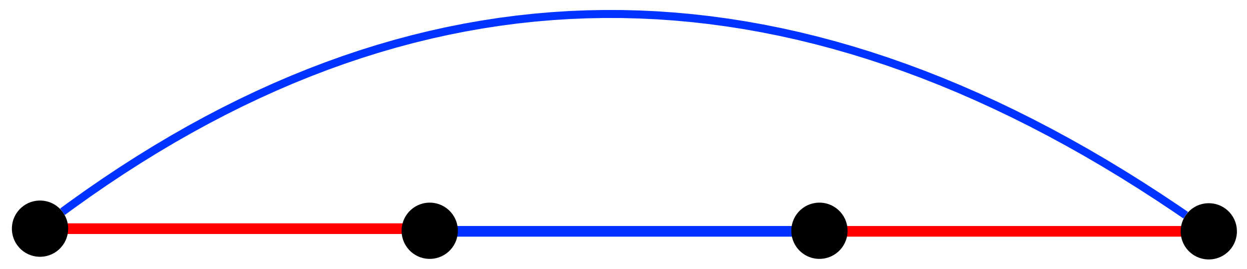
at least one color i is available at u, hence represented at v.

Likewise, at least one color j is available at v and represented at u.

Consider the subgraph $H_{ij} = G[M_i \cup M_j]$.

Because u has degree one in H_{ij}, the component containing u is an ij-path P.





This ij-path P does not terminate at v.

For if it did, it would be of even length, starting with an edge colored j and ending with an edge colored i, and $P + e$ would be a cycle of odd length in G, contradicting the hypothesis that G is bipartite.

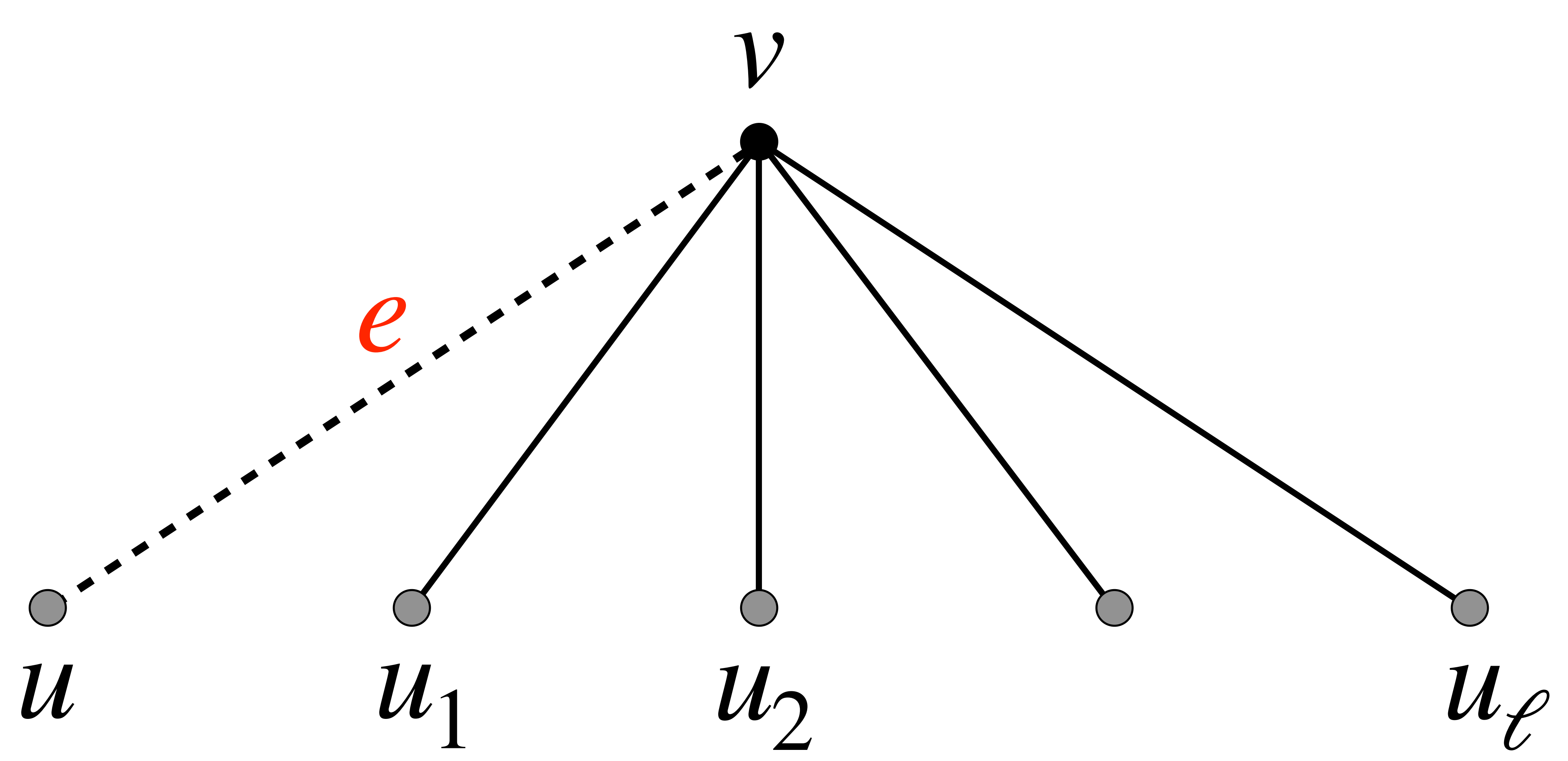
Interchanging the colors on P, we obtain a new Δ-edge-coloring of H with respect to which the color j is available at both u and v.

Assigning color j to e, we obtain a Δ-edge-coloring of G.

Theorem 22(Vizing). For any simple graph G with $\Delta(G) = \Delta$,

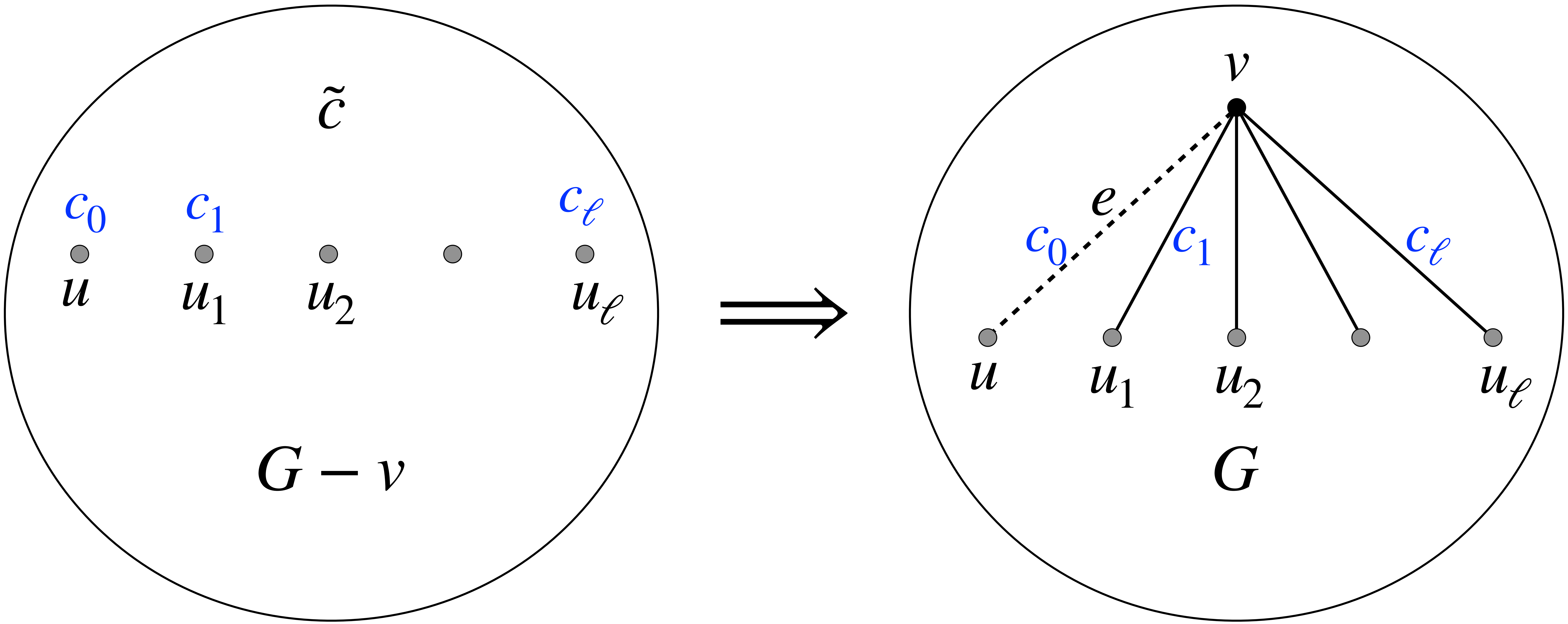
$$\chi'(G) \leq \Delta + 1.$$

Lemma 4. Let G be a simple graph with $\Delta(G) = \Delta$, v a vertex of G, e an edge of G incident to v, and k an integer, $k \geq \Delta$.
Suppose that $G \backslash e$ has a k-edge-coloring c for which every neighbor of v in G has at least one available color. Then G is k-edge-colorable.



