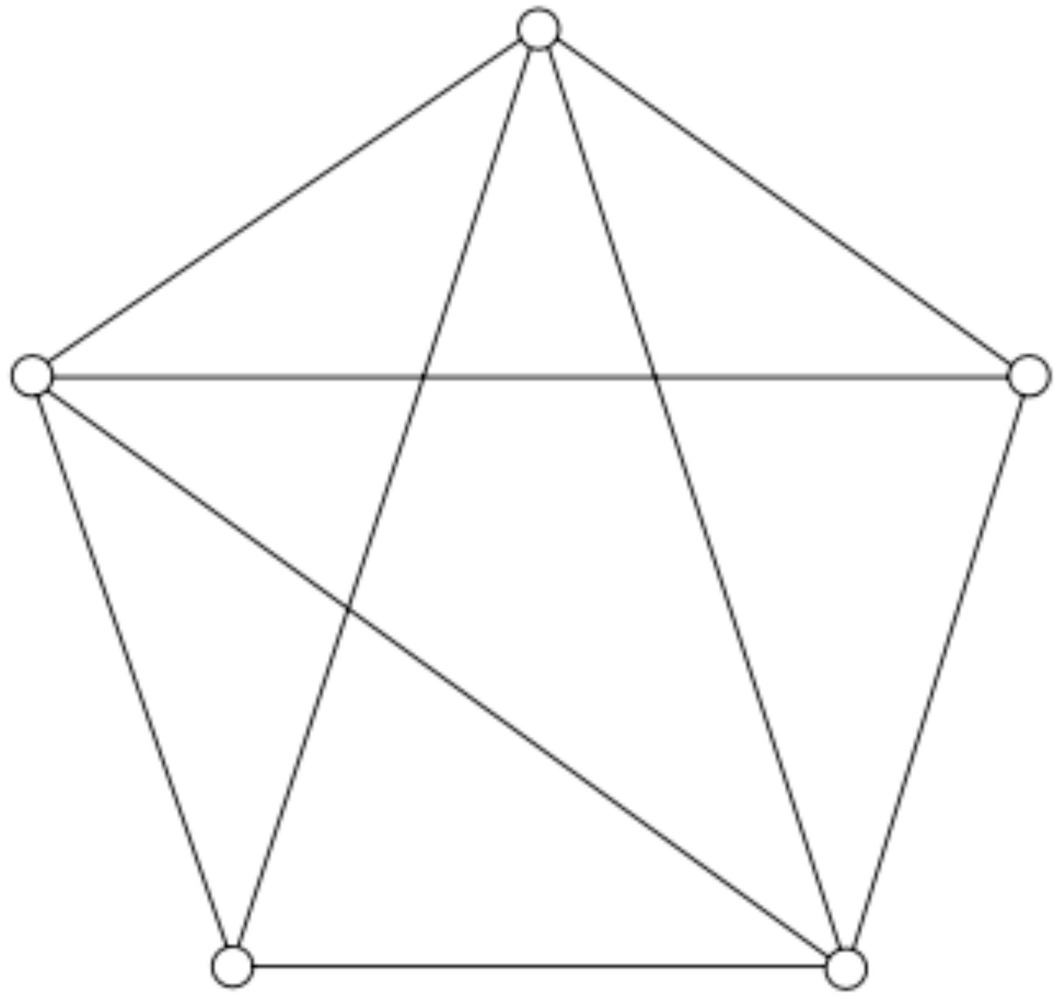
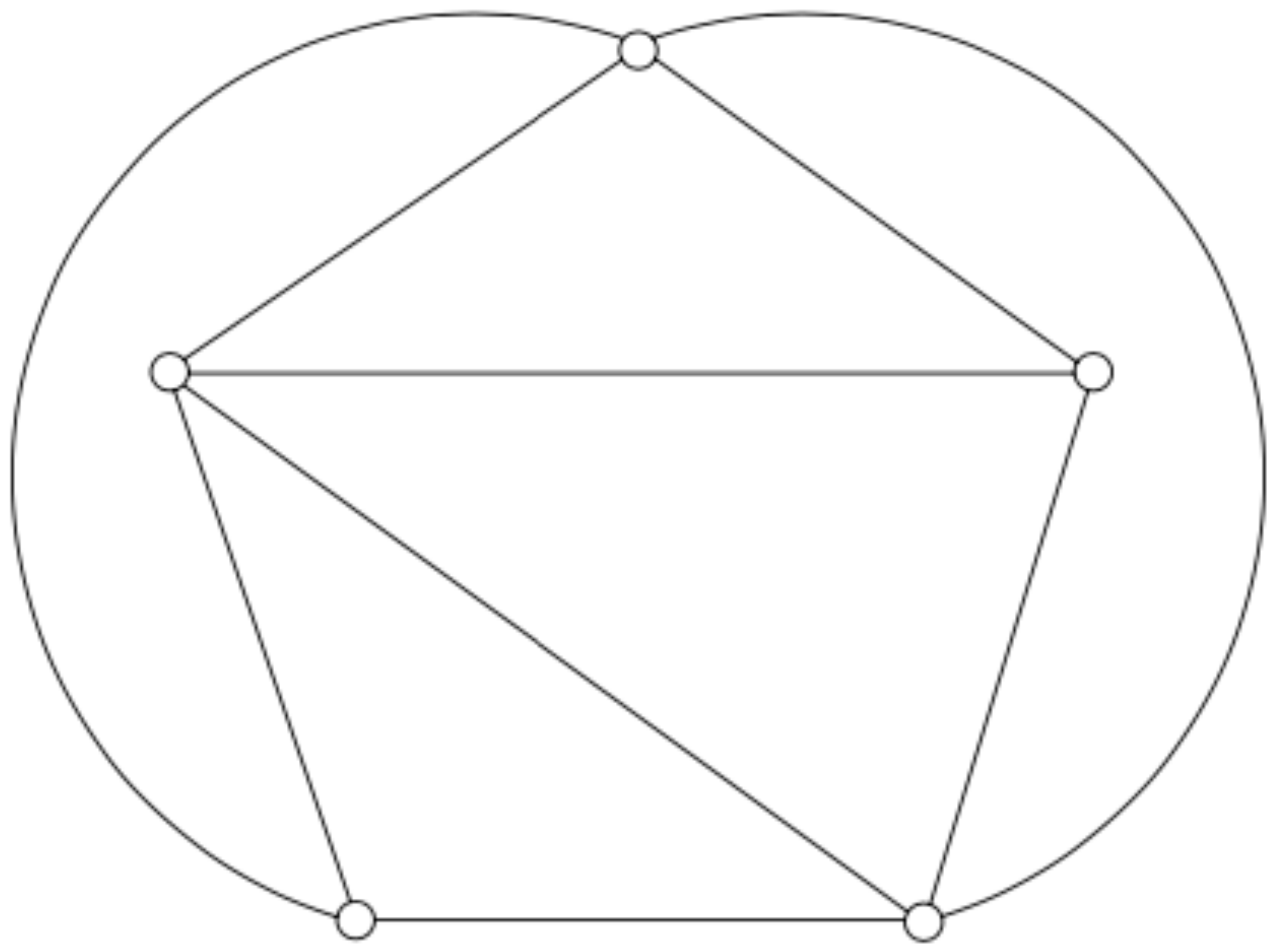
Planar Graphs

A graph is said to be embeddable in the plane, or planar (可平面的), if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a planar embedding of the graph.



