Nonlinear Dynamics in Physical Models: Simple Feedback-Loop Systems and Properties
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Source: *Computer Music Journal*, Vol. 23, No. 3, Recent Research at IRCAM (Autumn, 1999), pp. 18-34
Published by: The MIT Press
Stable URL: https://www.jstor.org/stable/3681237

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Nonlinear Dynamics in Physical Models: Simple Feedback-Loop Systems and Properties

This work establishes a new approach to the functioning of musical instruments that is oriented toward sound synthesis by computer, and its application to musical creation and composition. In this work, we only consider musical instruments that produce sustained sound. Our approach, which relies on the theory of nonlinear dynamical systems, provides theoretical results regarding instruments, their models, a class of equations with delay, and sound synthesis itself. Experimental and practical results open new sonic possibilities in terms of sound material and the control of sound synthesis, both of which are important to performers and composers of contemporary music. Out of these results, new possibilities also arise for the control of chaotic sounds and the control of the proportion of nonperiodic components introduced in sound by chaotic behavior. Therefore, new artificial instruments can be designed that fulfill the fundamental properties of a musical instrument: richness of the sonic space, expressivity, flexibility, predictability, and ease of control of sonic results. A model of brass instruments simulated on a workstation and played in real time which exemplifies these remarkable properties is presented in a second article in this issue, entitled "Nonlinear Dynamics in Physical Models: From Basic Models to True Musical-Instrument Models."

The notion of a "model" is essential for a better comprehension and use of the properties of sound analysis and synthesis methods. Many physical models of musical instruments have been proposed and studied by various authors (for an overview, see Smith's [1996] work, for instance). The approach described in this article is based on some advantages and difficulties specific to physical models. From a fundamental point of view, it appears that the complexity of physical models comes in part from their nonlinear nature. Therefore, their study should rely on the increasingly rich theory of nonlinear dynamical systems. However, developers should not merely build models and deliver them to musicians; rather, they should help musicians understand the models by conceiving abstractions of the models and offering useful explanations. Above all, it is necessary for the user to understand the structure of the space of instrumental sounds. In particular, this comprehension is indispensable for elaborating the control of synthesis models that are at the same time efficient and musically pertinent.

To fulfill the requirements mentioned above, we have developed models as archetypes, i.e., models that retain the essence of the behavior of a class of instruments while disregarding all details that are not useful for understanding what is typical of that class. The existence of delayed-feedback loops in the equations of the models is a characteristic common to instruments that have sustained sounds. This leads to a family of differential and integral delay equations that are particularly difficult, and are not well understood. To better introduce such systems, the first section presents Chua's circuit, one of the simplest examples of a dynamical system, but one that exhibits a rich behavior. By adding a delay element in a feedback loop, Chua's circuit is transformed into a remarkable musical system. The next section presents the now-classical feedback-loop formulation of physical models. Then, a basic feedback-loop formulation is proposed, and its dynamical behavior is approached in terms of stability and oscillation. The following section examines in greater detail the role of some nonlinearities, in particular,

Computer Music Journal, 23:3, pp. 18–34, Fall 1999

Figure 1. Chua's Circuit.

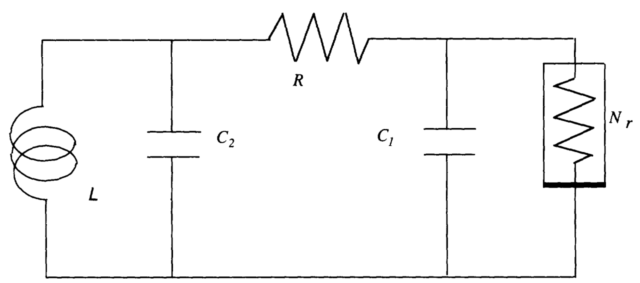
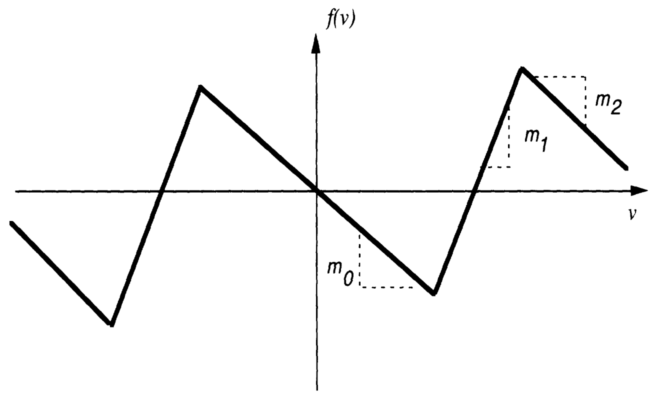


Figure 2. Chua's diode characteristic.



piecewise linear and rational functions. Finally, the class of single feedback-loop systems with a *memoryless* nonlinearity is defined in the last section, and precise stability and oscillation conditions are given. Other features relevant for the properties of sound produced by such systems are described as well. In particular, the control of the number of stable and unstable oscillating solutions is shown to be important, and some solution for this control is proposed.

Chua's Circuit and Time-Delayed Chua's Circuit

As a start, we present here Chua's circuit, one of the simplest dynamical systems, but one that is endowed with an extremely rich behavior (Chua 1992). However, we find that it is not well suited as a musical instrument. We then examine an extension of this dynamical system with a time-delay term, which by contrast leads to an extraordinary musical instrument. This is all the more interesting, because delayed-feedback loops are found in every sustained-sound instrument except the voice, as we will show in further sections.

Chua's Circuit

Chua's circuit is a very simple electronic circuit with remarkable properties with respect to nonlinear dynamical systems. Madan (1992) underlines these properties as follows: "Chua's circuit (Chua and Lin 1990) is endowed with such an unusually rich repertoire of nonlinear dynamical phenomena, including all of the standard bifurcations and routes to chaos, that it has become a *universal paradigm for chaos....*" It is also the minimum circuit—it has the minimum number of states—with such properties. It contains (see Figure 1) three linear energy-storage elements (an inductor L and two capacitors C_1 and C_2), a linear resistor R, and a single nonlinear resistor N_r called Chua's diode (Chua, Komuro, and Matsumoto 1986). The v-i characteristic f of N_r is shown in Figure 2. For conciseness, let us skip the state equations, as they can be written directly from the circuit (see the work of Chua and Lin [1990] for these expressions), and give only their most simple representation. Chua and Lin (1990) show that these state equations for Chua's circuit can be rewritten in a remarkably simple dimensionless form as:

$$\begin{aligned}\dot{x} &= \alpha\big(y - x - f(x)\big)x \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y\end{aligned}$$

where x, y, and z are the new state variables, α and β are two parameters derived from L, C_1, C_2, and R, and the only nonlinear function is f.

As shown by Shuxian (1987), the circuit has a broad set of trajectories and periodic orbits. Similarly, signals of sustained sounds from musical instruments are generally periodic to a very good approximation. Therefore, it is tempting to consider one of the state variable values as an acoustic signal to be amplified and sent to loudspeakers (Rodet 1992, 1993c). We have done a real-time simulation of the circuit on a digital computer with audio capabilities, and we have controlled it

Figure 3. *Time-delayed Chua's Circuit.*

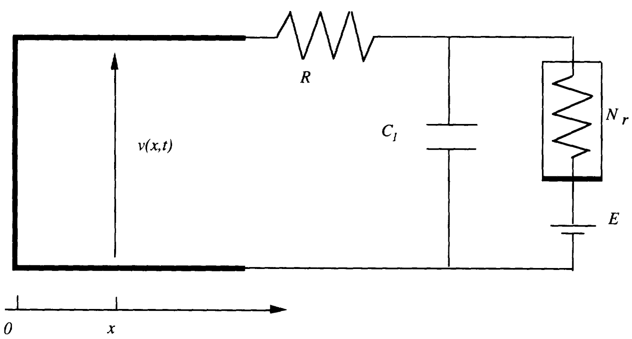
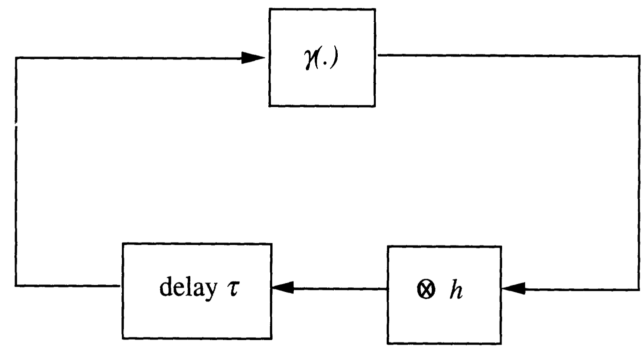


Figure 4. A single delayed feedback loop nonlinear system, also a basic clarinet model.



interactively. According to the parameter values, many harmonic and chaotic (Madan 1992) sounds can be obtained.

However, for musically interesting use, a synthesis algorithm has to provide control parameters allowing for expressive timbre modifications, i.e., essentially spectrum-content modifications as required by the performer. In this respect, Chua's circuit happens to be relatively difficult to control, and does not offer as rich a palette of timbres as wished for in musical applications. The reasons for these limitations are explained in terms of feedback and acoustic modes, in the last section. We now consider a slightly modified circuit known as the time-delayed Chua's circuit (Sharkovsky, Mastrenko, Deregel, and Chua 1993), which offers a richer instrument model and better flexibility.

