Square with relative degree one

Leonid Fridman

Sirius

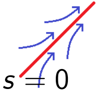
Sliding Mode Existence Conditions

- Scalar Linear Control

$$\dot{s} = -s$$

$$\lim_{s \to 0^+} \dot{s} = 0, \quad \lim_{s \to 0^-} \dot{s} = 0$$

$$V(s) = s^2$$
$$\dot{V}(s) = 2s\dot{s} = -2s^2 < 0$$
$$\dot{V}(s) = -2V$$
$$s(t) = s(0)e^{-t}$$

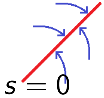


- Scalar Sliding Mode Control

$$\dot{s} = -\mathrm{sign(s)}$$

$$\lim_{s \to 0^+} \dot{s} = -1, \quad \lim_{s \to 0^-} \dot{s} = 1$$

$$V(s) = |s|$$
$$\dot{V}(s) = -[\mathrm{sign(s)}]^2 = -1 < 0$$
$$T_f = |s(0)|$$



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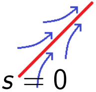

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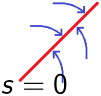
$$\dot{V}(s) = -2V^{1/2}$$

$$T_f = \sqrt{V(0)} = s(0)$$

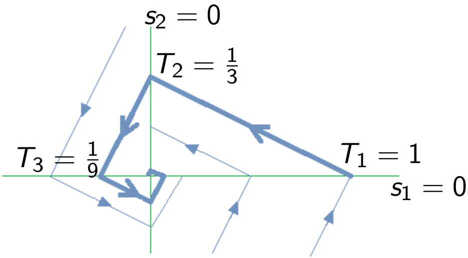


MIMO Systems

Zeno Phenomenon

- Vector Control

$$\dot{s}_1 = -\mathrm{sign}(s_1) - 2\mathrm{sign}(s_2)$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) + \mathrm{sign}(s_2)$$



$$\sum_{i=1}^{\infty} T_i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Piece-Wise Lyapunov Function I

Motion Equations:

$$\dot{s}_1 = -\mathrm{sign}(s_1) - 2\mathrm{sign}(s_2)$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) + \mathrm{sign}(s_2)$$

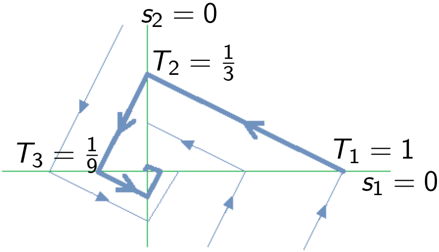
Lyapunov Function:

$$V = s^T \mathrm{Sign}(s), \quad s^T = (s_1, s_2),$$

$$\mathrm{Sign}(s) = \begin{bmatrix} \mathrm{sign}(s_1) \\ \mathrm{sign}(s_2) \end{bmatrix},$$

Time Derivative

$$\dot{V} = \frac{\partial V}{\partial s_1}\dot{s}_1 + \frac{\partial V}{\partial s_2}\dot{s}_2 = -2.$$



Sliding mode exists only in $s_1 = s_2 = 0$.

Piece-Wise Lyapunov Function II

Motion Equations:

$$\dot{s}_1 = -2\mathrm{sign}(s_1) - \mathrm{sign}(s_2)$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) + \mathrm{sign}(s_2)$$

Lyapunov Function:

$$V = s^T P \mathrm{Sign}(s), \quad s^T = (s_1, s_2),$$

$$\mathrm{Sign}(s) = \begin{bmatrix} \mathrm{sign}(s_1) \\ \mathrm{sign}(s_2) \end{bmatrix},$$
$$P = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix},$$

$$V(s) = 4|s_1| + |s_2|$$

Time Derivative

$$\begin{aligned} \dot{V} =& \frac{\partial V}{\partial s_1}\dot{s}_1 + \frac{\partial V}{\partial s_2}\dot{s}_2 \\ =& 4\mathrm{sign}(s_1)\dot{s}_1 + \mathrm{sign}(s_2)\dot{s}_2 \\ =& 4\mathrm{sign}(s_1)\left(-2\mathrm{sign}(s) - \mathrm{sign}(s_2)\right) + \\ &+\mathrm{sign}(s_2)\left(-2\mathrm{sign}(s) + \mathrm{sign}(s_2)\right) \\ =& -7 - 6\mathrm{sign}(s_1 s_2) < 0 \end{aligned}$$

Piece-Wise Lyapunov Function II

$$\dot{s}_1 = -2\mathrm{sign}(s_1) - \mathrm{sign}(s_2)$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) + \mathrm{sign}(s_2)$$

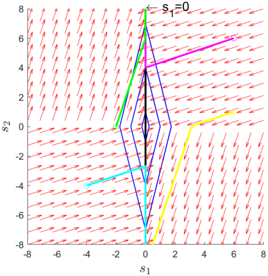
- Equivalent control for $s_1 = 0$

$$0 = -2\left[\mathrm{sign}(s_1)\right]_{eq} - \mathrm{sign}(s_2)$$
$$\therefore \left[\mathrm{sign}(s_1)\right]_{eq} = -\frac{1}{2}\mathrm{sign}(s_2)$$

- sliding dynamics for $s_1 = 0$:

$$\dot{s}_2 = -2\left[\mathrm{sign}(s_1)\right]_{eq} + \mathrm{sign}(s_2)$$
$$= 2\mathrm{sign}(s_2)$$

sliding dynamics on for $s_1 = 0$ is unstable!



Piece-wise Lipschitz continuous Lyapunov functions without detailed analysis are unable to ensure stability!

Extended invariance principle for nonautonomous switched systems

IEEE Transactions on Automatic Control 48(8) 2003, pp. 1448-1452

Yuri Orlov

Generalized Invariant Principle [Orlov 2003]

Basic definitions

- A continuous scalar function $v(x)$ is *positive definite* iff $v(0) = 0$ and $v(x) > 0$ for all $x \neq 0$.

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- $V(x, t)$ is *radially unbounded* iff $v(x)$.

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- $V(x,t)$ is *positive semidefinite* iff $V(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^{n+1}$.
- $V(x,t)$ is *negative definite (semidefinite)* iff $-V(x,t)$ is positive definite (semidefinite).

Generalized Invariant Principle [Orlov 2003]

Stability and Lipschitz Lyapunov Functions

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.

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- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.
- Special attention to the behavior of the composed function $V(x(t),t)$ on sliding manifolds and on nondifferentiability sets of $V(x,t)$.

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- By applying standard Lyapunov arguments, the system stability is guaranteed.
- Asymptotic stability is additionally to be studied (Barbalat lemma, extended invariance principle...).

Generalized Invariant Principle [Orlov 2003]

Stability and Lipschitz Lyapunov Functions

$V(x,t)$ is Lipschitz continuous, $x(t)$ is a solution of $\dot{x} = \varphi(x,t)$

$$\Downarrow$$

The composite function $V(x(t),t)$ is absolutely continuous and

$$\frac{d}{dt}V(x(t),t) = \frac{d}{dh}V(x(t)+h\dot{x}(t),t+h)|_{h=0}$$

almost everywhere.

Generalized Invariant Principle [Orlov 2003]

Lyapunov approach: Extension to VSS

Theorem

There exists a Lipschitz-continuous, positive definite, decrescent function $V(x,t)$ such that its time derivative

$$\frac{d}{dt}V(x(t),t) = \frac{d}{dh}V(x(t)+h\dot{x}(t),t+h)|_{h=0} \leq 0$$

for almost all t and for all trajectories $x(t)$ of the VSS $\dot{x} = \varphi(x,t)$, initialized within some B_δ.

$$\Downarrow$$

The VSS is uniformly stable.

Generalized Invariant Principle [Orlov 2003]

Corollary

The stability of the VSS $\dot{x} = \varphi(x, t)$ remains in force if the time derivative $\frac{d}{dt}V(x(t), t)$ is nonpositive at the points of the nondifferentiability set N_v of $Vx, t)$ and in the continuity domain of the function $\varphi(x, t)$ where it is expressed in the standard form

$$\frac{d}{dt}V(x, t) = \frac{\partial V(x, t)}{\partial t} + gradV(x, t) \cdot \varphi(x, t), \ (x, t) \in \mathbb{R}^{n+1} \backslash (N \bigcup N_v)$$

Generalized Invariant Principle [Orlov 2003]

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- Indeed, at the discontinuity points $(x,t) \in N$ of the function $\varphi(x,t)$, the right-hand side $\Phi(x,t))$ of the corresponding differential inclusion $\dot{x} \in \Phi(x,t)$ is obtained by closure of the graph of $\varphi(x,t)$ and by passing over to a convex hull.

