Equations with Discontinuous Right Hand Side

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Outline

1. Preliminaries
   - Absolute Continuity

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Absolute Continuity

Definition

Let $\mathcal{I}$ be an interval in the real line $\mathbb{R}$. A function $f : \mathcal{I} \to \mathbb{R}$ is absolutely continuous on $\mathcal{I}$ if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k; y_k)$ of $\mathcal{I}$ satisfies

$$\sum_k (y_k - x_k) < \delta$$

then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon$$

The collection of all absolutely continuous functions on $\mathcal{I}$ is denoted $AC(\mathcal{I})$.

Absolute continuity of functions

Equivalent Definitions

1. f is absolutely continuous
2. f has a Lebesgue integrable derivative f' almost everywhere and

$$f(x) = f(a) + \int_a^x f'(t)dt; \quad \forall x \in [a; b]$$

3. there exists a Lebesgue integrable function g on $[a; b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt; \quad \forall x \in [a; b]$$

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental

Absolute continuity of functions

Properties

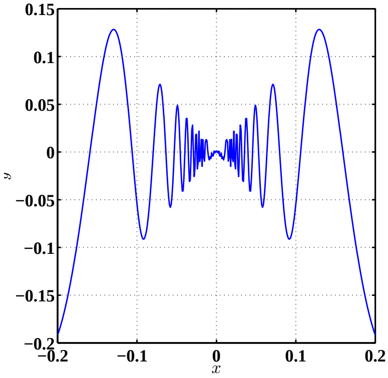
1. If $f, g \in AC(\mathcal{I})$, then $f \pm g$ is absolutely continuous.
2. If $\mathcal{I}$ is a bounded closed interval and $f, g \in AC(\mathcal{I})$, then fg is also absolutely continuous.
3. If $\mathcal{I}$ is a bounded closed interval, $f \in AC(\mathcal{I})$ and $f \neq 0$ then $\frac{1}{f}$ is absolutely continuous.
4. Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
5. If $f : \mathcal{I} \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a; b]$.

$$f(x) = x\sin\frac{1}{x}$$

$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0 \ \rightarrow \ f(x)$ is continuous !

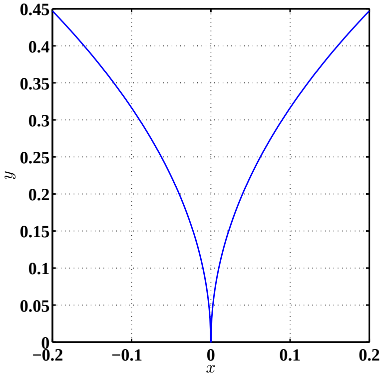
$$f'(x) = x\sin\frac{1}{x} - \frac{\cos(\frac{1}{x})}{x}, \ x \neq 0$$

At zero it is not differentiable and the lateral derivatives do not exist!



$$f(x) = \sqrt{|x|} = 2\int_0^x \frac{1}{\sqrt{|t|}}dt$$

At zero it is not differentiable and the lateral derivatives do not exist!



The function is still absolutely continuous