The planar graph $K_5 \backslash e$.



A planar embedding of $K_5 \backslash e$.

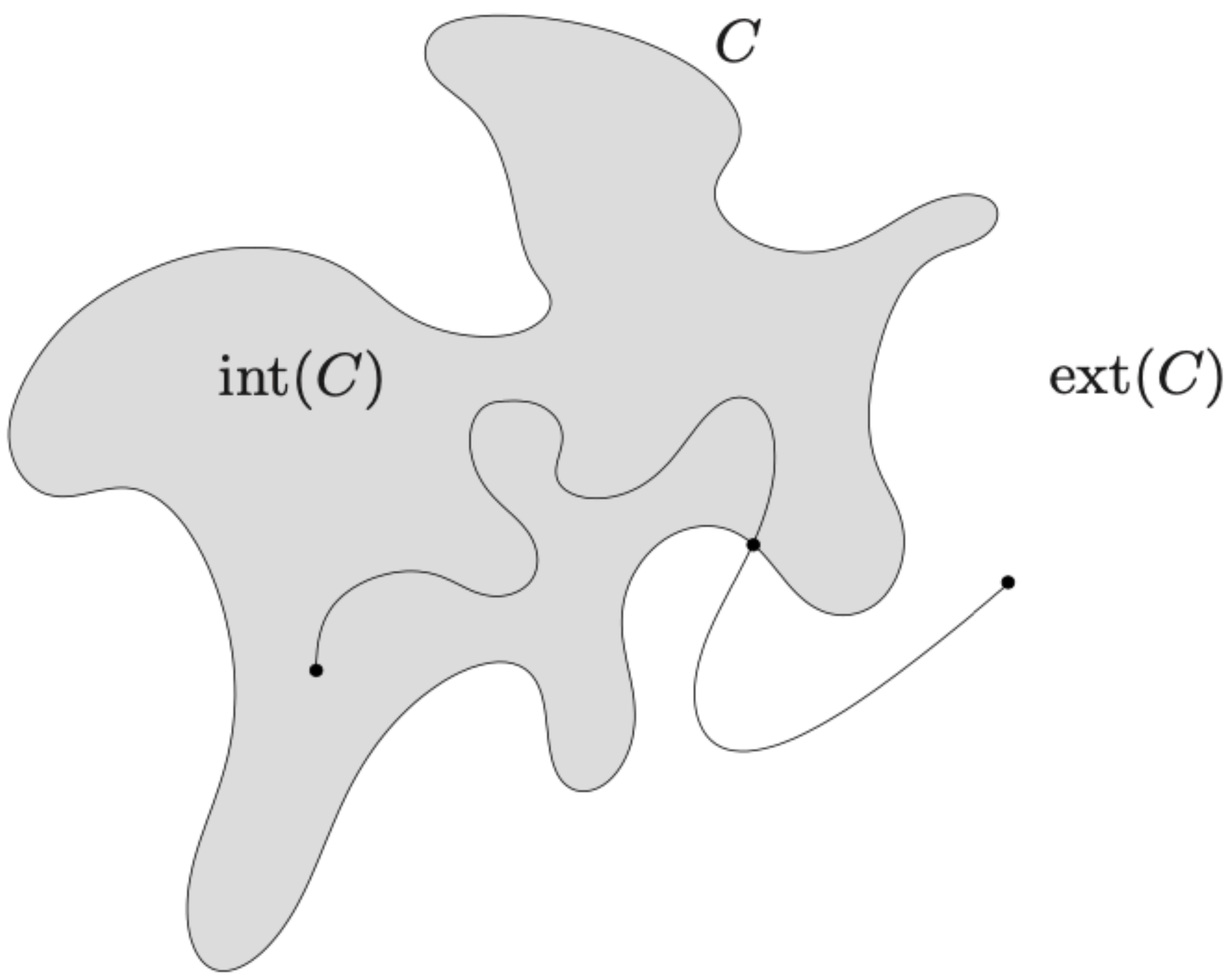
By a curve, we mean a continuous image of a closed unit line segment.
A closed curve is a continuous image of a circle.
A curve or closed curve is simple if it does not intersect itself (in other words, if the mapping is one-to-one).
Properties of such curves come into play in the study of planar graphs because cycles in plane graphs are simple closed curves.

A subset of the plane is arcwise-connected (弧连通的) if any two of its points can be connected by a curve lying entirely within the subset.

The basic result of topology that we need is the Jordan Curve Theorem.

Theorem 23(Jordan). Any simple closed curve C in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets.

The two open sets into which a simple closed curve C partitions the plane are called the interior and the exterior of C, denote them by $\mathrm{int}(C)$ and $\mathrm{ext}(C)$, and their closures by $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$, respectively ($\mathrm{Int}(C) \cap \mathrm{Ext}(C) = C$). The Jordan Curve Theorem implies that every arc joining a point of $\mathrm{int}(C)$ to a point of $\mathrm{ext}(C)$ meets C in at least one point.



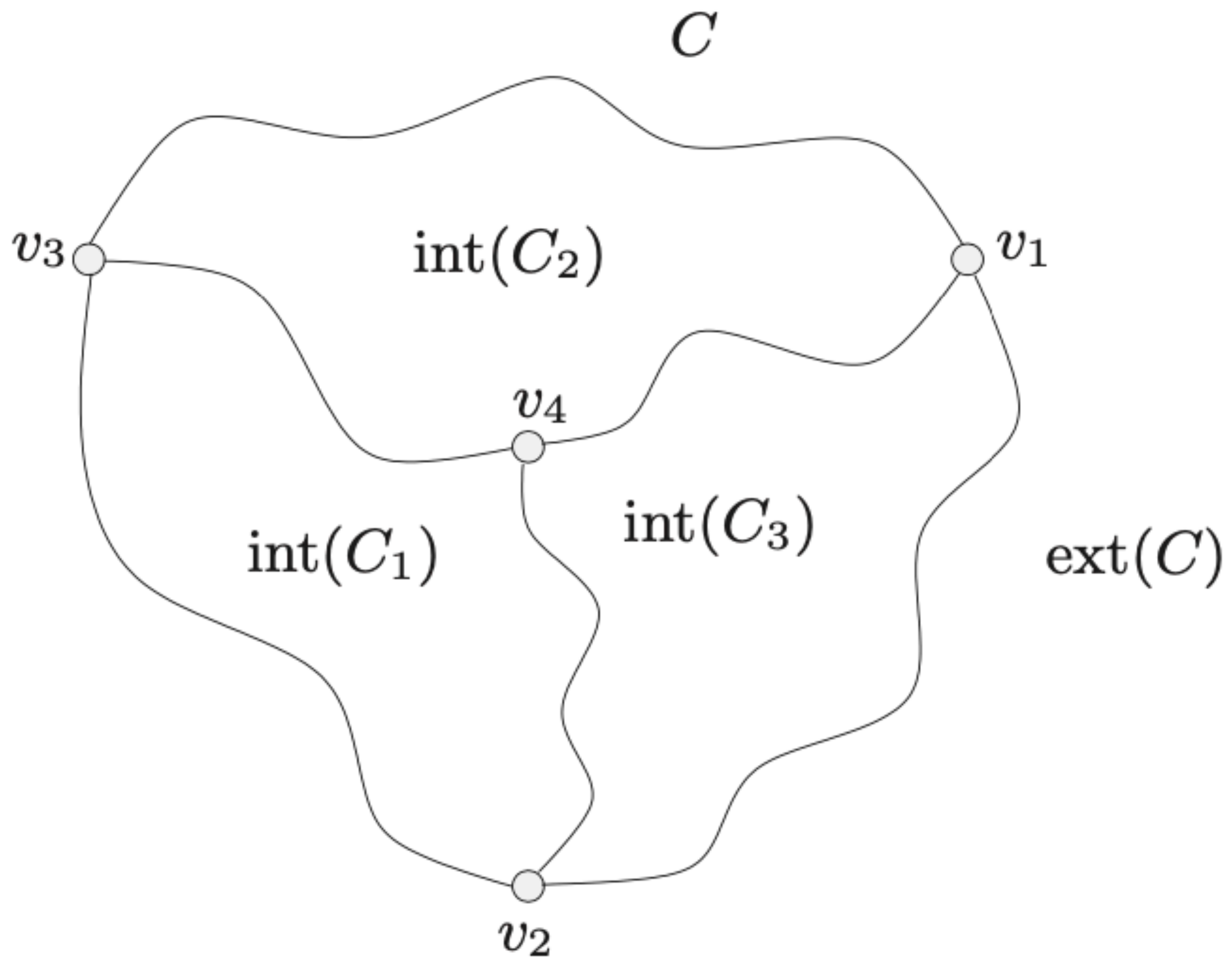
Theorem 24. K_5 is nonplanar.

