High Order Sliding Mode Control based on Lyapunov Functions

UNAM

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The Sliding Mode Control Problem

As the relative degree is well define, the system (1)-(2) can be transformed in the form

$$\Sigma : \begin{cases} \dot{x}_i = x_{i+1} & i = 1, \ldots, \rho - 1, \\ \dot{x}_\rho = a(t,z) + b(t,z)u, \end{cases} \tag{1}$$

$$\Xi : \quad \dot{\eta} = \phi_0(t, \eta, x), \tag{2}$$

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where $x := [x_1, \ldots, x_\rho]^T = [\sigma, \ldots, \sigma^{(\rho-1)}]^T$. These functions are assume uniformly bounded $\forall z \in \mathbb{R}$ and $\forall t \geq 0$, meaning

$$0 < K_m \leq b(t,z) \leq K_M, \ |a(t,z)| \leq C, \tag{3}$$

for some known positive constants C, K_m, K_M.

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for some known positive constants C, K_m, K_M.

The vector $\dot{\eta} \in \mathbb{R}^{(n-\rho)}$ is the state of the reduce dynamic (or the zero dynamic).

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where $[a, b] \subset \mathbb{R}$ represents the interval of the points between a and b, $a \leq b$.

The Sliding Mode Control Problem

The following can be pointed out about the differential inclusion.

- For a differential inclusion $\dot{x} \in F(t,x)$, the right-side $F(t,x)$ is a *set* for every point (x,y).
- If $F(t,x)$ satisfies certain *standard conditions*, the existence of solutions can be assured.
- Σ_{DI} is not dependent of the particular properties of the original system and the differential inclusion only conserves the constants ρ, C, K_m, K_M.

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Given a suitable σ, the problem of the control is solved designing a control law (3) $u = \phi(x)$ for (7), that stabilize the point $x = 0$ in finite time. If ϕ is continuous it can no achieve the stabilization of the point $x = 0$. If the control ϕ is continuous, it most occur that $\phi(0) = 0$, so the second equation of (7), becomes

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if, $C > 0$, the point $x = 0$ in (7) can not be an equilibrium point.

The Sliding Mode Control Problem

We can conclude the following:

- Classic methods of design of *continuous* controls can not be used for our propose.
- The implementation of the control law (3) requires the estimation of the variables x in finite time.
- The movement of the set $\sigma = \cdots = \sigma^{\rho-1} = 0$, or $x = 0$, that consists on the Filippov trajectories, is called *sliding mode of order* ρ.
- If $\rho = 1$, the classic SMC, $x = \sigma \in \mathbb{R}$ is a scalar and the control law (3) reduces to the sign function $u = \phi(x) = ksign(x)$.
- For $\rho \geq 2$ the determination of the functions ϕ is a harder task. So the use of ϕ*homogeneous* functions, simplifies the problem of finding these solutions.

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Notation: For the real manifold $z \in \mathbb{R}$ and the real number $p \in \mathbb{R}$ the symbol

$$\lfloor z \rceil^{\rho} = |z|^{\rho} sign(z)$$

represents the power p of z. In particular, $\lfloor z \rceil^{0} = sign(z)$ is the sign function.

For each vector $\mathbf{x} = [x_1, \ldots, x_n]^T$ and each scalar $\varepsilon > 0$ we have the following definitions:

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The *dilatation* operator is defined as

$$\Delta_\varepsilon^r \mathbf{x} := [\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n]^T$$

where $r_i > 0$ are the *weights* of the coordinates x_i, and $\mathbf{r} = [r_i, \ldots, r_n]^T$ is defined as the *vector of weights*.

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A scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is said that is $\mathbf{r}$-*homogeneous of degree* $l \in \mathbb{R}$ if the identity

$$V(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^l V(\mathbf{x})$$

holds $\forall \varepsilon > 0$ and $\forall x \in \mathbb{R}$

Example: $f(x,y) = x^2 + y^2$

Suppose the following function $f(x,y) = x^2 + y^2$. The domain of the function is the vector $\mathbf{v} = [x,y]^T$. The dilatation vector is in the form

$$\Delta_\varepsilon^r \mathbf{v} = [\varepsilon^{r_1} x, \varepsilon^{r_2} y]^T$$

The vector of the weights is selected as $\mathbf{r} = [r_1, r_2]^T = [2,2]^T$, so deg $x = 2$ and deg $y = 2$. Having this the function is written as

$$f(\varepsilon^2 x, \varepsilon^2 y) = (\varepsilon^2 x)^2 + (\varepsilon^2 y)^2$$

$$f(\varepsilon^2 x, \varepsilon^2 y) = \varepsilon^4 x^2 + \varepsilon^4 y^2$$

$$f(\varepsilon^2 x, \varepsilon^2 y) = \varepsilon^4 (x^2 + y^2)$$

$$f(\Delta_\varepsilon^r \mathbf{v}) = \varepsilon^l f(\mathbf{v})$$

And the function is said to be $\mathbf{r}$-homogeneous of degree $l = 4$

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Example: $f(x_1, x_2) = x_1^3 - x_2^2\lceil x_1\rfloor^0 + 2|x_1|^{3/2}x_2$

