Quantum Mechanics Questions

Sadasivan MJ

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1. **Consider $\psi(x) = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$ where $|\phi_1\rangle and |\phi_2\rangle$ are the ground state and the first excited state wave functions of a particle in a deep potential well of length 'L'. Then $\langle\phi_1|\hat{P_x}|\phi_2\rangle$ is?**

$$\begin{aligned}
\langle\phi_1|\hat{P_x}|\phi_2\rangle &= \int_0^L \sqrt{\frac{2}{L}}sin(\frac{\pi x}{L}) \cdot -i\hbar\frac{\partial}{\partial x}(\sqrt{\frac{2}{L}}sin(\frac{2\pi x}{L}))dx \\
&= -\frac{2i}{L}\int_0^L sin(\frac{\pi x}{L}) \cdot cos(\frac{2\pi x}{L}) \cdot \frac{2\pi}{L}dx \\
&= -\frac{4\pi i}{L^2}\int_0^L sin(\frac{\pi x}{L}) \cdot (1 - 2sin^2(\frac{\pi x}{L}))dx \\
&= -\frac{4\pi i}{L^2}\int_0^L sin(\frac{\pi x}{L}) \cdot 2\int_0^L sin^3(\frac{\pi x}{L})dx
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^L sin^3(\frac{\pi x}{L})dx &= \int_0^L sin^2(\frac{\pi x}{L}) \cdot sin(\frac{\pi x}{L})dx \\
&= \int_0^L (1 - cos^2(\frac{\pi x}{L})) \cdot sin(\frac{\pi x}{L})dx
\end{aligned}$$

Substituting $1 - cos^2(\frac{\pi x}{L}) = u \implies du = -2sin(\frac{\pi x}{L}) \cdot \frac{\pi}{L}dx$ or $sin(\frac{\pi x}{L})dx = -\frac{L}{2\pi}du$ and our equation simplifies to

$$\begin{aligned}
-\frac{L}{2\pi}\int udu &= -\frac{L}{4\pi}u^2 \\
&= \left[-\frac{L}{2\pi}(1 - cos^2(\frac{\pi x}{L}))^2\right]_0^L
\end{aligned}$$

2. **The motion of a quantum particle in one - dimension is governed by the Hamiltonian $H = \frac{p^2}{2} + gx$, The expectation value od the e force on the particle is**

   The expectation value a quantum measurement correspond to the classical analogue of it. Here Energy = $\frac{p^2}{2} + gx$ and force = $-\frac{\partial H}{\partial x} = -g$

3. **A particle is confined in the region 0 ≤ x ≤ a and its wave function is**

$$\psi(x,t) = sin(\frac{\pi x}{a})e^{-i\omega t}$$

   The probability of finding the electron in the interval $\frac{a}{4} \le x \le \frac{3a}{4}$ is

   The given wavefunction is not normalised, We know

$$\int_{-\infty}^{\infty}\psi^*\psi = 1$$

$$\begin{aligned}
A^2\int_0^a sin^2(\frac{\pi x}{a}) &= \frac{A^2}{2}\int_0^a 1 - cos(\frac{2\pi x}{a})dx \\
&= \frac{A^2}{2}\left[x - sin(\frac{2\pi x}{a}) \cdot \frac{a}{2\pi}\right]_0^a \\
&= \frac{A}{2}a = 1
\end{aligned}$$

   or A = $\sqrt{\frac{2}{a}}$

$$
\begin{aligned}
P &= 4a^2 \int_{\frac{a}{4}}^{\frac{3a}{4}} \sqrt{\frac{2}{a}} sin(\frac{\pi x}{a}) e^{i\omega t} \cdot \sqrt{\frac{2}{a}} sin(\frac{\pi x}{a}) e^{-i\omega t} dx \\
&= \frac{2}{a} \int_{\frac{a}{4}}^{\frac{3a}{4}} sin^2(\frac{\pi x}{a}) dx \\
&= \frac{1}{a} \int_{\frac{a}{4}}^{\frac{3a}{4}} 1 - cos(\frac{2\pi x}{a}) dx \\
&= \frac{1}{a} \left[x - sin(\frac{2\pi x}{a}) \cdot \frac{a}{2\pi} \right]_{\frac{a}{4}}^{\frac{3a}{4}} \\
&= \frac{1}{a} \left[\frac{a}{2} + \frac{a}{2\pi} \right] \\
&= \frac{\pi + 2}{2\pi}
\end{aligned}
$$