Equations with Discontinuous Right Hand Side

L. Fridman

Sirius

Outline

1. Preliminaries
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   - Upper semi-continuity
2. Equations with Discontinuous RHS
   - Historical Remarks
3. Regularization Procedure for ODE with Discontinuous RHS
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4. Disturbed Systems and Extended Differential Inclusions
   - Disturbances and Differential Inclusions
5. Existence of Solutions
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Outline

Absolute Continuity

Definition

Let $\mathcal{I}$ be an interval in the real line $\mathbb{R}$. A function $f : \mathcal{I} \to \mathbb{R}$ is absolutely continuous on $\mathcal{I}$ if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k; y_k)$ of $\mathcal{I}$ satisfies

$$\sum_k (y_k - x_k) < \delta$$

then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon$$

The collection of all absolutely continuous functions on $\mathcal{I}$ is denoted $AC(\mathcal{I})$.

Absolute continuity of functions

Equivalent Definitions

1. f is absolutely continuous
2. f has a Lebesgue integrable derivative f' almost everywhere and

$$f(x) = f(a) + \int_a^x f'(t)dt; \quad \forall x \in [a; b]$$

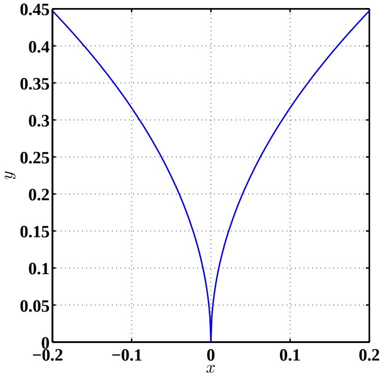
3. there exists a Lebesgue integrable function g on $[a; b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt; \quad \forall x \in [a; b]$$

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental

$$f(x) = \sqrt{|x|} = 2\int_0^x \frac{1}{\sqrt{|t|}}dt$$

At zero it is not differentiable and the lateral derivatives do not exist!



The function is still absolutely continuous

$$f(x) = x\sin\frac{1}{x}$$

$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0 \; \rightarrow \; f(x)$ is continuous !

$$f'(x) = \sin\frac{1}{x} - \frac{\cos(\frac{1}{x})}{x}, \;\; x \neq 0$$

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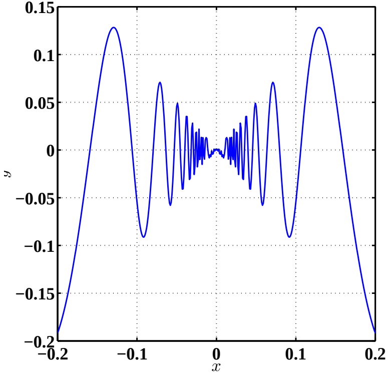

$$f(x) = x^2 sin\frac{1}{x}$$

$$\lim_{x\to 0} x^2 sin(\tfrac{1}{x}) = 0 \ \rightarrow \ f(x)$$ is continuous !

$$f'(x) = 2xsin\frac{1}{x} - cos(\frac{1}{x}), \ x \neq 0$$

At zero it is not differentiable and the lateral derivatives do not exist!

$f'(x)$ is bounded for $x \neq 0$!

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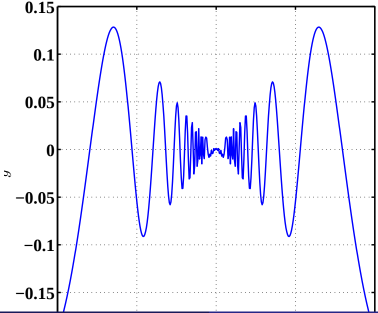

Absolute continuity of functions

Properties

1. If $f, g \in AC(\mathcal{I})$, then $f \pm g$ is absolutely continuous.
2. If $\mathcal{I}$ is a bounded closed interval and $f, g \in AC(\mathcal{I})$, then fg is also absolutely continuous.
3. If $\mathcal{I}$ is a bounded closed interval, $f \in AC(\mathcal{I})$ and $f \neq 0$ then $\frac{1}{f}$ is absolutely continuous.
4. Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
5. If $f : \mathcal{I} \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a; b]$.