As before suppose a function in the form $f(x_1, x_2) = x_1^3 - x_2^2\lceil x_1\rfloor^0 + 2|x|^{3/2}x_2$ with domain $\mathbf{x} = [x_1, x_2]^T$. The vector $\mathbf{r} = [2, 3]^T$, so we have deg $x_1 = 2$ and deg $x_2 = 3$. We write

$$f(\varepsilon^2 x_1, \varepsilon^3 x_2) = (\varepsilon^2 x_1)^3 - (\varepsilon^3 x_2)^2\lceil \varepsilon^2 x_1\rfloor^0 + 2|\varepsilon^2 x_1|^{3/2}\varepsilon^3 x_2$$

$$f(\varepsilon^2 x_1, \varepsilon^3 x_2) = \varepsilon^6 x_1^3 - \varepsilon^6 x_2^2\lceil x_1\rfloor^0 + 2\varepsilon^6|x_1|^{3/2}x_2$$

$$f(\varepsilon^2 x_1, \varepsilon^3 x_2) = \varepsilon^6(x_1^3 - x_2^2\lceil x_1\rfloor^0 + 2|x|^{3/2}x_2)$$

$$f(\Delta_\varepsilon^r x) = \varepsilon^l f(x)$$

And the function is said to be $\mathbf{r}$-homogeneous of degree $l = 6$

Simple Rules of Homogeneous arithmetic

- Homogeneity degree of zero is not defined.
- Let A and B be some homogeneous functions of $\mathbf{x} \in \mathbb{R}^n$ different from identical zero, and let λ be a real number; then.
  1) The sum of A and B is a homogeneous function only if deg $A =$ deg B.
  2) $\forall \lambda \neq 0$ deg $\lambda = 0$.
  3) deg $AB =$ deg $A +$ deg B
  4) deg $(A/B) =$ deg $A -$ deg B
  5) deg $(\lambda A) =$ deg A
  6) deg $\frac{\partial}{\partial x_i} A =$ deg $A -$ deg x_i, if $\frac{\partial}{\partial x_i} A$ is not identical zero

To verify 6) with $A = f(\Delta_\varepsilon^r \mathbf{x})$ and $f(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^{\text{deg }(f(\Delta_\varepsilon^r \mathbf{x}))} f(\mathbf{x})$ it can be seen that

$$\begin{aligned} \frac{\partial}{\partial \varepsilon^{r_i} x_i} f(\Delta_\varepsilon^r \mathbf{x}) &= \varepsilon^{-r_i} \frac{\partial}{\partial x_i} \varepsilon^{\text{deg }(f(\Delta_\varepsilon^r \mathbf{x}))} f(\mathbf{x}) \\ &= \varepsilon^{-r_i} \varepsilon^{\text{deg }(f(\Delta_\varepsilon^r \mathbf{x}))} \frac{\partial}{\partial x_i} \varepsilon^{\text{deg }(f(\Delta_\varepsilon^r \mathbf{x}))} f(\mathbf{x}) \\ &= \varepsilon^{[\text{deg }(f(\Delta_\varepsilon^r \mathbf{x})) - r_i]} \frac{\partial}{\partial x_i} f(\mathbf{x}) \end{aligned}$$

with $\text{deg }(f(\Delta_\varepsilon^r \mathbf{x})) = l$

$$\frac{\partial}{\partial \varepsilon^{r_i} x_i} f(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^{(l - r_i)} \frac{\partial}{\partial x_i} f(\mathbf{x})$$

The Sliding Mode Control Problem

- Methodology: Functions and Homogeneous Systems

A vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$, is called *homogeneous of degree* $q \in \mathbb{R}$ with the dilation

$$\Delta_\varepsilon^r \mathbf{x} := [\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n]^T$$

and the dilation
and written as deg $\mathbf{f} = q$, if all its components f_i are homogeneous and the identities

$$\text{deg } f_i = \text{deg } x_i + \text{deg } \mathbf{f} = \text{deg } (x_i) + q \quad i = 1, 2, \ldots, n \tag{5}$$

hold.

Consider the differential equation

$$\dot{x} = f(x) \tag{6}$$

and introduce a homogeneity weight of the time variable.

Let deg $t = -q$, $q \in \mathbb{R}$; then we can take an element of (9) in the form $\dot{x}_i = dx_i/dt$ (Refer to Appendix A for a better explanation). Using property 6) equation (8) is read as

$$\text{deg } \dot{x}_i = \text{deg } (x_i) - \text{deg } (t) = \text{deg } (f_i) \tag{7}$$

This means (9) is invariant with respect to the linear time-coordinate transformation

$$G_k : \ (t, \mathbf{x}) \mapsto (\varepsilon^{-q} t, \Delta_\varepsilon^r \mathbf{x}), \ \ \varepsilon > 0 \tag{8}$$

For equation (8), we use the dilatation operator with the vector field in the following way

$$\Delta_\varepsilon^{-r}\mathbf{f}(\mathbf{x}) = [\varepsilon^{-r_1} f_1(\mathbf{x}), \dots, \varepsilon^{-r_n} f_n(\mathbf{x})]^T$$

$$\Delta_\varepsilon^{-r}\mathbf{f}(\Delta_\varepsilon^r\mathbf{x}) = [\varepsilon^{-r_1} f_1(\Delta_\varepsilon^r\mathbf{x}), \dots, \varepsilon^{-r_n} f_n(\Delta_\varepsilon^r\mathbf{x})]^T$$

$$\Delta_\varepsilon^{-r}\mathbf{f}(\Delta_\varepsilon^r\mathbf{x}) = [\varepsilon^{(l_1-r_1)} f_1(\mathbf{x}), \dots, \varepsilon^{(l_n-r_n)} f_n(\mathbf{x})]^T$$

