What is an LMI?

$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0$ strict LMI

$F_i = F_i^T \in \mathbb{R}^{n\times n} \quad i \in \{0,\ldots,m\}$ given symmetric matrices

$x_i \qquad i \in \{1,\ldots,m\}$ decision variables

$S = \{x \in \mathbb{R}^m, F(x) > 0\}$ **convex** feasible set (fundamental property)

Immediate: $\{x, y \in S\} \Rightarrow \{z \triangleq \alpha x + (1-\alpha) y \in S, \alpha \in [0,1]\}$

F is an **affine function** of the decision variables

Also non strict LMI: $F(x) \geq 0$

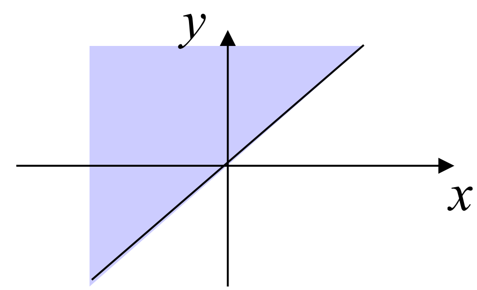
Recap: $F(x) > 0$ positive definite = strictly positive eigenvalues

$F(x) \geq 0$ positive semidefinite = non negative eigenvalues

Simple LMI in a plane, LMI sets

$y > x \qquad -x + y > 0$

$f_0 = 0,\ f_1 = -1,\ f_2 = 1$



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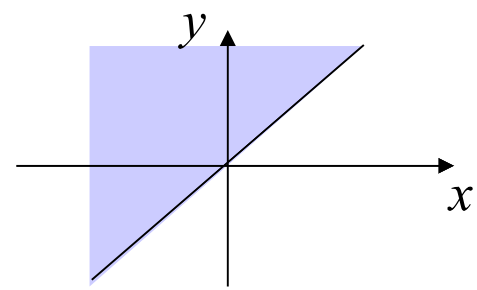
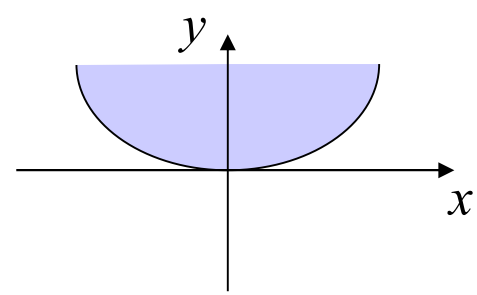


$y > x^2$ $\qquad \begin{bmatrix} 1 & x \\ x & y \end{bmatrix} > 0$

$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



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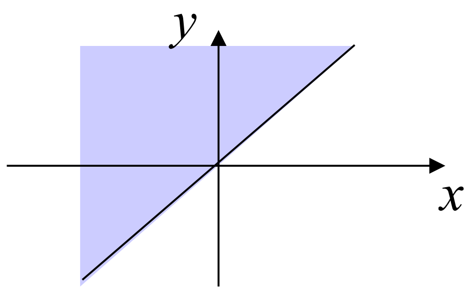

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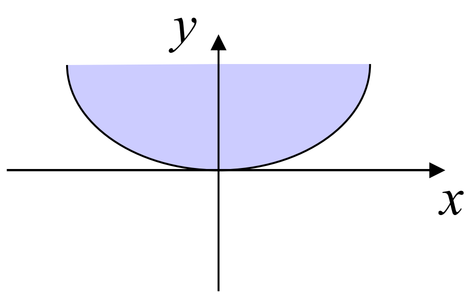
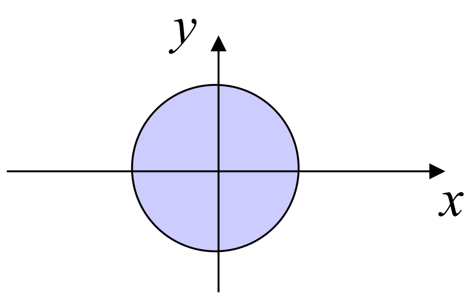