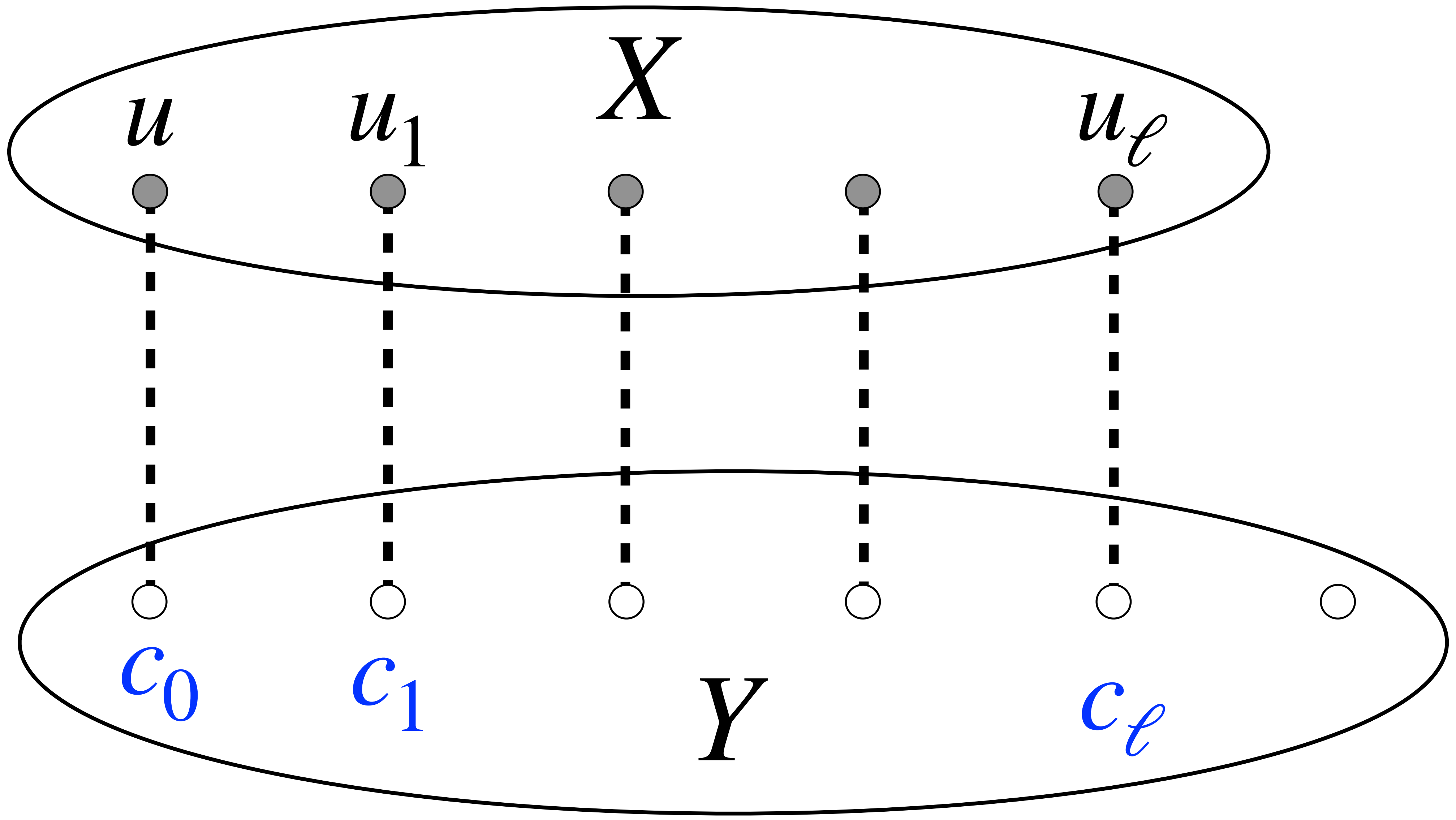
Proof. Consider the restriction $\tilde{c}$ of c to $G - v$.

If c_0 is available at u and c_i is available at u_i such that $c_i \neq c_j$ for $0 \leq i < j \leq \ell$, then

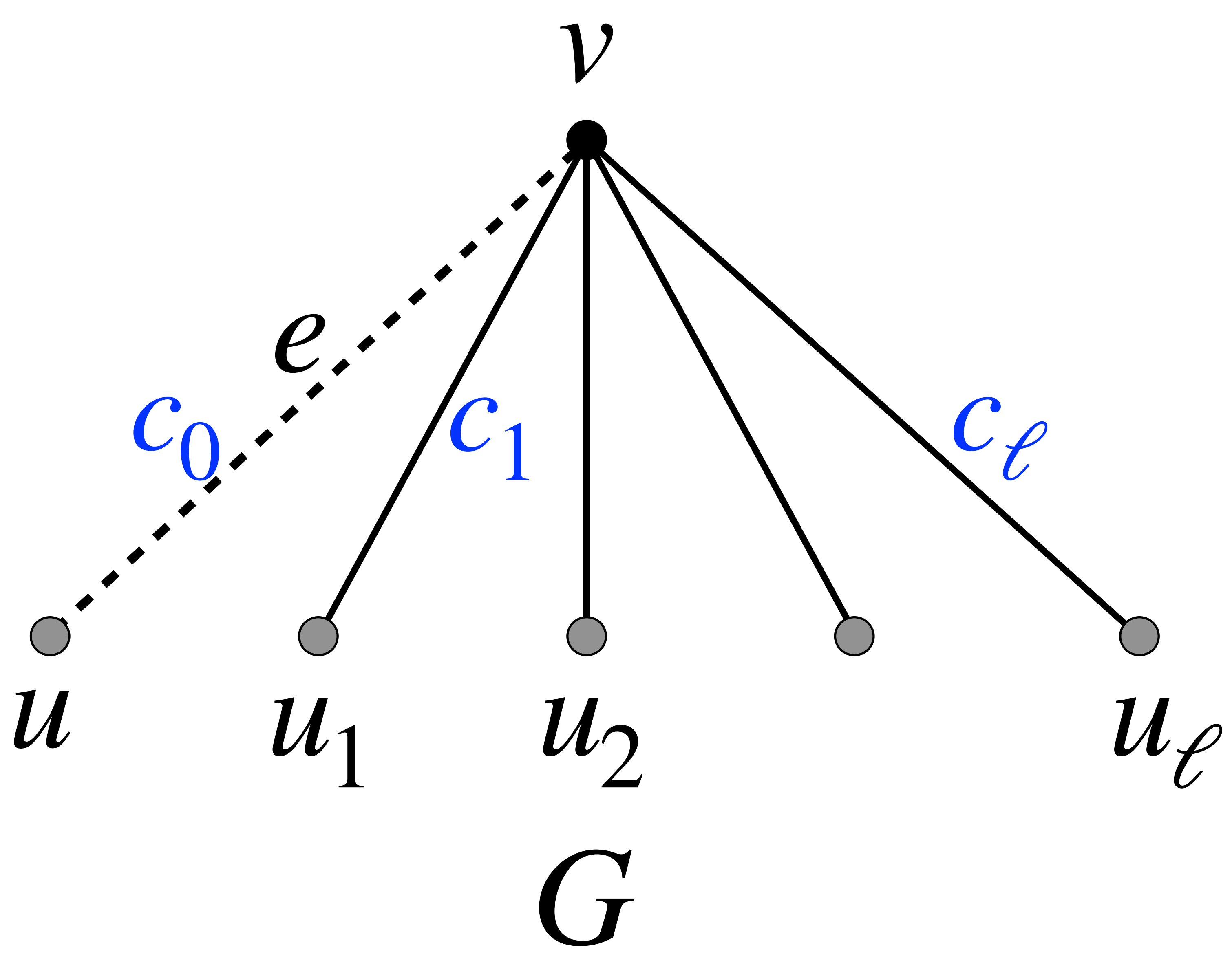


For convenience, study the bipartite graph $H[X, Y]$, where $X = N_G(v)$ and $Y = \{1,2,\ldots,k\}$, and vertices $x \in X$ and color $i \in Y$ being adjacent if color i is available at vertex x in the restriction $\tilde{c}$ of c to $G - v$.