For identity (8) we know that deg $(f_i) = l_i$ and deg $(x_i) = r_i$, so we have $q = l_i - r_i$, yielding

$$\Delta_\varepsilon^{-r}\mathbf{f}(\Delta_\varepsilon^r\mathbf{x}) = \varepsilon^q\mathbf{f}(\mathbf{x})$$

The equation (8) can be written in vector form as:

$$\mathbf{f}(\Delta_\varepsilon^r\mathbf{x}) = \varepsilon^q\Delta_\varepsilon^r\mathbf{f}(\mathbf{x}) \quad \text{or} \quad \mathbf{f}(\mathbf{x}) = \varepsilon^{-q}\Delta_\varepsilon^{-r}\mathbf{f}(\Delta_\varepsilon^r\mathbf{x})$$

Example: Scalar case

In the scalar case $x \in \mathbb{R}$ any function can be consider as a vector field, but the homogeneity degrees of the function f and the vector field $\mathbf{f}$ are different. Suppose

$f(x) = x^2$ with deg $(x) = 1$

$f(\Delta_\varepsilon^r) = \varepsilon^2 x^2,\ \ $ deg $(f(x)) = 2$

deg $(\mathbf{f}(x)) =$ deg $(f(x))$ - deg $(x) = 2 - 1 = 1$

deg $(\mathbf{f}(x)) \neq$ deg $(f(x))$

Example: Scalar case

The ambiguity disappears if we speak about the homogeneity of the differential equation

$\dot{x} = x^2$

deg $(x^2) =$ deg $(x)-$ deg (t)

deg $(t) =$ -(2 - 1) $=$ -1

for equation (10) we know that the def $(\dot{x}) =$ deg $(f(x)) = 1$

Any linear time-invariant homogeneous differential equation is indeed homogeneous with homogeneity degree 0, and deg $x_i = 1$, $i = 1, 2, \ldots, n$.

Example: A differential equation

Obtain the homogeneity of the system of differential equations, with weights of x_1 and x_2 be 3 and 2 respectively.

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = -x_1^{1/3} - \lfloor x_2 \rceil^{1/2} \end{cases} \tag{9}$$

The vector of weights is $\mathbf{r} = [3, 2]^T$, and we need to check the homogeneity of each function of the vector field, so we have

$\dot{x}_1(\Delta_\varepsilon^r) = \varepsilon^2 x_2$

deg $(\dot{x}_1) = l_1 = 2$

Example: A differential equation

For the second function we have

$\dot{x}_2(\Delta_\varepsilon^r) = -(\varepsilon^3 x_1)^{1/3} - (\varepsilon^2)^{1/2}\lfloor x_2 \rceil^{1/2}$

$\dot{x}_2(\Delta_\varepsilon^r) = \varepsilon(-x_1^{1/3} - \lfloor x_2 \rceil^{1/2})$

deg $(\dot{x}_2) = l_2 = 1$

Using identity (8), it yields

deg $(\dot{\mathbf{x}}) =$ deg $(\dot{x}_i)$ - deg (x_i)

$q = 2 - 3 = -1 \quad i = 1$

$q = 1 - 2 = -1 \quad i = 2$

The differential equation (vector field) is homogeneous of the degree -1, deg $(t) = 1$.

A vector-set field $F(\mathbf{x}) \subset \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, and the differential inclusion

$$\dot{\mathbf{x}} \in F(\mathbf{x}) \tag{10}$$

are called homogeneous of the degree $q \in \mathbb{R}$ with the dilation

$$\Delta_\varepsilon^r \mathbf{x} := [\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n]^T$$

which is written as deg $(F) = q$, if (13) is invariant with respect to time-coordinate transformation

$$G_k : \ (t, \mathbf{x}) \mapsto (\varepsilon^{-q} t, \Delta_\varepsilon^r \mathbf{x}), \ \ \varepsilon > 0 \tag{11}$$

The invariance is equivalent to the identity

$$F(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^q \Delta_\varepsilon^r F(\mathbf{x}) \quad \text{or} \quad F(\mathbf{x}) = \varepsilon^{-q} \Delta_\varepsilon^{-r} F(\Delta_\varepsilon^r \mathbf{x}) \tag{12}$$

Indeed, the necessary equivalence of

$$\dot{\mathbf{x}} \in F(\mathbf{x}) \ \text{ and } \ \frac{d(\Delta_\varepsilon^r \mathbf{x})}{d(\varepsilon^{-q} t)} \in F(\Delta_\varepsilon^r \mathbf{x})$$

implies (15).

Example: A differential inequality

Let once more the weights of x_1 and x_2 be 3 and 2, respectively. Then the differential inequality

$$|\dot{x}_1| + \dot{x}_2^{4/3} \leq x_1^{4/3} + x_2^2$$

corresponds to the homogeneous differential inclusion $(\dot{\mathbf{x}} \in F(\mathbf{x}))$

$$(\dot{x}_1, \dot{x}_2) \in \left\{ (z_1, z_2) \mid |z_1| + z_2^{4/3} \leq x^{4/3} + x_2^2,\ z_1, z_2 \in \mathbb{R}^2 \right\}$$

$\mathbf{r} = [3, 2]^T$, $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{z} = [z_1, z_2]^T$, $f(\mathbf{x}) = x_1^{4/3} + x_2^2$, and $f(\mathbf{z}) = |z_1| + z_2^{4/3}$

