Chladni Pattern

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Abstract

The report aimed at mathematically describing the Chladni Pattern for a plate based on the equation of motion in two dimensions. It began with deriving the equation of motion of a single point from Newton's laws, and later derived the equation of motion for the vibration in two dimension, and gave the normal modes through the method of separation of variables. The next was deriving the nodal pattern when the initial conditions and boundary conditions were given. At the same time, the wave pattern and nodal pattern of the vibrating plate was drawn by Matlab as a visual help.

The equations of nodal patterns were found at last, and the patterns depended on the nodal numbers m and n which were related with the frequency of the driving force. Also in many situations the patterns were a mixture of some basic forms of patterns.

Acknowledgement

I could not have the courage to write a report about wave equations without the encouragement and continuous help of my supervisor Anders Madsen. His experience in mathematics is like a key to the mathematical world, and he can always explain the complicated techniques in simple language that is comprehensible for me.

Also I want to thank the PHD students Jon Papini and Jesper Christensen, who have introduced me some books as an introduction of partial differential equations, and also helped greatly with the problems on Matlab.

Peter Frederikson, who is the supervisor of my opponent group, gives valuable suggestions about the overall layout of the report.

Contents

Figure 1: The figure is a demonstration of Chladni pattern on a rectangle plate, achieved by drawing a violin bow at the rim of the plate. [Rees T., 2010]

1 Introduction

1.1 About Chladni Pattern

Chladni Pattern is the nodal pattern formed on a plate when an external driving source is given to make the plate vibrate. Figure 1 is a demonstration of how Chladni uses simple devices to visualize sound. On a horizontal plate he put some fine sand, and then give the plate a certain frequency of vibration at some point. As a response, all the points on the plate start to vibrate but each point vibrates with different amplitude. Particularly there are some points that always stand still. Therefore, the sand at the neighbourhood moves towards the "silent" points, which is how the sand pattern is formed. For the same plate, the pattern varies with the frequency of the driving source.

Ernst Chladni (1756-1827) was not the first person who got the idea of spreading sand on a vibrating surface, but he was the first making demonstration to the public and systematically recording the nodal pattern. He is recognized as a physicist, musician and musical instrument maker, and is mostly associated with "Chladni patterns"[Ullmann D. 2007]. Through his comprehensive study into the patterns of vibrations on surfaces, he has got visualized description of music.

Although the "myth" of Chladni's demonstration was not totally understood even by himself, he travelled to many countries in Europe to show the wonderful patterns, and his device even drew the French king Napoleon's interest. Later Napoleon raise a price of one kilo gold for anyone who could give the equation of the wave pattern. [Stöckmann 2007] When the deadline came two years later, nobody showed an satisfactory result to the judge, so the competition was extended for another two years, and the same thing happened. At the third trial a French female mathematician Sophie Germain handed in a paper involving an wave equation that the could make the judge satisfied at last.[University of St Andrews, Scotland, 1996]

I have been interested in the topic about sound since the second semester, when I did an experiment in thermoacoustics to pump heat by sound. The experiment was a success, but I failed at explaining why the sound in everyday life can do that kind of magic, especially at understanding the mathematical part. I start to realize sound is not easy to comprehend although it seems normal. Therefore, I want to learn about the movement of sound or more generally the movement of waves, as sound is a kind of plane wave. Chladni Pattern deals with the movement of wave, which is an old topic and still seems interesting even today. Through the study of this problem, it would be easier for me to read some really complicated equations about waves in future, so as to get into the world of sound.

It is a pity that I do not have my own experiment in this report. The original plan is that I work with another student from the second semester. He is responsible for the experiment part, and I should work on the mathematics. However, due to the the problem that he cannot stick to the work schedule, I have to give up including his part in this report. Instead I take the experiment result from a group in RUC 2007 who has done the same kind of experiment on Chladni Pattern. Luckily, I got the same supervisor of that group, so he passes on valuable experience to me.

1.2 Problem formulation

Main question:
How do we mathematically describe the Chladni patterns from the wave equation in two dimensions?

Sub-questions:

- What is the Wave Equation / Partial Differential Equation (PDE) for the vibration of the plate?
- How to find out the solutions of PDE?
- What is the nodal equation corresponding to a wave equation, so as to describe a Chladni pattern?
- How to use Matlab to draw the graph of wave equations?

► How to use Matlab to draw the graph of nodal equations?

1.3 Fulfilment of semester theme requirement

As there is no requirement in the fourth semester theme. The reason why I choose this topic is that I want to continue to study maths and physics in my third year for bachelor. To be specialized in both subjects, deriving a equation of motion and analysing partial differential differential equation is important to learn. Therefore, I want to take the chance and get myself familiar with both subjects.

The techniques I plan to gain through this project work are:

- Try to use Latex to write a report with many equations.

- Use complex representation to describe waves.

- Learn how to do separation of variables for solving PDE.

- Learn to use Matlab to solve PDE.

1.4 Target group

Chladni oscillation is the first example I choose to look upon, in order to learn about partial differential equation. Therefore, this report is targeted to students of my level, and it starts out with the fundamental knowledge about wave equation and PDE, based on my own understanding.

1.5 Methodology

Wave equation is one of the representative example in the learning of partial differential equations, while the study of Chladni Pattern is actually the study of the mechanical movement of an elastic plate when there is a driving source.

To derive the equations for Chladni Pattern, the strategy is straight forward. First the wave equation for an elastic plate should be derived, and then based on the wave equation, get the equations for nodal lines. For the first task, the theory of Newton's laws should be applied to get the equation of motion. Here forces involved with wave motion is analysed, and partial differential equations to describe the physical problem are formulated. In that part the general form for the solution of wave equation (partial differential equation) and is get through the method of separation of variables.

In the second half of the report, the solution of the wave equation is reached when specific initial condition and boundary condition is taken into consideration. The solution shows the displacement of any point on the plate at any time. Therefore, from that result, the equation for the points with the property that the displacement is always zero can be derived, and it is the equation of the nodal lines.
At the same time, some results from former experiments and Matlab simulations are shown, as a compliment of the wave equation and nodal line equation.

2 Historical Contributions

2.1 About Chladni

As the vertical moment of the plate is rather small, it is hard to make direct observation with naked eyes, so Chladni invented the clever experiment by showing the vibration of the plate by revealing the sand pattern. Now with the help of Matlab, we can make an animation of the vibration (see the Matlab program in Appendix 1, which is an excerption from the book "Advanced mathematics and mechanics applications using MATLAB" [Wilson et al. 2003]). Figure 2 shows at some moments the state of the vibrating plate. Suppose the vertical displacement is u, and u should be a function of any given point (x, y) on the plate and any given time t. More specifically in this program the plate is made to fit into the initial condition and boundary conditions as following:

Initial condition: $u(x, y, 0) = 0$, which means the vertical displacement is zero at any point (x, y) on the plate at time $t = 0$.

Boundary conditions: $u(0.5, 0.5, t) = f(t)$, which means the center of the plate $(0.5, 0.5)$ is moving up and down following the function $f(t)$, and here $f(t)$ is a harmonic motion as in a sine or cosine function. Also the edge of the plate is hold still, which can be expressed as $u(\triangle\Omega, t) = 0$, and $\triangle\Omega$ means the boundary of the plate.

Well, the first drawing of figure 2 shows the plate just start to move at the center point, and the other three drawings show the instantaneous state of the plate.

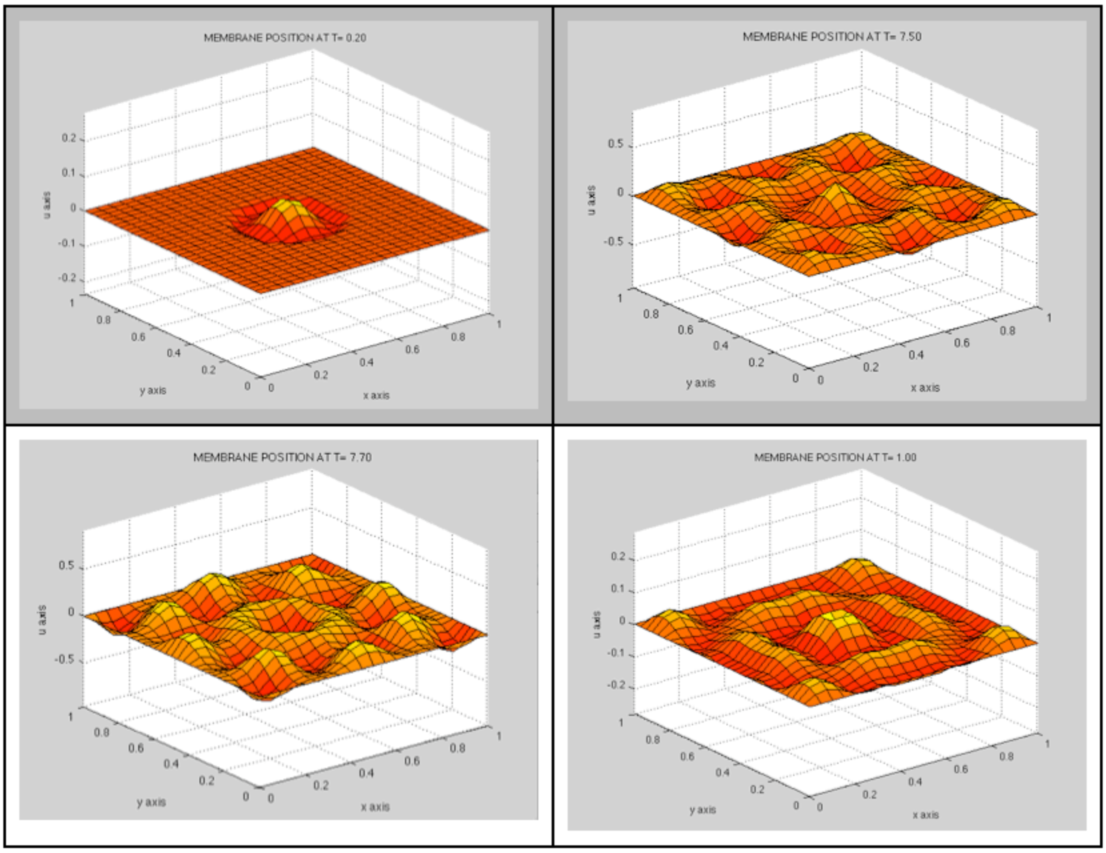

Figure 2: The figure shows snapshots of the animation of vibrating plate made by Matlab.

