Sliding Mode Observers

Theory and Practice

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Outline

1. Conventional Sliding Mode Observers
2. Higher Order Sliding Mode Observers
3. Cascaded HOSM Observers for Linear systems with unknown inputs
4. Super-twisting based Observers for Mechanical Systems
5. HOSM based Observers for Nonlinear Systems
6. Output-feedback finite-time stabilization of disturbed LTI systems
7. Unknown input identification
8. Parameter Identification

Outline

Conventional Sliding Mode Observers

Observer Purpose: To estimate the unmeasurable states of a system based only on:

- the measured outputs and inputs;
- mathematical model of the system, driven by the input of the system together with a signal representing the difference between the measured system and observer outputs
- First Observer: Luenberger
- Drawbacks of Luenberger Observer in the presence of uncertainties
  a. Unable to force the output estimation error to zero
  b. The observer states do not converge to the system states
- Solution: sliding mode observer if the uncertainties are bounded.
- Advantages:
  a. Force the output estimation error to converge to zero in *finite time*
  b. Observer states converge asymptotically to the system states
  c. Disturbances can be reconstructed

Drakunov-Utkin Observer (reduced order SM observer)

- Consider a nominal linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$

- Assume C has full row rank
- Necessary and sufficient condition: (C, A) is observable
- Observability condition will be assumed to hold.

Outline

1. Conventional Sliding Mode Observers
   - A Simple Sliding Mode Observer
   - LTI systems with unknown inputs without need of differentiation
   - Walcot-Zak Observes

Coordinate transformation

- $x \mapsto z = T_c x$

$$T_c = \begin{bmatrix} C^{\perp} \\ C \end{bmatrix} \quad (3)$$

where $N_c \in \mathbb{R}^{n \times (n-p)}$ spans the null-space of C.

- By construction $\det(T_c) \neq 0$
- Applying the change of coordinates

$$T_c A T_c^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_c B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C T_c^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix} \quad (4)$$

where $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_1 \in \mathbb{R}^{(n-p)\times q}$.

$$\dot{z}_1(t) = A_{11} z_1(t) + A_{12} z_2(t) + B_1 u, \;\; \dot{z}_2(t) = A_{21} z_1(t) + A_{22} z_2(t) + B_2 u$$

Drakunov-Utkin observer

Transformed System

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + B_1 u,\ \dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + B_2 u$$

Drakunov-Utkin observer

$$\dot{\hat{z}}_1(t) = A_{11}\hat{z}_1(t) + A_{12}\hat{z}_2(t) + B_1 u, + L\nu,\ \dot{\hat{z}}_2(t) = A_{21}\hat{z}_1(t) + A_{22}\hat{z}_2(t) + B_2 u - \nu,$$

Error of Observer

$$e(t) := \hat{z}(t) - z(t),$$

$$\hat{z}(t) = (\hat{z}_1(t), \hat{z}_2(t),\ z(t) = (z_1(t), z_2(t),\ e_y(t) := \hat{y}(t) - y(t) = \hat{y}(t) - y(t)$$

$$e = \mathrm{col}(e_1, e_y), e_1 \in \mathbb{R}^{n-p}$$

System describing an error of observer

$$\dot{e}_1(t) = A_{11}e_1(t) + A_{12}e_y(t) + L\nu,\ \dot{e}_y(t) = A_{21}e_1(t) + A_{22}e_y(t) - \nu$$

$$G_n = \begin{bmatrix} L \\ -I_p \end{bmatrix} \quad (5)$$

where $L \in \mathbb{R}^{(n-p)\times p}$ represents the design freedom

- ν is designed to be discontinuous with respect to the sliding surface $\mathcal{S} = \{e : Ce = 0\}$ to force the trajectories of $e(t)$ onto $\mathcal{S}$ in finite time.
- Component-wise discontinuous injection

$$\dot{e}_{y,i}(t) = A_{21,i}e_1(t) + A_{22,i}e_y(t) - \rho\ \nu_i,$$

$$\nu_i = \rho \mathrm{sign}(e_{y,i}),\ i = 1, 2, ..., p$$

  $A_{21,i}$ and $A_{22,i}$ represent the ith rows of A_{21}, A_{22}, $e_{y,i}$ represents the ith component of e_y

- Equivalent injection $\nu_{eq} = A_{21}e_1(t)$
- Sliding dynamics $\dot{e}_1(t) = (A_{11} + LA_{21})e_1(t) \rightarrow$ Reduced Order Luenberger Observer

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LTI systems with unknown inputs

The system

Consider

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Dw, \quad x(0) = x_0, \\ y = Cx, \end{cases} \tag{6}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the unknown input;
- $u(t) \in \mathbb{R}^q$ is the control, $y(t) \in \mathbb{R}^p$ is the measured output.

Strong Observability:

The system is strongly observable if for any $x(0)$ and $w(t)$ it follows from $y(t) \equiv 0 \ \forall\, t \geq 0$ that $x(t) \equiv 0$ [Hautus: 83].

Strong Detectability:

The system is strongly detectable if for any $x(0)$ and $w(t)$ it follows from $y(t) \equiv 0 \ \forall\, t \geq 0$ that $x(t) \to 0$ as $t \to \infty$ [Hautus: 83].

LTI systems with unknown inputs

Invariant zeros

The Rosenbrock of (A, C, D):

$$R(s) = \left[\begin{array}{cc} sI - A & -D \\ C & 0 \end{array} \right].$$

The values $s_0 \in \mathbb{C}$ such that $rank\, R(s_0) < n + m$ are called invariant zeros of (A, C, D).

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LTI systems with unknown inputs

State reconstruction without differentiation[Hautus: 1983]

- The system does not have invariant zeroes.
- All the matrices are known i.e.,A, B, C, D.
- C and D are full rank matrices.
- If $rank(C) = p$ and $rank(D) = m$, then $p \geq m$.
- $rank(CD) = m$ Relative degree 1 condition.

State Estimation and Unknown Inputs Reconstruction

Walcot-Zak Observes: Canonical form

$$\begin{pmatrix} dy^{\perp}/dt \\ dy_1/dt \\ dy_2/dt \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} y^{\perp} \\ y_1 \\ y_2 \end{pmatrix} + Bu + \begin{pmatrix} 0 \\ 0 \\ w(t) \end{pmatrix}$$

y_1 non contaminated outputs,
y_2 contaminated outputs,
$y^{\perp}$ unmeasured states

- The pair $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, (0, I_{p-m})$ is observable
- $y^{\perp}$ can be observed with Luenberger or Utkin SM observer
- Unknown Inputs can be reconstructed from Equivalent Outputs Injection $\nu = \rho e_2/||e_2||, \rho > 0$

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2. Higher Order Sliding Mode Observers
   - Strong Observability - Invariant Zeros - Relative Degree
   - Relation of concepts for SUISO Systems
   - Methodology: SM based differentiators

Need of differentiation

Mechanical system

1DOF mechanical system:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + P(\dot{q}) + G(q) + \Delta(t,q,\dot{q}) = \tau$$

State space form $x_1 = q$, $x_2 = \dot{q}$, $u = \tau$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = f(t,x_1,x_2,u) + w(t,x_1,x_2); \qquad y = x_1$$

Relative degree condition(linearized case)

$$C = [1 \quad 0], \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad CD = 0.$$

Need of differentiation

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Relative degree condition(linearized case)

$$C = [1 \quad 0], \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad CD = 0.$$

Remark:

When the relative degree of $w(t)$ w.r.t. $y(t)$ is higher than one, i.e. $rank(CD) < m$, output differentiations are necessary.

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Single Unknown Input- Single Output Case

Strong Observability - Invariant Zeros Relation

Strong observability requires that for any input w, the equality $y \equiv 0$ implies $x \equiv 0$. The existence of invariant zeros s_0 implies the existence of inputs $w(s_0)$ such that $y \equiv 0$ for $x \neq 0$

Single Unknown Input- Single Output Case

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Absence of invariant zeros is sufficient and necessary condition for strong observability(Haustus,1983)

Single Unknown Input- Single Output Case

Strong Observability - Relative Degree Relation

Taking the first $n-1$ derivatives of the output

$$
\begin{aligned}
y &= Cx \\
\dot{y} &= C\dot{x} = CAx(t) + CDw \\
&\vdots \\
y^{(n-1)} &= CA^{n-1}x + CA^{n-2}Dw + ... + CDw^{(n-2)}
\end{aligned}
$$

Single Unknown Input- Single Output Case

Strong Observability - Relative Degree Relation

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y &= Cx \\
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y^{(n-1)} &= CA^{n-1}x + CA^{n-2}Dw + ... + CDw^{(n-2)}
\end{aligned}$$

Relative degree n is required

to obtain a set of n equations independent on w:

$$\begin{bmatrix} CD \\ CAD \\ \vdots \\ CA^{n-2}D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Single Unknown Input- Single Output Case

Invariant zeros - Relative Degree Relation

Rosenbrock matrix for the tuple (A, C, D):

$$R(s) = \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix}.$$

Determinant of the Rosenbrock matrix
$\det(R) = (s^{n-1} + a_n s^{n-2} + ... + a_2)CD+$
$(s^{n-2} + a_n s^{n-3} + ... + a_3)CAD + ... + (s + a_n)CA^{n-2}D + CA^{n-1}D$

Single Unknown Input- Single Output Case

Invariant zeros - Relative Degree Relation

Rosenbrock matrix for the tuple $(A,\ C,\ D)$:

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$(s^{n-2} + a_n s^{n-3} + \ldots + a_3)CAD + \ldots + (s + a_n)CA^{n-2}D + CA^{n-1}D$

Relative degree n is necessary:

The determinant does not dependent on s iff:

$$\begin{bmatrix} CD \\ \vdots \\ CA^{n-2}D \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

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2. Higher Order Sliding Mode Observers
   - Strong Observability - Invariant Zeros - Relative Degree
   - Relation of concepts for SUISO Systems
   - Methodology: SM based differentiators

Differentiation Problem

- Signal to differentiate: $f(t)$
- Assume $|\dot{f}(t)| \leq M$
- Find an observer for $\dot{f}(t)$
- Observer

$$\dot{x} = Ksign[e(t)], x(0) = f(0)$$

  Error $e = f - x(t)$
- Equation

$$\dot{e} = \dot{f}(t) - Ksign[e(t)], K > M, e(0) = 0$$

- $\dot{f}(t)$ bounded perturbation

Filtration of $sign[e(t)]$ needed!!!

Robust Exact Differentiation Problem

Robust Exact Differentiator, Levantovsky(1998)
Signal to differentiate: $f(t)$, $|\ddot{f}(t)| \leq L$
Find an observer for $f(t)$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{f}, \quad y = x_1,$$

$\ddot{f}(t)$ bounded perturbation!!!
STA observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1|\hat{x}_1 - y|^{\frac{1}{2}} sign(\hat{x}_1 - y) + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -k_2 sign(\hat{x}_1 - y) \end{aligned}$$

Convergence of STA assures:$(f - \hat{x}_1) = (\dot{f} - \hat{x}_2) = 0$ after finite time.

Uniform(Fixed-Time) Robust Exact First-Order Differentiator [Cruz et. al. 11].

$$\dot{z}_0 = -k_1\phi_1(z_0 - y_j) + z_1, \quad \dot{z}_1 = -k_2\phi_2(z_0 - y_j),$$

where

$$\begin{aligned} \phi_1(\sigma_0) &:= \lceil \sigma_0 \rfloor^{1/2} + \lceil \sigma_0 \rfloor^{3/2}, \\ \phi_2(\sigma_0) &:= 0{,}5\,\mathrm{sign}(\sigma_0) + 2\sigma_0 + 1{,}5\lceil \sigma_0 \rfloor^{2}. \end{aligned}$$

and $\lceil v \rfloor^p := |v|^p\,\mathrm{sign}(v)$.

Remarks.

- the differentiator is uniform with respect to the initial differentiation error;
- useful for hybrid systems with strictly positive dwell-time;
- arbitrary order uniform exact differentiator [Angulo et al.: Automatica].

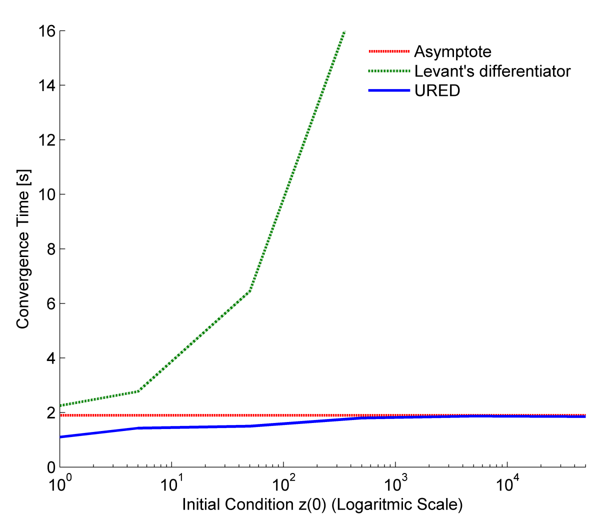

Figura: Convergence time of the Uniform Robust Exact Differentiator (URED).

Methodology

The Unknown Input Observer (UIO) design problem for strongly observable systems is reduced to evaluate in real-time derivatives of the output.

The k-th order HOSM differentiator for y_j

$$\begin{aligned}
\dot{z}_0 &= \nu_0 = -\lambda_k L^{\frac{1}{k+1}} |z_0 - y_j|^{\frac{k}{k+1}} \operatorname{sign}(z_0 - y_j) + z_1, \\
\dot{z}_1 &= \nu_1 = -\lambda_{k-1} L^{\frac{1}{k}} |z_1 - \nu_0|^{\frac{k-1}{k}} \operatorname{sign}(z_1 - \nu_0) + z_2, \\
&\vdots \\
\dot{z}_{k-1} &= \nu_{k-1} = -\lambda_1 L^{\frac{1}{2}} |z_{k-1} - \nu_{k-2}|^{\frac{1}{2}} \operatorname{sign}(z_{k-1} - \nu_{k-2}) + z_k, \\
\dot{z}_k &= -\lambda_0 L \operatorname{sign}(z_k - \nu_{k-1}),
\end{aligned} \tag{7}$$

$\lambda_0 = 1{,}1,\ \lambda_1 = 1{,}5,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 5,\ \lambda_5 = 8.$

Convergence of the HOSM differentiator [Levant:03].

If the gain L satisfies $L > |y_j^{(k+1)}(t)|$ for all t, then $z_i = y_j^{(i)}$ after finite-time.

Arbitrary Order Uniform HOSM differentiator [Angulo et.al: 13, Automatica].

Given a signal $y_j(t)$ to be differentiated $(n-1)$-times, the differentiator

$$
\begin{aligned}
\dot{\hat{x}}_i &= -\lambda_i \lceil y_j - \hat{x}_1 \rfloor^{\frac{n-i}{n}} - k_i \lceil y_j - \hat{x}_1 \rfloor^{\frac{n+\alpha i}{n}} + \hat{x}_{i+1}, \quad i = 1, \dots, n-1, \\
\dot{\hat{x}}_n &= -\lambda_n \operatorname{sign}(y_j - \hat{x}_1) - k_n \lceil y_j - \hat{x}_1 \rfloor^{1+\alpha},
\end{aligned} \tag{8}
$$

with $\{\lambda_i\}_{i=1}^n$ chosen as Levant's, $\alpha > 0$ small enough and $\{k_i\}_{i=1}^n$ such that the polynomial

$$P(s) = k_n s^{n-1} + k_{n-1} s^{n-2} + \cdots + k_2 s + k_1, \quad \text{is stable}$$

provides:

- uniform finite-time estimation, i.e., $\exists T$ independent of $|\hat{x}_i(0) + -y_j^{i-1}(0)|$, $i = 1, \dots, n$, such that

$$\hat{x}_i(t) = y_j^{(i-1)}(t), \quad \forall t \geq T, \quad i = 1, \dots, n;$$

- the best precision under measurement noise [Kolmogorov:62].

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   - Cascaded HOSM Observers for LTI systems with unknown inputs
   - Cascaded HOSM Observers for LTV systems with Unknown inputs
   - Cascaded Observers for strong detectable LTI systems with unknown inputs
   - Cascaded Functional HOSM Observers for linear systems

4. Super-twisting based Observers for Mechanical Systems

5. HOSM based Observers for Nonlinear Systems

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LTI systems with unknown inputs

The system

Consider

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Dw, \quad x(0) = x_0, \\ y = Cx, \end{cases} \tag{9}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the unknown input;
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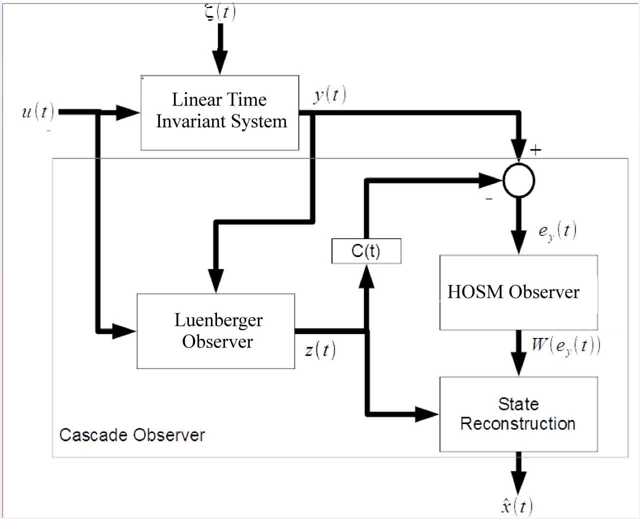
Relative degree

The integer r_i such that

$$\begin{aligned} c_i A^j D &= 0, \; j = 0, \ldots, r_i - 2, \quad c_i A^{r_i - 1} D \neq 0, \\ r_i &\leq n - 1 \end{aligned}$$

where c_i is the i-th row of C.