Time-Delayed Chua's Circuit

Let us look at Chua's circuit displayed in Figure 1. In the work of Sharkovsky and colleagues (1993), a DC-bias voltage source is added in series with the Chua's diode, then the capacitor C_2 and the inductance L are replaced by a lossless transmission line. The time-delayed Chua's circuit is shown in Figure 3. In a first simplification, the slopes m_0 and m_2 of the characteristic f of N_r are set equal (see Figure 4). The study of this dynamical system is difficult, but with $C_1 = 0$, it reduces to a nonlinear difference equation. The solution consists of the sum of an incident wave $a(t - x/v)$ and a reflected wave $b(t + x/v)$ such that:

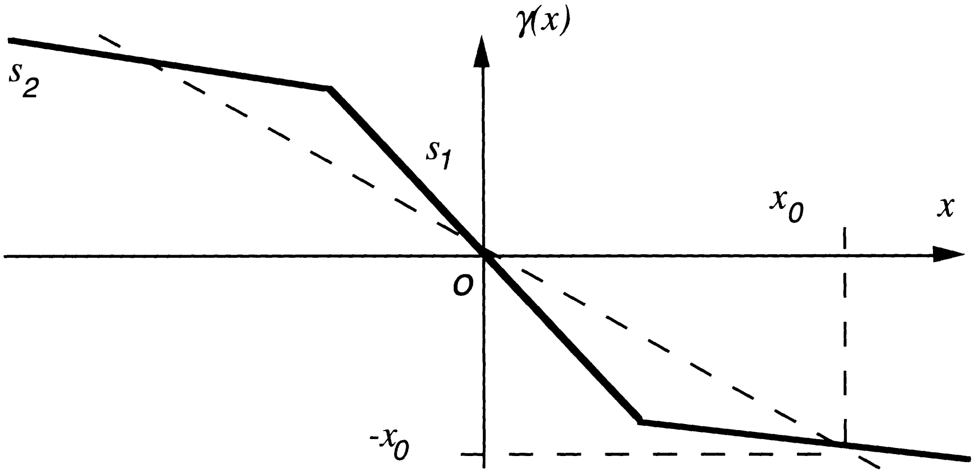
$$\begin{aligned} a(t+x/v) &= -b(t-x/v) = \Phi(t-x/v) \\ \Phi(t) &= \gamma(\Phi(t-2T)) \end{aligned} \tag{1}$$

where T is the time delay in the transmission line, and γ is a piecewise-linear 1-D map that can be computed from the parameters of the circuit (Sharkovsky et al. 1993). Remarkably, the time-delayed Chua's circuit leads to the same Equation 1 as a basic clarinet-like model (Equation 3 in the next section), except that a filter h is added in the feedback loop of the clarinet model. The choice of γ is also different, but we find interesting analogies between these maps. We have implemented a digital simulation of this system. It exhibits a surprisingly large variety of sounds and behaviors, while being always easy to control and relatively predictable (Rodet 1993e).

Feedback-Loop Formulation of Physical Models

Many physical models of musical instruments with sustained sounds (strings, brass, reeds, flutes, and voice) can be described by an autonomous nonlinear system of integral and differential equations with delays and convolution terms (McIntyre, Schumacher and Woodhouse 1983; Smith 1986; Saneyoshi, Teramura, and Yoshikawa 1987; Rodet and Depalle 1990, 1992). Complicated models are required to take into account all the subtleties of real instruments (Keefe 1992). Unfortunately, these equations are extremely difficult to solve, and have not received as much attention as have more common differential equations.

Figure 5. A piecewise-linear map with slopes s_1 *and* s_2.



However, formulation of these equations as feedback-loop systems (Mees 1981) provides some insight into their properties. We can derive the simplest example of a feedback-loop dynamical system in a basic clarinet model. Let us examine the reed of a clarinet-like instrument coupled to the bore (Schumacher 1981). Following the work of McIntyre, Schumacher, and Woodhouse (1983), let us call the outgoing and incoming pressure waves in the bore q_o and q_i, respectively, the pressure in the player's mouth p, and the characteristic impedance of the bore z. The system can be described in a simplified way by the equations:

$$\begin{aligned} q_0(t) - q_i(t) &= zF\big(q_i(t) + q_0(t) - p(t)\big) \\ q_i(t) &= r * q_0(t) = h * q_0(t - \tau) \end{aligned} \qquad (2)$$

where $h(t - \tau)=r(t)$ is the *reflection function* of the bore, and F:R → R is a nonlinear function at the reed.

The most important assumption here is that the reed has no mass, which leads to a *memoryless* nonlinearity F. Let us suppose now that the function F is such that, for t fixed, the above system has a unique solution. Then $q_o(t)$ is uniquely defined from $q_i(t)$. Since h is the known impulse response, $q_i(t)$ is uniquely defined from $q_o(t - \tau)$. Therefore, we have:

$$q_0(t) = \gamma\big(h * q_0(t - \tau)\big) \qquad (3)$$

for some function γ : R → R.

This simple example shows how the system of equations describing the behavior of an instrument can be reformulated as a feedback-loop system (see Figure 4). The first advantage of this formulation is that some theorems, as we show in the following, can be applied to prove stability and oscillation of the system. Another advantage is that this formulation is more intuitive and gives some insight into the behavior of the feedback loop. Note the similitude between this model and the time-delayed Chua's circuit presented earlier. Note also that, by using $x(t)=h^* q_0(t)$, Equation 3 can be written as well in the form:

$$x(t) = h * \gamma\big(x(t - \tau)\big), \qquad (4)$$

where $x \in$ R, γ:R → R is a memoryless nonlinear function (e.g., see Figure 5), $t \in$ R is a time delay, h:R → R is an impulse response, and * is the convolution operator.

Here, h represents the impulse response of the whole instrument considered as a linear passive system. Usually, for a musical instrument, h is a rather complicated impulse response.

Even in the case of this simplified Equation 4, the solutions and their stability are known only partially, and in restricted cases (Chow and Green 1983; Hale 1991; Ivanov and Sharkovsky 1992; Dieckmann, van Gils, Lunel, and Walther 1995; see next section). For musical use, we would like to guarantee which solution is obtained, among

Figure 6. *Iterated map γ showing divergence.*

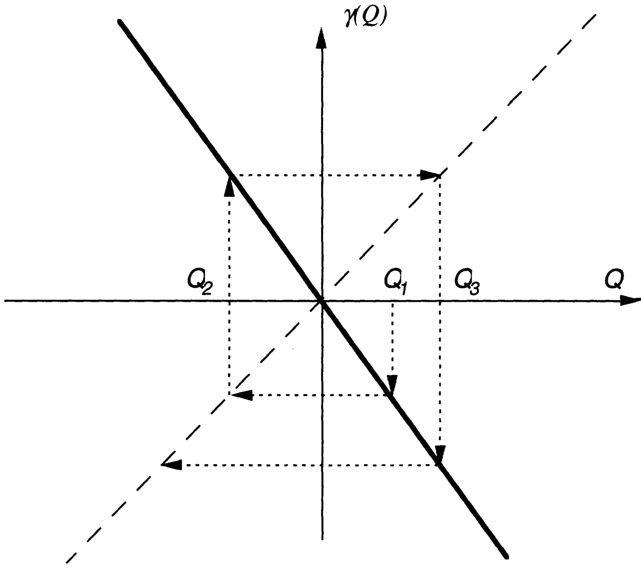
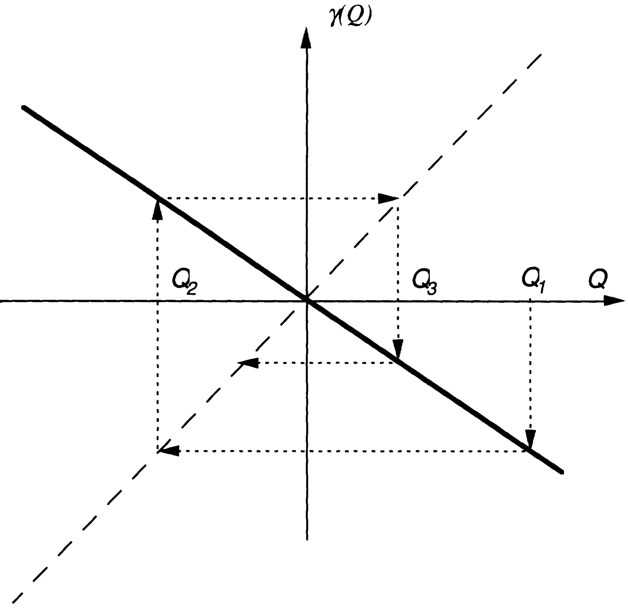


Figure 7. *Iterated map γ showing convergence.*



possibly several stable solutions. For this purpose, we will specify some classes of functions g and of filters h, provided they are flexible enough.

First Properties of a Basic Feedback-Loop System

Equation 4 can be considered as the most basic feedback-loop system, especially if a simple non-linear function γ is used. Because of this simplicity, some properties of the behavior of the system can be easily studied and controlled, with interesting musical applications.