$x^2 + y^2 < 1 \quad \begin{bmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} > 0$

$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

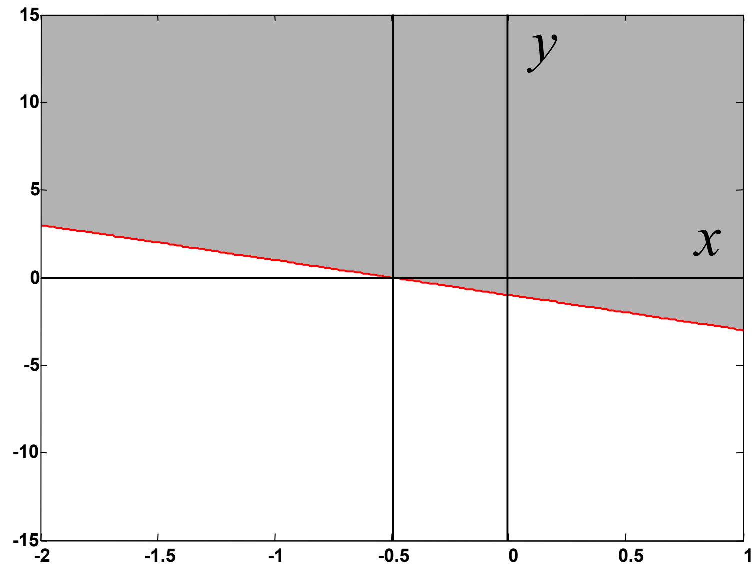


Simple LMI in a plane, LMI Sets

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} > 0 \quad \Leftrightarrow \quad F(x) = \begin{bmatrix} 1+2x+y & -x \\ -x & 1+2x \end{bmatrix} > 0$$

First principal minor:

$$1+2x+y>0$$



Simple LMI in a plane, LMI sets

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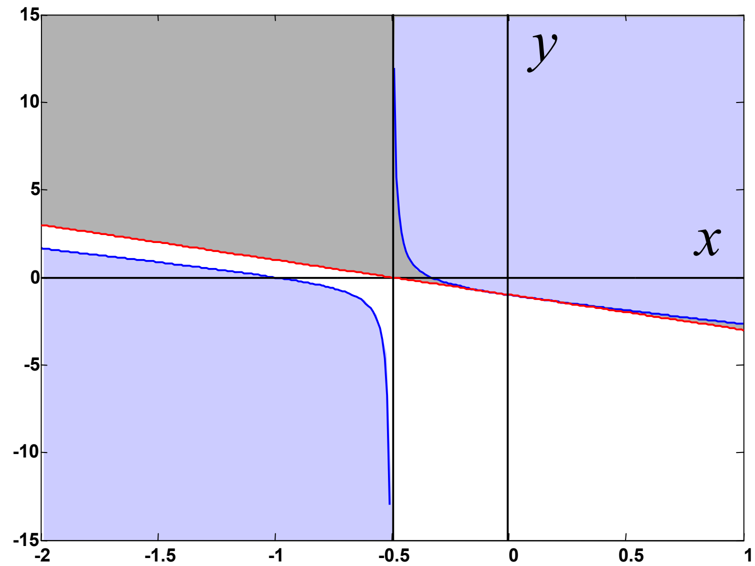
First principal minor:

$$1 + 2x + y > 0$$

Second principal minor:

