Dated: 3/March/2022 Ex: 3.1

In this section we discuss 3 - linear Models;

i) Growth & Decay ii) Newton's law of cooling/warming
iii) Series Circuit

* Growth & Decay:
  - An initial value problem;

$$\boxed{\frac{dx}{dt} = kx(t)}$$

$x = x(t)$: no. of population
if $-k \rightarrow$ decay
$+k \rightarrow$ growth

Example # 1:

P_0: no. of bacteria ; $P(t)$

1) $P(0) = P_0 \longrightarrow c = ?$
2) $P(1) = \frac{3}{2}P_0 \longrightarrow k = ?$

$$\boxed{\frac{dP}{dt} = kP(t)}$$

3) $P(t) = 3P_0$

$$\Rightarrow \frac{dP}{dt} = KP \Rightarrow \frac{dP}{dt} - KP = 0$$

$$I.F = e^{\int -k\,dt} = \boxed{e^{-kt}}$$

$$\frac{d}{dt}\left[e^{-kt}.P\right] = 0 \rightarrow e^{-kt}.P = c \rightarrow \boxed{P(t) = ce^{kt}} - (1)$$

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1) $P(0) = P_0$ $\Rightarrow t = 0,\ P = P_0$

$P_0 = ce^{k(0)}$ $\Rightarrow$ $\boxed{P_0 = C}$ $\longrightarrow$ put in (1)

$\hookrightarrow$ $\boxed{P(t) = P_0 e^{kt}}$ — (2)

2) $P(1) = \frac{3}{2} P_0$ $\Rightarrow P(1) = ;\ P = \frac{3}{2} P_0,\ t = 1.$

$\frac{3}{2} P_0 = P_0 e^{k}$ $\Rightarrow$ $\boxed{\ln\left(\frac{3}{2}\right) = k}$

$\Rightarrow P(t) = P_0 e^{0.405t}$

3) $P(t) = 3P_0$ $\Rightarrow P = 3P_0\ ;\ t = t = ?$

$\Rightarrow 3P_0 = P_0 e^{0.405t}$

$\Rightarrow \frac{\ln 3}{0.405} \simeq t$ $\boxed{t \simeq 2.71h}$

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Q.1 $P =$ no. of people

$\frac{dP}{dt} = kP$; $P(t) = ?$

Conditions:

1) $P(0) = P_0$
2) $P(5) = 2P_0$
3) $P(t) = 3P_0$
4) $P(t) = 4P_0$

$\frac{dP}{dt} - kP = 0 \Rightarrow \boxed{I.F = e^{-kt}}$

$\frac{d}{dx}\left[e^{-kt} \cdot P\right] = 0$

$\boxed{P = ce^{kt}}$ — (1)

sub condition 1:

$P_0 = ce^{k(0)}$

$\boxed{P_0 = c}$ — (2)

$\Rightarrow$ sub (2) in (1)

$\boxed{P(t) = P_0 e^{kt}}$ — (2)

2) $P(5) = 2P_0$; $t = 5$, $P = 2P_0$

sub in (2): $\rightarrow$ $2P_0 = P_0 e^{5k}$ $\Rightarrow 2 = e^{5k}$

$\Rightarrow \ln e^{5k} = \ln 2$

$\Rightarrow 5k = \ln 2 \Rightarrow \boxed{k = \frac{\ln 2}{5}}$

$\rightarrow$ sub value of k in (2)

$\boxed{P = P_0 e^{0.139t}}$ — (3)

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3) $P(t) = 3P_0$

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$P_0 e^{0.139t} = 3P_0$

$e^{0.139t} = 3$

$\ln e^{0.139t} = \ln 3$

$0.139t = \ln 3$

$\boxed{t = \frac{\ln 3}{0.139t}}$ → Time it will take to triple

4) $P(t) = 4P_0$

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ii) Newton's Law of cooling / Warming :-