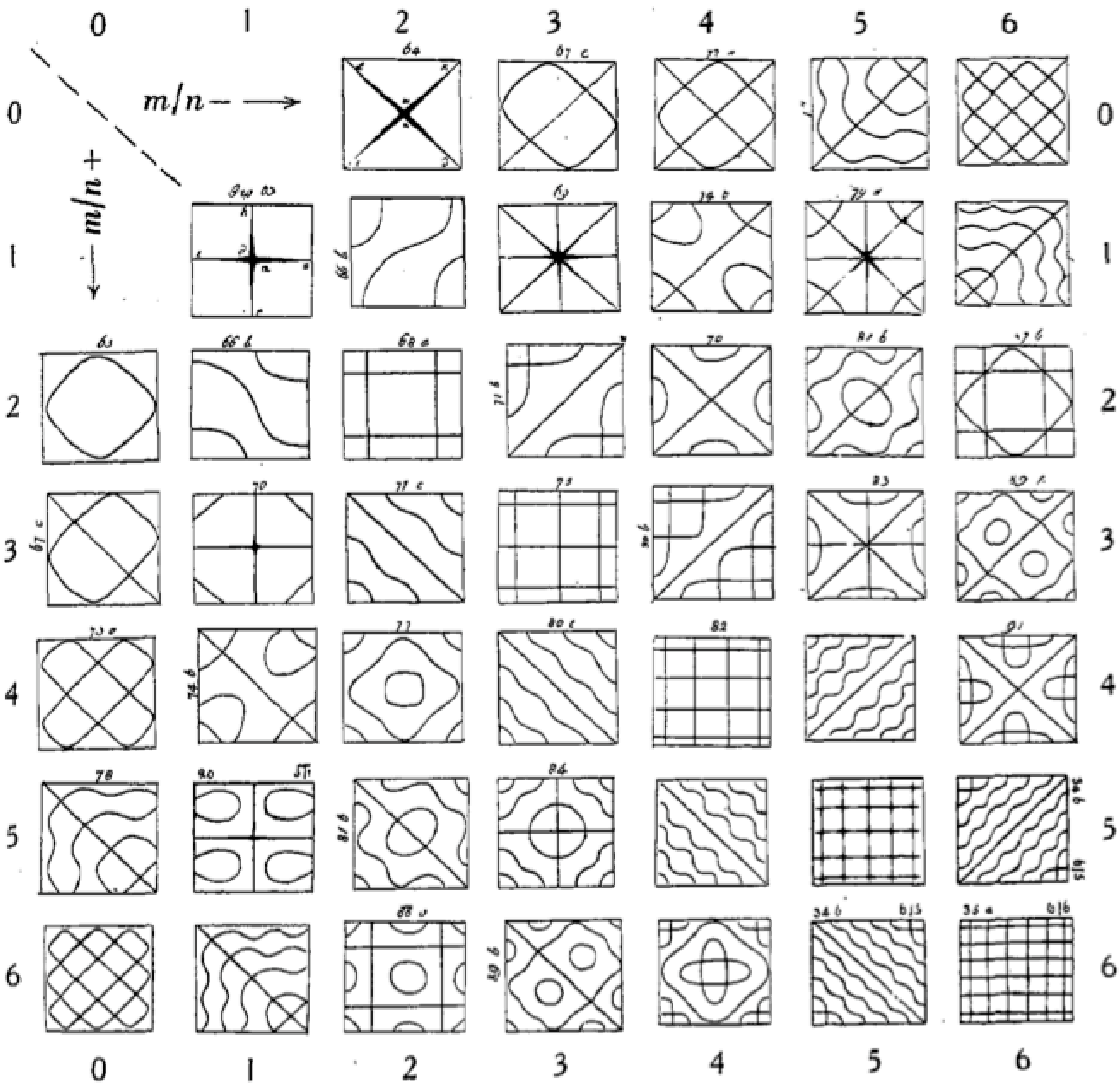

Figure 3: The figure shows Chladni pattern. [Waller 1939]

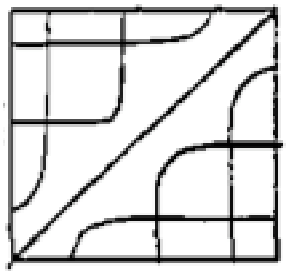

Figure 4: A case from Chladni pattern, to show the number of m and n, Here $m = 4$ and $n = 3$. [Waller M. D. 1939]

From a intuitive guess, for the same plate, different frequencies of vibration give different pattern, and Chladni has made drawing of those pattern in a systematic way, see Figure 3. Later on, Chladni postulated that there is a relationship between the frequency and nodal numbers, by saying $f \sim (m + 2n)^2$ [Kverno and Nolen 2010], where the integers m and n are the number of node lines by counting the diametric node number a and side node number n, and $m + n = a$. For example, in figure 4, $m + n = 7$ and $n = 4$, so $m = 3$. Therefore, according to Chladni's hypothesis, the frequency is $f \sim 11^2$.

2.2 Sophie Germain

Sophie Germain (April 1, 1776 – June 27, 1831) was a female mathematician. She was fascinated by Chladni's experiment with elastic plates, and took part in a contest arranged by the Paris Academy of Science, with the aim "to give the mathematical theory of the vibration of an elastic surface and to compare the theory to experimental evidence". After years of trying, Sophie Germain submitted her paper "Recherches sur la théorie des surfaces élastique" and won the prize at last.[Petrovich V.C. 1999] In the paper, her equation for the vibration was:

$$N^2(\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4}) + \frac{\partial^2 z}{\partial t^2} = 0 \quad (1)$$

where N^2 is a constant. [Case and Leggett 2005]

2.3 Lord Rayleigh

Lord Rayleigh is another scientist who shows great interest into sound. In his book "The theory of sound" there are two chapters "Vibrations of Membranes", and "Vibrations of Plates" [Rayleigh and Strutt 1945, P306-394] that are referred many times in this report.

2.4 Mary Waller

Mary D. Waller is also a female scientist, who has made contributions to the theoretical study of Chladni plate. She modified Chladni's hypothesis of $f \sim (m + 2n)^2$. Instead, she agrees that $f \sim (m + bn)^2$, but b increases from 2 to 5 as m gets larger. [Kverno and Nolen 2010]

2.5 A project work in 2007

In 2007 a group in Natbas RUC did experiment about Chladni oscillation, and made their analysis of the equation:

$$\frac{f}{c} = \sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}} = \sqrt{\frac{p^2}{L_x^2} + \frac{q^2}{L_y^2}} \quad (2)$$

where f is frequency, c is wave speed, L_x and L_y are the dimensions of the plate, [Anarajalingam, Duch langpap and Holm 2007]. We will talk that equation in detail later.

In their report they say that the results from the experiments are not always pure model, and always a combination of two pure ones, where the two sets of numbers can be obtained from Equation (2). Figure 5 is an example

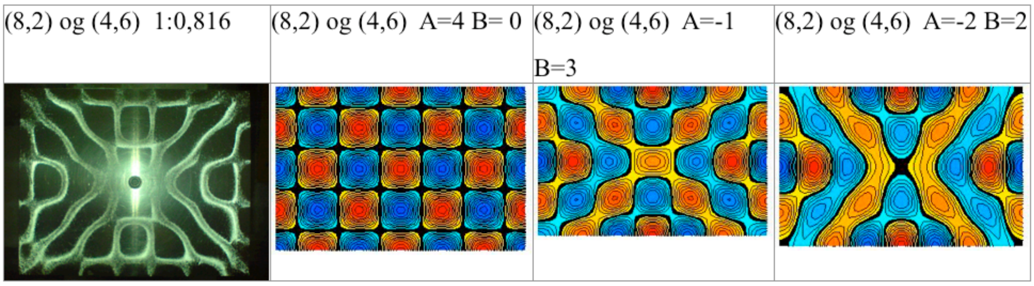

Figure 5: An example showing the experiment result is a combination of two patterns. [Anarajalingam et al. 2007]

from their analysis. Here the ratio between the length and width of the plate is 1 : 0.816, from which they get $m = 8, n = 2$ and $p = 4, q = 6$. The last three drawings from Figure 5 is made by making a combination of the two sets of numbers, but with different weighing of each set.

In this report, the mathematics about Chladni pattern will be discussed. Start with deriving the wave equation, which is a general equation for vibration on a membrane, and further the initial condition which tells at a given time t_0 the position or velocity of the plate and boundary condition which tells what happens on the boundary at all the time t will be added to solve for a particular situation. At last, the nodal line equation will be introduced and discussed in the end.

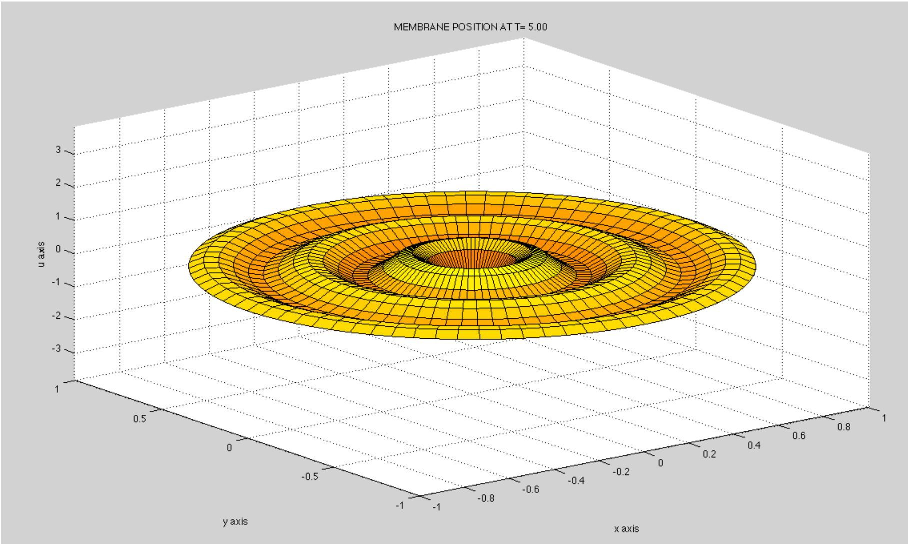

Figure 6: An drawing made by Matlab shows the vibration of a disk, driven at the center.