$$(1+2x)y + (1+x)(1+3x) > 0$$



Simple LMI in a plane, LMI sets

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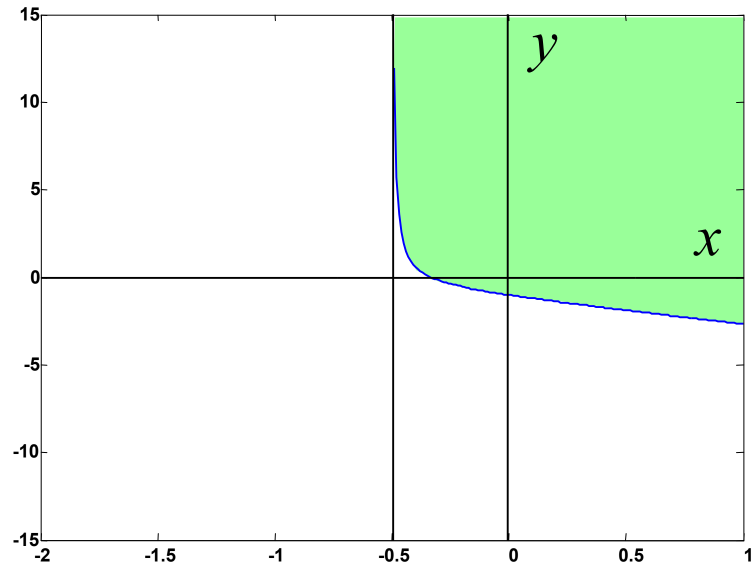

Simple LMI in a plane, LMI sets

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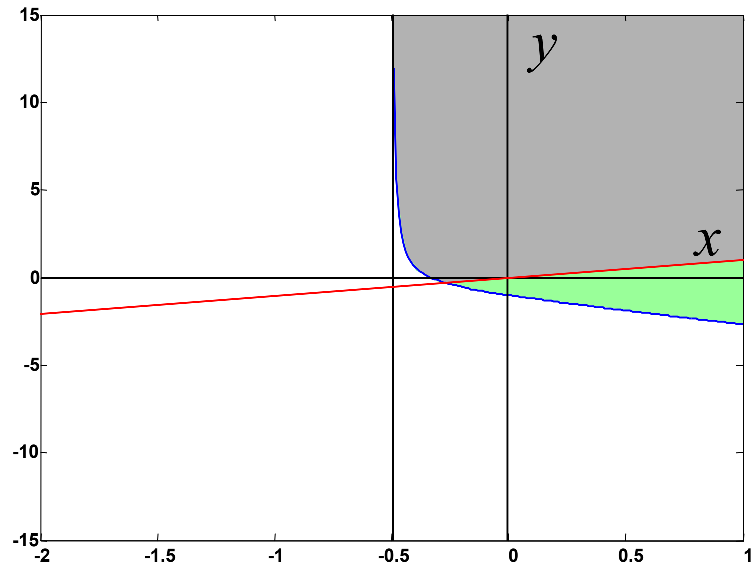

Intersection with LMI sets is also a LMI set: $F(x)>0,\ y>x$

$$\begin{bmatrix} F(x) & 0 \\ 0 & y-x \end{bmatrix} > 0$$

What is a LMI?

Remarks:

- only very simple case can be treated analytically
- need for numerical algorithms
- of course, $F(x) < 0, F(x) \leq 0$ are defined similarly

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Remarks:

- only very simple case can be treated analytically
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The LMI **feasibility** problem:

Test whether there exists $x_1, \ldots x_m$ such that $F(x) < 0$

$F(x) < 0$ is feasible if and only if $\min_x \lambda_{\max}(F(x)) < 0$

therefore involves minimizing the function: $f : x \to \lambda_{\max}(F(x)) < 0$

Possible because this function is convex!

Efficient algorithms (ellipsoid, interior point)

LMI feasibility problem

The LMI **feasibility** problem:

Test whether there exists $x_1, \ldots x_m$ such that $F(x) < 0$

Introduce $\gamma \in \mathbb{R}$ and solve the optimization LMI:

$$\min_{\gamma \in \mathbb{R},\, x \in \mathbb{R}^m} \gamma$$
$$\text{s.t.} \quad F(x) - \gamma I \leq 0$$

Remarks:

let (γ^*, x^*) be the optimal solution of the LMI problem:

a) if $\gamma^* > 0$ no feasible solution to the original problem

b) if $\gamma^* = 0$, then x^* is a feasible solution to the original problem $(F(x) \leq 0)$

c) if $\gamma^* < 0$, then x^* is a **strictly feasible** solution to the original problem $(F(x) < 0)$

LMI optimization problem

The LMI **optimization** problem:

Minimize $c_1 x_1 + \ldots + c_m x_m$ over all $x_1, \ldots x_m$ that satisfy $F(x) < 0$

Various formulations: known also as semi definite programming (**SDP**)
In fact:

LMI optimization = *generalization of linear programming (LP) to* ***cone*** *of positive semidefinite matrices* = semidefinite programming (**SDP**)

Remark: multiple LMI can be put in one

$$
\begin{aligned} &F^{(1)}(x) > 0, \\ &\ldots, \\ &F^{(p)}(x) > 0 \end{aligned} \quad \Leftrightarrow \quad \text{one LMI} \quad \begin{bmatrix} F^{(1)}(x) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & F^{(p)}(x) \end{bmatrix} > 0
$$

Solvers normally require to generate the $F^{(i)}(x)$

Standard problems

$A(x), B(x), C(x)$ supposed to be affine in x

For MATLAB LMI Toolbox:

Feasibility problem: find $x \in \mathbb{R}^N$ $\quad A(x) < B(x)$

corresponding solver *feasp*

Optimization problem: $\min c^T x,\ x \in \mathbb{R}^N$ $\quad A(x) < B(x)$

corresponding solver *mincx*

Generalized eigenvalue
minimization problem: $\min \lambda,\ x \in \mathbb{R}^N$ $\quad \begin{cases} C(x) < 0 \\ 0 < B(x) \\ A(x) < \lambda B(x) \end{cases}$

corresponding solver *gevp*

Standard problems

Semi definite $B(x) \geq 0$ in

Generalized eigenvalue

minimization problem: min λ, $x \in \mathbb{R}^N$ $\quad \begin{cases} C(x) < 0 \\ 0 \leq B(x) \\ A(x) < \lambda B(x) \end{cases}$

Technically $B(x) > 0$ is required. Nevertheless writing:

$$B(x) = \begin{bmatrix} B_1(x) & 0 \\ 0 & 0 \end{bmatrix}, B_1(x) > 0$$

and replace: $A(x) < \lambda B(x),\ B(x) > 0$

with: $A(x) < \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}, Z < \lambda B_1(x),\ B_1(x) > 0$

Matrices as variables

Matrices as variables:

Example Lyapunov stability for linear models: $P > 0, \quad A^T P + PA < 0$

Basis of symmetric matrices: $P_1, \ldots, P_m \in \mathbb{R}^{n \times n} \quad m = \frac{1}{2} n(n+1)$

Thus: $P = x_1 P_1 + \ldots + x_m P_m$ then: $F_0 = 0, \quad F_i = -A^T P_i - P_i A$

$$\left(\begin{array}{l} \text{for instance: } m = 2 \\ P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_3 = \sum_{i=1}^{3} x_i P_i \\ A^T P + PA = \sum_{i=1}^{3} x_i \left(A^T P_i + P_i A \right) \end{array} \right)$$

For sake of convenience and notation we will refer as to:

LMI $\quad A^T P + PA < 0 \quad$ in *P (decision variable)*

Schur complement for LMI

Goal: converts non linear inequalities to linear inequalities

Direct applications to symmetric matrices. Consider $Y(x) = Y(x)^T$, $R(x) = R(x)^T$ and $X(x)$ matrices depending affinely on the variable x

Lemma: the two following statements are equivalent

$$\begin{cases} Y(x) - X(x)R^{-1}(x)X(x)^T > 0 \\ R(x) > 0 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} Y(x) & X(x) \\ X(x)^T & R(x) \end{bmatrix} > 0$$

Remark: $(\Leftarrow)$ can be recovered using a congruence with row full rank matrix: $\begin{bmatrix} I & -X(x)R^{-1}(x) \end{bmatrix}$ i.e.:

$$\begin{bmatrix} I & -X(x)R^{-1}(x) \end{bmatrix} \begin{bmatrix} Y(x) & X(x) \\ X(x)^T & R(x) \end{bmatrix} \begin{bmatrix} I \\ -R^{-1}(x)X^T(x) \end{bmatrix} = Y(x) - X(x)R^{-1}(x)X(x)^T$$