Cascaded observers for LTI systems with unknown inputs



Cascaded observers for LTI systems with unknown inputs

Cascaded HOSM observer [Fridman et al. 2007]

Observer for the strongly observable case and $\sum_{i=1}^{m} r_i = n$

$$\begin{aligned} \dot{z} &= Az + Dw + L(y - Cz), \\ \dot{v} &= W(y - Cz,\, v), \\ \hat{x} &= z + Kv \end{aligned} \tag{10}$$

- $L \in \mathbb{R}^n$ is the correction term chosen such that $A - LC$ is Hurwitz;
- for i=1, ..., m:

$$K^{-1} = \begin{bmatrix} P_1 \\ \vdots \\ P_m \end{bmatrix}, \quad P_i = \begin{bmatrix} c_i \\ \vdots \\ c_i(A - LC)^{n_i - 1} \end{bmatrix};$$

.

- $W(\cdot)$ is a nonlinear HOSM term.

Cascaded observers for LTI systems with unknown inputs

Cascaded HOSM observer [Fridman et al. 2007]

Nonlinear HOSM term $W^T = [\ v_1 \quad v_2 \quad \dots \quad v_n\ ]$

$$
\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_n N^{1/n}|v_1 - (y - Cz)|^{(n-1)/n} sign(v_1 - (y - Cz)) + v_2 \\
\dot{v}_2 &= w_2 = -\alpha_{(n-1)} N^{1/(n-1)}|v_2 - w_1|^{(n-2)/(n-1)} sign(v_2 - w_1) + v_3 \\
&\vdots \\
\dot{v}_{n-1} &= w_{n-1} = -\alpha_2 N^{1/2}|v_{n-1} - w_{n-2}|^{1/2} sign(v_{n-1} - w_{n-2}) + v_n \\
\dot{v}_n &= -\alpha_1 N sign(v_n - w_{n-1})
\end{aligned}
\tag{11}
$$

where $N > |C(A - LC)^{n-1} Dw(t)|$.

Cascaded observers for LTI systems with unknown inputs

Canonical form of the estimation error $x - z$

The form is composed for Brunovsky blocks.

$$
\begin{bmatrix} \dot{\xi}_{1_1} \\ \dot{\xi}_{1_2} \\ \vdots \\ \dot{\xi}_{1_{r_1-2}} \\ \dot{\xi}_{1_{r_1-1}} \\ \vdots \\ \vdots \\ \dot{\xi}_{m_1} \\ \dot{\xi}_{m_2} \\ \vdots \\ \dot{\xi}_{m_{r_m-2}} \\ \dot{\xi}_{m_{r_m-1}} \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \ddots & & \cdots & \cdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & 1 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
* & * & \cdots & * & * & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 & \vdots & & & \vdots & \ddots & & & \vdots & & \vdots & \\
 & \vdots & & & \vdots & & \ddots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & & \cdots & \cdots & & \vdots & & \ddots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\
* & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * & *
\end{bmatrix}
\begin{bmatrix} \xi_{1_1} \\ \xi_{1_2} \\ \vdots \\ \xi_{1_{r_1-2}} \\ \xi_{1_{r_1-1}} \\ \vdots \\ \vdots \\ \xi_{m_1} \\ \xi_{m_2} \\ \vdots \\ \xi_{m_{r_m-2}} \\ \xi_{m_{r_m-1}} \end{bmatrix}
+
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
* & \cdots & * \\
 & \vdots & \\
 & \vdots & \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
* & \cdots & *
\end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{m-1} \\ w_m \end{bmatrix}
$$

Cascaded observers for LTI systems with unknown inputs

HOSM Observer [Fridman et al. 2007]

Advantages:

1. Finite-time theoretically exact observation of the system states
2. The cascade structure of observer allows to use any pre-filters or stabilizers

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Question

What can we do when the system is strongly observable but $\sum_{i=1}^{m} r_i < n$?

Weakly unobservable subspace

Invariant zeros

The Rosenbrock of $(A,\ C,\ D)$:

$$R(s) = \left[\begin{array}{cc} sI - A & -D \\ C & 0 \end{array} \right].$$

The values $s_0 \in \mathbb{C}$ such that $rank\, R(s_0) < n + m$ are called invariant zeros of $(A,\ C,\ D)$.

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The weakly unobservable subspace $\mathcal{V}^*$

A state $x_0 \in \mathcal{X}$ is called weakly unobservable, if there exist an input w such that the corresponding output $y_w(t, x_0) = 0$ for all $t \geq 0$. The set of all the weakly unobservable points is denoted by $\mathcal{V}^*$ and it is called the weakly unobservable subspace of the system.

Molinari Decoupling Algorithm [Molinari: 1976]

Step 1

- $y_{\mathrm{e}}(t) = \underbrace{C}_{M_1} e(t)$

Molinari Decoupling Algorithm [Molinari: 1976]

Step 1

- $y_{\mathrm{e}}(t) = \underbrace{C}_{M_1} e(t)$

Step 2

- $\dot{y}_{\mathrm{e}}(t) = M_1 \tilde{A} e(t) + M_1 D w(t)$

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- $y_\mathrm{e}(t) = \underbrace{C}_{M_1} e(t)$

Step 2

- $\dfrac{d}{dt}\left[(M_1 D)^\perp y(t)\right] = (\ CD)^\perp CAx(t)$

Step 3

Molinari Decoupling Algorithm [Molinari: 1976]

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- $y_{\mathrm{e}}(t) = \underbrace{C}_{M_1} e(t)$

Step 2

- $\dfrac{d}{dt}\left[(M_1 D)^{\perp} y(t)\right] = (\ CD)^{\perp} CAx(t)$
- $\underbrace{\begin{bmatrix} \left(\frac{d}{dt} M_1 D\right)^{\perp} M_1 \tilde{A} \\ C \end{bmatrix}}_{M_2} e(t) = \frac{d}{dt} \begin{bmatrix} (M_1 D)^{\perp} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_{\mathrm{e}}(t) \\ \int y_{\mathrm{e}} dt \end{bmatrix}$

Step 3

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Step 3

- $\frac{d}{dt} M_2 e(t) = M_2 \tilde{A} e(t) + M_2 D w(t)$

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Step 3

- $\frac{d}{dt}(M_2 D)^{\perp} M_2 e(t) = (M_2 D)^{\perp} M_2 \tilde{A} e(t)$

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- $\underbrace{\begin{bmatrix} \left(\frac{d}{dt} M_1 D\right)^{\perp} M_1 \tilde{A} \\ C \end{bmatrix}}_{M_2} e(t) = \frac{d}{dt} \begin{bmatrix} (M_1 D)^{\perp} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_{\mathrm{e}}(t) \\ \int y_{\mathrm{e}} dt \end{bmatrix}$

Step 3

- $\frac{d}{dt}(M_2 D)^{\perp} M_2 e(t) = (M_2 D)^{\perp} M_2 \tilde{A} e(t)$
- $\underbrace{\begin{bmatrix} \frac{d}{dt}(M_2 D)^{\perp} M_2 \tilde{A} \\ C \end{bmatrix}}_{M_3} e = \frac{d^2}{dt^2} \begin{bmatrix} J_2 & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_{\mathrm{e}} \\ \int y_{\mathrm{e}} \\ \int\int y_{\mathrm{e}} d\tau dt \end{bmatrix}$

State recovering

There exist a $k \leq n$ such that $rank\,(M_k) = n$, *(Molinari, 1976)*

Step k

- $$\underbrace{\begin{bmatrix} \frac{d}{dt}\left(M_{k-1}D\right)^{\perp} M_{k-1}\tilde{A} \\ C \end{bmatrix}}_{M_k} e = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} \underbrace{\begin{bmatrix} y_\mathrm{e} \\ \int y_\mathrm{e} \\ \int \ldots \int y_\mathrm{e} d\tau dt \end{bmatrix}}_{Y_{[k-1]}}$$

Then,

$$e\,(t) = \frac{d^{k-1}}{dt^{k-1}} \underbrace{M_k^{+} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]}}_{\Theta(t)}$$

Molinari Decoupling Algorithm [Molinari: 1976]

Theorem

$$ker\ M_n = \mathcal{V}^*$$

Molinari Decoupling Algorithm [Molinari: 1976]

Theorem

$$ker\ M_n = \mathcal{V}^*$$

Importance of the Molinari's algorithm

The algorithm gives an explicit algebraic relation between the output, and its derivatives, and the state.

$$v(y,\ \dot{y},\ \ldots,\ y^{(n)}) = M_n x \tag{12}$$

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Importance of the Molinari's algorithm

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$$v(y,\ \dot{y},\ ...,\ y^{(n)}) = M_n x \tag{12}$$

Relations for strong observability

i) The system is strongly observable;

ii) the triplet (A, C, D) does not have invariant zeros;

iii) $\mathcal{V}^*$ contains only the zero vector, i.e. $\mathcal{V}^* = \{0\}$.

Observers for LTI strongly observable systems with unknown inputs

Canonical form

Let $\sum_{i=1} mr_i = r_p < n$. The canonical form is composed by Brunovsky blocks and a w dependent block.

$$
\begin{bmatrix} \dot{\xi}_{1_1} \\ \dot{\xi}_{1_2} \\ \vdots \\ \dot{\xi}_{1_{r_1-2}} \\ \dot{\xi}_{1_{r_1-1}} \\ \vdots \\ \vdots \\ \dot{\xi}_{p_1} \\ \dot{\xi}_{p_2} \\ \vdots \\ \dot{\xi}_{p_{r_p-2}} \\ \dot{\xi}_{p_{r_p-1}} \\ \dot{\xi}_{r_p+1} \\ \vdots \\ \dot{\xi}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \ddots & & \cdots & \cdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & 1 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
* & * & \cdots & * & * & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 & \vdots & & & \vdots & \ddots & & & \vdots & & \vdots & \\
 & \vdots & & & \vdots & & \ddots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & & \cdots & \cdots & & \vdots & & \ddots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\
* & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * & * \\
* & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * & * \\
 & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & \\
* & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * & *
\end{bmatrix}
\begin{bmatrix} \xi_{1_1} \\ \xi_{1_2} \\ \vdots \\ \xi_{1_{r_1-2}} \\ \xi_{1_{r_1-1}} \\ \vdots \\ \vdots \\ \xi_{p_1} \\ \xi_{p_2} \\ \vdots \\ \xi_{p_{r_p-2}} \\ \xi_{p_{r_p-1}} \\ \xi_{r_p+1} \\ \vdots \\ \xi_n \end{bmatrix}
+
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
* & \cdots & * \\
 & \vdots & \\
 & \vdots & \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
* & \cdots & * \\
* & \cdots & * \\
\vdots & & \vdots \\
* & \cdots & *
\end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{m-1} \\ w_m \end{bmatrix}
$$

Cascaded observers for LTI strongly observable systems with unknown inputs

Observer in [Fridman et al. 2007]

- ✓ The states are estimated exactly after finite time;
- ✓ The cascade structure of observer allows to use a pre-filter.

Outline

3. Cascaded HOSM Observers for Linear systems with unknown inputs
   - Cascaded HOSM Observers for LTI systems with unknown inputs
   - Cascaded HOSM Observers for LTV systems with Unknown inputs
   - Cascaded Observers for strong detectable LTI systems with unknown inputs
   - Cascaded Functional HOSM Observers for linear systems

LTV System with Unkown inputs

The system

Consider

$$\Sigma : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)w(t), \quad x(t_0) = x_0, \\ y(t) = C(t)x(t), \end{cases} \tag{13}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the unknown input;
- $u(t) \in \mathbb{R}^q$ is the control, $y(t) \in \mathbb{R}^p$ is the measured output.

LTV Observability

Theorem (Rugh 1993)

Let the given matrix functions $A(t)$ and $C(t)$ of the LTV system be $n-2$ and $n-1$ times continuously differentiable respectively, on the non-degenerate time interval $\mathcal{T}$. The observability matrix is defined by

$$\mathcal{O}_{(A,C),n}(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_{n-1}(t) \end{bmatrix} \in \mathbb{R}^{pn\times n}, \qquad (14)$$

where $N_0(t) = C(t)$ and $N_i(t) = N_{i-1}(t)A(t) + \frac{dN_{i-1}(t)}{dt}$ for $i = 1, \ldots, n-1$. Then, the pair $(A(t), C(t))$ is observable on the time interval $\mathcal{T}$ if and only if $\text{rank}(\mathcal{O}_{(A,C),n}(t)) = n$, for all $t \in \mathcal{T}$.

LTV Strong Observability

Definition (Observability index (Rugh 1993))

Let the observability index l_o be the minimum integer such that

$$\operatorname{rank}(\mathcal{O}_{(A,C),l_o}(t)) = n, \text{ for all } t \in \mathcal{T}$$

LTV Strong Observability

Definition (Observability index (Rugh 1993))

Let the observability index l_o be the minimum integer such that

$$\operatorname{rank}(\mathcal{O}_{(A,C),l_o}(t)) = n, \text{ for all } t \in \mathcal{T}$$

Definition (LTV Strong observability (Kratz and Liebscher 1998))

The triplet $(A(t), C(t), D(t))$ is called strongly observable in the non-degenerate interval $\mathcal{T}$, if $\dot{x}(t) = A(t)x + D(t)\zeta(t)$, $C(t)x(t) \equiv 0$, for some unknown input $\zeta(t)$, with $D(t)\zeta(t)$ being a continuous function, implies that $x(t) \equiv 0$, for all $t \in \mathcal{T}$.

LTV Strong Observability

Theorem (Kratz and Liebscher 1998 (1/3))

Let the elements of matrices $A(t)$, $D(t)$, and $C(t)$ be $l_o - 2$, $l_o - 2$, $l_o - 1$ times continuously differentiable, respectively, in the time interval $t \in \mathcal{T}$, and define the matrices $\mathcal{D}_{\mu,\nu} = \mathcal{D}_{(A,C,D),\mu,\nu}(t)$, recursively by

$$\begin{array}{lll} \mathcal{D}_{\mu,\mu-1} := C(t)D(t) & \text{for} & 2 \leq \mu \leq l_o, \\ \mathcal{D}_{\mu,1} := N_{\mu-2}D(t) + \frac{d\mathcal{D}_{\mu-1,1}}{dt} & \text{for} & 3 \leq \mu \leq l_o, \\ \mathcal{D}_{\mu,\nu} := \mathcal{D}_{\mu-1,\nu-1} + \frac{d\mathcal{D}_{\mu-1,\nu}}{dt} & \text{for} & 3 \leq \nu < \mu \leq l_o. \end{array}$$

where $N_i(t) = N_{i-1}(t)A(t) + \frac{dN_{i-1}(t)}{dt}$ and l_o is the observability index.

LTV Strong Observability

Theorem (Kratz and Liebscher 1998 (2/3))

Define the matrix functions $S : \mathcal{T} \to \mathbb{R}^{pl_o \times [n+(l_o-1)m]}$ *and* $S^* : \mathcal{T} \to \mathbb{R}^{(pl_o+n) \times [n+(l_o-1)m]}$ *as*

$$S(t) := \begin{bmatrix} \mathcal{O}_{(A,C),l_o}(t) & \mathcal{J}_{(A,C,D),l_o}(t) \end{bmatrix},$$
$$S^*(t) := \begin{bmatrix} I_n & 0 \\ \mathcal{O}_{(A,C),l_o}(t) & \mathcal{J}_{(A,C,D),l_o}(t) \end{bmatrix}, \tag{15}$$

with

$$\mathcal{J}_{(A,C,D),l_o}(t) := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{D}_{2,1} & 0 & \cdots & 0 \\ \mathcal{D}_{3,1} & \mathcal{D}_{3,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{l_o,1} & \mathcal{D}_{l_o,2} & \cdots & \mathcal{D}_{l_o,l_o-1} \end{bmatrix},$$

and I_n *the* $n \times n$ *identity matrix, where the matrix* $\mathcal{O}_{(A,C),l_o}$ *is the observability matrix with* $n = l_o$.

LTV Strong Observability

Theorem (Kratz and Liebscher 1998 (3/3))

Then the triplet $(A(t), C(t), D(t))$ is strongly observable on $\mathcal{T}$ if and only if

$$rank\ S(t) = rank\ S^*(t)$$

for all $t \in \mathcal{T}$.

LTV Strong Observability: State reconstruction

Corollary (Kratz and Liebscher 1998)

Assume that the matrices $A(t)$, $D(t)$, and $C(t)$ are $l_o - 2$, $l_o - 2$ and $l_o - 1$ times continuously differentiable, respectively, on the time interval $\mathcal{T}$; suppose that $D(t)\zeta(t)$ is continuously differentiable and $y(t)$ is $l_o - 1$ continuously differentiable on $\mathcal{T}$.
Let $K(t) \in R^{pl_o \times pl_o}$ such that $\ker K_{(A,C,D)}(t) = Im\mathcal{J}_{(A,C,D),l_o}$. Define

$$\mathcal{H}_{(A,C,D)}(t) = \mathcal{O}^T_{(A,C),l_o}(t) K^T_{(A,C,D)}(t) K_{(A,C,D)}(t) \mathcal{O}_{(A,C),l_o}(t).$$

Then $\mathcal{H}_{(A,C,D)}(t)$ is invertible, and

$$x(t) = \mathcal{H}^{-1}_{(A,C,D)}(t) \mathcal{O}^T_{(A,C),l_o}(t) K^T_{(A,C,D)}(t) K_{(A,C,D)}(t) \hat{y}(t)$$

with $\hat{y}(t) = \left[\begin{array}{cc} y^T(t), & \dots, y^{(l_o-1)T}(t) \end{array}\right]^T$ for all $t \in \mathcal{T}$.