Proof. Let G be a planar embedding of K_5, with vertices v_1, v_2, v_3, v_4, v_5. Because G is complete, any two of its vertices are joined by an edge.

In particular, the cycle $C = v_1 v_2 v_3$ is a simple closed curve in the plane, and the vertex v_4 must lie either in $\text{int}(C)$ or in $\text{ext}(C)$.

Without loss of generality, we may suppose that $v_4 \in \text{int}(C)$. Then the edges $v_1 v_4, v_2 v_4, v_3 v_4$ all lie entirely in $\text{int}(C)$, too (apart from their ends v_1, v_2, v_3).

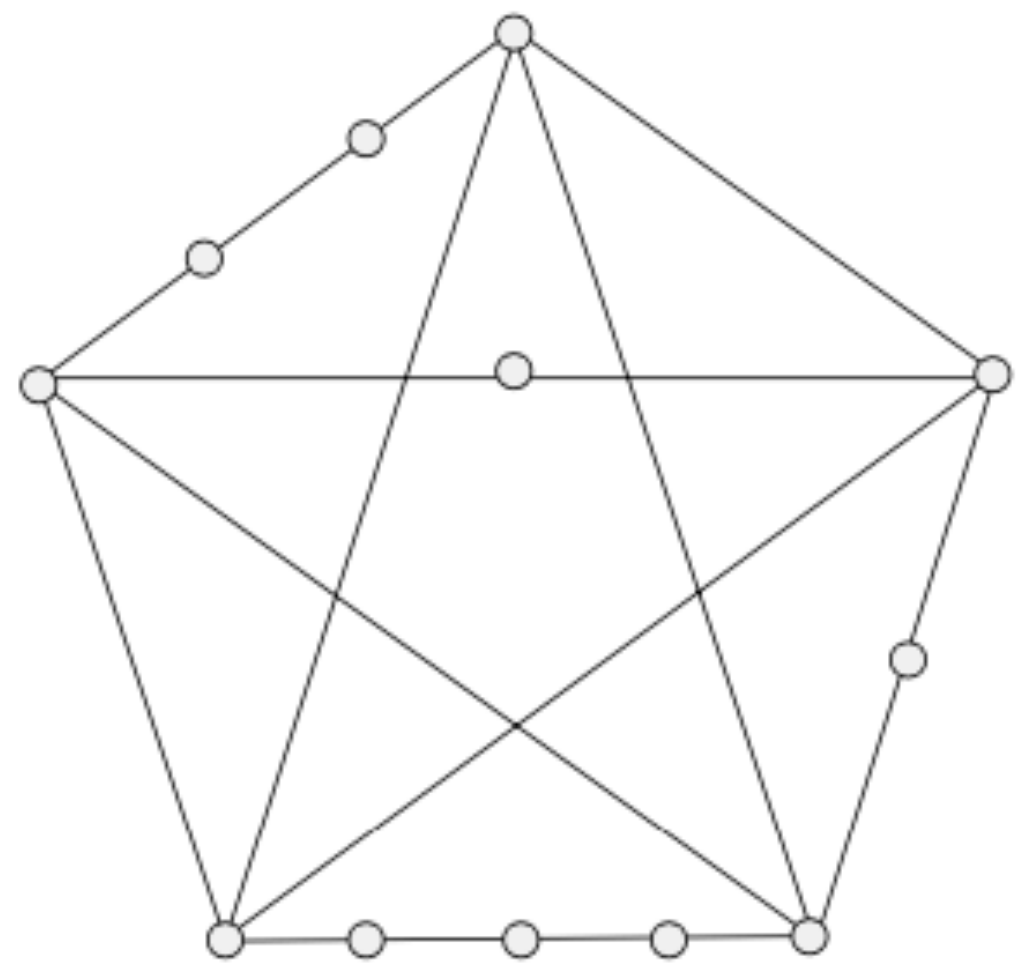
Let $C_1 = v_2 v_3 v_4$, $C_2 = v_3 v_1 v_4$, and $C_3 = v_1 v_2 v_4$.
Observe that $v_i \in \text{ext}(C_i)$ for $i = 1,2,3$.