3 The Equations of Motion and Normal Modes

3.1 Basic characterization

Here is some basic concepts to know about the oscillating plate, and I choose the example in Figure 6. A disk that is initially at rest, and then made to vibrate by giving its center point a harmonic motion. At the same time, hold the edge still (fixed boundary).

3.1.1 The driving point at the center

The following parameters describe the movement of the center driving point:

- Amplitude A: The maximum displacement of an oscillator from its equilibrium position.

- Period T: The time needed for finishing a cycle.

- Frequency f: The number of cycles happened in one unit time, which is $\frac{1}{T}$.

- Angular frequency ω: Frequency converted into radius.

3.1.2 Motion along the radius

As for the transmition of wave pattern along the radius of the disk:

- Wave Speed v: How fast the wave pattern is transmitted along the radius r.

- Wave Length λ: How far the wave pattern is transmitted within one period time. $v = \lambda/T$

Mathematically, there are standard form of wave equations that is convenient to apply for different purposes. In this chapter, we will start with the easiest simply harmonic motion, and then look into the equation of motion for a vibrating string. At last, we will try to derive the equation of motion for a plate.

3.2 Wave equation for a simple harmonic motion

3.2.1 Derive the equation of motion

Now we take the simplest form of vibration, simple harmonic motion of a mass point, which is the base to derive complex vibrations, for example, the Chladni vibration. Taking the spring-mass system as example, a massless spring is fixed one end to the wall and the other to a mass m. The spring constant is k. The equilibrium point is the position x_{eq} where the mass receives no force from the spring, and x is the displacement that m is away from the equilibrium position.

Apply Newton's second law $F = ma$, we can write:

$$-kx = m\frac{\partial^2 x}{\partial t^2} \tag{3}$$

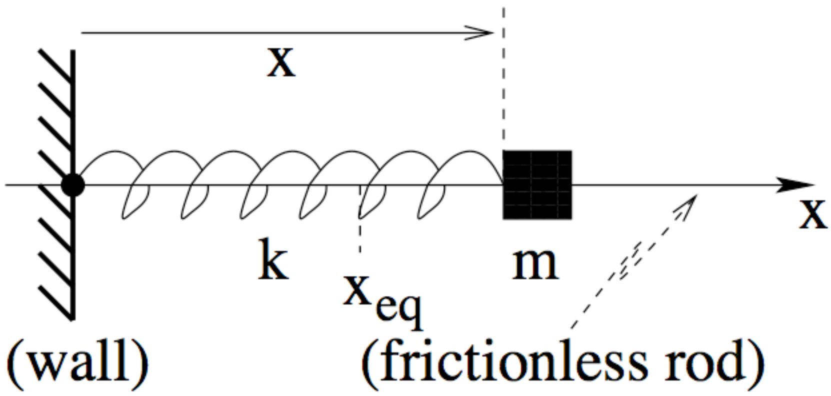

Figure 7: The figure shows the mass-spring system. [Tomas Arias, 2003]

where $-kx$ is the net force that m receives, as the normal force from the table and gravity cancel each other, and the negative sign means that the spring force has opposite direction with the displacement x. $\frac{\partial^2 x}{\partial t^2}$ is the acceleration a in Newton's second law.

Equation (3) is called equation of motion for a spring-mass system, as it only contains the position, the second derivative of position, and constants.

3.2.2 Find a general solution

In order to know where m is at time t, that is to express x in explicit mathematical formula. From experience, we know that $x = sin(\omega t)$ and $x = cos(\omega t)$ are functions that have similar forms after deriving twice with respect to t, where ω is a constant, and can leave a negative sign in front of the second derivative. In steps:

$$x = sin(\omega t) \tag{4}$$

$$\frac{\partial x}{\partial t} = \omega cos(\omega t) \tag{5}$$

$$\frac{\partial^2 x}{\partial t^2} = -\omega^2 sin(\omega t) \tag{6}$$

In similar way for $x = cos(\omega t)$

$$x = cos(\omega t) \tag{7}$$

$$\frac{\partial x}{\partial t} = -\omega sin(\omega t) \tag{8}$$

$$\frac{\partial^2 x}{\partial t^2} = -\omega^2 cos(\omega t) \tag{9}$$

Inserting the above results for $x = sin(\omega t)x$ into Equation (3), we have:

$$-ksin(\omega t) = m[-\omega^2 sin(\omega t)] \tag{10}$$

$$\omega^2 = \frac{k}{m} \tag{11}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{12}$$

Also we can get the same value of ω when use $x = cos(\omega t)$. Remember that $x(t)$ is a displacement about the equilibrium point x_{eq}. Therefore, the general solution for the configuration of a spring-mass system can be written as:

$$x(t) = x_{eq} + C_1 sin(\omega t) + C_2 cos(\omega t) \tag{13}$$

Equation (13) should be a general solution for all simple harmonic motion, as it includes all the combinations of possible solutions. We can make the solution look shorter by using trigonometric trick:

x(t)=x$_{eq} + C_1 sin(\omega t) + C_2 cos(\omega t)$

x(t)=x$_{eq} + \sqrt{C_1^2 + C_2^2}[\frac{C_1}{\sqrt{C_1^2+C_2^2}} sin(\omega t) + \frac{C_2}{\sqrt{C_1^2+C_2^2}} cos(\omega t)]$

Let:

$$\frac{C_1}{\sqrt{C_1^2 + C_2^2}} = -sin\phi \tag{14}$$

$$\frac{C_2}{\sqrt{C_1^2 + C_2^2}} = cos\phi \tag{15}$$

Then

$$x(t) = x_{eq} + \sqrt{C_1^2 + C_2^2} cos(\omega t + \phi) \tag{16}$$

3.2.3 Find the solution by complex representation approach

Actually, there is a more simple way to reach the solution, which is by using the complex representation:
Equation (16) can be written as:

$$x(t) = x_{eq} + \sqrt{C_1^2 + C_2^2} \Re(e^{(\omega t + \phi)i}) \tag{17}$$

where $\Re$ means taking the real part of $e^{(\omega t + \phi)i}$

That is because of Euler's equation $e^{x+yi} = e^x(cosy + isiny)$. Using the complex representation can shorten the writing. Now we try to get the general solution by using complex expression:

From the wave equation $\frac{d^2u}{dt^2} = -\omega^2 u$, where ω is real number (adapted from the previous wave equation to make some conveniences in notation). Immediately we know the answer is $u = e^{\lambda t}$, where $\lambda^2 = -\omega^2$ should hold, which means $\lambda = \pm i\omega$. Therefore,

$$u = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{18}$$

is the answer, where C_1 and C_2 are complex numbers.

As u is a real number, from real number property $u = \overline{u}$. Then:

$$C_1 e^{i\omega t} + C_2 e^{-i\omega t} = \overline{C_1} e^{-i\omega t} + \overline{C_2} e^{i\omega t} \tag{19}$$

$$\Rightarrow C_2 = \overline{C_1} \tag{20}$$

Let $C = C_1$, then $u = Ce^{i\omega t} + \overline{Ce^{-i\omega t}}$, so

$$u = \Re(2Ce^{i\omega t}) \tag{21}$$

where $\Re$ means take the real part of $(2Ce^{i\omega t})$. Because any complex number $2C$ can be written as $2C = Ae^{i\phi}$ (if $C = C_x + iC_y$, A=$2\sqrt{C_x^2 + C_y^2}, \phi = arctan\frac{C_y}{C_x}$) :

If the equilibrium position is not zero, then $u = A_{eq} + Acos(\omega t + \phi)$.

3.2.4 Find a particular solution

A particular solution is a specific form of Equation (13) and made to suit specific conditions. As there are two unknown constants C_1 and C_2, two conditions are required to reach a particular solution. Suppose that at time $t = 0$, $x = x_0$ and the velocity $\frac{\partial x}{\partial t} = v_0$. Then we need to put the two conditions into the general solution:

$$x_0 = x_{eq} + C_1 sin(\omega * 0) + C_2 cos(\omega * 0) \tag{22}$$

$$x_0 = x_{eq} + C_2 \tag{23}$$

$$C_2 = x_0 - x_{eq} \tag{24}$$

and

$$\frac{\partial x(t)}{\partial t} = C_1 \omega * cos(\omega t) - C_2 \omega * sin(\omega t) \tag{25}$$

$$v_0 = C_1 \omega cos(\omega * 0) - C_2 \omega sin(\omega * 0) \tag{26}$$

$$v_0 = C_1 \omega \tag{27}$$

$$C_1 = \frac{v_0}{\omega} \tag{28}$$

Now we know the particular solution in that case:

$$x(t) = x_{eq} + \frac{v_0}{\omega} sin(\omega t) + (x_0 - x_{eq}) cos(\omega t) \tag{29}$$

3.3 Wave equation for a string

As for a string, there are countless oscillators to be taken into consideration. Obviously, the result from last section will help, and we will develop the equation of motion for a string in a similar way. However, there are too many variations that a string can make, so we take an example as in Figure 8, where one end of a horizontal string is fixed to the wall and the other end is attached to a power source through a small hole.

To make a start, we need some assumptions:

First, the mass of the string is evenly distributed. If the string mass is M and the length is L, then the mass per unit is $\mu = \frac{M}{L}$.

Second, in the x direction, a way of approximation is used to make the analysis simple. That is suppose that there is no motion along the x direction for any small section in the string, and it only moves up and down.

Third, as for the tension in a string, for each segment, it is the pulling force from both of its joint neighbour. Assume that gravity affects the string's vibration very little compared with the tension force, so the direction of the tension is always along the tangent line on each short segment.