Schur complement for LMI

Of course: $\begin{cases} Y(x) - X(x)R^{-1}(x)X(x)^T > 0 \\ R(x) > 0 \end{cases} \Leftrightarrow \begin{bmatrix} R(x) & X(x)^T \\ X(x) & Y(x) \end{bmatrix} > 0$

$(\Leftarrow)$ congruence with $\begin{bmatrix} -X(x)R^{-1}(x) & I \end{bmatrix}$

Generalization:

$$\begin{cases} Y - XR^{-1}X^T - US^{-1}U^T > 0 \\ R > 0, S > 0 \end{cases} \Leftrightarrow \begin{bmatrix} Y & X & U \\ X^T & R & 0 \\ U^T & 0 & S \end{bmatrix} > 0$$

$(\Leftarrow)$ congruence with $\begin{bmatrix} I & -XR^{-1} & -US^{-1} \end{bmatrix}$

Schur complement for LMI

Examples:

$Z(x) \in \mathbb{R}^{p\times q}$ affine on x, $\|Z(x)\| < 1 \Leftrightarrow Z(x)Z(x)^T < I$

$$\Leftrightarrow I - Z(x)Z(x)^T > 0 \Leftrightarrow \begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0$$

$$trace\left(S(x)^T P^{-1}(x) S(x)\right) < 1, \quad P(x) > 0$$

"**push**" cost function to constraint using an **auxiliary** variable X

$$trace(X) < 1, \quad S(x)^T P^{-1}(x) S(x) < X, \quad P(x) > 0$$

Can be "omitted"

$$X - S(x)^T P^{-1}(x) S(x) > 0 \Leftrightarrow \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0$$

$$trace(X) < 1, \quad \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0$$

Finsler's lemma (1st version)

Finsler's theorem: let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^n$, $R \in \mathbb{R}^{m\times n}$ s.t. $rank(R) < n$

following statements are equivalent:

1. $x^T Qx < 0$ for all $x \neq 0$ s.t. $Rx = 0$
2. $R_\perp^T Q R_\perp < 0$ where $RR_\perp = 0$
3. $Q - \sigma R^T R < 0$ for some scalar $\sigma \in \mathbb{R}$
4. $Q + XR + R^T X^T < 0$ for some matrix $X \in \mathbb{R}^{n\times m}$

Remarks:

- $R_\perp$ is a basis for the null space of R
- $X = -\dfrac{\sigma}{2} R^T$ is feasible
- X must be unconstrained and independent of other variables

Finsler's lemma (1st version)

Example: $X = X^T > 0, Y$ unknown, A, B given

$$X > 0, \quad \underbrace{AX + XA^T}_{Q} + \underbrace{Y^T}_{X} \underbrace{B^T}_{R} + \underbrace{B}_{R^T} \underbrace{Y}_{X^T} < 0 \qquad \text{LMI in } X \text{ and } Y$$

$$\Leftrightarrow \quad B_\perp^T \underbrace{\left(AX + XA^T \right)}_{Q} B_\perp < 0 \quad \Leftrightarrow \quad \underbrace{AX + XA^T}_{Q} - \sigma \underbrace{B}_{R^T} \underbrace{B^T}_{R} < 0, \quad \text{for some } \sigma \in \mathbb{R}$$

Finsler's lemma (2nd version)

Example: $X = X^T > 0, Y$ unknown, A, B given

$$X > 0, \quad \underbrace{AX + XA^T}_{Q} + \underbrace{Y^T}_{X}\underbrace{B^T}_{R} + \underbrace{B}_{R^T}\underbrace{Y}_{X^T} < 0 \qquad \text{LMI in } X \text{ and } Y$$

$$\Leftrightarrow \quad B_\perp^T \underbrace{\left(AX + XA^T\right)}_{Q} B_\perp < 0 \quad \Leftrightarrow \quad \underbrace{AX + XA^T}_{Q} - \sigma \underbrace{B}_{R^T}\underbrace{B^T}_{R} < 0, \quad \text{for some } \sigma \in \mathbb{R}$$