Example: A differential inequality

$\Rightarrow$

$$F(\mathbf{x}) := \left\{(z_1, z_2) \mid f(\mathbf{z}) \leq f(\mathbf{x}), \ \ \mathbf{z} \in \mathbb{R}^2\right\}$$

$\Rightarrow$

$$f(\Delta_\varepsilon^r \mathbf{x}) = (\varepsilon^3 x_1)^{4/3} + (\varepsilon^2 x_2)^2$$

$$f(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^4 (x_1^{4/3} + x_2^2), \qquad f(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^4 f(\mathbf{x})$$

$\Rightarrow$

$$F(\Delta_\varepsilon^r \mathbf{x}) := \left\{(z_1, z_2) \mid f(\mathbf{z}) \leq f(\Delta_\varepsilon^r \mathbf{x}), \ \ \mathbf{z} \in \mathbb{R}^2\right\}$$

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Example: A differential inequality

$\Rightarrow$

$F(\Delta_\varepsilon^r \mathbf{x}) = \varepsilon^q \Delta_\varepsilon^r F(\mathbf{x})$

$\Rightarrow$

$\text{deg}\ (F(\Delta_\varepsilon^r \mathbf{x})) = \text{deg}\ (\varepsilon^q \Delta_\varepsilon^r F(\mathbf{x})) + \text{deg}\ (\Delta_\varepsilon^r F(\mathbf{x}))$

$\text{deg}\ (\varepsilon^4 f(\mathbf{x})) = \text{deg}\ (\varepsilon^q \Delta_\varepsilon^r F(\mathbf{x})) + \text{deg}\ (\varepsilon^4 f(\mathbf{x}))$

$4 = q + 3$

$q = 1$

So the vector-set field $\dot{\mathbf{x}}$ is homogeneous of degree 1

Example: A differential inequality

If we select the degree of t as deg $(t) = -q$ the following equivalence holds

$\frac{d(\Delta_\varepsilon^r \mathbf{x})}{d(\varepsilon^{-q} t)} \in F(\Delta_\varepsilon^r \mathbf{x})$

and the differential inclusion is seen as a vector-set field, which means that we can use the equation (8). $\Rightarrow$

deg $(\dot{x_i}) =$ deg $(x_i) +$ deg $(\dot{\mathbf{x}})$

deg $(\dot{x_1}) = 3 + 1 = 4;$ deg $(\dot{x_2}) = 2 + 1 = 3$

And the degrees of the function of $\dot{x}_1$ and $\dot{x}_2$ are

$f_1(\varepsilon^4 \dot{x}_1) = \varepsilon^4 |\dot{x}_1|;$ $f_2(\varepsilon^3 \dot{x}_2) = \varepsilon^4 \dot{x}_2^{4/3}$

then deg $(|\dot{x}_1|) =$ deg $(\dot{x}_2^{4/3}) =$ deg $(x_1^{4/3}) =$ deg $(x_2^2) = 4$

The homogeneous systems described by DE or DI, have especial and important properties:

- Local attractivity of the equilibrium point in $x = 0$ is equivalent to global and asymptotic stability.
- If the system has negative homogeneity degree, attractivity is equivalent to stability in finite time.
- Internal stability of a system with inputs is equivalent to the external stability.
- Asymptotic stabilization of homogeneous systems (continuous or discontinuous) can be studied by *homogeneous* Lyapunov functions.

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A. Levant has developed a family of homogeneous controllers. In these works the problem of stabilization is solved using a state feedback control law (3), bounded and $\mathbf{r}$-homogeneous of degree 0, meaning, $\forall\varepsilon > 0$, $\forall x \in \mathbb{R}^{\rho}$

$$u = \phi(x_1, x_2, \cdots, x_\rho) = \phi(\varepsilon^{r_1}, \varepsilon^{r_2}, \cdots, \varepsilon^{r_\rho} x_\rho)$$

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with $\mathbf{r} = (\rho, \rho - 1, \ldots, 1)$, which makes the origin $x = 0$ being *robust enough* stable in finite time for the system Σ_{DI} (7). The above is also true for observers.

Some of the advantages of SMC are the followings.

- *Exact* compensation of uncertainties and *matching* perturbations, meaning, modeled by (6).
- The system dynamic of the closed loop, after finite time, is given by (5) with $x = 0$, and its of reduced order, because evolves with a dimensional manifold $n - \rho$. For the FOSMC this dimension is 1 less than the state space of the system, but using the SOSMC the size of the manifold can be reduced in any number of degrees, up to the order of the system.
- The converging of the sliding manifold is obtained in finite time. If $\rho = n$, the origin of the system is reaching in finite time.

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However, some disadvantages are notable.

- The use of a *discontinuous* control law, for ensuring that the trajectories will be kept in the sliding surface, causes *chattering*, which is a switching control signal of high frequency. The chattering has undesirable effects in the system, it reduces the life time of the actuators, it excites the dynamic of the high frequency of the plant, usually non-modeled, and a high consume of energy.
- The stabilization of the origin occur asymptotically. Only the case of the SOSMC is performed with relative degree equal to the order of the system, meaning, $\rho = n$, the stabilization is reached in finite time.

