Assignment 2

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PROBLEM 1

Solve the following homogeneous linear differential equations with constant coefficient.

a.) $(D^4+6D^3+15D^2+20D+12)y=0$

$$m^4+6m^3+15m^2+20m+12=0$$

$$m=-2,-2,-1\pm\sqrt{2}i$$

$$\boxed{y=c_1e^{-2x}+c_2xe^{-2x}+c_3e^{-x}\cos(\sqrt{2}x)+c_4e^{-x}\sin(\sqrt{2}x)}$$

b.) $(D^3-27)y=0$

$$m^3-27=0$$

$$m^3-3^3=0$$

$$m=3,-\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i$$

$$\boxed{y=c_1e^{3x}+c_2e^{-\frac{3}{2}x}\cos\left(\frac{3\sqrt{3}x}{2}\right)+c_3e^{-\frac{3}{2}x}\sin\left(\frac{3\sqrt{3}x}{2}\right)}$$

Problem 2

Solve the differential equations with coefficient Superposition Approach.

a.) $(D^2-7D+12)y = e^{2x}(x^3-5x^2)$

$m^2-7m+12=0$

$m=3,4$

$y_c = c_1e^{3x}+c_2e^{4x}$

$y_p = (Ax^3+Bx^2+Cx+D)e^{2x}$

$y_p' = 2e^{2x}(Ax^3+Bx^2+Cx+D)+(3Ax^2+2Bx+C)e^{2x}$

$y_p'' = 4e^{2x}(Ax^3+Bx^2+Cx+D)+2e^{2x}(3Ax^2+2Bx+C)$
$+e^{2x}(6Ax+2B)+4e^{2x}(3Ax^2+2Bx+C)$

$y_p''-7y_p+12=0$

$4e^{2x}(Ax^3+Bx^2+Cx+D)+2e^{2x}(3Ax^2+2Bx+C)+4e^{2x}(3Ax^2+2Bx+C)$
$+e^{2x}(6Ax+2B)-7[(Ax^3+Bx^2+Cx+D)e^{2x}+e^{2x}(3Ax^2+2Bx+C)]$
$+12[e^{2x}(Ax^2+Bx^2+Cx+D)] = e^{2x}(x^3-5x^2)$

$2e^{2x}(Ax^3+Bx^2+Cx+D)-3e^{2x}(3Ax^2+2Bx+C)$
$+e^{2x}(6Ax+2B) = e^{2x}(x^3-5x^2)$

$2e^{2x}Ax^3+2e^{2x}Bx^2+2e^{2x}Cx+2e^{2x}D-9e^{2x}Ax^2-6e^{2x}Bx$
$-3e^{2x}C+6e^{2x}Ax+2e^{2x}B = x^3e^{2x}-5x^2e^{2x}$

$2e^{2x}Ax^3 = x^3e^{2x}$

$\boxed{A=\frac{1}{2}}$

$-6e^{2x}Bx+2e^{2x}(Cx+6Ax)=0$

$\frac{6}{4}+2C+\frac{6}{2}=0$

$\boxed{C=-\frac{9}{4}}$

$-9e^{2x}Ax^2+2e^{2x}Bx^2=-5x^2e^{2x}$

$x^2e^{2x}(-9A+2B)=-5x^2e^{2x}$

$-\frac{9}{2}+2B=-5$

$\boxed{B=-\frac{1}{4}}$

$2e^{2x}B-3e^{2x}C+2e^{2x}D=0$

$2\left(\frac{-1}{4}\right)+\frac{27}{4}+2D=0$

$\frac{25}{4}+2D=0$ ~~$2D=-\frac{27}{4}$~~ ~~$D=-\frac{27}{8}$~~

$2D=-\frac{25}{4}$

$\boxed{D=-\frac{25}{8}}$

$\therefore y = y_c+y_p$

$\boxed{y=c_1e^{3x}+c_2e^{4x}+\frac{xe^{3x}}{2}-\frac{xe^{2x}}{4}-\frac{9xe^{2x}}{4}-\frac{25e^{2x}}{8}}$

