Sliding manifold design in HOSM

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Participants

Outline

Consider

Linear system

$$\dot{x} = Ax + Bu \,, \quad x \in \mathbb{R}^n \,, u \in \mathbb{R}^m \qquad (\Sigma)$$

with $\operatorname{rank} B = m$ and (A, B) controllable

Linear control design

$\exists\ u = -Kx$ such that

$$\lambda(A - BK)$$

has desired eigenvalues

Sliding mode methodology

1. Sliding manifold design:

$$\{x : \bar{\sigma} = Cx = \bar{0}\}\,, \quad C \in \mathbb{R}^{m\times n}\,, \bar{\sigma} := \text{sliding vector}$$

2. Control law design

$$u = -\rho\,\mathrm{sign}(\bar{\sigma}) \quad \text{or} \quad u = -\rho\frac{\bar{\sigma}}{||\bar{\sigma}||}$$

Since rank $B = m$, B can be divided as follows

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{(n-m)\times m}, B_2 \in \mathbb{R}^{m\times m}, \det B_2 \neq 0.$$

1. Transformation into the regular form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx, \quad T = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix}$$

We obtain

Regular form

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + u \end{aligned}$$

(A, B) controllable $\Rightarrow$

(A_{11}, A_{22}) controllable

Sliding manifold design

2. Virtual control

Since (A_{11}, A_{22}) is controllable

- Using x_2 as a virtual control of the $(n-m)$-dimensional subsystem

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2$$

Such that

$$x_2 = -Kx_1$$

assign $n-m$ desired eigenvalues

Sliding manifold

$$\bar{\sigma} = x_2 - Kx_1 = \bar{0}$$

- **How to design the sliding variable $\sigma = Cx$ in Classical SM for SISO systems?**

- Ackermann-Utkin (A-U) formula

Ackermann formula SISO systems

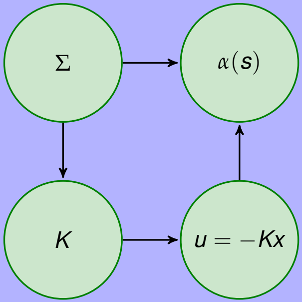
Linear system

$$\dot{x} = Ax + Bu \,, \qquad (1)$$

Control

$$u = -Kx$$

Algorithm



$$K = e_1 F^{-1} \alpha(A),$$

- $e_1 = [0\ 0 \cdots 0\ 1]$
- F : Controllability matrix
- $\alpha(s)$: desired polynomial

Such that:

$$\det(sI - A + BK) = \alpha(s)$$

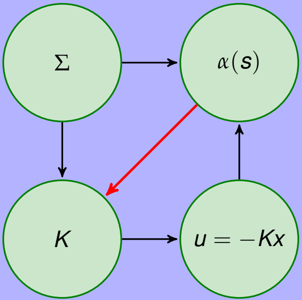
Linear system

$$\dot{x} = Ax + Bu\,, \qquad (1)$$

Control

$$u = -Kx$$

Algorithm



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Such that:

$$\det(sI - A + BK) = \alpha(s)$$

Ackermann formula

Linear perturbed system

$$\dot{x} = Ax + B(u + w) \quad (\Sigma_w)$$

Control

$$u = \mathrm{sign}(\sigma)\ , \quad \sigma = Cx$$

Clasiccal SM (1-SM)

$$\begin{tikzcd}
\Sigma_w \arrow[r] \arrow[d] & \gamma(s) \\
\sigma = Cx \arrow[r] & u \arrow[u]
\end{tikzcd}$$

$$C = e_1 F^{-1} \gamma(A)$$

- $\gamma(s)$:=desired polynomial of order $n-1$
- $\exists\ T : \forall t > T\ ,\ \sigma = 0$
- $x \to 0$ as $t \to \infty$.

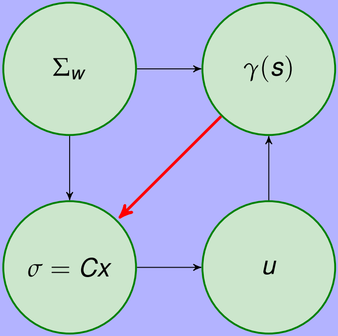
Linear perturbed system

$$\dot{x} = Ax + B(u + w) \quad (\Sigma_w)$$

Control

$$u = \text{sign}(\sigma)\ , \quad \sigma = Cx$$

Clasiccal SM (1-SM)



$$C = e_1 F^{-1} \gamma(A)$$

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- $x \to 0$ as $t \to \infty$.

