SMO 0115

Projekt 6. (Fourier)

$$f(t) = \begin{cases} t+2 & t \in \langle 0,1) \\ -t+2 & t \in \langle 1,2) \\ 3 & t \in \langle 2,3) \end{cases} \qquad T = 3$$

Funkce $f(t)$ splňuje dirichletovy podmínky

- periodická
- po částech spojitá
- $f \in L^2((0,T))$ $\Rightarrow$ existuje Fourierova řada

1) Fourierova řada (komplexní)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{3}t}$$

$$c_n = \frac{1}{3}\int_0^3 f(t)\, e^{-\frac{2\pi n}{3}t}$$

$$= \frac{1}{3}\left(\int_0^1 (t+2)\, e^{-i\frac{2\pi n}{3}t}\,dt + \int_1^2 (2-t)\, e^{-i\frac{2\pi n}{3}t}\,dt + \int_2^3 3\, e^{-i\frac{2\pi n}{3}t}\,dt\right)$$

(reálná)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{3}t\right) + b_n \sin\left(\frac{2\pi n}{3}t\right)\right)$$

$$a_0 = \frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}\left(\int_0^1 (t+2)\,dt + \int_1^2 (2-t)\,dt + \int_2^3 3\,dt\right)$$

$$= \frac{2}{3}\left(\underbrace{\left[\frac{1}{2}t^2 + 2t\right]_0^1}_{\frac{1}{2}+2-0} + \underbrace{\left[2t - \frac{1}{2}t^2\right]_1^2}_{(4-2)-(2-\frac{1}{2})} + \underbrace{\left[3t\right]_2^3}_{9-6}\right) = \frac{2}{3}\left(\frac{1}{2} + 2 + 4 - 2 - 2 + \frac{1}{2} + 3\right) = 4$$

$$a_n = \frac{2}{3}\int_0^3 f(t)\cos\left(\frac{2\pi n}{3}t\right)dt$$

$$= \frac{2}{3}\left(\int_0^1 (t+2)\cos\left(\frac{2\pi n}{3}t\right)dt + \int_1^2 (2-t)\cos\left(\frac{2\pi n}{3}t\right)dt + \int_2^3 3\cos\left(\frac{2\pi n}{3}t\right)dt\right)$$

$$= \frac{5\pi n \sin\left(\frac{2\pi n}{3}\right) + \frac{3}{2}\cos\left(\frac{2\pi n}{3}\right) - \frac{3}{2}}{\pi^2 n^2}$$

$$b_n = \frac{2}{3}\int_0^3 f(t)\sin\left(\frac{2\pi n}{3}t\right)dt$$

$$= \frac{2}{3}\left(\int_0^1 (t+2)\sin\left(\frac{2\pi n}{3}t\right)dt + \int_1^2 (2-t)\sin\left(\frac{2\pi n}{3}t\right)dt\right)$$

$$= \frac{1}{n^2\pi^2}\left(-12\pi n\cos\left(\frac{2\pi n}{3}\right) - 8\pi n + 6\sin\left(\frac{2\pi n}{3}\right)\right)\sin^2\left(\frac{\pi n}{3}\right)$$

jednostranné spektrum: $A_n = \sqrt{a_n^2 + b_n^2}$ $A_0 = \left|\frac{a_0}{2}\right|$

fázové spektrum: $\varphi_n = \operatorname{arctg}\left(\frac{b_n}{a_n}\right)$

n	0	1	2	3	4	5
a_n	4	1,15	-0,74	0	0,33	-0,28
b_n	X	-0,08	-0,33	0	-0,09	-0,11
A_n	2	1,15	0,81	0	0,34	0,30
φ_n	X	-0,07	-2,71	X	-0,27	-2,76

2. sinová Fourierova transformace

$$f_L(t) = \begin{cases} -f(-t) & t \in \langle -3;\ 0) \\ f(t) & t \in (0;\ 3\rangle \end{cases} \qquad T = 6$$

$$f_L(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{3} t\right) \qquad a_n = 0$$

$$b_n = \frac{2}{6}\int_{-3}^{3} f_L(t) \sin\left(\frac{2\pi n}{6} t\right) dt$$

$$= \frac{2}{3}\int_{0}^{3} f(t) \sin\left(\frac{\pi n}{3} t\right) dt$$

$$= \frac{2}{3}\left(\int_0^1 (2+t)\sin\left(\frac{\pi n}{3} t\right) dt + \int_1^2 (2-t)\sin\left(\frac{\pi n}{3} t\right) dt + \int_2^3 3\sin\left(\frac{\pi n}{3} t\right) dt\right)$$

$$= \frac{1}{\pi^2 n^2}\left(-2\pi n\left(3(-1)^n + 2\cos\left(\frac{\pi n}{3}\right) - 3\cos\left(\frac{2\pi n}{3}\right) - 2\right) + 12\sin\left(\frac{\pi n}{3}\right) - 6\sin\left(\frac{2\pi n}{3}\right)\right)$$