$\to y=c_1e^{3x}+c_2e^{4x}+\frac{x^3}{2}e^{2x}-\frac{x^2}{4}e^{2x}-\frac{9x}{4}e^{2x}-\frac{25}{8}e^{2x}$

b.) $y'' + y' - 2y = x^2 + 2\sin x - e^{3x}$

$m^2 + m - 2 = 0 \quad ; m = -2, 1$

$y_c = c_1 e^{-2x} + c_2 e^{x}$

$y_p = Ax^2 + Bx + C + E\cos x + F\sin x + Ge^{3x}$

$y_p' = 2Ax + B - E\sin x + F\cos x + 3Ge^{3x}$

$y_p'' = 2A - E\cos x - F\sin x + 9Ge^{3x}$

$y_p'' + y_p' - 2y_p = x^2 + 2\sin x - e^{3x}$

$2A - E\cos x - F\sin x + 9Ge^{3x} + 2Ax + B - E\sin x + F\cos x + 3Ge^{3x}$
$-2Ax^2 - 2Bx - 2C - 2E\cos x - 2F\sin x - 2Ge^{3x} = x^2 + 2\sin x - e^{3x}$

$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) + (-E + F - 2E)\cos x$
$+ (-F - E - 2F)\sin x + (9G + 3G - 2G)e^{3x} = x^2 + 2\sin x - e^{3x}$

$-2A\cancel{x^2} = \cancel{x^2}$

$\boxed{A = -\frac{1}{2}}$

$2A - 2B = 0$

$2\left(-\frac{1}{2}\right) - 2B = 0$

$\boxed{B = -\frac{1}{2}}$

$2A + B - 2C = 0$

$2\left(-\frac{1}{2}\right) - \frac{1}{2} - 2C = 0$

$\boxed{C = -\frac{3}{4}}$

$-3E + F = 0 \quad$ i.

$-E - 3F = 2 \quad$ ii.

$\times$ (i.) by 3

$-9E + \cancel{3F} = 0$

$-E \cancel{-3F} = +2$

$-10E = 2$

$\boxed{E = -\frac{1}{5}}$

(i.) $-3\left(-\frac{1}{5}\right) + F = 0$

$\boxed{F = -\frac{3}{5}}$

$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} - \frac{1}{5}\cos x - \frac{3}{5}\sin x - \frac{1}{10}e^{3x}$

$\therefore y = y_c + y_p$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} - \frac{1}{5}\cos x - \frac{3}{5}\sin x - \frac{1}{10}e^{3x}}$$

Problem 3

Solve differential using Variation of Parameters

a) $(D^2+1)y = \operatorname{cosec} x \Rightarrow f(x)$

The Homogeneous Differential Equation

$(D^2+1)y = 0$

It's auxillary Equation

$m^2+1=0$

$m^2 = -1$

$\boxed{m=\pm i} \Rightarrow m = 0 \pm i$ (α, β)

$y_c = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$

$y_c = c_1 e^0 \cos x + c_2 e^0 \sin x$

$\boxed{y_c = c_1\cos x + c_2 \sin x}$

$y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$W(y_1,y_2) = W(\cos x, \sin x)$

$W(y_1,y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$W(y_1,y_2) = \cos x\cos x + \sin x \sin x$

$W(y_1,y_2) = \cos^2 x + \sin^2 x$

$\boxed{W(y_1,y_2) = 1}$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \operatorname{cosec} x & \cos x \end{vmatrix}$

$W_1 = -\sin x \operatorname{cosec} x$

$W_1 = -\sin x \frac{1}{\sin x}$

$\boxed{W_1 = -1}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \operatorname{cosec} x \end{vmatrix}$

$W_2 = \cos x \cdot \operatorname{cosec} x$

$W_2 = \cos x \cdot \frac{1}{\sin x}$

$\boxed{W_2 = \cot x}$

$u_1' = \frac{W_1}{W}$

$u_1' = -\frac{1}{1}$

$u_1' = -1$

$\boxed{u_1 = -x}$

$u_2' = \frac{W_2}{W}$

$u_2' = \frac{\cot x}{1}$

$\boxed{u_2 = \ln \sin x}$

$\because y_p = u_1 y_1 + u_2 y_2$

$\boxed{y_p = -x\cos x + \ln\sin x \sin x}$

$\because y = y_c + y_p$

$$\boxed{\begin{aligned} y = {} & c_1\cos x + c_2 \sin x \\ & -x\cos x + \sin x \ln \sin x \end{aligned}}$$