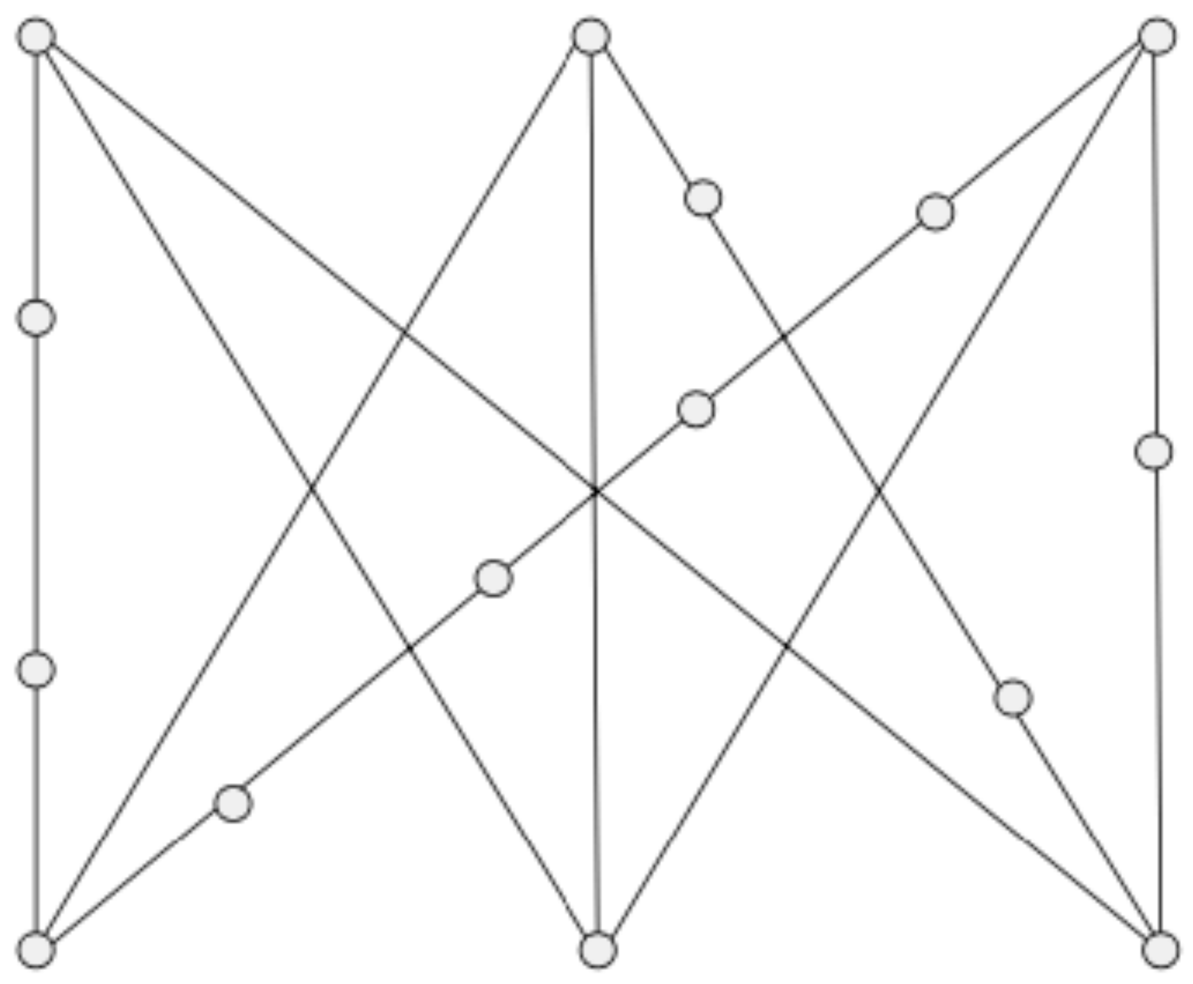
Because $v_i v_5 \in E(G)$ and G is a plane graph, it follows from the Theorem 23 that $v_5 \in \text{ext}(C_i)$, $i = 1,2,3$, too. Thus, $v_5 \in \text{ext}(C)$.
But now the edge $v_4 v_5$ crosses C, again by the Theorem 23, which contradicts the planarity of the embedding G.

Subdivisions (细分)

Any graph derived from a graph G by a sequence of edge subdivisions is called a subdivision of G or a G-subdivision.



A subdivision of K_5.



A subdivision of $K_{3,3}$.

Proposition 9. A graph G is planar if and only if every subdivision of G is planar.

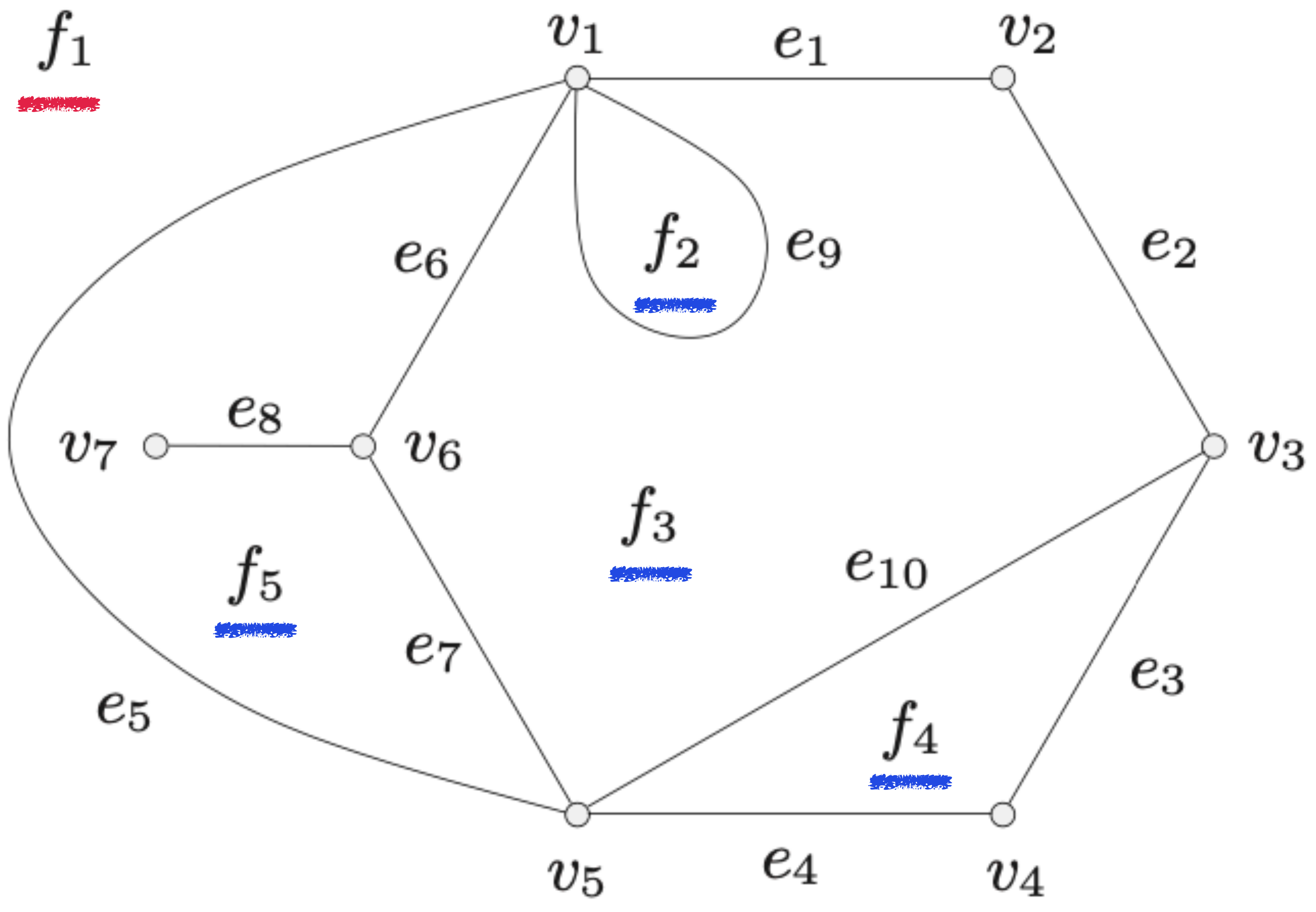
By the same method as in Theorem 24, one can show that $K_{3,3}$ is nonplanar. What graphs are planar?

Planarity being such a fundamental property, the problem of deciding whether a given graph is planar is clearly of great importance.
A major step towards this goal is provided by the following characterization of planar graphs, due to Kuratowski (1930).

