Main Ideas of Second Order Sliding Mode Control

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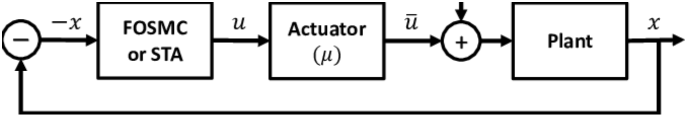
Idea of SOSMC.

Idea

Substitute a discontinuous control with a continuous one.

Presence of actuators.

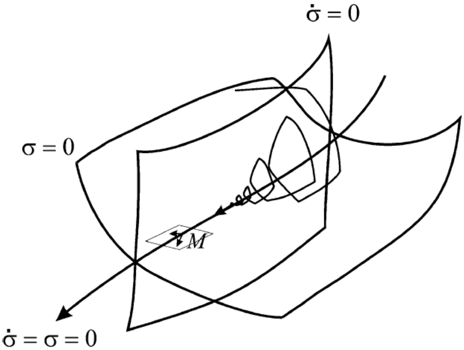
Presence of actuator is growing the relative degree of the system.



Idea of SOSM.

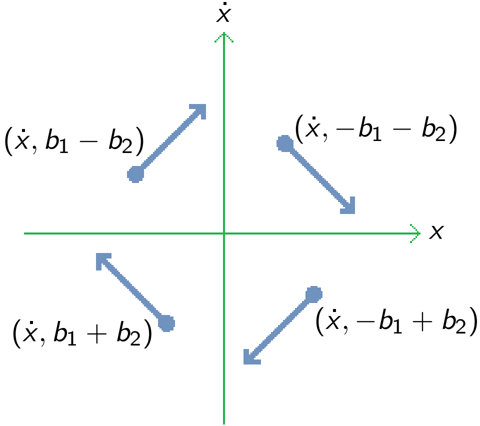
Main Idea

The idea is to reach $\sigma = 0$ and $\dot{\sigma} = 0$.



Twisting phase plane

$$\ddot{x} = -b_1 \mathrm{sign}(x) - b_2 \mathrm{sign}(\dot{x}), \quad b_1 > b_2 > 0$$



Twisting Algorithm Convergence

To describe the system behavior in the phase plane is convenient

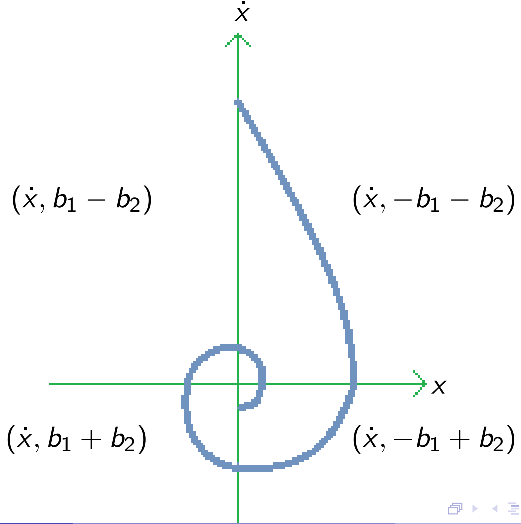
$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{d\dot{x}}{dx}\dot{x} = -b_1 \mathrm{sign}(\mathrm{x}) - b_2 \mathrm{sign}(\dot{\mathrm{x}})$$

or

$$\frac{d\dot{x}}{dx} = \frac{-b_1 \mathrm{sign}(\mathrm{x}) - b_2 \mathrm{sign}(\dot{\mathrm{x}})}{\dot{x}}$$

Trajectories of Twisting Agorithm

$$\ddot{x} = -b_1 \mathrm{sign}(x) - b_2 \mathrm{sign}(\dot{x}), \quad b_1 > b_2 > 0$$