b) $(D^2-1)y = \frac{2}{1+e^x}$

The Homogeneous Differential Equation

$(D^2-1)y=0$

It's auxillary Equation

$m^2-1=0$

$m^2=1$

$\boxed{m=\pm 1}$

$y_c = c_1 e^{\alpha x} + c_2 e^{\alpha x}$

$\boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$

$y_1 = e^x \qquad y_2 = e^{-x}$

$y_1' = e^x \qquad y_2' = -e^{-x}$

$W(y_1,y_2) = W(e^x, e^{-x})$

$$W(y_1,y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$W(y_1,y_2) = e^x \cdot e^{-x} - e^x \cdot e^{-x}$

$W(y_1,y_2) = -e^{\cancel{x}-\cancel{x}} - e^{\cancel{x}-\cancel{x}}$

$W(y_1,y_2) = -e^0 - e^0$

$W(y_1,y_2) = -1-1$

$\boxed{W(y_1,y_2) = -2}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{2}{1+e^x} & -e^{-x} \end{vmatrix}$$

$$\boxed{W_1 = -\frac{2}{1+e^x} \cdot e^{-x}}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ xe^x & \frac{2}{1+e^x} \end{vmatrix}$$

$$\boxed{W_2 = \frac{2}{1+e^x} e^x}$$

$$u_1' = \frac{W_1}{W} = \frac{\frac{\cancel{2}}{1+e^x} e^{-x}}{\cancel{2}} = \frac{1}{e^x(1+e^x)}$$

Integrate

$$\int u_1' = \int \frac{1}{e^x(1+e^x)} dx + c_1$$

Put $e^x = t \qquad dx = \frac{dt}{t}$

$e^x dx = dt$

$$u_1 = \int \frac{1}{t^2(1+t)} dt + C_1$$

$$u_1 = \int \left[\frac{1}{t^2} - \frac{1}{t} + \frac{1}{1+t}\right] dt + C_1$$

Using Partial fraction.

$u_1 = -\frac{1}{t} - \ln t + \ln(1+t) + C_1$

$\boxed{u_1 = -e^{-x} - x + \ln(1+e^x) + C_1}$

$$u_2' = \frac{W_2}{W} = \frac{\frac{\cancel{2}}{1+e^x} \cdot e^x}{-\cancel{2}} = -\frac{e^x}{1+e^x}$$

Integrate

$$\int u_2' = -\int \frac{e^x}{1+e^x} dx + C_2$$

$\boxed{u_2 = -\ln(1+e^x) + C_2}$

$y_p = u_1 y_1 + u_2 y_2$

$$\boxed{y_p = \left[-e^{-x} - x + \ln(1+e^x) + c_1\right] e^x + \left[-\ln(1+e^x) + c_2\right] e^{-x}}$$

$\therefore y = y_c + y_p$

$y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + e^x \ln(1+e^x) - e^{-x}\ln(1+e^x)$

$$\boxed{y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln(1+e^x)}$$

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PROBLEM 4

Solve the differential equations by method of Undetermine Coefficient-Annihilator Approach.

Formula:-

Operator	Annihilates
i) D^n	$a_{n-1}x^{n-1}+\dots\dots+a_1x+a_0$
ii) $D-a$	e^{ax}
iii) $(D-a)^n$	$x^{n-1}e^{ax}$
iv) $D^2-2\alpha D+(\alpha^2+\beta^2)$	$e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$
v) $(D^2-2\alpha D+(\alpha^2+\beta^2))^n$	$= x^{n-1}e^{\alpha x}\cos(\beta x)$, $x^{n-1}e^{\alpha x}\sin(\beta x)$