Newton's Law of cooling / warming of an object is given by the linear 1st order DE:-

$$\boxed{\frac{dT}{dt} = k(T - T_m)}$$

T_m : room temp

T = temp of an object

$T = T(t)$: Temp of an Object

T_m = Room temperature

t = time

Example #04:- DATA:-

Conditions:-

1) $T(0) = 300°F$ → $c = ?$

2) $T(3) = 200°F$

3) $T(t) = 70°F$ OR $T(t) = T_m = 70°F$

$$\frac{dT}{dt} - kT = -kT_m$$

$$\boxed{I.F = e^{-kt}}$$

$$\frac{d}{dx}\left[e^{-kt} \cdot T\right] = -e^{kt}(kT_m)$$

$$e^{-kt} \cdot T = \cancel{k} \, e^{kt}(T_m)$$

↳ $$\frac{dT}{dt} = \cancel{kT - 70k} \; k(T - 70)$$

$$\int \frac{1}{T-70} \, dT = k \int dt$$

$$\ln(T-70) = kt + c$$

$$T - 70 = ce^{kt}$$

$$\Rightarrow \boxed{T = 70 + ce^{kt}} \quad \text{— (1)}$$

1) $T(0) = 300$ → $t = 0$, $T = 300$

$$300 = 70 + ce^{k(0)}$$

$$c = 300 - 70$$

$$\boxed{c = 230}$$

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$T(t)$ or $\boxed{T = 70 + 230e^{kt}}$ — ②

2) $T(3) = 200$; $t = 3$, $T = 200$

$200 = 70 + 230e^{3k}$

$230e^{3k} = 130$

$e^{3k} = \frac{130}{230}$

$3k \ln e = \ln\left(\frac{130}{230}\right) \Rightarrow \boxed{k = \frac{1}{3}\ln\left(\frac{130}{230}\right)} \Rightarrow \boxed{k = -0.190}$

$\boxed{T = 70 + 230e^{-0.190t}}$

3) $70 = 70 + 230e^{0.190t}$

$e^{0.190t} = 0$

$0.190t \ln e \neq \ln(0)$ → cannot find any time value at 70°F thus we will find it at a value slightly higher ; for ex: 70.5

$70.5 = 70 + 230e^{0.190t}$

$230e^{0.190t} = 0.5$

$e^{0.190t} = \frac{0.5}{230}$

$0.190t \ln e = \ln\left(\frac{0.5}{230}\right)$

$t = \frac{1}{0.190}\ln\left(\frac{0.5}{230}\right)$

$\boxed{t = +32.26}$

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* Series Circuit :-

1) LR - Series Circuit :-

$$\boxed{L \frac{di}{dt} + Ri = E(t)}$$

L: Inductance

R: Resistance

E(t): Voltage

2) RC - Series Circuit:

$$\boxed{R \frac{dq}{dt} + \frac{1}{C} q = E(t)}$$

R = Resistance

q = Charge

C = capacitance

$i(t) = \frac{dq}{dt}$

Example # 7:

$$\frac{1}{2} \frac{di}{dt} + 10i = 12$$

$$\frac{di}{dt} + 20i = 24$$

$$\boxed{I.F = e^{20t}}$$

$$\frac{d}{dx}\left[e^{20t} \cdot i\right] = 24e^{20t}$$

$$e^{20t} \cdot i = \cancel{480e^{20t}} + c \Rightarrow e^{20t} \cdot i = \frac{6}{5} e^{20t} + c$$

$$\boxed{i = \cancel{480 + ce^{-20t}}}$$

$$\boxed{i = \frac{6}{5} + ce^{-20t}}$$

1) $i(0) = 0$

$$0 = \frac{6}{5} + ce^{-20(0)}$$

$$\boxed{c = -\frac{6}{5}}$$

$$\boxed{i = \frac{6}{5} - \frac{6}{5} e^{-20t}} \longrightarrow 2$$

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Dated: 16/3/21 Ex 3.2 Non-Linear Models!

i) Logistic Equation: {Advances of Growth & Decay} ↳ $\frac{dP}{dt} = kP(t)$

The IVP of the form:

$$\boxed{\frac{dP}{dt} = P(a-bP)} \quad ; \quad P(0) = P_0$$

is called logistic equation, and its solution $P(t)$ is called logistic function.

→ Solution of Logistic Equation:

$$\frac{dP}{dt} = P(a-bP) \quad ; \quad P(0) = P_0$$

$$\int \underbrace{\frac{1}{P(a-bP)}}_{\to \text{(partial F)}} dP = \int dt$$

$$P(t) = \frac{ac_1e^{at}}{1+bc_1e^{at}} = \frac{ac_1e^{at}}{e^{at}e^{-at}+bc_1e^{at}} = \frac{ac_1}{e^{-at}+bc_1}$$

Now using IVP:-

$$P(0) = P_0 \Rightarrow t_0 = 0, \ P = P_0$$

$$P_0 = \frac{ac_1}{1+bc_1} \Rightarrow \boxed{c_1 = \frac{P_0}{a-bP_0}}$$

$$\Rightarrow \boxed{P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}}$$ → Particular solution of logistic equation