If H has a matching covering X, then we can obtain a k-edge-coloring of G by combining this with $\tilde{c}$:

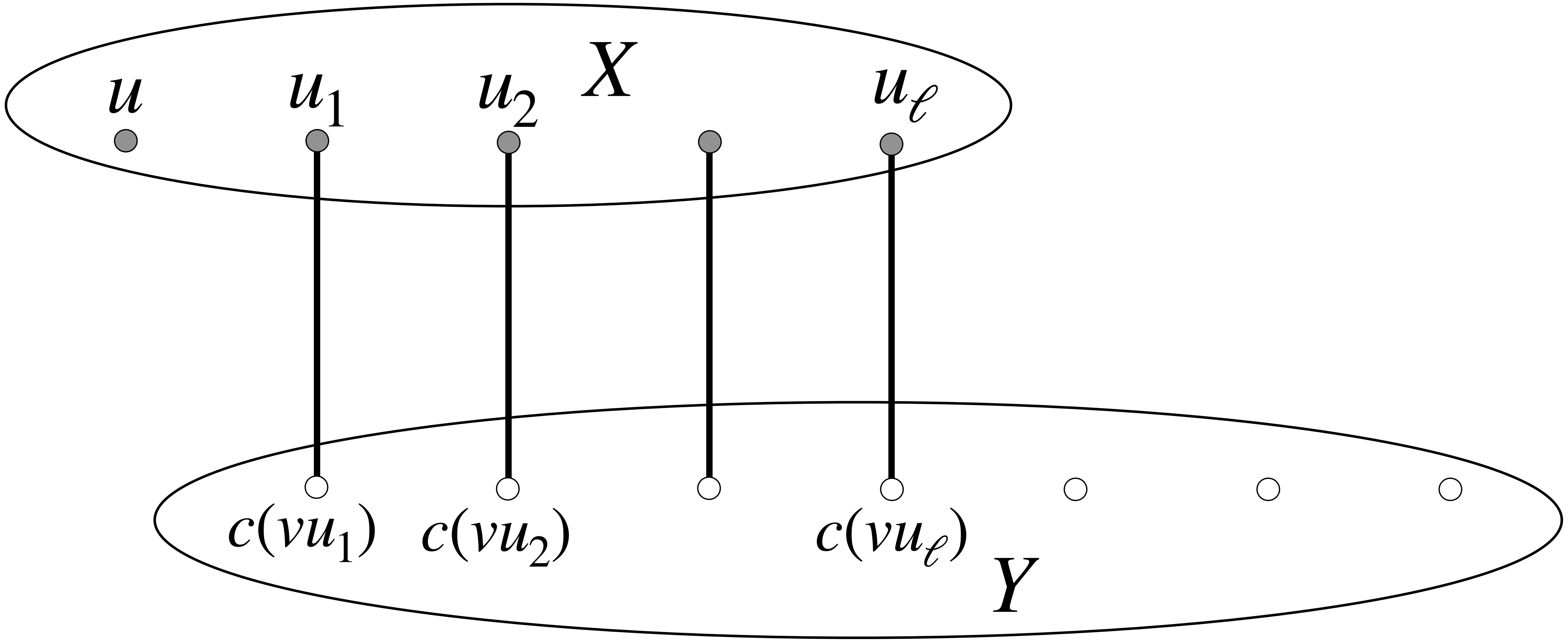


$\Longrightarrow$



In particular, for any u_i, the color of the edge vu_i is available at in $G - v$, so H contains the matching

$$M = \left\{(u_i, c(vu_i)) : 1 \leq i \leq \ell\right\}.$$



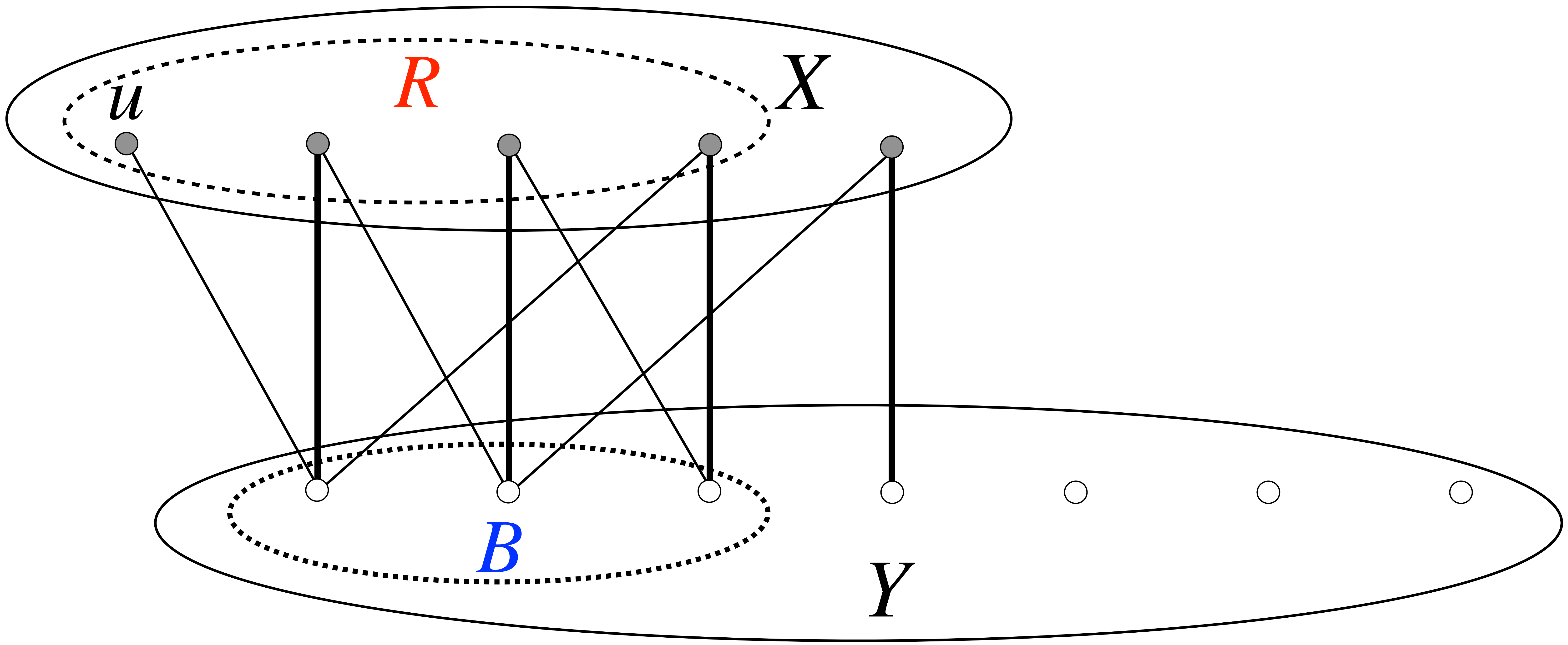
We may suppose that H has no matching covering X.
Our goal is to modify the coloring c to a coloring c' so that the corresponding bipartite graph H' does contain a matching covering X.

By hypothesis, each vertex u_i is incident with at least one edge of $H \backslash M$, and the vertex u is incident with at least one such edge as well, because

$$d_{G \backslash e}(u) = d_G(u) - 1 \leq \Delta(G) - 1 \leq k - 1.$$

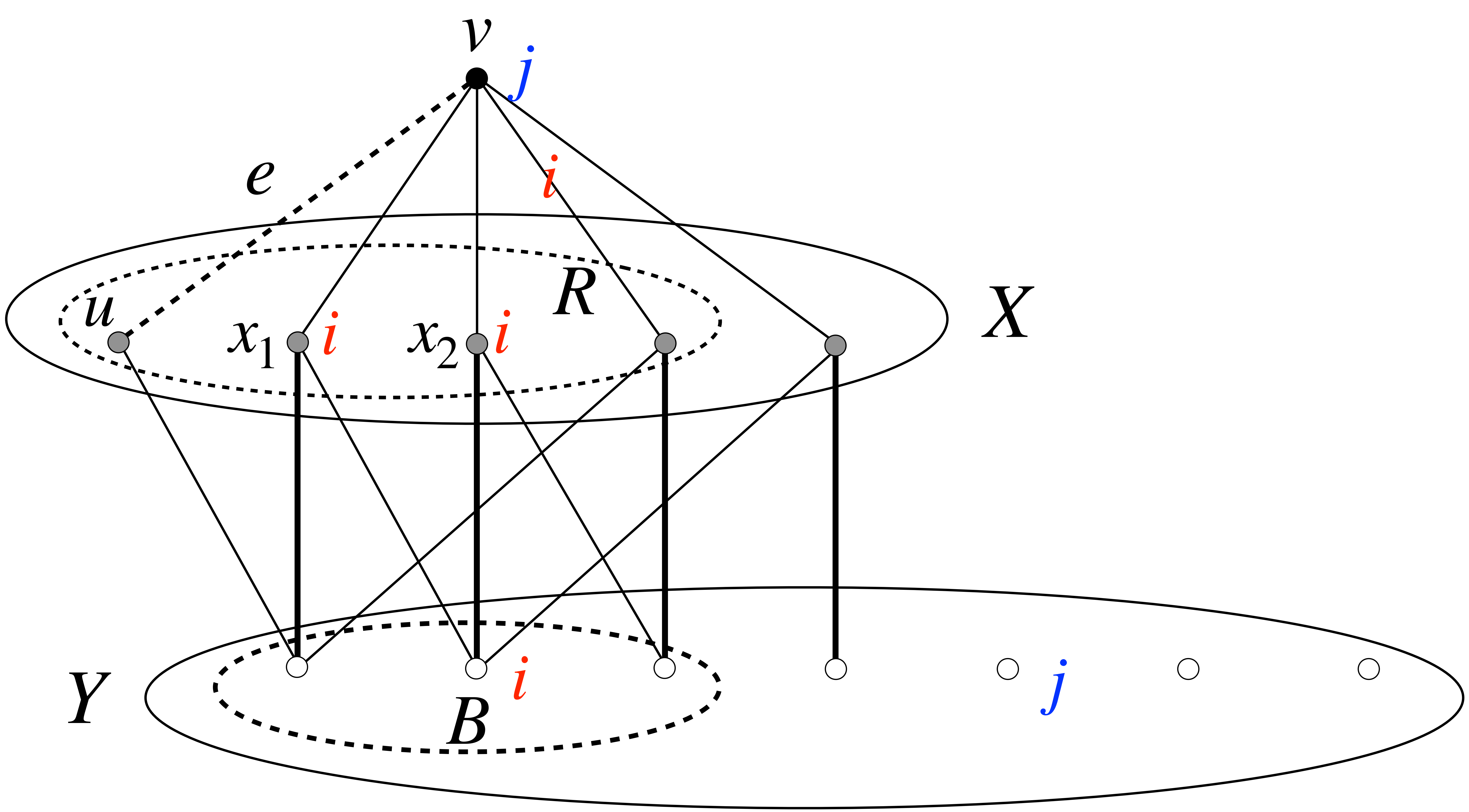
Therefore, each vertex of X is incident with at least one edge of $H \backslash M$.