Outline

Upper semi-continuity of set-valued functions

Introduce the following distances

$$\rho(x, \mathbf{M}) = \inf_{y \in \mathbf{M}} \|x - y\|, \quad x \in \mathbb{R}^n, \mathbf{M} \subset \mathbb{R}^n,$$

$$\rho(\mathbf{M}_1, \mathbf{M}_2) = \sup_{x \in \mathbf{M}_1} \rho(x, \mathbf{M}_2), \quad \mathbf{M}_1 \subset \mathbb{R}^n, \mathbf{M}_2 \subset \mathbb{R}^n,$$

In general, the distance ρ is not symmetric, $\rho(\mathbf{M}_1, \mathbf{M}_2) \neq \rho(\mathbf{M}_2, \mathbf{M}_1)$.

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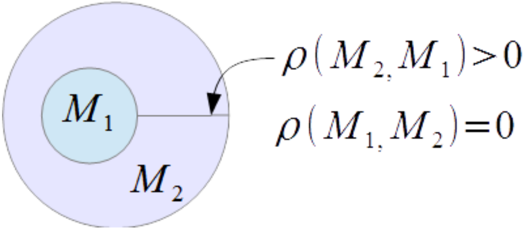


Definition

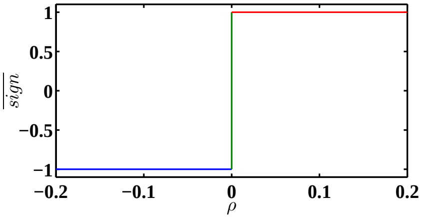
A set-valued function $F : R^{n+1} \to 2^{R^{n+1}}$ is said to be **upper semi-continuous** at a point $(t^*, x^*) \in \mathbb{R}^{n+1}$ if $(t, x) \to (t^*, x^*)$ implies

$$\rho(F(t, x), F(t^*, x^*)) \to 0.$$

Example (Upper semi-continuous set-valued function)

$$\overline{\text{sign}}[\rho] = \begin{cases} 1 & \text{if } \rho > 0 \\ -1 & \text{if } \rho < 0 \\ [-1, 1] & \text{if } \rho = 0 \end{cases}$$

is an upper semi-continuous set-valued function.



Outline

Historical Remarks

Differential Equations with Discontinuous RHS

$$\dot{x}(t) = f(t, x(t)),$$
$$t \in \mathbb{R},\ x \in \mathbb{R}^n,\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

- RHS Discontinuous with respect to the **time** variable (Caratheodory 1927)

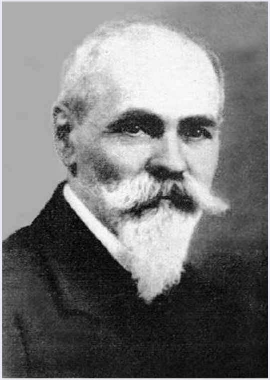
Constantin Caratheodory (1873-1950)

Differential Inclusions (Contingent Differential Equations)

$$\dot{x}(t) \in F(t, x(t)),$$

$$t \in \mathbb{R},\ \ x \in \mathbb{R}^n,\ \ F : \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$$

(Zaremba 1936, Marchaud 1938, Filippov 1960



Stanislaw Zaremba (1863-1942)

Differential Equations with Discontinuous RHS

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

- RHS Discontinuous with respect to the **state** variable (Filippov 1960, Utkin 1967, Aizerman & Pyatnitskii 1974)




Professors A. Filippov, E. Pyatnitskii, M. Aizerman and V. Utkin

Outline

ODE with Discontinuous RHS

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}, \; x \in \mathbb{R}^n, \; f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \qquad \text{(DiscRHS)}$$

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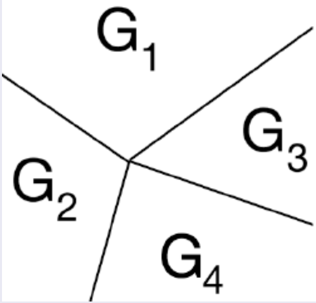
f is **piecewise continuous:**