a.) $y''+y'+\frac{1}{4}y=e^x(\sin 3x+\cos 3x)$

$y''+y'+\frac{1}{4}y=0$

$m^2+m+\frac{1}{4}y=0$

$m=-\frac{1}{2},-\frac{1}{2}$

$y_c=c_1e^{-x/2}+c_2e^{-x/2}x$

Annihilator:-

$D^2-2\alpha D+(\alpha^2+\beta^2)=e^x(\sin 3x+\cos 3x)$

$\alpha=1,\ \beta=3$

$D^2-2D+1^2+3^2$

$\boxed{D^2-2D+10}$; Annihilator

$y''+y'+\frac{1}{4}y=e^x(\sin 3x+\cos 3x)$

$(D^2+D+\frac{1}{4})y=e^x(\sin 3x+\cos 3x)$

Multiply b/s by $(D^2-2D+10)$

$(D^2-2D+10)(D^2+D+\frac{1}{4})=(D^2-2D+10)e^x(\sin 3x+\cos 3x)$

$(D^2-2D+10)(D^2+D+1/4)=0$

$m=\underset{\alpha}{1}\pm\underset{\beta}{3}i,-\frac{1}{2},-\frac{1}{2}$

$y=Ae^x\cos(3x)+Be^x\sin(3x)+ce^{-\frac{x}{2}}+Exe^{-\frac{x}{2}}$

$y_p=Ae^x\cos(3x)+Be^x\sin(3x)$

$y_p'=\cos 3x\cdot Ae^x+Ae^x(-3\sin 3x)+\sin 3x\ Be^x+Be^x(3\cos 3x)$

$y_p'=Ae^x\cos 3x-3Ae^x\sin 3x+Be^x\sin 3x+3Be^x\cos 3x$

$y_p'=(A+3B)e^x\cos 3x+(B-3A)e^x\sin 3x$

$$y_p'' = \cos 3x\,(A+3B)e^x + (A+3B)e^x(-3\sin 3x) + \sin 3x\,(B-3A)e^x$$
$$+ (B-3A)e^x(3\cos 3x)$$

$$y_p'' = (A+3B)e^x\cos 3x + (-3A-9B)e^x\sin 3x + (B-3A)e^x\sin 3x$$
$$+ (3B-9A)e^x\cos 3x$$

$$y_p'' = (A+3B+3B-9A)e^x\cos 3x + (-3A-9B+B-3A)e^x\sin 3x$$

$$y_p'' = (-8A+6B)e^x\cos 3x + (-6A-8B)e^x\sin 3x$$

$$y_p'' + y_p' + \frac{1}{4}y_p = e^x\sin 3x + e^x\cos 3x$$

$$(-8A+6B)e^x\cos 3x + (-6A-8B)e^x\sin x + (A+3B)e^x\cos 3x + (B-3A)e^x\sin 3x$$
$$+ \frac{1}{4}(Ae^x\cos(3x) + Be^x\sin 3x) = e^x\sin 3x + e^x\cos 3x$$

Compare:-

$$(-8A+6B)e^x\cos 3x + (A+3B)e^x\cos 3x + \left(\frac{1}{4}A\right)e^x\cos 3x = e^x\cos 3x$$

$$\left(-8A+6B+A+3B+\frac{1}{4}A\right)\cancel{e^x\cos 3x} = \cancel{e^x\cos 3x}$$

$$-8A+6B+A+3B+\frac{1}{4}A = 1$$

$$-\frac{27}{4}A + 9B = 1 \quad \text{—— (i)}$$

$$(-6A-8B)e^x\sin x + (B-3A)e^x\sin 3x + \left(\frac{1}{4}B\right)e^x\sin 3x = e^x\sin 3x$$

$$\left(-6A-8B+B-3A+\frac{1}{4}B\right)\cancel{e^x\sin x} = \cancel{e^x\sin x}$$

$$-6A-8B+B-3A+\frac{1}{4}B = 1$$

$$-9A - \frac{27}{4}B = 1 \quad \text{—— (ii)}$$

× eq (ii) by $\frac{3}{4}$ & equate with eq (i)

$$-\frac{27}{4}A + 9B = 1$$
$$-\frac{27}{4}A - \frac{81}{16}B = \frac{3}{4}$$
$$+\qquad\quad +\qquad -$$
$$B\left(\frac{225}{16}\right) = \frac{1}{4}$$
$$\boxed{B = \frac{4}{225}}$$