Twisting Algorithm Convergence

$$\dot{x}d\dot{x} = -(b_1 + b_2)dx$$

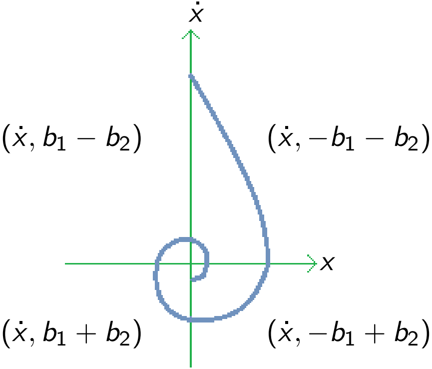
$$\frac{1}{2}\dot{x}^2 = -(b_1 + b_2)x - x_1$$

for $\dot{x} > 0$

$$x = x_1 - \frac{\dot{x}^2}{2(b_1 + b_2)}$$

for $\dot{x} \leq 0$

$$x = x_1 - \frac{\dot{x}^2}{2(b_1 - b_2)}$$



Twisting Algorithm Convergence

For $x = 0, \dot{x} = \dot{x}_0$

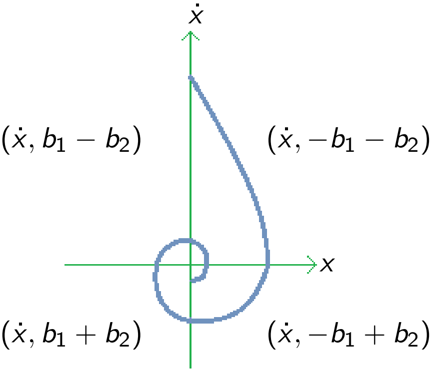
$$\begin{aligned} 0 =& x_1 - \frac{\dot{x}_0^2}{2(b_1 + b_2)} \\ \implies& \dot{x}_0^2 = 2(b_1 + b_2)x_1 \end{aligned}$$

For $x = 0, \dot{x} = \dot{x}_1$

$$\dot{x}_1^2 = 2(b_1 - b_2)x_1$$

Convergence rate

$$\frac{|\dot{x}_1|}{|\dot{x}_0|} = \sqrt{\frac{b_1 - b_2}{b_1 + b_2}} := q < 1$$

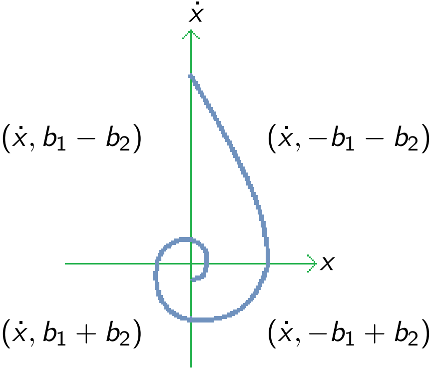


Twisting Algorithm Convergence

Extending the trajectory to $x < 0$ and using the same reasoning in successive crossing of $x = 0$ axis we obtained

$$\frac{|\dot{x}_{i+1}|}{|\dot{x}_i|} = q < 1$$

thus, the algorithm converge to origin. The real trajectory consist of infinite number of segments belonging to $x \geq 0$ and $x \leq 0$, the convergence time can be estimated.



Twisting Algorithm Convergence

$\dot{x} > 0, x > 0,$

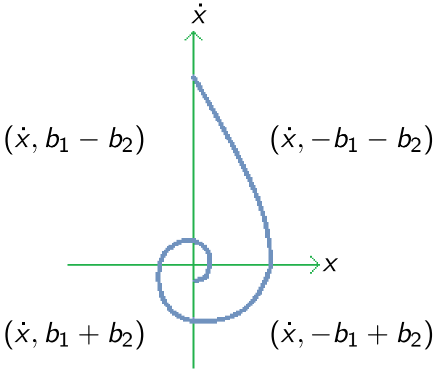
$\ddot{x} = -b_1 \mathrm{sign}(x) - b_2 \mathrm{sign}(\dot{x})$

$$\dot{x}(t) = -(b_1 + b_2)t + \dot{x}_0$$

$$\dot{x}(t_1^+) = 0 \implies t_1^+ = \tfrac{\dot{x}_0}{b_1+b_2}.$$

$$x(t) = -\frac{1}{2}(b_1 - b_2)t^2 + x_1$$

$$t_1^- = \sqrt{\frac{2x_1}{b_1 - b_2}}, \quad = \sqrt{\frac{1}{(b_1 - b_2)(b_1 + b_2)}}\dot{x}_0$$



