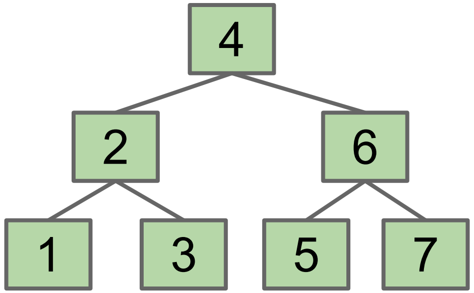
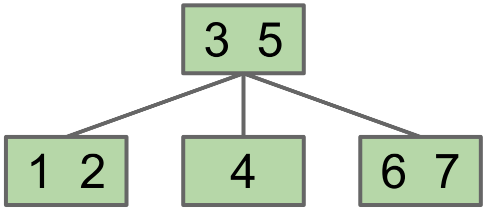




CS61B, 2020

Lecture 17: B-Trees

- BST Performance
- Big O ≠ Worst Case
- 2-3-4 and 2-3 Tree Operations (a.k.a. B-Trees)
- B-Tree Bushiness Invariants
- B-Tree Performance

BST Tree Height

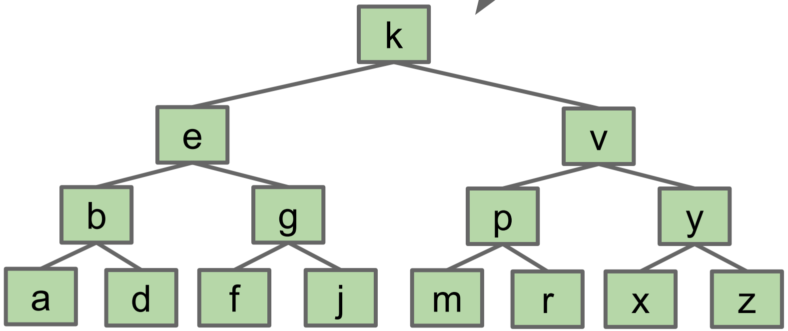
BST Tree Height

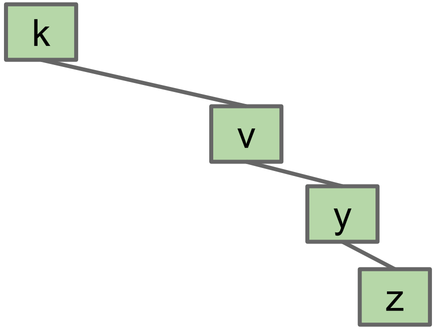
Let's start today by carefully discussing the height of binary search trees.

Trees range from best-case "bushy" to worst-case "spindly".

- Difference is dramatic!

H=3

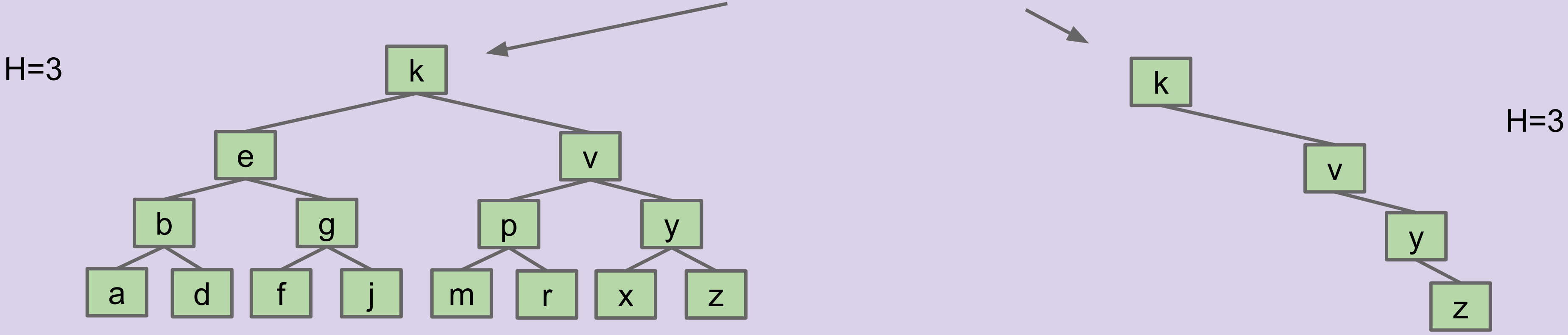




H=3

Tree Height: http://yellkey.com/?

Height varies dramatically between “bushy” and “spindly” trees.



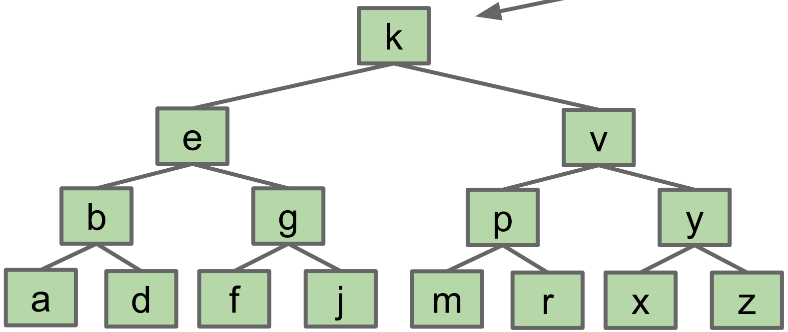
Let H(N) be the height of a tree with N nodes. Give H(N) in Big-Theta notation for “bushy” and “spindly” trees, respectively:

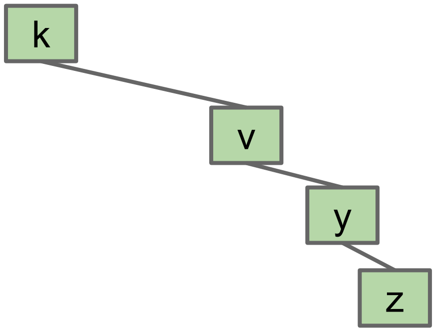
A. $\Theta(\log(N))$, $\Theta(\log(N))$
B. $\Theta(\log(N))$, $\Theta(N)$
C. $\Theta(N)$, $\Theta(\log(N))$
D. $\Theta(N)$, $\Theta(N)$

Tree Height

Height varies dramatically between “bushy” and “spindly” trees.