Denote by Z the set of all vertices of H reachable from u by M-alternating paths, and set $R = X \cap Z$ and $B = Y \cap Z$. As in the proof of Hall's Theorem, $N_H(R) = B$ and B is matched under M with $R \backslash \{u\}$, so $|B| = |R| - 1$.



Because at least one color is available at each vertex of R and $|B| = |R| - 1$, some color $i \in B$ is available at two vertices x_1, x_2 of R.

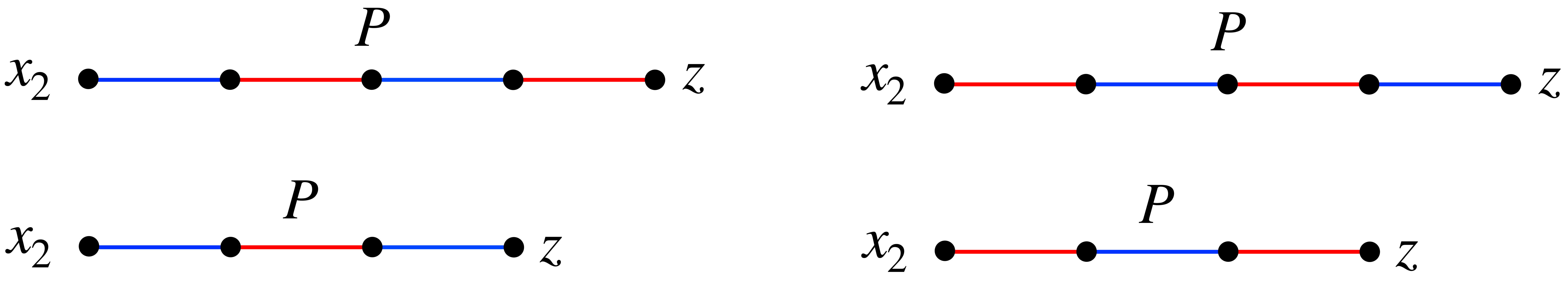
Note that every color in B is represented at v, because B is matched under M with $R \setminus \{u\}$. In particular, color i is represented at v.



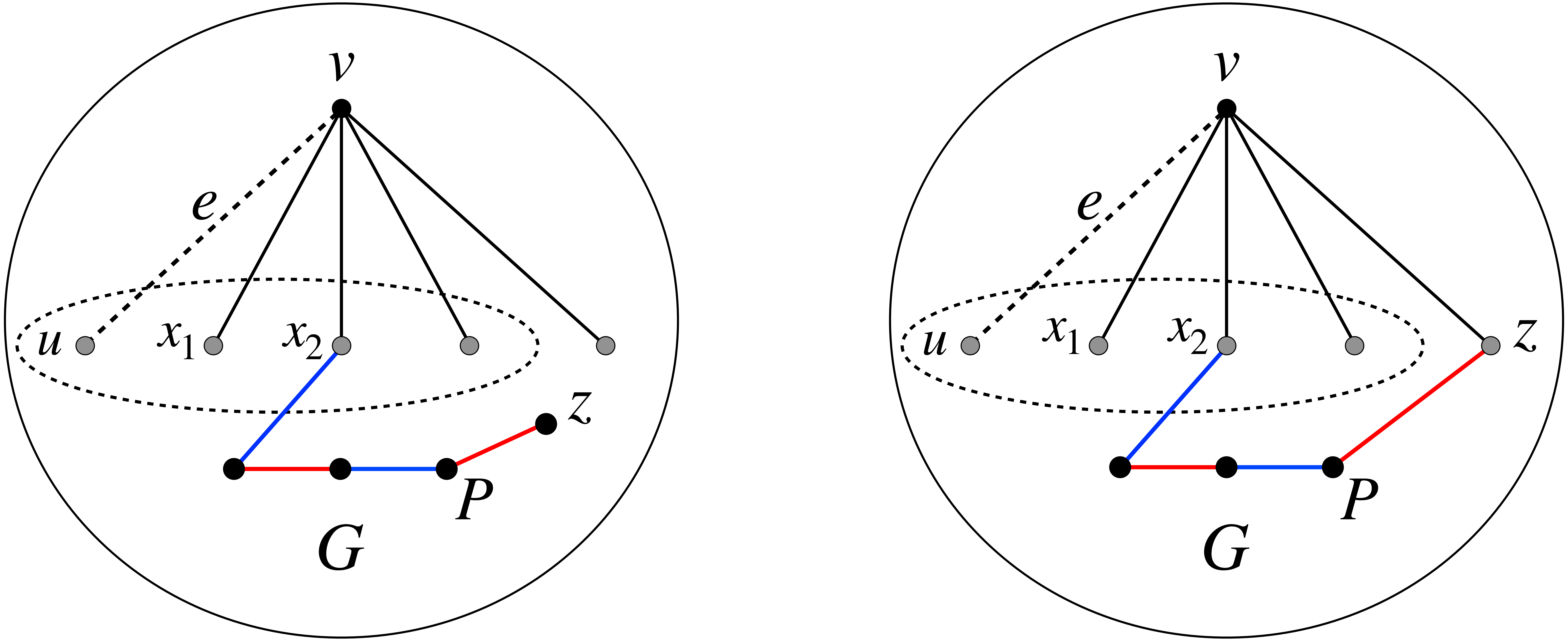
Because the $d_{G\setminus e}(v) \leq k-1$, some color j is available at v.
Observe that j is not in B because every color in B is represented at v.
Thus, j is represented at every vertex of R, in particular, at both x_1 and x_2.

Let us return to consider the graph $G\backslash e$. By the above observations, each of the three vertices v, x_1 and x_2, is an end of an ij-path in $G\backslash e$.

Consider the ij-path starting at v. Evidently, this path cannot terminate at both x_1 and x_2. We may suppose that the path starting at v does not terminate at x_2, and let z be the terminal vertex of the ij-path P starting at x_2.
Interchanging the colors i and j on P, we obtain a new coloring c' of $G\backslash e$.