Ackermann-Utkin formula

Example order $n = 3$ with A-U formula

System

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + w\end{aligned} \tag{n3}$$

Reduced order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3\end{aligned} \tag{n2}$$

A-U formula

$$C = [0\ 0\ 1] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \gamma(A)$$

$$= [1\ 3\ 2]$$

Desired polynomial

Suppose

$$\gamma(s) = (s+1)(s+2)$$

$$\begin{aligned} x_3 &= -K[x_1\ x_2]^T, \\ &= -(3x_1 + 2x_2) \end{aligned}$$

Example order $n = 3$ with Singular LQ

Singular performance index for (n3)

$$J = \frac{1}{2}\int_{t_1}^{\infty} \left(x^\top Q x \right) \mathrm{d}t \,, \quad Q = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \,,$$

- The performance index is singular with respect to the control $\Rightarrow$ *free-cost control*

Regular performance index

$$\bar{J} = \frac{1}{2}\int_{t_1}^{\infty} \left(x_{12}^\top \bar{Q} x_{12} + x_3 R x_3 \right) \mathrm{d}t \,, \quad \bar{Q} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \,, \quad R = q_3 \,,$$

which is no longer singular if $q_3 > 0$.

Solution of Regular LQR

$$x_3 = -K[x_1 \ x_2]^T \ , \quad K = R^{-1}\tilde{B}^\top P \ ,$$

where

P is the solution of the algebraic Riccati equation (ARE)

$$P\tilde{A} + \tilde{A}^\top P + \bar{Q} - P\tilde{B}R^{-1}\tilde{B}^\top P = 0$$

with $(\tilde{A}, \tilde{B})$ is the system matrix pair of the reduced order system (n2).

In briefly: How to design the manifold in FOSM in SISO systems?

$$\dot{x} = Ax + B(u + w) \qquad (\Sigma_w)$$

1. A-U formula

$$C = eF^{-1}\gamma(A) \qquad (2)$$

$\gamma(s)$ of order $n-1$

2. Singular LQ

$$J = \frac{1}{2}\int_{t_1}^{\infty} \left(x^\top Q x\right) \mathrm{d}t \,, \quad Q = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \,, \qquad (3)$$

$$Q > 0 \,.$$

Singularity of index one

Families of sliding-mode controllers

- Discontinuous
  - Twisting
  - Nested discontinuous
  - Quasi-continuous family
- Continuous
  - Super-twisting
  - Discontinuous integral
  - Continuous twisting
  - Continuous terminal

If (A, B) is controllable

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\vdots &= \vdots \\
y_n &= \sum_{i=1}^{n} \alpha_i y_i + u + w
\end{aligned}
$$

Finite time convergence

$\exists T : \forall t > T$ the set $(y_1, y_2, \dots, y_n) = 0$

HOSM

$u :=$ n-order QC

- Is it reasonable to use the higher order of sliding mode?
- What happens if the system is of relative degree $r < n$?

HOSM

Relative degree of σ	Codimension
1	$n-1$
2	$n-2$
$\vdots$	$\vdots$
r	$n-r$??!!
$r=n$	0

Formula de A-U [Ackermann and Utkin, 1998, Ackermann and Utkin, 1998]

Why HOSM?

SM order	QC control	Information needed	Accuracy
1-SM	$u = -\rho_0 \, \mathrm{sign}(\sigma)$	σ	1
2-SM	$u = -\rho_0 \frac{\left(\dot\sigma + \lvert\sigma\rvert^{1/2} \mathrm{sign}(\sigma)\right)}{\left(\lvert\dot\sigma\rvert + \lvert\sigma\rvert^{1/2}\right)}$	$\sigma, \dot\sigma$	2
3-SM	$u = -\rho_0 \frac{\ddot\sigma + 2\left(\lvert\dot\sigma\rvert + \lvert\sigma\rvert^{2/3}\right)^{-1/2} + \left(\dot\sigma + \lvert\sigma\rvert^{2/3} \mathrm{sign}(\sigma)\right)}{\lvert\ddot\sigma\rvert + 2\left(\lvert\dot\sigma\rvert + \lvert\sigma\rvert^{2/3}\right)^{1/2}}$	$\sigma, \dot\sigma, \ddot\sigma$	3

Accuracy of Derivatives Calculation

Accuracy of SMD Tracking

How to design the sliding variable $\sigma = Cx$ of relative degree $r < n$?

- Methodologies to design a sliding manifold of codimension $n - r$.

Table: Sliding manifold design in HOSM

Relative degree	Codimension	Method
r	$n-r$	**Generalized A-U formula** [1]
r	$n-r$	**Singular LQ** [2]

[1] [Hernández et al., 2014, Hernández et al., 2014]

[2] [Castillo et al., 2015, Castillo et al., 2015]

Methodologies in the reduced order system

1. Generalized A-U formula:
   - Zero dynamics, Isidori normal form [Isidori, 1996, Isidori, 1996].
   - Zero-placement problem: eigenvalue assignment
2. Singular LQ:
   - Optimal behavior
   - Singularity of the performance index [Utkin, 1992, Utkin, 1992]

Generalized Ackermann-Utkin formula

Consider again system Σ_w

$$\Sigma_w \left\{ \dot{x} = Ax + B(u + w) \right. ,$$

Assumption

1. (A, B) *controllable*
2. $\|w\| \leq \bar{w}$.