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n|$$

$$\varphi_n = \operatorname{arctg}\left(\frac{b_n}{a_n}\right)$$

n	0	1	2	3	4	5
a_n	0	0	0	0	0	0
b_n	X	2,11	-0,08	2,12	-0,33	0,29
φ_n	X	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$

3. cosinová Fourierova transformace

$$f_s(t) = \begin{cases} f(-t) & t \in \langle -3, 0) \\ f(t) & t \in (0, 3) \end{cases} \qquad T = 6$$

$$f_s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{6} t\right); \quad b_n = 0$$

$$a_0 = \frac{2}{6} \int_{-3}^{3} f_s(t)\, dt = \frac{2}{3} \int_0^3 f_s(t)\, dt = 4$$

$$a_n = \frac{2}{6} \int_{-3}^{3} f_s(t) \cos\left(\frac{2\pi n}{6} t\right) dt = \frac{2}{3} \int_0^3 f(t) \cos\left(\frac{\pi n}{3} t\right) dt$$

$$= \frac{2}{3}\left(\int_0^1 (2+t) \cos\left(\frac{\pi n}{3} t\right) dt + \int_1^2 (2-t) \cos\left(\frac{\pi n}{3} t\right) dt + \int_2^3 3 \cos\left(\frac{\pi n}{3} t\right) dt \right)$$

$$= \frac{1}{\pi^2 n^2}\left(2\pi n \left(2 \sin\left(\frac{\pi n}{3}\right) - 3 \sin\left(\frac{2\pi n}{3}\right)\right) + 12 \cos\left(\frac{\pi n}{3}\right) - 6 \cos\left(\frac{2\pi n}{3}\right) - 6 \right)$$

$$A_n = \sqrt{a_n^2 + b_n^2} = |a_n|$$

$$\varphi_n = \operatorname{arctg}\left(\frac{b_n}{a_n}\right) = 0$$

n	0	1	2	3	4	5
a_n	4	-0,24	1,15	-0,27	-0,7	0,12
b_n	X	0	0	0	0	0
φ_n	X	0	0	0	0	0

4. grafy jsou v přiloženém jupyter notebooku

SM00145

Projekt 6.1 (Laplace)

a) $y'' + y' - 6y = x^2 - 16$

$y(0) = 0 , \quad y'(0) = 1$

analýza $f(x) = (x^2 - 16)\,u(x)$ (u(x) Heavisideova funkce)

- $f(x) \in C(\langle 0 ; \infty))$
- $f(x)$; má omezený růst

$|x^2 - 16| \leq M e^{\alpha x}$ $\quad M = 16 \quad \alpha = 1$

pro $x < 0$

$\Downarrow$

existuje L-obraz $f(x)$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$

$\mathcal{L}\{x^2 - 16\} = \frac{2}{s^3} - \frac{16}{s}$

$$s^2 Y(s) - s y(0) - y'(0) + sY(s) - y(0) - 6Y(s) = \frac{2}{s^3} - \frac{16}{s}$$

$$Y(s)(s^2 + s - 6) = \frac{2}{s^3} - \frac{16}{s} + 1$$

$$Y(s) = \frac{s^3 - 16s^2 + 2}{s^3(s+3)(s-2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{E}{s+3} + \frac{D}{s-2}$$

Rozklad na parciální zlomky

$$A(s^2 + s - 6) + B(s^3 + s^2 - 6s) + C(s^4 + s^3 - 6s) + D(s^4 + 3s^3) + E(s^4 - 2s^3) = s^3 - 16s^2 + 2$$

$C + D + E = 0$ $\qquad E = -D - C$

$B + C + 3D - 2E = 1$ $\qquad C = \frac{1}{6}(A + B + 16) = \frac{281}{108}$

$A + B - 6C = -16$

$A - 6B = 0$ $\qquad B = \frac{1}{6}A = -\frac{1}{18}$

$-6A = 2$ $\qquad A = -\frac{1}{3}$

$$\left.\begin{array}{l} B + C + 3D + 2D + 2C = 1 \\ B + 3C + 5D = 1 \end{array}\right\} \Rightarrow D = \frac{1}{5}(1 - B - 3C)$$

$$= \frac{1}{5}\left(1 + \frac{1}{18} - 3\,\frac{281}{108}\right) = \frac{-1}{5}\,\frac{729}{108} = \frac{27}{20}$$