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Nested Sliding Controllers

In particular, the following families of discontinuous and quasi-continuous can be obtained

$$u_D = -k_\rho \phi_D(\mathbf{x}) = -k_\rho \lceil \sigma_\rho(\mathbf{x}) \rfloor^0 \tag{13}$$

$$u_C = -k_\rho \phi_C(\mathbf{x}) = -k_\rho \frac{\sigma_\rho(\mathbf{x})}{M(\mathbf{x})} \tag{14}$$

where $M(\mathbf{x})$ is any continuous function, $\mathbf{r}$-homogeneous and positive definite of degree α_ρ

Discontinuous controllers

Nested controllers: Some of the α are different.

$$u_{2D} = -k_2 \left\lceil \lceil x_2 \rfloor^{\alpha_2} + k_1^{\alpha_2} \lceil x_1 \rfloor^{\frac{\alpha_2}{2}} \right\rfloor^0$$

$$u_{3D} = -k_3 \left\lceil \lceil x_3 \rfloor^{\alpha_3} + k_2^{\alpha_3} \left\lceil \lceil x_2 \rfloor^{\frac{\alpha_2}{2}} + k_1^{\frac{\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{\alpha_3}{3}} \right\rfloor^{\frac{\alpha_3}{\alpha_2}} \right\rfloor^0 \tag{15}$$

Switching polynomial controllers: when $\alpha_\rho = \alpha_{\rho-1} = \cdots = \alpha_1 = \alpha \geq \rho$

$$
\begin{aligned}
u_{2R} &= -k_2 \left\lceil \lceil x_2 \rfloor^{\alpha} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{2}} \right\rfloor^{0} \\
u_{3R} &= -k_3 \left\lceil \lceil x_3 \rfloor^{\alpha} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{\alpha}{2}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{3}} \right\rfloor^{0}
\end{aligned}
\tag{16}
$$

where, if $\rho = 2$, $\bar{k}_1 = k_1^{\alpha}$; if $\rho = 3$, $\bar{k}_1 = k_2^{\alpha} k_1^{\frac{\alpha}{2}}$, $\bar{k}_2 = k_2^{\alpha}$; if $\rho > 3$, $\bar{k}_i = \prod_{j=i}^{\rho-1} k^{\frac{\alpha}{\rho-j}} - j$, for $i = 1, \cdots, \rho$.

Quasi-continuous controllers

Nested controllers: Some of the α_i are different. The parameters $\beta_i > 0$ are arbitrary

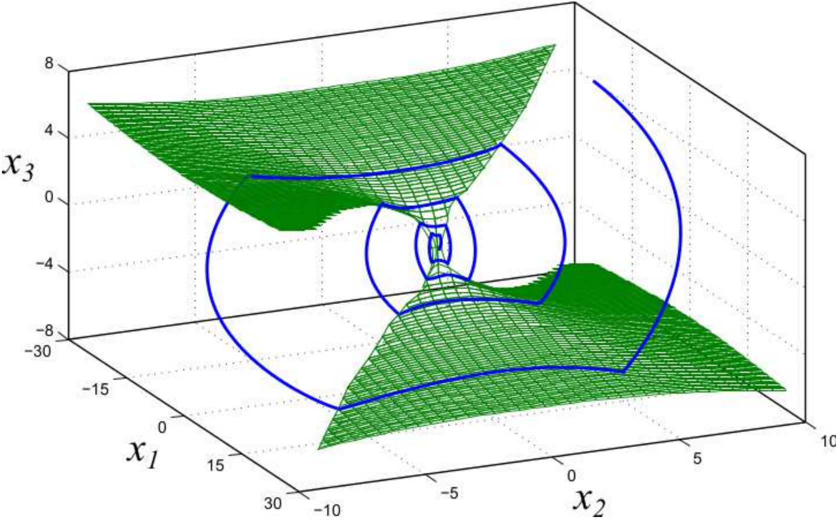
$$
\begin{aligned}
u_{2Q} &= -k_2 \frac{\lceil x_2 \rfloor^{\alpha_2} + k_1^{\alpha_2} \lceil x_1 \rfloor^{\frac{\alpha_2}{2}}}{|x_2|^{\alpha_2} + \beta_1 |x_1|^{\frac{\alpha_2}{2}}} \\
u_{3Q} &= -k_3 \frac{\lceil x_3 \rfloor^{\alpha_3} + k_2^{\alpha_3} \left\lceil \lceil x_2 \rfloor^{\frac{\alpha_2}{2}} + k_1^{\frac{\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{\alpha_3}{3}} \right\rfloor^{\frac{\alpha_3}{\alpha_2}}}{|x_3|^{\alpha_3} + \beta_2 |x_2|^{\frac{\alpha_3}{2}} + \beta_1 |x_1|^{\frac{\alpha_3}{3}}}
\end{aligned}
\tag{17}
$$

Quasi-continuous controllers

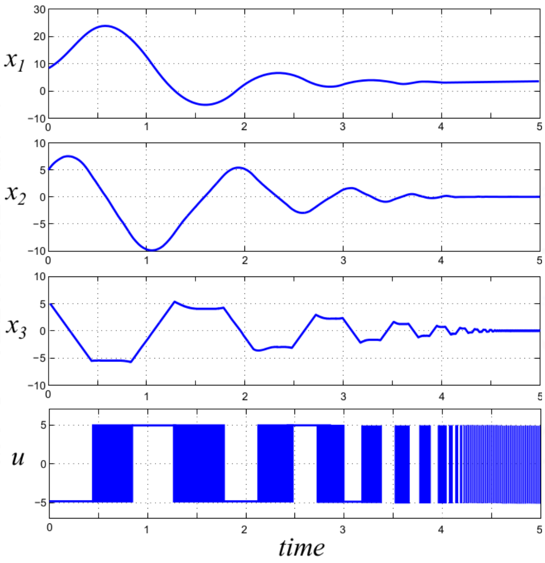
Switching Polynomial Controllers: when $\alpha_\rho = \alpha_{\rho-1} = \cdots = \alpha_1 = \alpha \geq \rho$