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Example # 1:

Total students = 1000

i) $x(0) = 1$

$\frac{dx}{dt} = x$ { where x is no of ~~non~~-infected students

$1000 - x$ = no. of non-infected students.

$$\boxed{\frac{dx}{dt} = kx(1000 - x)} \quad ; \quad x(0) = 1$$

i) $x(0) = 1$

ii) $x(6)$?

iii) $x(4) = 50$

Solve: $\frac{dP}{dt} = P(a - bP)$; $P(0) = P_0$

$$\Rightarrow \frac{dx}{dt} = x(1000k - kx) \quad ; \quad x(0) = 1$$

↳ Have converted it in the above form.

Now directly use this formula: ↓

$$\boxed{x(t) = \frac{a x_0}{b x_0 + (a - b x_0)e^{-at}}}$$

$$x(t) = \frac{1000k}{k + (1000k - k)e^{-1000kt}}$$

$$\boxed{x(t) = \frac{1000}{1 + 999e^{-1000kt}}}$$

→ Now use IVPs & solve further.

Dated:

Qs. 1 b) $N(t) =$ no. of supermarkets

$$\frac{dN}{dt} = N(1 - 0.0005N) \quad ; \quad \cancel{N_0} \; N(0) = 1$$

$$N(t) = \frac{1}{0.0005 + (1 - 0.0005)e^{-t}}$$

i) $N(10) = ?$

Solve $\rightarrow$ Ans $= 1834$

CHR # 04 — HIGHER ORDER D.E:

4.1 "Preliminary Theory - Linear Equations":

Example #03: A BVP can have, many, one or no solutions.

$$x'' + 16x = 0 \quad ; \quad x(0) = 0 \quad , \quad x\left(\frac{\pi}{2}\right) = 0$$

$$\boxed{x = c_1 \cos 4t + c_2 \sin 4t}$$

Apply BVP:-

i) $x(0) = 0$

$\boxed{c_1 = 0}$

$\boxed{x(t) = c_2 \sin 4t}$

ii) $x\left(\frac{\pi}{2}\right) = 0$

$0 = c_2 \sin 2\pi \qquad (c_1, c_2) = (0, c_2)$

$\therefore$ Given BVP has infinitely many solutions

b) $x'' + 16x = 0 \quad ; \quad x(0) = 0 \; ; \; x\left(\frac{\pi}{8}\right) = 0$

sub BVP:-

i) $x(0) = 0$

$\boxed{c_1 = 0}$

ii) $x\left(\frac{\pi}{8}\right) = 0$

$\boxed{0 = c_2} \qquad (c_1, c_2) = (0, 0) \rightarrow$ Given BVP has one solution.

$\boxed{x(t) = 0}$

↳ Given BVP has one solution and it has trivial solution.

c) $x(0) = 0$, $x(\frac{\pi}{2}) = 1$

$\downarrow$

$\boxed{c_1 = 0}$

ii) $x(\frac{\pi}{2}) = 1$

$1 = c_2 \sin 4(\frac{\pi}{2})$

$1 \neq 0 \rightarrow$ Given BVP has no solution.

* Homogeneous Linear DE:

i) nth order Linear DE:

$$\boxed{a_n(x)y^{(n)} + \dots + a_1 x y' + a_0(x)y = g(x)}$$

General form of n^{th} order LDE.

ii) nth-order homogeneous L.DE:-

$$\boxed{a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = 0}$$

NOTE: Every H.L.D.E must contain a trivial solution.

$\{0, y_1, y_2 \dots$ sols of HLDE$\}$

* Linear Combination:-

The set of functions $\{y_1, y_2, y_3\}$

$$\boxed{y = c_1 y_1 + c_2 y_2 + c_3 y_3}$$

Note:- If y_1, y_2 & y_3 are the solutions of given DE then their linear combination is also a solution, i $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

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* **Linear Dependence / Linear Independence :-**

· The set of solutions $\{y_1, y_2, y_3\}$ can be linearly independent or dependent if;

$$\boxed{c_1 y_1 + c_2 y_2 + c_3 y_3 = 0}$$

i) If all scalars $\{c_1, c_2, c_3\}$ are equals to zero then the set of solutions is linearly independent.

ii) If any scalar among all is not equal to zero, then the set of solutions is linearly dependent.