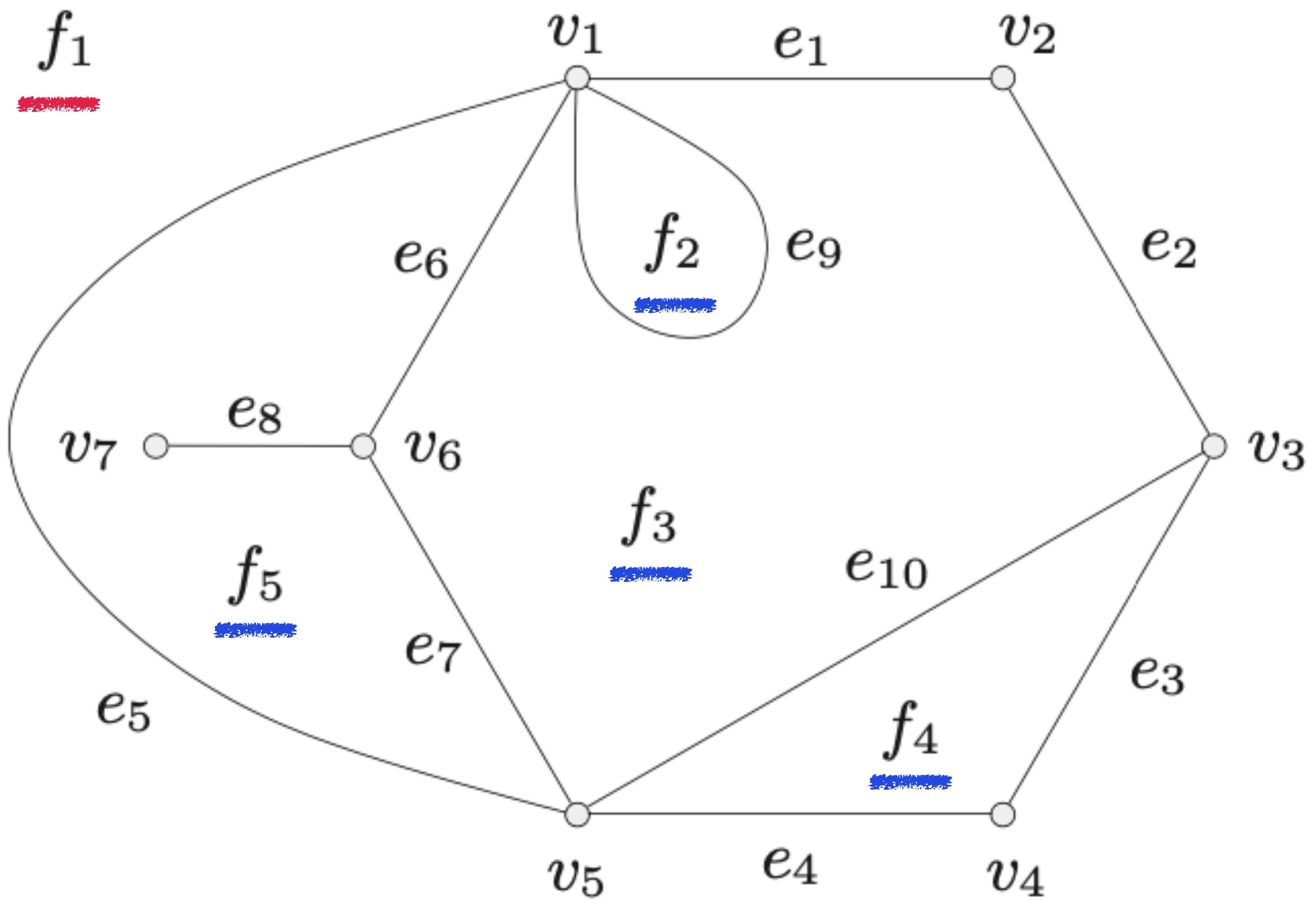
Theorem 25 (Kuratowski). A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.

Duality (对偶)

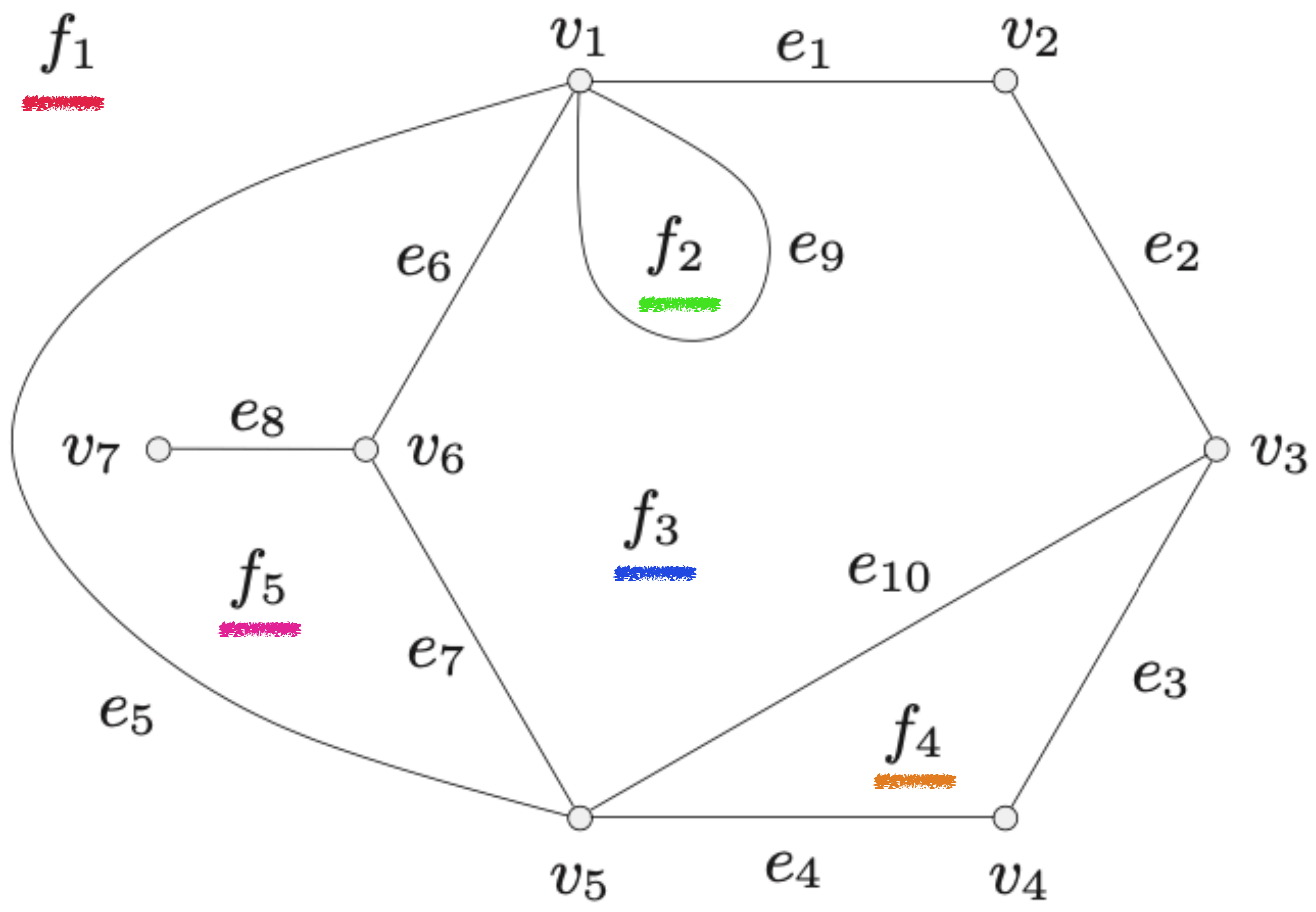
A plane graph G partitions the rest of the plane into a number of arcwise-connected open sets. These sets are called the faces of G. Each plane graph has exactly one unbounded face, called the outer face.

