Colorings of Planar Graphs

In a letter written to William Rowan Hamilton in 1852, Augustus De Morgan communicated the following Four-Color Problem, posed by Francis Guthrie.

> A student of mine [Frederick Guthrie, brother of Francis] asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be anyhow divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored—four colors may be wanted, but not more–the following is the case in which four colors are wanted. Query cannot a necessity for five or more be invented……

This surmise became known as the Four-Color Conjecture(4CC).

In order to translate the Four-Color Problem into the language of graph theory, we need the notion of a face coloring of a plane graph.
A *k*-face coloring of a plane graph is an assignment of k colors to its faces. The coloring is proper if no two adjacent faces are assigned the same color. A plane graph is k-face-colorable if it has a proper k-face coloring.

Conjecture 1 (4CC, Face Version). Every plane graph without cut edges is 4-face-colorable.

Conjecture 2 (4CC, Vertex Version).
Every loopless planar graph is 4-colorable.

The Four-Color Conjecture was verified, in 1977, by Appel and Haken.

If 4CC is false, then there is a planar graph G which is not 4-colorable.
Choose such a G that $v(G) + e(G)$ is as small as possible.
We call G a smallest counterexample to 4CC.

Proposition 11. Let G be a smallest counterexample to the 4CC. Then

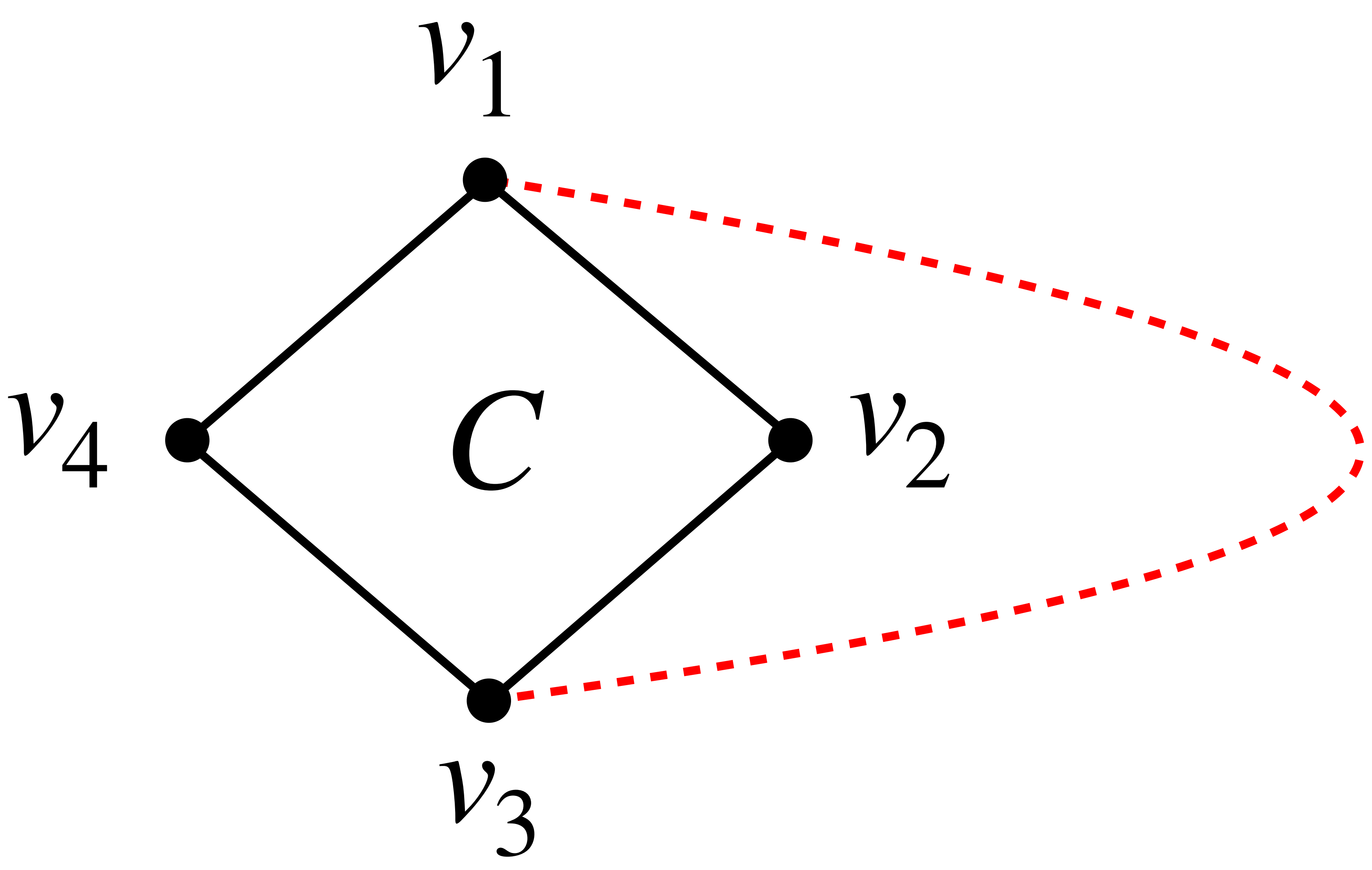
(1) G is 5-critical;

(2) G is a triangulation.