Finsler's theorem: let $x \in \mathbb{R}^n, Q = Q^T \in \mathbb{R}^n, R \in \mathbb{R}^{m\times n}, S \in \mathbb{R}^{p\times n}$

s.t. $rank\left(R\right) < n, rank\left(R\right) < n$. The following statements are equivalent:

1. $x^T Q x < 0$ for all $x \neq 0$ s.t. $Rx = 0, Sx = 0$
2. $R_\perp^T Q R_\perp < 0$ and $S_\perp^T Q S_\perp < 0$
3. $Q - \sigma R^T R < 0$ and $Q - \sigma S^T S < 0$ for some scalar $\sigma \in \mathbb{R}$
4. $Q + S^T X R + R^T X^T S < 0$ for some matrix $X \in \mathbb{R}^{n\times m}$

Examples

Example 1: Consider the following stable transfer function

$$G(s) = C\left(sI - A\right)^{-1} B, \qquad A \in \mathbb{R}^{6\times 6}, B \in \mathbb{R}^{6\times 4}, C \in \mathbb{R}^{4\times 6}.$$

with a set of input/output scaling matrices of the form

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & d_2 & d_3 \\ 0 & 0 & d_4 & d_5 \end{bmatrix}.$$

For robust stability, find, if any, a scaling matrix D such that the largest gain across frequency of $DG(s)D^{-1}$ is less than 1.

Examples

Equivalent LMI formulation: find symmetric matrices $X \in \mathbb{R}^{6\times 6}$, $S = D^T D \in \mathbb{R}^{4\times 4}$

such that

$$\begin{bmatrix} A^T X + XA + C^T SC & XB \\ B^T X & -S \end{bmatrix} < 0, \qquad X > 0, \qquad S > 1 \Leftrightarrow S - 1 > 0.$$

The previous LMI system can be solved via LMItoolbox in MATLAB

```
setlmis([])
X=lmivar(1,[6 1])        %Define structure of variable matrix X
S=lmivar(1,[2 0;2 1])    %Define structure of variable matrix S

lmiterm([1 1 1 X],1,A,'s') %Component (1,1) of 1st LMI: A^TX+XA
lmiterm([1 1 1 S],C',C)   %Component (1,1) of 1st LMI: C^TSC
lmiterm([1 1 2 X],1,B)     %Component (1,2) of 1st LMI: XB
lmiterm([1 2 2 S],1,1)     %Component (2,2) of 1st LMI: -S

lmiterm([-2 1 1 X],1,1)    %2nd LMI: X>0

lmiterm([-3 1 1 S],1,1)    %3rd LMI: S-1>0
lmiterm([-3 1 1 0],-1)     %3rd LMI: S-1>0

lmisys=getlmis
```

Examples

Example 2: Consider the following optimization problem:

Minimize tr(X) subject to $A^T X + XA + XBB^T X + Q < 0$

$$A = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -3 & -12 \\ 0 & -12 & -36 \end{bmatrix}.$$

Which by Schur's complement is equivalent to:

Minimize tr(X) subject to $\begin{bmatrix} A^T X + XA + Q & XB \\ B^T X & -I \end{bmatrix} < 0.$

This special class of problems (minimization), falls within the scope of the `mincx` solver and can be numerically solved as follows:

Examples

```
setlmis([]);
X=lmivar(1,[3 1]); %X is a full symetric 3x3 matrix

lmi1=newlmi;          %1st LMI is called lmi1
lmiterm([lmi1 1 1 X],1,A,'s');
lmiterm([lmi1 1 1 0],Q);
lmiterm([lmi1 1 2 X],1,B);
lmiterm([lmi1 2 2 0],-1);

LMIs=getlmis; %X is a full symetric 3x3 matrix
c=mat2dec(LMIs,eye(3)); %Select diagonal entries of X
options=[1e-5,0,0,0,0]; %1e-5 is the desired accuracy
[copt,xopt]=mincx(LMIs,c,options); %Solver
Xopt=dec2mat(LMIs,xopt,X); %Convert the decision
   variable X into a matrix format
```