× eq (i) by $\frac{3}{4}$ and equate eq (ii)

$$-\frac{81}{16}A + \frac{27}{4}B = \frac{3}{4}$$
$$-9A - \frac{27}{4}B = 1$$
$$A\left(-\frac{225}{16}\right) = \frac{7}{4}$$
$$\boxed{A = -\frac{28}{255}}$$

$$\therefore y_p = -\frac{28}{255}e^x\cos 3x + \frac{4}{225}e^x\sin 3x$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = c_1e^{-\frac{x}{2}} + c_2xe^{-\frac{x}{2}} - \frac{28}{255}e^x\cos 3x + \frac{4}{225}e^x\sin 3x}$$

b.) $y'' + 2y' + y = x^2e^{-x}$

$m^2 + 2m + 1 = 0$

$y_c = c_1e^{-x} + c_2xe^{-x}$

Annihilator : $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$

$[D^2 - 2(-1)D + (1 + 0^2)]^3$

$\boxed{(D+1)^3}$; Annihilator

Multiply b/s by $(D+1)^3$

$(D+1)^3(D^2+2D+1)y = (D+1)^3x^2e^{-x}$ $\qquad m = -1, -1$

$(D+1)^3(D^2+2D+1)y = 0 \quad \because y = c_1e^{-x} + c_2xe^{-x}$

$y_p = Ax^2e^{-x} + Bx^3e^{-x} + Cx^4e^{-x}$

$y_p' = 2Axe^{-x} - Ax^2e^{-x} + 3Bx^2e^{-x} - Bx^3e^{-x} + 4Cx^3e^{-x} - Cx^4e^{-x}$

$y_p'' = 2Ae^{-x} - \cancel{2Axe^{-x}} - 4Axe^{-x} + Ax^2e^{-x} + 6Bxe^{-x} - \cancel{3Bx^2}e^{-x}$
$-6Bx^2e^{-x} + Bx^3e^{-x} + 12Cx^2e^{-x} - \cancel{4Cx^3}e^{-x} - 8Cx^3e^{-x} + Cx^4e^{-x}$

$2Ae^{-x} - \cancel{4Axe^{-x}} + \cancel{Ax^2e^{-x}} + 6Bxe^{-x} - \cancel{6Bx^2e^{-x}} + \cancel{Bx^3e^{-x}}$
$+ 12Cx^2e^{-x} - \cancel{8Cx^3e^{-x}} + \cancel{Cx^4e^{-x}} = x^2e^{x}$

$2A\cancel{e^{-x}} + 6Bx\cancel{e^{-x}} + 12Cx^2\cancel{e^{-x}} = x^2\cancel{e^{-x}}$

$2A + 6Bx + 12Cx^2 = x^2$

$12C\cancel{x^2} = \cancel{x^2}$

$12C = 1$

$\boxed{C = \frac{1}{12}}$

$\because A = B = 0$

$y_p = Ax^2e^{-x} + Bx^3e^{-x} + Cx^4e^{-x}$

$y_p = \frac{x^4e^{-x}}{12}$

$y = y_c + y_p$

$\boxed{y = c_1e^{x} + c_2xe^{-x} + \frac{x^4e^{-x}}{12}}$

PROBLEM 5

Solve Cauchy Euler Equation $x^2y''+xy'-y=x^3e^x$

$$x^2y''+xy'-y=x^3e^x$$

$$y=x^m,\ y'=mx^{m-1},\ y''=(m^2-m)x^{m-2}$$

$$x^2(m^2-m)x^{m-2}+xmx^{m-1}-x^m=0$$

$$m^2-m+m-1=0$$

$$m^2-1=0$$

$$m=-1,\ 1$$

$$y_c=\underbrace{c_1x^{-1}}_{y_1}+\underbrace{c_2x}_{y_2}$$

$$y_1=x^{-1},\ y_2=x$$

$$W=\begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix}$$

$$W=x^{-1}+x^{-1}$$

$$\boxed{W=2x^{-1}}$$

$$y''+\frac{1}{x}y'-\frac{1}{x^2}y=xe^x$$

$$W_1=\begin{vmatrix} 0 & x \\ xe^x & 1 \end{vmatrix}$$

$$\boxed{W_1=-x^2e^x}$$

$$W_2=\begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & xe^x \end{vmatrix}$$