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- Indeed, at the discontinuity points $(x, t) \in N$ of the function $\varphi(x, t)$, the right-hand side $\Phi(x, t))$ of the corresponding differential inclusion $\dot{x} \in \Phi(x, t)$ is obtained by closure of the graph of $\varphi(x, t)$ and by passing over to a convex hull.
- These procedures do not increase the upper value of $\dot{V}(x, t)$ and hence the negative semidefiniteness of $\dot{V}(x, t)$ guarantees the negative definiteness of $\dot{V}(x, t)$ for all $(x, t) \in N$.

Kitchen Receipt

1. Find a Lipschitz continuous Lyapunov function $V(x) > 0$ such that it is positive definite and his first time derivative (when it exists) is negative semidefinite $\dot{V} \leq 0$.
2. If they are solutions in the invariant set $\Omega = \left\{ x \in \mathbf{R}^n : \dot{V} = 0 \right\}$ their stability need to be examined.
3. If they are solutions in the set $S = \left\{ x \in \mathbf{R}^n : \frac{dV}{dx} \text{ does not exist} \right\}$ their stability need to be examined

Existence Conditions: Lyapunov approach

- Square System: $\dot{s} = -D\mathrm{Sign(s)}\ s, u \in R^m$
- $\mathrm{Sign(s)}^T = [\mathrm{sign(s_1)}, \ldots, \mathrm{sign(s_m)}]$

Theorem

If $D + D^T > 0$ then there exists a sliding mode exists in manifold $s = 0$.

The statement of the theorem may be proven using sum of absolute values of s_i

$$V = s^T \mathrm{Sign(s)} \geq 0$$

as Lipschitz continuous Lyapunov candidate function.

Existence Conditions:

Proof

$$\begin{aligned}
V(s) &= \frac{1}{2}\mathrm{Sign(s)}^T s + \frac{1}{2} s^T \mathrm{Sign(s)} > 0, \\
\dot{V}(s) &= \frac{1}{2}\mathrm{Sign(s)}^T \dot{s} + \frac{1}{2} \dot{s}^T \mathrm{Sign(s)}, \\
\dot{V}(s) &= -\frac{1}{2}\mathrm{Sign(s)}^T D \mathrm{Sign(s)} - \frac{1}{2}\mathrm{Sign(s)}^T D^T \mathrm{Sign(s)}
\end{aligned}$$

For this Lyapunov function, even if $\dot{V} < 0$, it is necessary to examine the stability in all possible intersections of switching surfacess

Existence Conditions

$$D + D^T > 0$$

- Proof Utkin 1992

Quadratic Function

$$\dot{s} = -D\mathrm{Sign}(s)$$

$$V(s) = s^T s$$
$$\dot{V}(s) = s^T \dot{s} + \dot{s}^T s$$
$$\dot{V}(s) = -s^T D\mathrm{Sign}(s) - \mathrm{Sign}(s)^T D^T s$$

D is a matrix with a dominating diagonal by rows

$$d_{ii} > \sum_{i=1, i\neq j}^{i=m} |d_{ij}|$$

The stability of dynamics on the intersections of the switching surfaces does not need to be checked

$$\dot{s} = u$$
$$u = \mathrm{sign(s)}$$

Unstable

$$\dot{s} = u$$
$$u = \mathrm{sign(-s)}$$

Stable

What we have to do if $D + D^T$ is not positive definite?

$\Longrightarrow$ There are some unstable sliding surfaces!

Stabilizing Sliding Surfaces

$$\dot{s} = -D\mathrm{Sign}(s)$$

- New switching surface

$$\hat{s} = Ts$$

$$\dot{\hat{s}} = T\dot{s}$$

$$\dot{\hat{s}} = -TD\mathrm{Sign}(\hat{s})$$

- Lyapunov function:

$$V(\hat{s}) = sign(\hat{s})^T\hat{s}$$

$$V(\hat{s}) = \frac{1}{2}\mathrm{Sign}(\hat{s})^T\hat{s} + \frac{1}{2}\hat{s}^T\mathrm{Sign}(\hat{s}) > 0,$$

$$\dot{V}(\hat{s}) = \frac{1}{2}\mathrm{Sign}(\hat{s})^T\dot{\hat{s}} + \frac{1}{2}\dot{\hat{s}}^T\mathrm{Sign}(\hat{s}),$$

$$\dot{V}(\hat{s}) = -\frac{1}{2}\mathrm{Sign}(\hat{s})^T TD\mathrm{Sign}(\hat{s}) - -\frac{1}{2}\mathrm{Sign}(\hat{s})^T(TD)^T\mathrm{Sign}(\hat{s}),$$

$$\dot{V}(\hat{s}) = -\mathrm{Sign}(\hat{s})^T\frac{TD + (TD)^T}{2}\mathrm{Sign}(\hat{s})$$

Find such T that $\lambda_{min}\left\{\frac{TD+(TD)^T}{2}\right\} > 0$

- Sliding mode $\hat{s} = 0 \rightarrow s = 0$

Stabilizing Sliding Surfaces

Shaul Gutman

George Leitmann

Stabilizing Switching Surfaces $T = D^T \rightarrow \text{if } detD \neq 0 \rightarrow D^T D > 0$

Aletrnative solutions $detD \neq 0 \rightarrow T = D^{-1} \rightarrow D^{-1} D = I_m$

Stabilizing Sliding Surfaces: Example: $T = D^T$

System:

$$\dot{s} = -\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \mathrm{Sign}(s)$$

New sliding surfaces set

$$\hat{s} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_! \\ s_2 \end{bmatrix}$$

Motion equations:

$$\dot{\hat{s}} = -\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \mathrm{Sign}(\hat{s})$$

$$V(\hat{s}) = sign(\hat{s})^T \hat{s}$$

$$\dot{V}(\hat{s}) = -Sign(\hat{s})^T \hat{D} Sign(\hat{s})$$

$$\hat{D} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} > 0$$

$$\lambda\{\hat{D}\} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\dot{V}(\hat{s}) \leq -2\|Sign(\hat{s})\|^2 = -4$$

Stabilizing Sliding Surfaces: Example

- System dynamics:

$$\dot{\hat{s}}_1 = -5\mathrm{sign}(\hat{s}_1) - 3\mathrm{sign}(\hat{s}_2)$$
$$\dot{\hat{s}}_2 = -3\mathrm{sign}(\hat{s}_1) - 5\mathrm{sign}(\hat{s}_2)$$

- $\hat{s}_1 = 0$

$$[sign(\hat{s}_1)]_{eq} = -\frac{3}{5}\mathrm{sign}(\hat{s}_2)$$

- $\hat{s}_2$ dynamics:

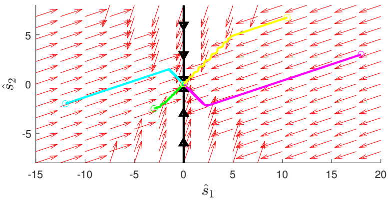
$$\dot{\hat{s}}_2 = -3\,[sign(\hat{s}_1)]_{eq} - 5\mathrm{sign}(\hat{s}_2)$$
$$\dot{\hat{s}}_2 = -\frac{16}{5}\mathrm{sign}(\hat{s}_2)$$

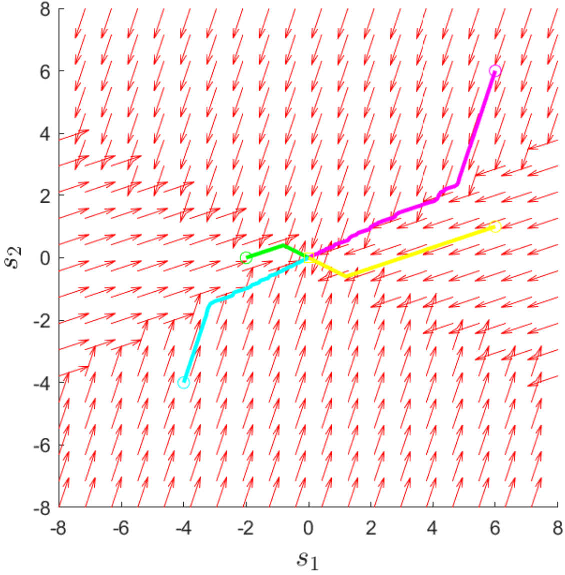
- $\hat{s}_2 = 0$

$$[\mathrm{sign}(\hat{s}_2)]_{eq} = -\frac{3}{5}\mathrm{sign}(\hat{s}_1)$$

- $\hat{s}_1$ dynamics:

$$\dot{\hat{s}}_1 = -5\mathrm{sign}(\hat{s}_1) - 3\,[\mathrm{sign}(\hat{s}_2)]_{eq}$$
$$\dot{\hat{s}}_1 = -\frac{16}{5}\mathrm{sign}(\hat{s}_1)$$