Isidori normal form

Consider system (Σ) and output $y = \sigma$. If $r > 0 \rightarrow$ there exists $\begin{bmatrix} \xi \\ \eta \end{bmatrix} = Tx$ [Isidori, 1996, Isidori, 1996]:

$$\left[\begin{array}{c} \dot{\eta} \\ \hline \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_{r-1} \\ \dot{\xi}_r \end{array}\right] = \left[\begin{array}{c} A_n\eta + B_n\xi \\ \hline \xi_2 \\ \vdots \\ \xi_r \\ CA^r x \end{array}\right] + \left[\begin{array}{c} 0 \\ \hline 0 \\ \vdots \\ 0 \\ CA^{r-1}B \end{array}\right] u \,,$$

$$\sigma = \xi_1 \,.$$

The dynamics $\dot{\eta} = A_0\eta$, $\eta \in \mathbb{R}^{n-r}$, are the *zero dynamics*

Theorem

Let $e_1 := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$ *and let* F *be the system's controllability matrix. If*

$$C = e_1 F^{-1} \gamma(A) \ , \tag{4}$$

with $\gamma(\lambda) = \lambda^{n-r} + \gamma_{n-r-1}\lambda^{n-r-1} + \cdots + \gamma_1 \lambda + \gamma_0$*, then* σ *is of relative degree* r *and the roots of* $\gamma(\lambda)$ *are the eigenvalues of the sliding-mode dynamics in the intersection of the planes* $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$.

Since $\sigma = Cx$ with C as in (4) is of relative degree r, we have

$$\sigma^{(r)} = CA^r x + CA^{r-1}B(u + w)$$

with $CA^{r-1}B \neq 0$. We can take

$$u = -\frac{v_r + CA^r x}{CA^{r-1}B},$$

where v_r is a HOSM control responsible for rejecting the perturbations.

A-U formula: Example

Consider the linearized model of a real inverted pendulum on a cart [Fantoni and Lozano, 2002, Fantoni and Lozano, 2002]

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1.56 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 46.87 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.97 \\ 0 \\ -3.98 \end{pmatrix} (u + w) \ , \tag{5}$$

- $x = [q_1, q_2, \dot{q}_1, \dot{q}_2]$
- $|w| \leq 1$
- Open-loop characteristic polynomial: $\lambda^2(\lambda + 6.85)(\lambda - 6.85)$

Sliding manifold design using the generalized A-U formula

1-SM	2-SM	3-SM
$z_i = -5,\ i = 1,2,3$	$z_i = -5,\ i = 1,2$	$z_1 = -5$
$\gamma(\lambda) = (\lambda+5)^3$	$\gamma(\lambda) = (\lambda+5)^2$	$\gamma(\lambda) = \lambda+5$
$C = [-3.2\ \ -1.9\ \ -4.5\ \ -0.7]$	$C = [-0.6\ \ -0.2\ \ -0.4\ \ 0]$	$C = [-0.1\ \ 0\ \ 0\ \ 0]$
$g(s) = \frac{(s+5)^3}{\det(sI-A)}$	$g(s) = \frac{(s+5)^2}{\det(sI-A)}$	$g(s) = \frac{(s+5)}{\det(sI-A)}$

Algorithm: A-U formula

$$
\begin{array}{c}
\boxed{r\text{-MD},\ 1 \leq r \leq 4} \\
\downarrow \\
\boxed{\lambda_i,\ 1 \leq i \leq n - r} \\
\downarrow \\
\boxed{\gamma(\lambda)} \\
\downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{r^{th}\text{-order SMC}}
\end{array}
$$

Singular LQ

Consider again system Σ_w

$$\Sigma_w \left\{ \dot{x} = Ax + B(u + w) \right. ,$$

Assumption

1. (A, B) *controllable*
2. $\|w\| \leq \bar{w}$
3. $x(0) = x_0$, $\|x_0\| \leq L$

$$J = \frac{1}{2} \int_{t_1}^{\infty} \left(x^\top Q x \right) \mathrm{d}t$$

$$z = Tx, \qquad \dot{z} = \bar{A}z + \bar{B}(u + w) \ ,$$

Controllable canonical form

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \vdots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix}, \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

Transformation of the performance index

$$\bar{Q} = \left(T^{-1}\right)^{T} Q T^{-1}$$

3. Order of the singularity

$$\bar{Q} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & \dots & 0 \\ \bar{Q}_{21} & \bar{Q}_{22} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \qquad 0}_{n-k \text{ columns}} & & \underbrace{0 \quad \dots \quad 0}_{k- \text{ zero columns}} \end{pmatrix}$$

$\bar{Q}_{11} = \bar{Q}_{11}^T \geq 0 \in \mathbb{R}^{(n-k-1)\times(n-k-1)}$

$\bar{Q}_{22} > 0, \ \bar{Q}_{22} \in \mathbb{R}$

Order of Singularity of the Performance Index: i

$i = k + 1$, such that $\bar{Q}_{22} > 0$ in $\bar{Q}$.