(3) G has no vertex of degree less than four.

Proof. It is easy to see that both (1) and (3) hold.
(2) To see that G is a triangulation, suppose that it has a face whose boundary is a cycle C of length greater than three.

Because G is planar, there must exist two vertices x and y of C which are nonadjacent in G.



The graph $G/\{x, y\}$ obtained by identifying x and y into a single vertex z is a planar graph with fewer vertices than G, and the same number of edges, hence has a 4-coloring c. The coloring of G derived from c by assigning the color $c(z)$ to both x and y is then a 4-coloring of G, a contradiction.

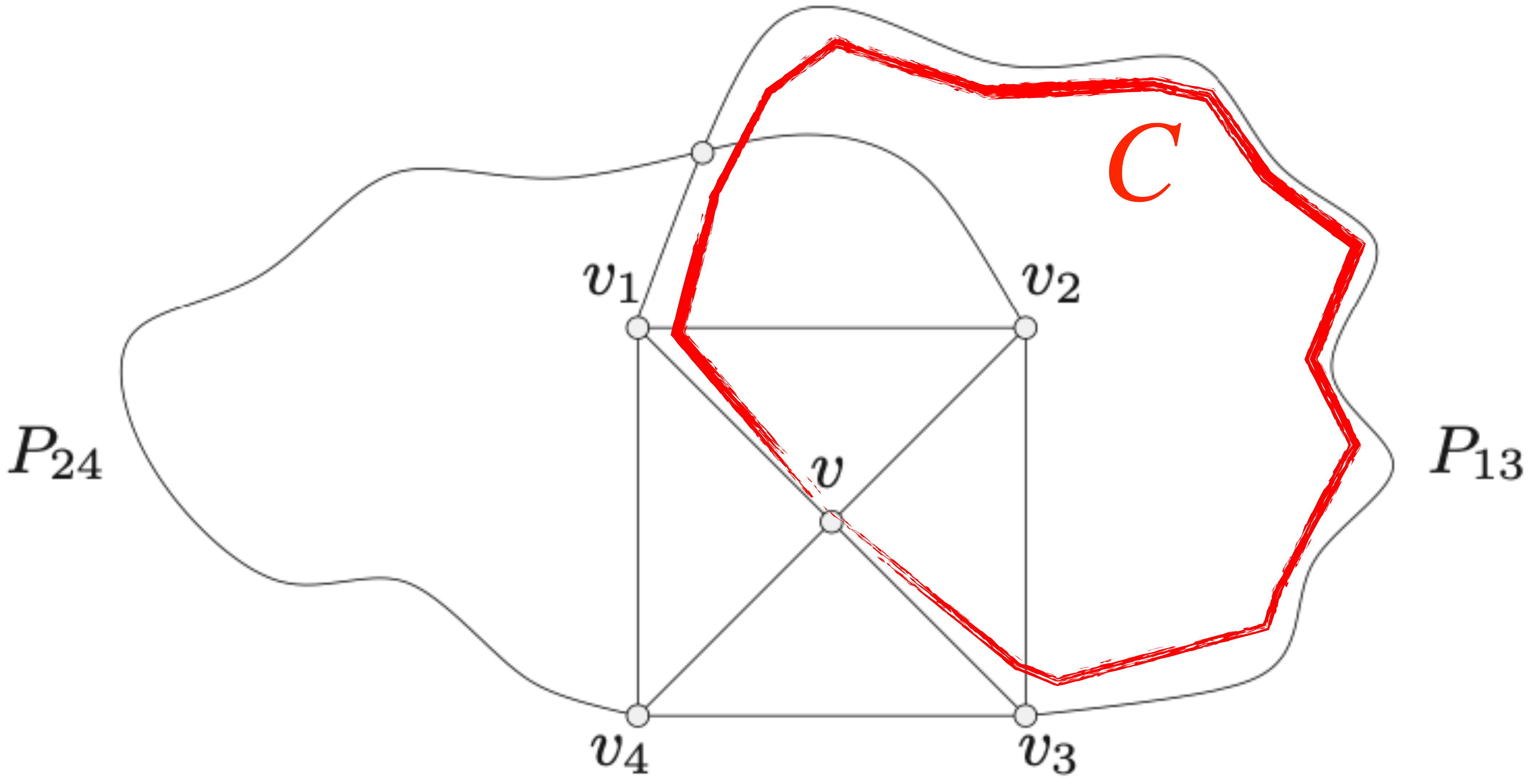
Theorem 27 (Kempe). Let G be a smallest counterexample to 4CC. Then G has no vertex of degree four.

Proof. Assume that v is a vertex of degree four in G, and let $\{V_1, V_2, V_3, V_4\}$ be a 4-coloring of $G - v$; such a coloring exists because G is 5-critical.

Because G is not 4-colorable, v must be adjacent to a vertex of each color. Therefore, we can assume that the neighbors of v in clockwise order around v are v_1, v_2, v_3, and v_4, where $v_i \in V_i$ for $1 \leq i \leq 4$.

Let $G_{ij} = G[V_i \cup V_j]$. Then v_i and v_j belong to the same component of G_{ij}.
If not, consider the component of G_{ij} that contains v_i.
By interchanging the colors i and j in this component, we obtain a new 4-coloring of $G - v$ in which only three colors (all but i) are assigned to the neighbors of v. This means G is 4-colorable, a contradiction.

