Assignment 1

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Problem 1:-

(a) $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial z}\right)^2 + ux^3 + uy^2 + uz = 0$

Order : 3

Degree: 2

Type: PDE

Non-Linear

(b) $\left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2} + y\right)^{\frac{3}{2}}$

Order: 2

Degree: 2

Type : ODE

Non-Linear

Problem 2 :

INITIAL/BOUNDARY VALUE PROBLEMS

(a.) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$, $y(0) = -2$, $y'(0) = 6$ (Initial Value)

$y = c_1e^{4x} + c_2e^{-3x}$

diff wrt x

$y' = 4c_1e^{4x} - 3c_2e^{-3x}$

again diff wrt x

$y'' = 16c_1e^{4x} + 9c_2e^{-3x}$

$y = c_1e^{4x} + c_2e^{-3x}$; $y(0) = -2$

$c_1e^{0} + c_2e^{-0} = -2$

$\boxed{c_1 + c_2 = -2}$ — (i)

$y' = 4c_1e^{4x} - 3c_2e^{-3x}$; $y'(0) = 6$

$6 = 4c_1e^{0} - 3c_2e^{-0}$

$\boxed{4c_1 - 3c_2 = 6}$ — (ii)

x eq(i) by 4 and subtract

$\cancel{4c_1} + 4c_2 = -8$

$-\cancel{4c_1} \mp 3c_2 = -6$

$7c_2 = -14$

$c_2 = -\frac{14}{7}$

$\boxed{c_2 = -2}$

(i) $c_1 + c_2 = -2$

$c_1 - 2 = -2$

$c_1 = -\cancel{2} + \cancel{2}$

$\boxed{c_1 = 0}$

$y = 0e^{4x} + (-2)e^{-3x}$

$\boxed{y = -2e^{-3x}}$

Solution of given D.E:-

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

$(16c_1e^{4x} + 9c_2e^{-3x}) - (4c_1e^{4x} - 3c_2e^{-3x})$

$-12(c_1e^{4x} + c_2e^{-3x}) = 0$

$16\cancel{c_1e^{4x}} + \cancel{9c_2e^{-3x}} - 4c_1e^{4x} + 3\cancel{c_2e^{-3x}}$

$-12\cancel{c_1e^{4x}} \cancel{-12c_2e^{-3x}} = 0$

$\boxed{0 = 0}$ solution of

$\because$ Indicated function is the given $\boxed{D.E}$

(b.) $x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$; $y(2)=2$
$y'(2)=2$
$y''(2)=6$

$y = c_1x + c_2x^2 + c_3x^3$

diff wrt x

$y' = c_1 + 2c_2x + 3c_3x^2$

again diff wrt x

$y'' = 2c_2 + 6c_3x$

again diff wrt x

$y''' = 6c_3$

Solution of given D.E :-

$x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$

$x^3(6c_3) - 3x^2(2c_2 + 6c_3x) + 6x(c_1 + 2c_2x + 3c_3x^2) - 6(c_1x + c_2x^2 + c_3x^3) = 0$

$\cancel{6c_3x^3} - \cancel{6c_2x^2} - \cancel{18c_3x^3} + \cancel{6c_1x} + \cancel{12c_2x^2} + \cancel{18c_3x^3} - \cancel{6c_1x} - \cancel{6c_2x^2} - \cancel{6c_3x^3} = 0$

$0 = 0$

$\therefore$ Indicated function is the given solution of D.E

$y(2) = 2$

$2c_1 + 4c_2 + 8c_3 = 2$ — (i)

$y'(2) = 2$

$c_1 + 4c_2 + 12c_3 = 2$ — (ii)

$y''(2) = 6$

$2c_2 + 18c_3 = 6$ — (iii)