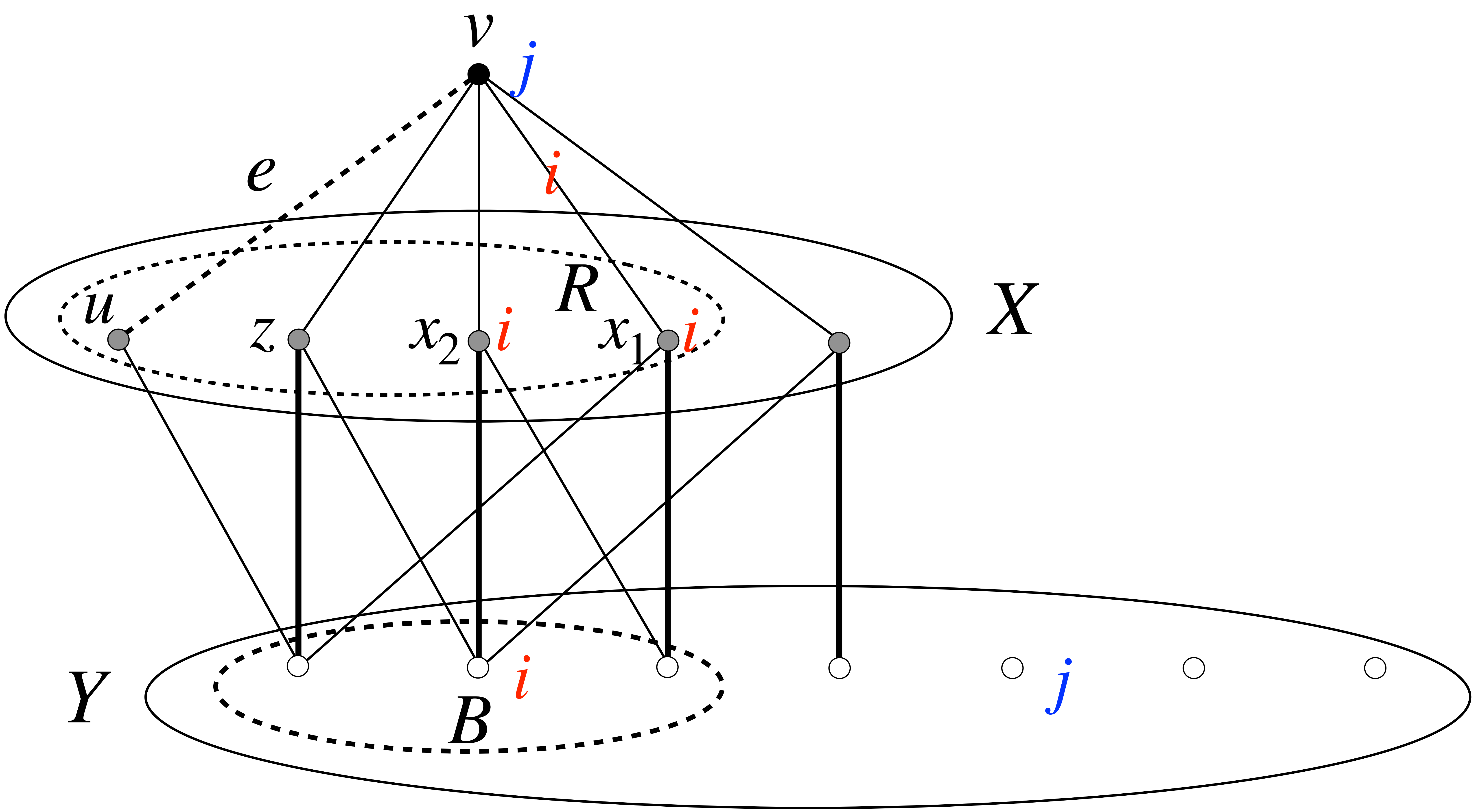


Let $H'[X, Y]$ be the bipartite graph corresponding to c'. The only differences in the edge sets of H and H' occur at x_2 and possibly z (if $z \in X$). Moreover, because v does not lie on P, the matching M is still a matching in H'.

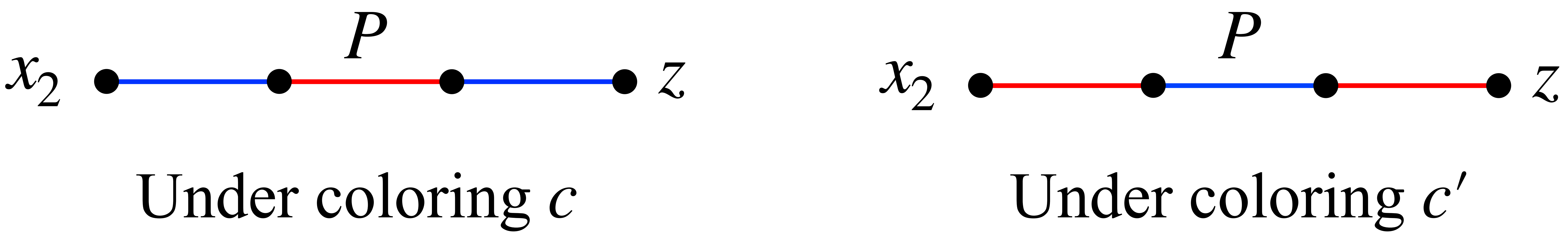


Consider the alternating ux_2-path Q in H.

If z lies on Q, then $z \in R$ and the path uQz is still an M-alternating path in H', as it terminates with an edge of M.



Also, because j is not in B, the path P must have originally terminated at z in an edge of color j, and now terminates in an edge of color i.

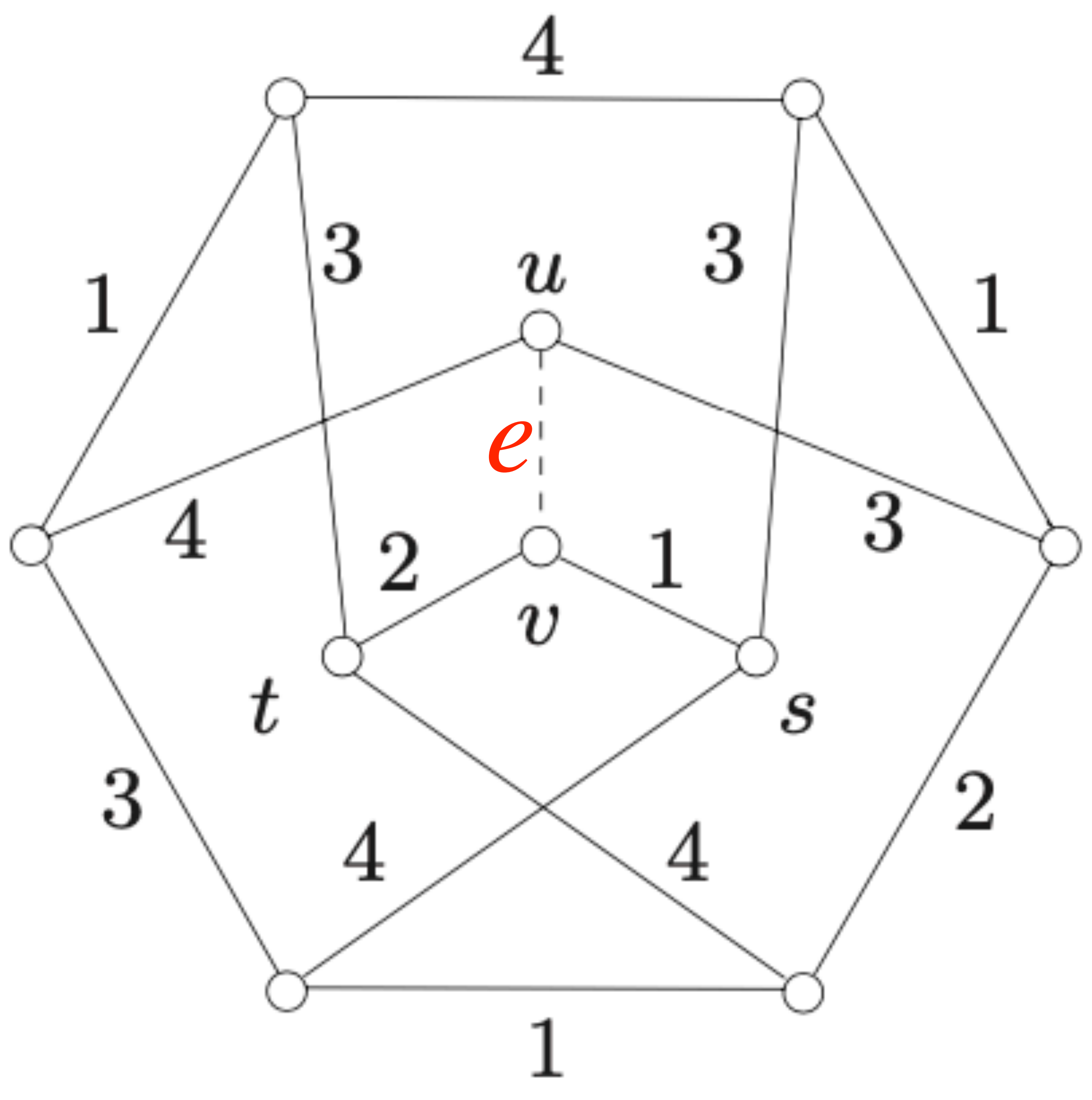


With respect to the coloring c', the color j is therefore available at z, and $Q' = uQzj$ is an M-augmenting path in H'.
If z does not lie on Q, then $Q' = uQx_2j$ is an M-augmenting path in H'.

Set $M' = M \, \Delta \, E(Q')$.
Then M' is a matching in H' which covers every vertex in X.