Cascaded observers for strongly observable LTV systems R. Galvan et al, 2017

Assumptions

1. $A(t)$, $D(t)$ and $C(t)$ are $l_o - 2$, $l_o - 2$ and $l_o - 1$ continuously differentiable functions,

$$\|A(t)^{(i)}\| \leq k_{i1}; \quad \|D(t)^{(i)}\| \leq k_{i2}; \quad \|C(t)^{(j)}\| \leq k_{j3} \;, \forall i = 0, ..., n{-}2;$$

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2. $\|\zeta(t)^{(i)}\| \leq \zeta_i^+, \qquad \forall i = 0, ..., n-1$

Cascaded observers for strongly observable LTV systems R. Galvan et al, 2017

Assumptions

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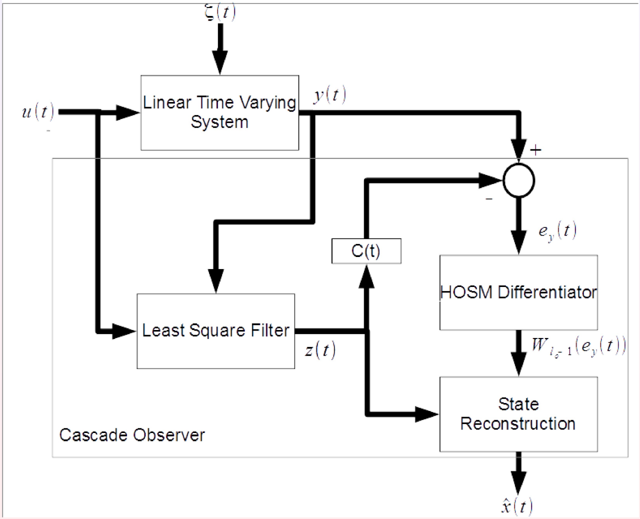
$$\|A(t)^{(i)}\| \leq k_{i1}; \quad \|D(t)^{(i)}\| \leq k_{i2}; \quad \|C(t)^{(j)}\| \leq k_{j3} \ , \forall i = 0, ..., n-2;$$

2. $\|\zeta(t)^{(i)}\| \leq \zeta_i^+, \qquad \forall i = 0, ..., n-1$
3. $(A(t), D(t), C(t))$ is strongly observable.

Cascaded Observers for strongly observable LTV systems R. Galvan et al, 2017



Observers for strongly observable LTV systems R. Galvan et al, 2017

- Observer Form

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)u(t) + L(t)\left(y(t) - C(t)z(t)\right), \quad (15) \\ \hat{x}(t) &= z(t) + F(t)W_{l_o}(e_y(t)); \quad (16) \end{aligned}$$

Observers for strongly observable LTV systems R. Galvan et al, 2017

- Observer Form

$$
\begin{aligned}
\dot{z}(t) &= A(t)z(t) + B(t)u(t) + L(t)\left(y(t) - C(t)z(t)\right), &(15)\\
\hat{x}(t) &= z(t) + F(t)W_{l_o}(e_y(t)); &(16)
\end{aligned}
$$

- $e_y(t) = y(t) - C(t)z(t)$

Observers for strongly observable LTV systems R. Galvan et al, 2017

- Observer Form

$$
\begin{aligned}
\dot{z}(t) &= A(t)z(t) + B(t)u(t) + L(t)\left(y(t) - C(t)z(t)\right), && (15) \\
\hat{x}(t) &= z(t) + F(t)W_{l_o}(e_y(t)); && (16)
\end{aligned}
$$

- $e_y(t) = y(t) - C(t)z(t)$
- $L(t) = P^{-1}(t)C^T(t)$, with $P(t) = P^T(t)$ positive definite

$$
\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + 2C^T(t)C(t) - Q(t), \quad P(t_0) = P_0 \qquad (17)
$$

with a symmetric positive definite matrix $Q(t)$

Observers for strongly observable LTV systems R. Galvan et al, 2017

- Observer Form

$$\dot{z}(t) = A(t)z(t) + B(t)u(t) + L(t)\left(y(t) - C(t)z(t)\right), \quad (15)$$
$$\hat{x}(t) = z(t) + F(t)W_{l_o}(e_y(t)); \quad (16)$$

- $e_y(t) = y(t) - C(t)z(t)$
- $L(t) = P^{-1}(t)C^T(t)$, with $P(t) = P^T(t)$ positive definite

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + 2C^T(t)C(t) - Q(t), \quad P(t_0) = P_0 \quad (17)$$

with a symmetric positive definite matrix $Q(t)$

- $F(t) = \mathcal{H}^{-1}_{(\tilde{A},C,D)}(t)\mathcal{O}^T_{(\tilde{A},C),l_o}(t)K^T_{(\tilde{A},C,D)}(t)K_{(\tilde{A},C,D)}(t)$

Observers for strongly observable LTV systems R. Galvan et al, 2017

- $W_{l_o-1}\left(e_y(t)\right) = \begin{bmatrix} e_y(t)_1 \\ \vdots \\ e_y(t)_p \\ D^1_{l_o-1}\lfloor e_y(t)_1 \rceil \\ \vdots \\ D^1_{l_o-1}\lfloor e_y(t)_p \rceil \\ \vdots \\ D^{l_o-1}_{l_o-1}\lfloor e_y(t)_1 \rceil \\ \vdots \\ D^{l_o-1}_{l_o-1}\lfloor e_y(t)_p \rceil \end{bmatrix}$., $e_y(t)_i$ the i-th row of $e_y(t)$.

Observers for strongly observable LTV systems R. Galvan et al, 2017

- $W_{l_o-1}\left(e_y(t)\right) = \begin{bmatrix} e_y(t)_1 \\ \vdots \\ e_y(t)_p \\ D^1_{l_o-1}\lfloor e_y(t)_1 \rceil \\ \vdots \\ D^1_{l_o-1}\lfloor e_y(t)_p \rceil \\ \vdots \\ D^{l_o-1}_{l_o-1}\lfloor e_y(t)_1 \rceil \\ \vdots \\ D^{l_o-1}_{l_o-1}\lfloor e_y(t)_p \rceil \end{bmatrix}$., $e_y(t)_i$ the i-th row of $e_y(t)$.
- Differentiators gains:
  $\Gamma > 2\zeta_0^+ ||N_{l_o-1}|| ||D(t)|| \frac{||P(t)||}{||Q(t)||} + \sum_{j=1}^{l_o-1} ||\mathcal{D}_{l_o,j}|| \zeta_{j-1}^+$

Cascaded Observers for strongly observable LTV systems R. Galvan et al, 2017

Theorem

Let the LTV system be affected by the unknown inputs $\zeta(t)$, satisfying Assumptions A1-A3. The observer provides global exact convergence of the estimation error $e = x - \hat{x}$ to zero after a finite-time transient, i.e. $e \to 0$ after a finite-time transient, therefore $\hat{x} \to x$ after a finite-time transient.

Observers for strongly observable LTV systems R. Galvan et al, 2017

Highlights

- ✓ The states are reconstructed exactly after finite time;

Observers for strongly observable LTV systems R. Galvan et al, 2017

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- ✓ Deterministic Least Square Filter is combined with a HOSM differentiator

Observers for strongly observable LTV systems R. Galvan et al, 2017

Highlights

- ✓ The states are reconstructed exactly after finite time;
- ✓ Deterministic Least Square Filter is combined with a HOSM differentiator
- ✓ The LTV can be unstable

Outline

3. Cascaded HOSM Observers for Linear systems with unknown inputs
   - Cascaded HOSM Observers for LTI systems with unknown inputs
   - Cascaded HOSM Observers for LTV systems with Unknown inputs
   - Cascaded Observers for strong detectable LTI systems with unknown inputs
   - Cascaded Functional HOSM Observers for linear systems

Strong detectability

Question

If the system is not strongly observable, but strongly detectable, can we design an observer?

Strong detectability

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If the system is not strongly observable, but strongly detectable, can we design an observer?

Relations for strong detectability

i) The system is strongly detectable;

ii) the triplet (A, C, D) is minimum phase, i.e. the invariant zeros of the triplet (A, C, D) satisfy $Re\, s < 0$;

iii) all the trajectories belonging to $\mathcal{V}^*$ converges to zero asymptotically.

State space

State space for LSUI

The state space is divided in strongly observable and the weakly unobservable sub-spaces. But the weakly unobservable subspace contains the systems unobservable subspace.

State space

State space for LSUI

The state space is divided in strongly observable and the weakly unobservable sub-spaces. But the weakly unobservable subspace contains the systems unobservable subspace.

State space for systems with unknown inputs.

Cascaded Observers for strongly detectable LTI systems

A canonical form.

$$\dot{x} = \begin{bmatrix} A_{11} & D_1 A_{12} & D_1 A_{13} \\ A_{21} C_1 & A_{22} & 0 \\ A_{31} C_1 & A_{32} C_2 & A_{33} \end{bmatrix} x + \begin{bmatrix} D_1 \\ 0 \\ 0 \end{bmatrix} w,$$

$$y = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} x,$$

1. the triplet (A_{11}, C_1, D_1) is strongly observable;
2. the pair (A_{22}, C_2) is observable;
3. A_{33} is stable (i.e. the invariant zeros of the system).

Cascaded Observers for strongly detectable LTI systems

An observer [Fridman et al. 2011]

- x_1 can be recovered using y_1 and its derivatives (strong observability);
- x_2 is estimated asymptotically using Luenberger observer because it is not contaminated

$$\dot{\hat{x}}_2 = A_{21}y_1 + A_{22}\hat{x}_2 + L_2C_2(x_2 - \hat{x}_2);$$

- for x_3 a copy of the system without injection (A_{33} is stable)

$$\dot{\hat{x}}_3 = A_{31}y_1 + A_{32}y_2 + A_{33}\hat{x}_3.$$

Remarks.

- it turns out that every strongly detectable (linear) system can be written in the previous form [Moreno:01];
- strong detectability [Hautus:83] is equivalent to the asymptotic distinguishability of the state trajectory from the output [Moreno:01]

Question

What happens when the system is not strongly observable. Is still possible to do something?

Outline

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Functional Unknown Input Observers

Problem formulation

Given a linear system Σ, estimate a linear combination $z = Ex$ using only output measurements.

Motivation

- in many output based control problems, it is not necessary to estimate the whole state but a linear combination Kx;
- this is particularly true for output based sliding-mode control: only the the surface is required;
- despite a UIO does not exist (i.e. the system is not strongly detectable), the required functional UIO may exist.

Functional Unknown Input Observers

Theorem. [Sanuti and Saberi: 87]

Every linear system, with $p \geq m$, can be written as

$$\dot{x} = \begin{bmatrix} A_{11} & D_1 A_{12} & D_1 A_{13} & D_1 A_{14} \\ A_{21} C_1 & A_{22} & 0 & 0 \\ A_{31} C_1 & A_{32} C_2 & A_{33} & 0 \\ A_{41} C_1 & A_{42} C_2 & 0 & A_{44} \end{bmatrix} x + \begin{bmatrix} D_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w,$$

$$y = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \end{bmatrix} x, \quad z = \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \end{bmatrix} x,$$

and the following properties hold:

1. (A_{11}, C_1, D_1) is strongly observable and (A_{22}, C_2) is observable;
2. A_{33} are the stable invariant zeros and A_{44} the unstable ones.

Remark. [Angulo et.al: JFI14]

A functional UIO exists iff $E_4 = 0$.

Outline

Outline

4. Super-twisting based Observers for Mechanical Systems
   - Super-twisting based Observers for Mechanical systems a-priori bounded Coriolis term

Super-twisting based Observers for Mechanical systems with a-priori bounded Coriolis term

Formulation of the problem:

Estimate the velocity using the position, under the hypothesis of a-priori bounded Coriolis term

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(x_1, x_2, t) + w, \quad y = x_1.$$

A simple observer [Davila et.al. 05]

The observer

$$\begin{aligned}
\dot{\hat{x}}_1 &= -1{,}5\sqrt{L}|y - \hat{x}_1|^{\frac{1}{2}} \operatorname{sign}(y - \hat{x}_1) + \hat{x}_2, \\
\dot{\hat{x}}_2 &= f(x_1, \hat{x}_2, t) - 1{,}1L \operatorname{sign}(y - \hat{x}_1),
\end{aligned}$$

$$|f(x_1, \hat{x}_2, t) - f(x_1, x_2, t) + w| < L$$

- finite-time estimation of x_2, i.e., $\hat{x}_2(t) = x_2(t), \forall t \geq T$;

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

Consider the following system from mechanical system

$$\Sigma : \begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = \psi(\xi_1, \xi_2) + \alpha(\xi_1)\xi_2^2 + u + F_t + w_1(t, \xi_1, \xi_2), \\ y = \xi_1, \end{cases}$$

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

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Difficulties:

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

Consider the following system from mechanical system

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Difficulties:

- Non linearity $\psi(\cdot)$.

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Difficulties:

- Non linearity $\psi(\cdot)$.
- Quadratic term $\alpha(\xi_1)\xi_2^2$.

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Consider the following system from mechanical system

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Difficulties:

- Non linearity $\psi(\cdot)$.
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- Discontinuous signal F_t.

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Difficulties:

- Non linearity $\psi(\cdot)$.
- Quadratic term $\alpha(\xi_1)\xi_2^2$.
- Discontinuous signal F_t.
- Uncertain term $w_1(\cdot)$.

Goal

Observer converging exactly in finite time for non BIBS systems.

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

State transformation $T \to \alpha(\xi_1)\xi_2^2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1(\xi_1) \\ T_2(\xi_1, \xi_2) \end{bmatrix} = T(\xi) := \begin{bmatrix} \int_a^{\xi_1} \Upsilon(\mu) d\mu \\ \Upsilon(\xi_1)\xi_2 \end{bmatrix}$$

$$\Upsilon(y) := \exp\left(-\int_a^y \alpha(v) dv\right),$$

This is based by one proposed in [Krener-1985].

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

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$$\Upsilon(y) := \exp\left(- \int_a^y \alpha(v) dv \right),$$

This is based by one proposed in [Krener-1985].

Transformed system without quadratic term

$$\begin{cases} \dot{x}_1 = & x_2, \\ \dot{x}_2 = & \Upsilon(y_T)\psi(y_T, (\Upsilon(y_T))^{-1}x_2) + \Upsilon(y_T)u + \Upsilon(y_T)w(t, \xi_1, \xi_2), \end{cases}$$

$$y_T := T_1^{-1}(x_1), \;\; w(t, \xi_1, \xi_2) := F_t + w_1(t, \xi_1, \xi_2)$$

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

ASSUMPTIONS

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

ASSUMPTIONS

A-1 The uncertain term is bounded, $|w(t, \xi_1, \xi_2)| \leq L_w$.

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

ASSUMPTIONS

A-1 The uncertain term is bounded, $|w(t, \xi_1, \xi_2)| \leq L_w$.

A-2 In $\alpha(\xi_1)\xi_2^2$

$$\left| \int_a^y \alpha(v) dv \right| \leq L_\alpha, \quad \text{is satisfied for any } y.$$

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

ASSUMPTIONS

A-1 The uncertain term is bounded, $|w(t, \xi_1, \xi_2)| \leq L_w$.

A-2 In $\alpha(\xi_1)\xi_2^2$

$$\left| \int_a^y \alpha(v) dv \right| \leq L_\alpha, \quad \text{is satisfied for any } y.$$

A-3 There exists $\{q, s, r\}$, with $q < 0$, such that the nonlinearity

$$\Gamma(v_1, v_2, v_3) := \psi(v_1, v_2 + v_3) - \psi(v_1, v_2)$$

is $\{q, s, r\}-$dissipative, i.e.

$$\begin{bmatrix} \Gamma(v_1, v_2, v_3) \\ v_3 \end{bmatrix}^T \begin{bmatrix} q & s \\ s & r \end{bmatrix} \begin{bmatrix} \Gamma(v_1, v_2, v_3) \\ v_3 \end{bmatrix} \geq 0, \ \forall\, v_1, v_2, v_3 \in \mathbb{R}.$$

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

Theorem

If assumptions are satisfied, then following system

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - k_1\phi_1(\hat{x}_1 - x_1), \\ \dot{\hat{x}}_2 = \Upsilon(y_{_T})\,\psi\left(y_{_T}, (\Upsilon(y_{_T}))^{-1}\left(\hat{x}_2 + k_3\phi_1(\hat{x}_1 - x_1)\right)\right) + \\ \qquad\qquad\qquad + \Upsilon(y_{_T})u - k_2\phi_2(\hat{x}_1 - x_1), \end{cases}$$

with injection terms

$$\phi_1(\cdot) := \mu_1\lceil\cdot\rfloor^{1/2} + \mu_2\lceil\cdot\rfloor, \qquad \lceil\cdot\rfloor^p := |\cdot|^p\, sign(\cdot),$$

$$\phi_2(\cdot) := \frac{{\mu_1}^2}{2}\lceil\cdot\rfloor^0 + \frac{3\mu_1\mu_2}{2}\lceil\cdot\rfloor^{1/2} + {\mu_2}^2\lceil\cdot\rfloor.$$

is an observer converging exactly in finite time to the states of transformed system for some constants $k_1, k_2, k_3, \mu_1, \mu_2$.

Dissipative approach to Super-twisting Observers Design for Mechanical systems(Apaza et al, Automatica, 2018)

Lemma (gain design)

For constants L_w, q, s, r with $L_w \geq 0$, $q < 0$ there exist a matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0$ and constants $k_i, \mu_i, \theta_i > 0$, $i = 1, 2$, $\epsilon > 0$ and k_3 such that the following matrix inequality is satisfied

$$\begin{bmatrix} A^T P + PA + \theta_1 H_E + \theta_2 H_C + \epsilon I & PB & PB \\ B^T P & \theta_1 q & 0 \\ B^T P & 0 & \theta_2 \left(\frac{\mu_1{}^2}{2d_2} \right)^2 \end{bmatrix} \leq 0,$$

where $A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} k_3 & 1 \end{bmatrix}$,

$H_E = E^T \left(\frac{|s| + \sqrt{-qr + s^2}}{\sqrt{-q}\mu_2} \right)^2 E$ and $H_C = C^T L_w^2 C$.