$$E = -D - C = +\frac{27}{20} - \frac{281}{108} = -\frac{169}{135}$$

Inverzní Transformace

$$Y(s) = -\frac{1}{3s^3} - \frac{1}{18s^2} + \frac{281}{108\,s} - \frac{169}{135(s-2)} - \frac{27}{20(s+3)}$$

$$\mathcal{L}^{-1}(Y) = \underline{-\frac{1}{6}x^2 - \frac{1}{18}x + \frac{281}{108} - \frac{169}{135}e^{-3x} - \frac{27}{20}e^{2x}} = y(x)$$

$$y'(x) = -\frac{1}{3}x - \frac{1}{18} + \frac{507}{135}e^{-3x} - \frac{54}{20}e^{2x}$$

$$y''(x) = -\frac{1}{18} - \frac{1521}{135}e^{-3x} - \frac{108}{20}e^{2x}$$

$$-6y(x) = x^2 + \frac{1}{3}x - \frac{281}{18} + \frac{1014}{135}e^{-3x} + \frac{162}{20}e^{2x}$$

$$y(0) = \frac{281}{108} - \frac{169}{135} - \frac{27}{20} = \underline{0}$$

$$y'(0) = -\frac{1}{18} + \frac{507}{135} - \frac{54}{20} = \underline{1}$$

$$x^2 + \underbrace{\left(\frac{1}{3} - \frac{1}{3}\right)}_{=0}x + \underbrace{\left(\frac{-1}{18} + \frac{-6}{18} + \frac{-281}{18}\right)}_{=-16} + \underbrace{\left(\frac{+507}{135} - \frac{1521}{135} + \frac{1014}{135}\right)}_{=0}e^{-3x} + \underbrace{\left(\frac{162}{20} - \frac{54}{20} - \frac{108}{20}\right)}_{=0}e^{2x} = \underline{x^2 - 16}$$

b) $y'' + y' - 2y = 18x - 2$

$y(1) = 2 \quad y'(1) = 2$

substituce $x - 1 = t \Rightarrow x = t + 1$

$v'' + v' - 2v = 18t + 16$

$v(0) = 2 \quad v'(1) = 2$

analýza $f(x) = (18x - 2)u(t)$ (Heavisideova funkce)

- $f(x) \in C((0, \infty))$
- $f(x)$ má omezený růst

$|18x - 2| \leq Me^{\alpha x} \quad M = -2 \quad \alpha = 1$ pro $x < 0$

$\Downarrow$

pak existuje L-obraz $f(x)$

L-obraz

$\mathcal{L}\{v'\} = sV(s) - v(0)$

$\mathcal{L}\{v''\} = s^2V(s) - sv(0) - v'(0)$

$\mathcal{L}\{18t + 16\} = \frac{18}{s^2} + \frac{20}{s}$

$s^2V(s) - \cancel{s}v(0) - v'(0) + sV(s) - v(0) - 2V(s) = \frac{18}{s^2} + \frac{16}{s}$

$V(s)(s^2 + s - 2) = \frac{18}{s^2} + \frac{16}{s} + 2s + 4$

$V(s) = \frac{18 + 16s + 4s^2 + 2s^3}{s^2(s-1)(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-1} + \frac{D}{s+2}$

Rozklad na parciální zlomky

$A(s^2 + s - 2) + B(s^3 + s^2 - 2s) + C(s^3 + 2s^2) + D(s^3 - s^2) = 2s^3 + 4s^2 + 16s + 18$

$B + C + D = 2$

$A + B + 2C - D = 4$

$A - 2B = 16$

$-2A = 18$

$A = -9$

$B = -\frac{25}{2}$

$D = 2 - C - B$

$= 2 - \frac{40}{3} + \frac{25}{2} = \frac{12 - 80 + 75}{6} = \frac{7}{6}$

$A + B + 2C - 2 + C + B = 4$

$A + 2B + 3C = 6 \Rightarrow C = \frac{1}{3}(6 - 2B - A) = \frac{1}{3}(6 + 25 + 9) = \frac{40}{3}$

$V(s) = \frac{-9}{s^2} + \frac{-25}{2s} + \frac{40}{3(s-1)} + \frac{7}{6(s+2)}$

Zpětná transformace

$v(t) = \mathcal{L}^{-1}\{V(s)\} = -9t - \frac{25}{2} + \frac{40}{3}e^{t} + \frac{7}{6}e^{-2t}$