Therefore, v_i and v_j indeed belong to the same component of G_{ij}. Let P_{ij} be a (v_i, v_j)-path in G_{ij}, and let C denote the cycle $vv_1P_{13}v_3v$ as shown below:



Because C separates v_2 and v_4, it follows from the Jordan Curve Theorem that the path P_{24} meets C in some point. Since G is a plane graph, this point must be a vertex. But this is impossible, because the vertices of P_{24} have colors 2 and 4, whereas no vertex of C has either of these colors.

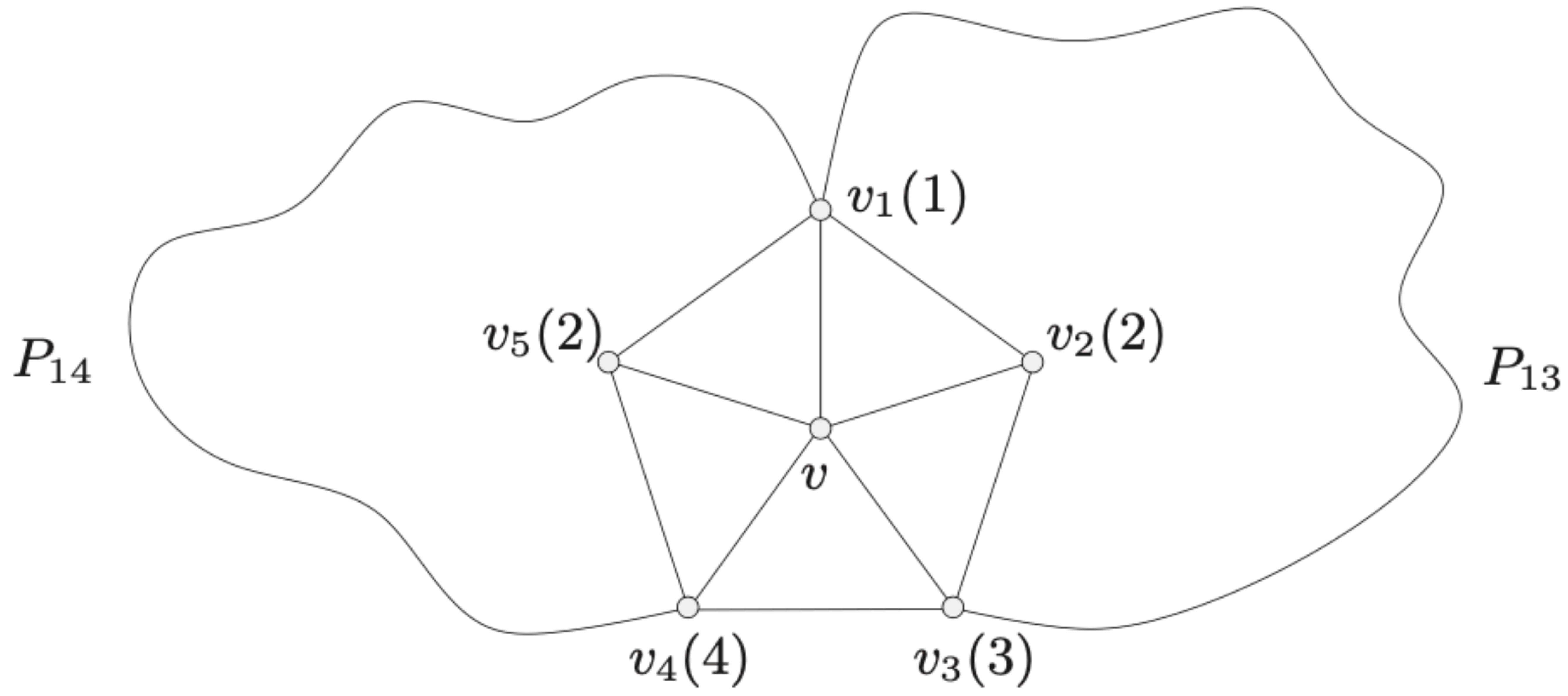
Kempe's (1879) false proof for 4CC

Suppose that G has a vertex v of degree five with $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. By the minimality of G, the subgraph $G - v$ has a 4-coloring $\{V_1, V_2, V_3, V_4\}$.

Kempe's aim is to find such a 4-coloring in which at most three colors are assigned to the neighbors of v; the vertex v can then be colored with one of the remaining colors, resulting in a 4-coloring of G.

Because G is not 4-colorable, v must be adjacent to a vertex of each of the four colors 1,2,3,4.
Therefore, we can assume that the neighbors of v in clockwise order around v are v_1, v_2, v_3, v_4, and v_5, where $v_i \in V_i$ for $1 \le i \le 4$, and $v_5 \in V_2$.