1. $\mathbb{R}^{n+1} = \bigcup_{j=1}^{N} \bar{G}_j$, where G_j-open connected sets $G_i \cap G_j \neq \emptyset$, $i \neq j$;
2. $\mathcal{S} \bigcup_{j=1}^{N} \partial G_j$ is of measure zero;
3. $f(t, x)$ is continuous in each G_j and $\forall (t, x) \in \partial G_j : \exists f^j(t, x) \in \mathbb{R}^n$

$$f^j = \lim_{(t^k, x^k) \to (t, x)} f(t^k, x^k),$$

$$(t^k, x^k) \in G_j, \quad (t, x) \in \partial G_j$$



Filippov Regularization

$$\dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R} \tag{DiffInc}$$

$$F(t,x) = \begin{cases} \{f(t,x)\} & \text{if } (t,x) \in \mathbb{R}^{n+1} \backslash \mathcal{S}, \\ \mathrm{co}\left(\bigcup\limits_{j \in \mathcal{N}(t,x)} \{f^j(t,x)\} \right) & \text{if } (t,x) \in \mathcal{S}, \end{cases}$$

$$\mathcal{N}(t,x) = \{j \in \{1, 2, \ldots, N\} : (t,x) \in \partial G_j\}.$$

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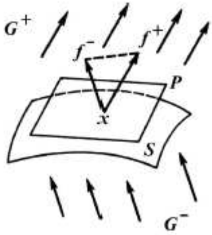
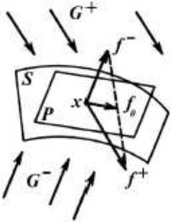
Definition (Filippov 1960)

An absolutely continuous function $x : \mathcal{I} \to \mathbb{R}^n$ defined on some interval or segment $\mathcal{I}$ is called a solution of (DiscRHS) if it satisfies the differential inclusion (DiffInc) almost everywhere on $\mathcal{I}$.

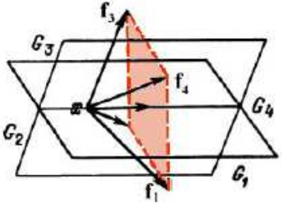
Illustration of Filippov regularization



(a) Switching case.



(b) Sliding mode.



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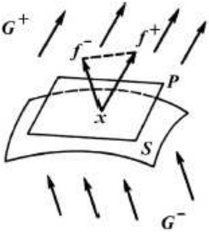

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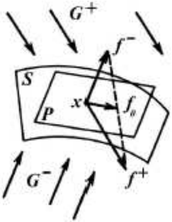

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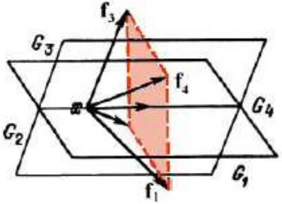

Example

$$\dot{x}(t) = -\operatorname{sign}[x(t)] + d(t), t > 0,$$

where $x(\cdot) \in \mathbb{R}, \ \|d\|_{\mathbb{C}} \leq d_0 < 1.$

$$\operatorname{sign}[\rho] = \begin{cases} 1 & \text{if } \rho > 0 \\ -1 & \text{if } \rho < 0 \\ 0 & \text{if } \rho = 0 \end{cases}$$

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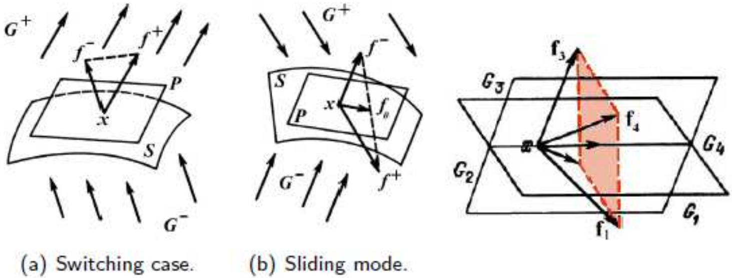

Example

$$\dot{x}(t) \in -\overline{\text{sign}}[x(t)] + d(t), t > 0,$$

where $x(\cdot) \in \mathbb{R}$, $\|d\|_{\mathbb{C}} \leq d_0 < 1$.