H=3





H=3

$H = \Theta(\log(N))$

$H = \Theta(N)$

Performance of operations on spindly trees can be just as bad as a linked list!

- Example: contains(“z”) would take linear time.

Statements about Tree Height: http://yellkey.com/?

Which of these statements are true?

A. Worst case BST height is $\Theta(N)$.
B. BST height is $O(N)$.
C. BST height is $O(N^2)$.

Statements about Tree Height

Which of these statements are true?

A. **Worst case BST height is Θ(N).**
B. **BST height is O(N).**
C. **BST height is O(N^2).**

All are **true**!

- A worst case (spindly tree) has a height that grows exactly linearly - Θ(N).
- All BSTs have a height that grows linearly or better - O(N).
- All BSTs have a height that grows quadratically or better - O(N^2).

Statements about Tree Height: http://yellkey.com/?

Which of these statements is more informative?

A. Worst case BST height is $\Theta(N)$.
B. BST height is $O(N)$.
C. They are equally informative.

Statements about Tree Height

Which of these statements is more informative?

A. Worst case BST height is Θ(N).
B. BST height is O(N).
C. They are equally informative.

Saying that the worst case has order of growth N is more informative than saying the height is O(N).

Let's see an analogy.

Question: http://yellkey.com/?

Which statement gives you more information about a hotel?

A. The most expensive room in the hotel is $639 per night.
B. Every room in the hotel is less than or equal to $639 per night.

Question: http://yellkey.com/?

Which statement gives you more information about a hotel?

A. The most expensive room in the hotel is $639 per night.

B. Every room in the hotel is less than or equal to $639 per night.

Most expensive room: $639/nt

THE RITZ-CARLTON
LAKE TAHOE

All rooms <= $639/nt

THE RITZ-CARLTON
LAKE TAHOE

(A nice place to stay!)

BST Height

BST height is all four of these:

- $O(N)$.
- $\Theta(\log N)$ in the best case ("bushy").
- $\Theta(N)$ in the worst case ("spindly").
- $O(N^2)$.

The middle two statements are more informative.

- Big O is NOT mathematically the same thing as "worst case".
  - e.g. BST heights are $O(N^2)$, but are not quadratic in the worst case.
  - … but Big O often used as shorthand for "worst case".

The Usefulness of Big O

Big O is still a useful idea:

- Allows us to make simple blanket statements, e.g. can just say "binary search is O(log N)" instead of "binary search is Θ(log N) in the worst case".
- Sometimes don't know the exact runtime, so use O to give an upper bound.
  - Example: Runtime for finding shortest route that goes to all world cities is $O(2^N)^*$. There might be a faster way, but nobody knows one yet.
- Easier to write proofs for Big O than Big Theta, e.g. finding runtime of mergesort, you can round up the number of items to the next power of 2 (see A level study guide problems for Asymptotics2 lecture). A little beyond the scope of our course.

*: Under certain assumptions and constraints not listed.

BST Height

BST height is both of these:

- Θ(log N) in the best case (“bushy”).
- Θ(N) in the worst case (“spindly”).

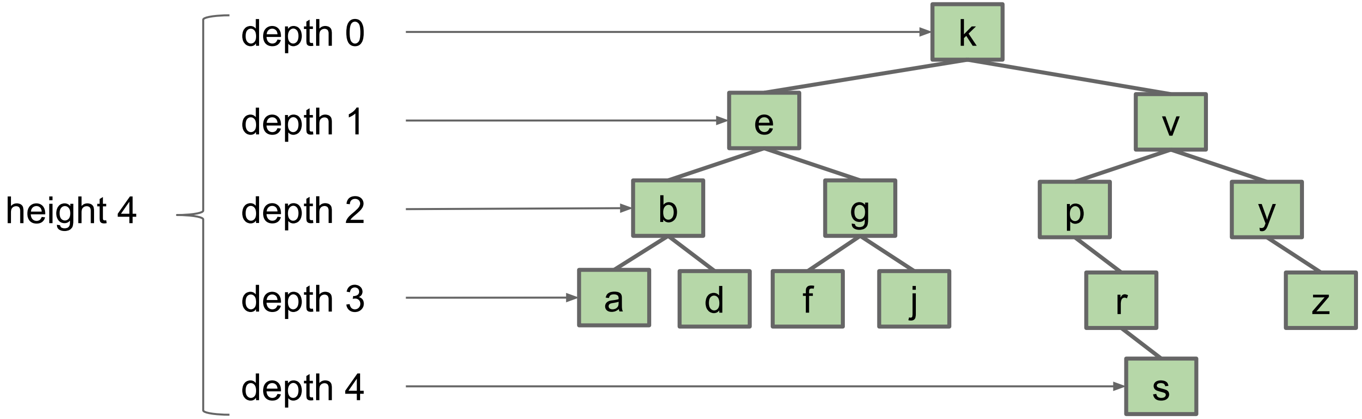
Let’s now turn to understanding the performance of BST operations.