$$W_2=\cancel{x^{-1}}\cancel{x^{+1}}e^x$$

$$\boxed{W_2=e^x}$$

$$u_1'=\frac{W_1}{W}$$

$$u_1'=\frac{-x^2e^x}{2x^{-1}}$$

$$u_1'=-\frac{x^3e^x}{2}$$

Integrate

$$u_1=-\frac{x^3e^x}{2}+\int\frac{3x^2}{2}e^x\,dx$$

$$u_1=-\frac{x^3e^x}{2}+\frac{3}{2}x^2e^x-\int 3xe^x$$

$$\boxed{u_1=-\frac{x^3e^x}{2}+\frac{3x^2e^x}{2}-3xe^x+3e^x}$$

$$u_2'=\frac{W_2}{W}=\frac{e^x}{2x^{-1}}$$

$$u_2'=\frac{xe^x}{2}$$

Integrate

$$\boxed{u_2=\frac{xe^x}{2}-\frac{e^x}{2}}$$

$$y_p=u_1y_1+u_2y_2$$

$$y_p=x^{-1}\left(-\frac{x^3e^x}{2}+\frac{3x^2e^x}{2}-3xe^x+3e^x\right)+x\left(\frac{xe^x}{2}-\frac{e^x}{2}\right)$$

$$y_p=-\frac{x^2e^x}{2}+\frac{3xe^x}{2}-3e^x+\frac{3e^x}{x}+\frac{x^2e^x}{2}-\frac{xe^x}{2}$$

$$\boxed{y_p=\frac{3xe^x}{2}-\frac{xe^x}{2}-3e^x+\frac{3e^x}{x}}$$

$\because\ y=y_c+y_p$

$$\boxed{y=c_1x^{-1}+c_2x+\frac{3xe^x}{2}-\frac{xe^x}{2}-3e^x+\frac{3e^x}{x}}$$

BONUS PROBLEM 6

a.) Determine whether the given set of function is linearly independent or linearly dependent on $(-\infty, \infty)$

i.) $y_1 = \cos 2x$, $y_2 = 1$, $y_3 = \cos^2 x$

$$W = \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -2\sin x \cos x \\ -4\cos 2x & 0 & \cos x(-2\cos x) + (-2\sin x)(-\sin x) \end{vmatrix}$$

$$W = \cos 2x \begin{vmatrix} 0 & -2\sin x \cos x \\ 0 & -2\cos^2 x + 2\sin^2 x \end{vmatrix} - 1 \begin{vmatrix} -2\sin 2x & -2\sin x \cos x \\ -4\cos 2x & -2\cos^2 x + 2\sin^2 x \end{vmatrix} + \cos^2 x \begin{vmatrix} -2\sin 2x & 0 \\ -4\cos 2x & 0 \end{vmatrix}$$

$$W = \cos 2x(0) - 1[4\sin 2\cos^2 x - 4\sin^2 x \sin 2x - 8\sin x \cos x \cos 2x] + \cos^2 x(0)$$

$$W = -1[4\sin 2x \cos^2 x - 4\sin^2 x \sin 2x - 8\sin x \cos x \cos 2x]$$

$$W = -1[4\sin 2x(\cos^2 x - \sin^2 x) - 4(2\sin x \cos x)\cos 2x]$$ $\because \cos 2x = \cos^2 x - \sin^2 x$

$\because \sin 2x = 2\sin x \cos x$

$$W = -1[4\sin \cancel{2x}\cancel{\cos 2x} - 4\sin \cancel{2x}\cancel{\cos 2x}]$$

$$W = -1(0)$$

$\boxed{W = 0}$; linearly dependent.

ii.) $y_1 = x$, $y_2 = x^{-2}$, $y_3 = x^2 \ln x$

$$W = \begin{vmatrix} x & x^{-2} & x^2 \ln x \\ 1 & -2x^{-3} & 2x\ln x + x \\ 0 & 6x^{-4} & 2\ln x + 3 \end{vmatrix}$$

$$W = x \begin{vmatrix} -2x^{-3} & 2x\ln x + x \\ 6x^{-4} & 2\ln x + 3 \end{vmatrix} - 1 \begin{vmatrix} x^{-2} & x^2 \ln x \\ 6x^{-4} & 2\ln x + 3 \end{vmatrix}$$

$$W = x[(-2x^{-3})(2\ln x + 3) - (6x^{-4})(2x\ln x + x)] - [(x^{-2})(2\ln x + 3) - (6x^{-4})(x^2 \ln x)]$$

$$W = x[-4x^{-3}\ln x - 6x^{-3} - (12x^{-3}\ln x + 6x^{-3})] - [2x^{-2}\ln x + 3x^{-2} - (6x^{-2}\ln x)]$$

$$W = -4\cancel{x^{-2}\ln x} - 6x^{-2} - 12x^{-2}\ln x - 6x^{-2} - 2\cancel{x^{-2}\ln x} - 3x^{-2} + 6\cancel{x^{-2}\ln x}$$

$$\boxed{W = -\frac{(12\ln x + 15)}{x^2}}$$; linearly independent.