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Outline

Discontinuous Control Systems

Let us consider the system

$$\dot{x}(t) = f(t, x(t), u(t, x(t))), t \in \mathbb{R}, \qquad \text{(DisContSys)}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $f \in \mathbb{C}$ and

$$u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m, \quad u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x))^T$$

is a **piecewise continuous** feedback control

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Assumption

Each component $u_i(t, x)$ is discontinuous only on a surface

$$\mathcal{S}_i = \{(t, x) \in \mathbb{R}^n : s_i(t, x) = 0\},$$

where functions $s_i : \mathbb{R}^{n+1} \to \mathbb{R}$ are smooth, i.e. $s_i \in \mathbb{C}^1(\mathbb{R}^{n+1})$.

Utkin Regularization

$$\dot{x}(t) = f(t, x(t), U(t, x(t))), t \in \mathbb{R},$$

where $U(t,x) = (U_1(t,x), U_2(t,x) \dots, U_m(t,x))^T$ and

$$U_i(t,x) = \begin{cases} \{u_i(t,x)\}, & s_i(t,x) \neq 0 \\ \operatorname{co}\left\{ \lim\limits_{\substack{(t_j,x_j)\to(t,x) \\ s_i(t_j,x_j)>0}} u_i(t_j,x_j), \lim\limits_{\substack{(t_j,x_j)\to(t,x) \\ s_i(t_j,x_j)<0}} u_i(t_j,x_j) \right\}, & s_i(t,x) = 0 \end{cases} \tag{ValFunc}$$

The set $f(t,x,U(t,x))$ is **non-convex** in general case.

Example (Utkin Regularization)

$$u(x) = -\operatorname{sign}[x] \quad \text{and} \quad U(x) = \overline{\operatorname{sign}}[x]$$

$$\operatorname{sign}[\rho] = \begin{cases} 1 & \text{if } \rho > 0 \\ -1 & \text{if } \rho < 0 \\ 0 & \text{if } \rho = 0 \end{cases}, \quad \overline{\operatorname{sign}}[\rho] = \begin{cases} 1 & \text{if } \rho > 0 \\ -1 & \text{if } \rho < 0 \\ [-1, 1] & \text{if } \rho = 0 \end{cases}$$

Equivalent Control (Utkin Solution)

Definition

An absolutely continuous function $x : \mathcal{I} \to \mathbb{R}^n$ defined on some interval or segment $\mathcal{I}$ is called a solution of (DisContSys) if there exists a measurable function $u_{eq} : \mathcal{I} \to \mathbb{R}^m$ such that $u_{eq}(t) \in U(t, x(t))$ and $\dot{x}(t) = f(t, x(t), u_{eq}(t))$ almost everywhere on $\mathcal{I}$.

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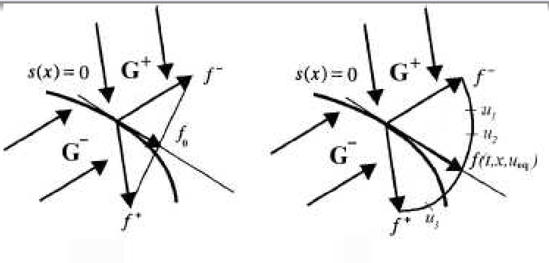

(a) Filippov definition. (b) Utkin definition.

Equivalent control (Utkin 1967): $s(x) = 0$ and $\frac{\partial s(x)}{\partial x} f(t, x, u_{eq}) = 0$