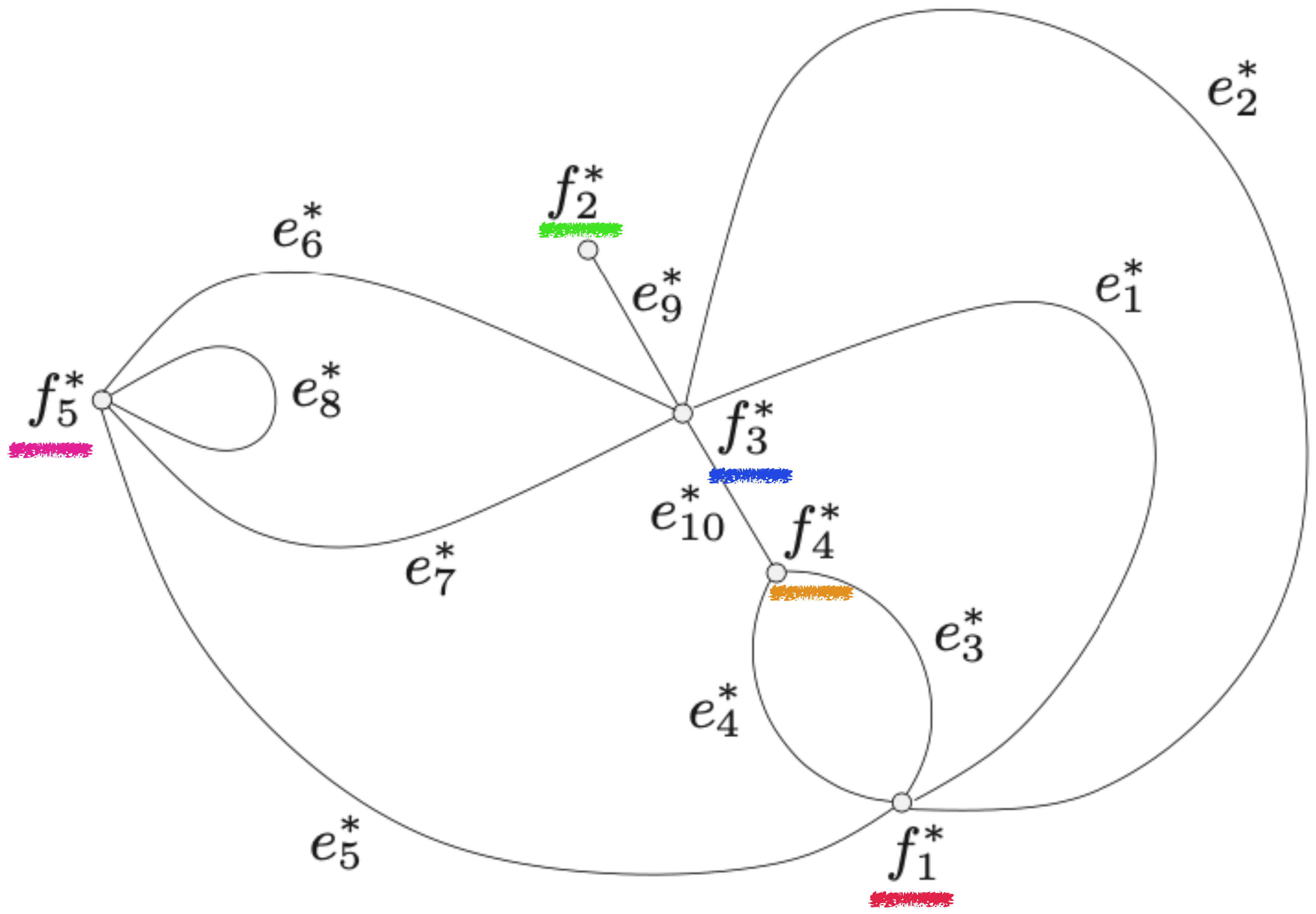


The boundary of a face f
is the boundary of the open set f in the usual topological sense.
A face is said to be incident with the vertices and edges in its boundary, and two faces are adjacent if their boundaries have an edge in common.
We denote the boundary of a face f by $\partial(f)$.

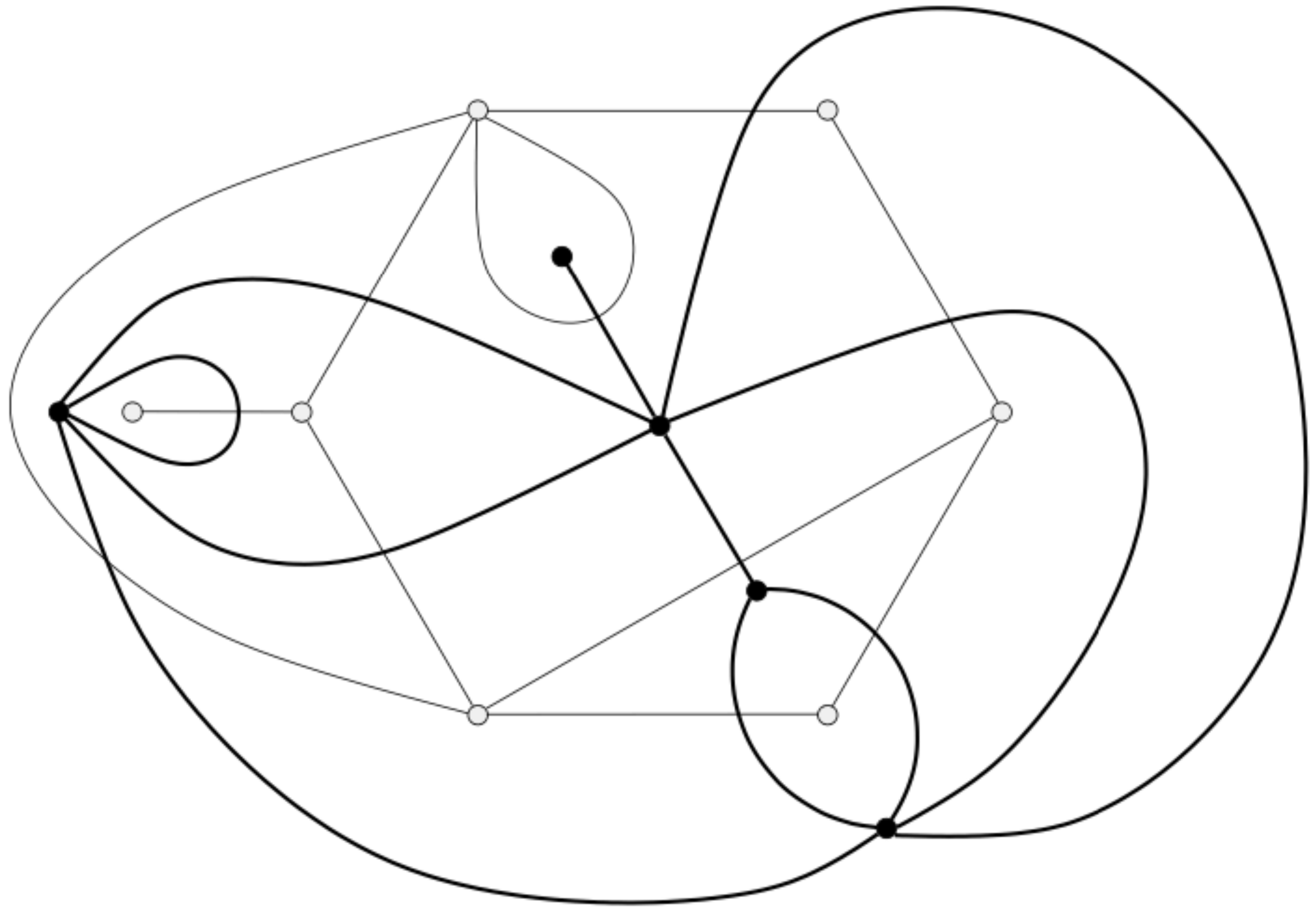


DUAL: Given a plane graph G, one can define a second graph G^* as follows. Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*.
Two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G.