Oscillation Process

Let us explain the main oscillation process in Equations 1, 3, and 4. To better explain this behavior, McIntyre, Scumacher, and Woodhouse (1983) note that if $h(t)$ is simplified into a dirac-impulse-generalized function δ_t, then:

$$x(t) = h * \gamma\left(x\left(t - \tau\right)\right),$$

and similarly for q_i. In this formulation, the sign inversion that occurs at the open end of the bore is included in γ The signal value $q_o(\tau)$ depends only on the value at t− τ. If $q_o(t) = Q_o$ is constant on $[-\tau, 0]$, then it is constant on any interval $[(n-1)\tau, n\tau]$, n-integer, with a value:

$$Q_n = \gamma\left(Q_{n-1}\right).$$

Therefore, we have found an iterated map as a model of the basic oscillation process of a clarinet-like instrument. We now examine the behavior of this iterated map. If the pressure in the mouth of the player p is zero, then the instrument should remain steady, i.e., q_o should remain zero. Thus the origin O is a *fixed* or *equilibrium* point of the map. The iteration on the map is shown on Figures 6 and 7. Starting from, say Q_1, we get Q_2 by taking the value of γ at Q_1. Then we trace a horizontal line at this ordinate Q_2 to reach the line $Q_n = Q_{n-1}$, which gives us the abscissa Q_2, so that the process can be done again. If γ is linear, then Figures 6 and 7 show that the iteration can lead to Q_n growing to infinity or decaying to zero. In the first case, the equilibrium point (the origin) is stable, and in the second case it is unstable. This is controlled by the slope of γ being negative and less than or greater than 1 in modulus.

In order for the system to oscillate around O (as we expect a musical instrument to do), the slope s_1 of a map γ, which is *smooth* in the neighborhood of O, has to be less than -1 (see Figure 5). In order for the signal not to grow to infinity, the slope of γ has to become greater than -1, say s_2, at some distance from O. Let us choose γ as three segments with slopes $s_1 < -1$ around O, $s_2 > -1$ elsewhere, and γ symmetric around O (see Figure 5). This (s_1, s_2) map is also justified by control considerations of the basic clarinet-like instrument, as explained below. We thus have shown that the basic time-delayed feedback-loop system is a model of an interesting class of musical instruments, namely those, like the clarinet, consisting of a massless reed coupled to a linear system.

The most important assumption here is that the reed has no mass, which leads to a memoryless nonlinearity. But for other systems ([Rodet 1995a]; see also the companion article in this issue [Rodet and Vergez 1999]), it may or may not happen that the nonlinearity can be extracted as a memoryless element.

Roles of Slopes s_1 and s_2

Let us consider a map γ with the origin O as an equilibrium point, a slope s_1 about O, and a slope s_2 at some distance from O. From Figure 6 it can easily be seen that the modulus $|s_1|$ controls the *transient-onset velocity* (for example, the attack portion of a note): the greater $|s_1|$, the faster the onset. We have here a clear control parameter for the onset behavior of our instrument. If $h(t) = \delta_t$, the signal is a square wave. If $h(t)$ is a low-pass impulse response, then the signal is rounded. This rounding can also be controlled by $|s_2|$: the closer $|s_1|$ and $|s_2|$ are to unity, the fewer high frequencies are in q_o, and are hence in the output signal of the instrument. This rounding can also be viewed as follows: in the square-wave case, the system uses only two points of the map; and in the rounded case, it uses more points spread on the map. The amount of high frequencies versus low frequencies is sometimes referred to as *richness* of the sound, because more high frequencies in the signal induce the perception of a richer sound.

As a first result, we have found that two important characteristics of the sound, transient-onset velocity and richness, are controlled by the slopes s_1 and s_2. Therefore, we have reached one of our main goals: the control of the sound characteristics of our instrument model. We address the topic of a smooth map, polynomial and rational, in a following section.

Special Cases

For strings, reed-woodwinds, or brass, the delay term, including the filter h, plays an essential role (Fletcher and Rossing 1991). In a simple clarinet model, for instance, the propagation of the sound wave and its reflection at the extremity is represented by a feedback through a delay line and a filter as in Equation 4. Unfortunately, with the linear element h, the solutions to these equations and their stability are known only partially and in restricted cases (Chow, Diekmann and Mallet-Paret 1985; Ivanov and Sharkovsky 1992; Walther 1989; Hale 1991).

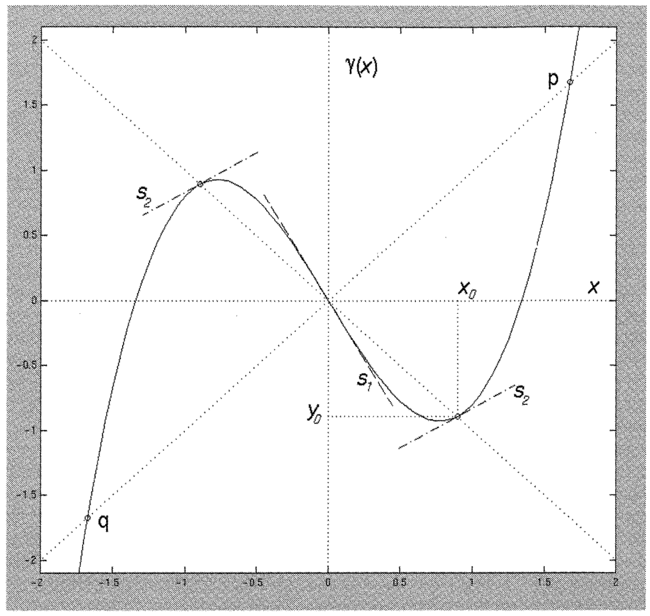
Chow, Diekmann and Mallet-Paret (1985) consider the particular case where h is constant on an interval $[-\varepsilon, +\varepsilon]$ and zero elsewhere. It is also restricted to functions $\gamma \in C^2$ which are odd, and such that $\gamma(1) = -1$, $\gamma'(x) < 0$, and $\gamma''(x) > 0$ for $x > 0$. In this case, Chow, Diekmann, and Mallet-Paret have proven that Equation 4 has a stable period-2 solution composed of odd harmonics only. This is an interesting result, since it corresponds to the usual playing condition of a clarinet. However, we feel the need for a more general result, since we show in a following section that the odd nature of and the odd harmonicity of the solution are in some ways nongeneric.

Ivanov and Sharkovsky (1992) consider the solutions to singularly perturbed delay equations such as:

$$a\,\dot{x}(t) + x(t) = \gamma\big(x(t-\tau)\big). \qquad (5)$$

Note that Equation 5 is essentially a particular case of Equation 4. But the term $\dot{x}(t)$ is far too restrictive compared to the convolution with an impulse response h, as examined in the last section.

Figure 8. *A smooth polynomial map* $\gamma(x) = a_3x^3+s_1x$ *with slopes* s_1 *and* s_2.



Some Results Regarding the Role of the Nonlinearity

The shape of the nonlinear function γ has important consequences on the behavior of the system considered as a musical instrument. In the following, it is shown that a polynomial or rational map should be preferred to a piecewise-linear map.

Piecewise-Linear and Polynomial Functions

Let us first consider the case of the time-delayed circuit without the filter h:

$$x(t) = \gamma(x(t-\tau)). \qquad (6)$$

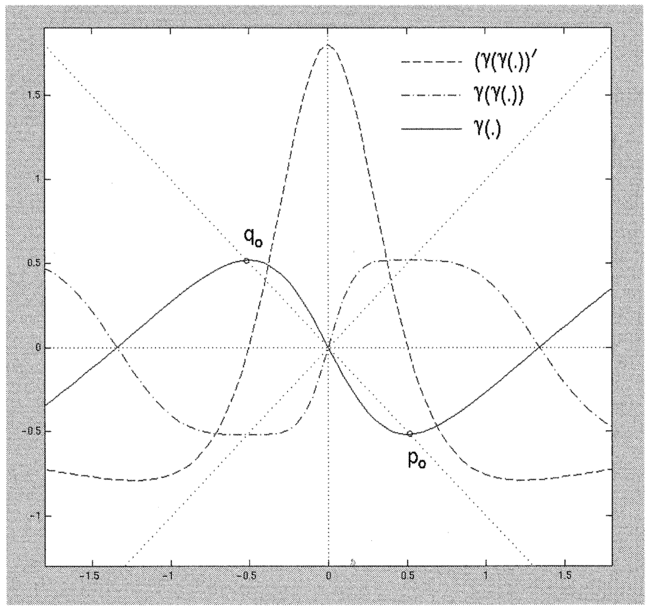
With γ a piecewise-linear function, Sharkovsky and colleagues (1993) have shown analytically that this system, the time-delayed Chua's circuit, exhibits a remarkable period-adding phenomenon. In some regions of the space (s_1, s_2), the system has a stable limit cycle with period respectively 2, 3, 4, etc. In between every two consecutive periodic regions, the system exhibits a chaotic behavior. Chaotic signals obtained in our simulations are examined in the companion article (Rodet and Vergez 1999).