8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8
s_1
s_2

Stabilizing Sliding Surfaces: Example: $T = D^{-1}$

System:

$$\dot{\sigma} = -\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \mathrm{Sign}(\sigma)$$

New sliding surfaces set

$$\hat{s} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Motion equations:

$$\dot{\hat{s}} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{Sign}(\hat{s})$$

$$V(\hat{s}) = Sign(\hat{\sigma})^T \hat{\sigma}$$

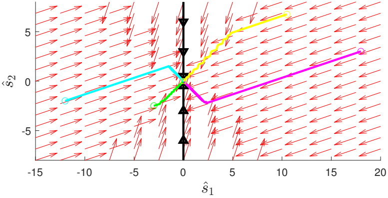
$$\dot{V}(\hat{s}) = -Sign(\hat{\sigma})^T \hat{D} Sign(\hat{\sigma})$$

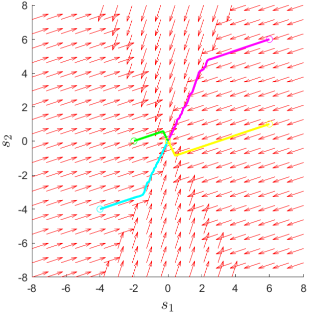
$$\hat{D} = I > 0$$

$$\lambda\{\hat{D}\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\dot{V}(\hat{\sigma}) \leq -\|Sign(\hat{\sigma})\|^2 = -2$$

Stabilizing Sliding Surfaces: Example: $T = D^{-1}$





Unit Control

Shaul Gutman

George Leitmann

Unit Control

Introductory example

$$\dot{\sigma}_1 = u_1,\, \dot{\sigma}_2 = u_2$$
$$u_1 = -\frac{\sigma_1}{\|\sigma\|},\, u_2 = -\frac{\sigma_2}{\|\sigma\|}$$

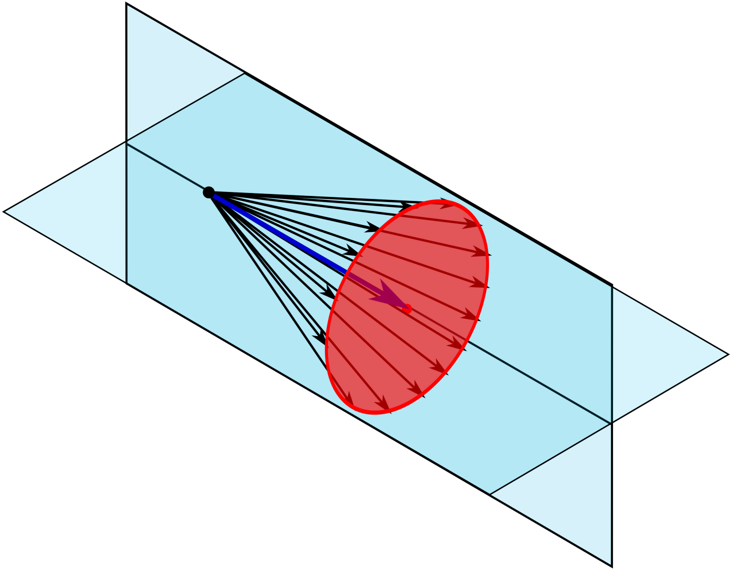
$$\|u\| = \sqrt{u_1^2 + u_2^2} = \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{\|\sigma\|^2}}$$
$$= \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} = 1$$

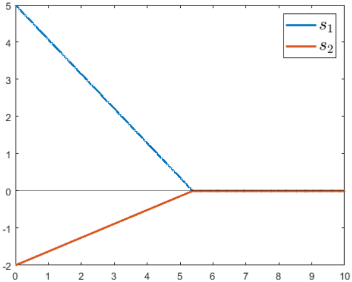
$\|u\| = u_1^2 + u_2^2 = 1 \rightarrow$ Unit Circle

Lyapunov function:

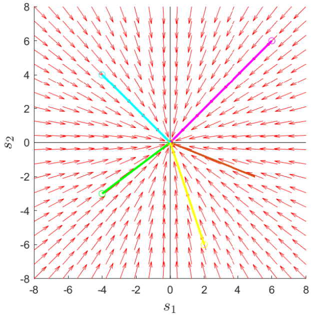
$$V = \frac{1}{2} s^T s$$
$$\dot{V} = s^T \dot{s} = -\frac{\sigma^T s}{\|s\|}$$
$$\|s\|^2 = s^T s = 2V$$
$$\dot{V} = -\sqrt{2} V^{\frac{1}{2}}$$

Unit control in the space





Surfaces



Phase Portrait of Surfaces

Unit Control

- Square system

$$\dot{\sigma} = Du, \ \ \sigma, u \in R^m$$

- Controller

$$u = -\frac{\sigma}{||\sigma||},$$

- Lyapunov function $V = \sigma^T \sigma$, s.t,

$$\dot{V} = -\sigma^T D \frac{\sigma}{||\sigma||} - \left(\frac{\sigma}{||\sigma||}\right)^T D^T \sigma \leq -min\lambda(D + D^T)||\sigma||$$

Unit Control

- Square system

$$\dot{\sigma} = Du, \;\; \sigma, u \in R^m$$

- Controller

$$u = -\frac{\sigma}{||\sigma||},$$

- Lyapunov function $V = \sigma^T \sigma$, s.t,

$$\dot{V} = -\sigma^T D \frac{\sigma}{||\sigma||} - \left(\frac{\sigma}{||\sigma||}\right)^T D^T \sigma \leq -min\lambda(D + D^T)||\sigma||$$

$D + D^T > 0$ sufficient conditions for SM on $\sigma = 0$

Some discussion about unit control

Advantages

- Discontinuity only in $\sigma = 0$
- Each component of σ converges at the same time

Disadvatages

- If a channel fails, all the control fails
- For computer implementation of unit controllers usually some positive constant in the denominator is used. It destroys final precision.

Relay Control with perturbations

$$\dot{s} = d(t,s) + u \quad u = -\rho(t,s)\mathrm{Sign(s)}$$

$$\begin{aligned} V &= \frac{s^T s}{2} \\ \dot{V} &= \frac{s^T \dot{s}}{2} + \frac{\dot{s}^T s}{2} = s^T \dot{s} \\ \dot{V} &= s^T d(t,s) - s^T \rho(t,s) Signs \\ \dot{V} &= s^T d(t,s) - \rho(t,s)\|s\|_1 \end{aligned}$$

$$\rho(t,s) > \|d(t,s)\|_1 \implies \gamma = min_{t,s}\left(\rho(t,s) - \|d(t,s)\|_1\right)$$

$$\dot{V} \leq -\gamma V^{1/2}$$

Consequently, $s = 0$ is reached in finite-time.

Vector relay control with perturbations

$$\dot{\sigma} = D(u + d(t,\sigma)), \quad u = -\begin{bmatrix} \alpha_1 \mathrm{sign}(\sigma_1) \\ \alpha_2 \mathrm{sign}(\sigma_2) \\ \vdots \\ \alpha_m \mathrm{sign}(\sigma_\mathrm{m}) \end{bmatrix}, \quad d(t,x) = \begin{bmatrix} d_1(t,\sigma) \\ d_2(t,\sigma) \\ \vdots \\ d_n(t,\sigma) \end{bmatrix}$$

$$\begin{aligned} V &= \frac{\sigma^T \sigma}{2} \\ \dot{V} &= \frac{\sigma^T \dot{\sigma}}{2} + \frac{\dot{\sigma}^T \sigma}{2} = s^T \dot{s} \\ &= \frac{1}{2} s^T D(u + d(t,s)) + \frac{1}{2}(u + d(t,s))^T D^T s \\ &= \frac{1}{2} s^T \left(D + D^T\right) d(t,s) + \frac{1}{2} s^T \left(D + D^T\right) u \end{aligned}$$

$$\alpha_i > |d_i(t,s)|, \implies \exists \gamma_1 \to \dot{V} \leq -\gamma_1 V^{1/2}$$

Vector control VS Unit control

System

$$\dot{s} = d(s,t) + u, \quad d(s,t) = \begin{bmatrix} 2 - \cos(3t) \\ 0.5 - 2\cos(3t) \end{bmatrix}$$