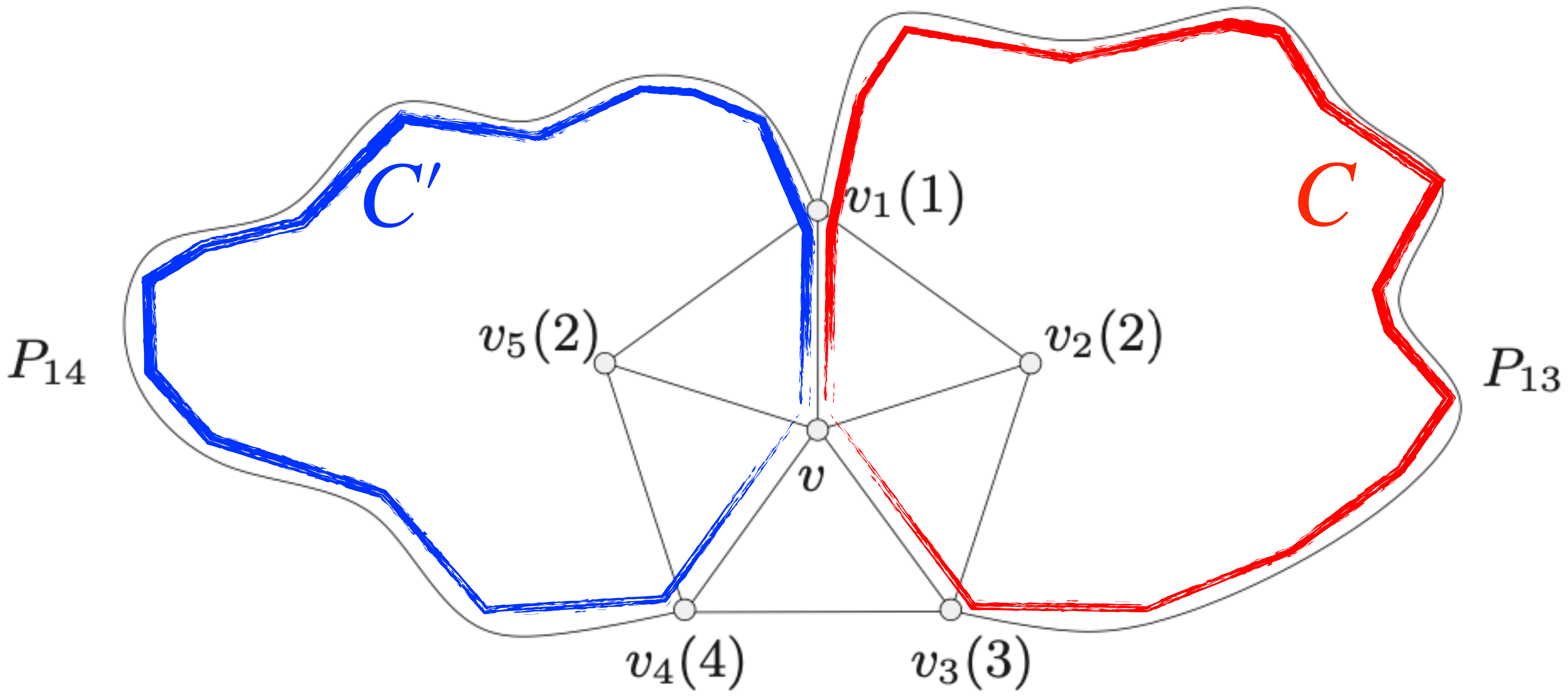
Let $G_{ij} = G[V_i \cup V_j]$.
We may assume that v_1 and v_3 belong to the same component of G_{13}, and that v_1 and v_4 belong to the same component of G_{14}, otherwise the colors could be switched in the component of G_{13} or G_{14} containing v_1, respectively, resulting in a 4-coloring in which only three colors are assigned to the neighbors of v.



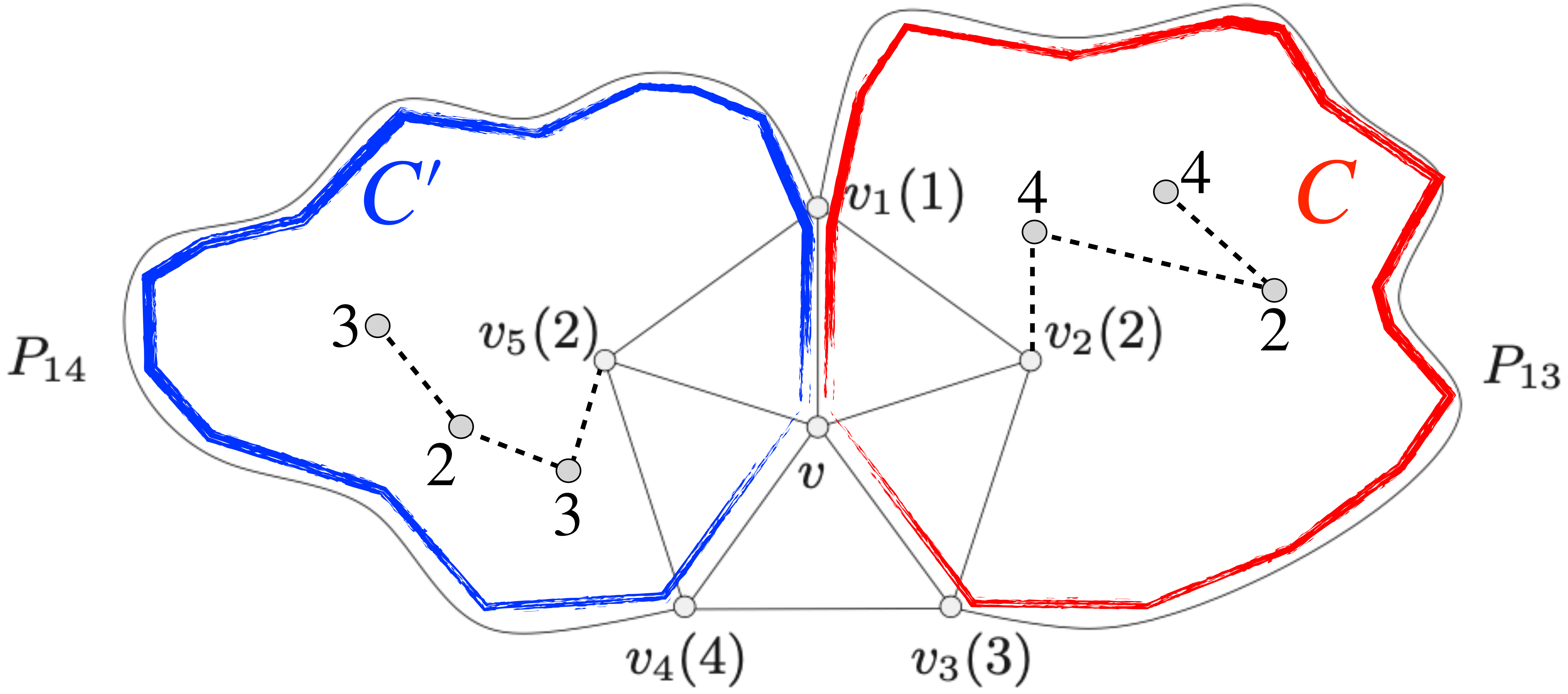
Let P_{13} be a (v_1, v_3)-path in G_{13} and P_{14} a (v_1, v_4)-path in G_{14}.