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b.) Solve differential equation using reduction of order then varify by formulation $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

i.) $9y'' - 12y' + 4y = 0$, $y_1 = e^{2x/3}$

$y = ue^{\frac{2x}{3}}$

$y' = \frac{2}{3}e^{\frac{2x}{3}}u + e^{\frac{2x}{3}}u$

$y'' = e^{2x/3}u'' + \frac{4}{3}e^{2x/3}u' + \frac{4}{9}e^{\frac{2x}{3}}u$

$9y'' - 12y' + 4y = 0$

$9\left(e^{2x/3}u'' + \frac{4}{3}e^{\frac{2x}{3}}u' + \frac{4}{9}e^{\frac{2x}{3}}u\right) - 12\left[\frac{2}{3}e^{\frac{2x}{3}}u + e^{\frac{2x}{3}}u'\right] + 4\left[ue^{\frac{2x}{3}}\right] = 0$

$9e^{\frac{2x}{3}}u'' + \cancel{12e^{\frac{2x}{3}}u'} + \cancel{4e^{\frac{2x}{3}}u} - \cancel{8e^{\frac{2x}{3}}u} - \cancel{12e^{\frac{2x}{3}}u'} + \cancel{4ue^{\frac{2x}{3}}} = 0$

$9e^{\frac{2x}{3}}u'' = 0$

$u'' = 0$

$\int u'' = \int 0$

$\int u' = \int c_1$

$u = c_1x + c_2$

$c_1 = 1,\ c_2 = 0,\ u = x$

Put u in eq

$y = xe^{\frac{2x}{3}}$

$y = (c_1x + c_2)e^{\frac{2}{3}x}$ $\because y_2 = xe^{2/3x}$

$\boxed{y = c_1xe^{\frac{2}{3}x} + c_2e^{\frac{2}{3}x}}$

Varifying :-

$y_2 = e^{2x/3}\int \frac{e^{-\int -\frac{12}{9}dx}}{(e^{2x/3})^2}$

$y_2 = e^{2x/3}\int \frac{e^{\frac{4x}{3}}}{e^{\frac{4x}{3}}}dx$

$y_2 = e^{2x/3}\int 1\, dx$

$y_2 = xe^{2x/3}$

$\boxed{y = e^{2x/3} + xe^{2x/3}}$

ii) $y'' - 3y' + 2y = 5e^{3x}$, $y_1 = e^{x}$

$y = ue^{x}$

$y' = u'e^{x} + ue^{x}$

$y'' = u''e^{x} + u'e^{x} + u'e^{x} + ue^{x}$

$y'' = u''e^{x} + 2u'e^{x} + ue^{x}$

$y'' - 3y' + 2y = 5e^{3x}$

$u''e^{x} + 2u'e^{x} + ue^{x} - 3(u'e^{x} + ue^{x}) + 2ue^{x} = 5e^{3x}$

$u''e^{x} + 2u'e^{x} + \cancel{3ue^{x}} - 3u'e^{x} - \cancel{3ue^{x}} = 5e^{3x}$

$u''e^{x} - u'e^{x} = 5e^{3x}$

$(u'' - u)e^{x} = 5e^{3x}$

$u'' - u = 5e^{2x}$

let,

$w = u'$

$w' = u''$

$w' - w = 5e^{2x}$

$\frac{dw}{dx} - w = 5e^{2x}$; Bernulli/ Separable can be applied

$P(x) = -1$

$\mu(x) = e^{\int -1\,dx}$

$= e^{-x}$

$e^{-x}\frac{dw}{dx} - e^{-x}w = (5e^{2x})(e^{-x})$