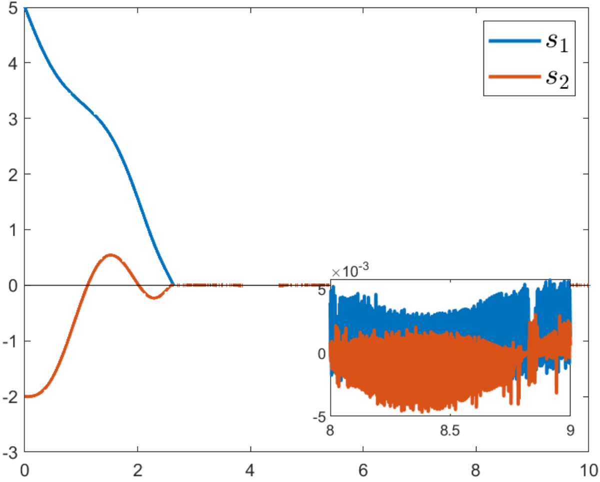
$$u_{pw} = -\begin{bmatrix} \alpha_1 \mathrm{sign}(s_1) \\ \alpha_2 \mathrm{sign}(s_2) \end{bmatrix}$$

- $\alpha_1 = 3$
- $\alpha_2 = 2.5$

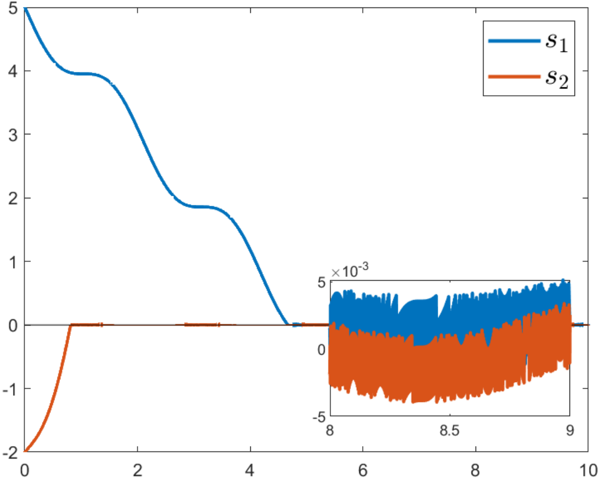
$$u_{unit} = -\rho(t,x)\frac{s}{\|s\|}$$

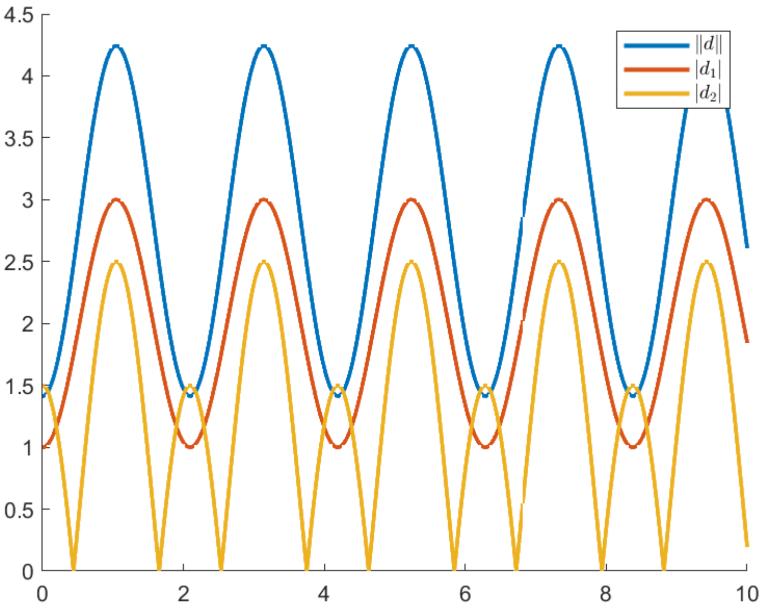
- $\rho = \sqrt{3^2 + 2.5^2} = 3.9$

Unit control $\rho = 3.9$



Relay control $\alpha_1 = 3$, $\alpha_2 = 2.5$





4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
$||d||$
$|d_1|$
$|d_2|$

Compensation of nominal equivalent control

$$\dot{\sigma} = f(t,\sigma) + d(t,s) + u, \quad u = -\rho(t,\sigma)\frac{\sigma}{\|\sigma\|}$$

$f(t,\sigma)$ known dynamics, $d(t,\sigma)$ unknown perturbations

- No compensation $u_{eq_{nom}} = 0$

$$V = \frac{\sigma^T\sigma}{2}$$
$$\dot{V} = \sigma^T\dot{\sigma}$$
$$\dot{V} = \sigma^T(d(t,\sigma) + f(t,\sigma)) - \rho(t,\sigma)\|\sigma\|$$
$$\rho(t,\sigma) > \|d(t,\sigma)\| + \|f(t,\sigma)\|$$

- Compensation

$$u_{eq_{nom}} = -f(t,\sigma)$$

$$V = \frac{\sigma^T\sigma}{2}$$
$$\dot{V} = \sigma^T\dot{\sigma}$$
$$\dot{V} = s^T(d(t,\sigma)) - \rho(t,\sigma)\|\sigma\|$$
$$\rho(t,\sigma) > \|d(t,\sigma)\|$$

Compensation of nominal equivalent control

$$\dot{s} = f(t,s) + d(t,s) + u, \quad u = u_{eq_{nom}} - \rho(t,s)\frac{s}{\|s\|}$$

$$f(t,s) = 1 + sin(3t) - 2s, \quad d(t,s) = \frac{1}{2} + \frac{1}{2}sin(3t) + 3s$$

First case

$$u_{eq_{nom}} = -1 - sin(3t) + 2s$$

Closed loop: $\dot{s} = d(t,s) - \rho(t,s)\frac{s}{\|s\|}$

$$V = \frac{s^2}{2}$$

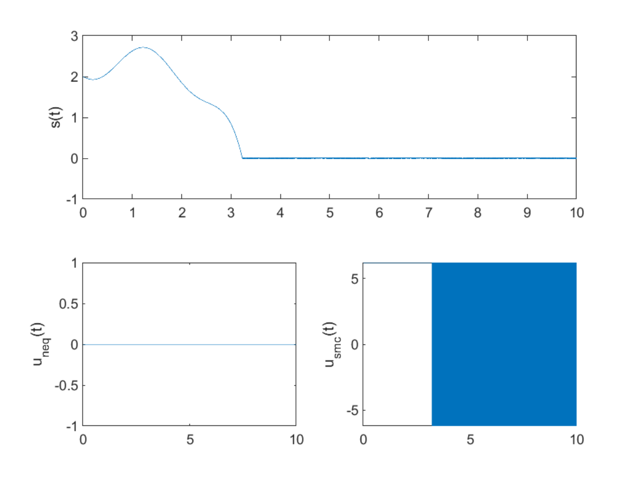
$$\dot{V} = s\dot{s} = s\left(d(s,t) - \rho(t,s)\frac{s}{\|s\|}\right)$$

$$\dot{V} = sd(s,t) - \rho(s,t)\|s\|$$

$$\rho > \|d(t,s)\| \quad \forall s \in \mathbb{R},\ t \in \mathbb{R}^+$$

Compensation of nominal equivalent control

First case: $\rho = 6.8$



Compensation of nominal equivalent control

Second case

$$u_{eq_{nom}} = -1 - \sin 3t$$

Closed loop: $\dot{s} = -2s + d(x,t) - \rho(t,x)\frac{s}{\|s\|}$

$$V = \frac{s^2}{2}$$

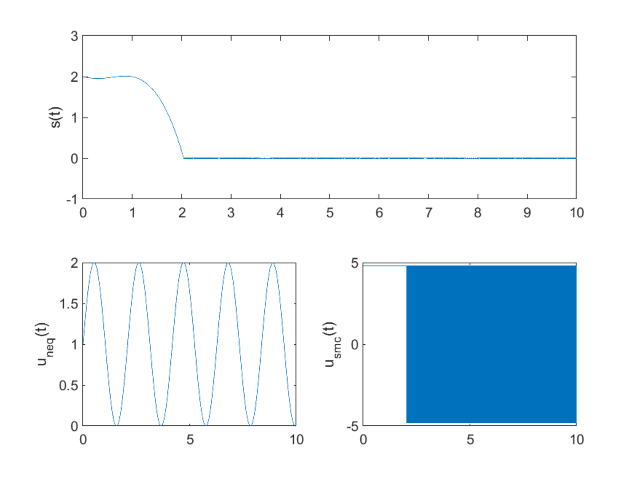
$$\dot{V} = s\dot{s} = s\left(-2s + d(s,t) - \rho(t,s)\frac{s}{\|s\|}\right)$$