However, the piecewise-linear function used in Chua's circuit has a drawback. Consider the onset of the signal, i.e. the transient from zero. Observe that before a certain amplitude is reached, only a linear part of γ is used. The system behaves therefore like a linear system: there is no change in the short-time spectrum of the signal other than an amplitude growth (this can be observed easily in the short-time spectrum display of our real-time implementation). On the contrary, the nonlinearity of the reed of a real instrument can be more realistically approximated by a quadratic function (Fletcher and Rossing 1991). Therefore, during the transient, there is a constant transfer of energy between frequency components. As a consequence, we favor a quadratic or cubic nonlinearity of the form $\gamma(x) = a_2x^2 + s_1x$ or $a_3x^3 + s_1x$ (see Figure 8). In the last case, for instance, the value of a_3 is determined according to the slopes s_1 at O and s_2 at the point (x_0, y_0) such that $y_0 = x_0 = a_3x_0^3 + s_1x_0$. Note that as we vary s_1, we determine the amplitude and spectral richness of the sound simultaneously, since s_2 at (x_0, y_0) varies with the coefficient s_1. Then a greater amplitude leads to a richer sound (i.e., more high-frequency components, with larger amplitude) as generally happens with natural instruments. However, by varying a_3 and s_1, we can still provide independent control of the two first sound qualities mentioned above, amplitude and richness.

The case where $\gamma(\bullet)$ is a polynomial has other interesting properties. The first one is that a polynomial is easy to implement and can be efficiently computed on a digital processor. The control of the important characteristics of γ is easy. Also, in presence of a filter h, if m_2 is the size of the support of the impulse response h, Equation 4 in discrete form can lead to a system of $m_2 + T - 1$ polynomial equations in $m_2 + T - 1$ variables (Rodet 1994).

It should be noted that such a polynomial function forces the existence of equilibrium points (e.g., p and q in Figure 8) other than the origin, thereby complicating the dynamics of the circuit. (Such other equilibrium points exist for natural instruments. For a high enough blowing pressure, the reed of a clarinet will keep the mouthpiece closed, but this is usually an unwanted effect.) An ideal function for our purpose should have the origin as the only equilibrium point. To guarantee this, the function should not cross the line $y = x/H(0)$, where $H(0)$ is the value of the transfer function of h at DC. This is examined in the next section.

Figure 9. Rational functions $\gamma(.)$, $\gamma(\gamma(.))$ and derivative $\gamma(\gamma(.))'$ for $a_1 = -1.8$ and $b_2=2$.



A Rational-Function Nonlinearity

We have proposed elsewhere (Rodet 1994) that the nonlinearity be implemented as a rational function. This avoids any spurious functioning point, guarantees that the oscillation of the system does not grow to infinity, and allows it to be studied with the same tools that we are using for the polynomials. Furthermore, this rational-function system exhibits a chaotic behavior with the simultaneous presence of harmonic sinusoidal components and of nonsinusoidal components that are heard as noise added to the harmonic tone.

The most simple polynomial for our purpose is the one that we looked at in the previous sections, $\gamma(x) = x^3 + a_1 x$, where $a_1 < 0$. Without loss of generality, we can suppose that $H(0) = 1$. Then, apart from the origin, this polynomial intersects the line $y = x/H(0) = x$ at two points:

$$x = \pm\sqrt{1-a_1}\,. \tag{7}$$

At these points, $\gamma(x) = x$, so that they are equilibrium points of the system 6 and, in general, are unstable equilibrium points. Furthermore, as $x \to \infty$, the slope of $\gamma(x) \to \infty$, implying that the system can become unstable and escape to infinity. This is rather annoying for a musical instrument. Note that the polynomial could as well contain a second-order term to avoid the function being odd, as suggested in the last section: $\gamma(x) = x^3 + a_2x^2 + a_1x$.

To correct the growing of the slope of γ we divide the polynomial by another polynomial of degree less than or equal to 3. With an even degree, the odd symmetry may be kept:

$$\gamma(x) = \left(x^3 + a_1 x\right)/\left(1 + b_2 x^2\right). \tag{8}$$

With this nonlinearity, the system 6 has no spurious equilibrium points, provided $b_2 > 1$, and it can be studied with the same tools as the polynomial case. This rational function γ is shown in Figure 9 for $b_2 = 2$ and $a_1 = -1.8$. To keep things simple, let us consider the case of Equation 4 when there is no filtering in the feedback loop. Then the first 2-periodic solutions oscillate between points p_0 and q_0, with abcissa x_0 and $-x_0$ such that $\gamma(x_0) = -x_0$; i.e.,

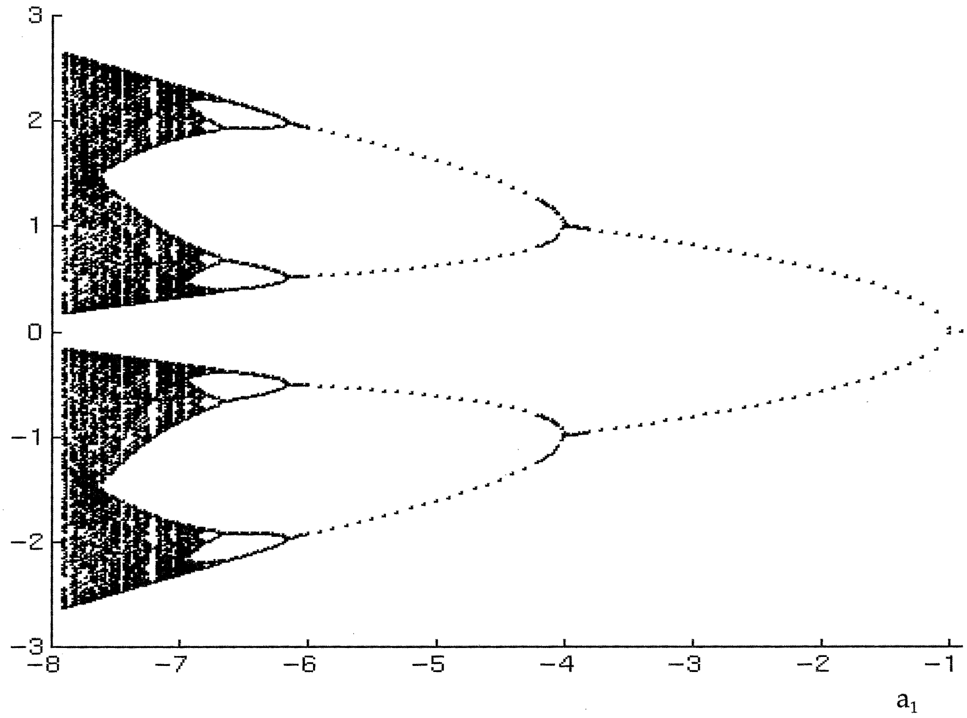
$$x_0 = \sqrt{\left(-1-a_1\right)/\left(1+b_2\right)}\,. \tag{9}$$

Such a 2-periodic solution is stable if the derivative of $\gamma \circ \gamma(x)$:

$$\left(\gamma^\circ\, \gamma(x)\right)' = \gamma'\left(\gamma(x)\right)\gamma'(x) = \gamma'(-x) = -\left(\gamma'\ (x)\right)^2 \tag{10}$$

is less than 1 in modulus at p_0 and q_0. This happens when $-4 \le a_1 \le -1$. Figure 9 shows $\gamma(x)$, $(\gamma^\circ\gamma(x))$, and $(\gamma^\circ\gamma(x))'$ for $a_1 = -1.8$. For smaller values of a_1, for approximately $-6.15 \le a_1 \le -4$, we get two 2-periodic solutions. Finally chaotic solutions (Rodet 1995b) can be obtained for approximately $a_1 \le -7.2$, as shown in Figure 10.

Figure 10. Bifurcation diagram of the rational function nonlinearity of section 5.2 versus parameter a_1.



Single Feedback Loop Systems with a Memoryless Nonlinearity

By generalizing the type of system examined in the preceding discussion, we define a class of systems that are simple to understand and control, but have interesting musical possibilities.

Stability

Let us consider our system with a filter h included in the feedback loop and described by Equation 4. Figure 4 shows that the system can be decomposed into a memoryless nonlinearity γ and a linear element including h and a delay. We define the class of single feedback-loop systems of which we can easily determine the stability and some oscillation properties. Such systems are composed of a unique memoryless nonlinearity and a linear feedback loop. Note that the only restriction on the linear element is that its impulse response be stable (Vidyasagar 1978). In particular, the transfer function of the linear element need not be a rational function, and thus can include delays. Many systems can be redesigned to fall into this class. For more complex systems, a larger class is studied in the companion article (Rodet and Vergez 1999).