Example (Equivalent Control)

$$\dot{x}_1 = u$$
$$\dot{x}_2 = (2u^2 - 1)x_2$$

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Stable

Outline

Aizerman-Pyatniskii Regularization (Filippov 1988)

$$\dot{x} \in \mathrm{co}(f(t, x, U(t, x)), \quad t \in \mathbb{R}$$

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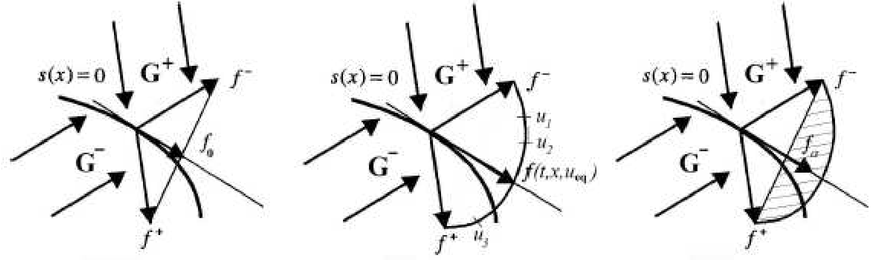

(a) Filippov (b) Utkin (c) Aizerman-Pyatnitskii

Example (Utkin 1970s)

$$\dot{x}_1 = 0.3x_2(t) + x_1(t)u(t), \qquad u(t) = -\operatorname{sign}[x_1(t)s(t)],$$
$$\dot{x}_2 = -0.7x_1(t) + 4x_1^3(t)u(t), \qquad s(t) = x_1(t) + x_2(t),$$

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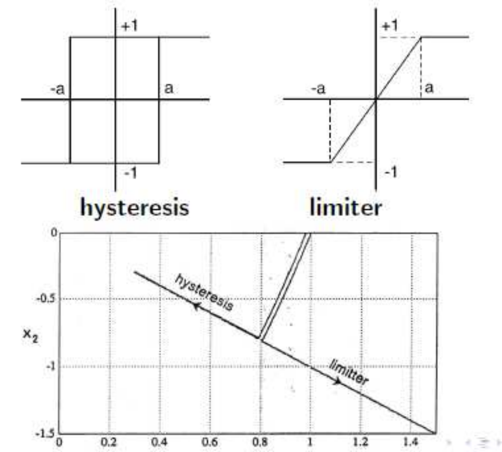

Presence of fast actuators (Fridman 2001,2002)

Actuators

2nd order actuator

1st order actuator

$$\mu \dot{z}_1 = -2z_1 - u(s),$$

$$\mu \dot{z}_1 = z_2,$$
$$\mu \dot{z}_2 = -2z_1 - 3z_2 - u(s),$$

Plant

$$\dot{s} = z_1, \quad \dot{x} = z_1^4 - z_1^2 + \beta x, \quad u(s) = \mathrm{sign}[s],$$

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Reduced Order System

$$\mu = 0 \Rightarrow z_1 = -u(s)/2, \quad \dot{s} = -u/2, \quad \dot{x} = (u^4 - u^2 + \beta)x,$$

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Sliding Dynamics (Filippov=Utkin)

$$\dot{x} = \beta x$$

1 st order actuator $(z(t), s(t)) \to 0$ Sliding dynamics $\dot{x} = \beta x \Rightarrow$ Unstable

2nd order actuator

$$\exists \left(z_0\left(\frac{t}{\mu}\right), s_0\left(\frac{t}{\mu}\right) \right) - \text{Periodic Solution}$$

$$\exists \bar{\beta}(\mu) : \forall \beta < \bar{\beta}(\mu) \exists \gamma :$$

$$-\gamma = \int_0^T \left[(2z_1(\tau))^4 - (2z_1(\tau))^2 \right] d\tau$$

$$\Rightarrow \dot{x} = -(\gamma - \beta)x$$

Could be stable

Equivalence of Definitions

Theorem (Utkin 1992, Zolezzi 2002)

Let a right-hand side of the system (DiscRHS) *be affine with respect to control:*

$$f(t,x,u(t,x)) = a(t,x) + b(t,x)u(t,x),$$

where $a : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $b : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times m}$, $a, b \in \mathbb{C}$ *and* $u : \mathbb{R}^{n+1} \to \mathbb{R}^m$ *is a piecewise continuous function* $u(t,x) = (u_1(t,x), \ldots, u_m(t,x))^T$, *such that* u_i *has a unique switching surface* $s_i(x) = 0$, $s_i \in \mathbb{C}^1(\mathbb{R}^n)$.