Height, Depth, and Performance

Height and Depth

Height and average depth are important properties of BSTs.

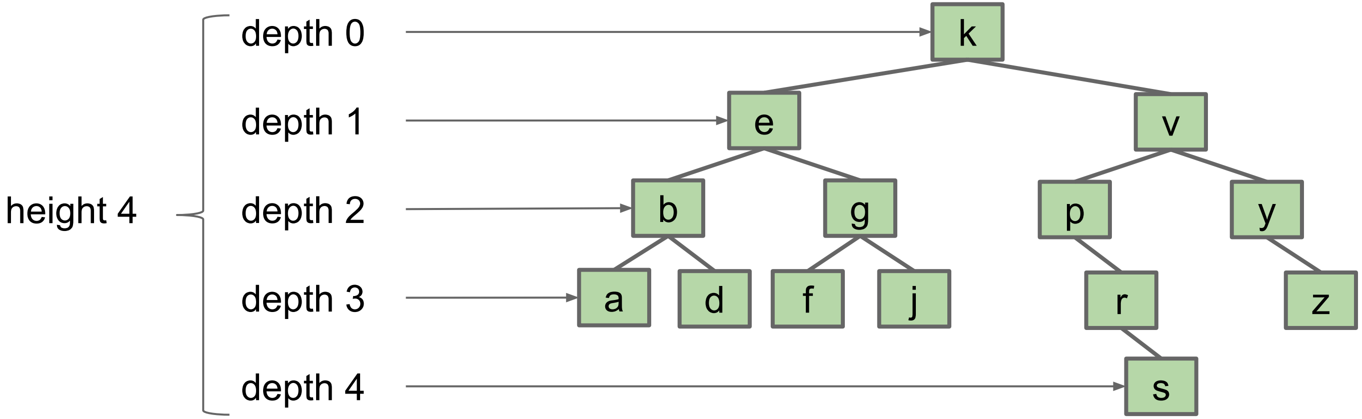
- The **“depth” of a node** is how far it is from the root, e.g. depth(g) = 2.
- The **“height” of a tree** is the depth of its deepest leaf, e.g. height(T) = 4.
- The **“average depth”** of a tree is the average depth of a tree’s nodes.
  - (**0**x1 + **1**x2 + **2**x4 + **3**x6 + **4**x1)/(1+2+4+6+1) = 2.35



Height, Depth and Runtime

Height and average depth determine runtimes for BST operations.

- The **“height” of a tree** determines the worst case runtime to find a node.
  - Example: Worst case is contains(s), requires 5 comparisons (height + 1).
- The **“average depth”** determines the average case runtime to find a node.
  - Example: Average case is 3.35 comparisons (average depth + 1).

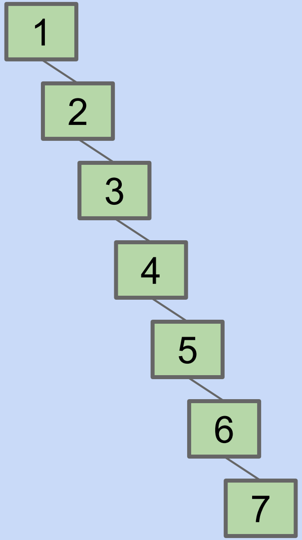


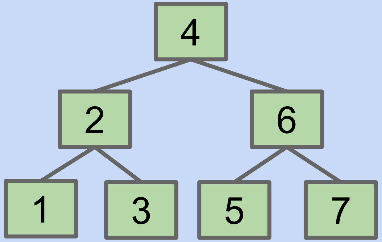
BSTs in Practice

Suppose we want to build a BST out of the numbers 1, 2, 3, 4, 5, 6, 7.

Give an example of a sequence of add operations that results in:

- A spindly tree.
- A bushy tree.



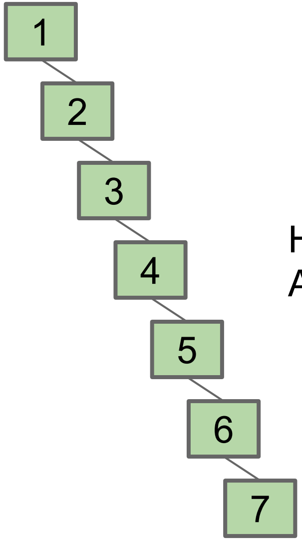


datastructur.es

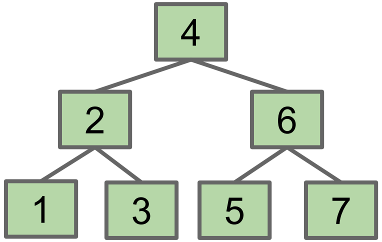
BSTs in Practice

Give an example of a sequence of add operations that results in:

- A spindly tree.
  - add(1), add(2), add(3), add(4), add(5), add(6), add(7)
- A bushy tree.
  - add(4), add(2), add(1), add(3), add(6), add(5), add(7)



Height: 6
Average Depth: 3



Height: 2
Average Depth: 1.43

Important Question: What about Real World BSTs?

BSTs have:

- Worst case $\Theta(N)$ height.
- Best case $\Theta(\log N)$ height.