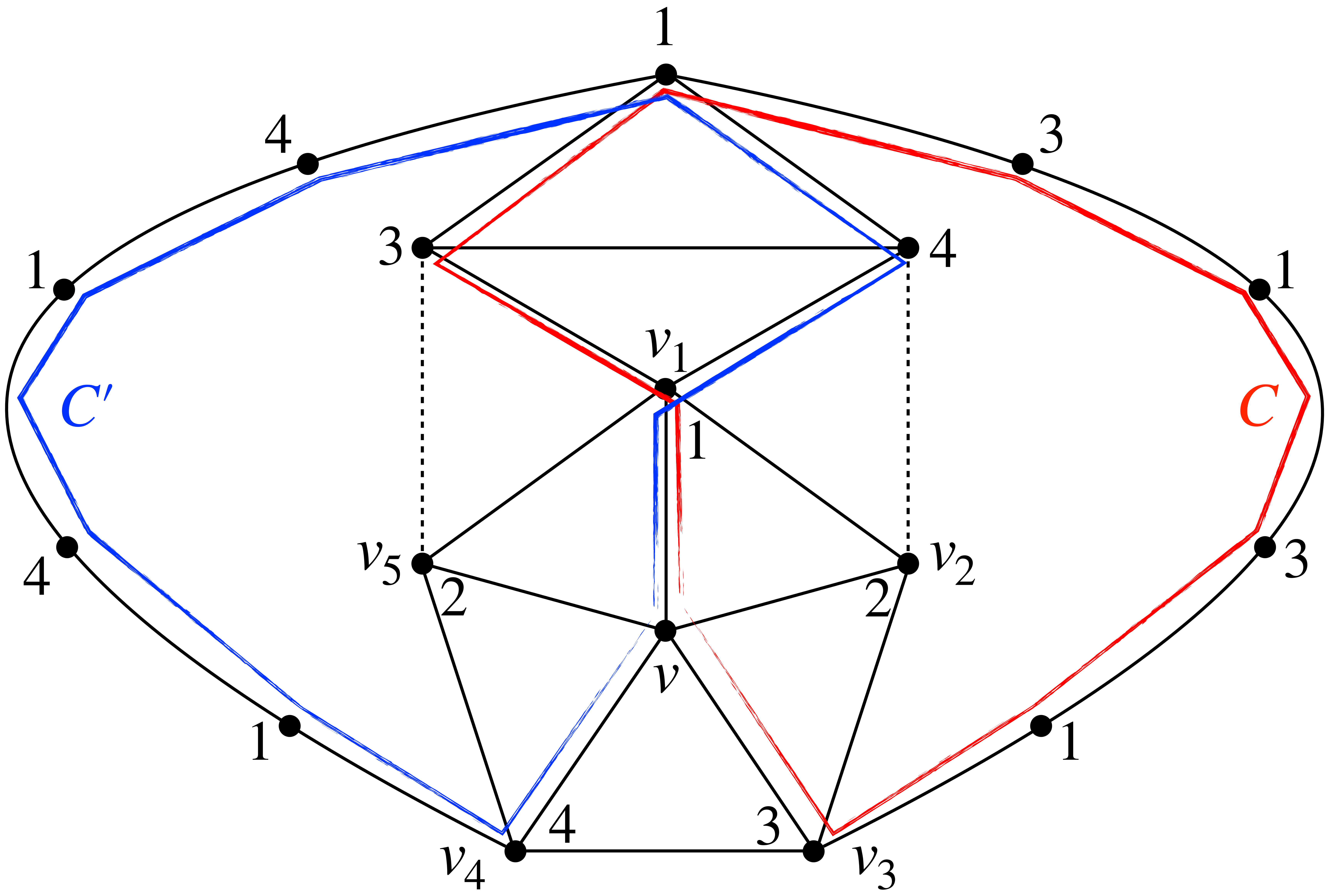
The cycle $C = vv_1P_{13}v_3v$ separates vertices v_2 and v_4; thus v_2 and v_4 belong to different components of G_{24}.