$$
\begin{aligned}
u_{2QR} &= -k_2 \frac{\lceil x_2 \rfloor^{\alpha} + k_1 \lceil x_1 \rfloor^{\frac{\alpha}{2}}}{|x_2|^{\alpha} + \beta_1 |x_1|^{\frac{\alpha}{2}}} \\
u_{3QR} &= -k_3 \frac{\lceil x_3 \rfloor^{\alpha} + \bar{k}_2 \lceil x_2 \lfloor^{\frac{\alpha}{2}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{2}}}{|x_3|^{\alpha} + \beta_2 |x_2|^{\frac{\alpha}{2}} + \beta_1 |x_1|^{\frac{\alpha}{3}}}
\end{aligned}
\tag{18}
$$

Nested Sliding Controllers



Nested Sliding Controllers



Lyapunov Design of High Order Sliding Mode Controllers

Given a relative degree $\rho \geq 2$, we assign the wights of the homogeneity $r_i = \rho - i + 1$ to the variables x_i, obtaining the vector $\mathbf{r} = (\rho, \rho - 1, \ldots, 1)$. Define also an arbitrary sequence not decreasing of real positive numbers α_i such that $\rho \leq \alpha_i \ldots \leq \alpha_{\rho_i} \leq \alpha_\rho$. Additionally, define in a recursive way, for $i = 2, \cdots, \rho$, the functions $\mathbf{r}$-homogeneous and $\mathcal{I}^1$

$$\sigma(\bar{x}_1) = \lfloor x_1 \rceil^{\frac{\alpha_1}{\rho}}, \cdots, \sigma_i(\bar{x}_i) = \lfloor x_i \rceil^{\frac{\alpha_i}{\rho - i + 1}} + k_{i-1}^{\frac{\alpha_i}{\rho - i + 1}} \lfloor \sigma_{i-1}(\bar{x}_{i-1}) \rceil^{\frac{\alpha_i}{\alpha_{i-1}}}$$

with certain constants $k_i > 0$ for designing, where $\bar{\mathbf{x}}_i = [x_1, \cdots, x_i]^T$.

For any constant $m \geq max_{(1\leq i\leq \rho)}\{\rho + 1 + 1\alpha_{i_1} - i\}$, define in a recursive way, for $i = 2, \cdots, \rho$, the functions $\mathbf{r}$-homogeneous and $\mathcal{I}^1$

$$V_1(x_1) = \frac{\rho}{m}^{\frac{m}{\rho}}, \ldots, V_i(\bar{\mathbf{x}}) = \gamma_{i-1} V_{i-1}(\bar{\mathbf{x}}_i) + W(\bar{\mathbf{x}}_i) \tag{19}$$

$$W_i(\bar{\mathbf{x}}_i) = \frac{r_i}{m}|x_i|^{\frac{m}{r_i}} - \lfloor \nu_{i_1}(\bar{\mathbf{x}}_{i-1}) \rceil^{\frac{m-r_i}{r_i}} x_i + \left(1 - \frac{r_i}{m}\right) |\nu_{i-1}(\bar{\mathbf{x}}_{i-1})|^{\frac{m}{r_i}}, \tag{20}$$

$$\nu(x_1) = -k_1 \lfloor \sigma(x_1) \rceil^{\frac{r_2}{\alpha_1}}, \ldots, \nu_i(\bar{\mathbf{x}}_i) = -k_i \lfloor \sigma_i(\bar{\mathbf{x}}_i) \rceil^{\frac{r_i+1}{\alpha_i}} \tag{21}$$

with arbitrary constants $\gamma_i > 0$. $\sigma(\bar{\mathbf{x}}_i), \nu_i(\bar{\mathbf{x}}_i)$ and $V_i(\bar{\mathbf{x}}_i)$ are $\mathbf{r}$-homogeneous with degree α_i, r_{i+1} and m, respectively.

$V_c(\bar{\mathbf{x}}_i) = V_\rho(\bar{\mathbf{x}}_i)$ is Lyapunov smooth control function and $\mathbf{r}$-homogeneous of the uncertain system (7) ,for the proper gains k_i.

Lyapunov Design of High Order Sliding Mode Controllers

Theorem (1)

For any $\rho \geq 2$, each of the discontinuous and quasi-continuous controller in (16)-(17)*, with arbitrary parameters $\rho \leq \alpha_1 \cdots \leq \alpha_\rho$, is $\mathbf{r}$-homogeneous, and for bigger enough the sliding mode of order ρ is established in $\mathbf{x} = \mathbf{0}$ in finite time for the uncertain system* (7) *if the gains $k_1, \cdots, k_{\rho-1}$ and $\beta_1, \cdots, \beta_{\rho-1}$ are selected appropriately.*

Theorem (2)

Consider the uncertain plant (7)*, whit any of the state feedback controllers* (16)-(17)*, and suppose the hypothesis of Theorem 1 are satisfy. Assume that the control is achieve in a discrete interval* τ*. In that case the state* $\mathbf{x}$ *reaches after a finite time a neighborhood of the origin, named "real sliding mode", and characterize by*

$$|x_1(t)| \leq \delta_1 \tau^{\rho}, \cdots, |x_i(t)| \leq \delta_i \tau^{\rho-(i-1)}, \; |x_\rho(t)| \leq \delta_\rho \tau$$

and stays in there for all future time.