Twisting Algorithm Convergence

Then the time

$$t_1 := t_1^+ + t_1^- = \eta \dot{x}_0, \ con\ \eta = \frac{1}{b_1 + b_2} + \sqrt{\frac{1}{(b_1 - b_2)(b_1 + b_2)}}$$

corresponds to the trajectory $\dot{x}_0 x_1 \dot{x}_1$. In the same way, the time

$$t_i = \eta |\dot{x}_{i-1}| = \eta q^{i-1} \dot{x}_0$$

corresponds to the trajectory $\dot{x}_{i-1} x_i \dot{x}_i$. In this case, the total convergence time is

$$T = \sum_{i=1}^{\infty} t_i = \sum_{i=1}^{\infty} \eta |\dot{x}_{i-1}| = \sum_{i=1}^{\infty} \eta q^{i-1} \dot{x}_0 = \frac{\eta \dot{x}_0}{1 - q}$$

Zeno phenomenon

Twisting Algorithm for Perturbed Systems

$$\ddot{x} = a(t,x) + b(t,x)u, \quad |a(t,x)| \le C, \quad 0 < K_m \le b(t,x) \le K_M,$$

The control

$$u = -b_1 \mathrm{sign}(\mathrm{x}) - b_2 \mathrm{sign}(\dot{\mathrm{x}}), \quad b_1 > b_2 > 0$$

Lemma

Let b_1 and b_2 satisfy the conditions

$$K_m(b_1 + b_2) - C > K_M(b_1 - b_2) + C, \; K_m(b_1 - b_2) > C.$$

Then, the controller u provides for the appearence of a 2-sliding mode $x = \dot{x} = 0$ attracting the trajectories of the system in finite time.

Twisting Algorithm for Perturbed Systems

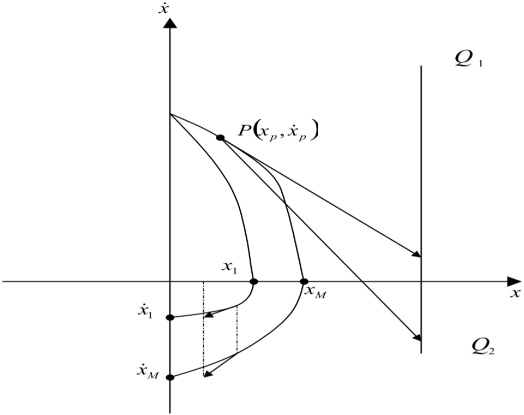


Figure: Majorant curves of the twisting controller.

Suboptimal Algorithm

Consider

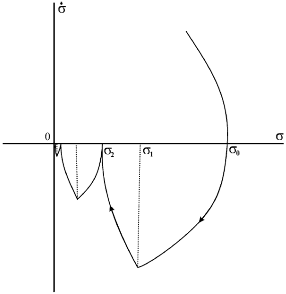
$$\ddot{\sigma} \in [-C, C] + [K_m, K_M]\, u,$$

The suboptimal controller is given by

$$u = r_1 \operatorname{sign}(\sigma - \sigma^*/2) + r_2 \operatorname{sign}(\sigma^*),$$
$$r_1 > r_2 > 0,$$

where

$$r_1 - r_2 > \frac{C}{K_m},$$
$$r_1 + r_2 > \frac{4C + K_M(r_1 - r_2)}{3K_m},$$



Suboptimal Algorithm

Figure: Bartolini's workgroup.

Terminal Sliding Mode Surface

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u(x),$$
$$u(x) = -\alpha \,\mathrm{sign}(s(x)),$$
$$s(x) = x_2 + \beta\sqrt{|x_1|}\,\mathrm{sign}(x_1).$$

Figure: Prof. Z. Man

Terminal Sliding Variable

Time derivative of the switching surface

$$\dot{s}(x) = \dot{x}_2 + \beta \frac{x_2}{2\sqrt{|x_1|}} = -\alpha\,\mathrm{sign}(s(x)) + \beta \frac{x_2}{2\sqrt{|x_1|}}.$$

- $s(x)$ is singular for $x_1 = 0$, and **the relative degree of the switching surface does not exist**
- On $x_2 = -\beta\sqrt{|x_1|}\,\mathrm{sign}(x_1)$

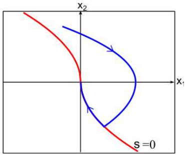
$$\dot{s} = -\alpha\,\mathrm{sign}(s(x)) - \frac{\beta^2}{2}\,\mathrm{sign}(x_1).$$

- Two types of behavior for the solution of the system are possible

Two types of behavior

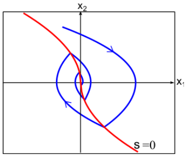
Terminal mode:

- $\beta^2 < 2\alpha$,
- Trajectories of the system reach the surface $s(x) = 0$ and remain there.