We first consider the condition for oscillation around an equilibrium point when such a filter h, with transfer function H, is introduced in the feedback loop. Without loss of generality, we can assume this equilibrium point to be the origin. The open-loop transfer function, i.e., the transfer function of the linear part in Figure 5, is:

$$G(j\omega) = e^{-j\omega\tau} H(j\omega).$$

Since this represents the transfer function of the physical instrument, we naturally suppose that its impulse response belongs to L_1, i.e., is measurable and absolutely integrable over $[0, +\infty)$. We can therefore apply the *graphical stability test* (Vidyasagar 1978, p. 270). As before, we call s_1 the slope of γ at the origin. The graphical stability test

Figure 11. Nyquist plot of G(jw) *for the Graphical Stability Test. The minimum real value –q is obtained on the first mode, at a frequency of 121 Hz.*

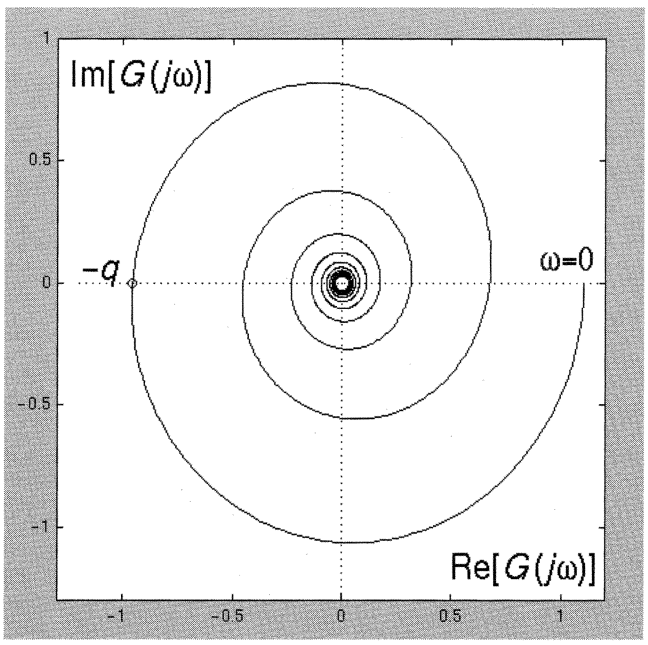
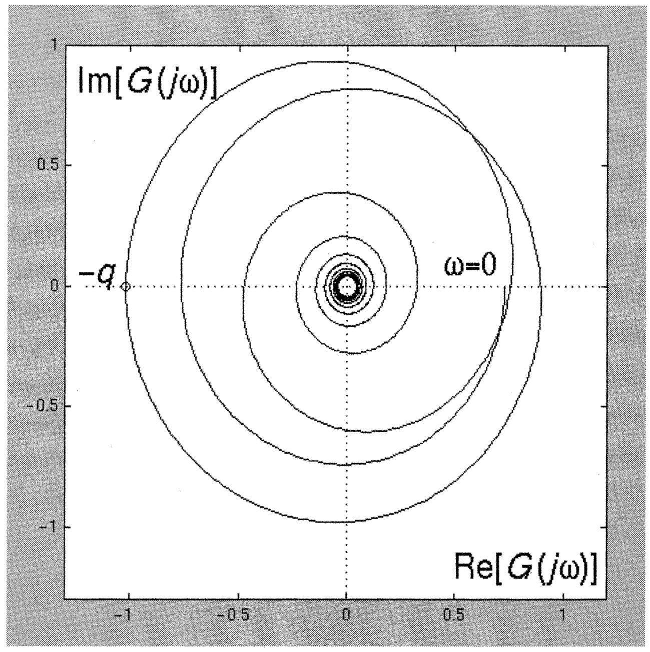


Figure 12. Nyquist plot of another G(jw) *for the Graphical Stability Test. The minimum real value –q is obtained on a mode higher than the first, at a frequency of 325 Hz.*

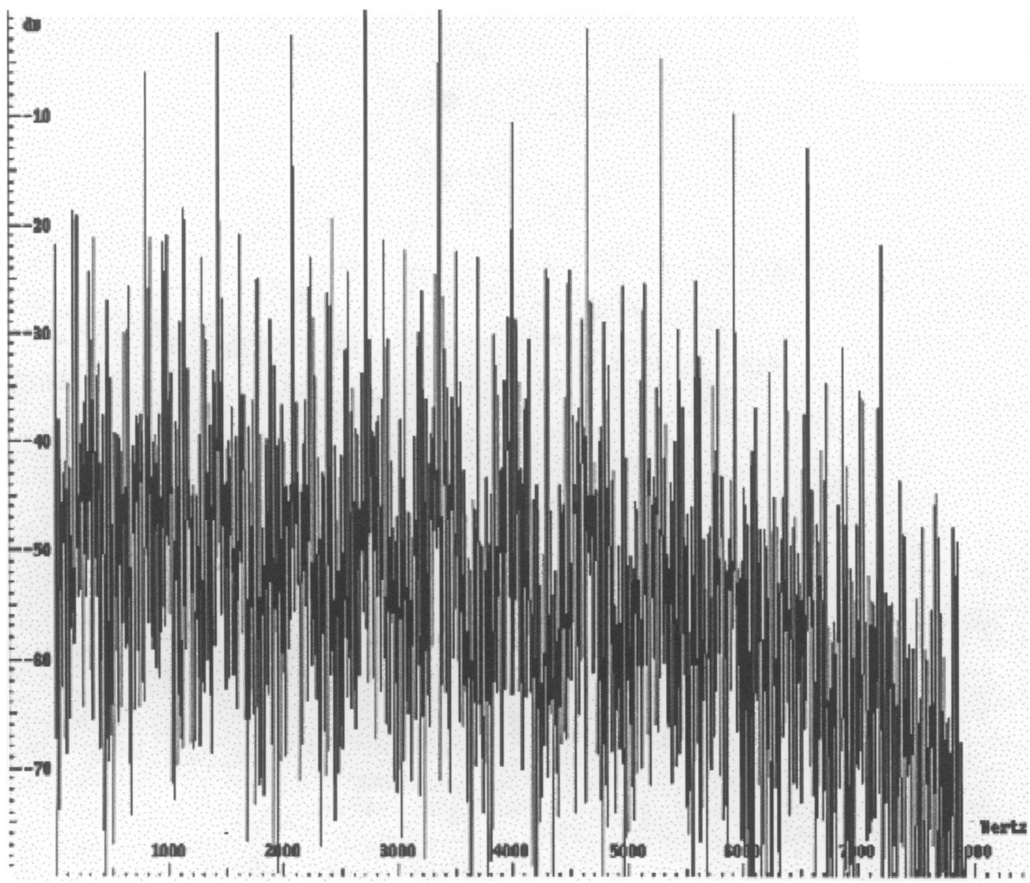


provides the value of the slope s_1 above which the system is stable at the equilibrium point. Let us call p the variable in the complex plane. By assuming that $G(p)$ has no poles in the closed right half-plane C_+, the test is simplified in the following way: for the system to be stable at the equilibrium point, the limit value $1/s_1$ should lie to the left of all intersections of the Nyquist plot of $G(j\omega)$ with the real axis (see Figure 11). (The Nyquist plot is the plot of the complex values $G(j\omega)$ in the complex plane for all the real values of $\omega \in [0, +\infty)$.) Let us use the notation $r + j0$ to designate a point on the real axis. Let $-q + j0$ denote the intersection point with the smallest value, and let ω_q be the value such that $G(j\omega_q) = -q$. Then the system becomes unstable when $s_1 < -1/q$. Note that this only indicates that the system could oscillate. A possible proof that it actually oscillates is more involved, and is postponed to the companion article in this issue (Rodet and Vergez 1999).

The Modes' Position and Fundamental Oscillation

We can extract more information from the Nyquist plot of $G(j\omega)$. Suppose for the moment that $H(j\omega)$ is real positive (without loss of generality, we can at least choose the delay τ such that for a given ω_0, $G(j\omega_0) = -q$). Then the intersection of $G(j\omega)$ with the negative real axis occurs for $\omega_k\tau = \pi + 2k\pi$, i.e., for frequencies $f_k = (1 + 2k)/2\tau$. Observe that $f_0 = 1/2\tau$ is the frequency corresponding to twice the delay t necessary for a sound wave to propagate from the reed to the end of the bore and back to the reed. The values f_k, $k = 0, 1, 2, \ldots$, are the frequencies of the modes of the instrument. Therefore, $G(j\omega_q)$ and $\omega_q/2\pi$ can be simply interpreted as the amplitude and the frequency of the strongest mode (see Figure 11) of the instrument. (In the case of the trumpet, for instance, the mouthpiece acts as a resonator that boosts some modes numbered greater than 1, thereby allowing an easy oscillation at the frequency of one of these modes [Fletcher and Rossing 1991].) Observe that the frequencies f_k are the odd harmonic partials of the fundamental f_0, but the oscillation frequency may be different from f_0, since the oscillation frequency generally is the frequency of the strongest mode $\omega_q/2\pi$. Suppose for simplicity that the oscillation frequency is $\omega_q/2\pi$ and is equal to f_0. Assume now that the argument of $G(j2k\omega_q)$ is different

Figure 13. Spectrum of a signal with complicated structure.



from zero: the consequence is that the modes are usually moved away from harmonic positions.

This case, where the argument of $H(j2k\omega_q)$ is different from zero, can lead to surprising results which are like quasi-periodicity (Rodet 1993e). For instance, Figures 13 and 14 display the spectrum of such a signal. The enlarged portion displayed in Figure 14 shows that a very low fundamental frequency (17 Hz) has been obtained even though the delay τ corresponds to a relatively high frequency (200 Hz). A similar phenomenon appears in Figures 15 and 16 with corresponding frequencies of 6 Hz and 300 Hz.