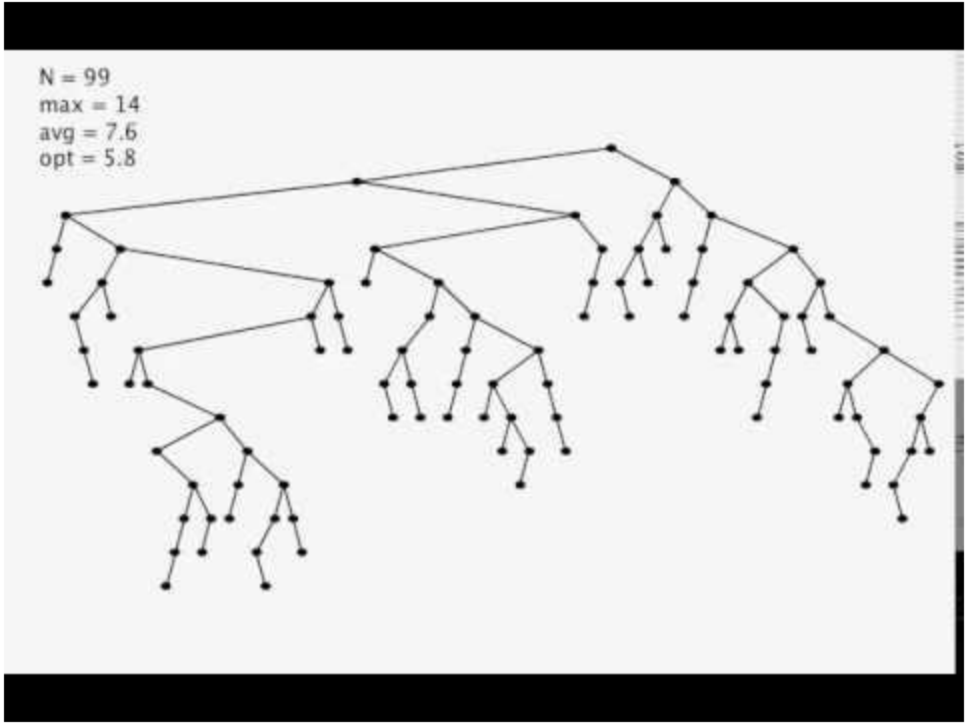
… but what about trees that you'd build during real world applications?

One way to approximate this is to consider randomized BSTs.

Simulation: Trees Built from Random Inserts

max: height + 1

avg: average depth + 1



Nice Property. Random trees have Θ(log N) average depth and height.

- In other words: Random trees are bushy, not spindly.

Video courtesy of Kevin Wayne (Princeton University)

Randomized Trees: Mathematical Analysis

Average Depth. If N distinct keys are inserted into a BST, the expected average depth is ~ 2 ln N = Θ(log N).

- Thus, average runtime for contains operation is Θ(log N) on a tree built with random inserts.
- Will discuss this proof briefly closer to the end of this course.

Tree Height. If N distinct keys are inserted in random order, expected tree height is ~ 4.311 ln N (see Reed, 2003).

- Thus, worst case runtime for contains operation is Θ(log N) on a tree built with random inserts.
- Proof is well beyond the scope of the course (and is 27 pages long!).

Note: ~ is the same thing as Big Theta, but you don't drop the multiplicative constants.

Important Question: What about Real World BSTs?

BSTs have:

- Worst case Θ(N) height.
- Best case Θ(log N) height.
- Θ(log N) height if constructed via random inserts.

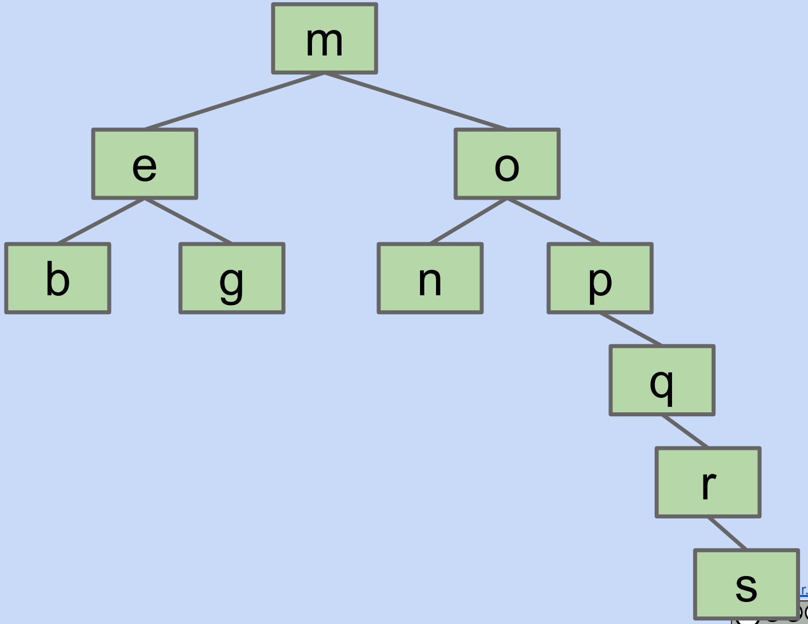
In real world applications we expect both insertion and deletion.

- See extra slides for more on simulations of trees including deletion.
- Can show that random trees including deletion are still Θ(log N) height.

Good News and Bad News

Good news: BSTs have great performance if we insert items randomly. Performance is Θ(log N) per operation.

Bad News: We can't always insert our items in a random order. Why?