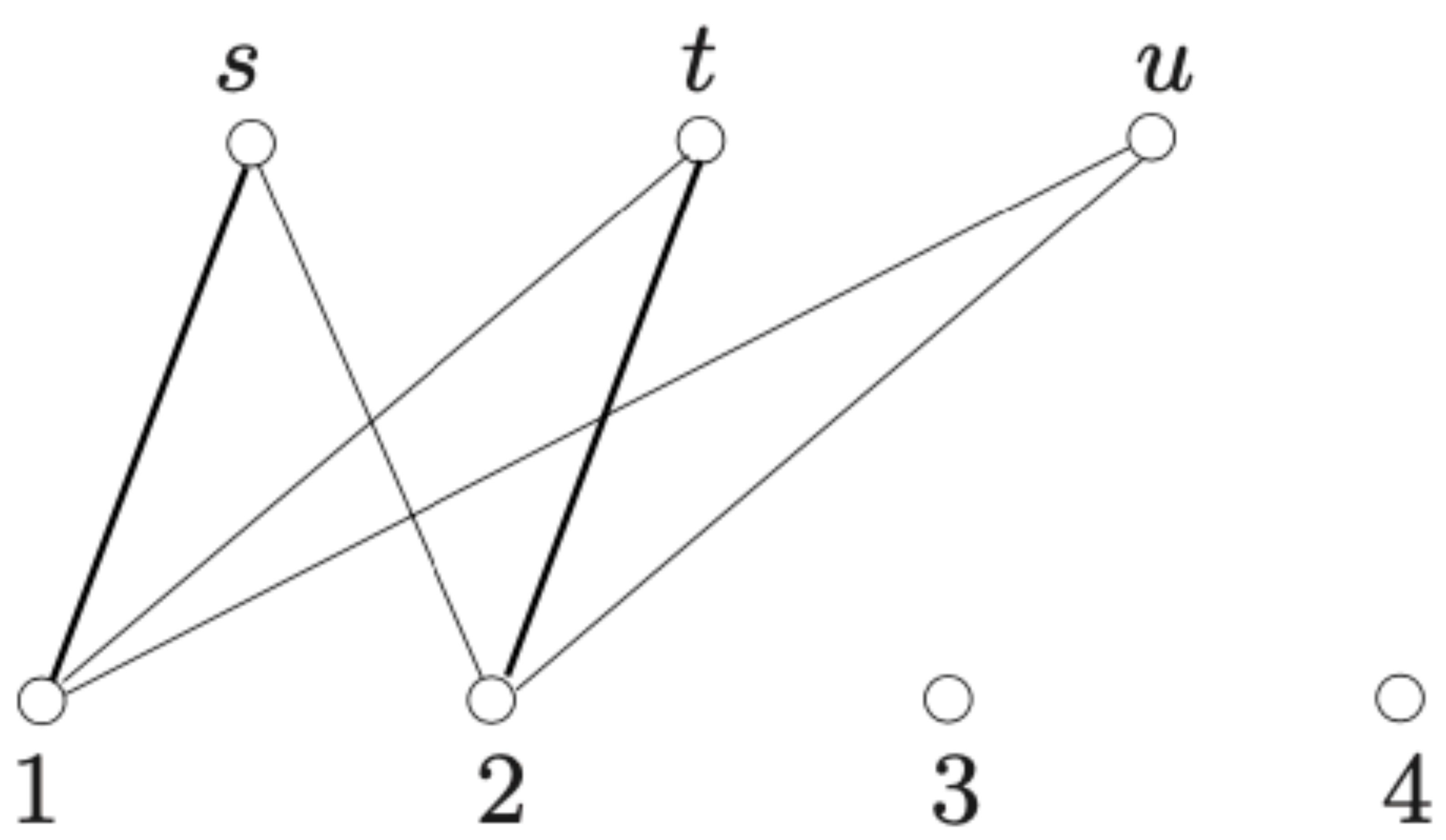
Example 4. 4-edge coloring of Petersen graph

A 4-edge coloring c of $G\backslash e$



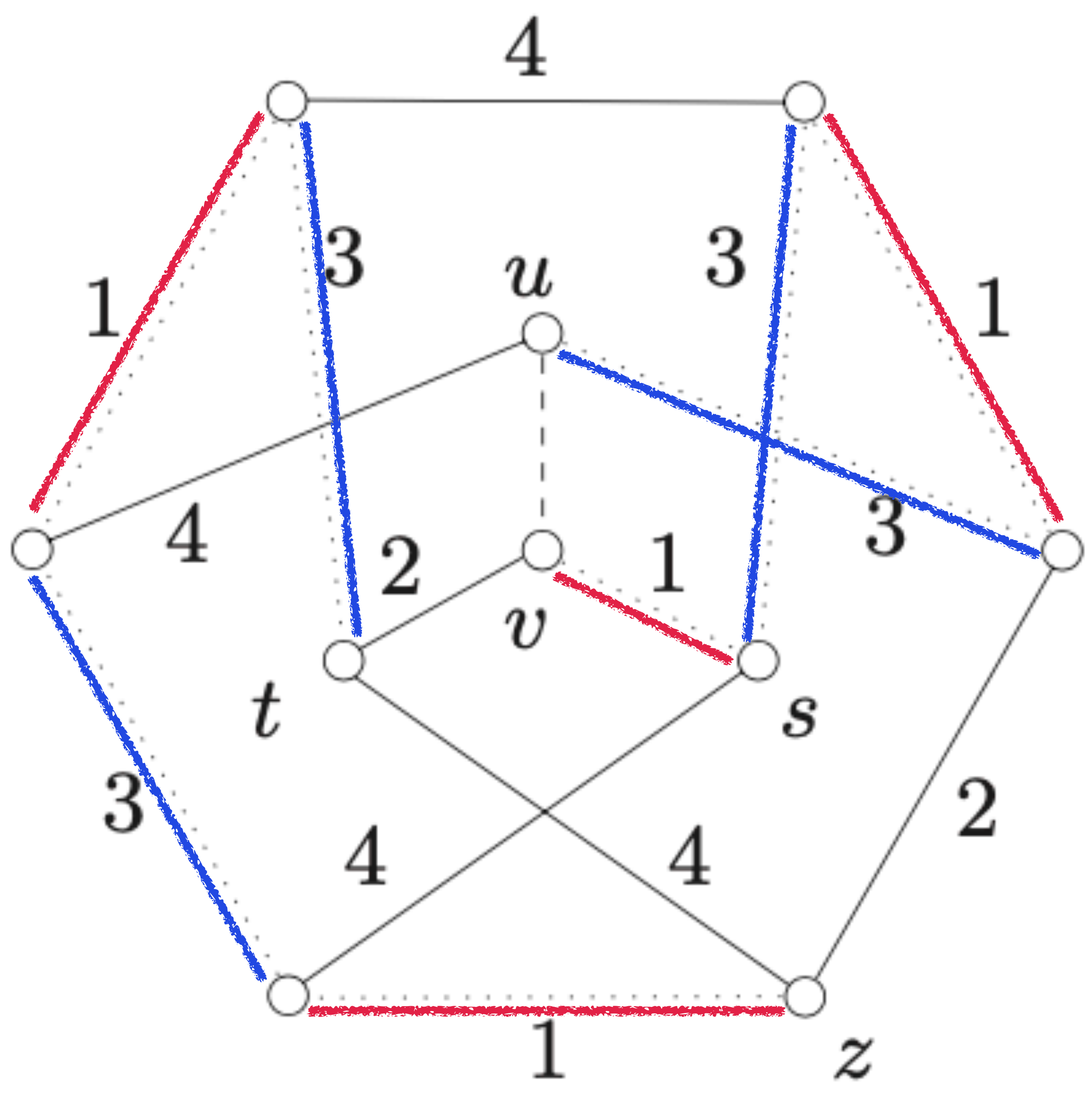
(a)