$$\dot{V} = -2s^2 + sd(s,t) - \rho(s,t)\|s\|$$

$$\rho > \|d(t,x)\| \quad \forall s \in \mathbb{R},\ t \in \mathbb{R}^+$$

Compensation of nominal equivalent control

Second case: $\rho = 4.8$



Compensation of nominal equivalent control

Third case

$$u_{eq_{nom}} = 0$$

Closed loop: $\dot{s} = -2s + d(s,t) + 1 + sin(3t) - \rho(t,s)\frac{s}{\|s\|}$

$$V = \frac{s^2}{2}$$

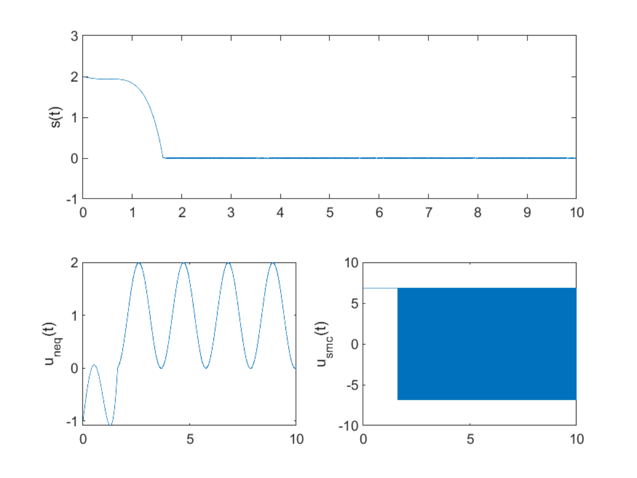
$$\dot{V} = s\dot{s} = s\left(-2s + d(s,t) + 1 + sin(3t) - \rho(t,s)\frac{s}{\|s\|}\right)$$

$$\dot{V} = -2s^2 + s(d(s,t) + 1 + sin(3t)) - \rho(x,t)\|s\|$$

$$\rho > \|d(t,x) + 1 + sin(3t)\| \quad \forall x \in \mathbb{R},\ t \in \mathbb{R}^+$$

Compensation of nominal equivalent control

Third case $\rho = 6.2$



Sliding Mode Control Design Invariance

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t)$$

- $h(x,t)$ is disturbance vector
- Matching Condition
  $h(x,t) \in range(B)$

B. Drazenovic,
The invariance conditions in variable structure systems, *Automatica*, v.5, No.3,Pergamon Press, 1969.

Design of Control Under Uncertainty:

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t),$$

Invariance condition(Matching Conditions)

$$\begin{array}{ll} h(x,t) \in range(B), & \Longrightarrow \\ \exists \lambda(x,t): \quad h = B\lambda, \quad \lambda \in \mathbb{R}, & \Longrightarrow \\ \dot{x} = f(x,t) + B(x,t)(u+\lambda), & (*) \end{array}$$

$s(x) = 0$ is a sliding manifold

Design of Control Under Matching Conditions

Employing Equivalent Control Method:

$$\dot{s} = 0 \implies \{u + \lambda\}_{eq}$$

$$\dot{s} = Gf + GB\{u + \lambda\}_{eq} = 0, \quad G := \frac{\partial s}{\partial x}$$

$$\{u + \lambda\}_{eq} = -(GB)^{-1} Gf \implies (*)$$

$$(*) \implies \dot{x} = f(x,t) - B(x,t)(GB)^{-1} Gf$$

$\dot{x}$ does not depend on disturbance $h(x,t)$.

UNMATCHED UNCERTAINTIES

$$\exists\lambda(x,t): \quad h = B\lambda_1 + B^{\perp}\lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}, ->$$
$$\dot{x} = f(x,t) + B(x,t)(u+\lambda_1) + B^{\perp}\lambda_2,$$

$s(x) = 0$ is a sliding manifold

UNMATCHED UNCERTAINTIES

$$\dot{s} = 0 \implies u_{eq}:$$
$$\dot{s} = Gf + GB(u_{eq} + \lambda_1) + B^{\perp}\lambda_2 = 0, \; G := \left\{\frac{\partial s}{\partial x}\right\}$$
$$u_{eq} + \lambda_1 = -(GB)^{-1}[Gf + B^{\perp}\lambda_2]$$
$$\dot{x} = f(x,t) - B(x,t)(GB)^{-1}[Gf + B^{\perp}\lambda_2].$$

In this case invariance does not exists.

Unmatched Uncertainties vs Output Tracking

$$\dot{x}_1 = x_2 + f_1(t), \quad \dot{x}_2 = f_2(t) + u \implies \text{Unmatched Uncertainties!}$$

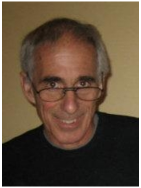
$$\ddot{x}_1 = \dot{x}_2 + \dot{f}_1(t) = f_2(t) + \dot{f}_1(t) + u$$

For the Output Tracking Uncertainties are Matched!

- It is impossible to grab two watermelons with just one hand

Lyapunov Redesign

Prof. Shaul Gutman



Prof. Ross Barmish

Prof. Martin Corless

Prof. George Leitmann

Prof. Raymond DeCarlo

Lyapunov Redesign

Barmish, B. R., Corless, M., & Leitmann, G. (1983). A new class of stabilizing controllers for uncertain dynamical systems. SIAM Journal on Control and Optimization, 21(2), 246-255.

System

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t)$$

- $h(x,t) = B(x,t)\gamma(x,t)$

Control law

$$u = u_{nom} + u_{smc}$$

- u_{nom}: Nominal control
- u_{smc}: Compensator

Lyapunov Redesign

Nominal System: $\dot{x} = f(x,t) + B(x,t)u_{nom}$

- $V(x) > 0$
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \mathrm{grad(V)}^T \dot{x} = \mathrm{grad(V)}^T \left(f(x,t) + B(x,t)u_{nom}\right) < 0$

- Disturbed system:

$$\dot{V}(x) = \mathrm{grad(V)}^T \left(f(x,t) + B(x,t)u_{nom}\right) + \mathrm{grad(V)}^T B(x,t)\left(u_{smc} + \gamma(x,t)\right)$$

- Sliding surface $w = B^T \mathrm{grad(V)}$. $V_0 = \mathrm{grad(V)}^T \left(f(x,t) + B(x,t)u_{nom}\right)$

$$\dot{V}(x) = V_0 + w^T\left(u_{smc} + \gamma(x,t)\right)$$

Lyapunov Redesign

- Unit control:

$$u_{smc} = -\rho(x,t)\frac{w}{\|w\|_2}$$

$$\dot{V}(x) = V_0 + w^T\gamma(x,t) - \rho(x,t)\|w\|_2$$
$$\dot{V}(x) < V_0 + \|w\|_2\left(\gamma(x,t) - \rho(x,t)\right)$$

$$\rho(x,t) > \|\gamma(x,t)\|_2$$

- Relay control:

$$u_{smc} = -\rho(x,t)\mathrm{Sign(w)}$$

$$\dot{V}(x) = V_0 + w^T\gamma(x,t) - \rho(x,t)\|w\|_1$$
$$\dot{V}(x) < V_0 + \|w\|_1\left(\gamma(x,t) - \rho(x,t)\right)$$

$$\rho_i(x,t) > \|\gamma_i(x,t)\|_1$$

Lyapunov Redesign

$$\dot{x} = Ax + B[u + \phi(t, x, u)]$$

Nominal control $u = -Kx$,

$$\dot{x} = (A - BK)x \implies (A - BK) = \bar{A} \text{ is Hourwitz}$$

Lyapunov function $V = x^T P x$:

$$\bar{A}^T P + P\bar{A} = -Q \implies \dot{V} = -x^T Q x$$

New control law $u = -Kx + v$

$$\|\phi(t, x, -Kx + v)\|_2 \le p(t, x) + \kappa_0 \|v\|_2, \quad 0 \le \kappa_0 < 1.$$

$$\dot{x} = \bar{A}x + B[v + \phi(t, x, -Kx + v)],$$

$$\dot{V} = -x^T Qx + 2x^T PB[v + \phi(t, x, -Kx + v)]$$

defining $w^T = 2x^T PB$

$$\dot{V} \leq -x^T Qx + w^T v + w^T \phi$$

and using the control $v = -\eta(t, x)\frac{w}{\|w\|_2}$

$$w^T v + w^T \phi \leq -\eta\|w\|_2 + p\|w\|_2 + \kappa_0\eta\|w\|_2 = -\eta(1 - \kappa_0)\|w\|_2 + p\|w\|_2$$

taking $\eta \geq p/(1 - \kappa_0)$, implies

$$w^T v + w^T \phi \leq -\frac{p}{1 - \kappa_0}(1 - \kappa_0)\|w\|_2 + p\|w\|_2 \leq 0$$