3.3.1 Derive the equation of motion

First take a look at the free-body diagram which illustrates all the forces on a small section of a vibrating string, Figure 9. As we have restricted that any

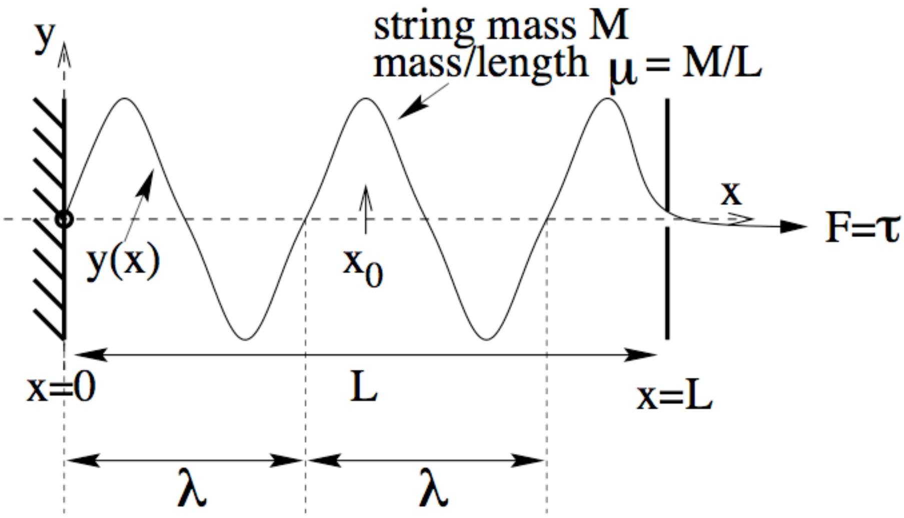

Figure 8: The figure shows the free-body diagram for a small section of a vibrating string. [Tomas Arias, 2001]

small section of the string does not move along the x axis, which means the velocity of the string in the x direction is zero and the net force in x axis for any section is also zero, according to Newton's second law. Set the tension in the x direction to $T_x = \tau$.

Now the task is left to find the displacement in the y direction, for any point at x at any time. To use Newton's second law, we need to find the net force in y direction in Figure 9.

Therefore, $F_y = T_y(x + \Delta) - T_y(x)$. Now we are ready to apply Newton's

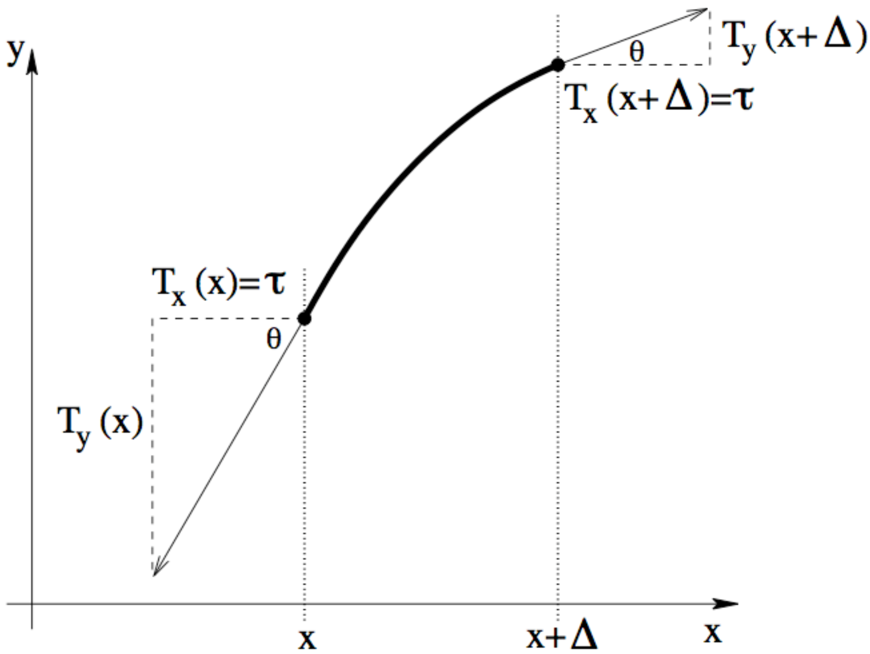

Figure 9: The figure shows the free-body diagram for a small section from a vibrating string. [Tomas Arias, 2001]

second law:

$$F_y = ma_y \tag{30}$$

$$T_y(x+\Delta) - T_y(x) = (\mu\Delta)a_y \tag{31}$$

$$\frac{T_y(x+\Delta) - T_y(x)}{\Delta} = \mu a_y \tag{32}$$

When Δ is really small, $\frac{T_y(x+\Delta)-T_y(x)}{\Delta}$ is actually $\frac{\partial T_y}{\partial x}$, and a_y is $\frac{\partial^2 y(x,t)}{\partial t^2}$ by definition. Equation (32) can be rewritten as:

$$\frac{\partial T_y}{\partial x} = \mu\frac{\partial^2 y(x,t)}{\partial t^2} \tag{33}$$

In order to construct the equation of motion, the part $\frac{\partial T_y}{\partial x}$ needs to be converted into some form only expressed with y,x or t. There is something special about force in a string, which is the force generated inside a string is always a pulling force and can only be along the tangent line for the string curve. The tangent Figure 9 at point x is $\frac{\partial y}{\partial x}$ and the tension in the x direction is τ, so

$$\frac{T_y}{\tau} = \frac{\partial y}{\partial x} \tag{34}$$

$$T_y = \tau\frac{\partial y}{\partial x} \tag{35}$$

Insert quation (35) into Equation (33), and we can get:

$$\frac{\partial}{\partial x}(\tau\frac{\partial y}{\partial x}) = \mu\frac{\partial^2 y(x,t)}{\partial t^2} \tag{36}$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 y(x,t)}{\partial x^2} \tag{37}$$

Equation (37) is the equation of motion for a string. Similar with the case of a mass point, we can get the solution form:

$$y = A\frac{sin}{cos}\}kx\frac{sin}{cos}\}\omega_n t \tag{38}$$

where A is a constant and $\frac{sin}{cos}$ means either sin or cos could be in the solution depending on the initial conditions and boundary conditions. [Pain 2005, P245]

3.4 Wave equation for a plate

3.4.1 Equation of motion

The wave equation for a plate can be seen as a combination of two independent string waves. As an assumption, first write the wave equation of a vibrating plate similar with Equation (37):

$$\frac{1}{c^2}\frac{\partial^2 U(x,y,t)}{\partial t^2} = \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2} \tag{39}$$

[Pain H. J., 2005 P:246], where $U(x,y,t)$ is a function describing the displacement in the z direction of each point (x,y) on the plate at any given time t.

The proof of the wave equation in two dimensions can be found in Lord Rayleigh's book "The theory of sound", where he examines the tension force in the neighbourhood of any point P on the membrane, and he also gives another method, that is by looking at the potential energy. [Rayleigh and Strutt 1945, P306-307]

The wave equation in two dimensions above can be expended into n dimensions:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} \tag{40}$$

or we can simply write:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{41}$$

where $u = U(x_1, x_2, ..., x_0, t)$, and the notation $\nabla^2 u$ is the Laplacian of u. [Logan 1998, P34]

3.4.2 Separation of variables

Suppose x, y, and t are mutually independent variables, and those three variables are in the following functions respectively:

$$X = X(x) \tag{42}$$

$$Y = Y(y) \tag{43}$$

$$T = T(t) \tag{44}$$

Then the displacement $U(x, y, t)$ is:

$$U(x, y, t) = X(x)Y(y)T(t) \tag{45}$$

From Equation (45), we can get:

$$\frac{\partial^2 U(x, y, t)}{\partial t^2} = X(x)Y(y)\frac{\partial^2 T(t)}{\partial t^2} \tag{46}$$

$$\frac{\partial^2 U(x, y, t)}{\partial x^2} = Y(y)T(t)\frac{\partial^2 X(x)}{\partial x^2} \tag{47}$$

$$\frac{\partial^2 U(x, y, t)}{\partial y^2} = X(x)T(t)\frac{\partial^2 Y(y)}{\partial y^2} \tag{48}$$

Take the results from Equation (46), (47) and (48), and plug them into Equation (39). We get:

$$\frac{1}{c^2}X(x)Y(y)\frac{\partial^2 T(t)}{\partial t^2} = Y(y)T(t)\frac{\partial^2 X(x)}{\partial x^2} + X(x)T(t)\frac{\partial^2 Y(y)}{\partial y^2} \tag{49}$$

Divide Equation (49) by $X(x)Y(y)T(t)$, and it becomes:

$$\frac{\frac{1}{c^2}\frac{\partial^2 T(t)}{\partial t^2}}{T(t)} = \frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)} + \frac{\frac{\partial^2 Y(y)}{\partial y^2}}{Y(y)} \tag{50}$$

The left side of Equation (50) is only about the variable t, and the right is about x, and y. As we know x, y and t are independent variables, so each side of Equation (50) must equal to the same constant, $-k^2$ say. Now we can separate Equation (50) into two equations:

$$\frac{\frac{1}{c^2}\frac{\partial^2 T(t)}{\partial t^2}}{T(t)} = -k^2 \tag{51}$$

$$\frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)} + \frac{\frac{\partial^2 Y(y)}{\partial y^2}}{Y(y)} = -k^2 \tag{52}$$

In the same way,Equation (52) can be separated into two, and the terms about x and y differ by only a constant $-k^2$:

$$\frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)} = -k_1^2 \tag{53}$$

$$\frac{\frac{\partial^2 Y(y)}{\partial y^2}}{Y(y)} = -k_2^2 \tag{54}$$

$$-(k_1^2 + k_2^2) = -k^2 \tag{55}$$

Now we have put the three independent variables t, x and y into three equations with only t, x or y respectively. The task left is to solve Equation (51), (53) and (54) for T, X and Y, and that could be easily done by recalling what we did in Chapter 2:

$$T(t) = a_1 e^{\pm ickt} \tag{56}$$

$$X(x) = a_2 e^{\pm ik_1 x} \tag{57}$$

$$Y(y) = a_3 e^{\pm ik_2 y} \tag{58}$$

In the same manner, we can get the solution form:

$$u = A\frac{sin}{cos}\}k_1 x\frac{sin}{cos}\}k_2 y\frac{sin}{cos}\}ckt \tag{59}$$

where A is a constant and $k^2 = k_1^2 + k_2^2$. [Pain 2005, P247]

4 The nodal line equation

4.1 In a closed boundary situation

Until now we have derived the equation of motion in two dimensions, and get the normal form of the solution. The job next is to look into the problem when specific initial conditions and boundary conditions are given, in a fixed boundary case:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, 0 < x < a, 0 < y < b, t > 0 \tag{60}$$

$$u(\triangle\ \Omega, t) = 0, t > 0 \tag{61}$$

$$u(x, y, 0) = 0, 0 < x < a, 0 < y < b \tag{62}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = 0, 0 < x < a, 0 < y < b \tag{63}$$

- ▶ Equation (60) is the equation of motion we have derived.
- ▶ Equation (61) gives the boundary condition, saying the points within the boundary $\triangle\ \Omega$ are not moving at any time.
- ▶ Equation (62) is the initial condition, saying the plate is flat initially.
- ▶ Equation (63) is another initial condition, saying the initial velocity of each element on the plate is zero.