Preposition

The controllers (16)-(17), in closed loop with (7), make the state trajectory starts at $\mathbf{x}_0 = \mathbf{x}(0) \in \mathbb{R}$ reaches $\mathbf{x} = \mathbf{0}$ in a finite time lower than

$$T(\mathbf{x}_0) \leq m\eta_\rho V^{\frac{1}{m}}\rho(\mathbf{x}_0) \tag{22}$$

where η_ρ is a function fo the gains $k_1, \cdots, k_\rho, K_m$ and C.

Theorem (3)

Suppose that in (7) $C = \bar{C} + \Theta(t, \mathbf{z})$, where the function $\Theta(t, \mathbf{z}) \geq 0$ is known. Then, the controllers (16)-(17), with k_ρ can be replaced with the gain variable $(K(T, \mathbf{z}) + k_\rho)$, stabilize in the origin $\mathbf{x} = \mathbf{0}$ in finite time, if the gain k_ρ is sufficient big and $K_m K(t, \mathbf{z}) \geq \Theta(t, \mathbf{z})$.

Gain scaling: If the vector of gains $\mathbf{K} = (k_1, \cdots, k_\rho)$ stabilize, then $\forall L \geq 1$ the scale gain vector $\mathbf{k}_L = (L^{\frac{1}{\rho}} k_1, \cdots, L^{\frac{1}{\rho+1-i}} k_i, \cdots, L k_\rho)$ also can stabilize.

For the switching polynomial controllers

$\bar{k}_i = \prod_{j=1}^{\rho-1} k_j^{\frac{\rho+\alpha}{\rho-j}}$, for $i = 1, \cdots, \rho$, are scale as

$\bar{k}_i \to L^{\frac{(\rho-i)\alpha}{\rho-i+1}} \bar{k}_i$.

The gains k_i can be calculated in a recursive way $k_1 > 0$

$$k_{i+1} > G_{i+1}(k_1, \cdots, k_i), \cdots, k_\rho > \frac{1}{K_m}\left(G_\rho(k_1, \cdots, k_{\rho-1}) + C\right),$$

where the functions G_i are obtained with the Lyapunos function $V_c(\mathbf{x})$ and depend of ρ, γ_i and α_i. The gains can also be obtained in function of k_1 as

$$k_1 > 0, \cdots, k_i = \mu_i k_1^{\overline{\frac{\rho}{\rho-(i-1)}}}, \; k_\rho > \frac{1}{K_m}(\mu_\rho k_1^\rho + C) \tag{23}$$

for some positive constants μ_i which depend of ρ, γ_i and α_i.

Robust Exact Differentiator

Suppose a signal $f(t) t \in [0, \infty)$ that can be discomposed in two signals $f(t) = f_0(t) + \mathbf{v}(t)$

where

$f_0(t)$ is the signal for differentiation belonging to a class $\mathcal{L}_L^n$, $(n-1)$ differentiable and $|f_0^{(n)}(t)| \leq L$

$\mathbf{v}(t)$ is a noise signal, with $|\mathbf{v}(t)| \leq \varepsilon$, $\forall t \geq 0$.

For the estimation of $f_0(t)$ the following differentiator is proposed

$$\begin{aligned} \dot{x}_i &= -\lambda_i L^{\frac{i}{n}} \lceil x_1 - f \rfloor^{\frac{n-1}{n}} + x_{i+1}, \qquad i = 1, \cdots, n-1 \\ \dot{x}_n &= -\lambda_n L \lceil x_1 - f \rfloor^0. \end{aligned} \tag{24}$$

Robust Exact Differentiator

The differentation error is define as $e_i \triangleq x_i - f_0^{(i-1)}$ and with a scaling

$$z_1 = \frac{e_1}{L}, \cdots, z_i = \frac{e_i}{\lambda_{i-1} L}, \ i = 1, \cdots, n,$$

The error dynamic is

$$\begin{aligned} \dot{z}_i &= -\tilde{\lambda}_i \left(\lceil z_1 + \tilde{\mathbf{v}}(t) \rfloor^{\frac{n-i}{n}} - z_{i+1} \right), \ i = 1, \cdots, n-1 \\ \dot{z}_n &\in -\tilde{\lambda}_n \left(\lceil z_1 + +\mathbf{v}\tilde{(t)} \rfloor^0 + \frac{1}{\lambda_n} [-1, 1] \right) \end{aligned} \quad (25)$$

where $\tilde{\mathbf{v}}(t) = \frac{v(t)}{L} \in [-1, 1] \frac{\varepsilon}{L} \Rightarrow$

$$\lambda_0 = 1, \ \tilde{\lambda}_i = \frac{\lambda_i}{\lambda_{i-1}} \ i = 1, \cdots, n$$

Robust Exact Differentiator

A Lyapunov function for (28) is $V(\mathbf{z}) = V_1(\mathbf{z})$ recursive define (backwards) by

$$V_n(z_n) = \frac{\beta_n}{p_n}|z_n|^{p_n}, \cdots, V_i(\underline{\mathsf{z}}_i) = \beta_i Z_i(z_i, z_{i+1}) + V_{i+1}^{\frac{p_i}{p_i+1}}(\underline{\mathsf{z}}_{i+1}) \qquad (26)$$

where $\underline{\mathsf{z}}_i \triangleq (z_i, \cdots, z_n)$, $\beta_i > 0$, $i = 1, \cdots, n$, are arbitrary and

$$Z_i(z_i, z_{i+1}) = \frac{r_i}{p_i}|z_i|^{\frac{p_i}{r_i}} - z_i\lceil z_{i+1}\rfloor^{\frac{p_i - r_i}{n-i}} + (\frac{p_i - r_i}{p_i})|z_{i+1}|^{\frac{p_i}{n-1}} \qquad (27)$$

$V(\mathbf{z})$ is **r**-homogeneous of degree p_1, continually differentiable, and positive definite.