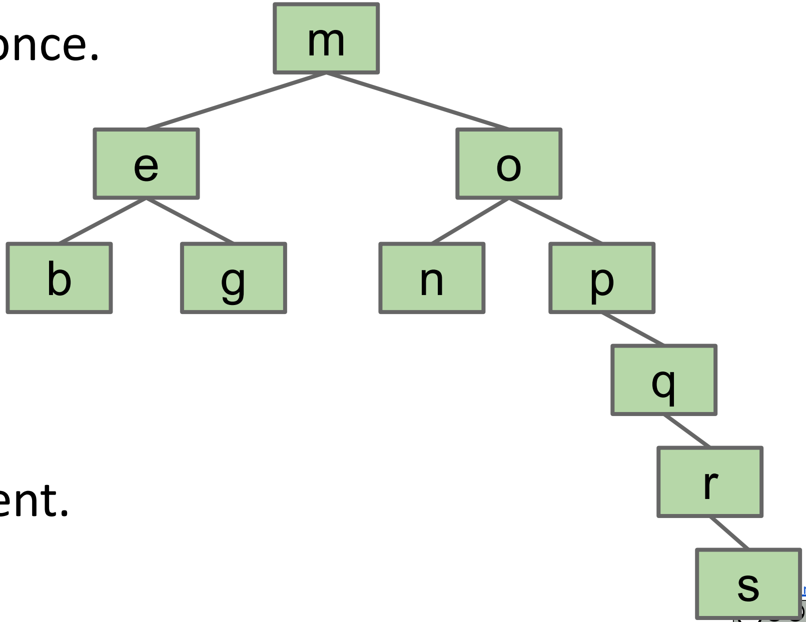


Good News and Bad News

Good news: BSTs have great performance if we insert items randomly. Performance is Θ(log N) per operation.

Bad News: We can't always insert our items in a random order. Why?

- Data comes in over time, don't have all at once.
  - Example: Storing dates of events.
    - add("01-Jan-2019, 10:31:00")
    - add("01-Jan-2019, 18:51:00")
    - add("02-Jan-2019, 00:05:00")
    - add("02-Jan-2019, 23:10:00")



In this lecture, we'll do something totally different.

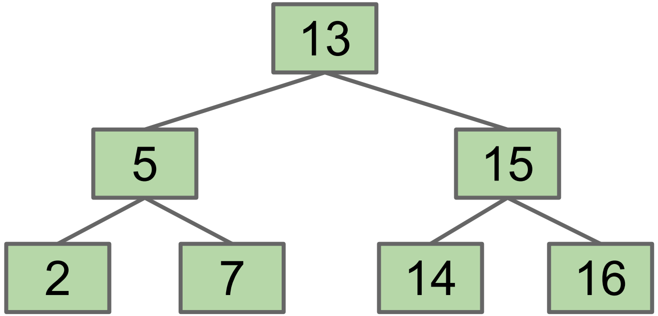
B-trees / 2-3 trees / 2-3-4 trees

Avoiding Imbalance

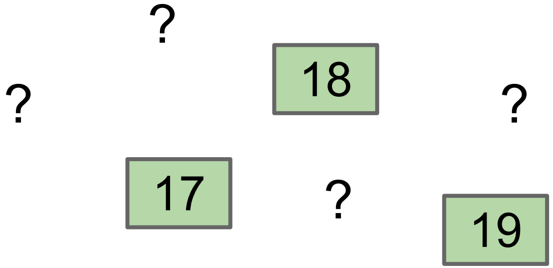
The problem is adding new leaves at the bottom.

Crazy idea: Never add new leaves at the bottom.

- Tree can never get imbalanced.



Q: What do we do with incoming keys?

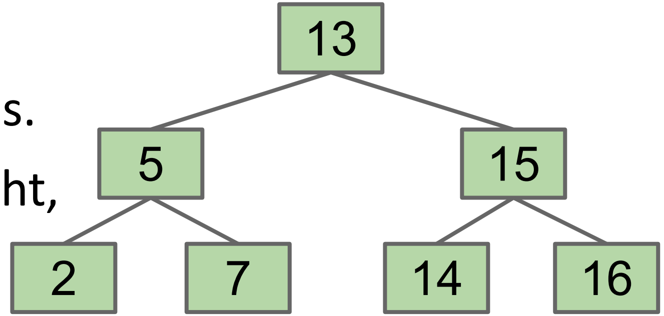


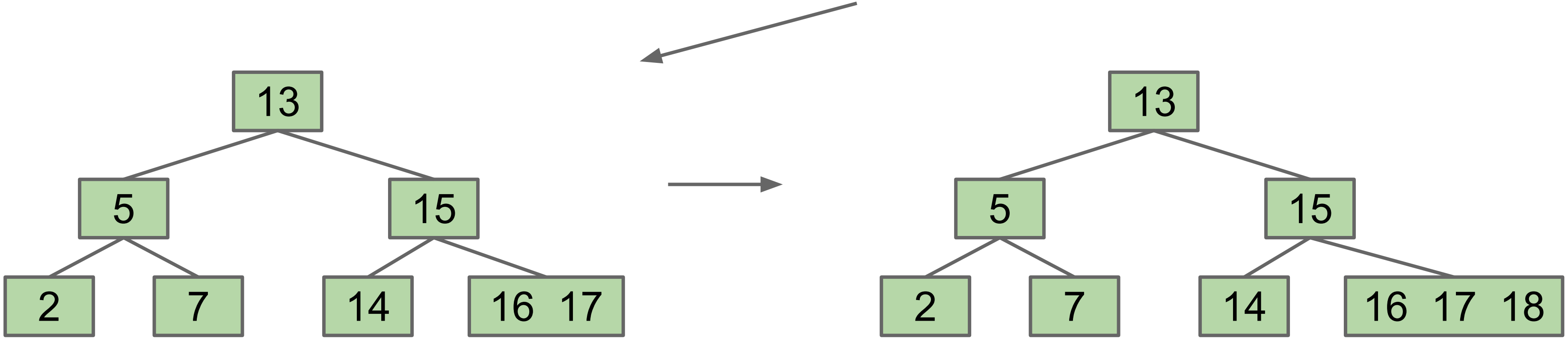
Avoiding Imbalance through Overstuffing

The problem is adding new leaves at the bottom.