Similarly, the cycle $C' = vv_1P_{14}v_4v$ separates v_3 and v_5, so v_3 and v_5 belong to different components of G_{23}.

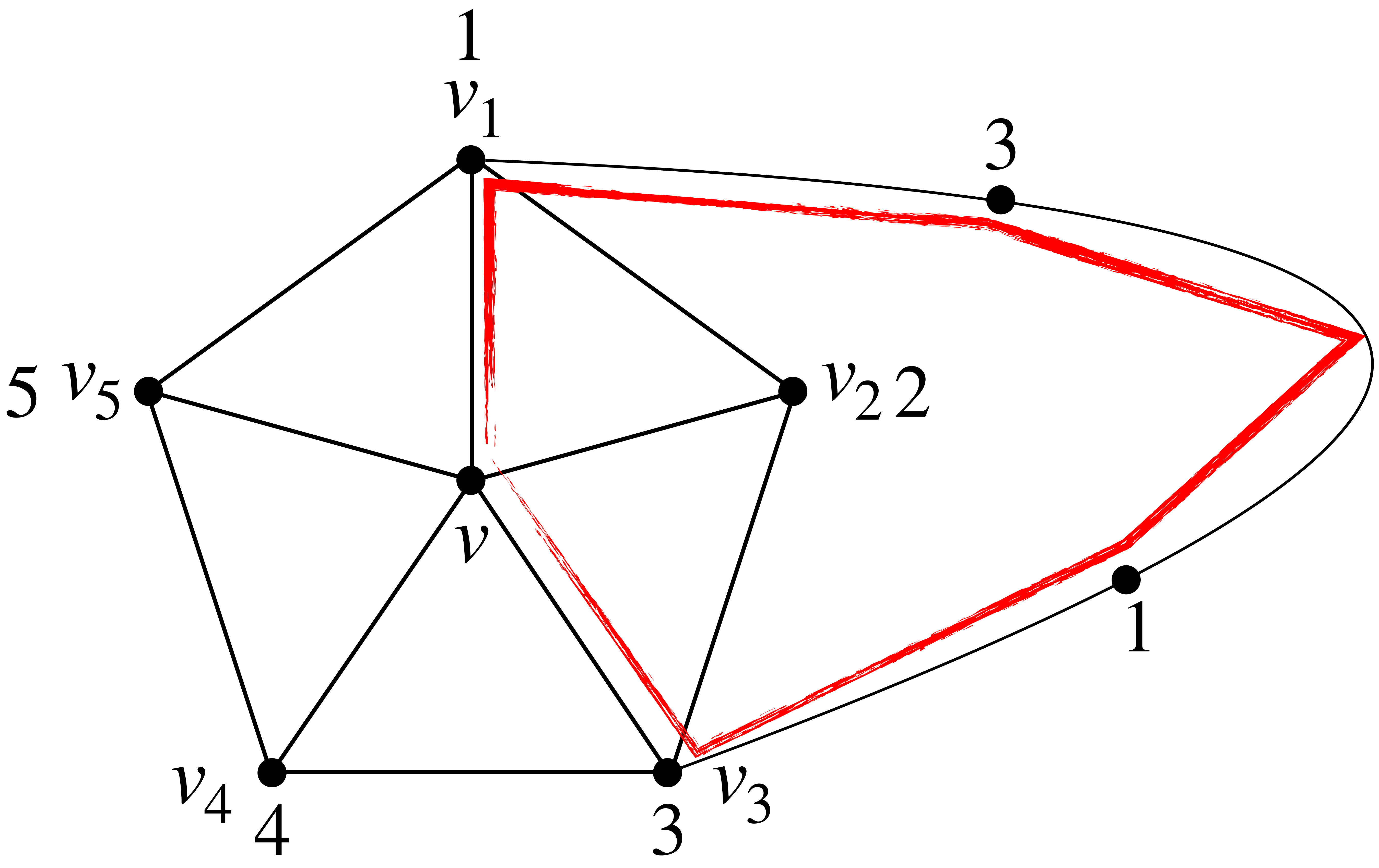


In light of these observations, Kempe argued that the colors 2 and 4 in the component of G_{24} containing v_2, and the colors 2 and 3 in the component of G_{23} containing v_5, could be interchanged to produce a 4-coloring of $G - v$ in which just the three colors 1, 3, and 4, are assigned to the neighbors of v. On assigning color 2 to v, a 4-coloring of G would then be obtained.




1
4
3
1
1
3
4
v_1
C'
C
1
v_5
v_2
2
2
4
3
1
1
v
4
3
v_4
v_3

Theorem 28 (Heawood). Every loopless planar graph is 5-colorable.



List Colorings of Planar Graphs

Thomassen (1994) given another proof of the Five-Color Theorem for planar graphs, which is very elegant. To see this, we need the following concept.

A near-triangulation is a plane graph all of whose inner faces have degree 3.