Twisting mode

- $\beta^2 > 2\alpha$
- Trajectories do not slide on the surface $s(x) = 0$



How to overcame the Singularity?

Singularity can be overcome by rewriting the function s

$$\bar{s}(x) = \beta^2 x_1 + x_2^2 \operatorname{sign}(x_2).$$

Quasi-Continuous Algorithm

This algorithm is given by

$$u = -\alpha \frac{\dot{\sigma} + \beta|\sigma|^{1/2}\mathrm{sign}(\sigma)}{|\dot{\sigma}| + \beta|\sigma|^{1/2}}$$

where

$$\alpha, \beta > 0, \quad \alpha K_m - C > 0,$$

and the inequality

$$\alpha K_m - C - 2\alpha K_m \frac{\beta}{\rho + \beta} - \frac{1}{2}\rho^2 > 0,$$

must be satisfied for some positive $\rho > \beta$.

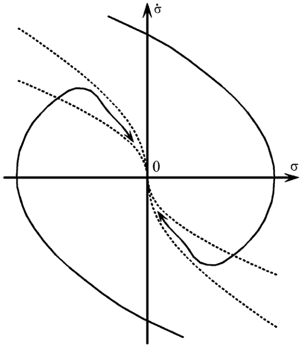


Figure: Trajectories of the quasi-continuous controller.

Anti-chattering Strategy

$$\dot{X} = F(t, X) + G(t, X)u, X \in R^n, u \in R, \; |F| < F^+,$$

The switching variable $\sigma(X)$: $\dot{\sigma} = f(\sigma, t) + g(\sigma, t)u.$

Anti-chattering strategy:

Add an Integrator in control input:

If $\dot{u} = v = -a\,\mathrm{sign}(\dot{\sigma}(t)) - b\,\mathrm{sign}(\sigma(t))$, so u is a Lipschitz continuous control signal ensuring finite-time convergence to $\sigma = 0$

Anti-chattering Strategy

Criticism(1987) If it is possible to measure $\dot{\sigma} = f(t,\sigma) + g(t,\sigma)u$, then the uncertainty $f(t,\sigma) = \dot{\sigma} - g(t,\sigma)u$ is also known and can be compensated without any discontinuous control!

Counter-argument

If g is uncertain so $\ddot{\sigma}$ depends on u through uncertainty! The anti-chattering strategy is reasonable for the case of uncertain control gains.

Discussion about SOSM

Advantages of SOSM

1. Allows to compensate bounded matched uncertainties for the systems with relative degree two with discontinuous control signal
2. Allows to compensate Lipschitz matched uncertainties with continuous control signal using the first derivative of sliding inputs
3. Ensures quadratic precision of convergence with respect to the sliding output
4. For one degree of freedom mechanical systems: the sliding surface design is no longer needed.
5. For systems with relative degree r: the order of the sliding dynamics is reduced up to $(r-2)$. The design of the sliding surface of order $(r-2)$ is still necessary!

OPEN PROBLEMS:EARLY 90th

- To reduce the chattering substituting **discontinuous control signal with continuous one** the derivative of the sliding input still needed!
- The problem of exact finite-time stabilization and exact disturbance compensation for SISO systems with arbitrary relative degree remains open. More deep decomposition is still needed
- Theoretically exact differentiators are needed to realize theoretically exact compensation of the Lipschitz matched uncertainties

The basic differential inclusion

Defining $x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T, \sigma^{(i)} = \frac{d}{dt}h(z,t)$

- The regular form

$$\Sigma_T : \begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \ldots, \rho - 1, \\ \dot{x}_\rho = w(t,z) + b(t,z)u, & x_0 = x(0), \\ \dot{\zeta} = \phi(\zeta, x) & \zeta_0 = \zeta(0), \end{cases}$$

$0 < K_m \leq b(t,z) \leq K_M, \ |w(t,z)| \leq C.$

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- Reduced Dynamics Asymptotically stable, i.e., $\dot{\zeta} = \phi(\zeta, 0) \ \zeta_0 = \zeta(0),$

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- The basic Differential Inclusion (DI)

$$\Sigma_{DI} : \begin{cases} \dot{x}_i = x_{i+1}, \ i = 1, \dots, \rho - 1, \\ \dot{x}_\rho \in [-C, C] + [k_m, K_M]u, \end{cases}$$

The basic differential inclusion

If x_1 is a plain output then the zero dynamics does not exists.
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