Avoid new leaves by "overstuffing" the leaf nodes.

- "Overstuffed tree" always has balanced height, because leaf depths never change.
  - Height is just max(depth).

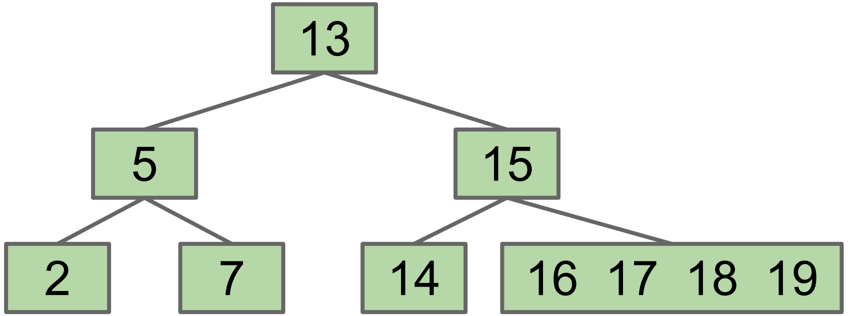




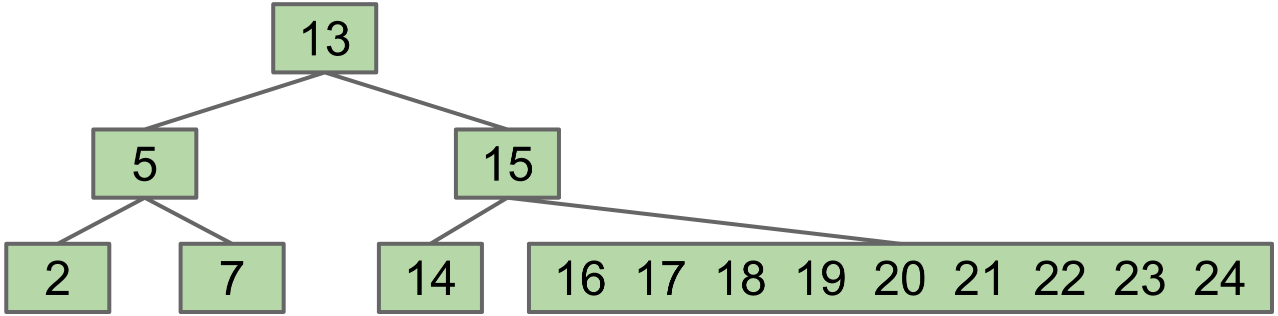
Avoiding Imbalance through Overstuffing

Overstuffed trees are a logically consistent but very weird data structure.

- contains(18):
  - Is 18 > 13? Yes, go right.
  - Is 18 > 15? Yes, go right.
  - Is 16 = 18? No.
  - Is 17 = 18? No.
  - Is 18 = 18? Yes! Found it.



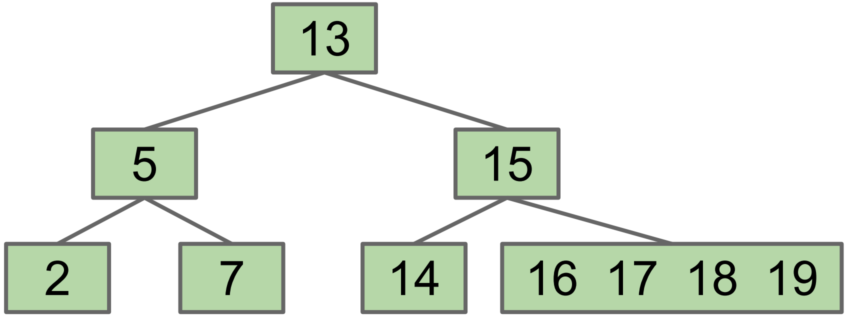
Q: What is the problem with this idea?



Revising Our Overstuffed Tree Approach

Height is balanced, but we have a new problem:

- Leaf nodes can get too juicy.

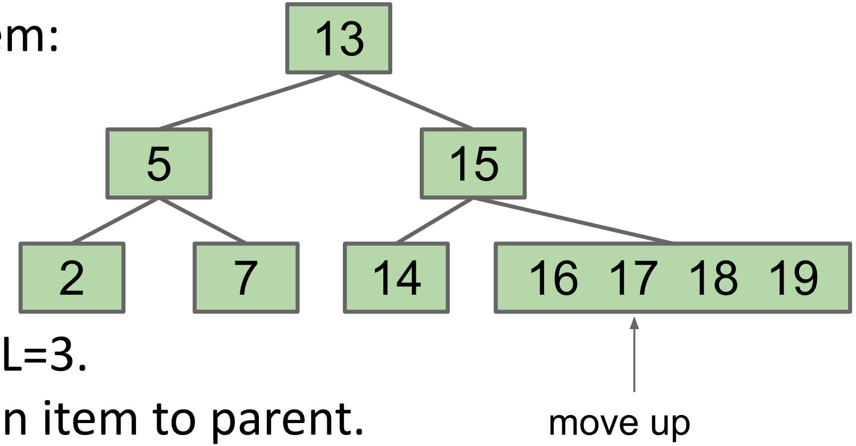


Solution?