Corresponding bipartite graph H



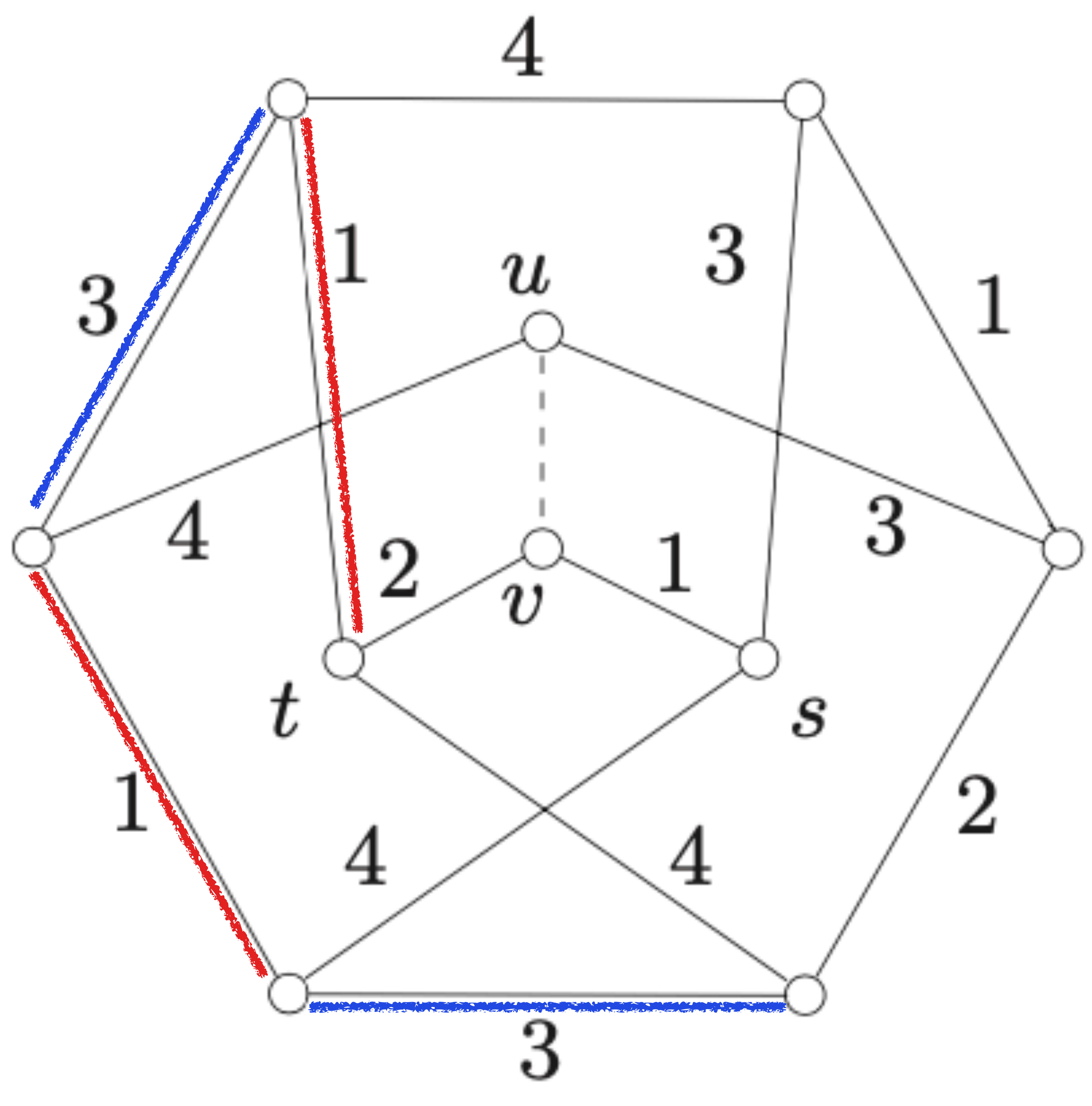
(b)

The ij-path from v (to u) and the ij-path P from t to z



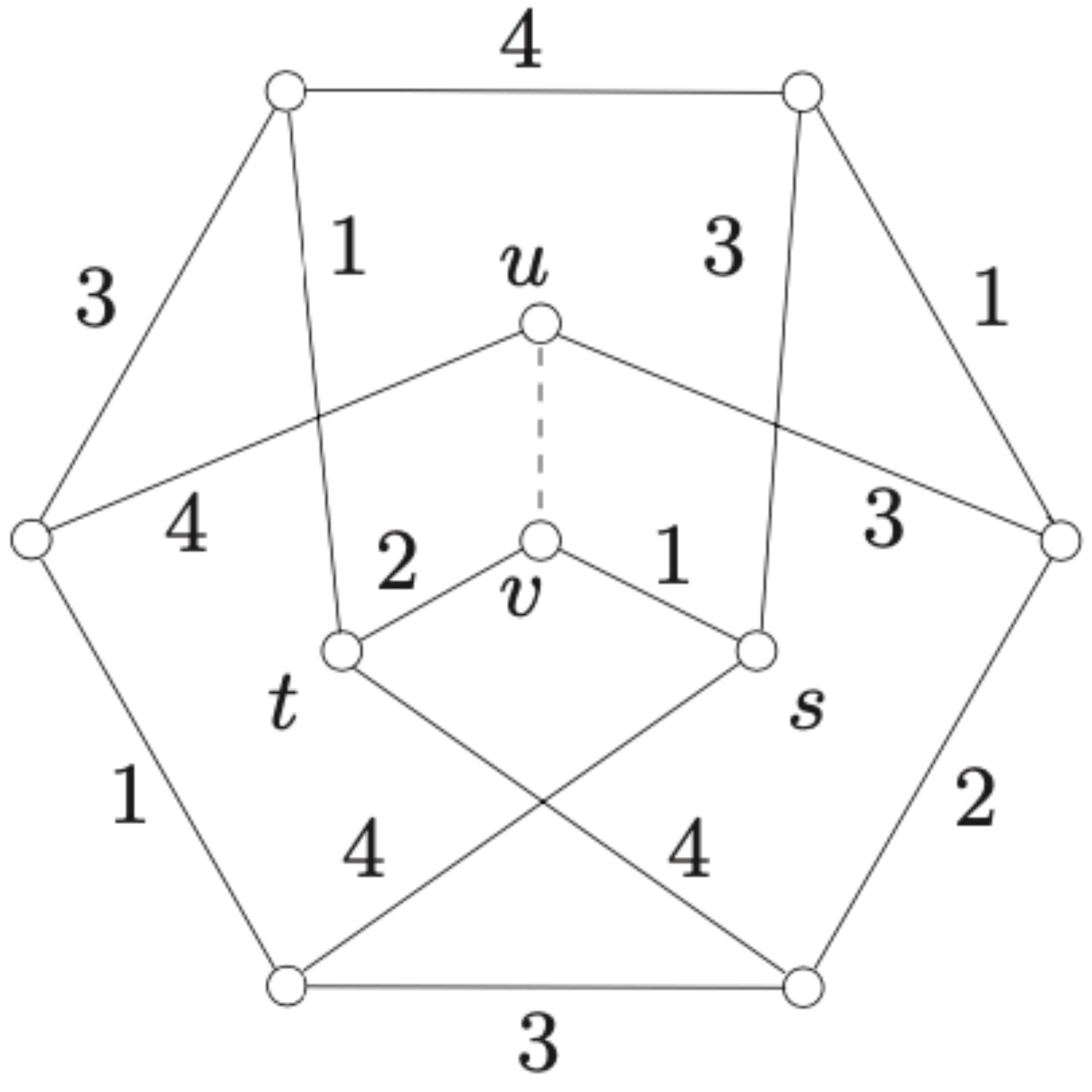
(c)

A 4-edge coloring c' of $G\backslash e$



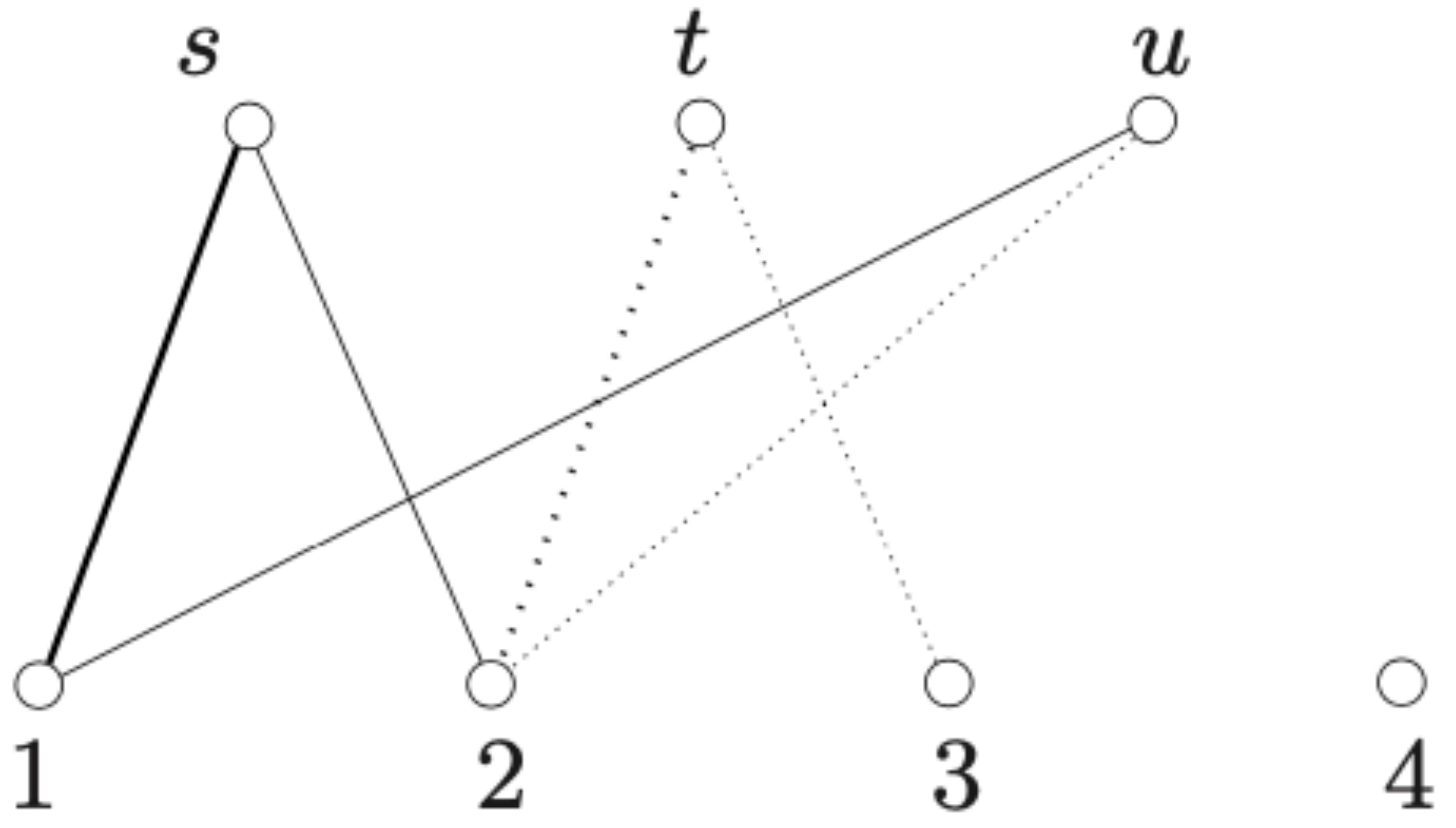
(d)

