Optimal Control: LQ Regulator

Brief Introduction

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Linear Quadratic Regulator

- The plant is **linear**.
- The performance index is **quadratic**, also called cost function.

Consider the following plant:

$$\Sigma := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

Performance Index

Goal

We have to find the control $u(t)$ which minimizes the quadratic performance index:

$$J(x,\ u) = \int_0^{\infty} x^T Qx + u^T Ru\, dt.$$

also called *inifinte horizon cost function*. The "*weights*" needs to fulfil $Q = Q^T \succeq 0$ and $R = R^T \succ 0$.

Hamiltonian

The Hamiltonian function for this optimal control is depicted by:

$$H(x, u, \lambda) = x^T Qx + u^T Ru + \lambda^T (Ax + Bu)$$

where λ is the Lagrange multiplier.

Hamilton-Jacobi-Bellman (HJB) partial differential equation

The objective is to find the optimal function $J^*(x)$ which satisfies the HJB:

$$\forall x \quad \min_u \left[\underbrace{x^T Q x + u^T R u + \frac{\partial J^*}{\partial x}(Ax + Bu)}_{H(x,u,\lambda)} \right] = 0$$

Notice that the multiplier Lagrange represents the partial derivative w.r.t the state of the optimal function J^*, i.e., $\lambda^T = \frac{\partial J^*}{\partial x}$

If $J(x,u)$ is a quadratic function and also convex by virtue of $R \succ 0$, then we can denoted the optimal function as

$$J^*(x) = x^T P x, \quad P = P^T \succeq 0, \implies \nabla J^*(x) = 2x^T P = 0.$$

How to minimize HJB?

Solution.- We can find the minimum explicitly by finding the solution of

$$\frac{\partial H(x, u, \lambda)}{\partial u} = 2u^T R + 2x^T PB = 0$$

LQ Regulator

$$u^* = -R^{-1}B^T Px$$

Using u^* into the HJB partial diff equation we obtain the following

$$x^T \left(Q + PA + A^T P - PBR^{-1}B^T P \right) x = 0$$

Hamilton-Jacobi-Bellman

$$
\begin{aligned}
0 &= x^T Qx + u^{*T} Ru^* + 2x^T P\left(Ax + Bu^*\right) \\
0 &= x^T Qx + x^T P^T BR^{-1}B^T Px + 2x^T P\left(Ax - BR^{-1}B^T Px\right) \\
0 &= x^T\left(Q + 2PA + A^T P - PBR^{-1}B^T P\right)x \\
0 &= x^T\left(Q + PA + A^T P - PBR^{-1}B^T P\right)x.
\end{aligned}
$$

Algebraic Riccati Equation

$$
Q + PA + A^T P - PBR^{-1}B^T P = 0 \qquad \text{(ARE)}
$$

Hamilton-Jacobi-Bellman

$$
\begin{aligned}
0 &= x^T Q x + {u^*}^T R u^* + 2x^T P \left(Ax + Bu^* \right) \\
0 &= x^T Q x + x^T P^T B R^{-1} B^T P x + 2x^T P \left(Ax - BR^{-1} B^T P x \right) \\
0 &= x^T \left(Q + 2PA + A^T P - PBR^{-1} B^T P \right) x \\
0 &= x^T \left(Q + PA + A^T P - PBR^{-1} B^T P \right) x.
\end{aligned}
$$

Algebraic Riccati Equation

$$
Q + PA + A^T P - PBR^{-1} B^T P = 0 \qquad \text{(ARE)}
$$

Then, what about the stability of Σ?

Lyapunov Stability

Consider the "optimal" function J^* as a LCF, that is

$$V(x) = x^T P x$$

thus, the time derivative along the trajectories of Σ is

$$\begin{aligned}\dot{V}(x) &= 2x^T P \dot{x} \\ &= 2x^T P \left[Ax - R^{-1} B^T P x \right] \\ &= x^T \left[2PA - 2PBR^{-1} B^T P \right] x\end{aligned}$$

From (ARE) we know that $PA + A^T P = PBR^{-1} B^T P - Q$, thus

$$\dot{V}(x) = x^T \left[-PBR^{-1} B^T P - Q \right] x, \quad \therefore \dot{V} < 0, \quad \forall x$$