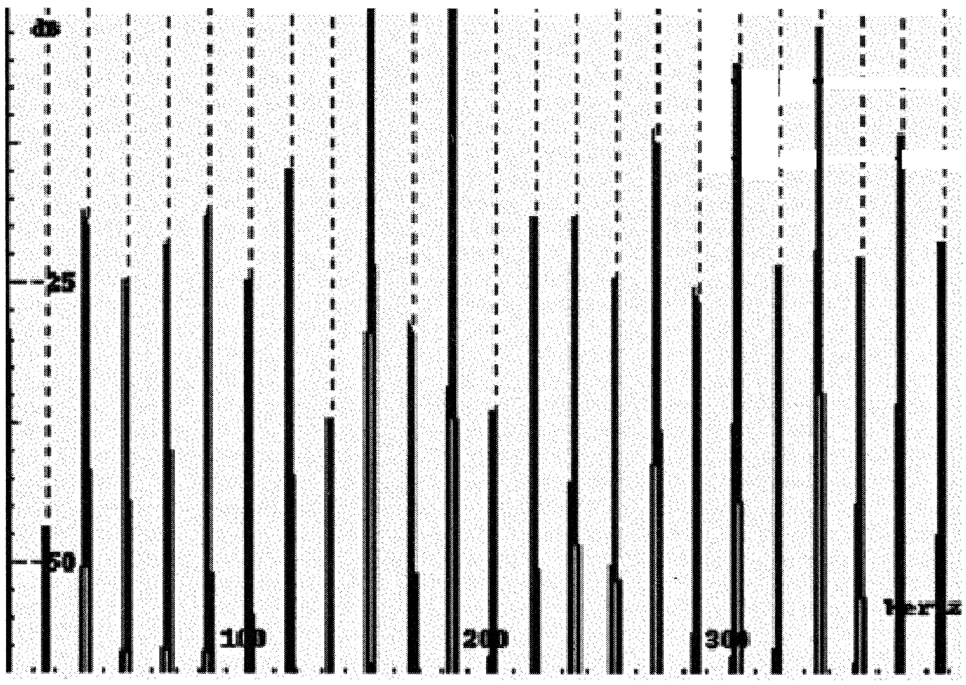
The oscillation frequency may be moved from one mode to another just by a little change of $H(j\omega)$, which makes another mode stronger than the initial one; this modification is also easily viewed on the Nyquist plot of $G(j\omega)$ in Figure 12 (see also the works of Matignon, Depalle, and Rodet [1992] and Rodet [1993a]).

Absence of the *n*th Partials

In the case where there is no filter in the loop, it is relatively obvious that when the system has a stable limit cycle with period 2, of duration 2τ, the limit cycle $x(t)$ has the symmetry $x(t + \tau) = x(-t)$; i.e., it has no even harmonics. What is more remarkable is that this result generalizes also to integers k greater than 2. In any region where the system has a stable limit cycle $x(t)$ with period k, the harmonics k, $2k$, $3k$, etc., are absent. The spectrum of such a limit cycle is given by Rodet (1993e, fig. 4) for $k = 3$.

To prove this result, let us take $k = 3$, for instance. Then the system has a period 3 of duration

Figure 14. Detail of Figure 13 for frequencies up to 400 Hz.



3τ, and only three values of γ–(.) are used—say, c_i, where i = 1, 2, and 3. If $x(t) = c_i$, then $x(t + \tau) = c_{i+1}$ and $x(t + 2\tau) = c_{i+2}$, where i + 1 and i + 2 are taken modulo 3. Therefore, each value c_i is represented during exactly a third of the total period 3τ, and the value of the $3k^{th}$ Fourier coefficient is zero. This is a rather interesting result from a musical point of view as well, since harmonic or pseudo-harmonic partials are essential for the musical perception and musical use of a great majority of sounds.

In our experiments, we have observed that neither the non-odd character of γ nor the inclusion of a filter h in the feedback loop are sufficient alone to produce stable solutions with even harmonic partials. When γ is not odd symmetric and there is simultaneously a filter h, even partials can appear (see Figure 17). Note that a very slight breaking of the symmetry of γ is sufficient. This is why, as we mentioned in the earlier section entitled "Special Cases," we tend to consider the result of Chow, Diekmann and Mallet-Paret (1985) to have little applicability in our case, and all the less so since natural instruments will not have perfectly odd-symmetric nonlinearities. Finally, it seems also that when γ is not very far from odd symmetry, even harmonic partials are of small amplitude (clarinet) if the argument of $H(j2k\omega_q)$ is zero (see Figure 17), and can be of large amplitude (saxophone) when the argument of $G(j2k\omega_q)$ is different from zero (see Figure 18).

Number of Stable and Unstable Solutions

Real instruments and computer simulations often have more than one oscillating solution for a given setting of their parameters (Idogawa, Shimizu and Iwaki 1992; Välimäki, Karjalainen, and Laakso 1993; Rodet 1993e). On one hand, it can be argued that this is part of a musical instrument's richness. On the other hand, unpredictable solutions can render the usage of a physical model rather difficult in a real-time musical performance. Only stable solutions should be considered, because unstable ones do not

Figure 15. Spectrum of a signal with complicated structure.

Figure 16. Detail of Figure 15 for frequencies between 0 and 700 Hz.

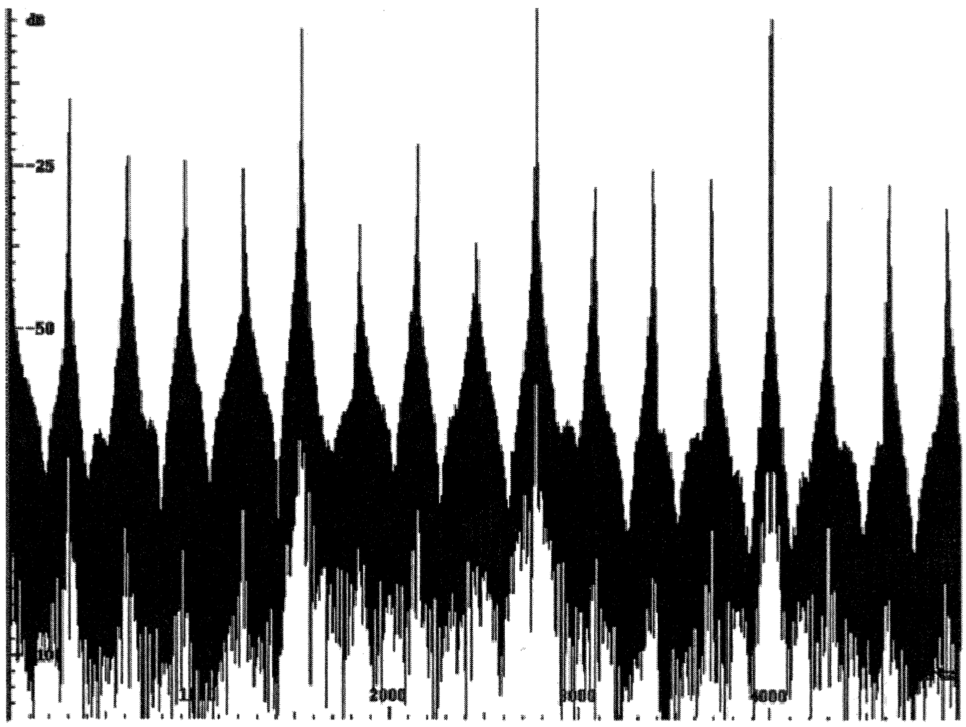


Fig. 15

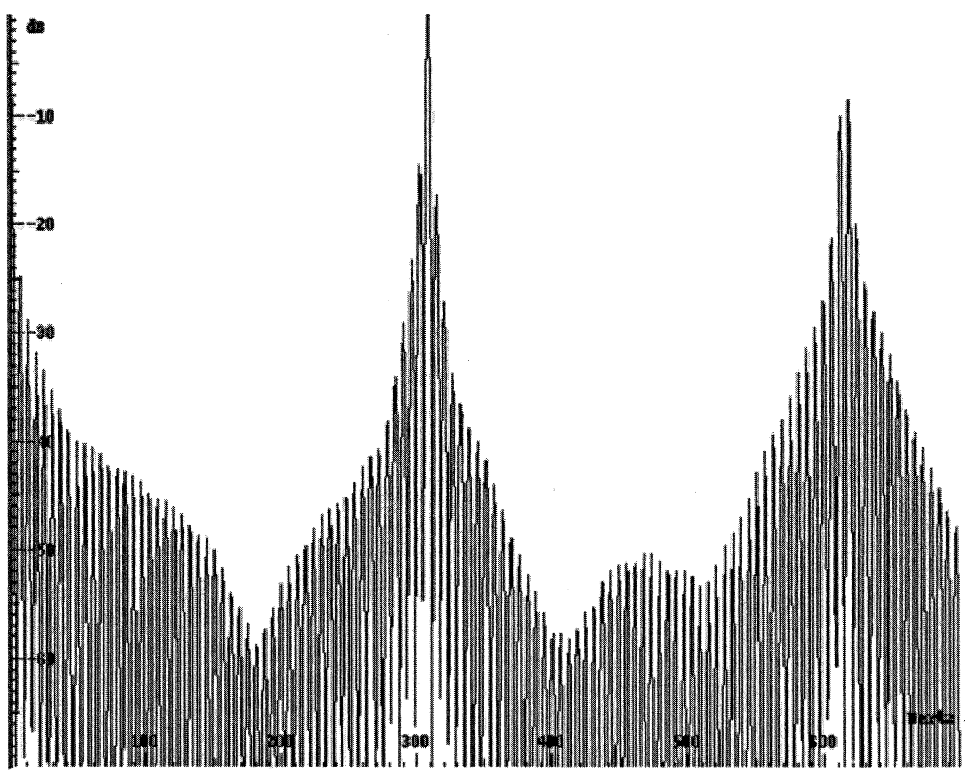


Fig. 16

Figure 17. Spectrum of a signal obtained with a non-odd nonlinearity and a zero-phase filter.