A 4-edge coloring c' of $G\backslash e$



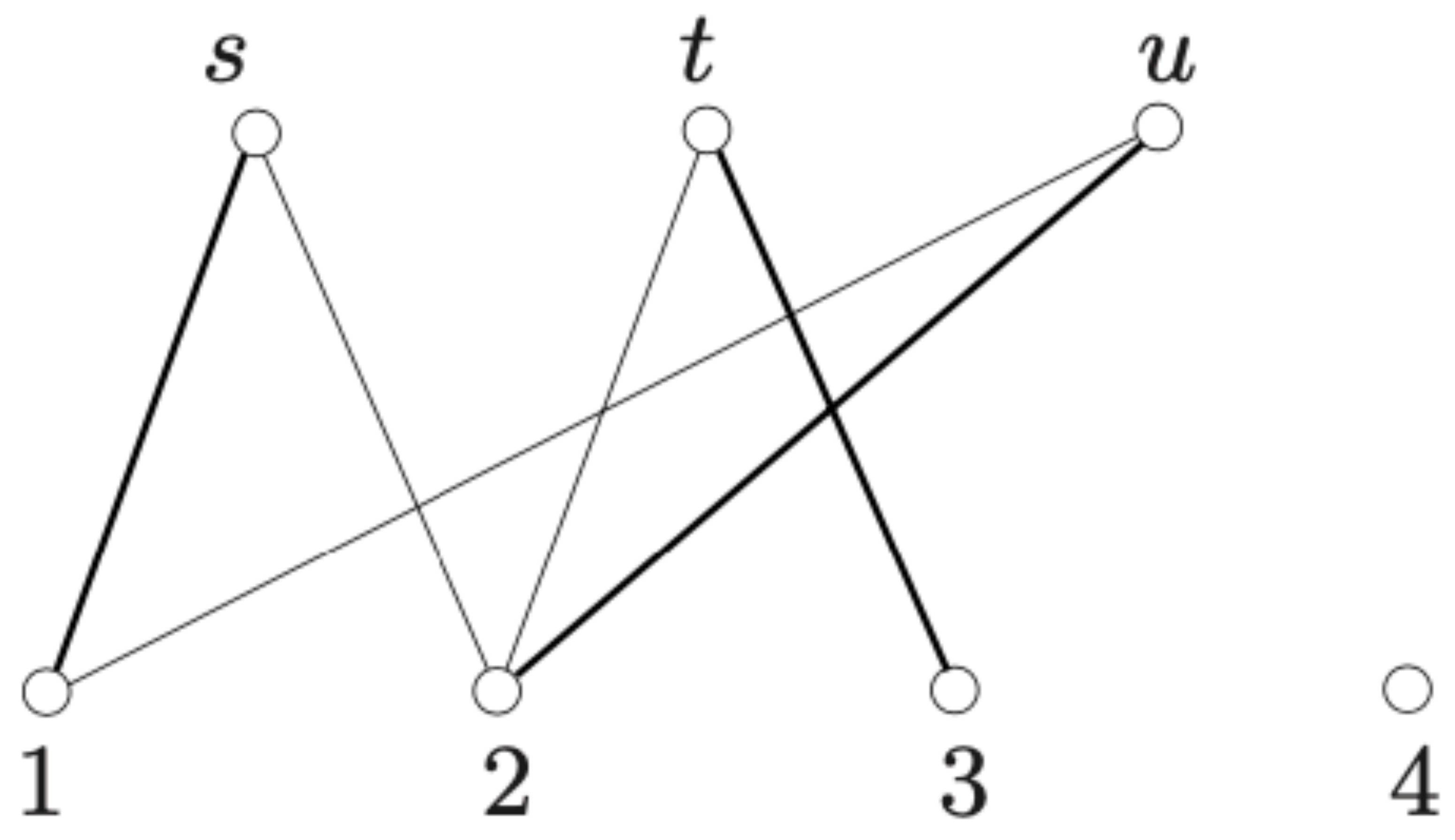
(d)

Corresponding bipartite graph H', an M-augmenting u-path Q



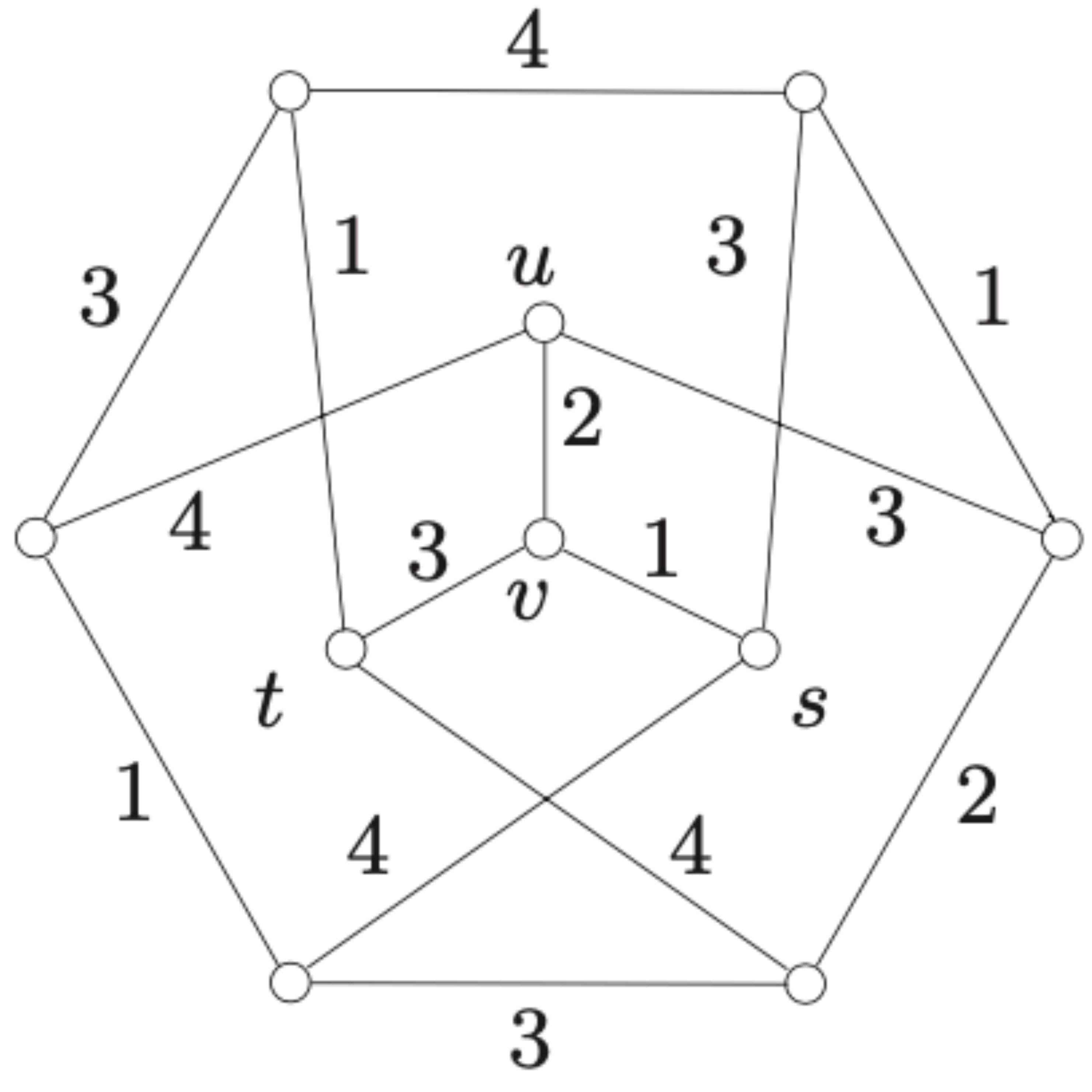
(e)

A matching in H' covering X



(f)

The resulting 4-edge coloring of G



(g)

Exercise 7.

1. Show that $\chi'(G) = 4$ if G is Petersen graph.

2. Let G be a graph of oder n and size m with maximum degree Δ. Show that if $m > \lfloor n/2 \rfloor \Delta$, then $\chi'(G) = \Delta + 1$.