Revising Our Overstuffed Tree Approach: Moving Items Up

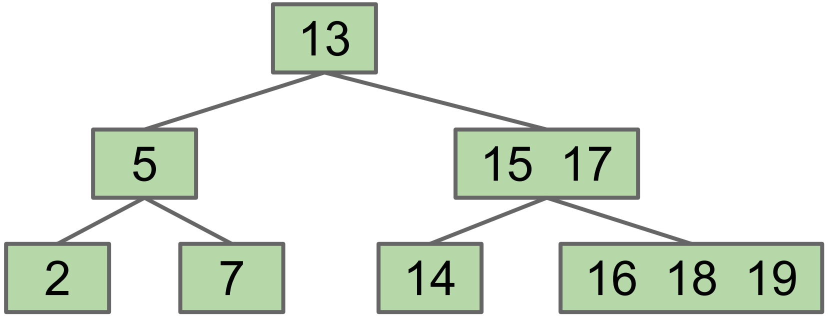
Height is balanced, but we have a new problem:

- Leaf nodes can get too juicy.



Solution?

- Set a limit L on the number of items, say L=3.
- If any node has more than L items, give an item to parent.
    - Which one? Let's say (arbitrarily) the left-middle.



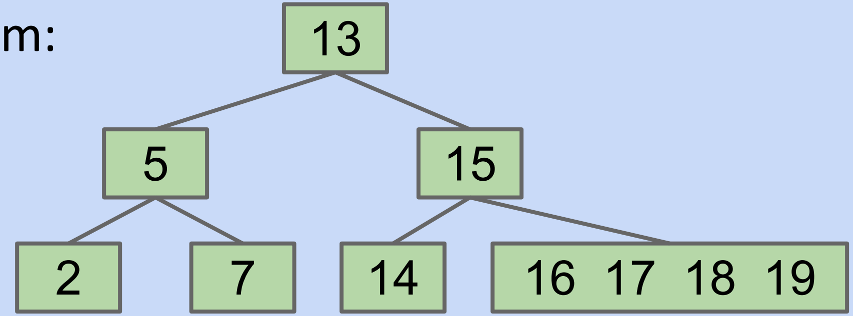
Q: What's the problem now?

- 16 is to the right of 17.

Revising Our Overstuffed Tree Approach

Height is balanced, but we have a new problem:

- Leaf nodes can get too juicy.



Solution?

- Set a limit L on the number of items, say L=3.
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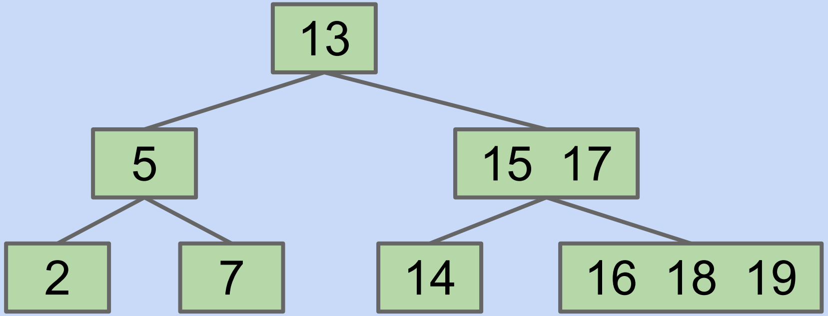


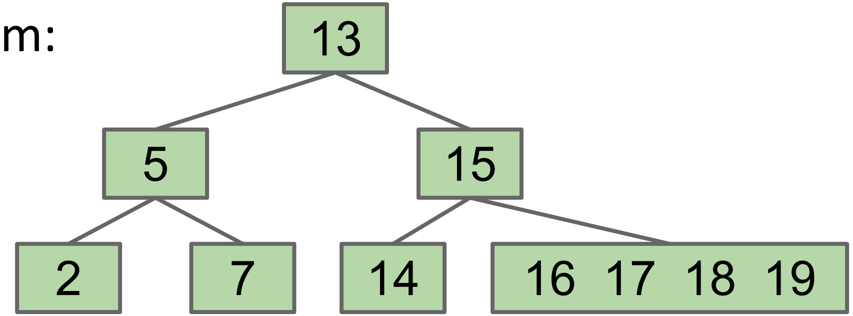
Challenge for you:

- How can we tweak this idea to make it work?

Revising Our Overstuffed Tree Approach: Node Splitting

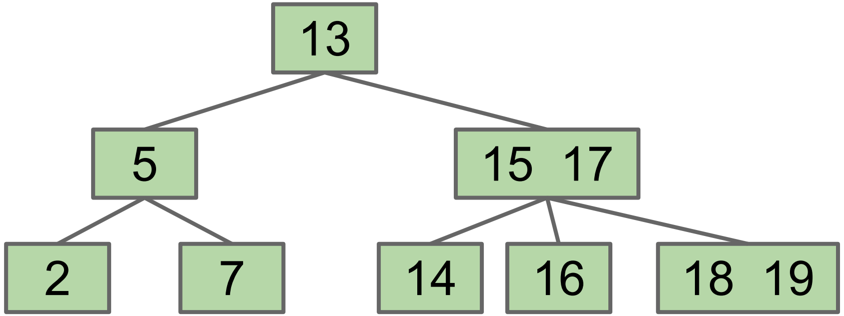
Height is balanced, but we have a new problem:

- Leaf nodes can get too juicy.



Solution?