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Definitions of Filippov, Utkin and Aizerman-Pyatnitskii are equivalent iff

$$\det\left(\nabla^T s(x) b(t,x)\right) \neq 0 \;\; \text{if} \;\; (t,x) \in \mathcal{S},$$

where $s(x) = (s_1(x), s_2(x), ..., s_m(x))^T$, $\nabla s(x) \in R^{n\times m}$ *is the matrix of partial derivatives* $\frac{\partial s_i}{\partial x_i}$ *and* $\mathcal{S}$ *is a discontinuity set of* $u(t,x)$.

Example (Neimark 1961)

$$\dot{x} = Ax(t) + cu_1(t) + bu_2(t), \qquad u_1(t) = -\operatorname{sign}[x_1(t)],$$
$$t > 0, \quad x(\cdot) = (x_1(\cdot), x_2(\cdot))^T \in \mathbb{R}^2, \qquad u_2(t) = -\operatorname{sign}[x_1(t)],$$
$$A \in \mathbb{R}^{2\times 2}, \; b = (0,1)^T, \qquad c = (1,0)^T,$$

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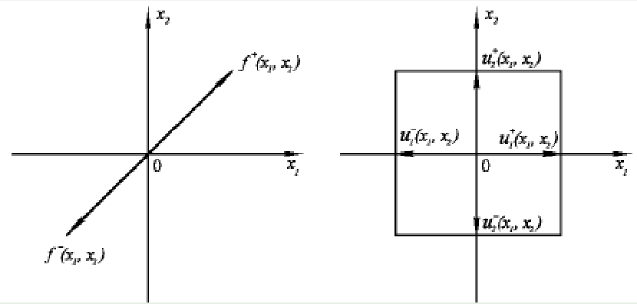
Filippov definition

$$\dot{x} \in \{Ax\} \dot{+} (b + c) \cdot \overline{\operatorname{sign}}[x_1]$$

Utkin definition

$$\dot{x} \in \{Ax\} \dot{+} b \cdot \overline{\operatorname{sign}}[x_1] + c \cdot \overline{\operatorname{sign}}[x_1].$$



Outline

Disturbances and Differential Inclusions

Models of sliding mode control systems usually have the form

$$\dot{x}(t) = f(t, x(t), u(t, x(t)), d(t)), \quad t \in \mathbb{R},$$

- $x(\cdot) \in \mathbb{R}^n$ is the vector of system states,
- $u(\cdot, \cdot) \in \mathbb{R}^m$ is the vector of control inputs,
- $d(\cdot) \in R^k$ is the vector of disturbances,
- the function $f : \mathbb{R}^{n+m+k+1} \to R^n$ is assumed to be continuous,
- the control function $u : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is piecewise continuous,
- the vector-valued function $d : R \to \mathbb{R}^k$ is assumed to be locally measurable and bounded as follows:

$$d_i^{\min} \leq d_i(t) \leq d_i^{\max}$$

where $d(t) = (d_1(t), d_2(t), \ldots, d_k(t))^T$, $t \in \mathbb{R}$.

Example (Disturbed sliding mode system)

Consider the simplest disturbed sliding mode system

$$\dot{x}(t) = -d_1(t)\,\mathrm{sign}[x(t)] + d_2(t), \tag{Ex1}$$

where $x \in \mathbb{R}$, unknown functions $d_i : \mathbb{R} \to \mathbb{R}$ are bounded by

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Obviously, all solutions of the system (Ex1) belong to a solution set of the following extended differential inclusion

$$\dot{x}(t) \in -\left[d_1^{\min}, d_1^{\max}\right] \cdot \overline{\mathrm{sign}}[x(t)] + \left[d_2^{\min}, d_2^{\max}\right]. \tag{Ex2}$$

Stability of the system (Ex2) implies the same property for (Ex1). In particular, for $d_1^{\min} > \max\{|d_2^{\min}|, |d_2^{\max}|\}$ both these systems have **asymptotically stable origin**.