Theorem 28 (Thomassen). Let G be near-triangulation whose outer face is bounded by a cycle C, and let x and y be consecutive vertices of C. Suppose that $L : V \mapsto 2^{\mathbb{N}}$ is an assignment of lists of colors to the vertices of G such that:

(i) $|L(x)| = |L(y)| = 1$, where $L(x) \neq L(y)$,

(ii) $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{x, y\}$,

(iii) $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$.

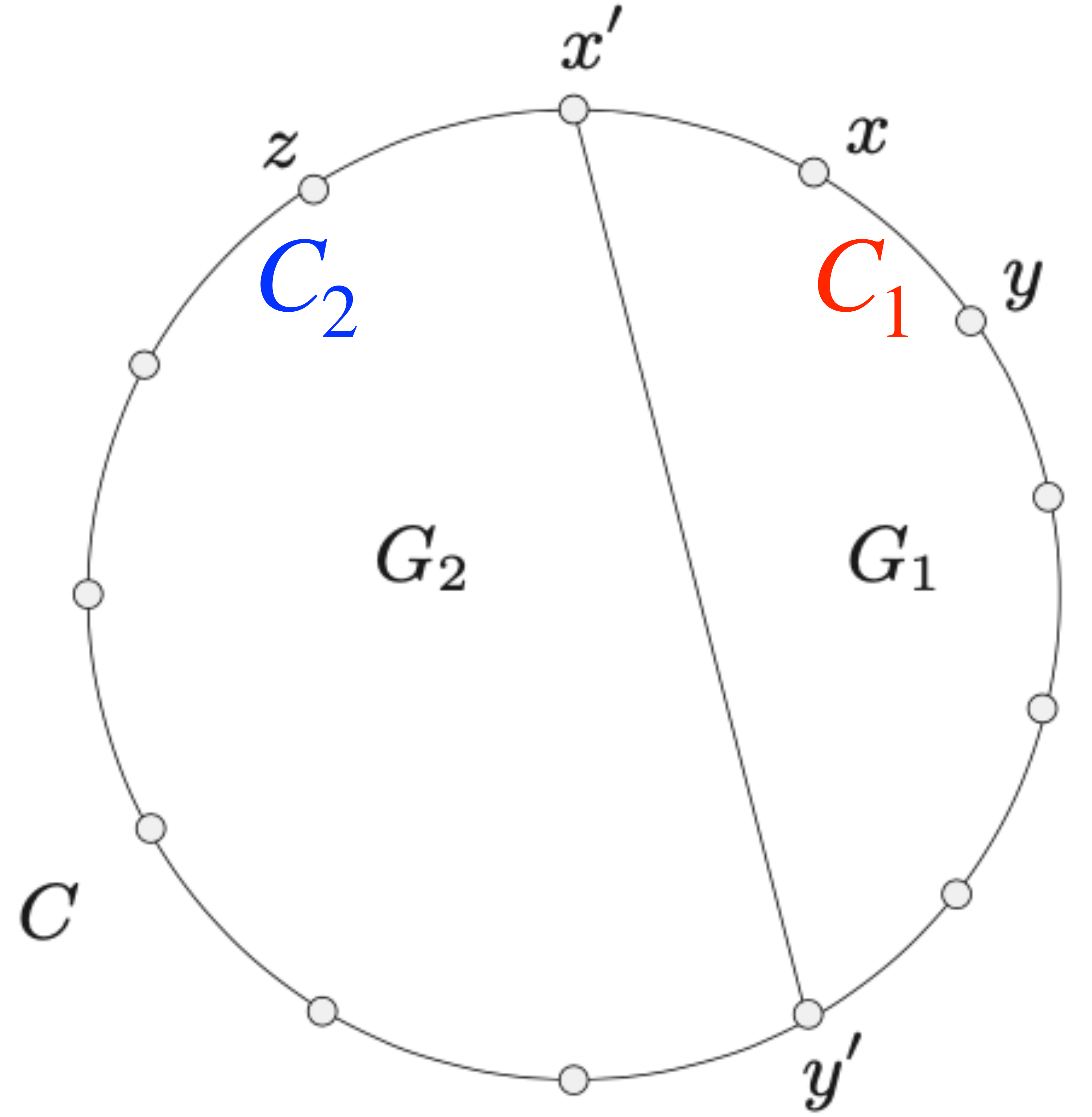
Then G is L-colorable.

Proof. By induction on $v(G)$. If $v(G) = 3$, then $G = C$ and the statement is trivial. So we may assume that $v(G) > 3$.

Let z and x' be the immediate predecessors of x on C.
Consider first the case where x' has a neighbor y' on C other than x and z.

$C_2 = x'zCy'x'$

G_2 :
A near-triangulation consisting of C_2 with its interior.



$C_1 = x'xCy'x'$

G_1 :
A near-triangulation consisting of C_1 with its interior.

Let L_1 denote the restriction of L to $V(G_1)$.
By induction, G_1 has an L_1-coloring c_1.