From before we know that the solution of the wave equation takes the form $u = \frac{sin}{cos}\}k_1x\frac{sin}{cos}\}k_2y\frac{sin}{cos}\}ckt$.

Put the solution form into the boundary condition $u(\triangle\ \Omega, t) = 0$ we get $u = \frac{sin}{cos}\}k_1a\frac{sin}{cos}\}k_2b\frac{sin}{cos}\}ckt = 0$. As $\frac{sin}{cos}\}ckt$ is not always zero, while

$\frac{sin}{cos}\}k_1 a \frac{sin}{cos}\}k_2 b$ is a constant, we have to make $\frac{sin}{cos}\}k_1 a \frac{sin}{cos}\}k_2 b$ be zero, that is

$$\frac{sin}{cos}\}k_1 a = 0, \tag{64}$$

$$\frac{sin}{cos}\}k_2 b = 0 \tag{65}$$

The simplest form we can get from Equation (64) and (65) is

$$sink_1 a = 0, k_1 = \frac{m\pi}{a} \tag{66}$$

$$sink_2 b = 0, k_2 = \frac{n\pi}{b} \tag{67}$$

where m and n are integers. so the solution form is $u = sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \frac{sin}{cos}\}pt$, where p is a constant.

Now consider the initial condition $\frac{\partial u}{\partial t}(x, y, 0) = 0, 0 < x < a, 0 < y < b$. Regardless of the part involved with x and y, consider $\frac{\partial}{\partial t} \frac{sin}{cos}\}pt = 0$ at $t = 0$. Then the simplest form is $\frac{\partial}{\partial t} cospt$, and until now the solution form is $u = sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} cospt$. Plug that result into the wave equation, we get:

$$p^2(-u) = c^2(-\frac{m^2\pi^2}{a^2}u - \frac{n^2\pi^2}{b^2}u) \tag{68}$$

that is:

$$p^2 = c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2}) \tag{69}$$

Therefore, the solution form is

$$u = sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} cospt \tag{70}$$

where $p^2 = c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$.

From this the general solution can be derived. So

$$u = \Sigma_{m=1}^{m=\infty} \Sigma_{n=1}^{n=\infty} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \{A_{mn} cospt + B_{mn} sinpt\} \tag{71}$$

[Rayleigh and Strutt 1945, P307]

where

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b u_0 sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy \tag{72}$$

$$B_{mn} = \frac{4}{abp}\int_0^a\int_0^b \dot{u_0} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy \tag{73}$$

[Rayleigh and Strutt 1945, P308]

The coefficients A_{mn} and B_{mn} come from any arbitrary initial conditions which defines the initial positions u_0 and initial velocity $\dot{u_0}$, and they can be calculated according to Fourier expansion.

No matter what function of u_0 about x and y is, it can be expressed with a Fourier series about x, within the domain from 0 to a

$$Y_1 sin\frac{\pi x}{a} + Y_2 sin\frac{2\pi x}{a} + ... \tag{74}$$

where Y_1, Y_2 ... are functions of y, and they are the Fourier coefficients calculated as

$$Y_m = \frac{2}{a}\int_0^a u_0 sin\frac{m\pi x}{a} dx \tag{75}$$

[Logan 1998, P103]

In the same manner, each of the Y functions can be expanded within the range from 0 to b

$$C_1 sin\frac{\pi y}{b} + C_2 sin\frac{2\pi y}{b} + ... \tag{76}$$

where C_1, C_2... are constants, that can be calculated as

$$C_n = \frac{2}{b}\int_0^b Y_m sin\frac{n\pi y}{b} dy \tag{77}$$

[Logan 1998, P103]

Combining equations (75) and (77), we get

$$C_{mn} = \frac{4}{ab} \int_0^a \int_0^b u_0 sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy. \tag{78}$$

From equation (71), put $t = 0$, and we know that

$$u_0 = \Sigma_{m=1}^{m=\infty} \Sigma_{n=1}^{n=\infty} A_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \tag{79}$$

Therefore, $A_{mn} = \frac{4}{ab} \int_0^a \int_0^b u_0 sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy.$

Another arbitrary initial condition is the velocity $\dot{u_0}$, which is $\frac{\partial u}{\partial t}$ when t is zero. Since from equation (71), we can get the expression for $\dot{u_0}$

$$\dot{u} = \Sigma_{m=1}^{m=\infty} \Sigma_{n=1}^{n=\infty} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} p\{-A_{mn} sinpt + B_{mn} cospt\} \tag{80}$$

then, when $t = 0$

$$\dot{u_0} = \Sigma_{m=1}^{m=\infty} \Sigma_{n=1}^{n=\infty} pB_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \tag{81}$$

Use the same trick for getting A_{mn}, we have the expression
$B_{mn} = \frac{4}{abp} \int_0^a \int_0^b \dot{u_0} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy$.

From the general solution equation (71), we know that if for any point on the plate the part about x and y is constantly zero, even though t varies, the displacement of that point is always zero. That is the character of a nodal point. Thus, set

$$sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} = 0 \tag{82}$$

where the relationship between x and y depends on the values of m and n. As before we have derived the expression
$p^2 = c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$, where p is the frequency of the applied load [Rayleigh

and Strutt 1945, P311]. In some cases, there should be other pair of integers, for example, p and q that can fit into the equation, and they give some other nodal pattern, so the general pattern that appears in the experiment is a mixture of all the "pure" individual patterns.

$$Csin\frac{m\pi x}{a}sin\frac{n\pi y}{b} + Dsin\frac{p\pi x}{a}sin\frac{q\pi y}{b} = 0 \tag{83}$$

Particularly, when a and b are equal and m and n are not equal, we can simply switch m and n to get the other pair p and q, and the nodal line expression is the mix of two patterns

$$Csin\frac{m\pi x}{a}sin\frac{n\pi y}{a} + Dsin\frac{n\pi x}{a}sin\frac{m\pi y}{a} = 0 \tag{84}$$

where C and D are the weight of each pattern. However, because we are not certain about values of the constants C, D, m and n, it is impossible to separate x and y. If the explicit nodal line expression is necessary, we have to be given those values, and Rayleigh has made some sample calculation in his book "The theory of sound" [Rayleigh and Strutt 1945, P311-315]. One particular example could be when $m = 2$, $n = 1$, $a = b = 1$, and we use the Matlab program from Appendix 3, but change the *cos* into *sin* functions. Thus, here are some shoots of different mixtures of both patterns, see Figure 10.

So far, we can get the nodal lines of a vibrating plate in a closed boundary situation, when provided with the numbers m, n, C and D.

4.2 In an open boundary situation

Calculating the expression of nodal line in an open boundary situation is a little tricky that the boundary is also moving up and down. Recall what we did in deriving the wave equation for a string, theoretically we assume that

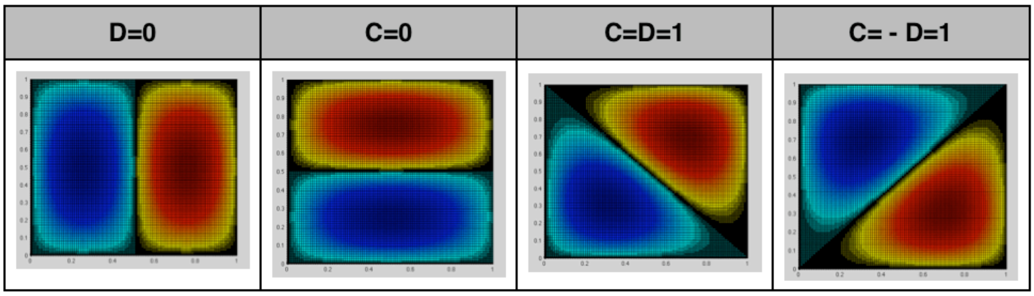

Figure 10: The figure shows how the nodal pattern looks like when we tune the weight of each "pure" pattern in a closed boundary situation, the black lines are the nodal lines. Here $m = 2$, $n = 1$, and $a = b = 1$.

the acceleration of the boundary elements is only horizontal. [Arias 2001]

In the end, we should be able to derive the nodal line expression in an open boundary situation like this

$$C cos\frac{m\pi x}{a} cos\frac{n\pi y}{b} + D cos\frac{p\pi x}{a} cos\frac{q\pi y}{b} = 0 \tag{85}$$

[Waller, 1939]

The above equation is in the program in Appendix 3, and we give the same constants as we did in the closed boundary situation, to get the following patterns in Figure 11.

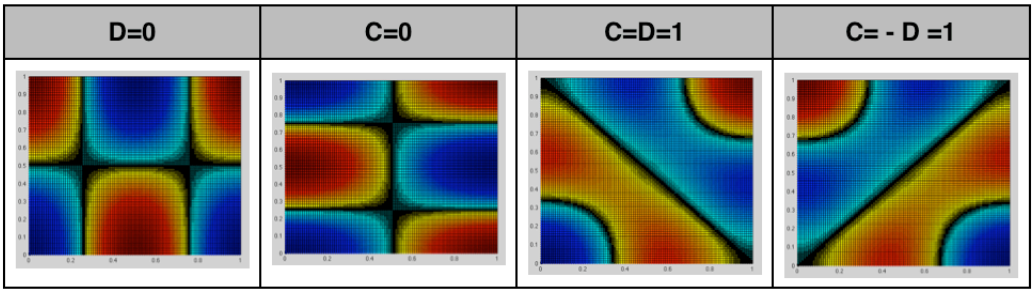

Figure 11: The figure shows how the nodal pattern looks like when we tune the weight of each "pure" pattern in an open boundary situation, the black lines are the nodal lines. Here $m = 2$, $n = 1$, and $a = b = 1$.