Figure 18. Spectrum of a signal obtained with a non-odd nonlinearity and a non-zero-phase filter.

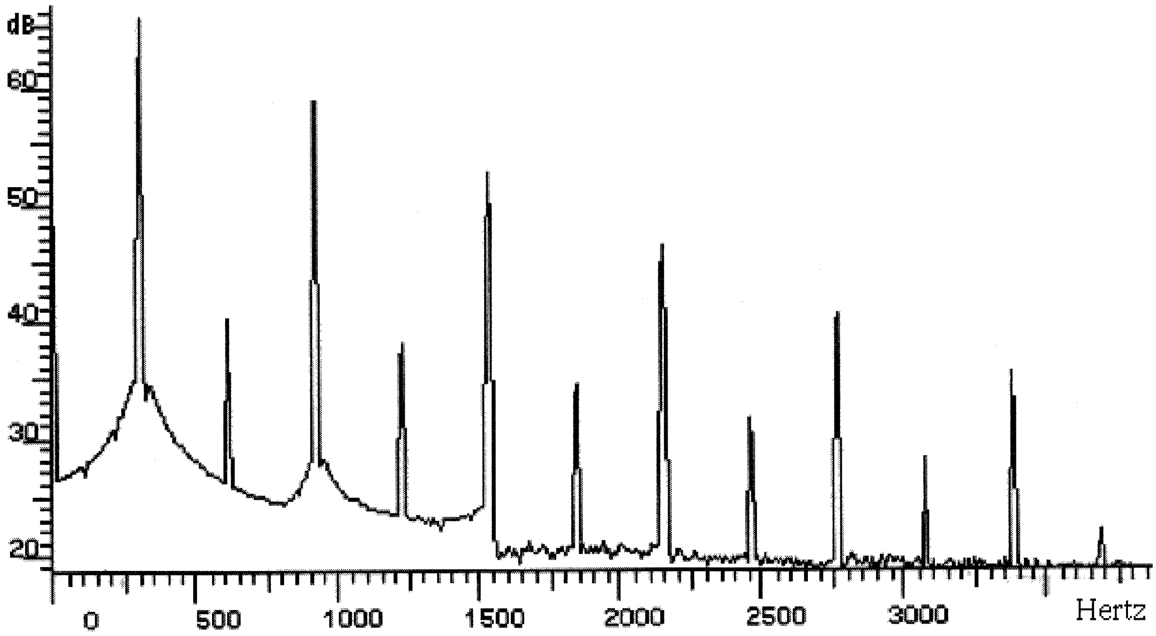


Fig. 17

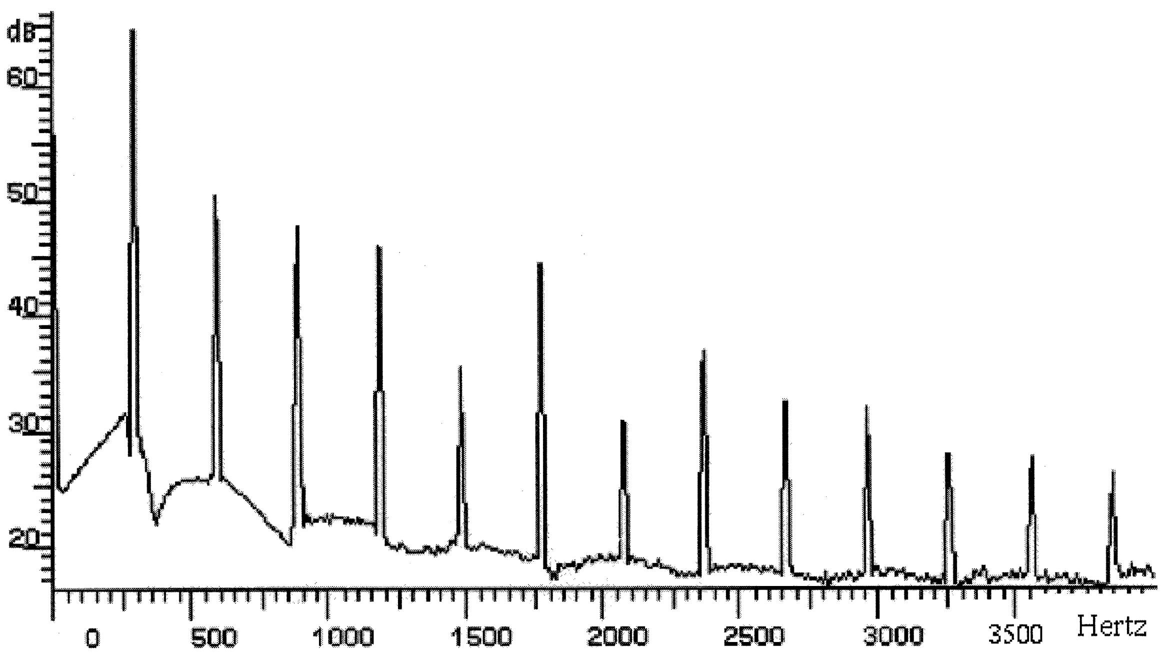


Fig. 18

persist. Therefore, in an earlier work (Rodet 1994) we studied the multiple solutions of simple physical models and their stabilities. We found that a path toward the reduction of solutions could be based upon the low-pass character of the linear element. As a second goal, it would be an interesting achievement to design a system that would model the usual playing behavior of an instrument but that could avoid the other behaviors if requested. By the other behaviors, we mean those corresponding to stable solutions other than the usual playing solution.

Bergen and Franks (1971) found that the low-pass character above the oscillation frequency appeared among the sufficient conditions for oscillation, according to the describing function method. However, it does not guarantee the uniqueness of the solution. The low-pass character appears also in the formulation of the Hopf theorem in the frequency domain (Mees and Chua 1979; Rodet 1993d; Rodet and Vergez 1999), which gives sufficient conditions for oscillation and uniqueness of the solution.

Similarly, the linear part of Chua's circuit (Chua and Lin 1990), at least for some values of the parameters, can be viewed as a low-pass filter with cutoff frequency around the frequency of a pair of conjugate poles, this frequency being approximately the oscillation frequency. This suggests some relation to the dominant mode of an acoustic instrument (Rodet 1993b). This idea has also been evoked by Wawrzynek (1987), but is not fully developed. It can be shown that there is a precise correspondence between Chua's circuit and a model of an acoustic instrument where the feedback loop is limited to the first mode, or the first few modes (the first mode could easily be replaced here by the dominant mode in the same way). Naturally, the system so obtained has a unique oscillatory solution in a certain range of parameter values.

Finally, in a previous work (Rodet 1994), we first show how the periodical solutions of our system using a rational nonlinearity are the solutions of a system of polynomial equations. In general, one can follow the evolution of these solutions by a homotopy method (Chow, Mallet-Parret, and Yorke 1978). We then show how the number of solutions and of stable solutions decreases when more and more low-pass filtering is introduced in the feedback loop.

Conclusion

We have studied here some topics stemming from physical models of musical instruments for the purpose of sound synthesis. In particular, we have shown that a new research strategy using the theory of nonlinear dynamical systems was needed and successful. We have found that models of musical instruments with sustained sounds have in common the characteristic of being made up of a passive linear element coupled with a nonlinear map. We have shown the advantages of a formulation that exhibits a feedback loop closed on the map. An essential feature appears to be the presence of a delay term in the loop.

We have explained the role of the delay feedback loop. It has a stabilization effect, and because it constitutes a delay line, it has a large information content. The interaction of the information contained in the delay line with the rich dynamics of the nonlinear map provides extremely wide and interesting sound possibilities.

Since many systems can be reformulated as a delay feedback loop closed on a memoryless nonlinearity, we have studied the corresponding integral delay equations by using a feedback-loop formulation. We have been able to determine the main behavior of such systems and the regions of the parameter space where this behavior occurs. The first behaviors are stability and the bifurcation from stability, which usually lead to periodic oscillation of the instrument. The periodic oscillation is generally used in normal playing conditions, but the question of a slight proportion of chaos arises. Quasi-periodic oscillations are also observed with specific reflection functions, in particular when the open-loop phase-transfer function is not linear. Finally, chaotic behavior is exhibited, in particular for some nonlinear functions.

We have found how the parameters of single feedback-loop systems closed on a memoryless nonlinearity can be used to control the main characteristics of the sounds they produce. Pitch, for periodic as well as for chaotic sounds, is essentially controlled by the duration of the delay. Richness is controlled by the transfer function of the linear element and the slopes of the nonlinear map. Transient-onset velocity is controlled by the slope of the nonlinear map around the equilibrium point.