Now let L_2 be the function on $V(G_2)$ defined by

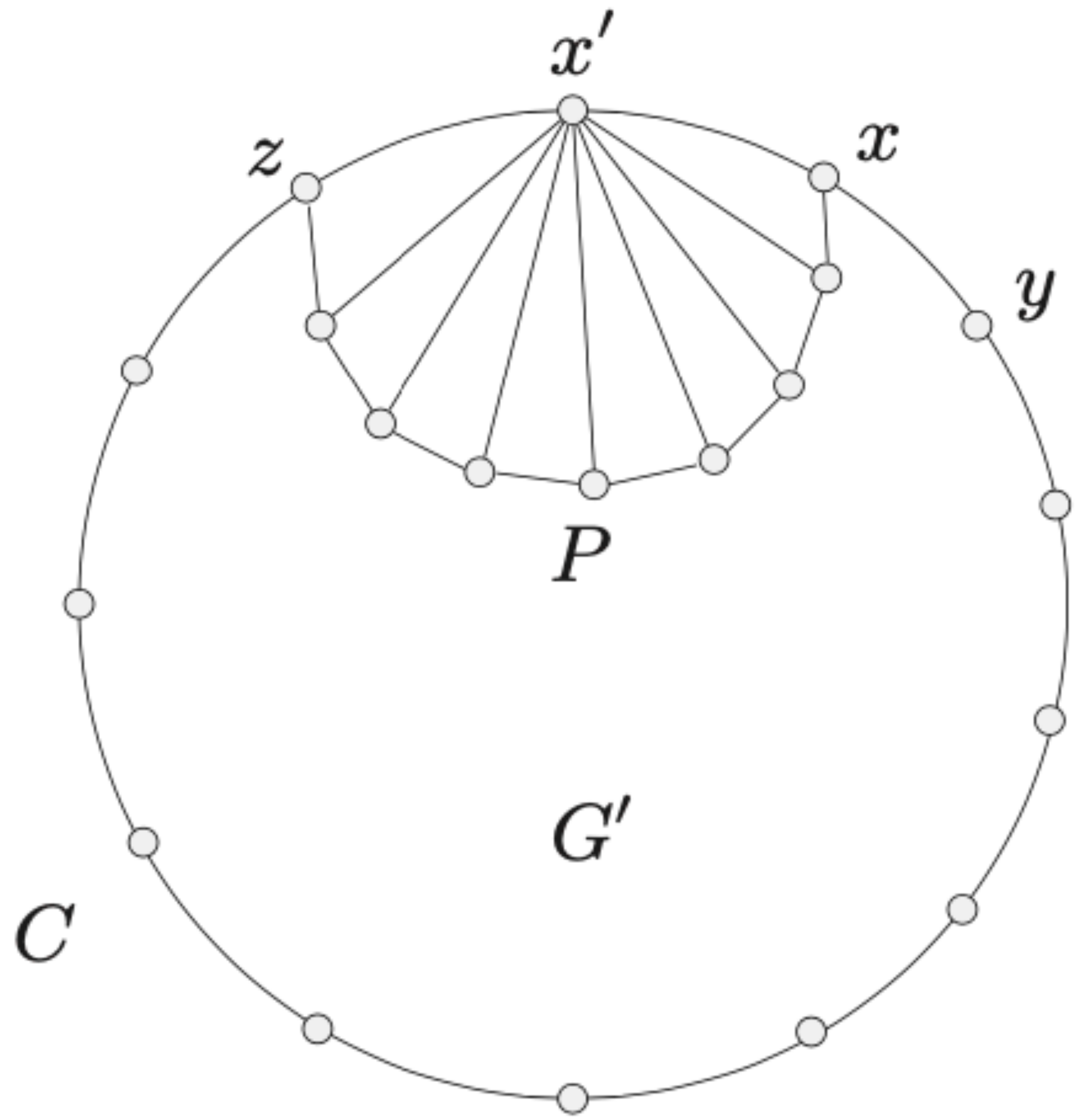
$$L_2(x') = \{c_1(x')\}, L_2(y') = \{c_1(y')\}, \text{ and } L_2(v) = L(v) \text{ for } v \in V(G_2) \backslash \{x', y'\}.$$

By induction (with x' and y' playing the roles of x and y, respectively), there is an L_2-coloring c_2 of G_2.

By the definition of L_2, the colorings c_1 and c_2 assign the same colors to x and y, the two vertices common to G_1 and G_2.

Thus the function c defined by $c(v) = c_1(v)$ for $v \in V(G_1)$, and $c(v) = c_2(v)$ for $v \in V(G_2) \backslash V(G_1)$ is an L-coloring of G.

Suppose now that the neighbors of x' lie on a path xPz internally disjoint from C, as shown below.



In this case, $G' = G - x'$ is a near-triangulation whose outer face is bounded by the cycle $C' = xCzPx$.

Let α and β be two distinct colors in $L(x')\backslash L(x)$.
Consider the function L' on $V(G')$ defined by

$$L'(v) = L(v)\backslash\{\alpha, \beta\} \text{ for } v \in V(P)\backslash\{x, z\}, \text{ and } L'(v) = L(v) \text{ otherwise.}$$

By induction, there is an L'-coloring c' of G'.

Since one of the colors α and β is different from $c'(z)$, by assigning that color to x', the coloring c' is extended to an L-coloring c of G.

Corollary 12. Every loopless planar graph is 5-list-colorable.

Corollary 13. Every loopless planar graph is 5-colorable.

Exercise 9.

1. Show that any Hamiltonian planar graph is 4-face-colorable.