Outline

Observer for nonlinear system with unknown inputs

MIMO locally stable system [Fridman et al.: 2008].

Consider

$$\begin{aligned} \dot{x} &= f(x) + g(x)\varphi(t), \quad x(0) = x_0, \\ y &= h(x), \end{aligned} \tag{15}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $\varphi(t) \in \mathbb{R}^m$ is the disturbance;
- $y(t) \in \mathbb{R}^m$ is the measured output.

Formulation of the problem:

Estimate of $x(t)$ and $\varphi(t)$ based on output measurements only $y(t)$.

Assumption

The system is locally weakly observable
The system is BIBS

Outline

5. HOSM based Observers for Nonlinear Systems
   - Observers for BIBS nonlinear systems with unknown inputs

BIBS nonlinear systems with unknown inputs using coordinate transformation

Coordinate transformation.

New coordinates

$$
\begin{aligned}
\xi^i &= \begin{pmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{pmatrix} = \begin{pmatrix} h_i(x) \\ L_f h_i(x) \\ \vdots \\ L_f^{r_i-1} h_i(x) \end{pmatrix}, \quad i = 1, ..., m; \; r = r_1 + ... + r_m \\
\xi &= \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^m \end{pmatrix}; \; \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n-r} \end{pmatrix}
\end{aligned}
$$

Local diffeomorphism.

There exist a local diffeomorphism such that

$$x = \Phi^{-1}(\xi, \eta)$$

Observer for nonlinear system with unknown inputs using coordinate transformation

Transformed system.

New coordinates

$$
\begin{aligned}
\dot{\xi}^i &= \Lambda_i \xi^i + \psi^i(\xi,\eta) + \lambda^i(\xi,\eta,\varphi(x)) \\
\eta &= q(\xi,\eta)
\end{aligned}
$$

where

$$
\Lambda_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; \Phi_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ L_f^{r_i} h_i(x) \end{bmatrix}
$$

$$
\lambda_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \end{bmatrix}, \quad \forall\, i = 1, \ldots, m
$$

Observer for nonlinear system with unknown inputs using coordinate transformation

State estimation [Fridman et al.: 2008].

HOSM Differentiator

$$
\begin{aligned}
\dot{v}_0^i &= w_0^i = -\alpha_{r_i} N^{1/r_i} |v_0^i - y_i|^{(r_i-1)/r_i} sign(v_0^i - y_i) + v_1^i \\
\dot{v}_1^i &= w_1^i = -\alpha_{(r_i-1)} N^{1/(r_i-1)} |v_1^i - w_0^i|^{(r_i-2)/(r_i-1)} sign(v_1^i - w_0^i) + v_2^i \\
&\vdots \\
\dot{v}_{r_j-1}^i &= w_{r_i-1}^i = -\alpha_2 N^{1/2} |v_{r_i-1}^i - w_{r_i-2}^i|^{1/2} sign(v_{r_i-1}^i - w_{r_i-2}^i) + v_{r_i}^i \\
\dot{v}_{r_i}^i &= -\alpha_1 N sign(v_{r_i}^i - w_{r_i-1})
\end{aligned}
\tag{16}
$$

Estimation by construction

$$
\begin{array}{llll}
\hat{\xi}_1^1 = v_0^1, & \ldots & \hat{\xi}_{r_1}^1 = v_{r_1-1}^1, & \dot{\hat{\xi}}_{r_1}^1 = v_{r_1}^1, \\
& \vdots & & \\
\hat{\xi}_1^m = v_0^m, & \ldots & \hat{\xi}_{r_1}^m = v_{r_1-1}^m, & \dot{\hat{\xi}}_{r_1}^m = v_{r_1}^m,
\end{array}
\tag{17}
$$

Observers for BIBS systems with unknown inputs using coordinate transformation

The system

Consider

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)w, \quad x(0) = x_0, \\ y = h(x), \end{cases} \tag{18}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the unknown input;
- $y(t) \in \mathbb{R}^p$ is the measured output.

Formulation of the problem:

Estimate $x(t)$ based on output measurements only $\{y(t), t \in [0, T]\}$.

Remark.

When the relative degree of y is higher than one, output differentiations are necessary [Hautus: 83].

Observers for BIBS systems with unknown inputs without system transformation(Davila et al, IJC 2009)

Nonlinear system with unknown inputs

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + \mathbf{D}w(x,\ t) \\ y &= h(x) \end{aligned}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}$ is the bounded unknown input;
- $\mathbf{D}$ is a known distribution matrix.
- $y(t) \in \mathbb{R}^p$ is the measured output.

Assumption

$$dL^i_{f(x)}h(x)\mathbf{D} = 0, \qquad i = 0,\ \ldots,\ n-2.$$

Observer design

Observer structure

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u$$
$$\hat{y} = h(\hat{x}) \tag{19}$$

where

- $\hat{x} \in \mathbb{R}^n$ observed state vector.
- $\hat{y} \in \mathbb{R}$ observed output variable.
- $g(\hat{x}) = M^{-1}(\hat{x}) \cdot [0, 0, \ldots, 1]^T$.

Matrix computation

Distribution matrix

Define the following n-th order square matrix

$$M(z) = \begin{bmatrix} dh(z) \\ dL_{f(z)}h(z) \\ \vdots \\ dL_{f(z)}^{n-2}h(z) \\ dL_{f(z)}^{n-1}h(z) \end{bmatrix}$$

where $L_{f(z)}h(z)$ is sometimes called the Lie derivative of $h(z)$ along $f(z)$, i.e., $L_{f(z)}h(z) = \frac{\partial h(z)}{\partial z} f(z)$ and the kth derivative of $h(z)$ along $f(z)$ is defined as $L_{f(z)}^{k}h(z) = \frac{\partial L_{f(z)}^{k-1}h(z)}{\partial z} f(z)$.

Assumption

Matrix $M(z)$ in (81) is nonsingular for every possible value of z.

Observers for systems with unknown inputs

Definition: Strong Observability. [Angulo et. al.: Automatica 13]

Σ is strongly observable if there exists a function F and integer k such that

$$x = F(y, \dot{y}, \cdots, y^{(k)})$$

- equivalent to the distinguishability of the state trajectory using only the output when f, g and h are meromorphic [Angulo et.al: 10];
- also equivalent when Σ is linear [Hautus: 83]. In such a case, F is linear and therefore

$$x = \frac{d^k}{dt^k} M \begin{bmatrix} y \\ \int_0^t y(s)ds \\ \vdots \end{bmatrix}, \quad M \text{ is a matrix.} \tag{20}$$

- function F or matrix M can be computed using an algorithm [Angulo et. al: 10, Bejarano et.al.: 11, Davila et.al:11].

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Consider the system

$$\Sigma : \begin{cases} \dot{x}(t) & = A\,x(t) + B\,\left(\psi(x(t)) + w(t)\right) + \varphi(u(t), y(t)), \\ y(t) & = Cx(t), \end{cases} \tag{21}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C^T := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

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- $x(t) \in \mathbb{R}^n$ state.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Consider the system

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- $x(t) \in \mathbb{R}^n$ state.
- $y(t) \in \mathbb{R}$ measured output.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Consider the system

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- $x(t) \in \mathbb{R}^n$ state.
- $y(t) \in \mathbb{R}$ measured output.
- $\psi(\cdot)$, $\varphi(\cdot, \cdot)$ non linearities.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

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where

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C^T := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

- $x(t) \in \mathbb{R}^n$ state.
- $y(t) \in \mathbb{R}$ measured output.
- $\psi(\cdot)$, $\varphi(\cdot,\cdot)$ non linearities.
- $u \in \mathbb{R}$ control input.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Consider the system

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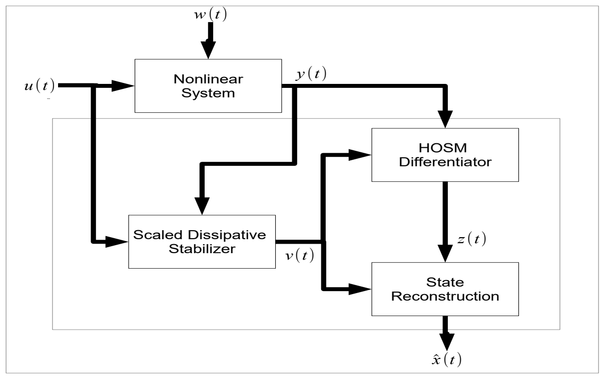
- $u \in \mathbb{R}$ control input.
- w unknown input.
- (21) is forward complete.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Goal

Building a global theoretically exact finite time observer for (21).

Idea of the proposed observer:



Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Proposed observer: HOSM observer with SDS

$$\dot{v}(t) = Av(t) + B\psi(v(t) + N(Cv(t) - y(t))) + \varphi(u, y) + \Delta_l K(Cv(t) - y(t)), \tag{22a}$$

$$\dot{z}(t) = W(z(t), Cv(t) - y(t)), \tag{22b}$$

$$\hat{x}(t) = v(t) - \mathcal{O}^{-1} z(t), \tag{22c}$$

where

$$W(z(t), Cv(t) - y(t)) := \begin{bmatrix} -\alpha_1 L_f^{1/n} \lceil z_1(t) - Cv(t) + y(t) \rfloor^{(n-1)/n} + z_2(t) \\ -\alpha_2 L_f^{1/(n-1)} \lceil z_2(t) - \dot{z}_1(t) \rfloor^{(n-2)/(n-1)} + z_3(t) \\ \vdots \\ -\alpha_{n-1} L_f^{1/2} \lceil z_{n-1}(t) - \dot{z}_{n-2}(t) \rfloor^{1/2} + z_n(t) \\ -\alpha_n L_f \lceil z_n(t) - \dot{z}_{n-1}(t) \rfloor^{0} \end{bmatrix},$$

with $\lceil \circ \rfloor^s := |\circ|^s \, sign(\circ)$, and

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

$$\mathcal{O} = \begin{bmatrix} C \\ C(A+\Delta_l KC) \\ \vdots \\ C(A+\Delta_l KC)^{n-1} \end{bmatrix}, \quad \text{y} \quad \Delta_l := \begin{bmatrix} l & 0 & \cdots & 0 \\ 0 & l^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l^n \end{bmatrix}$$

and Δ_l, K, N, L_f are design and α_i's are designed as in [Levant-2003].

Assumption 1

There are $q < 0$, $S \in \mathbb{R}^{1\times(n-1)}$ and $R \in \mathbb{R}^{(n-1)\times(n-1)}$ such that the non linearity

$$\Psi(x,h) := \psi(x_1, x_2 + h_1, ..., x_n + h_{n-1}) - \psi(x_1, x_2, ..., x_n), \tag{23}$$

is $\{q, S, R\}-$disipative, i.e.,

$$\begin{bmatrix} \Psi(x,h) \\ h \end{bmatrix}^T \begin{bmatrix} q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Psi(x,h) \\ h \end{bmatrix} \geq 0, \ \forall\, x \in \mathbb{R}^n,\ h \in \mathbb{R}^{n-1}.$$

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Theorem

If the assumption 1 is satisfied, then

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Theorem

If the assumption 1 is satisfied, then

i) there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ with $P, Q \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^n$, $\tilde{N} = \begin{bmatrix} N_2 & \cdots & N_3 \end{bmatrix}^T \in \mathbb{R}^{(n-1)}$ and scalars $\epsilon > 0$, $l_0 \geq 1$ such that the inequality

$$\begin{bmatrix} l\left(PA_K + A_K^T P + \epsilon Q + \frac{1}{l^{2n}}(\tilde{I}_n\Delta_l)_{l\tilde{N}}^T R(\tilde{I}_n\Delta_l)_{l\tilde{N}}\right) & \star \\ B^T P + \frac{1}{l^{2n-1}} S(\tilde{I}_n\Delta_l)_{l\tilde{N}} & lq \end{bmatrix} \leq 0,$$

is satisfied for all $l \geq l_0$, where

$$\tilde{I}_n := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n}, \quad A_K := A + KC, \quad \text{y}$$

$(\tilde{I}_n\Delta_l)_{l\tilde{N}} := \tilde{I}_n\Delta_l + l\tilde{N}C.$

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

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ii) the estimation error is finally uniform bounded with final bound

$$\|e_v\| < \frac{b}{l}, \qquad \text{for all} \quad l \geq l_0, \tag{24}$$

where $b := \frac{2\|PB\|\varrho_w}{\epsilon\lambda_{\text{mín}}(Q)}\sqrt{\frac{\lambda_{\text{máx}}(P)}{\lambda_{\text{mín}}(P)}}n$.

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iii) the $n-$th time derive $e_{vy} = Cv - y$ is bounded, with bound

$$L_f = \left\{ \frac{2\|PB\| + \delta}{\epsilon\lambda_{\text{mín}}(Q)} \left[\left\| (A + KC)^n G^T diag \left\{ \frac{\lambda_{\text{máx}}(P)}{\lambda_i(P)} \right\} \right\| \sqrt{n} + \right. \right.$$
$$\left. \left. + \ \varrho_\psi \left\| (I_n + \tilde{N}C) \frac{\Delta_l}{l^{n+1}} G^T \right\| \right] + 1 \right\} \varrho_w, \tag{25}$$

$\varrho_\psi = \frac{1}{\sqrt{-q}} \left(\sqrt{-\lambda_{\text{máx}}(R)q + \lambda_{\text{máx}}(S^TS)} + \{\lambda_{\text{máx}}(S^TS)\}^{1/2} \right)$, G a orthogonal matrix $D = diag\{\lambda_i(P)\}$ where $\lambda_i(P)$ denotes the eigenvalue of matrix P for $i = 1, ..., n$ such that $P = G^TDG$.

Global observers for nonlinear systems with unknown inputs without BIBS property (Apaza et al, IJC 2018)

Algorithm: design of HOSM observer with SDS

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f) Define L_f in (25) and the matrix $\mathcal{O}^{-1}$.

Outline

Output-feedback stabilization of disturbed systems

$$\dot{x} = Ax + B[u + w], \quad y = Cx, \quad \|w(t)\| \le W_1\|x(t)\| + W_2.$$

[Angulo et.al.: IJSS 2011, Automatica 2012,13

- under strong observability, use a HOSM observer to estimate x in finite-time;
- by controllability there exists an output $\zeta(t) \in \mathbb{R}^m$ (not necessary the measured one) with vector relative degree;
- using ζ_i and its derivatives, write the system as m integrator chains;
- use (non-homogeneous) HOSM controllers to obtain (robust) finite-time state stability

$$v_i = -\alpha_i(k_{i,1}\|x\| + k_{i,2})H_{r_i}(\zeta_i, \dot{\zeta}_i, \cdots, \zeta_i^{(r_i-1)}), \quad i = 1, \ldots, m,$$

H_{r_i} is a r_i-th order SM algorithm, e.g.,

$$\dot{\zeta}_i + \beta|\zeta_i|^{\frac{1}{2}}\mathrm{sign}\ \zeta_i$$

Output-feedback stabilization of disturbed systems

Moreover

- the previous requirement of a measured output with vector relative degree is not necessary;
- robust finite-time stabilization of the whole state: useful for switching systems [Angulo, Fridman and Levant: IJSS11];
- the same idea can be used for strongly observable nonlinear systems that are flat (not necessarily w.r.t. the measured output) [Angulo et. al.: AUTOMATICA,2013];
- adapt the gain of the differentiator and control: reduce chattering;
- Separation Principle: (robust) on-line detection of the convergence of the differentiators;
- until now, this was done by waiting enough time.

Outline

Unknown input identification

Problem formulation.

Given

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)w, \quad x(0) = x_0, \\ y = h(x), \end{cases} \tag{26}$$

estimate the input $w(t)$ using only the measured output.

DEFINITION: static left-inverse.

Σ has a static left-inverse if there exists a function G and integer k such that

$$w = G(y, \dot{y}, \cdots, y^{(k)}).$$

Remarks.

- if Σ is strongly observable and $\operatorname{rank} g = m$ then Σ has a static left-inverse [Angulo et.al: Automatica,13];
- however, neither strong observability nor strong detectability are necessary for Σ to have a static left-inverse (see, e.g., [Bejarano et.

Unknown input identification

Recall the canonical form

$$\dot{x} = \begin{bmatrix} A_{11} & D_1 A_{12} & D_1 A_{13} & D_1 A_{14} \\ A_{21} C_1 & A_{22} & 0 & 0 \\ A_{31} C_1 & A_{32} C_2 & A_{33} & 0 \\ A_{41} C_1 & A_{42} C_2 & 0 & A_{44} \end{bmatrix} x + \begin{bmatrix} D_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w,$$

$$y = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \end{bmatrix} x,$$

Theorem. [Bejarano et. al: 09]

- obviously w can be estimated iff x_1 can be estimated and rank $D_1 = m$;
- it is not necessary strong observability nor strong detectability;
- the system has a static left-inverse if and only if the invariant zeros that do not belong to the set of unobservable eigenvalues are stable.

Outline

Regressor Form

System in regressor form

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{x}_2, \\
\dot{\mathbf{x}}_2 &= F(t, \mathbf{x}_1, \mathbf{x}_2, u) + \theta(t)\varphi(t, \mathbf{x}_1, \mathbf{x}_2, u), \quad u = U(t, \mathbf{x}_1, \mathbf{x}_2), \\
\mathbf{y} &= \mathbf{x}_1,
\end{aligned}$$

- $F(t, \mathbf{x}_1, \mathbf{x}_2, u) \in \mathbb{R}^n$ - completely known part of the system.