Adding (i), (ii) and (iii) and x (ii) by -2 and x (iii) by 2

$\cancel{2c_1} + \cancel{4c_2} + 8c_3 = 2$

$-\cancel{2c_1} - \cancel{8c_2} - 24c_3 = -4$

$\cancel{4c_2} + 36c_3 = 12$

$20c_3 = 10$

$c_3 = \frac{10}{20} = \frac{1}{2}$

$2c_2 + 18\left(\frac{1}{2}\right) = 6$

$2c_2 + 9 = 6$

$2c_2 = 6 - 9$

$2c_2 = -3$

$\boxed{c_2 = \frac{-3}{2}}$

$c_1 + 4\left(\frac{-3}{2}\right) + 12\left(\frac{1}{2}\right) = 2$

$c_1 - \cancel{6} + \cancel{6} = 2$

$\boxed{c_1 = 2}$

$\therefore \boxed{y = 2x - \frac{3}{2}x^2 + \frac{1}{2}x^3}$

(c.) $\frac{d^2y}{dx^2}+y=0$, $y(0)=1$, $y'(\frac{\pi}{2})=-1$ (Boundary Value)

$y=c_1\sin x+c_2\cos x$

diff wrt x

$y'=c_1\cos x-c_2\sin x$

$y''=-c_1\sin x-c_2\cos x$

Solution of the given D.E :-

$\frac{d^2y}{dx^2}+y=0$

$-c_1\cancel{\sin x}-c_2\cancel{\cos x}+c_1\cancel{\sin x}+c_2\cancel{\cos x}=0$

$0=0$

∵ Indicated function is the given solution of D.E

<u>By using First Boundary Condition :-</u>

$y=c_1\sin x+c_2\cos x$; $y(0)=1$

$c_1\sin 0+c_2\cos 0=1$

$\boxed{c_2=1}$

<u>By using Second Boundary Condition :-</u>

$y'=c_1\cos x-c_2\sin x$; $y'(\frac{\pi}{2})=-1$

$c_1\cos(\frac{\pi}{2})-c_2\sin\frac{\pi}{2}=-1$

$\cancel{-}c_2=\cancel{-}1$

$\boxed{c_2=1}$

∵ $c_1=1$

$c_2=1$

$\boxed{y=\sin x+\cos x}$

Problem 3 ELIMINATE ARBITARY CONSTANT

(a.) $x^3+y^3=3cxy$

$$\frac{x^3+y^3}{xy}=3c$$

$$\frac{xy\frac{d}{dx}(x^3+y^3)-(x^3+y^3)\frac{d}{dx}xy}{(xy)^2}=0$$

$$\frac{xy\left(3x^2+3y^2\frac{dy}{dx}\right)-(x^3+y^3)\left[y\frac{d}{dx}x+x\frac{d}{dx}y\right]}{x^2y^2}=0$$

$$xy\left(3x^2+3y^2\frac{dy}{dx}\right)-(x^3+y^3)\left(y+x\frac{dy}{dx}\right)=0$$

$$3x^3y+3xy^3\frac{dy}{dx}-\left(x^3y+x^4\frac{dy}{dx}+y^4+xy^3\frac{dy}{dx}\right)=0$$