An alternative idea using a relay control law

Suppose now that

$$\|\phi(t,x,-Kx+v)\|_\infty \leq p(t,x) + \kappa_0\|v\|_\infty\,, \quad 0 \leq \kappa_0 < 1$$

then

$$w^T v + w^T\phi \leq w^T v + \|w\|_1\|\phi\|_\infty \leq w^T v + \|w\|_1\left[\rho(t,x) + \kappa_0\|v\|_\infty\right]$$

Choose

$$v = -\eta(t,x)\mathrm{Sign(w)}\,, \quad \mathrm{Sign(w)}^T = \begin{bmatrix}\mathrm{sign(w_1)} & \dots & \mathrm{sign(w_p)}\end{bmatrix}$$

Take $\eta(t,x) \geq \rho(t,x)/(1-\kappa_0)$ for all (t,x), then

$$\begin{aligned} w^T v + w^T\phi &\leq -\eta\|w\|_1 + \rho\|w\|_1 + \kappa_0\eta\|w\|_1 \\ &= -\,\eta(1-\kappa_0)\|w\|_1 + \rho\|w\|_1 \leq 0 \end{aligned}$$

which implies that $\Rightarrow \dot{V} \leq -\alpha_3(\|x\|)$

Example

Consider the linearized 3-tank water system DST200,

$$\dot{x} = \begin{bmatrix} 0.0148 & 0 & 0.0148 \\ 0 & -0.0213 & 0.0145 \\ 0.0148 & 0.0145 & -0.0293 \end{bmatrix} x + \begin{bmatrix} 0.0032 & 0 \\ 0 & 0.0032 \\ 0 & 0 \end{bmatrix} (u + f)$$

with $f = \mathrm{Sign}(\mathrm{x}_1) + \sin(0.1t)$ which can be a failure in the actuator.
Designed nominal control

$$u = -\begin{bmatrix} 20.10 & 0.68 & 4.10 \\ 2.40 & 15.58 & 6.25 \end{bmatrix} x, \quad \text{s.t. } eig\{\bar{A}\} = \begin{bmatrix} -0.03 \\ -0.05 \\ -0.07 \end{bmatrix}$$

from the Algebraic Lyapunov equation, setting $Q = I_{3\times 3}$, P is

$$P = \begin{bmatrix} 10.2074 & -0.8813 & 2.1452 \\ -0.8813 & 7.1400 & -0.2554 \\ 2.1452 & -0.2554 & 18.0220 \end{bmatrix}$$

setting $\eta(t, x) = \|x\| + 5$ in both cases.

Simulations of Lyapunov redesign with unit control

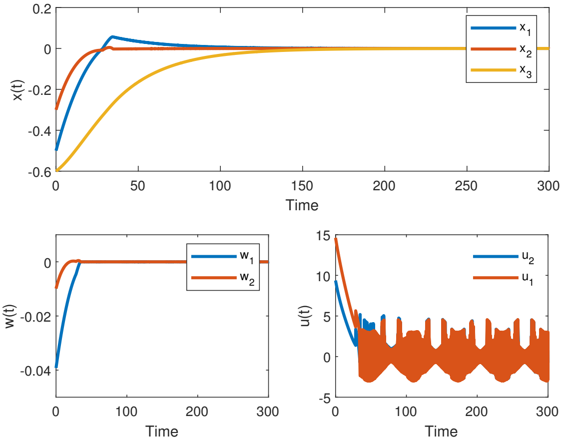


Figure: Control, variable w and states of the Lyapunov Redesign.

Advantages

- Discontinuity only in $w = 0$
- Each component of w converges at the same time

Disadvatages

- If a channel fails, all the control fails
- The gain should compensate the 2-norm of the perturbation vectors, chattering could be bigger .

Simulations of Lyapunov redesign with relay control

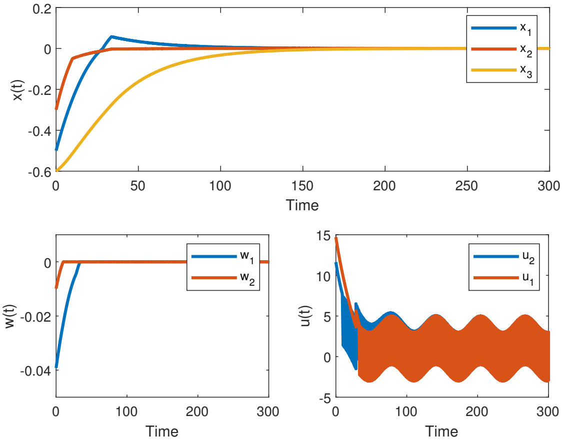


Figure: Control, variable w and states of the Lyapunov Redesign.

Advantages

- It compensates the perturbation component-wise
- The gains can be chosen component-wise too, which allows to decrease the Chattering effect in some channels

Disadvatages

- It is discontinuous in every coordinate, i.e. when $w_i = 0$

Integral Sliding Mode with Matched Perturbations

Prof. Raymond DeCarlo, 1988

G.P. Matthews, R.A. DeCarlo, Decentralized tracking for a class of interconnected nonlinear systems using variable structure control. Automatica 24, 187–193 (1988)

Integral Sliding Mode: Problem formulation

- Consider
$$\dot{x} = f(x) + B(x)\,u + \phi(x,t)\,, \tag{Unc}$$
- $x \in \mathbb{R}^n$ state
- $u \in \mathbb{R}^m$ control input
- $x(0) = x_0$
- $\phi(x,t)$ uncertainties
- Nominal System
$$\dot{x}_0 = f(x_0) + B(x_0)\,u_0. \tag{Nom}$$
- u_0 nominal control

If $\phi \neq 0$ the trajectories of (2) and (Nom) are different.

Integral Sliding Mode: Problem formulation

Assumptions

A1) $\operatorname{rank} B(x) = m$ for all $x \in \mathbb{R}^n$;

A2) $\phi(x,t) = B(x)\,\xi(x,t)$;

A3)

$$\|\xi(x,t)\| \leq \xi^+(x,t).$$

Control design objective

Objective

$x(0) = x_0(0)$, and $x(t) = x_0(t)$ for all $t \geq 0$.

Solution

- Design $u = u(t)$ as

$$u(t) = u_0(t) + u_1(t),$$

- $u_0(t)$ nominal control
- $u_1(t)$ compensates $\phi(x, t)$, for all $t \geq 0$.

Linear case

LTI system

$$\dot{x}(t) = Ax(t) + B\left(u_0(t) + u_1(t)\right) + B\xi(t, x), \tag{LTI}$$

Nominal system

$$\dot{x}_0(t) = Ax(t) + B\left(u_0(t)\right), \tag{NOM}$$

Difference between plant and nominal system

$$\epsilon(t) = x(t) - x_0(t)$$

$$\dot{\epsilon}(t) = \dot{x}(t) - \dot{x}_0(t)$$

$$\epsilon(t) = \int_0^t \dot{x}(\tau)d\tau - \int_0^t \dot{x}_0(\tau)d\tau$$

$$\epsilon(t) = x(t) - x(0) - \int_0^t Ax(\tau) + B\left(u_0(\tau)\right) d\tau$$

Sliding dynamics

$$\sigma(x) = G\epsilon = G\left(x(t) - x(0)\right) - G\int_0^t \left(Ax(\tau) + Bu_0(\tau)\right) d\tau,$$

- $G \in \mathbb{R}^{m\times n}$ satisfies $\det GB \neq 0$.
- $\dot{\sigma}(x) = GB(u_1 + \xi)$

Linear case

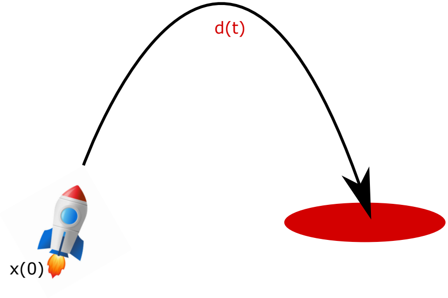
Control Design

- $$u_1 = -\rho(t,x)\frac{(GB)^T\sigma}{\|(GB)^T\sigma\|}, \quad \rho(t,x) > \|\xi^+(t,x)\|$$
- $V = \frac{1}{2}\sigma^T\sigma$,
- $$\begin{aligned} \dot{V} &= \sigma^T GB(u_1 + \gamma) \\ &\leq -\|(GB)^T\sigma\|(\rho(t,x) - \xi^+) < 0. \end{aligned}$$
- Integral sliding mode is guaranteed.