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- $\theta(t) \in \mathbb{R}^{n \times l}$ - uncertain parameters.
- $\varphi(t, \mathbf{x}_1, \mathbf{x}_2, u) \in \mathbb{R}^l$ - known nonlinear functions vector.

Observer

State observer

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_1 &= \hat{\mathbf{x}}_2 + \alpha_2 \lambda(\tilde{\mathbf{x}}_1) sign(\tilde{\mathbf{x}}_1)\\ \dot{\hat{\mathbf{x}}}_2 &= F(t, \mathbf{x}_1, \hat{\mathbf{x}}_2, u) + \bar{\theta}(t)\varphi(t, \mathbf{x}_1, \hat{\mathbf{x}}_2, u) + \alpha_1 sign(\tilde{\mathbf{x}}_1),\end{aligned}$$

- $\bar{\theta} \in \mathbb{R}^{n \times l}$ -nominal values of the parameters matrix $\theta(t)$.

Error dynamics

Error dynamics

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_1 &= \tilde{\mathbf{x}}_2 - \alpha_2\lambda(\tilde{\mathbf{x}}_1)sign(\tilde{\mathbf{x}}_1)\\ \dot{\tilde{\mathbf{x}}}_2 &= (\theta(t) - \bar{\theta}(t))\varphi(t, \mathbf{x}_1, \mathbf{x}_2, u) - \alpha_1 sign(\tilde{\mathbf{x}}_1)\end{aligned}$$

- Equivalent output injection:

$$\bar{z}_{eq}(t) = \alpha_1 sign(\tilde{\mathbf{x}}_1) = (\theta - \bar{\theta})\varphi(t, \mathbf{x}_1, \mathbf{x}_2, u)$$

Parameter identification. Time Invariant Case

- Parametric uncertainty: $\Delta_\theta := \theta - \bar{\theta}$.

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- Time invariant parameters identification (dynamic least-square method):

$$\dot{\widehat{\Delta_\theta}} = \left[-\widehat{\Delta_\theta}\varphi(t) + \bar{z}_{eq}(t)\right]\varphi^T(t)\Gamma(t).$$

Where

$$\dot{\Gamma}(t) = -\Gamma(t)\varphi(t)\varphi^T(t)\Gamma(t)$$

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Parameter identification. Time Variant Case

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- Time variant parameters identification (forgetting factor method):

$$\dot{\hat{\vartheta}}(t) = \left(\bar{z}_{eq}(t) - \hat{\vartheta}(t)\varphi(t)\right)\varphi^T(t)\Gamma(t).$$

Where

$$\dot{\Gamma}(t) = -\left(\Gamma(t)\varphi(t)\varphi^T(t) + \ln R\right)\Gamma(t).$$

The matrix $R = R^T \in \mathbb{R}^{n\times n}$ is called the matrix forgetting factor, satisfying the conditions:

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The matrix $R = R^T \in \mathbb{R}^{n\times n}$ is called the matrix forgetting factor, satisfying the conditions:

  - $|\lambda_{min}(R)| > 0$,
  - $||R|| = \varrho < 1$.

Outline

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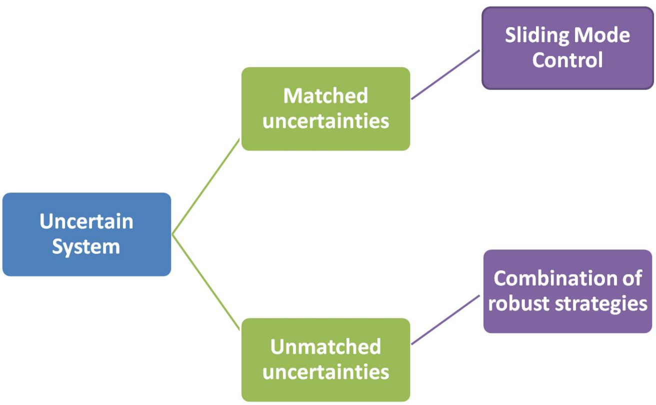
9. Output-based stabilization of disturbed systems
   - Estimation of the disturbance.

Motivation


Uncertain System
Matched uncertainties
Unmatched uncertainties
Sliding Mode Control
Combination of robust strategies

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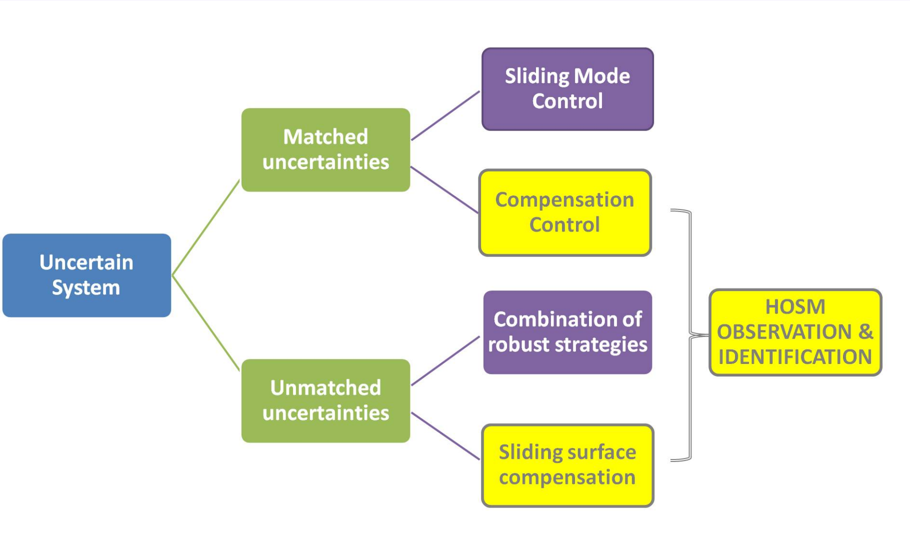
Uncertain System
Matched uncertainties
Unmatched uncertainties
Sliding Mode Control
Compensation Control
Combination of robust strategies
Sliding surface compensation
HOSM OBSERVATION & IDENTIFICATION

Theoretically exact uncertainty compensation control

Sliding mode control

- Advantages:
  - Robustness (insensitivity) against matched uncertainties/disturbances.
  - Finite-time reaching of the transient.
  - Reduction of the system dynamics order on the sliding surface
- Shortcomings:
  - Lack of robustness against unmatched uncertainties/disturbances.
  - Chattering !!

Questions

- How can we preserve the insensitivity against matched disturbances without the noxious effect of chattering?
- How can we deal against unmatched disturbances?

Starting point . . .

High-order sliding-mode observers: A powerful tool

HOSM observers ([Davila et al. 2006], [Fridman et al. 2007], [Bejarano-Fridman 2010]...)

- Ensure robustness in the presence of disturbances (unknown inputs)
- Provide, theoretically exact, state estimation and disturbances (unknown inputs) identification
- Offer finite-time convergence

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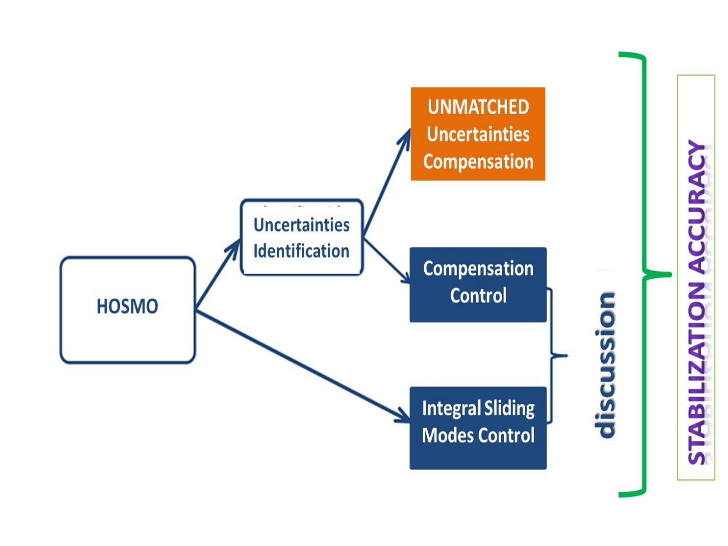
Output-feedback stabilization of disturbed systems

Using HOSMO to estimate in finite-time:

- the state to design a robust controller
- the state and identify the disturbance to compensate it through the control signal

Output-feedback stabilization of disturbed systems

HOSMO-based control approaches to deal with matched and unmatched disturbances



Exact *matched* disturbances compensation

With or without chattering?

$$\dot{x} = Ax + B[u + w], \quad y = Cx.$$

Problem formulation

Design a controller

$$u(t) = u_n + u_c$$

- $u_n(t)$ nominal control $(i.e., \forall w = 0)$
- $u_c(t)$ compensator

Question:

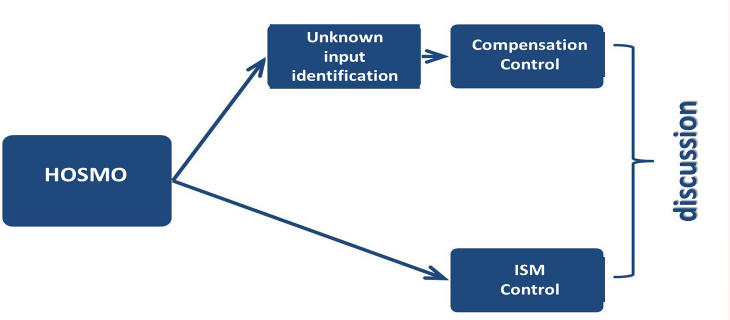
How to design the compensation term u_c?

Exact *matched* disturbances compensation

With or without chattering?

Two exact compensation approaches [Ferreira et al. 2011]

- HOSMO based identification and compensation control *(continuous)*
- Output integral sliding mode control *(discontinuous)*



Exact *matched* disturbances compensation

With or without chattering?

$$\dot{x} = Ax + B[u + w], \quad y = Cx.$$

$$x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^p,\ w(t) \in \mathbb{R}^q$$

Assumptions

A1. (A, C, D) strongly observable

A2. $w(t)$ bounded with a constant w^+ such that $\|w(t)\| \leq w^+ \ \forall t \geq 0$

A3. $w(t)$ satisfies $\left\|w^{(i)}(t)\right\| \leq w^+$ for $i = 1, \ldots, \alpha \ \forall t \geq 0$.

HOSM algebraic observer

Luemberger observer (bounding of the error)

$$\dot{\tilde{x}} = A\tilde{x} + Bu + L\left(y - \tilde{y}\right), \qquad \tilde{y} = C\tilde{x}$$

Observer error dynamics, $e\left(t\right) := x - \tilde{x}$

$$\begin{aligned} \dot{e} &= \left(A - LC\right)e + Dw \\ y_e &= Ce \end{aligned}$$

$$\left\|e\left(t\right)\right\| \leq e^{+}, \text{ for all } t > T$$

Disturbance decoupling (Molinari) and state recovering

Remark [Molinari 1976]

There exists a $k \le n$ such that $rank\,(M_k) = n$,

Step k

- $$\underbrace{\begin{bmatrix} \frac{d}{dt}\left(M_{k-1}D\right)^{\perp} M_{k-1}\tilde{A} \\ C \end{bmatrix}}_{M_k} e = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} \underbrace{\begin{bmatrix} y_{\mathrm{e}} \\ \int y_{\mathrm{e}} \\ \int \ldots \int y_{\mathrm{e}} d\tau dt \end{bmatrix}}_{Y_{[k-1]}}$$

Error state recovering

$$e\,(t) = \frac{d^{k-1}}{dt^{k-1}} \underbrace{M_k^{+} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]}}_{\Theta(t)}$$

HOSM differentiation

HOSM differentitator, [Levant 2003]

$$
\begin{aligned}
\dot{z}_0 &= \lambda_0 \Lambda^{\frac{1}{i+1}} \left\| z_0 - \Theta(t) \right\|^{\frac{i}{i+1}} \operatorname{sgn}\left(z_0 - \Theta(t)\right) + z_1 \\
&\vdots \\
\dot{z}_{i-1} &= \lambda_{i-1} \Lambda^{\frac{1}{2}} \left\| z_{i-1} - \dot{z}_{i-2} \right\|^{\frac{1}{2}} \operatorname{sgn}\left(z_{i-1} - \dot{z}_{i-2}\right) + z_i \\
\dot{z}_i &= \lambda_i \Lambda \operatorname{sgn}\left(z_i - \dot{z}_{i-1}\right)
\end{aligned}
$$

Remark

Under $A3$, the higher differentiation order possible is

$$i = \alpha + \kappa - 1$$

$$\frac{d^i \Theta(t)}{dt^i} = z_i \quad \forall t \geq t_f$$

State estimation and disturbance identification

State estimation, $\forall t \geq T$

$$\hat{x}(t) = \tilde{x}(t) + M_k^+ z^{(k-1)}$$

Perturbation identification, $\forall t \geq T$

$$\hat{w}(t) = D^+(\underbrace{\dot{e}}_{z_k} - (A - LC)\underbrace{e}_{z_{k-1}})$$

Estimation and identification accuracy

Measured output

$$\Theta(t) = \Theta_0(t) + \eta(t)$$

- η *Deterministic noise signal* $\|n(t)\| \leq \eta$
- δ *Sampling step*
- Δ *Combined effect of deterministic noise and sampling time*

Differentiator accuracy [Angulo et al.: 2011]

Error	δ
Observation	$O\left(\delta^{\alpha+1}\right)$
Identification	$O\left(\delta^{\alpha}\right)$
Differentiator	$O\left(\delta\right)$

Estimation and identification accuracy

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- δ *Sampling step*
- Δ *Combined effect of deterministic noise and sampling time*

Differentiator accuracy [Angulo et al.: 2011]

Error	δ	η
Observation	$O\left(\delta^{\alpha+1}\right)$	$O\left(\nu^{\frac{\alpha+1}{\alpha+k}}\right)$
Identification	$O\left(\delta^{\alpha}\right)$	$O\left(\nu^{\frac{\alpha}{\alpha+k}}\right)$
Differentiator	$O\left(\delta\right)$	$O\left(\nu^{\frac{1}{\alpha+k}}\right)$

Estimation and identification accuracy

Measured output

$$\Theta(t) = \Theta_0(t) + \eta(t)$$

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Differentiator accuracy [Angulo et al.: 2011]

Error	δ	η	Δ
Observation	$O\left(\delta^{\alpha+1}\right)$	$O\left(\nu^{\frac{\alpha+1}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha+1}\right)$
Identification	$O\left(\delta^{\alpha}\right)$	$O\left(\nu^{\frac{\alpha}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha}\right)$
Differentiator	$O\left(\delta\right)$	$O\left(\nu^{\frac{1}{\alpha+k}}\right)$	$O\left(\Delta\right)$

Exact *matched* disturbances compensation control

With or without chattering?

$$\dot{x} = Ax + B[u + w], \qquad y = Cx$$

HOSM based compensation control *(continuous)*

$$u(t) = -K\hat{x} - \hat{w}$$

Output-based integral sliding mode control *(discontinuous)*

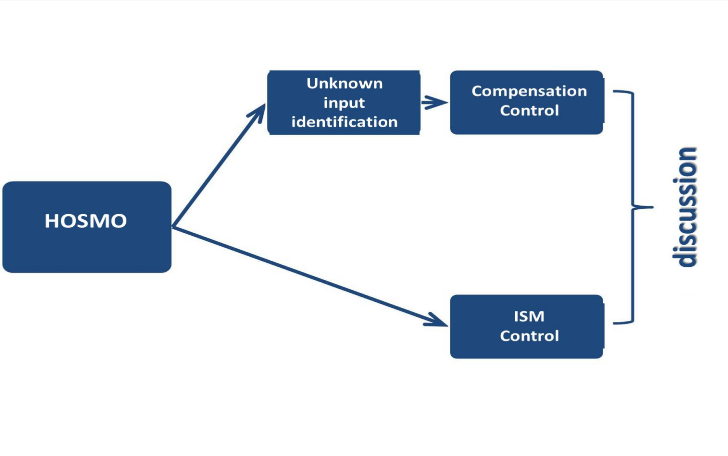
$$\begin{aligned} u(t) &= -K\hat{x}(t) - \rho \frac{s(\hat{x}, t)}{\|s(\hat{x}, t)\|} \\ s(\hat{x}, t) &= B^{+}\left[\hat{x}(t) - x(T) - \int_T^t [A\hat{x}(\tau) + Bu_n(\tau)]\, d\tau\right] \qquad \rho > w^{+} \end{aligned}$$

Exact *matched* disturbances compensation

With or without chattering?

Discussion

Which approach we should use?



Closed-loop system accuracy ϵ

HOSM identification and compensation control

$$\epsilon = \underbrace{O\left(\Delta^{\alpha+1}\right)}_{\text{Observation}} + \underbrace{O\left(\Delta^{\alpha}\right)}_{\text{Identification}} + \underbrace{O\left(\mu\right)}_{\text{Execution}}$$

Output-based integral sliding mode control

$$\epsilon = \underbrace{O\left(\Delta^{\alpha+1}\right)}_{\text{Observation}} + \underbrace{O\left(\mu\right)}_{\text{Execution}}$$

Where μ is the actuator time constant with an execution error $O(\mu)$ [Fridman: 02].

Exact *matched* disturbanceds compensation

With or without chattering?