$$3xy^3\frac{dy}{dx}-x^4\frac{dy}{dx}-xy^3\frac{dy}{dx}=-3x^3y+x^3y+y^4$$

$$(2xy^3-x^4)\frac{dy}{dx}=-2x^3y+y^4$$

$$\frac{dy}{dx}=\frac{-2x^3y+y^4}{2xy^3-x^4}$$

$$\boxed{\frac{dy}{dx}=\frac{y(y^3-2x^3)}{x(2y^3-x^3)}}$$

(b.) $3y=\frac{4x^3}{x^2+1}+\frac{3c}{x^2+1}$

$$3y(x^2+1)=4x^3+3c$$

$$3y(x^2+1)-4x^3=3c$$

$$3yx^2+3y-4x^3=3c$$

$$3\left[x^2\frac{dy}{dx}+2xy\right]+3\frac{dy}{dx}-12x^2=0$$

$$3x^2\frac{dy}{dx}+6xy+3\frac{dy}{dx}-12x^2=0$$

$$\frac{dy}{dx}(3x^2+3)=12x^2-6xy$$

$$\frac{dy}{dx}=\frac{12x^2-6xy}{3x^2+3}$$

$$\frac{dy}{dx}=\frac{\cancel{6}x(2x-y)}{\cancel{3}(x^2+1)}$$

$$\boxed{\frac{dy}{dx}=\frac{x(2x-y)}{(x^2+1)}}$$

Bonus Problem 4

(a.) SEPARATION OF VARIABLE:-

$$(xy+2x+y+2)dx + (x^2+2x)dy = 0$$

$$(xy+y+2x+2)dx + (x^2+2x)dy = 0$$

$$y(x+1)+2(x+1)dx + (x^2+2x)dy = 0$$

$$(x+1)(y+2)dx + (x^2+2x)dy = 0$$

$$(x^2+2x)dy = -(x+1)(y+2)dx$$

$$\frac{1}{y+2}dy = -\frac{(x+1)}{(x^2+2x)}dx$$

Taking Integral on bothsides

$$\int \frac{1}{y+2}dy = -\frac{1}{2}\int \frac{1}{x^2+2x}(2x+2)dx$$

$$\ln(y+2) = -\frac{1}{2}\ln(x^2+2x)+c$$

$$\ln(y+2) = \ln(x^2+2x)^{-\frac{1}{2}}+c$$

Taking e on b/s

$$e^{\ln(y+2)} = e^{\ln(x^2+2x)^{-\frac{1}{2}}}+c$$

$$\boxed{y+2 = (x^2+2x)^{-\frac{1}{2}}+c}$$

(b.) LINEAR DIFFERENTIABLE EQUATION:-

$$\frac{dy}{dx} + \frac{y}{x\ln x} = \frac{3x^2}{\ln x}$$

Compare:-

$$y' + P(x)y = Q(x)$$

$$P(x) = \frac{1}{x\ln x}$$

$$Q(x) = \frac{3x^2}{\ln x}$$

$$I(x) = e^{\int P(x)dx}$$

$$I(x) = e^{\int \frac{1}{\ln x}\cdot\frac{1}{x}dx}$$

$$I(x) = e^{\ln(\ln x)}$$

$$I(x) = \ln x$$

$$y = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + c\right]$$

$$y = \frac{1}{\ln x}\left[\int \ln x \frac{3x^2}{\ln x}\,dx + c\right]$$

$$y = \frac{1}{\ln x}\left[3\int x^2\,dx + c\right]$$

$$y = \frac{1}{\ln x}\left[\cancel{3}\frac{x^3}{\cancel{3}} + c\right]$$

$$\boxed{y = \frac{x^3}{\ln x} + \frac{c}{\ln x}}$$

(c.) Solve the exact DE:

EXACT DIFFERENTIAL EQUATION:-

$$e^x\left[y - 3(e^x+1)^2\right]dx + (e^x+1)\,dy = 0 \quad ; \; y(0) = 4$$

$M = e^x y - 3e^x(e^x+1)^2$

$N = e^x + 1$

$M_y = e^x - 0$

$N_x = e^x$

$M_y = e^x$

$M_y = N_x$

$\therefore$ Differential Equation is Exact

$$\int e^x y - 3e^x(e^x+1)^2\,dx$$

$$\int e^x y - 3e^x(e^{2x} + 2e^x + 1)$$

$$\int (e^x y - 3e^{3x} - 6e^{2x} - 3e^x)\,dx$$

$$e^x y - \frac{3e^{3x}}{3} - \frac{6e^{2x}}{2} - 3e^x$$

$$e^x y - e^{3x} - 3e^{2x} - 3e^x \quad \text{--- (i.)}$$

$$\int (e^x+1)\,dy$$

$$e^x y + y \quad \text{--- (ii.)}$$

Combining terms (i) & (ii) we get:.

$$\boxed{e^x y - e^{3x} - 3e^{2x} - 3e^x + y = c}$$