Differential Equations (DE). Date: 02-02-22

MT-1006.

Reference Book: Differential Equations with Applications (11th Edition); Dennis G. Zill.

Chapter #1. Introduction To Differential Equations (1.1, 1.2)

1.1. Definitions And Terminology.

Differential Equations (DE):- An equation containing the derivatives of 1 or more variables (dependant), wrt 1 or more independant variables is said to be DE.

$\frac{dy}{dx}$ / y', $\frac{d^2y}{dx^2}$ / y'', $\frac{d^4y}{dx^4}$ / $y^{(4)}$, $\frac{d^ny}{dx^n}$ / $y^{(n)}$

prime notation

e.g. $\frac{dy}{dx} + 7y = 16x$

Classification of Differential Equations: We shall classify DE according to:
- → Type
- ↳ Order
- ↳ Linearity

Type: ① Ordinary (ODE) ② Patial (PDE).

Linearity: ① Linear ② Non-linear

Classification By Type: ODE: It contains $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$

leibneiz notation

PDE: $\frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2} \ldots \frac{\partial^n y}{\partial x^n}$

↳ dava (∂

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PDE: $\dfrac{\partial^4 w}{\partial z\,\partial y\,\partial x}$, $\dfrac{\partial^3 u}{\partial^2 x\,\partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}\left\{\dfrac{\partial u}{\partial y}\right\}\right]$

↳ can also be written as.

w_{xyz} u_{yxx}

subscript Notation.

Classification By Order: The order of a ODE is the order of the highest derivative eg. $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3 - 4y = e^x$ => ODE, 2nd order.

Format Of first Order ODE: A 1st order ODE is sometimes written as $\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$

↳ Multi variable function

only for 1st order ODE.

To solve the equation we must convert this form into a rationalized form. We can assume dy as an independent variable (not necessary, but preferrable). Divide both sides by dependant " (dx)

eg. $(y-x)\,dx + 4x\,dy = 0$

$$\frac{y-x\,dx}{dx} + \frac{4x\,dy}{dx} = \frac{0}{dx}$$

$$y - x + 4x\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + y = x$$ => 1st order ODE.

Normal Form Of an ODE: We have to solve the ODE for higher order derivative. $\dfrac{d^n y}{dx^n}$ must be on the Left Hand Side.

eg. $\dfrac{dy}{dx} = \dfrac{x-y}{4x}$ ↳ Normal form of 1st order ODE.

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$\frac{dy}{dx} = f(x,y)$ => General format of Normal form of 1st Order ODE

$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$ => Normal form of 2nd Order ODE.

$\frac{d^2y}{dx^2} = \frac{dy}{dx} - 6y$

$\frac{d^2y}{dx^2} = f(x, y, y')$ => General format of Normal form of 2nd Order ODE

Classification By linearity: — Linear (function is linear when it is not multiplied by itself)
— Non-Linear

General form of n^{th} order linear ODE.

$$a_0(x)\, y + a_1(x)\frac{dy}{dx} \dots + a_n(x)\frac{d^ny}{dx^n} = \underbrace{g(x)}_{\text{purely function of } x}.$$

Ways in which a linear eq. loses its linearity:
- multiplied by same variable
- power on derivative
- If it is an angle of $\underbrace{\text{transcendental}}_{\text{all trignometric, exponential, logarithmic.}}$ function/Non-Linear function

$a_n(x)$ => It can be a constant or an equation of x and it can also be non linear.

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ODE

Linearity:

$(1-y)y' + 2y = e^x$ 1st order Non linear ODE

$\frac{d^2y}{dx^2} + \sin y = 0$ 2nd order " " "

$\frac{d^4y}{dx^4} + y^2 = 0$ 4th " " " "

$x^2 \frac{d^4y}{dx^4} + \sin x \frac{d^3y}{dx^3} = e^{x^2}$ 4th order Linear ODE.

<u>Solutions of ODE</u> (Finding dependant variable (y)).

General Solution, Particular Solution, Trivial Solution ($y=0$).

$\frac{dy}{dx} = 4x^3$

taking $\int dx$ on both sides

$\int \frac{dy}{dx} dx = \int 4x^3$

$y = x^4$

$y = x^4 + 17$

$y = x^4 + \frac{17\sqrt[3]{22}}{\pi}$ } can also be sol. of this eq, so we say

$y = x^4 + C$

A single DE has infinite solutions, families/curve of solutions.

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General Solution / nth-parameter family of solutions:

Family of solutions that contains one or more than paramaters c, or ($c_1, c_2, \dots, c_n$), depending upon order of DE is called general solution.

Particular Solution:

A solution of an ODE (y) which is free of parameters c is called particular solution.

Trivial Solution:

$y=0$ is called trivial solution.

Verification Of Solutions:

Verify that the the indicated function is a solution of the given DE on the interval $(-\infty,\infty)$

$$y''-2y'+y=0 \quad ; y=xe^x$$

$$y=xe^x$$

$$uv'+vu'$$

$$y'=xe^x+e^x$$

$$y''=xe^x+2e^x$$

$$xe^x+2e^x-2(xe^x+e^x) = 0+xe^x=0$$

$$0=0$$

~~It is~~ $y=xe^x$ is the explicit and particular solution of ODE.