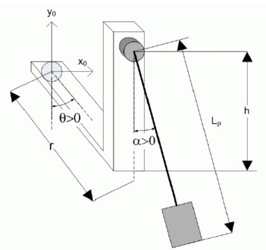
Controller approach selection [Ferreira et al. 2011]

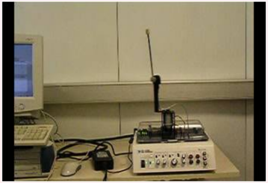
$$\underbrace{O\left(\Delta^{\alpha}\right)}_{\textit{Identification}} << \underbrace{O\left(h\right)}_{\textit{Execution}}$$ Compensation control

$$\underbrace{O\left(\Delta^{\alpha+1}\right)}_{\textit{Observation}} << \underbrace{O\left(h\right)}_{\textit{Execution}} << \underbrace{O\left(\Delta^{\alpha}\right)}_{\textit{Identification}}$$ ISM control

$$\underbrace{O\left(h\right)}_{\textit{Execution}} << \underbrace{O\left(\Delta^{\alpha+1}\right)}_{\textit{Observation}}$$ Compensation control

Experiments





$$x = \left[\theta, \alpha, \dot{\theta}, \dot{\alpha}\right]^T$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 82{,}4 & 1{,}31 & 0 \\ 0 & 56{,}81 & 0{,}37 & 0 \end{bmatrix},$$

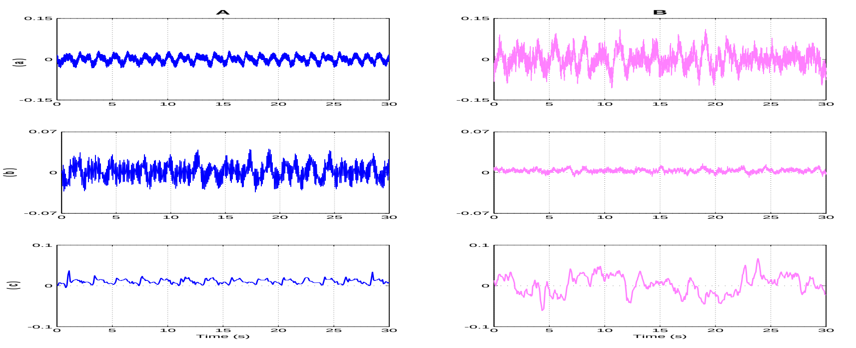
$$B = \begin{bmatrix} 0 \\ 0 \\ 46{,}75 \\ 13{,}20 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 46{,}75 \\ 13{,}20 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$w(t) = 0{,}4\mathrm{sen}(2{,}5t) + 0{,}5.$$

Compensation control

ISM Control

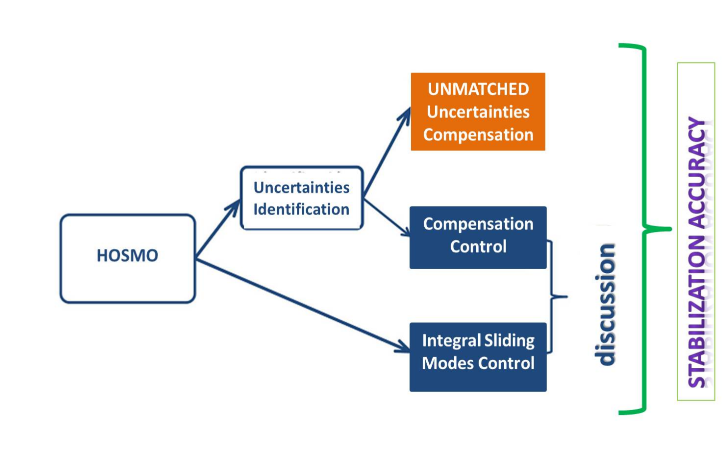


Remarks

- identification error $\ll$ execution error: remove perturbation by identification (a);
- execution error $\ll$ identification error: use ISM (b);

Output-feedback stabilization of disturbed systems

HOSMO-based control approaches to deal with matched and unmatched disturbances



Question

How to deal with *unmatched* disturbances?

Exact *unmatched* disturbances compensation control

System with unmatched disturbances, i.e., $\mathrm{span}\{D\} \subset \mathrm{span}\{B^{\perp}\}$

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw, \\ y &= Cx \end{aligned}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ $(1 \leq p < n)$

Identification and compensation approach [Ferreira et al. 2013]

- Providing (A, C, D) is strongly observable, use a HOSM observer to estimate x and identify w.
- Compensate the unmatched disturbances through the sliding surface.

Exact *unmatched* disturbances compensation control

$$\dot{x} = Ax + Bu + Dw, \qquad y = Cx$$

Coordinate transformation $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \mapsto T_r x$

$$T_r = \begin{bmatrix} B^\perp \\ B^+ \end{bmatrix}$$

where $B^\perp B = 0$, $B^+ = \left(B^T B\right)^{-1} B^T$.

- By construction $\det(T_r) \neq 0$.
- Applying the change of coordinates yields to

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + D_1 w \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + D_2 w + u \end{aligned}$$

where $x_1 \in \mathbb{R}^{n-m}$ and $x_2 \in \mathbb{R}^m$.

Exact *unmatched* disturbances compensation control

The regularized system

$$
\begin{aligned}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + D_1 w \\
\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + D_2 w + u \\
y &= CT_r x
\end{aligned}
$$

$$x_1 \in \mathbb{R}^{n-m},\ x_2 \in \mathbb{R}^m,\ u \in \mathbb{R}^m,\ w \in \mathbb{R}^q.$$

Assumptions [Ferreira et al.: 13]

A1. (A, B) controllable.

A2. (A, C, D) strongly observable.

A3. $w(t)$ and its derivatives up to order $\alpha + 1$ are bounded by the same constant w^+, i.e. $\|w^{(\alpha+1)}\| < w^+$ with $\alpha \geq 0$.

A4. $\mathrm{span}\{D_1\} \subset \mathrm{span}\{A_{12}\}$.

Exact *unmatched* disturbances compensation control

The regularized system

$$\begin{aligned}\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + D_1 w\\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + D_2 w + u\\ y &= CT_r x\end{aligned}$$

$$x_1 \in \mathbb{R}^{n-m},\ x_2 \in \mathbb{R}^m,\ u \in \mathbb{R}^m,\ w \in \mathbb{R}^q.$$

Problem formulation

- Use a HOSMO to estimate the state x and identify the disturbances w.
- Compensate the unmatched disturbances through the sliding surface (exploiting x_2 as a pseudo-control to stabilize x_1).
- Maintain the remained states trajectories bounded.

Exact *unmatched* disturbances compensation control

Sliding surface design

$$s = K\hat{x}_1 + \hat{x}_2 + G\hat{w}$$

- $K \in \mathbb{R}^{m\times(n-m)}$ determines the behavior of the reduced order dynamics.
- $G\hat{w}$ compensates the unmatched disturbances, $G = A_{12}^{+}D_1$.

Control law

$$u = u_n - \rho(x)\frac{s(t)}{\|s(t)\|}$$

- u_n is the nominal controller
- $-\rho(x)\frac{s(t)}{\|s(t)\|}$ with $\rho(x) \in \mathbb{R}$, drives the state trajectories to the sliding surface despite the disturbances w.

Exact *unmatched* disturbances compensation control

First, design u such that $s = \dot{s} = 0$

- Sliding surface dynamics

$$\dot{s} = \Phi x + (KD_1 + D_2)w + G\dot{w} + u$$

  where $\Phi \in \mathbb{R}^{m \times n}$ is a known matrix.

- Proposing

$$u = -\Phi x - \rho \frac{s}{\|s\|}, \qquad \rho > (\|KD_1 + D_2\| + \|G\|)w^+\gamma$$

  with $\gamma > 0$.

- Using $V = 0{,}5s^T s$ produces $\dot{V} \leq -\gamma V^{1/2}$. Therefore the sliding mode is established after $t > t_r$.

Exact *unmatched* disturbances compensation control

On the sliding surface $s = 0$

$$\hat{x}_2 = -K\hat{x}_1 - G\hat{w}$$

The reduced order dynamics becomes

$$\dot{x}_1 = (A_{11} - A_{22}K)\,x_1 + D_1 w - A_{12}G\hat{w}$$

Due to $A4$, G may be designed as $G = A_{12}^{+}D_1$ yielding to

$$\dot{x}_1 = (A_{11} - A_{22}K)\,x_1 \qquad \lambda(A_{11} - A_{22}K) < 0$$

Exact *unmatched* disturbances compensation control

The compensated dynamics

$$\begin{aligned} \|x_1(t)\| &\leq \alpha \|x_1(0)\| e^{-\beta t} & \qquad \alpha, \beta > 0 \\ \|x_2(t)\| &\leq \|K\| \|x_1(t)\| + \|G\| w^+ & \end{aligned}$$

In summary [Ferreira et al 2013]

- Use a HOSMO to estimate x and identify w.
- Compensate the unmatched disturbances through the sliding surface.
- The unmatched disturbances must be *matched* to the pseudo-control x_2, i.e., $\text{span}\{D_1\} \subset \text{span}\{A_{12}\}$.

Question

How to deal with the case when $\text{span}\{D_1\} \not\subset \text{span}\{A_{12}\}$?

Exact *unmatched* disturbances compensation control

$$\dot{x} = Ax + Bu + Dw, \qquad y = Cx$$

$$x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, \text{ and } y(t) \in \mathbb{R}^p \ (1 \leq p < n)$$

Assumptions [Ferreira et al. 2015]

A1. (A, C, D) strongly observable.

A2. $w(t)$ and its derivatives up to order r are bounded by the same constant w^+, i.e. $\|w^{(r)}\| < w^+$ with $r > 0$.

A3. $\mathrm{span}\{D\} \subset \mathrm{span}\{B^\perp\}$.

Problem formulation

Design an output feedback sliding mode controller u allowing x_1 (i.e., output to be controlled) to track a smooth signal x_d despite system disturbances w.

Exact *unmatched* disturbances compensation control

$$\dot{x} = Ax + Bu + Dw, \qquad y = Cx$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ $(1 \leq p < n)$

Strict-feedback form [Loukianov 1993]

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1(x_2 + \Gamma_1 w) \\ \dot{x}_i &= A_i \bar{x}_i + B_i(x_{i+1} + \Gamma_i w) \\ \dot{x}_r &= A_r \bar{x}_r + B_r(u + \Gamma_r w) \end{aligned}$$

$i = \overline{2, r-1}$, $\bar{x}_i = [\ x_1^T \quad \ldots \quad x_i^T\ ]^T$, $x_i \in \mathbb{R}^{n_i}$, $n_i = rank(B_i)$, $\sum_{i=1}^{r} n_i = n$

- Sub-systems comprising $i = \overline{1, r-1}$ represent the sub-actuated dynamics.
- $i = r$ corresponds to the actuated dynamics, $x_r \in \mathbb{R}^m$.

Exact *unmatched* disturbances compensation control

Identification and compensation approach [Ferreira et al. 2015]

- Use a HOSMO to estimate the state and identify the distrubance and their successive derivatives until $r-2$-th order.
- Construct a dynamic sliding surface in $r-1$-*steps*.
- Propose a sliding mode controller at $r-$ *th-step*.

Exact *unmatched* disturbances compensation control

Dynamic sliding surface design

- *Step 1)* Exploit x_2 as a pseudo-control for x_1,

$$\dot{x}_1 = A_1 x_1 + B_1(x_2 + \Gamma_1 w)$$

$$\phi_1 = -\Gamma_1 \hat{w} - B_1^{\dagger}(A_1 x_1 - \hat{A}_1(x_1 - x_d) - \dot{x}_d)$$

Exact *unmatched* disturbances compensation control

Dynamic sliding surface design

- *Step i)* Exploit x_{i+1} as a pseudo-control for x_i, it is $x_{i+1} := \phi_i$

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1(x_2 + \Gamma_1 w) \\ \dot{x}_i &= A_i \bar{x}_i + B_i(x_{i+1} + \Gamma_i w) \end{aligned}$$

$$\phi_i = -\Gamma_i \hat{w} - B_i^{\dagger}(A_i \bar{x}_i - \hat{A}_i(x_i - \phi_{i-1}) + X_{i-1}(x_{i-1} - \phi_{i-2}) - \dot{\phi}_{i-1})$$

Exact *unmatched* disturbances compensation control

Dynamic sliding surface design

- *Step $r-1$)* Finally, the sliding surface is designed as

$$s = x_r - \phi_{r-1}$$

$$\begin{aligned}\phi_{r-1} = & - \Gamma_{r-1}\hat{w} - B_{r-1}^{\dagger}(A_{r-1}\bar{x}_{r-1} - \hat{A}_{r-1}(x_{r-1} - \phi_{r-2}) \\ & + X_{r-2}(x_{r-2} - \phi_{r-3}) - \dot{\phi}_{r-2})\end{aligned}$$

Exact *unmatched* disturbances compensation control

The control law becomes

$$u = -\underbrace{-\Gamma_r \hat{w} - B_r^{\dagger}(A_r \bar{x}_r - \dot{\bar{\phi}}_r + \dots)}_{u_n} - \underbrace{B_r^{\dagger}\nu}_{u_c}$$

where $\nu \in \mathbb{R}^m$ may be a sliding mode control, for instance super-twisting controller

$$\nu = K_1 \frac{s}{\|s\|^{1/2}} + K_2 \int_{t_f}^{t} \frac{s}{\|s\|}.$$

Remarks

- An output tracking control for MIMO systems subjected to matched and unmatched disturbances based on HOSMO.
- The exact compensation of the disturbances is tackled through a dynamic sliding surface design
- A sliding mode controller drives the states to the sliding-surface and diminishes the computational complexity (system order reduction)

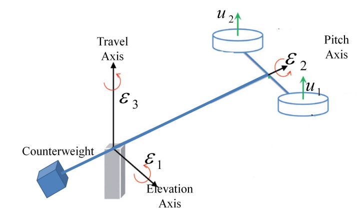
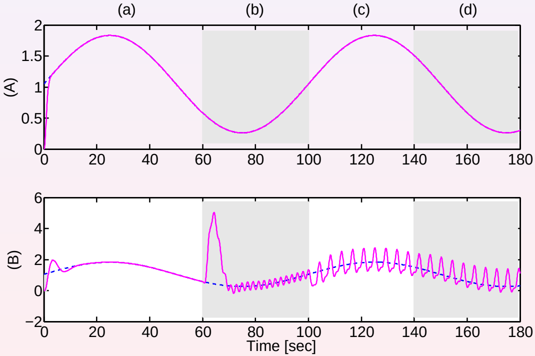


Figura: Schematic diagram of a 3-DOF helicopter.

A linearized model around $\epsilon_2^* = 0$ is given by

$$\begin{aligned}
\ddot{\epsilon}_1 &= 0{,}45\,((u_1 + f_1) + (u_2 + f_2)) \\
\ddot{\epsilon}_2 &= 3{,}05\,((u_1 + f_1) - (u_2 + f_2)) \\
\ddot{\epsilon}_3 &= -0{,}49\epsilon_2 + \nu
\end{aligned}$$

Sliding Mode Compensation vs H_∞



Travel tracking performance: (A) Exact unmatched compensation control vs (B) H_∞ control.

Outline

Outline

10. Switched Systems
    - Observation of the continuous and discrete state of switched systems
    - Linear Switched Systems

Problem Statement

Consider the switched system:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + B_{\sigma(t)}u(t) + D_{\sigma(t)}w(t), \tag{27}$$

$$y(t) = h_{\sigma(t)}(x(t)) \tag{28}$$

$x \in \mathcal{X} \subseteq \mathbb{R}^n$, $y \in \mathcal{Y} \subseteq \mathbb{R}^m$, $u \in \mathcal{U} \subseteq \mathbb{U}^p$, $w \in \mathcal{W} \subseteq \mathbb{R}^m$,
$\sigma \in \mathcal{Q} \in \aleph$,
$f_{\sigma(t)} : \mathcal{X}_\sigma \to \mathbb{R}^n$, $B_{\sigma(t)} \in \mathbb{R}^{n\times p}$, $D_{\sigma(t)} \in \mathbb{R}^{n\times m}$, $h_{\sigma(t)} : \mathcal{X}_\sigma \to \mathcal{Y}$,

- **Autonomous.** The dynamics changes discontinuously when $x(t)$ hits certain boundaries, i.e. σ depends on $x(t)$.
- **Non-Autonomous.** The dynamics changes abruptly in response to a control command or an exogenous signal, i.e. σ does not depends on $x(t)$.

Main Objective

To develop some observation approaches for estimating the continuous and discrete states in certain classes of SS.

Main idea: use HOSM Observers as a bank of observers (continuous state) and the equivalent injection (discrete state/unknown inputs).

J. Davila et al. AsJC 2012

- Finite time continuous state estimation for Nonlinear Switched Systems (Non-Autonomous, i.e. $\sigma(t)$);
- Finite time discrete state estimation using Equivalent Injection;

H. Ríos et al. JFI 2012

- Finite time continuous and discrete state estimation for Nonlinear Switched Systems (Autonomous, i.e. $\sigma(t)$);
- Approximated unknown input identification using Equivalent Injection;

H. Ríos et al. IJACSP 2014

- Finite time continuous and discrete state estimation for Linear Switched Systems (Autonomous, i.e. $\sigma(t)$);
- Approximated unknown input identification using Equivalent Injection;

Outline

10. Switched Systems
    - Observation of the continuous and discrete state of switched systems
    - Linear Switched Systems

Problem Statement

Consider the following class of SS

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t), \\ y(t) &= C_{\sigma(t)}x(t), \end{aligned} \tag{29}$$

$$\sigma(t) = \begin{cases} 1, & \forall x(t) \mid \quad Hx(t) \in \mathcal{H}_1, \\ 2, & \forall x(t) \mid \quad Hx(t) \in \mathcal{H}_2, \\ \vdots & \\ q, & \forall x(t) \mid \quad Hx(t) \in \mathcal{H}_q, \end{cases}$$

$x \in \Re^n$, $y \in \Re^m$, $w \in \Re^m$, $\|w(t)\| \leq w^+ < \infty$,
$A_\sigma \in \Re^{n\times n}$, $C_\sigma \in \Re^{m\times n}$ and $E_\sigma \in \Re^{n\times m}$,
$\sigma \in \mathcal{Q} = \{1, \ldots, q\}$,
$H \in \Re^{1\times n}$ known,
$mathcalH_1$, $\mathcal{H}_2, ..., \mathcal{H}_q \in \Re$ known and disjoint intervals.