Robust Trajectory Tracking for the initial moment

V.I. Utkin, J. Shi, Integral sliding mode in systems operating under uncertainty conditions, in Proceedings of the 35th IEEE-CDC, Kobe, Japan, 1996

- Robust trajectory tracking for the initial time moment
- Initial conditions $x(0)$ are known $\Longrightarrow$ ISM starts from the initial moment



Linearized model of an inverted pendulum on a cart

$$\dot{x} = Ax + B\left(u_0 + u_1\right) + \phi, \qquad \text{(InvPen)}$$

- x_1 and x_2 are the cart and pendulum positions respectively
- x_3 and x_4 respective velocities.
- $$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.25 & 0 & 0 \\ 0 & 7.55 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0.19 \\ 0.14 \end{bmatrix},$$

Linearized model of an inverted pendulum on a cart

- $u_0 = u_0^*$ minimizes

$$J(u_0) = \int_0^\infty x_0^T(t)\, Q x_0(t) + u_0^T(t)\, R u_0(t)\, dt$$

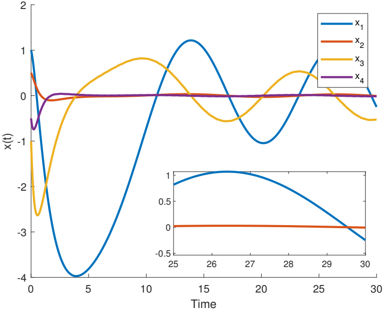
- $u_0^*(x) = -R^{-1}B^T P x = -Kx$,
- P symmetric positive definite matrix solution of

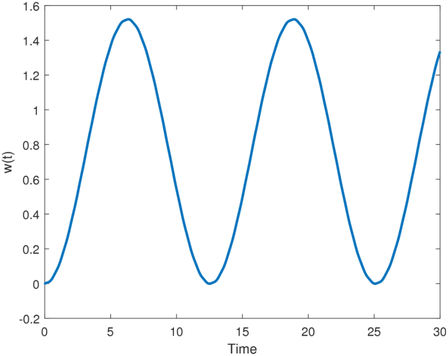
$$A^T P + PA - PBR^{-1}B^T P = -Q$$

- $Q = I$ and $R = 1$,
- Simulation: $\phi = B\xi$ with $\xi = 2\sin(0.5t) + 0.1\cos(10t)$
- $G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

Linearized model of an inverted pendulum on a cart

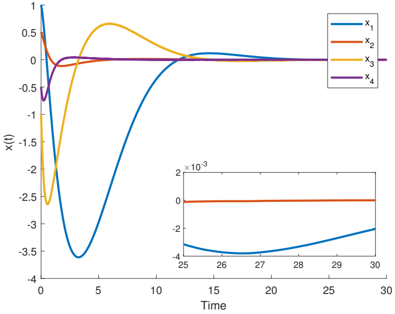
Without unit control in presence of matched disturbances

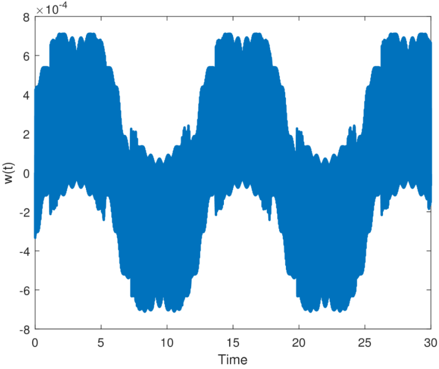




Linearized model of an inverted pendulum on a cart

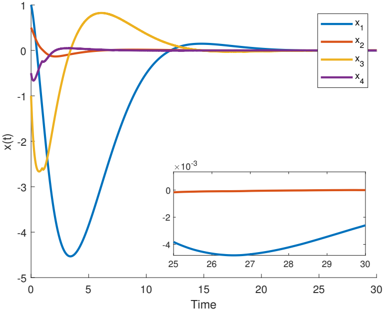
ISM compensation in presence of matched disturbances

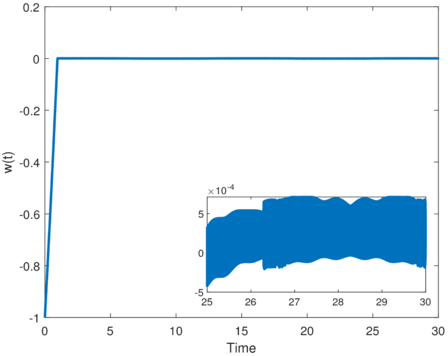




Linearized model of an inverted pendulum on a cart

ISM compensation in presence of matched disturbances, IC unknown in ISM





ISM compensation of unmatched disturbances

Analysis and design of integral sliding manifolds for systems with unmatched perturbations. IEEE Transactions on Automatic Control 2006, 51(5),pp. 853 - 858

Dr. Fernando Castaños Luna y Dr. Leonid Fridman

ISM compensation of unmatched disturbances

$$\dot{x} = f(x) + B(x)u + \phi(x,t), \tag{2}$$

Main Idea

- Combination of ISM and other robust techniques.
- Ensure the compensation
  1. Does not amplify unmatched disturbances.
  2. Minimizes the effect of the unmatched disturbances.

Assumptions

A1) $\mathrm{rank} B(x) = m$ for all $x \in \mathbb{R}^n$;

A2) $\phi(x,t)$ is not assumed matched

A3)

$$\|\phi(x,t)\| \leq \phi^{+}(x,t).$$

ISM compensation of unmatched disturbances

It is convenient to project $\phi(x,t)$ into matched and unmatched spaces.

Projection

- $B^{\perp} \in \mathbb{R}^{n\times(n-m)}$ full rank matrix.
- $B^T B^{\perp} = 0$.
- $\begin{bmatrix} B & B^{\perp} \end{bmatrix}$ is nonsingular.
- $\operatorname{rank}(I - BB^{+}) = n - m$
- $\xi(x,t) \in \mathbb{R}^m$ and $\mu(x,t) \in \mathbb{R}^{n-m}$

$$\begin{bmatrix} \xi(x,t) \\ \mu(x,t) \end{bmatrix} = \begin{bmatrix} B & B^{\perp} \end{bmatrix}^{-1} \phi(x,t).$$

- System:

$$\dot{x} = Ax + B(u_1 + u_0) + B\xi + B^{\perp}\mu. \tag{M\&UUnc}$$

ISM compensation of unmatched disturbances

Sliding Dynamics

- $$\sigma(x) = G(x(t) - x(0)) - G\int_0^t (Ax(\tau) + Bu_0(\tau))\, d\tau,$$
- $$\dot{\sigma} = GB(u_1 + \xi) + GB^{\perp}\mu.$$

Equivalent Control

- Assume GB is nonsingular.
- $\rho \geq \xi^{+} + (GB)^{+} GB^{\perp}\mu.$
- By solving $\dot{\sigma} = 0$

$$u_{1_{\mathrm{eq}}} = -\xi - (GB)^{-1} GB^{\perp}\mu.$$

ISM compensation of unmatched disturbances

Sliding Motion Equation

- $$\dot{x} = Ax + Bu_0 + (I - B\,(GB)^{-1}\,G)B^{\perp}\mu.$$
- T
$$\bar{d} = (I - B\,(GB)^{-1}\,G)$$

Is it possible to select G to ensure $\|\bar{d}\| < \|B^{\perp}\mu\|$?

ISM compensation of unmatched disturbances

Proposition

The set of matrices $\left\{G = QB^T : \; Q \in \mathbb{R}^{m\times m} \text{ and } \det Q \neq 0\right\}$ is the solution of the optimization problem

$$G^* = \arg\min_{G\in\bar{G}} \|(I - B\left(GB\right)^{-1} G)B^{\perp}\mu, \mu \neq 0\|$$

where $\bar{G} = \{G \in \mathbb{R}^{m\times n} : \det GB \neq 0\}$.

ISM compensation of unmatched disturbances

Proof

- Since $B^{\perp}\mu$ and $B\,(GB)^{-1}\,GB^{\perp}\mu$ are orthogonal vectors.