The relation $x^2+y^2=25$ is an implicit solution of the DE $\frac{dy}{dx}=\frac{-x}{y}$. prove.

$$\frac{dy}{dx}=\frac{-x}{y}$$

$$x^2+y^2=25$$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y} \text{ proved.}$$

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Show that
$$x = c_1 \cos 4t$$
$$x = c_2 \sin 4t$$
$$x = c_1 \cos 4t + c_2 \sin 4t$$
are solutions of given ODE.

$$x'' + 16x = 0$$

$x = c_1 \cos 4t$

$x' = -4c_1 \sin 4t$

$x'' = -16c_1 \cos 4t$

$-16c_1 \cos 4t + 16 \cancel{x = 0} (c_1 \cos 4t) = 0$

$0 + 0 = 0$

$x = c_2 \sin 4t$

$x' = 4c_2 \cos 4t$

$x'' = -16c_2 \sin 4t$

$-16c_2 \cos 4t + 16(c_2 \sin 4t) = 0$

$0 = 0$

$x = c_1 \cos 4t + c_2 \sin 4t$

$x' = -4c_1 \sin 4t + 4c_2 \cos 4t$

$x'' = -16c_1 \cos 4t - 16c_2 \sin 4t$

$-16c_1 \cos 4t - 16c_2 \sin 4t + 16(c_1 \cos 4t + c_2 \sin 4t)$

$0 = 0.$

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1.2. Initial-Value Problem.

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(IVP).

(Initial conditions are given (values of x, y))

1- First Order IVP

$$\frac{dy}{dx} = f(x,y). \; ; \; y(x_0) = y_0 \Rightarrow \text{general form of 1st order IVP}$$

Order of derivate = Number of conditions given

2- 2nd Order IVP

$$\frac{d^2y}{dx^2} = f(x,y,y') \; ; \; y(x_0) = y_0 \qquad y'(x_0) = y'_0$$

3- 3rd Order IVP

$$\frac{d^3y}{dx^3} = f(x,y,y',y'') \; ; \; y(x_0) = y_0 \qquad y'(x_0) = y'_0 \qquad y''(x_0) = y''_0$$

4- 4th Order IVP

$$\frac{d^4y}{dx^4} = f(x,y,y',y'',y''') \; ; \; y(x_0) = y_0 \qquad y'(x_0) = y'_0 \qquad y''(x_0) = y''_0 \qquad y'''(x_0) = y'''_0$$

IVP

To solve $\frac{d^ny}{dx^n} = f(x,y,y', \dots y^{(n-1)})$ subject to

$$y(x_0) = y_0, \; y'(x_0) = y'_0 \; \dots \; y^{(n-1)}(x_0) = y_0^{(n-1)}$$

eg.

$y' = y$; $y(0) = 3$ has 1-parameter family of solutions; $\underline{y = ce^x}$ A

Find a solution of 1st order ODE by using IVP.

$y = 3, \; x = 0$

$3 = ce^0$

$c = 3$; $y = 3e^x$

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$y' = y \quad ; \; y(1) = -2$

$x = 1, \; y = -2$

$-2 = ce^{1}$

$c = \frac{-2}{e}$

$c = -2e^{-1}$

$y = \frac{-2}{e} e^{x}$

$y = -2e^{x-1}$

$x = c_1 \cos 4t + c_2 \sin 4t$ is a 2-parameter family of solutions of 2nd order ODE. $x'' + 16x = 0$. Find a solution of given ODE by using.

IVP :- $x'' + 16x = 0 \; ; \; x(\frac{\pi}{2}) = -2$ $\quad t = \pi/2 \quad x = -2$

$x'(\frac{\pi}{2}) = 1 \quad x = 1$

$-2 = c_1 \cos 2\pi + c_2 \sin 2\pi$

$-2 = c_1 \, 1 + 0$

$c_1 = -2$

$x' = 4c_1 \sin 4t + 4c_2 \cos 4t$

$1 = -4c_1 \sin 2\pi + 4c_2 \cos 2\pi$

$1 = 0 + 4c_2$

$c_2 = 1/4$

$x = -2\cos 4t + 1/4 \sin 4t$

Q3-C. $y = \frac{1}{x^2 + c}$ is a 1-parameter family of solutions of 1st order ODE

$y' + 2xy^2 = 0$ Find solutions by using following IVP's.