Autonomous Switchings

Problem Statement

Consider the following class of SS

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t),\\ y(t) &= C_{\sigma(t)}x(t),\end{aligned} \tag{29}$$

$$\sigma \in \mathcal{Q} = \{1, \ldots, q\},$$

$x \in \Re^n$, $y \in \Re^m$, $w \in \Re^m$, $\|w(t)\| \leq w^+ < \infty$,
$A_\sigma \in \Re^{n\times n}$, $C_\sigma \in \Re^{m\times n}$ and $E_\sigma \in \Re^{n\times m}$,
$\sigma(t)$ Exogenous Signal.

Non-autonomous Switchings

Problem Statement

Consider the following class of SS

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t), \\ y(t) &= C_{\sigma(t)}x(t), \end{aligned} \tag{29}$$

$$\sigma \in \mathcal{Q} = \{1, \ldots, q\},$$

$x \in \Re^n$, $y \in \Re^m$, $w \in \Re^m$, $\|w(t)\| \leq w^+ < \infty$,
$A_\sigma \in \Re^{n\times n}$, $C_\sigma \in \Re^{m\times n}$ and $E_\sigma \in \Re^{n\times m}$,
$\sigma(t)$ Exogenous Signal.

The objective is to estimate the continuous $x(t)$ and discrete $\sigma(t)$ states, respectively.

Assumptions: Continuous State

1. The pair (A_σ, C_σ), $\forall \sigma \in \mathcal{Q}$, is detectable.
2. The output has a *r.d.v.* $(r_1, \ldots, r_m)$ w.r.t. w, such that $r_1 + \cdots + r_m = n_\mathcal{V}$, $\forall \sigma \in \mathcal{Q}$.

Assumptions: Continuous State

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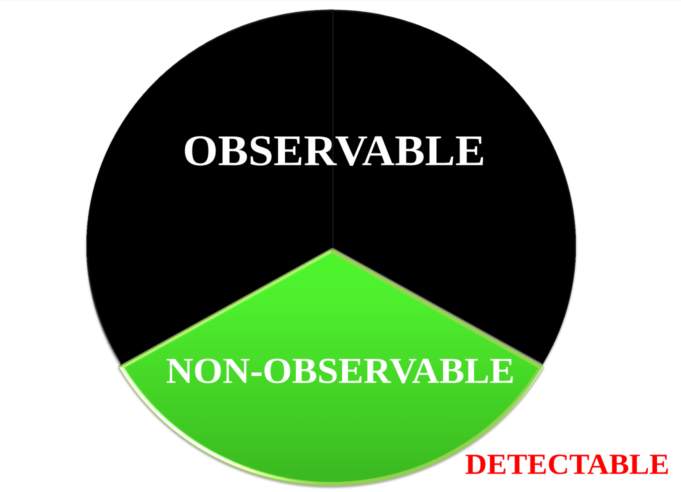
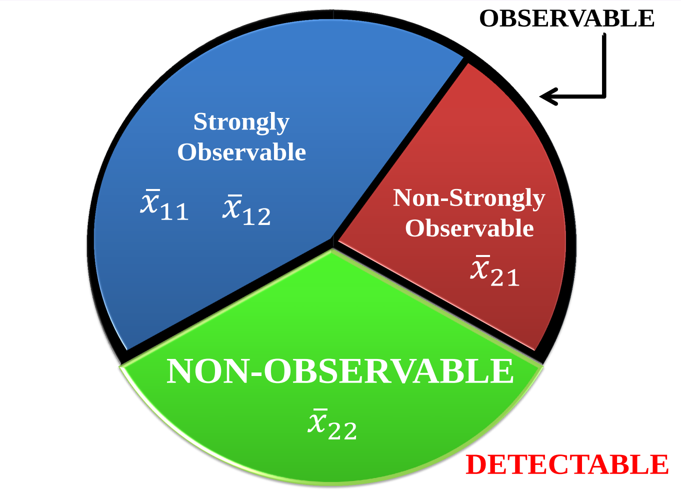


Assumptions: Continuous State

1. The pair (A_σ, C_σ), $\forall \sigma \in \mathcal{Q}$, is detectable.
2. The output has a *r.d.v.* $(r_1, \ldots, r_m)$ w.r.t. w, such that $r_1 + \cdots + r_m = n_{\mathcal{V}}$, $\forall \sigma \in \mathcal{Q}$.



Assumptions: Continuous State

1. The pair (A_σ, C_σ), $\forall \sigma \in \mathcal{Q}$, is detectable.
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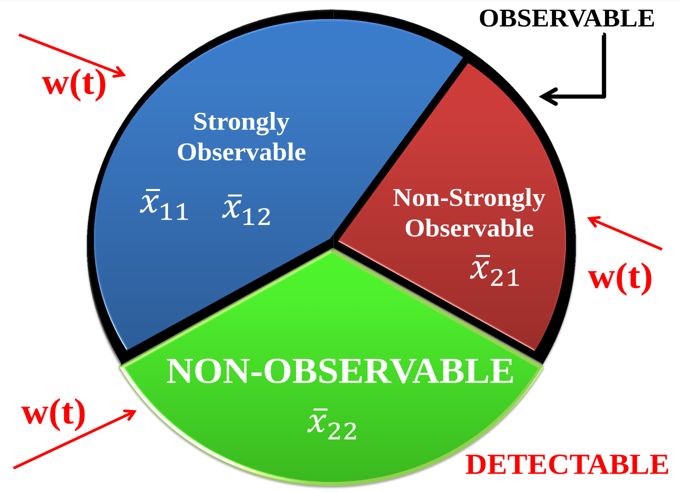


Transformation

$$T := \left[\begin{array}{cccc} U_1^T & U_2^T & (V_{\bar{\mathcal{N}}}{}^{+})^T & (N^{+})^T \end{array} \right]^T \Rightarrow \bar{x}(t) = Tx(t)$$

FOR EACH $\sigma = 1, \ldots, q$!!.

$$\begin{bmatrix} \dot{\bar{x}}_{11}(t) \\ \dot{\bar{x}}_{12}(t) \\ \dot{\bar{x}}_{21}(t) \\ \dot{\bar{x}}_{22}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \\ \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} \bar{w}(t),$$

$$\begin{aligned} y(t) &= C_1 \left[\begin{array}{cc} \bar{x}_{11}^T(t) & \bar{x}_{12}^T(t) \end{array} \right]^T, \\ \bar{w}(t) &= w(t) - K_1^* \bar{x}_{21}(t), \end{aligned}$$

Estimation for $\bar{x}_{11}$ and $\bar{x}_{12}$ with $\sigma(t) = \sigma^* = cte.$

Consider $\bar{x}_1 = \left[\begin{array}{cc} \bar{x}_{11}^T & \bar{x}_{12}^T \end{array}\right]^T$ **Strongly Observable**,

$$\dot{\bar{x}}_1(t) = A_{1_{\sigma^*}}\bar{x}_1(t) + B_{12_{\sigma^*}}u(t) + E_{1_{\sigma^*}}\bar{w}(t), \quad y(t) = C_{1_{\sigma^*}}\bar{x}_1(t). \quad (30)$$

Estimation for $\bar{x}_{11}$ and $\bar{x}_{12}$ with $\sigma(t) = \sigma^* = cte.$

Consider $\bar{x}_1 = \left[\begin{array}{cc} \bar{x}_{11}^T & \bar{x}_{12}^T \end{array}\right]^T$ **Strongly Observable**,

$$\dot{\bar{x}}_1(t) = A_{1\sigma^*}\bar{x}_1(t) + B_{12\sigma^*}u(t) + E_{1\sigma^*}\bar{w}(t), \quad y(t) = C_{1\sigma^*}\bar{x}_1(t). \tag{30}$$

Observer **[L. Fridman et. al IJSS2007]**

$$\begin{aligned}
\hat{x}_{1_{\sigma^*}}(t) &= z_{1_{\sigma^*}}(t) + P_{1_{\sigma^*}}^{-1}\nu_{\sigma^*}(t), \\
\dot{z}_{1_{\sigma^*}}(t) &= A_{1\sigma^*}z_{1\sigma^*}(t) + B_{12\sigma^*}u(t) + L_{1\sigma^*}\left(y(t) - C_{1\sigma^*}z_{1\sigma^*}(t)\right), \\
\dot{\nu}_{\sigma^*}(t) &= W_{\sigma^*}(y(t) - C_{1\sigma^*}z_{1\sigma^*}(t), \nu_{\sigma^*}(t)),
\end{aligned}$$

$A_{L_{1\sigma^*}} \Rightarrow (A_{1\sigma^*} - L_{1\sigma^*}C_{1\sigma^*})$ Hurwitz.

$$\begin{aligned}
P_{1\sigma^*} = \Big[\; & c_{1_{\sigma^*}}^T, \quad (c_{1\sigma^*}A_{L_{1\sigma^*}})^T, \quad \cdots, \quad (c_{1\sigma^*}A_{L_{1\sigma^*}}^{r_{1_{\sigma^*}}-1})^T, \quad \cdots, \\
& c_{m_{\sigma^*}}^T, \quad (c_{m\sigma^*}A_{L_{1\sigma^*}})^T, \quad \cdots, \quad (c_{m\sigma^*}A_{L_{1\sigma^*}}^{r_{m_{\sigma^*}}-1})^T \;\Big]^T \\
& \Rightarrow \operatorname{rank}(P_{1\sigma^*}) = n_{\mathcal{V}_{\sigma^*}}.
\end{aligned}$$

Estimation for $\bar{x}_{11}$ and $\bar{x}_{12}$ with $\sigma(t) = \sigma^* = cte.$

The correction terms

$$\nu_{\sigma^*}(t) = \left(\nu_{1_{1_{\sigma^*}}}, \ldots, \nu_{1_{r_{1_{\sigma^*}}}}, \ldots, \nu_{m_{1_{\sigma^*}}}, \ldots, \nu_{m_{r_{m_{\sigma^*}}}}\right)$$

Estimation for $\bar{x}_{11}$ and $\bar{x}_{12}$ with $\sigma(t) = \sigma^* = cte$.

The correction terms

$$\nu_{\sigma^*}(t) = \left(\nu_{1_{1_{\sigma^*}}}, \ldots, \nu_{1_{r_{1_{\sigma^*}}}}, \ldots, \nu_{m_{1_{\sigma^*}}}, \ldots, \nu_{m_{r_{m_{\sigma^*}}}}\right)$$

HOSM Differentiator **[Levant and Livne TAC2012]**

$$\dot{\nu}_{k_{1_{\sigma^*}}} = \nu_{k_{2_{\sigma^*}}} - \alpha_{k_{1_{\sigma^*}}} M_{k_{\sigma^*}}(t)^{\frac{1}{r_{k_{\sigma^*}}}} \left\lceil \nu_{k_{1_{\sigma^*}}} - e_{y_{k_{\sigma^*}}} \right\rfloor^{\frac{r_{k_{\sigma^*}}-1}{r_{k_{\sigma^*}}}},$$

$$\dot{\nu}_{k_{i_{\sigma^*}}} = \nu_{k_{(i+1)_{\sigma^*}}} - \alpha_{k_{i_{\sigma^*}}} M_{k_{\sigma^*}}(t)^{\frac{1}{r_{k_{\sigma^*}}-i+1}} \left\lceil \nu_{k_{i_{\sigma^*}}} - \dot{\nu}_{k_{(i-1)_{\sigma^*}}} \right\rfloor^{\frac{r_{k_{\sigma^*}}-i}{r_{k_{\sigma^*}}-i+1}},$$

$$\forall i = 2, \ldots, r_{k_{\sigma^*}} - 1,$$

$$\dot{\nu}_{k_{r_{k_{\sigma^*}}}} = -\alpha_{k_{r_{k_{\sigma^*}}}} M_{k_{\sigma^*}}(t) \left\lceil \nu_{k_{r_{k_{\sigma^*}}}} - \dot{\nu}_{k_{r_{k_{\sigma^*}}-1}} \right\rfloor^{0}, \ \forall k = 1, \ldots, m,$$

where $e_{y_{k_{\sigma^*}}}(t) = y_k(t) - c_k z_{1_{\sigma^*}}(t)$, the constants $\alpha_{k_{i_{\sigma^*}}}$ and the function $M_{k_{\sigma^*}}(t)$ are chosen recursively and sufficiently large.

Estimation for $\bar{x}_{21}$ with $\sigma(t) = \sigma^* = cte.$

Consider the **Non-Strongly Observable but Observable** part, i.e.

$$\dot{\bar{x}}_{21}(t) = A_{31_{\sigma^*}}\bar{x}_{11}(t) + A_{32_{\sigma^*}}\bar{x}_{12}(t) + A_{33_{\sigma^*}}\bar{x}_{21}(t) + B_{3_{\sigma^*}}u(t) + E_{21_{\sigma^*}}\bar{w}(t). \quad (31)$$

Estimation for $\bar{x}_{21}$ with $\sigma(t) = \sigma^* = cte.$

Consider the **Non-Strongly Observable but Observable** part, i.e.

$$\dot{\bar{x}}_{21}(t) = A_{31_{\sigma^*}}\bar{x}_{11}(t) + A_{32_{\sigma^*}}\bar{x}_{12}(t) + A_{33_{\sigma^*}}\bar{x}_{21}(t) + B_{3_{\sigma^*}}u(t) + E_{21_{\sigma^*}}\bar{w}(t). \quad (31)$$

Let $\hat{x}_{21_{\sigma^*}}(t)$ be the estimated state

$$\begin{aligned}
\hat{x}_{21_{\sigma^*}}(t) &= z_{2_{\sigma^*}}(t) + L_{2_{\sigma^*}}\hat{x}_{12_{\sigma^*}}(t), \\
\dot{z}_{2_{\sigma^*}}(t) &= \bar{A}_{1_{\sigma^*}}\hat{x}_{11_{\sigma^*}}(t) + \bar{A}_{2_{\sigma^*}}\hat{x}_{12_{\sigma^*}}(t) + A_{L_{2\sigma^*}}\hat{x}_{21_{\sigma^*}}(t) + \bar{B}_{2_{\sigma^*}}u(t),
\end{aligned}$$

$$\begin{aligned}
\bar{A}_{1_{\sigma^*}} &= A_{31_{\sigma^*}} - L_{2_{\sigma^*}}A_{21_{\sigma^*}}, \\
\bar{A}_{2_{\sigma^*}} &= A_{32_{\sigma^*}} - L_{2_{\sigma^*}}A_{22_{\sigma^*}}, \\
A_{L_{2\sigma^*}} &= A_{33_{\sigma^*}} - K^*_{1\,\sigma^*}E_{21_{\sigma^*}} + L_{2_{\sigma^*}}E_{12_{\sigma^*}}K^*_{1\,\sigma^*}, \\
\bar{B}_{2_{\sigma^*}} &= B_{3_{\sigma^*}} - L_{2_{\sigma^*}}B_{2_{\sigma^*}}.
\end{aligned}$$

Estimation for $\bar{x}_{22}$ with $\sigma(t) = \sigma^* = cte.$

Consider the **Non-Observable** part, i.e.

$$\dot{\bar{x}}_{22}(t) = A_{41_{\sigma^*}}\bar{x}_{11}(t) + A_{42_{\sigma^*}}\bar{x}_{12}(t) + A_{43_{\sigma^*}}\bar{x}_{21}(t) + A_{44_{\sigma^*}}\bar{x}_{22}(t) + B_{4_{\sigma^*}}u(t) + E_{22_{\sigma^*}}\bar{w}(t). \quad (32)$$

Estimation for $\bar{x}_{22}$ with $\sigma(t) = \sigma^* = cte.$

Consider the **Non-Observable** part, i.e.

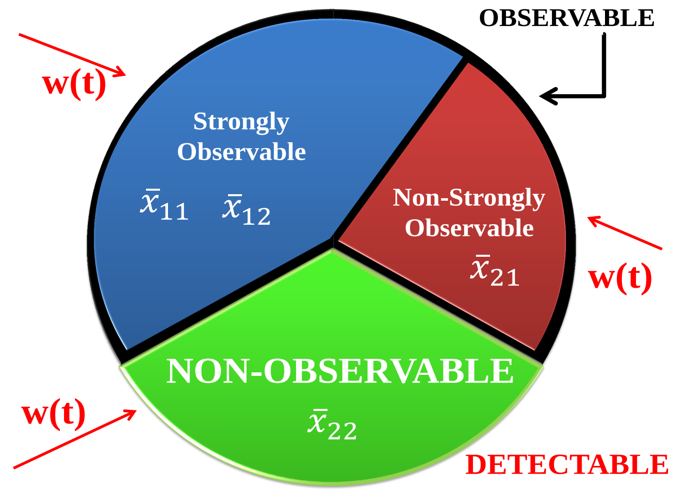
$$\dot{\bar{x}}_{22}(t) = A_{41_{\sigma^*}}\bar{x}_{11}(t) + A_{42_{\sigma^*}}\bar{x}_{12}(t) + A_{43_{\sigma^*}}\bar{x}_{21}(t) + A_{44_{\sigma^*}}\bar{x}_{22}(t) + B_{4_{\sigma^*}}u(t) + E_{22_{\sigma^*}}\bar{w}(t). \quad (32)$$

Let $\hat{x}_{22_{\sigma^*}}(t)$ be the estimated state

$$\hat{x}_{22_{\sigma^*}}(t) = z_{3_{\sigma^*}}(t) + E_{22_{\sigma^*}}E_{2_{\sigma^*}}^{+}\begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{21}(t) \end{bmatrix},$$

$$\begin{aligned} \dot{z}_{3_{\sigma^*}}(t) = {} & A_{41_{\sigma^*}}\hat{x}_{11_{\sigma^*}}(t) + A_{42_{\sigma^*}}\hat{x}_{12_{\sigma^*}}(t) + \\ & A_{43_{\sigma^*}}\hat{x}_{21_{\sigma^*}}(t) + A_{44_{\sigma^*}}\hat{x}_{22_{\sigma^*}}(t) + B_{4_{\sigma^*}}u(t) \\ & -E_{22_{\sigma^*}}E_{2_{\sigma^*}}^{+}\begin{bmatrix} A_{21_{\sigma^*}}\hat{x}_{11_{\sigma^*}}(t) + A_{22_{\sigma^*}}\hat{x}_{12_{\sigma^*}}(t) + B_{2_{\sigma^*}}u(t) \\ A_{31_{\sigma^*}}\hat{x}_{11_{\sigma^*}}(t) + A_{32_{\sigma^*}}\hat{x}_{12_{\sigma^*}}(t) + A_{33_{\sigma^*}}\hat{x}_{21_{\sigma^*}}(t) + B_{3_{\sigma^*}}u(t) \end{bmatrix}, \end{aligned}$$

Partial Results: The Worst Case!!