Extended Differential Inclusion

All further considerations deal with the extended differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R},$$

where

$$F(t, x) = \mathrm{co}\{f(t, x, U(t, x), D)\},$$

the set-valued function $U(t, x)$ is defined by (ValFunc) and

$$D = \begin{pmatrix} \left[d_1^{\min}, d_1^{\max}\right] \\ \left[d_2^{\min}, d_2^{\max}\right] \\ \vdots \\ \left[d_k^{\min}, d_k^{\max}\right] \end{pmatrix}$$

Outline

Local existence conditions

Theorem (Filippov 1960)

Let

- $F : \mathbf{G} \to 2^{R^n}$ *be* **upper semi-continuous** *at each point of the set*

$$\mathbf{G} = \{(t,x) \in \mathbb{R}^{n+1} : |t - t_0| \leq a \text{ and } \|x - x_0\| < b,$$

  where $a, b \in \mathbb{R}_+$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$*;*

- $F(t,x)$ *be* **nonempty, compact and convex** *for* $(t,x) \in \mathbf{G}$*;*
- *there exists* $K > 0$ *such that* $\rho(0, F(t,x)) < K$ *for* $(t,x) \in \mathbf{G}$*;*

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- $F(t,x)$ *be* **nonempty, compact and convex** *for* $(t,x) \in \mathbf{G}$;
- *there exists* $K > 0$ *such that* $\rho(0, F(t,x)) < K$ *for* $(t,x) \in \mathbf{G}$;

then $\exists x : \mathbb{R} \to \mathbb{R}^n$ - **absolutely continuous** *and defined at least on* $[t_0 - \alpha, t_0 + \alpha]$, $\alpha = \min\{a, b/K\}$, *such that* $x(t_0) = x_0$ *and the inclusion*

$$\dot{x}(t) \in F(t, x(t))$$

holds almost everywhere on $[t_0 - \alpha, t_0 + \alpha]$.

On existence of Utkin Solutions

Lemma (Filippov 1959)

Let

- *a function* $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ *be continuous;*
- *a set-valued function* $U : \mathbb{R}^{n+1} \to 2^{R^m}$ *be defined and upper-semicontinuous on an open set* $\mathcal{I} \times \Omega$, *where* $\Omega \subseteq \mathbb{R}^n$*;*
- $U(t, x)$ *be nonempty, compact and convex for every* $(t, x) \in \mathcal{I} \times \Omega$.
- *a function* $x : \mathbb{R} \to \mathbb{R}^n$ *be absolutely continuous on* $\mathcal{I}$, $x(t) \in \Omega$ *for* $t \in \mathcal{I}$,
- $\dot{x}(t) \in f(t, x(t), U(t, x(t)))$ *almost everywhere on* $\mathcal{I}$*;*

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Then there exists a measurable function $u_{eq} : R \to \mathbb{R}^m$ *such that*

$$u_{eq}(t) \in U(t, x(t)) \quad \text{and} \quad \dot{x}(t) = f(t, x(t), u_{eq}(t))$$

almost everywhere on $\mathcal{I}$.

Non-local existence conditions

Theorem (Gelig et al. 1978)

Let a set-valued function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be defined and upper-semicontinuous in* $\mathbb{R}^{n+1}$.
Let $F(t, x)$ *be nonempty, compact and convex for any* $(t, x) \in \mathbb{R}^{n+1}$.
If there exists a real valued function $L : \mathbb{R}_+ \cup \{0\} \to R_+ \cup \{0\}$ *such that*

$$\rho(0, F(t, x)) \leq L(\|x\|) \quad \textit{and} \quad \int_0^{+\infty} \frac{1}{L(r)} dr = +\infty,$$

then for any $(t_0, x_0) \in \mathbb{R}^{n+1}$ *the system* (DiffInc) *has a solution* $x(t) : x(t_0) = x_0$ *defined for all* $t \in R$.

Summary

- Stability property of ODE with discontinuous RHS depends on definition of a solution.
- Stability of Aizerman-Pyatnitskii solutions always implies stability of Filippov and Utkin solutions.
- All introduced definitions may be equivalent in the case of affine control systems with discontinuous input.
- Analysis of the disturbed systems can be reduced to differential inclusions.