All of these results constitute a contribution for sound synthesis and musical and artistic production. The complexity of the parameter-space structure provides a family of sounds that are extremely rich and varied. All the sound-space structures that we have mentioned here are fascinating sources of inspiration for musical creation and composition which musicians have already started to use. In particular, new simulated instruments have been designed, as is explained in the companion article (Rodet and Vergez 1999).

References

Bergen, A. R., and R. L. Franks. 1971. "Justification of the Describing Function Method." *SIAM Journal of Control* 9(4):568–589.

Chow, S. N., O. Diekmann, and J. Mallet-Paret. 1985. "Stability, Multiplicity, and Global Continuation of Symmetric Periodic Solutions of a Nonlinear Volterra Integral Equation." *Japan Journal of Applied Mathematics* 2(2):433–469.

Chow, S. N., and D. Green, Jr. 1983. "Some Results on Singular Delay-Differential Equations." In P. Fischer and W. R. Smith, eds. *Chaos, Fractals and Dynamics.* New York: Marcel Dekker, pp. 161–182.

Chow, S. N., J. Mallet-Paret, and J. A. Yorke. 1978. "A Homotopy Method for Locating all the Zeros of a System of Polynomials." In *Functional Differential Equations and Approximation of Fixed Points,* volume 730 of *Lecture Notes in Mathematiques.* Berlin: Springer-Verlag, pp. 77–88.

Chua, L. O. 1992. "A Zoo of Strange Attractors from the Chua's Circuit." *Proceedings of the 35th Midwest Symposium on Circuits and Systems.* Washington, DC, pp. 916–926.

Chua, L. O., M. Komuro, and T. Matsumoto. 1986. "The Double Scroll Family." *IEEE Transactions on Circuits and Systems* 33(11):1073–1118.

Chua, L. O., and G.-N. Lin. 1990. "Canonical Realization of Chua's Circuit Family." *IEEE Transactions on Circuits and Systems* 37(7):885–902.

Dieckmann, O., S. A. van Gils, S. M. V. Lunel, and H. O. Walther. 1995. "Delay Equations. Functional, Complex and Nonlinear Analysis." *Applied Mathematical Sciences*, vol. 110. Berlin: Springer-Verlag.

Fletcher, N. H., and T. D. Rossing. 1991. *The Physics of Musical Instruments.* Berlin: Springer-Verlag.

Hale, J. K. 1991. "Dynamics and Delays." In S. Busenberg and M. Martelli, eds. *Delay Differential Equations and Dynamical Systems*, vol. 1475 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag.

Idogawa, T., M. Shimizu, and M. Iwaki. 1992. "Acoustical Behaviors of an Oboe and a Soprano Saxophone Artificially Blown." *Proceedings of the Symposium on Some Problems on the Theory of Dynamical Systems in Applied Science.* Singapore: World Scientific, pp. 71–93.

Ivanov, A. F., and A. N. Sharkovsky. 1992. "Oscillations in Singularly Perturbed Delay Equations." In C. K. R. T. Jones, U. Kirchgraber, and H. O. Walther, eds. *Dynamics Reported.* Berlin: Springer-Verlag, pp. 164–224.

Keefe, D. 1992. "Physical Modeling of Wind Instruments." *Computer Music Journal* 16(4):57–73.

Madan, R. N. 1992 (Washington, DC, 9–12 Aug.). "Observing and Learning Chaotic Phenomena from Chua's Circuit." *Proceedings of the 35th Midwest Symposium on Circuits and Systems*, pp. 736–745.

Matignon, D., Ph. Depalle, and X. Rodet. 1992 (San Jose, California). "State-Space Models for Wind-Instrument Synthesis." *Proceedings of the 1992 International Computer Music Conference.* San Francisco: International Computer Music Association, pp. 273–276.

McIntyre, M. E., R. T. Schumacher, and J. Woodhouse. 1983. "On the Oscillations of Musical Instruments." *Journal of the Acoustical Society of America* 74(5):1325–1345.

Mees, A. I. 1981. *Dynamics of Feedback Systems.* New York: Wiley.

Mees, A. I., and L. Chua. 1979. "The Hopf Bifurcation Theorem and Its Applications to Nonlinear Oscillations in Circuits and Systems." *IEEE Transactions on Circuits and Systems* 26(4):235–254.

Rodet, X. 1992 (San Jose, California). "Nonlinear Oscillator Models of Musical Instrument Excitation." *Proceedings of the 1992 International Computer Music Conference.* San Francisco: International Computer Music Association, pp. 412–413.

Rodet, X. 1993a (Hamburg, 16–20 August). "Nonlinear Oscillations in Sustained Musical Instruments: Models and Control." *Proceedings of the 1993 European Nonlinear Oscillation Conference Euromech.* Berlin: Akademie Verlag.

Rodet, X. 1993b (Hawaii). "Applications of Chua's Circuit to Sound, Music and Musical Instruments." *Proceedings of the 1993 International Symposium on Nonlinear Theory and its Applications.* Hawaii: IEICE, pp. 105–110.

Rodet, X. 1993c. "Sound and Music from Chua's Circuit." *Journal of Circuits, Systems and Computers* 3(1):49–61. (Special issue: Chua's Circuit: A Paradigm for Chaos.)

Rodet, X. 1993d. "Models of Musical Instruments from Chua's Circuit with Time Delay." *IEEE Transactions on Circuits and Systems,* 40(10) 696–701. (Special issue: Chaos in Nonlinear Electronic Circuits.)

Rodet, X. 1993e (Tokyo). "Flexible yet Controllable Physical Models: A Nonlinear Dynamics Approach." *Proceedings of the 1993 International Computer Music Conference.* San Francisco: International Computer Music Association.

Rodet, X. 1994 (Copenhagen). "Stability/Instability of Periodic Solutions and Chaos in Physical Models of Musical Instruments." *Proceedings of the 1994 International Computer Music Conference.*: San Francisco: International Computer Music Association, pp. 386–393.

Rodet, X. 1995a (Banff, Canada). "One- and Two-Mass Model Oscillation for Voices and Instruments." *Proceedings of the 1995 International Computer Music Conference.* San Francisco: International Computer Music Association, pp. 207–214.

Rodet, X. 1995b (Las Vegas, Nevada). "Oscillation, Chaos and Control of Nonlinear Dynamical Models of Musical Instruments with Sustained Sound." *Proceedings of the Nonlinear Theory and Applications Conference.* Las Vegas: IEICE, pp. 367–370.

Rodet, X., and Ph. Depalle. 1990. "Modèles de Signaux et Modèles Physiques d'Instruments." *Proceedings of the Colloquium Modèles Physiques, Création Musicale et Ordinateur, Grenoble, 1990.* Paris: Editions de la Maison des Sciences de l'Homme, 1994, pp. 351–370.

Rodet, X., and Ph. Depalle. 1992 (San Jose, California). "A Physical Model of Lips and Trumpet." *Proceedings of the 1992 International Computer Music Conference.* San Francisco: International Computer Music Association, pp. 132–135.

Rodet. X., and C. Vergez. 1999. "Nonlinear Dynamics in Physical Models: From Basic Models to True Musical-Instrument Models." *Computer Music Journal* 23(3):35–49.

Saneyoshi, J., H. Teramura, and S. Yoshikawa. 1987. "Feedback Oscillations in Reed Woodwind and Brasswind Instruments." *Acustica* 62:194–210.

Schumacher, R. T. 1981. "Ab Initio Calculations of the Oscillations of a Clarinet." *Acustica* 48:71–85.

Sharkovsky, A. N., Yu. Mastrenko, Ph. Deregel, and L. O. Chua. 1993. "Dry Turbulence from a Time-Delayed Chua's Circuit." *Journal of Circuits, Systems and Computers* 3(2). (Special issue: Chua's Circuit: A Paradigm for Chaos.)

Shuxian, W. 1987. "Chua's Circuit Family." *Proceedings of the IEEE* 75(8):1022–1032.

Smith, J. O. 1986. "Efficient Simulation of the Reed-Bore and Bow-String Mechanism." *Proceedings of the 1986 International Computer Music Conference.* San Francisco: Computer Music Association, pp. 275–280.

Smith, J. O. 1996. "Physical Modeling Synthesis Update." *Computer Music Journal* 20(2):44–56.

Välimäki, V., M. Karjalainen, and T. I. Laakso. 1993 (Tokyo). "Modeling of Woodwind Bores with Finger Holes." *Proceedings of the 1993 International Computer Music Conference.* San Francisco: International Computer Music Association, pp.32–39.

Vidyasagar, M. 1978. *Nonlinear System Analysis.* Englewood Cliffs, New Jersey: Prentice-Hall.

Walther, H. O. 1989. "Hyperbolic Periodic Solutions, Heteroclinic Connections, and Transversal Homoclinic Points in Autonomous Differential Delay Equations." *Memoirs of the American Mathematical Society* 79(402):104.

Wawrzynek, J. C. 1987. "VISI Concurrent Computation for Music Synthesis." PhD dissertation, California Institute of Technology, Pasadena, California.