5 compare Matlab simulations with experiment results

5.1 Nodal pattern from the vibration

Before we have seen the Matlab simulation of an elastic plate in a closed boundary situation, but the nodal lines are not shown clearly. Therefore, Jesper Christensen has helped to modify the Matlab program of Appendix 1, and made the program to draw the nodal lines on the basis of the vibration, see Appendix 2.

From the previous analysis we know that these nodal lines follow the pattern given by the equation
$Csin\frac{m\pi x}{a}sin\frac{n\pi y}{b} + Dsin\frac{n\pi x}{a}sin\frac{m\pi y}{b} = 0$, where C and D are constants.

Now let us see two examples from the program, both are under closed boundary condition and the frequency of the loaded force is chosen as one of their

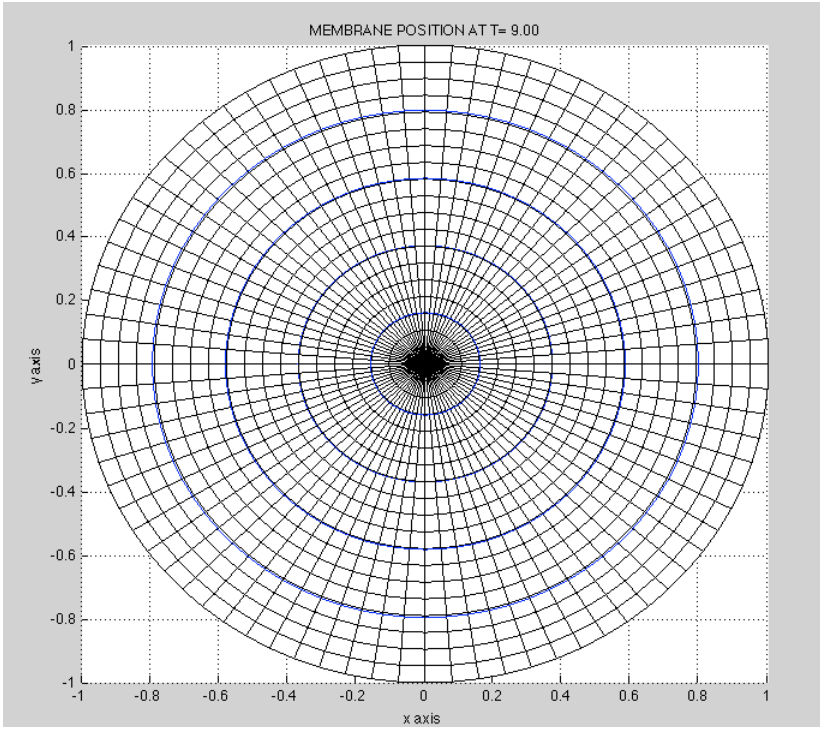

Figure 12: The figure shows how the nodal pattern of a disk looks like when it vibrates under one of its natural frequencies. The blue lines are the nodal lines.

natural frequencies, which are provided within the program.
Figure 12 shows the nodal lines (the blue lines) of a disk and Figure 13 shows the nodal lines of a square plate.

Notice in Figure 13 which describes the nodal lines that is the result of the vibration, so the pattern lines are not parallel to the borders of the square, and are already ”mixed”.

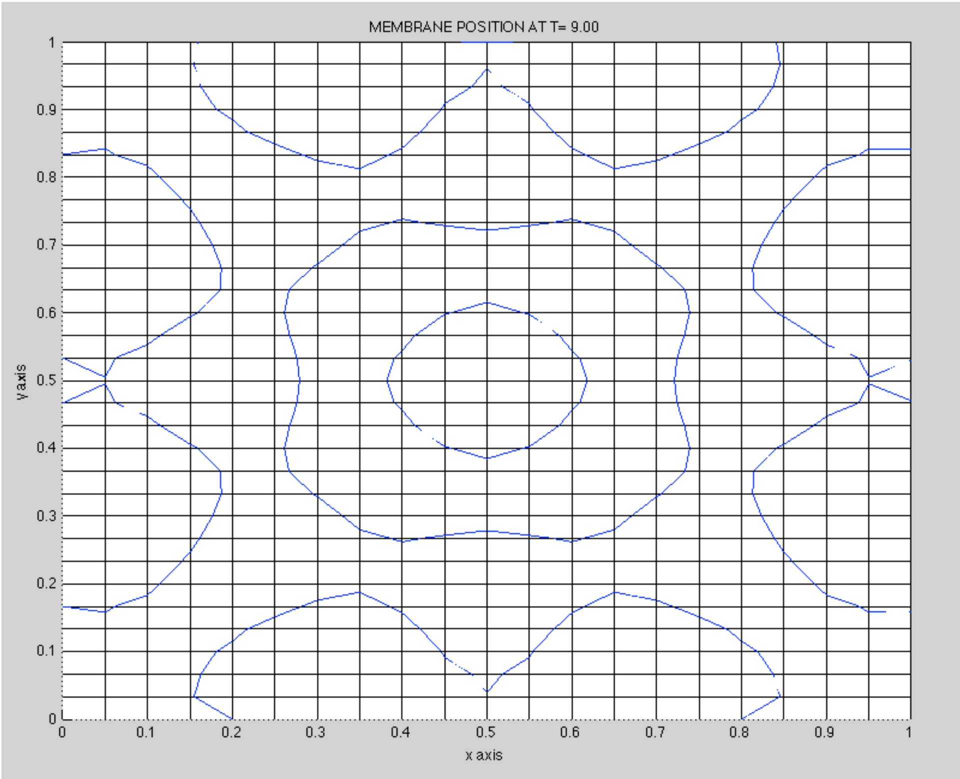

Figure 13: The figure shows how the nodal pattern of a square looks like when it vibrates under one of its natural frequencies. The blue lines are the nodal lines.

5.2 Nodal pattern from the nodal equation in open boundary condition

This situation is the focus of the report from the group in 2007 [Anarajalingam, Duch and Holm 2007]. They have discussed about when the sides of the square are not equal, and then the "blending ratio" (C and D) also varies. The method they use is mainly by comparing the experiment result with the drawings from computer, which are calculated from the equation $Ccos\frac{m\pi x}{a}cos\frac{n\pi y}{b} + Dcos\frac{n\pi x}{a}cos\frac{m\pi y}{b} = 0$.

Appendix 3 is a Matlab program that we can use to mix the two patterns in our arbitrary style. The six constants in the above equation can be changed as we want.

However, there are some restrictions that we should follow about tuning the constants, judging from the analysis in previous chapter:

- C and D real numbers.
- m and m are positive integers.
- a and b are positive real numbers.

Therefore, we cannot give zero to any of the m or n values, but the 2007 group tried to include that kind of pattern into the blending type. The following example is from their report that they put n to zero: [Anarajalingam, Duch langpap and Holm 2007]

We cannot see there is a good match between the experiment result and the computer simulation, in Figure 14.

So let us see if there are other possibilities. In the example in Figure 14,

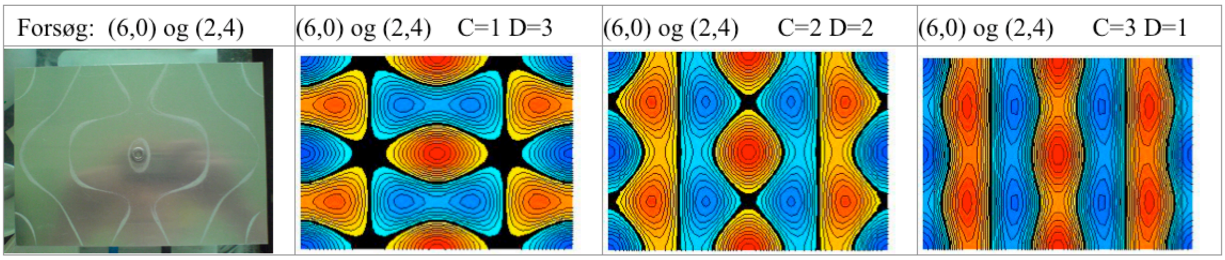

Figure 14: The figure is from the 2007 report, which is not appropriate because $n = 0$.

the ratio between the two sides is

$$1 : 0.7071$$

Because in equation $p^2 = c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$ we need the square of those constants, that is $\frac{a^2}{b^2} = \frac{1^2}{0.7071^2} \doteq \frac{2}{1}$, then $\frac{m^2}{2} + n^2$ is a constant if the frequency of vibration is fixed.

We can see from the experiment result in Figure 14 that it is possible that$m + n = 8$, but we cannot be sure that addition fits for all the pairs because the plate is not square, but there is only limited choices of the positive integer pairs even though we try a wider range than exactly 8. After try out the possible numbers, we get
$\frac{4^2}{2} + 5^2 = \frac{8^2}{2} + 1^2 = 33$ and
$\frac{5^2}{2} + 4^2 = \frac{7^2}{2} + 2^2 = 28\frac{1}{2}$

Then we use the two pairs of m and n to make Matlab simulation, by the program in Appendix 3:
First consider the pair $m = 4, n = 5$ and $m = 8, n = 1$. Next consider the pair $m = 5, n = 4$ and $m = 7, n = 2$. By comparison, the drawings from Figure 16 dose not seem to have a tendency of approaching the ex-

Figure 15: In this table there are calculations of $\frac{m^2}{2} + n^2$, when different values of m and n are tried.