Zpětná substituce

$y(x) = -9x - \frac{7}{2} + \frac{40}{3}e^{x-1} + \frac{7}{6}e^{-2(x-1)}$

Zkouška

$y'(x) = -9 + \frac{40}{3}e^{x-1} - \frac{14}{6}e^{-2(x-1)}$

$y''(x) = \frac{40}{3}e^{x-1} + \frac{28}{6}e^{-2(x-1)}$

$-2y(x) = 18x + 7 - \frac{80}{3}e^{x-1} - \frac{14}{6}e^{-2(x-1)}$

$y(1) = -9 - \frac{7}{2} + \frac{47}{3} + \frac{7}{6} = \underline{2}$

$y'(1) = -9 + \frac{40}{3} - \frac{14}{6} = \underline{2}$

$(-9 + 7) + (18x) + \underbrace{\left(\frac{40}{3} + \frac{40}{3} - \frac{80}{3}\right)}_{=0}e^{x-1} + \underbrace{\left(-\frac{14}{6} + \frac{28}{6} - \frac{14}{6}\right)}_{=0}e^{-2(x-1)} = \underline{18x - 2}$

c)

$x'' + x = f(t)$ $\quad f(t) = \begin{cases} 1 & t \in \langle 0,1) \\ -1 & t \in \langle 1,2) \\ 0 & t \in \langle 2,\infty) \end{cases}$ $\Leftrightarrow f(t) = u(t) - 2u(t-1) + u(t-2)$

$x(0) = x'(0) = 0$ $\quad$ $u(t)$ is Heavisideova funkce

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

$$\mathcal{L}\{f\} = \mathcal{L}\{u\} - 2\mathcal{L}\{u(t-1)\} + \mathcal{L}\{u(t-2)\}$$

$$\mathcal{L}\{u(t-1)\} = \int_0^\infty u(t-1)\, e^{-st}\,dt = \int_c^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_c^\infty = \frac{e^{-cs}}{s}$$

$$= \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

Analýza:
- $f(t) \in C(\langle 0,\infty)$
- pro $t<0$ definujeme $f(t)=0$
- $f(t)$ má omezený růst:
  $1 \leq Me^{\alpha t}$ $\quad M=0\ \alpha=0$ pro $t<0$

$\Rightarrow$ existuje L-obraz f

$$s^2X(s) + X(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}$$

$$X(s) = \frac{1-2e^{-s}+e^{-2s}}{s(s^2+1)}$$

Rozložíme na parciální zlomky

začneme s $\frac{1}{s(s^2+1)}$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$As^2 + A + Bs^2 + Cs$$

$$\left.\begin{aligned} A+B &= 0 \\ C &= 0 \end{aligned}\right\} \Rightarrow A = 1 \wedge B = -1$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{1}{s^2+1} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 - \cos t$$

Nyní využijeme vlastnosti posunutí pro zjištění dalších členů.

$$\mathcal{L}^{-1}\left\{-2e^{-s}\frac{1}{s(s^2+1)}\right\} = -2u(t-1)(1-\cos(t-1))$$

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s(s^2+1)}\right\} = u(t-2)(1-\cos(t-2))$$

a tedy

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1-\cos t) - 2u(t-1)(1-\cos(t-1)) + u(t-2)(1-\cos(t-2))$$

zkouška

$$x(t) = \begin{cases} 1 - \cos t & t \in \langle 0\ 1) \\ 1 - \cos t - 2 + 2\cos(t-1) & t \in \langle 1\ 2) \\ 1 - \cos t - 2 + 2\cos(t-1) + 1 - \cos(t-2) & t \in \langle 2\ \infty) \end{cases}$$

$$x'(t) = \begin{cases} -\sin t & t \in \langle 0, 1) \\ -\sin t + 2\sin(t-1) & t \in \langle 1, 2) \\ -\sin t + 2\sin(t-1) - \sin(t-2) & t \in \langle 2, \infty) \end{cases}$$

$$x''(t) = \begin{cases} \cos t & t \in \langle 0, 1) \\ \cos t - 2\cos(t-1) & t \in \langle 1, 2) \\ \cos t - 2\cos(t-1) + \cos(t-2) & t \in \langle 2, \infty) \end{cases}$$

$$x'' + x = \begin{cases} \cos t + 1 - \cos t = \underline{1} & t \in \langle 0, 1) \\ \cos t - 2\cos(t-1) - 1 - \cos t + 2\cos(t-1) = \underline{-1} & t \in \langle 1, 2) \\ \cos t - 2\cos(t-1) + \cos(t-2) - \cos t + 2\cos(t-1) - \cos(t-2) = \underline{0} & t \in \langle 2, \infty) \end{cases}$$

$$x(0) = 1 - \cos 0 = \underline{0}$$

$$x'(0) = -\sin 0 = \underline{0}$$