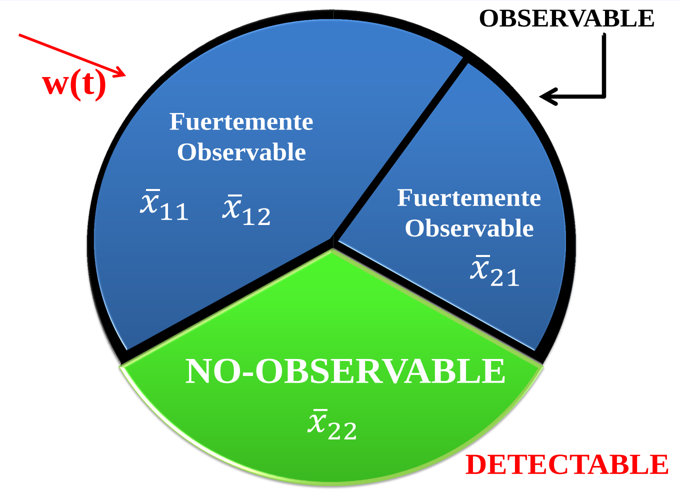


Partial Results: The Worst Case!!

Results: Continuous State $\forall t \in [t_{\sigma^*}, T_\delta)$

1. Exact and Finite Time estimation for the *Strongly Observable* states $x_{11}(t)$ and $x_{12}(t)$.
2. Bounded Estimation Error for the *Non-Strongly Observable* but *Observable* state $x_{21}(t)$.
3. Bounded Estimation Error for the *Non-Observable* state $x_{22}(t)$.

Partial Results: The Best Case!!

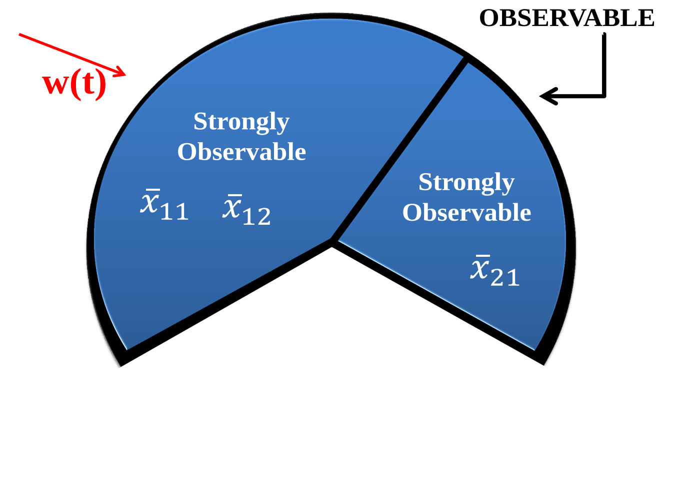


Partial Results: The Best Case!!

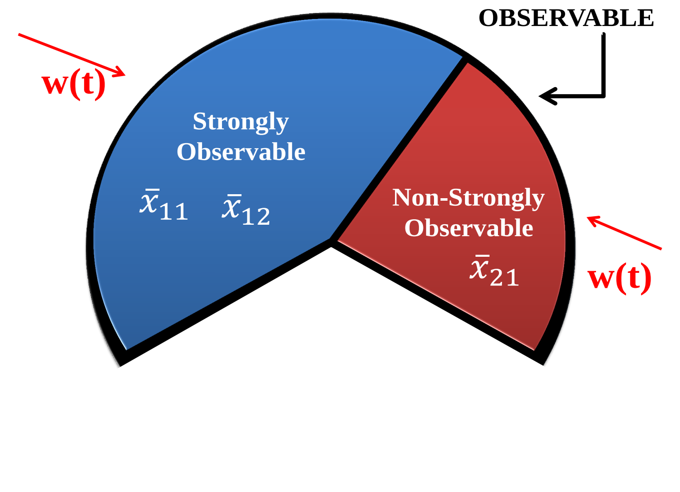
Results: Continuous State $\forall t \in [t_{\sigma^*}, T_\delta)$

1. Exact and Finite Time estimation for the *Strongly Observable* states $x_{11}(t)$ and $x_{12}(t)$.
2. Exponential estimation for the *Non-Observable* state $x_{22}(t)$.

Partial Results: The Best Case!!



Partial Results: The Best Case!!



Partial Results: The Best Case!!

Results: Continuous State $\forall t \in [t_{\sigma^*}, T_\delta)$

1. Exact and Finite Time estimation for the *Strongly Observable* states $x_{11}(t)$ and $x_{12}(t)$.
2. Bounded Estimation Error (adjustable) for the *Non-Strongly Observable* but *Observable* state $x_{21}(t)$.

Bank of Observers

The following Bank of Observers is proposed

$$\begin{aligned}
\tilde{x}_{1_{\hat{\sigma}}}(t) &= z_{1\hat{\sigma}}(t) + P_{1_{\hat{\sigma}}}^{-1}\nu_{\hat{\sigma}}(t),\\
\tilde{y}_{\hat{\sigma}}(t) &= C_{1\hat{\sigma}}\tilde{x}_{1_{\hat{\sigma}}}(t),\\
\tilde{x}_{21_{\hat{\sigma}}}(t) &= z_{2\hat{\sigma}}(t) + L_{2\hat{\sigma}}\tilde{x}_{12_{\hat{\sigma}}}(t),\\
\tilde{x}_{22_{\hat{\sigma}}}(t) &= z_{3\hat{\sigma}}(t) + E_{22\hat{\sigma}}E_{2_{\hat{\sigma}}}^{+}\begin{bmatrix} \tilde{x}_{12_{\hat{\sigma}}}(t) \\ \tilde{x}_{21_{\hat{\sigma}}}(t) \end{bmatrix},\\
\forall\hat{\sigma} &= 1,\ldots,q,
\end{aligned}$$

Bank of Observers

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&\tilde{x}_{1_{\hat{\sigma}}}(t) = z_{1\hat{\sigma}}(t) + P_{1_{\hat{\sigma}}}^{-1}\nu_{\hat{\sigma}}(t),\\
&\tilde{y}_{\hat{\sigma}}(t) = C_{1\hat{\sigma}}\tilde{x}_{1_{\hat{\sigma}}}(t),\\
&\tilde{x}_{21_{\hat{\sigma}}}(t) = z_{2\hat{\sigma}}(t) + L_{2\hat{\sigma}}\tilde{x}_{12_{\hat{\sigma}}}(t),\\
&\tilde{x}_{22_{\hat{\sigma}}}(t) = z_{3\hat{\sigma}}(t) + E_{22\hat{\sigma}}E_{2_{\hat{\sigma}}}^{+}\begin{bmatrix} \tilde{x}_{12_{\hat{\sigma}}}(t) \\ \tilde{x}_{21_{\hat{\sigma}}}(t) \end{bmatrix},\\
&\forall\hat{\sigma} = 1,\dots,q,
\end{aligned}
$$

Assumptions: Discrete State

3. Known initial Discrete State, i.e. $\sigma(0) = \hat{\sigma}(0) = \sigma^*$ known.
4. Strictly positive Dwell time, such that $t_{\sigma^*} < T_\delta$.
5. The eigenvalues of $A_{44_{\sigma^*}}$ and the i.c. $x(0)$, allow to satisfy $t_{\sigma*} < t_1$.

Discrete State Estimation

AUTONOMOUS SWITCHINGS

Consider the discrete state observer

$$\hat{\sigma}(t) = \begin{cases} 1, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_1, \\ 2, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_2, \\ \vdots & \\ q, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_q, \end{cases}$$

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Observability Matrix

$$Q_\sigma = \left[\; H^T, \quad (HA_\sigma)^T, \quad \cdots, \quad (HA_\sigma^{n-1})^T \; \right]^T.$$

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Observability Matrix

$$Q_\sigma = \left[\ H^T, \quad (HA_\sigma)^T, \quad \cdots, \quad (HA_\sigma^{n-1})^T\ \right]^T.$$

$$\mathcal{V}_\sigma^* \subset \ ker(Q_\sigma), \ \forall \sigma \in \mathcal{Q} = \{1, \ldots, q\}.$$

AUTONOMOUS SWITCHINGS

Consider the discrete state observer

$$\hat{\sigma}(t) = \begin{cases} 1, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_1, \\ 2, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_2, \\ \vdots & \\ q, & \forall \hat{x}(t) \mid \quad H\hat{x}(t) \in \mathcal{H}_q, \end{cases}$$

Observability Matrix

$$Q_\sigma = \left[\; H^T, \;\; (HA_\sigma)^T, \;\; \cdots, \;\; (HA_\sigma^{n-1})^T \; \right]^T .$$

$$\mathcal{V}_\sigma^* \subset \; ker(Q_\sigma), \; \forall \sigma \in \mathcal{Q} = \{1, \ldots, q\} .$$

$\Rightarrow \; \sigma(t)$ **is estimated Exactly and in Finite Time.**

Discrete State Estimation

NON-AUTONOMOUS SWITCHINGS

Consider the case $w(t) = 0$

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{1_{\sigma(t)}} & 0 \\ A_{2_{\sigma(t)}} & A_{3_{\sigma(t)}} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_{1_{\sigma(t)}} \\ B_{2_{\sigma(t)}} \end{bmatrix} u(t),$$

$$y(t) = C_{1_{\sigma(t)}} \bar{x}_1(t)^T,$$

Discrete State Estimation

NON-AUTONOMOUS SWITCHINGS

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$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{1_{\sigma(t)}} & 0 \\ A_{2_{\sigma(t)}} & A_{3_{\sigma(t)}} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_{1_{\sigma(t)}} \\ B_{2_{\sigma(t)}} \end{bmatrix} u(t),$$

$$y(t) = C_{1_{\sigma(t)}} \bar{x}_1(t)^T,$$

Assumptions: Discrete State

5a. The eigenvalues of $A_{3_{\sigma*}}$ and the i.c. $x(0)$, allow to satisfy $t_{\sigma*} < t_1$.

6.

$$A_{1_i} \neq A_{1_j} \;\vee\; B_{1_i} \neq B_{1_j},\; \forall i \neq j,\; \forall i, j \in \mathcal{Q}.$$

Discrete State Estimation

NON-AUTONOMOUS SWITCHINGS

Consider the case $w(t) = 0$

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{1_{\sigma(t)}} & 0 \\ A_{2_{\sigma(t)}} & A_{3_{\sigma(t)}} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_{1_{\sigma(t)}} \\ B_{2_{\sigma(t)}} \end{bmatrix} u(t),$$

$$y(t) = C_{1_{\sigma(t)}} \bar{x}_1(t)^T,$$

Output Equivalent Injection

$$\nu_{\hat{\sigma}^* eq}(t) = P_{1\hat{\sigma}^*} \left(\bar{x}_1(t) - z_{1\hat{\sigma}^*}\right) = 0,$$

$$r_j(t) = \nu^2_{1_{1_j}}(t) + \ldots + \nu^2_{m_{1_j}}(t) + \ldots + \nu^2_{m_{r_{m_j}}}(t), \ \forall j \in \mathcal{Q},$$

$$\hat{\sigma}(t) = \arg \min_j r_j(t).$$

$\Rightarrow$ $\sigma(t)$ **is estimated Exactly and in Finite Time.**

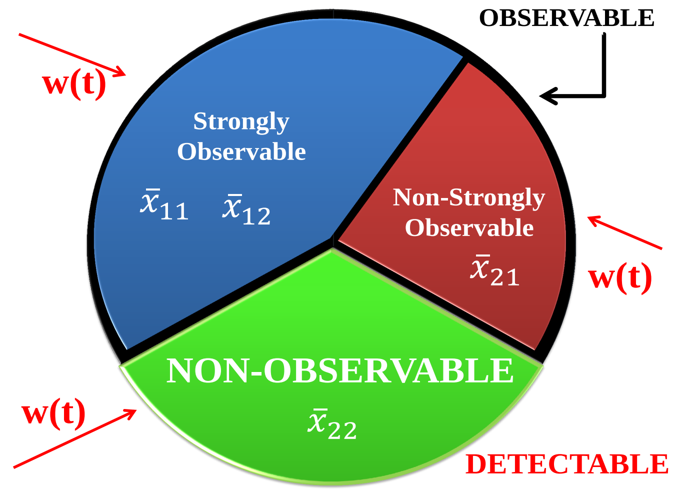
Switching Instants

Proposition: Reset Equations

The continuous state estimation is maintained in spite of the switchings if the following reset equations are implemented in the bank of observers, for all $\hat{\sigma}(t) \neq \hat{\sigma}^*$, i.e.

$$
\begin{aligned}
&\nu_{\hat{\sigma}(t_i{}^+)}(t_i{}^+) = 0,\\
&z_{1_{\hat{\sigma}(t_i{}^+)}}(t_i{}^+) = \tilde{x}_{1_{\hat{\sigma}^*}}(t_i),\\
&z_{2_{\hat{\sigma}(t_i{}^+)}}(t_i{}^+) = \tilde{x}_{21_{\hat{\sigma}^*}}(t_i) - L_{2_{\hat{\sigma}^*}}\tilde{x}_{12_{\hat{\sigma}^*}}(t_i),\\
&z_{3_{\hat{\sigma}(t_i{}^+)}}(t_i{}^+) = \tilde{x}_{22_{\hat{\sigma}^*}}(t_i) - E_{22\hat{\sigma}^*}E_{2_{\hat{\sigma}^*}}^{+}\begin{bmatrix} \tilde{x}_{12_{\hat{\sigma}^*}}(t_i) \\ \tilde{x}_{21_{\hat{\sigma}^*}}(t_i) \end{bmatrix}.
\end{aligned}
$$

Main Results: The Worst Case!!



Main Results: The Worst Case!!

Results $\forall t \in [t_{\sigma^*}, \infty)$

1. Exact and Finite Time estimation for the *Strongly Observable* states $x_{11}(t)$ and $x_{12}(t)$.
2. Ultimate Bounded Estimation Error for the *Non- Strongly Observable* but *Observable* state $x_{21}(t)$.
3. Ultimate Bounded Estimation Error for the *Non- Observable* state $x_{22}(t)$.
4. Exact and Finite Time estimation for the *Discrete State* $\sigma(t)$.

HOSM observers for switched systems

Remarks

- the HOSM differentiator needs a finite time to converge;
- therefore the switching system requires to have a positive dwell-time ;
- nevertheless, this is not sufficient: its convergence time grows with initial differentiator error;

Conclusions

- to apply HOSM observers for switching systems, a uniform HOSM differentiator is required;
- a first order uniform robust exact differentiator [Cruz et.al. IEEE TAC 2011];
- arbitrary order uniform robust exact differentiator [Angulo et. al. Aut 2013].

Outline

Outline

11. Fault detection
    - Fault detection using multi-model approach

Fault detection using multi-model approach

The system with fault

$$\begin{aligned} \dot{x} &= f(x) + Bu + F_i(x, u), \quad i = 1, \ldots, q \\ y &= h(x) \end{aligned} \tag{33}$$

where

- $x(t) \in \mathbb{R}^n$ is the continuous state;
- $u(t) \in \mathbb{R}^m$ is a known input;
- $y(t) \in \mathbb{R}^p$ is the measured output;
- Function and matrix (f, B) are known;
- $F_i(x, u)$ are faults (known a-priori) defined by:

$$F_i(x, u) \in \mathcal{F}, \quad \textit{where} \quad \mathcal{F} = \{F_1(x, u), \ldots, F_q(x, u)\}$$

i.e. q faulty cases that change the system properties (*plant faults*) or/and the dynamical input properties of the system (*actuator faults*).

Fault detection using multi-model approach

Problem statement

Given $\{y(s), s \in [0, t]\}$ identify and isolate the q possible faults (FDI).

Main idea: use q HOSM Observers as multi-models and the equivalent injection to carry out FDI.

Proposed Solution (H. Ríos et al. IJRNC 2014)

- HOSM differentiator applied to the estimation error dynamics;
- Finite-time reconstruction of continuous state;
- Fault detection and isolation using Equivalent Injection;

HOSM Observers

- provide theoretically exact observation and unknown inputs and fault estimation under sufficient and necessary conditions of the strong observability/ detectability of states or unknown inputs or faults;

HOSM Observers

- provide theoretically exact observation and unknown inputs and fault estimation under sufficient and necessary conditions of the strong observability/ detectability of states or unknown inputs or faults;
- provide best possible approximation w.r.t. discretization step and/or bounded deterministic noises:

HOSM Observers

- provide theoretically exact observation and unknown inputs and fault estimation under sufficient and necessary conditions of the strong observability/ detectability of states or unknown inputs or faults;
- provide best possible approximation w.r.t. discretization step and/or bounded deterministic noises:
- can ensure prescribed time convergence independent from any initial conditions.

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http://verona.fi-p.unam.mx/ lfridman/