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\| \geq \|B^{\perp}\mu\|$$

- Indeed,

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\|^2 = \|B^{\perp}\mu\|^2 + \|B\,(GB)^{-1}\,GB^{\perp}\mu\|^2$$

- That is,

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\| \geq \|B^{\perp}\mu\|$$

ISM compensation of unmatched disturbances

Proof

- If $\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\| = \|B^{\perp}\mu\|$, then

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\|$$

  is minimized with respect to G.
- Identity is obtained, if and only if
  1. $B\,(GB)^{-1}\,GB^{\perp}\mu = 0$.
  2. Since rank$B = m$, $GB^{\perp}\mu = 0$, i.e. $G = QB^{T}$, where Q is nonsingular.

Proposition

For an optimal matrix G^*, the Euclidean norm of the disturbance is not amplified, that is,

$$\|\phi(t)\| \geq \|(I - B(G^* B)^{-1} G^*) B^{\perp} \mu(t)\| \qquad \text{(P2)}$$

Proof

- From last proposition

$$\|(I - B(G^* B)^{-1} G^*) B^{\perp} \mu(t)\| = \|(I - BB^{+}) B^{\perp} \mu(t)\| = \|B^{\perp} \mu(t)\|$$

- Since $\phi(t) = B\xi + B^{\perp}\mu$, and $B^T B^{\perp} = 0$,

$$\|\phi(t)\|^2 = \|B\xi(t) + B^{\perp}\mu(t)\|^2 = \|B\xi(t)\|^2 + \|B\mu(t)\|^2 \geq \|B\mu(t)\|^2$$

- Comparing this two equations we obtain (P2).

Remark

- The control law itself is not modified to optimize the effect of the unmatched uncertainties
- The optimal solution for $G^* = B^+$ is simple.

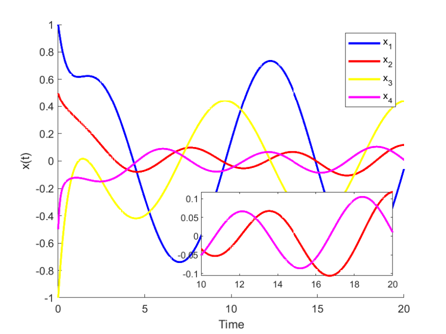
Linearized model of an inverted pendulum on a cart

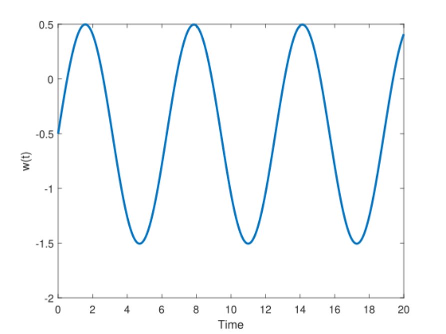
- $$\phi(x,t) = \begin{bmatrix} 0 & 0 & 2\sin(0.5t) + 0.1\cos(10t) & 0.1\sin(1.4t) \end{bmatrix}^T$$
- Project the perturbation $\phi(x,t)$
- Selecting

$$B^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.14 \\ 0 & 0 & 0.19 \end{bmatrix}.$$

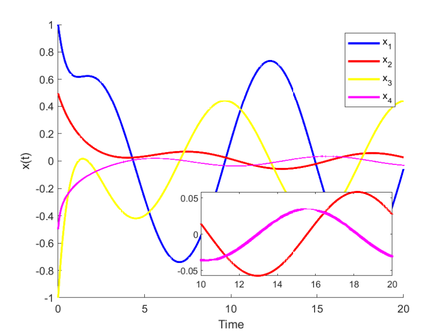
$$\begin{aligned} \xi &= 3.41(2\sin 0.5t + 0.1\cos 10t) + 2.51(0.1\sin 1.4t), \\ \mu &= \begin{bmatrix} 0 \\ 0 \\ 3.41\,[0.1\sin 1.4t] - 2.51\,[2\sin 0.5t + 0.1\cos 10t] \end{bmatrix}. \end{aligned}$$

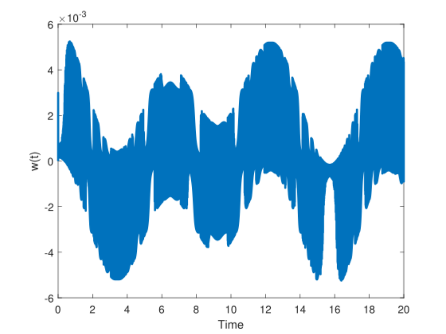
Case 1) No ISM



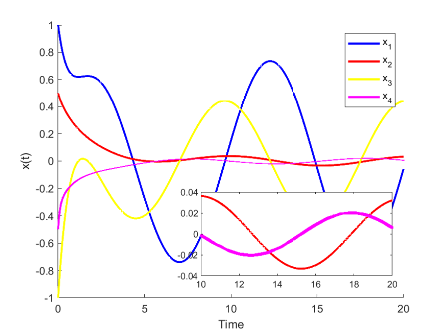


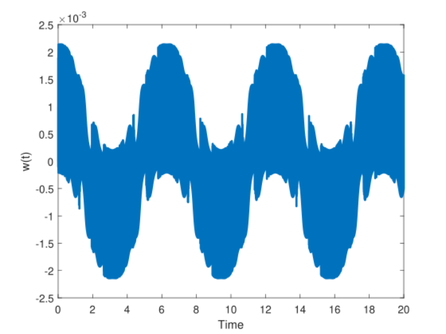
Case 2) Arbitrary G

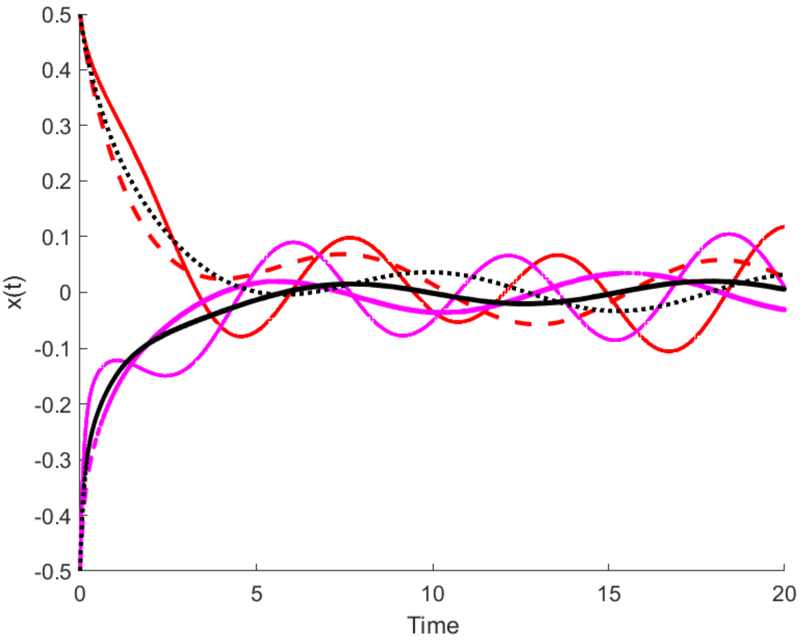




Case 3) $G = B^{+}$.







0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
x(t)
0
5
10
15
20
Time

The basic differential inclusion

Defining $x = (x_1, \dots, x_\rho)^T = (\sigma, \dot{\sigma}, \dots, \sigma^{(\rho-1)})^T, \sigma^{(i)} = \frac{d}{dt} h(z,t)$

- The regular form

$$\Sigma_T : \begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \dots, \rho - 1, \\ \dot{x}_\rho = w(t,z) + b(t,z)u, & x_0 = x(0), \\ \dot{\zeta} = \phi(\zeta, x) & \zeta_0 = \zeta(0), \end{cases}$$

$0 < K_m \le b(t,z) \le K_M, \; |w(t,z)| \le C.$

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- Reduced Dynamics Asymptotically stable, i.e., $\dot{\zeta} = \phi(\zeta, 0) \; \zeta_0 = \zeta(0),$

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- Reduced Dynamics Asymptotically stable, i.e., $\dot{\zeta} = \phi(\zeta, 0) \ \zeta_0 = \zeta(0),$
- The basic Differential Inclusion (DI)

$$\Sigma_{DI} : \begin{cases} \dot{x}_i = x_{i+1}, \ i = 1, \dots, \rho - 1, \\ \dot{x}_\rho \in [-C, C] + [k_m, K_M]u, \end{cases}$$

The basic differential inclusion

If x_1 is a plain output then the zero dynamics does not exists.
Defining $x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T, \sigma^{(i)} = \frac{d}{dt} h(z, t)$

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