(m,n)	m^2 /2 + n^2	(m,n)	m^2 /2 + n^2
(2, 5)	37	(1, 6)	36.5
(2, 6)	38	(3, 4)	20.5
(4, 3)	17	(3, 5)	29.5
(4, 4)	24	(3, 6)	40.5
(4, 5)	33	(5, 2)	16.5
(6, 1)	19	(5, 3)	21.5
(6, 2)	22	(5, 4)	28.5
(6, 3)	27	(7, 1)	25.5
(8, 1)	33	(7, 2)	28.5

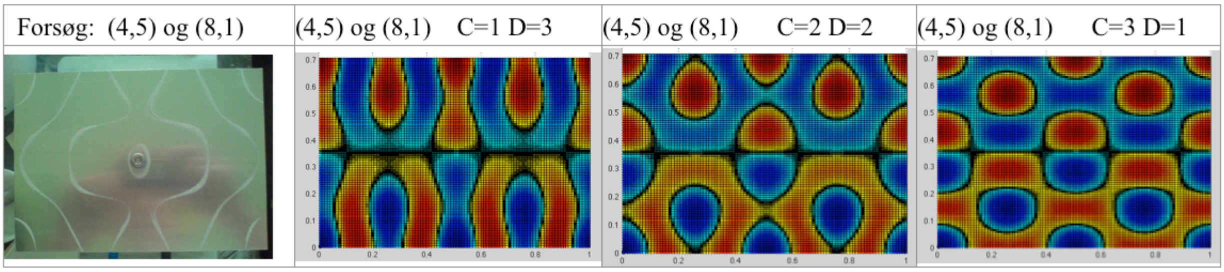

Figure 16: The figure is made to see the blending result from $m = 4, n = 5$ and $m = 8, n = 1$.

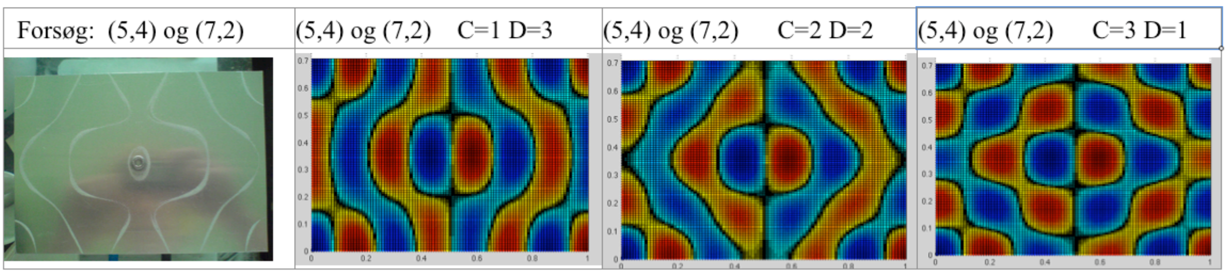

Figure 17: The figure is made to see the blending result from $m = 5, n = 4$ and $m = 7, n = 2$.

periment pattern, while the the drawings from Figure 17 which is under a different driving frequency dose seem to have a tendency of approaching the experiment pattern.

6 Discussion

6.1 The ultimate wave equation of an elastic plate

The wave equation I am working with is not considered as "perfect", because it uses the approximation that assume the displacement of the elements on the plate is only vertical. Also, Sophie German's wave equation was the best describing the movement of vibrating plate, still it is considered only works under particular circumstances.

The book called "Theory and analysis of elastic plates" is quite an advanced book, and "it is the only text to provide detailed coverage of classic and shear deformation plate theories and their solutions by analytical, as well as numerical methods, for bending, buckling, and natural vibrations".[Reddy 1999] In Chapter 3 "The classical theory of plates", the author gives the equation of motion in three dimensions [Reddy 1999, P120]. I would like to make the mark here in case I will have the chance to learn more about elastic plates.

6.2 Open boundary condition

The swift from closed boundary situation to open boundary situation is not shown in detail in the report. Another way instead of deriving from the Newton's laws, assuming the acceleration is only horizontal. I think we may start from the general form

$$u = \frac{sin}{cos}\}k_1x\frac{sin}{cos}\}k_2y\frac{sin}{cos}\}ckt \tag{86}$$

Since from the closed situation, we can fix the value of k_1 and k_2, and if $sink_1x$ or $sink_2y$ is chosen, the boundary is not open. Therefore, we should use the alternative form $cosk_1xcosk_2y$.

6.3 The theory used in Matlab program for the vibration of a plate

The wave equation we have used to get the nodal lines is sufficient to calculated the line expression, but there is no clue saying the source of the vibration. If we need to simulate the vibration, we have to add source term that drives the plate to vibrate:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(x, y, t), 0 < x < a, 0 < y < b, t > 0 \tag{87}$$

$$u(\triangle\ \Omega, t) = 0, t > 0 \tag{88}$$

$$u(x, y, 0) = 0, 0 < x < a, 0 < y < b \tag{89}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = 0, 0 < x < a, 0 < y < b \tag{90}$$

- ▶ Equation (87) is an adjusted form of the equation of motion we have derived, and the right side of the equation $f(x, y, t)$ is the source term, describing the motion of the center point which is moving up and down.

- ▶ Equation (88) gives the boundary condition, saying the points within the boundary $\triangle\ \Omega$ are not moving at any time.

- ▶ Equation (89) is the initial condition, saying the plate is flat initially.

- ▶ Equation (90) is another initial condition, saying the initial velocity of each element on the plate is zero.

The only difference between this set of equations and previous description is the right side of Equation (87), which describes "the applied normal load per unit area divided by the membrane tension per unit length". [Wilson and

Turcotte 2003, P341]. Well, in our case we concentrate the load at one point (x_0, y_0), which does not necessarily be the center of the plate. Then

$$p(x,y) = p_0\delta(x - x_0)\delta(y - y_0) \tag{91}$$

where δ is the Dirac delta function, and $\delta(x-x_0) = 1$ if $x = x_0$; $\delta(x-x_0) = 0$ if $x \neq x_0$.

Now we have an inhomogeneous problem as the previous one. Theoretically, the general strategy for solving the inhomogeneous problem is first finding the homogeneous solution and then add one of the inhomogeneous solution. [Logan 1998]

6.4 The nodal numbers

Before we talked about the pair of integers m and n, and some other pairs of integers can do the same job, if they can satisfy the equation:

$$p^2 = c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2}) \tag{92}$$

Rayleigh states that there are two groups that a^2 and b^2 fall, depending on if they are commensurable or not (commensurable means $\frac{a^2}{b^2}$ is rational):

- ► if they (a^2 and b^2) are incommensurable, there is only one pair of m and n can fit in the equation under the same frequency p.

- ► if they (a^2 and b^2) are commensurable, there are two or more pairs of m and n can fit in the equation under the same frequency p.

[Rayleigh and Strutt 1945, P311]

7 Conclusion

Above all, we have derived the wave equation in two dimensions, $\frac{1}{c^2}\frac{\partial^2 U(x,y,t)}{\partial t^2} = \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2}$, which is achieved by generalizing the wave equation for a mass point and one dimension case. In solving the equations for Chladni pattern, we have got the general solution, $u = \frac{sin}{cos}\}k_1x\frac{sin}{cos}\}k_2y\frac{sin}{cos}\}ckt$, by the separation of variables method.

Specifically, we have put the initial conditions and boundary conditions into consideration to get the general solution form in a closed boundary situation. That is,

$u = \Sigma_{m=1}^{m=\infty}\Sigma_{n=1}^{n=\infty} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}\{A_{mn}cospt + B_{mn}sinpt\}$,

from that we get the nodal lines equation as

$Csin\frac{m\pi x}{a}sin\frac{n\pi y}{b} + Dsin\frac{p\pi x}{a}sin\frac{q\pi y}{b} = 0$

where C, D, m, n, p, q a, and b are constants.

For an open boundary condition, we have given different boundary condition, and swift the nodal form from closed to open situation. That is,

$Ccos\frac{m\pi x}{a}cos\frac{n\pi y}{b} + Dcos\frac{p\pi x}{a}cos\frac{q\pi y}{b} = 0$

We have made drawings of a wave pattern and nodal pattern in a closed boundary situation, and the nodal pattern in an open boundary situation.

8 Perspectivation

The report focuses on maths about wave equation, but have not discussed much about the physics, except when is it necessary. While many of the coefficients in our equations indeed have some very important physical meanings, for example, the velocity c in the wave equation and the frequency p in the general solution of the wave equation.

Another issue from physics is the energy part. It is a topic that is always worth being considered. Also it reminds me of my second semester's project, about thermoacoustics, where the movement of air could transport energy. In order to understand that case, energy plays a critical part.

As I have not described much about the physics of vibrating plates, many of Mary Waller's theories about frequency and nodal pattern is not included in the report. It could be worthwhile to be able to predict the pattern when given a certain frequency to vibrate the plate.