Robust Exact Differentiator

Theorem

Robust Exact Differentiator Under the hypothesis made about the signal $f(t)$*, in absence of noise* $(\mathbf{v}(t) \equiv 0)$ *and fo some gains* $\lambda_i > 0$, $i = 1, \cdots, n$ *suitable selected, the origin* $\mathbf{z} = \mathbf{0}$ *of the dynamic of the error differentiator* (28) *is stable in finite time.* $V(\mathbf{z})$ *in* (29) *is a smooth Lyapunov function* $\forall p \geq 2n - 1$ *and* $\beta_i > 0$ *and satisfies the differential inequality*

$$\dot{V} \leq -\kappa V(\mathbf{z})^{\frac{p-1}{p}}, \tag{28}$$

for some constant $\kappa > 0$. *The time of convergence can be estimated by*

$$T(\mathbf{z}_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(\mathbf{z}_0). \tag{29}$$

Robust Exact Differentiator

Preposition

Given $n > 2$ select p_i such that $p_i > r_i + r_{i+1} = 2(n-i)+1$ and $p = p_1 > r_1 + r_2 = 2n - 1 > 1$:

1) The sequence of gains $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n$ can be calculated backwards as follows: (a) Select $\lambda_n > 1$, and $\tilde{\lambda}_n > 0$. (b) For $i = n$, $n-1, \cdots, 2$ choose $\tilde{\lambda}_{i-1}$ such that

$$\tilde{\lambda}_{i-1} > \omega_{i-1}(\tilde{\lambda}_i, \cdots, \tilde{\lambda}_n),$$

   ω_{i-1} is a function obtained of $V(\mathbf{z})$ and depends only of $(\tilde{\lambda}_i, \cdots, \tilde{\lambda}_n)$, p_i and $\beta_{i-1}, \cdots, \beta_n$.

2) For any $j = 1, \cdots, n-1$ and the gains $(\tilde{\lambda}_j, \cdots, \tilde{\lambda}_n)$ are suitable for the differentatior of order $n - j$

Robust Exact Differentiator

Preposition

For a noise signal uniformly bounded ($|\mathbf{v}(t)| \leq \varepsilon$) and the stabilizers gains of the differentiator, the error of differentiation
$x_i(t) - f_0^{i-1}(t),\ i = 1, \cdots, n$, satisfy the following inequalities after finite time

$$|x_i()t - f_0^{i-1}(t)| \leq \wp_i L^{\frac{i-1}{n}} |\varepsilon|^{\frac{n-i+1}{n}} \tag{30}$$

$\wp_i$ depends of the gains λ_i and of $\wp > 1$

Robust Exact Differentiator

Output feedback sliding mode control. Using the Lypuanov functions developed, is possible to prove the global convergence of the system in the output feedback closed loop.

$$\begin{aligned} \dot{x}_i &= x_{i+1} \\ \dot{x}_\rho &\in [-C, C] - k_\rho [K_m, K_M] \phi(\hat{\mathbf{x}}), \\ \dot{\hat{x}}_i &= -k_i L^{\frac{i}{p}} \lceil \hat{x}_1 - y \rfloor^{\frac{\rho - i}{\rho}} + \hat{x}_{i+1}, \\ \dot{\hat{x}}_\rho &= -k_\rho L \rceil \hat{x}_1 - y \rfloor^0 \end{aligned} \tag{31}$$

for any controller $\phi(\mathbf{x})$ discontinuous (16) o quasi-continuous (17).

Appendix A. Degree of a differential equation

As we know the dilatation vector is in the form $\Delta_\varepsilon^r \mathbf{x} = [\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n]^T$, so a differential vector has the form

$$\frac{d(\Delta_\varepsilon^r \mathbf{x})}{d(\varepsilon^{r_t} t)} = \left[\frac{d(\varepsilon^{r_1} x_1)}{d(\varepsilon^{r_t} t)}, \dots, \frac{d(\varepsilon^{r_n} x_n)}{d(\varepsilon^{r_t} t)} \right] \tag{A1}$$

If we take a component of (A1) and we use the property 6) it yields

$$\frac{d}{d\varepsilon^{r_t} t}(\varepsilon^{r_i} x_i) = \varepsilon^{(r_i - r_t)} \frac{d}{dt} x_i(t)$$

so we get

$$\text{deg } (\dot{x}_i) = \text{deg } (x_i) - \text{deg } (t)$$

If we select deg (t) as $-q$, that is the degree of the vector field in this case the differential equation vector, yields

$$\text{deg } (\dot{x}_i) = \text{deg } (x_i) + q = \text{deg } (f_i)$$

In conclusion, if we select the deg (t) as the negative degree of the vector field, a differential equation is seen as a function (is invariant of time), equation (8) holds and

$$\frac{d(\Delta_\varepsilon^r \mathbf{x})}{d(\varepsilon^{-q} t)} = \mathbf{f}(\Delta_\varepsilon^r \mathbf{x})$$