Recipes for Conversion of Orbital Elements

1. Cartesian ⇒ Kepler: Given $\vec{r}$, $\vec{v}$, determine $\{a, e, i, \Omega, \omega, M\}$

The alogorithm proceeds by following the numbered equations in order. Some additional text describes where these equations come frome

Eccentricity (e) In our proof that Newton's law of gravity leads to elliptical motion we were left with a vector constant of integration C that points from the center of the earth towards perigee and has magnitude equal to μe

$$\vec{C} = \vec{v} \times \vec{h} - \mu \frac{\vec{r}}{r}$$

We will use this to define the ***eccentricity vector*** ,

$$\vec{e} = \frac{1}{\mu}(\vec{v} \times \vec{h}) - \frac{\vec{r}}{r}$$

Then ***the algorithm to calculate the eccentricity*** is as follows

(1) $\vec{h} = \vec{r} \times \vec{v}$

(2) $\vec{e} = \frac{1}{\mu}(\vec{v} \times \vec{h}) - \frac{\vec{r}}{r}$

(3) $e = \sqrt{\vec{e} \cdot \vec{e}}$

Semi-major Axis (a) Since the orbital semi-parameter p satisfies

$$p = a(1 - e^2) = h^2 / \mu$$

we can ***calculate the semi-major axis***,

(4) $a = \dfrac{\vec{h} \cdot \vec{h}}{\mu(1 - e^2)}$

Inclination (i)Since the inclination is defined by either of the two (equivalent) defintions as (a) the angle between the orbit plane and the equatorial plane of the earth and (b) the angle between the orbit normal and the z-axis of the ECI coordinate system (i.e., the north polar axis), we conclude that since the angular momentum vector $\vec{h}$ is perpendicular to the plane of the orbit , the inclination is also the angle between the z-axis and angular momentum vector , so that

$$\hat{k} \cdot \hat{h} = \cos i$$

where the "hat" symbol denotes a unit vector and $\hat{k}$ is the standard unit vector oriented along the z-axis. Therefore

(5) $i = \cos^{-1}\left(\dfrac{\hat{k} \cdot \vec{h}}{h}\right)$ where $h = \sqrt{\vec{h} \cdot \vec{h}}$

The inclination is always chosen so that $0 \le i < \pi$,

Right Ascension of Ascending Node (Ω). You should convince yourself that the following ***node vector*** points from the center of the earth to towards the ascending node in the plane of the equator:

(6) $\vec{n} = \hat{k} \times \vec{h}$

and therefore Ω is the angle between $\vec{n}$ and the x-axis. Letting $\hat{i}$ be the usual unit vector parallel to the x-axis,

(7) $\Omega = \cos^{-1}\left(\frac{\hat{i} \cdot \vec{n}}{|\vec{n}|}\right)$ where $|\vec{n}| = \sqrt{\vec{n} \cdot \vec{n}}$

(8) If $\vec{n} \cdot \hat{j} < 0$, $\Omega \to 2\pi - \Omega$ (because the ArcCos function gives an angle between 0 and π; and if the line of nodes has a negative y-component, we need to get an angle between π and 2π.

Argument of Perigee (ω). Since the argument of perigee is defined as the angle between the line of ascending nodes and the perigee, it is also the angle between the vectors $\vec{n}$ and $\vec{e}$, and therefore

(9) $\omega = \cos^{-1}\left(\frac{\vec{n} \cdot \vec{e}}{|n|e}\right)$

(10) If $\vec{e} \cdot \hat{k} < 0$, $\omega \to 2\pi - \omega$ because the argument of perigee is between the angles of 0 and π only of the eccentricity vector is in the upper half plane.

True Anomaly (θ) The true anomaly is the angle measured in the orbit plane from perigee; thus it is the angle between the vectors $\vec{r}$ and $\vec{e}$:

(11) $\theta = \cos^{-1}\left(\frac{\vec{e} \cdot \vec{r}}{er}\right)$

(12) If $\vec{r} \cdot \vec{v} < 0$, $\theta \to 2\pi - \theta$ (convince yourself that this is true; an earlier version of these notes had the less-than sign reversed! To prove it, dot the equations for r and v on the next page).

Eccentric Anomaly (E) Recall that in our derivation of Kepler's equation we showed that

$$r = a(1 - e\cos E)$$

Also, the equation of an ellipse is polar coordinates is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

Equating these two equations

$$1 - e\cos E = \frac{1 - e^2}{1 + e\cos\theta}$$

$$e\cos E = 1 - \frac{1 - e^2}{1 + e\cos\theta} = \frac{1 + e\cos\theta - 1 + e^2}{1 + e\cos\theta} = \frac{e\cos\theta + e^2}{1 + e\cos\theta}$$

$$\cos E = \frac{\cos\theta + e}{1 + e\cos\theta}$$

Therefore

(13) $E = \cos^{-1}\left(\frac{e + \cos\theta}{1 + e\cos\theta}\right)$

(14) If $\pi < \theta < 2\pi$, $E \to 2\pi - E$

Mean Anomaly (M). By Kepler's equation

(15) $M = E - e\sin E$

2. Kepler ⇒ Cartesian: Given $\{a, e, i, \Omega, \omega, M\}$ compute $\vec{r}, \vec{v}$

(1) Determine E by solving Kepler's equation $M = E - e\sin E$ for E. By Newtons' method the procedure is:

$E_0 = M$

$\varepsilon = \textit{something small like } 10^{-15}$

$i = 0$

Repeat the following:

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M}{1 - e\cos E_i}$$

$$i \to i + 1$$

Until $|E_i - E_{i-1}| < \varepsilon$

E is the final value of E_i that you have calculated

(2) Compute the unit vectors $\vec{P}$ and $\vec{Q}$ along the axes of the PQW frame. From the notes on Kepler Elements, these are defined as

$$P_x = \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega$$

$$P_y = \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega$$

$$P_z = \sin\omega\sin i$$

$$Q_x = -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega$$

$$Q_y = -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega$$

$$Q_z \cancel{= \sin i\cos\Omega}$$

Should be $\sin i\cos\omega$

$$\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$$

$$\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$$

(3) In our derivation of Kepler's equation we determined that (eq. 12 on p. 20)

$$\vec{r} = a(\cos E - e)\vec{P} + a\sqrt{1-e^2}\sin E\vec{Q}$$

$$\vec{v} = -a(\sin E)\dot{E}\vec{P} + a\sqrt{1-e^2}(\cos E)\dot{E}\vec{Q}$$

To determine $\dot{E}$ we solve the expression that we had

$$\sqrt{\frac{\mu}{a^3}} = (1 - e\cos E)\dot{E}$$

and therefore

$$\vec{v} = -ae(\sin E)\frac{\sqrt{\mu}}{a^{3/2}(1 - e\cos E)}\vec{P} + a\sqrt{1-e^2}(\cos E)\frac{\sqrt{\mu}}{a^{3/2}(1 - e\cos E)}\vec{Q}$$

$$= \frac{1}{1 - e\cos E}\sqrt{\frac{\mu}{a}}\left(-e\sin E\vec{P} + \sqrt{1-e^2}\cos E\vec{Q}\right)$$