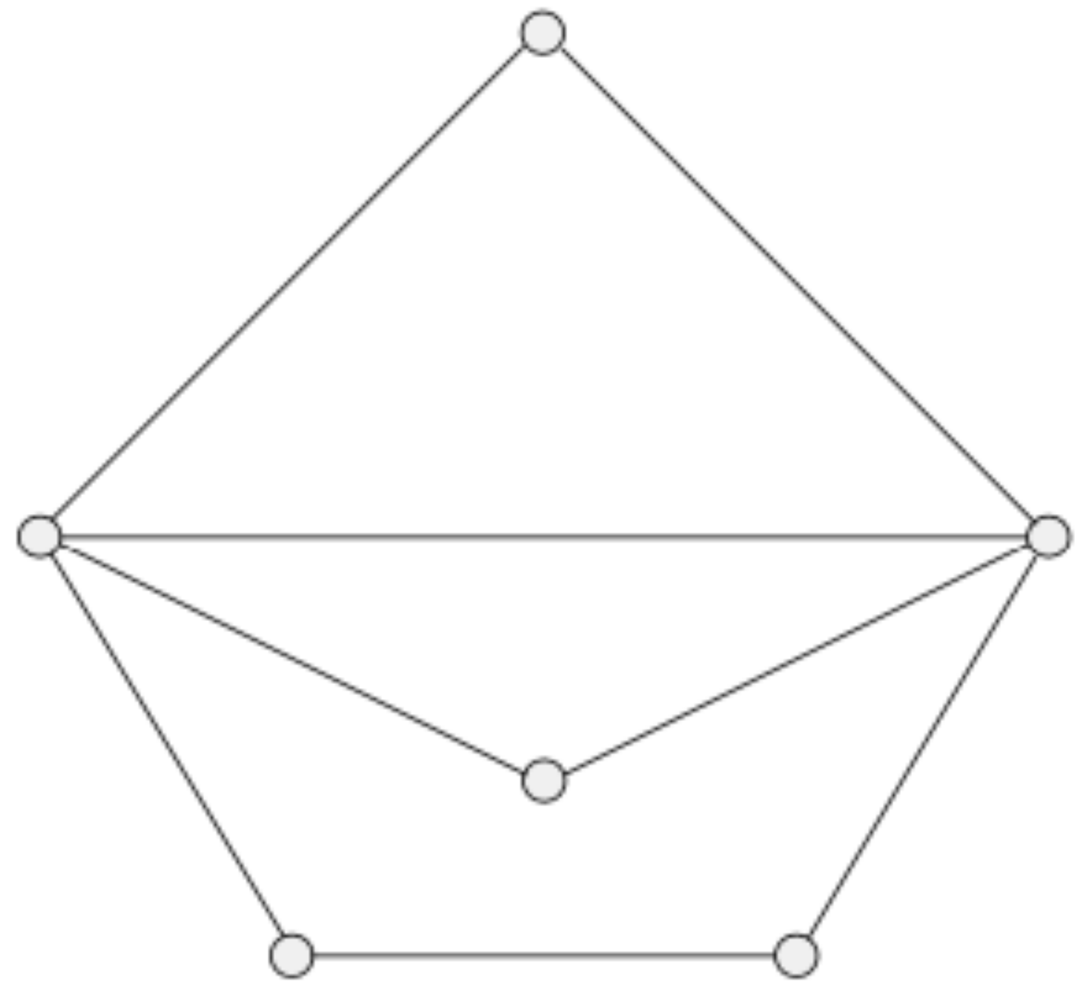
It is easy to see that the dual G^* of a plane graph G is itself a planar graph. In fact, there is a natural embedding of G^* in the plane. Place each vertex f^* in the corresponding face f of G, and then draw each edge e^* in such a way that it crosses the corresponding edge e of G exactly once (and crosses no other edge of G).

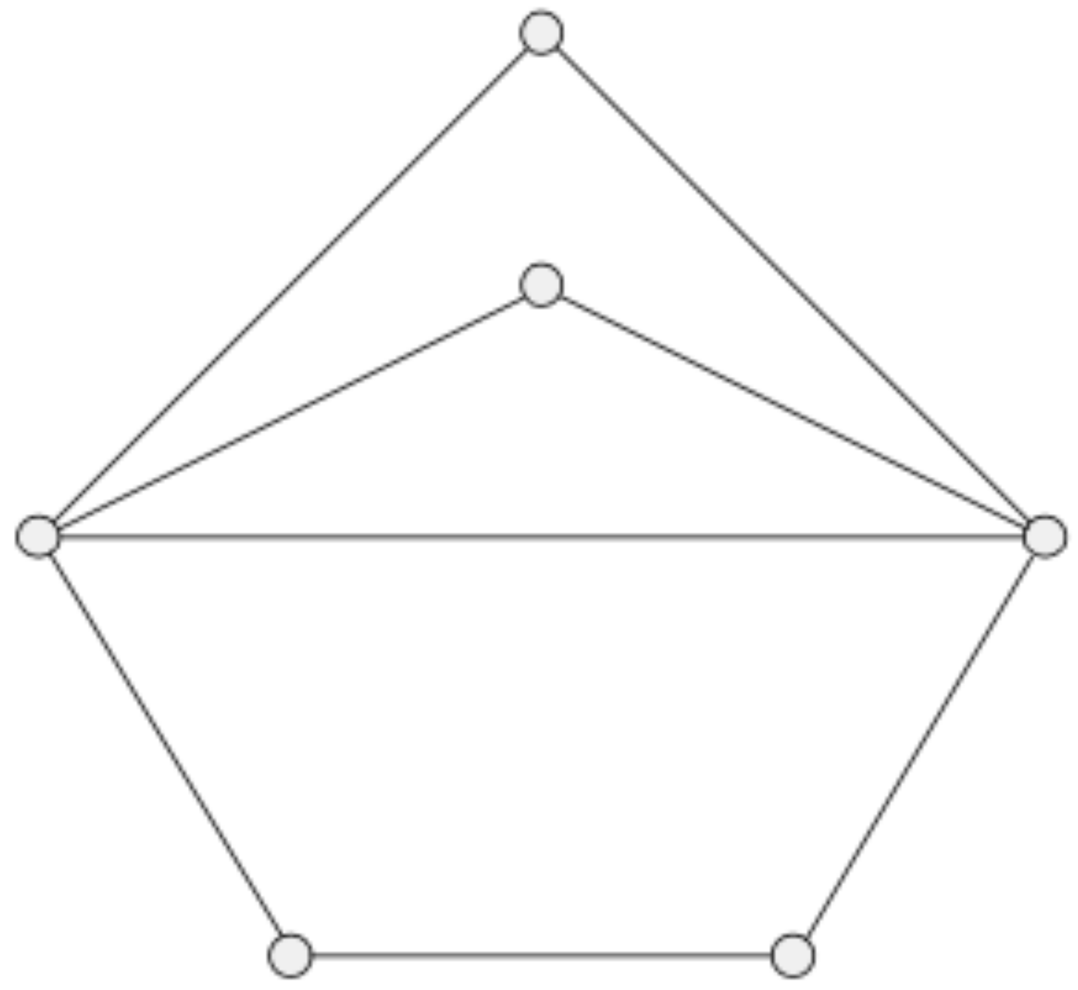


The degree $d_G(f)$ of a face f in a plane graph is the number of edges incident with f, where an cut edge in the boundary of f is counted twice.

$$v(G^*) = f(G), \qquad e(G^*) = e(G), \qquad d_{G^*}(f^*) = d_G(f).$$

It should be noted that two isomorphic plane graphs may have nonisomorphic duals.