$\frac{d}{dx}we^{-x} = 5e^{x}$

$\int we^{-x} = \int 5e^{x}\,dx$

$we^{-x} = 5e^{x} + c_1$

$w = \frac{5e^{x}}{e^{-x}} + \frac{c_1}{e^{-x}}$

$w = 5e^{2x} + c_1e^{x}$

$u' = 5e^{2x} + c_1e^{x}.$

$u = \frac{5}{2}e^{2x} + c_1e^{x} + c_2$

$y = ue^{x}$

$y = \left(\frac{5}{2}e^{2x} + c_1e^{x} + c_2\right)e^{x}$

$\boxed{y = \frac{5e^{3x}}{2} + c_1e^{2x} + c_2e^{x}}$

$\because y_1 = e^{x}$

$y_2 = e^{2x}$

Varifying :-

$y_2 = e^{x}\int \frac{e^{-\int -3dx}}{e^{2x}}\,dx$

$y_2 = e^{x}\int \frac{e^{3x}}{e^{2x}}\,dx$

$y_2 = e^{x}e^{x}$

$y_2 = e^{2x}$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 5e^{3x}$

$\boxed{A = \frac{5}{2}}$

$\boxed{y = e^{x} + e^{2x} + \frac{5}{2}e^{3x}}$

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BONUS PROBLEM 7:-

a.) $40 =$ $\quad$ $P = 40$

$y = 5$ $\quad$ $P = 104$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right)$$

$$P(t) = \frac{a40}{40b + (a - 40b)e^{-at}}$$

Limiting factor $= \frac{1}{t'/4000} = t4000$

$$P(t) = \frac{40a \div b}{\frac{40b}{b}\left(\frac{a}{b} - 40\frac{b}{b}\right)e^{-at}}$$

$$P(t) = \frac{40(4000)}{40 + (4000 - 40)e^{-5a}}$$

$$104 = \frac{40(4000)}{40 + (4000 - 40)e^{-5a}}$$

$$40 + 3960e^{-5a} = \frac{40(4000)}{104}$$

$$\ln(e^{-5a}) = \ln\left[\frac{\left(\frac{20000}{13} - 40\right)}{3960}\right]$$

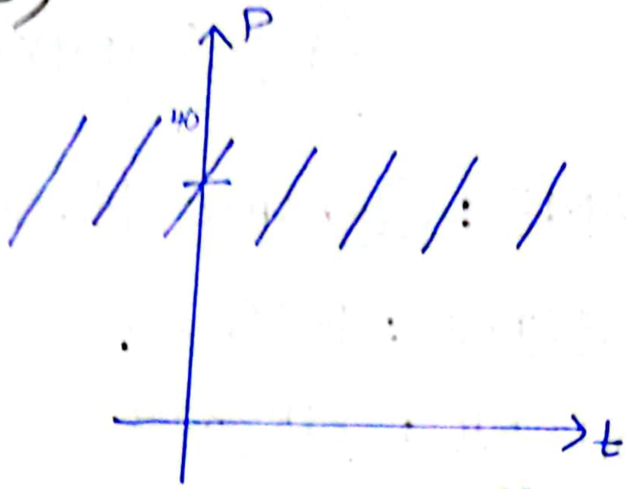
$$a = \frac{\ln\left[\frac{\left(\frac{20000}{13} - 40\right)}{3960}\right]}{\ln[e^{-5}]}$$

$$a = 0.19436$$

$$P(t) = \frac{1.6 \times 10^6}{40 + 3960e^{-0.194t}}$$

$$\boxed{P(t) = \frac{4000}{1 + 99e^{-0.194t}}}$$

b.)



$$P(t)' = \frac{7682.4e^{\frac{97x}{500}}}{\left(e^{\frac{97x}{500}} + 99\right)^2}$$

$$\boxed{P(0)' = 7.682}$$

c.)

$$P(15) = \frac{4000}{1 + 99e^{-0.194 \times 36 \times 15}}$$

$P(15) = \boxed{628.53}$ elKS

d.) As $t \to \infty$ $\quad P = \frac{4000}{1 + 99e^{-\infty}}$

$$P = \frac{4000}{1 + 99e^{-1.94t}}$$

$$P_{t \to \infty} = \boxed{4000}$$

As $t \to \infty$, $P \to 4000$