- System Partition

$$\begin{aligned}\bar{z}_1 &= \begin{bmatrix} z_1 & \dots & z_{n-i} \end{bmatrix}^T \\ \bar{z}_2 &= z_{n-i+1} \\ \bar{z}_3 &= \begin{bmatrix} z_{n-i+2} & \dots & z_n \end{bmatrix}^T\end{aligned}$$

Reduced Order System

$$\dot{\bar{z}}_1 = \bar{A}_{11}\bar{z}_1 + \bar{A}_{12}\bar{z}_2, \tag{6}$$

$$\left.\begin{matrix}\bar{A}_{11}\\ \bar{Q}_{11}\end{matrix}\right\} \in \mathbb{R}^{(n-i)\times(n-i)}$$

$$\bar{A}_{12} \in \mathbb{R}^{(n-i)}$$

Change of variable

Performance Index

$$\bar{J} = \frac{1}{2}\int_{t_0}^{\infty}\left(\bar{z}_1^T\bar{Q}_{11}\bar{z}_1 + 2\bar{z}_1^T\bar{Q}_{12}\bar{z}_2 + \bar{z}_2^T\bar{Q}_{22}\bar{z}_2\right)\mathrm{d}t$$

Change of variable

$$\nu = \bar{z}_2 + \left(\bar{Q}_{22}\right)^{-1}\bar{Q}_{12}^T\bar{z}_1$$

Regular conditions [Moore and Anderson, 1971, Moore and Anderson, 1971]

Performance Index

$$\bar{J} = \frac{1}{2}\int_{t_0}^{\infty} (\bar{z}_1^T (\underbrace{\bar{Q}_{11} - \bar{Q}_{12}\,(\bar{Q}_{22})^{-1}\,\bar{Q}_{12}^T}_{:=\hat{Q}})\bar{z}_1 + \nu^T\,\underbrace{\bar{Q}_{22}}_{:=\hat{R}}\,\nu)\,\mathrm{d}t$$

Reduced Order System

$$\dot{\bar{z}}_1 = (\underbrace{\bar{A}_{11} + \bar{A}_{12}(\bar{Q}_{22})^{-1}\bar{Q}_{12}^T}_{:=\hat{A}})\bar{z}_1 + \underbrace{\bar{A}_{12}}_{:=\hat{B}}\,\nu,$$

ARE

$$P\hat{A} + \hat{A}^T P + \hat{Q} - P\hat{B}\hat{R}^{-1}\hat{B}^T P = 0 \tag{7}$$

Singular LQ: Sliding Manifold Design Lemma

Lemma

If $\bar{Q}_{22} > 0$, with the *Order of Singularity i*, the pair (A, B) is controllable, and the pair $(\hat{A}, \bar{D})$ is observable, where $\bar{D}^T \bar{D} = \hat{Q}$, then the optimal vector playing as minimizing virtual control in (6) is

$$\bar{z}_2 = -K\bar{z}_1 \ , \quad K = (\hat{R})^{-1}(\hat{B}^T P + \bar{Q}_{12}^T)$$

where P is the unique positive definite solution to the Algebraic Riccati Equation (7).

Sliding variable

$$\sigma = \bar{z}_2 + K\bar{z}_1$$

Consider the same linearized model of the inverted pendulum (5).

FOSM

$$\bar{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Partition

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Consider the same linearized model of the inverted pendulum (5).

SOSM

$$\bar{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Partition

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Singular LQ: Example

Consider the same linearized model of the inverted pendulum (5).

TOSM

$$\bar{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Partition

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Conclusions

1. Old sliding manifold design:
   1. $r = 1 \rightarrow$ A-U formula.
   2. $r = n \rightarrow$ No sliding manifold design
2. New sliding manifold design ($1 < r < n$):
   1. Zero-placement: Generalized A-U formula
   2. Optimal stabilization: Singular LQ
3. Order of singularity $=$ Relative degree $=$ Order of sliding modes
4. HOSM := trade-off between accuracy and simplicity
5. A-U formula = Zero-placement problem = Choose the zero dynamics
6. Ackermann formula = Pole-placement problem = Eigenvalue assignment

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