Proposition 10. If G is plane graph of size m, then

$$\sum_{f \in F} d_G(f) = 2m\,.$$

Euler's Formula

Theorem 26. For a connected plane graph,

$$v(G) + f(G) - e(G) = 2.$$

Proof. By induction on the face number $f(G)$.

If $f(G) = 1$, each edge of G is a cut edge and so G, being connected, is a tree. In this case $e(G) = v(G) - 1$, and hence the assertion holds.

Suppose that it is true for all connected plane graphs with fewer than f faces, where $f \geq 2$, and let G be a connected plane graph with f faces.
Choose an edge e of G that is not a cut edge.
Then $G\backslash e$ is a connected plane graph with $f - 1$ faces,
because the two faces of G separated by e coalesce to form one face of $G\backslash e$.
By the induction hypothesis,

$$v(G\backslash e) + f(G\backslash e) - e(G\backslash e) = 2.$$

Note that $v(G\backslash e) = v(G)$, $e(G\backslash e) = e(G) - 1$, and $f(G\backslash e) = f(G) - 1$, we obtain

$$v(G) + f(G) - e(G) = 2.$$

Corollary 5. All planar embeddings of a connected planar graph have the same number of faces.

Corollary 6. Let G be a simple planar graph on $n \geq 3$ vertices and of size m. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of G is a triangulation.

Proof. It clearly suffices to prove the corollary for connected graphs.
Let $\tilde{G}$ be any planar embedding of G.
Because G is simple and connected and $n \geq 3$, $d(f) \geq 3$ for all $f \in F(\tilde{G})$.
Therefore,

$$2m = \sum_{f \in F(\tilde{G})} d(f) \geq 3f(\tilde{G}) = 3(m - n + 2)\,. \qquad (1)$$

Equivalently,

$$m \le 3n - 6. \tag{2}$$

Equality holds in (2) if and only if it holds in (1), if and only if $d(f) = 3$ for each face $f \in F(\tilde{G})$.

Corollary 7. Every simple planar graph has a vertex of degree at most 5.

The girth $g(G)$ of a graph G is the length of a shortest cycle in G.

Corollary 8. Let G be a simple planar graph on $n \ge 3$ vertices and of size m. If $g(G) = k$, then $m \le \dfrac{k}{k-2}(n-2)$.

Proof. It clearly suffices to prove the corollary for connected graphs.
Let $\tilde{G}$ be any planar embedding of G.
Because $g(G) = k$, $d(f) \geq k$ for all $f \in F(\tilde{G})$. Therefore, we have

$$2m = \sum_{f \in F(\tilde{G})} d(f) \geq k \cdot f(\tilde{G}) = k(m - n + 2)\,.$$

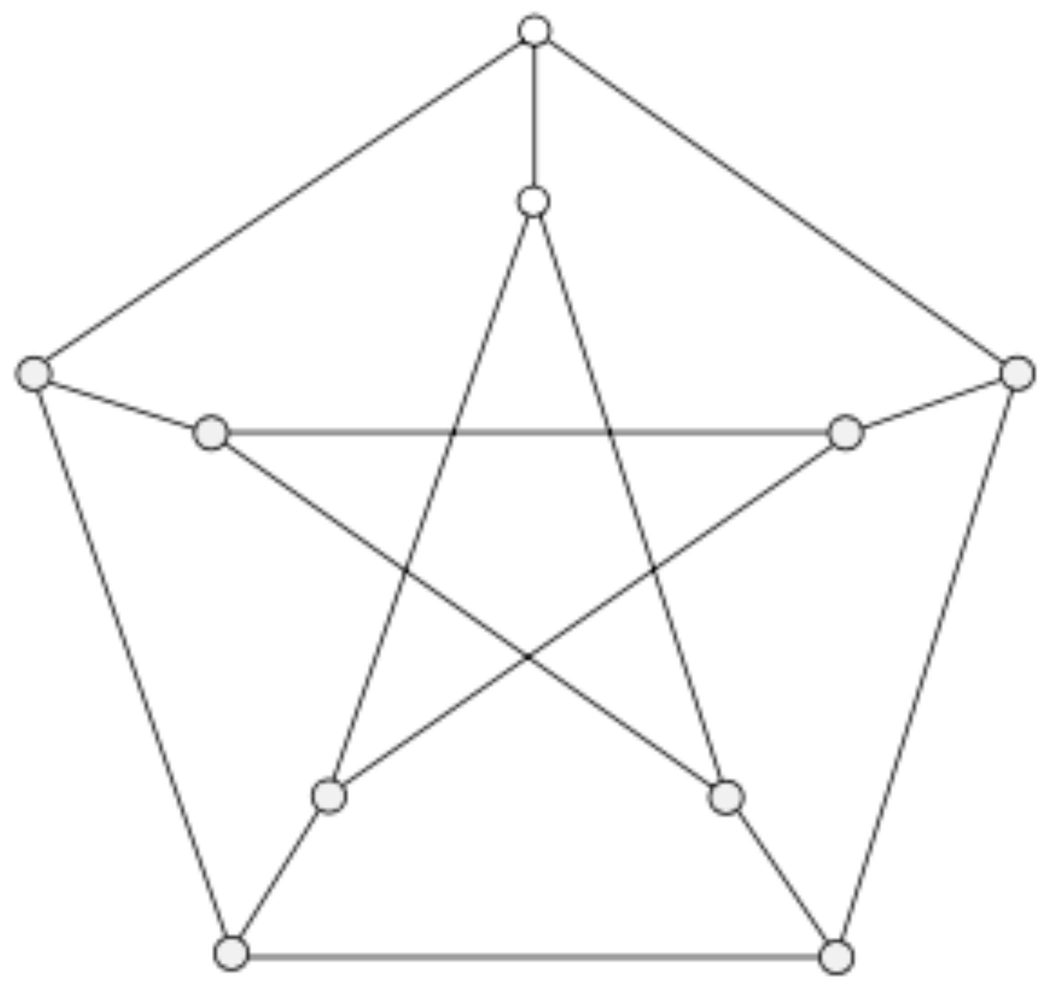
That is,

$$m \leq \frac{k}{k-2}(n-2)\,.$$

Corollary 9. K_5 is nonplanar.

Corollary 10. $K_{3,3}$ is nonplanar.

Corollary 11. Petersen graph is nonplanar.



Exercise 8.

1. Show that any planar graph 6-colorable.

2. Show that the complement of a simple planar graph on at least 11 vertices is nonplanar.

3. A plane graph is face-regular if all of its faces have the same degree. Characterize the plane graphs which are both regular and face-regular.