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10 Appendix

10.1 Appendix 1: Matlab program for the vibration of plate and disk

[Wilson et al. 2003]

```
function [u,x,y,t]= membwave(type,dims,alp,w,tmax)
%
%[u,x,y,t]=membwave(type,dims,alps,w,tmax)
disp('')
disp('WAVE MOTION IN A RECTANGULAR OR CIRCULAR')
disp('MEMBRANE HAVING AN OSCILLATING LOAD')
if nargin > 0% Data passes through the call list

    if type==1
        a=dims(1); b=dims(2); x0=dims(3);y0=dims(4);
        [u,x,y,t]=memrecwv(a,b,alp,w,x0,y0,tmax);
    else
        r0=dims(1);
    end
```

```
else
    disp(''), disp('Select the geometry type:')
    type =input(['Enter 1 for a rectangle, ',...
                 '2 for a circle > ?']);
        if type==1
            disp('')
            disp('Specify the rectangle dimensions;')
            s=input('Give values for a,b > ? ','s');
            s=eval(['[',s,']']); a=s(1); b=s(2);
            disp('')
            disp('Give coordinates(x0,y0) where the')
            s=input('force acts. Enter x0,y0 > ?','S');
            s=eval(['[',s,']']); x0=s(1); y0=s(2);
            disp(''), alp=input('Enter the wave speed > ? ');

            N=40; M=40; pan=pi/a*(1:N)'; pbm=pi/b*(1:M);
            W=alp*sqrt(repmat(pan.^2,1,M)+repmat(pbm.^2,N,1));
            wsort=sort(W(:)); wsort=reshape(wsort(1:42),6,7)';
            disp('')
            disp(['The first forty-two natural ',...
                  'frequencies are:'])
              disp(wsort)
              w=input(....
                  'Input the frequency of the forcing function ? ');
```

```
        else
            disp(''), disp(...
                'The circle radius equals one. GIve the radial')
            disp(....
                'distance r0 from the circle center to the')
            r0=input('force > ? ');

            disp(''), alp=input('Enter the wave speed >?');

            wsort=alp*[...
                2.4048  3.8317  5.1356  5.5201  6.3801  7.0156
                7.5883  8.4173  8.6537  8.7715  9.7611  9.9362
               10.1735 11.0647 11.0864 11.6199 11.7916 12.2251
               12.3385 13.0152 13.3237 13.3543 13.5893 14.3726
               14.4755 14.7960 14.8213 14.9309 15.5898 15.7002
               16.0378 16.2234 16.4707 16.6983 17.0037 17.2412
               17.6159 17.8014 17.9599 18.0711 18.2876 18.4335];
```

```
           disp(''), disp(['The first forty-two ',...
                           'natural frequencies are:'])
           disp(wsort)
           w=input(...
               'Input the frequency of the forcing function ? ');
        end
            disp('')
            disp('Input the maximum solution evaluation time.')
            tmax=input(' > ? ');
end

if type==1
    [u,x,y,t]=memrecwv(a,b,alp,w,x0,y0,tmax);
else
    th=linspace(0,2*pi,81); r=linspace(0,1,20);
    [u,x,y,t]=memcirwv(r,th,r0,alp,w,tmax);
end
```

```
membanim(u,x,y,t);

function [u,x,y,t]=memrecwv(a,b,alp,w,x0,y0,tmax)

if nargin==0
    a=2; b=1; alp=1; tmax=3; w=13; x0=1.5; y0=0.5;
end
if a<b
    nx=31; ny=round(b/a*21); ny=ny+rem(ny+1,2);
else
    ny=31; nx=round(a/b*21); nx=nx+rem(nx+1,2);
end
x=linspace(0,a,nx); y=linspace(0,b,ny);
```

```
N=40; M=40; pan=pi/a*(1:N)'; pbm=pi/b*(1:M);
W=alp*sqrt(repmat(pan.^2,1,M)+repmat(pbm.^2,N,1));
wsort=sort(W(:)); wsort=reshape(wsort(1:30),5,6)';
Nt=ceil(40*tmax*alp/min(a,b));
t=tmax/(Nt-1)*(0:Nt-1);

mat=sin(x0*pan)*sin(y0*pbm)./(w^2-W.^2);
sxn=sin(x(:)*pan'); smy=sin(pbm'*y(:)');

u=zeros(ny,nx,Nt);
for j=1:Nt
    A=mat.*(cos(w*t(j))-cos(W*t(j)));
    uj=sxn*(A*smy); u(:,:,j)=uj';
end
```

```
function [u,x,y,t,r,th]=memcirwv(r,th,r0,alp,w,tmax)

    if nargin==0
        r0=.4; w=15.5; th=linspace(0,2*pi,81);
        r=linspace(0,1,21); alp=1;
    end

    Nt=ceil(20*alp*tmax); t=tmax/(Nt-1)*(0:Nt-1);

    lam=besjroot(0:20,20,1e-3);

    [nj,nk]=size(lam); r=r(:)'; nr=length(r);
    th=th(:); nth=length(th); nt=length(t);
    N=repmat((0:nj-1)',1,nk); Nvec=N(:)';
    c=besselj(N,lam*r0)./(besselj(...
        N+1,lam).^2.*(lam.^2-w^2));
    c(1,:)=c(1,:)/2; c=c(:)';



    lamvec=lam(:)'; wlam=w./lamvec;
    c=cos(th*Nvec).*repmat(c,nth,1);
    rmat=besselj(repmat(Nvec',1,nr),lamvec'*r);
    u=zeros(nth,nr,nt);
    for k=1:nt
        tvec=-cos(w*t(k))+cos(lamvec*t(k));
        u(:,:,k)=c.*repmat(tvec,nth,1)*rmat;
    end
    u=2/pi*u; x=cos(th)*r; y=sin(th)*r;

    function rts=besjroot(norder,nrts,tol)
        if nargin<3, tol=1e-5; end
        jn=inline('besselj(n,x)','x','n');
        N=length(norder); rts=ones(N,nrts)*nan;
        opt=optimset('TolFun',tol,'TolX',tol);
        for k=1:N
            n=norder(k); xmax=1.25*pi*(nrts-1/4+n/2);
            xsrch=.1:pi/4:xmax; fb=besselj(n,xsrch);
            nf=length(fb); K=find(fb(1:nf-1).*fb(2:nf)<=0);
            if length(K)<nrts
                disp('Search error in function besjroot')
                rts=nan; return
```

```
    else
        K=K(1:nrts);
        for i=1:nrts
            interval=xsrch(K(i):K(i)+1);
            rts(k,i)=fzero(jn,interval,opt,n);
        end
    end
end

function membanim(u,x,y,t)

   if nargin==0;
       [u,x,y,t]=memrecwv(2,1,1,15.5,1.5,.5,5);
   end
   xmin=min(x(:)); xmax=max(x(:));
   ymin=min(y(:)); ymax=max(y(:));
   xmid=(xmin+xmax)/2; ymid=(ymin+ymax)/2;
   d=max(xmax-xmin,ymax-ymin)/2; Nt=length(t);
   range=[xmid-d,xmid+d,ymid-d,ymid+d,...
          3*min(u(:)),3*max(u(:))];
```

```
    while 1
        disp(''), disp('Press return for animation')
        dumy=input('or enter 0 to stop > ? ','s');
        if ~isempty(dumy)
          disp(''), disp('All done'), break
        end

        for j=1:Nt
            surf(x,y,u(:,:,j)), axis(range)
            xlabel('x axis'), ylabel('y axis')
            zlabel('u axis'), titl=sprintf(...
                'MEMBRANE POSITION AT T=%5.2f',t(j));
            title(titl), colormap(autumn)
            colormap(autumn)
            drawnow, shg, pause(.1)
        end
    end
```

10.2 Appendix 2: An adapted Matlab program of Appendix 1, shows the nodal lines of the vibrating surface

What we did is to replace the plotting part in the end of Appendix 1:

```
while 1
    disp(''), disp('Press return for animation')
    dumy=input('or enter 0 to stop > ? ','s');
    if ~isempty(dumy)
      disp(''), disp('All done'), break
    end

    for j=1:Nt
        surf(x,y,u(:,:,j)), axis(range)
        xlabel('x axis'), ylabel('y axis')
        zlabel('u axis'), titl=sprintf(...
            'MEMBRANE POSITION AT T=%5.2f',t(j));
        title(titl), colormap(autumn)
        colormap(autumn)
        drawnow, shg, pause(.1)
    end
end
```

with the following language:

```
while 1
    disp(''), disp('Press return for animation')
    dumy=input('or enter 0 to stop > ? ','s');
    if ~isempty(dumy)
      disp(''), disp('All done'), break
    end

    for j=1:Nt
        figure(1)
        surf(x,y,u(:,:,j)), axis(range)
        xlabel('x axis'), ylabel('y axis')
        zlabel('u axis'), titl=sprintf(...
            'MEMBRANE POSITION AT T=%5.2f',t(j));
        title(titl), colormap([1 1 1])
        colormap([255/255 1 255/255])
        drawnow, shg, hold on
        contour3(x,y,u(:,:,j),[0 0],'b-'), hold off
        pause(.1)
        end
end
```

10.3 Appendix 3:Matlab program for the nodal line in an open boundary situatio

Matlab program for the nodal line in an open boundary situation:

```
resolution = 100;
Lx = 1;
Ly = 1;
x = linspace(0,Lx,resolution);
y = linspace(0,Ly,resolution);
[X,Y] = meshgrid(x,y);
m = 6;
n = 0;
p = 2;
q = 4;
A = 3;
B = 1;

u = A .* cos((m*pi/Lx).*X) .* cos((n*pi/Ly).*Y) + B .* cos((p*pi/Lx).*X) .* cos((q*pi/Ly).*Y);
patterncmap = [0        0    0.5625;        0        0    0.5962;        0        0    0.6298;        0        0    0.6635;
0        0    0.6971;        0        0    0.7308;        0        0    0.7644;        0        0    0.7981;        0        0
0.8317;        0        0    0.8654;        0        0    0.8990;        0        0    0.9327;        0        0    0.9663;
0        0    1.0000;        0    0.0667    1.0000;        0    0.1333    1.0000;        0    0.2000    1.0000;        0
0.2667    1.0000;        0    0.3333    1.0000;        0    0.4000    1.0000;        0    0.4667    1.0000;        0
0.5333    1.0000;        0    0.6000    1.0000;        0    0.6667    1.0000;        0    0.7333    1.0000;        0
0.8000    1.0000;        0    0.8667    1.0000;        0    0.9333    1.0000;        0    1.0000    1.0000;        0
0.7500    0.7500;        0    0.5000    0.5000;        0    0.2500    0.2500;        0        0        0;    0.2500
0.2500        0;    0.5000    0.5000        0;    0.7500    0.7500        0;    1.0000    1.0000        0;    1.0000
0.9286        0;    1.0000    0.8571        0;    1.0000    0.7857        0;    1.0000    0.7143        0;    1.0000
0.6429        0;    1.0000    0.5714        0;    1.0000    0.5000        0;    1.0000    0.4286        0;    1.0000
0.3571        0;    1.0000    0.2857        0;    1.0000    0.2143        0;    1.0000    0.1429        0;    1.0000
0.0714        0;    1.0000        0        0;    0.9615        0        0;    0.9231        0        0;    0.8846        0
0;    0.8462        0        0;    0.8077        0        0;    0.7692        0        0;    0.7308        0        0;    0.6923
0        0;    0.6538        0        0;    0.6154        0        0;    0.5769        0        0;    0.5385        0        0;
0.5000        0        0]

figure(1)
set(figure(1),'Colormap',patterncmap)
surf(X,Y,-u)
view([0 0 1])
```