$y(2) = 1/3 \; , \; y(-2) = 1/2 \; , \; y(0) = 1 \; , \; y(1/2) = -4.$

$1/3 = \frac{1}{4+c}$

$c = -1$

$y = \frac{1}{x^2 - 1}$

$1/2 = \frac{1}{4+c}$

$c = -2$

$y = \frac{1}{x^2 - 2}$

$1 = \frac{1}{0+c}$

$c = 1$

$y = \frac{1}{x^2 + 1}$

$-4 = \frac{1}{1/2 + c}$

$-4 \times \frac{1+2c}{2} = 1$

$1 = -2 - 4c$

$3 = -4c$

$c = -3/4$

$y = \frac{1}{x^2 - 3/4}$

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Q.2-10 $x = c_1 \cos t + c_2 \sin t$ — (A)

$x'' + x = 0$

$x(0) = -1 \qquad x'(0) = 8$

$t = 0 \qquad x = -1 \qquad x' = 8$

$-1 = c_1 \cos 0 + c_2 \sin 0 \qquad\qquad x' = -c_1 \sin t + c_2 \cos t$

$-1 = c_1 + 0 \qquad\qquad 8 = c_2$

$c_1 = -1$

$x = -\cos t + 8 \sin t$

$x(\pi/4) = \sqrt{2}, \quad x'(\pi/4) = 2\sqrt{2}$

$t = \pi/4 \quad x = \sqrt{2}, \quad x' = 2\sqrt{2}$

$\sqrt{2} = c_1 \cos \pi/4 + c_2 \sin \pi/4 \qquad\qquad 2\sqrt{2} = -c_1 \sin \pi/4 + c_2 \cos \pi/4$

$c_1 = \dfrac{\sqrt{2} - c_2 \sin \pi/4}{\cos \pi/4} \qquad\qquad \cancel{2\sqrt{2} = -c_1 \sin \pi/4 +}$

$2\sqrt{2} = \dfrac{\sqrt{2} - c_2 \sin \pi/4}{\cos \pi/4} + c_2 \cos \pi/4$

$2\sqrt{2} = \dfrac{\sqrt{2} - c_2 \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} + c_2 \dfrac{\sqrt{2}}{2}$

$2\sqrt{2} =$

$2\sqrt{2} - \dfrac{c_2\sqrt{2}}{2} \times \dfrac{2}{\sqrt{2}} = \sqrt{2} - \dfrac{c_2\sqrt{2}}{2}$

$2\sqrt{2} - \dfrac{c_2 2\sqrt{2}}{2\sqrt{2}} = \sqrt{2} - \dfrac{c_2\sqrt{2}}{2}$

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Q. (1-18), (21-24) (31-36), (51) (11-14) - 1.1
(1-16) (39-44) 1.2

Q.21-24. Verify that the given family of solutions are solutions of given ODE's.

21. $\frac{dP}{dt} = P(1-P)$; $P = \frac{c_1e^t}{1+c_1e^t}$

$$\frac{dP}{dt} = \frac{(1+c_1e^t)(c_1e^t) - (c_1e^t)(c_1e^t)}{(1+c_1e^t)^2}$$

$$\frac{dP}{dt} = \frac{c_1e^t}{1+c_1e^t}\left\{\frac{(1+c_1e^t) - c_1e^t}{1+c_1e^t}\right\}$$

$$\frac{dP}{dt} = P\left\{\frac{1+c_1e^t}{1+c_1e^t} - \frac{c_1e^t}{1+c_1e^t}\right\}$$

$$\frac{dP}{dt} = P(1-P)$$

Q 31-34. Find values of m so that function $y = e^{mx}$ is solution of given ODE's.

32. $y'' - 5y' + 6y = 0$ $\qquad y = e^{mx}$ — (A)

$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$ $\qquad y' = me^{mx}$

~~$e^{mx} = 3$, $e^{mx} = 2$~~ $\qquad y'' = m^2e^{mx}$

~~$mx = \ln 3$~~ $e^{mx}(m^2 - 5m + 6) = 0$

~~$m =$~~ $e^{mx}((m-2)(m-3)) = 0$

$y = e^{mx}$ is a solution

$m = 2$, $m = 3$.

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Q15-16. Determine by inspection at least two solutions of the given 1st order IVP.

15 $y' = 3y^{2/3}$; $y(0)=0$

1st sol.: $\boxed{y=0}$ $\quad x=0, y=0$

$y=0, \; y'=0$

$0 = 3(0)^{2/3}$

$0=0$

2nd sol.

let $y=c\neq 0$; $y'=0$

$0 = 3c^{2/3}$

let $y=x\neq 0$; $y'=1$

$1 = 3c^{2/3}$

let $y=x^2$; $y'=2x$

$2x = 3x^{4/3}$

let $\boxed{y=x^3}$; $y'=3x^2$

$3x^2 = 3(x^3)^{2/3}$

$3x^2 = 3x^2$

Boundary Value Problem (BVP).

Q.39-44. $y = c_1\cos 2x + c_2 \sin 2x$ is a 2-parameter family of solutions of 2nd order ODE $y''+4y=0$, If possible find a solution of DE that satisfies the given side conditions.

$y(0)=0, \quad y(\pi/4)=3$

1/ By using first boundary conditions

$0 = c_1\cos 0 + c_2 \sin 0$

$c_1 = 0$

2/ By using second boundary condition

$3 = 0 + c_2 \sin \pi/2$

$c_2 = 3$

$y = 3\sin 2x.$

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