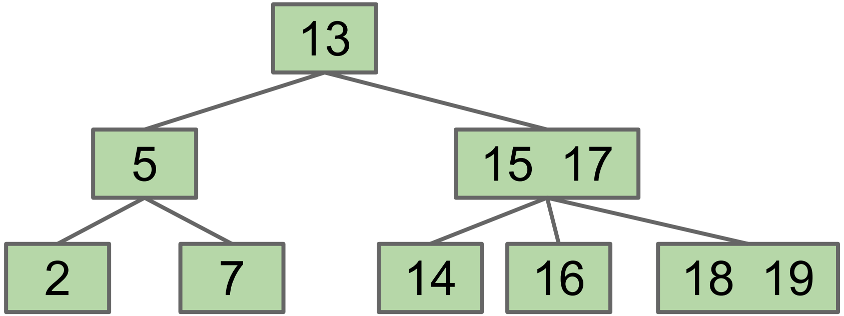
- Set a limit L on the number of items, say L=3.
- If any node has more than L items, give an item to parent.
  - Pulling item out of full node splits it into left and right.
  - Parent node now has three children!



Revising Our Overstuffed Tree Approach: Node Splitting

This is a logically consistent and not so weird data structure.

- contains(18):
  - 18 > 13, so go right
  - 18 > 15, so compare vs. 17
  - 18 > 17, so go right



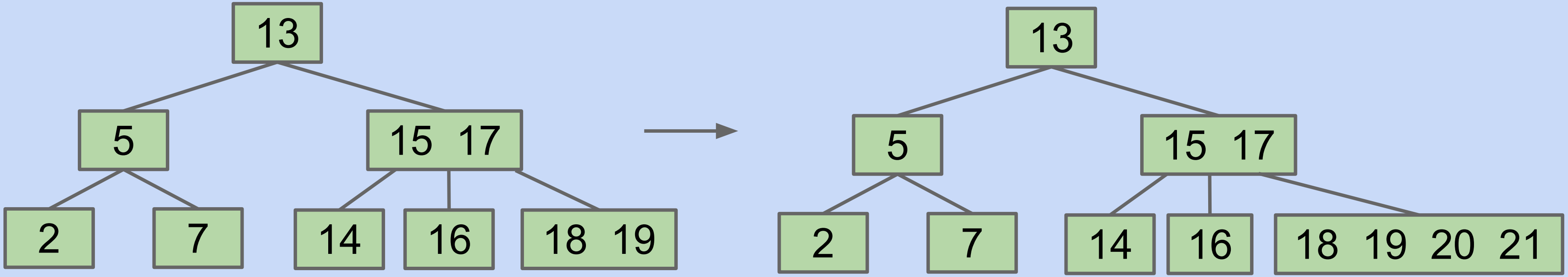
Examining a node costs us O(L) compares, but that's OK since L is constant.

What if a non-leaf node gets too full? Can we split that?

- Sure, we'll do this in a few slides, but first...

add Understanding Check

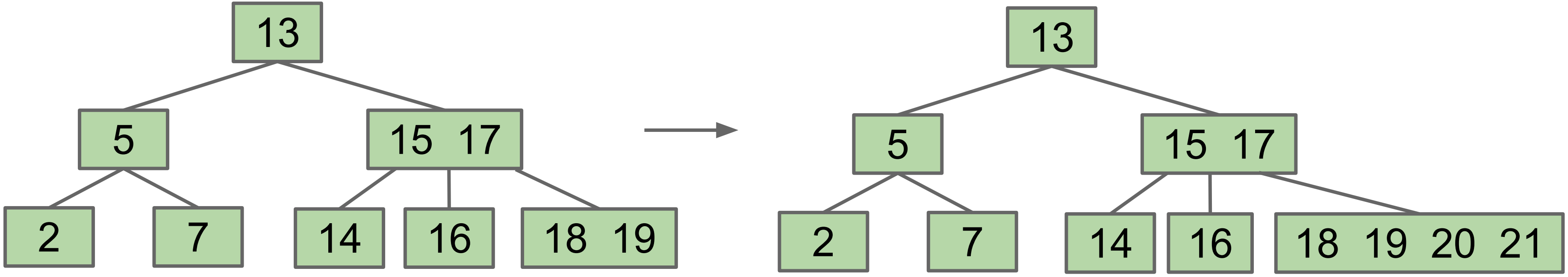
- Suppose we add 20, 21:



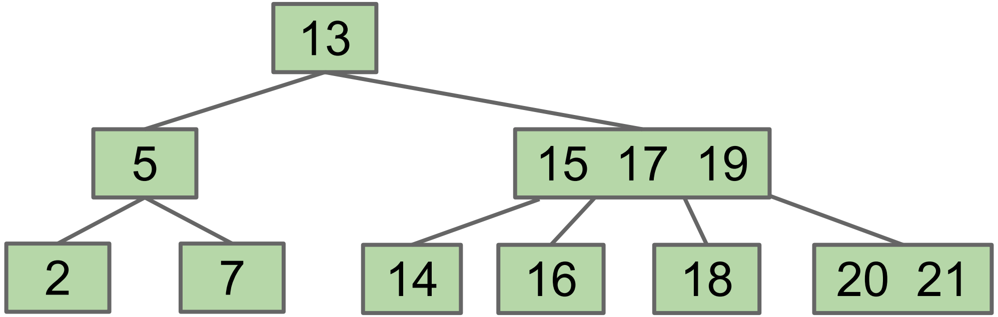
- Q: If our cap is at most L=3 items per node, draw post-split tree:

add Understanding Check

- Suppose we add 20, 21:

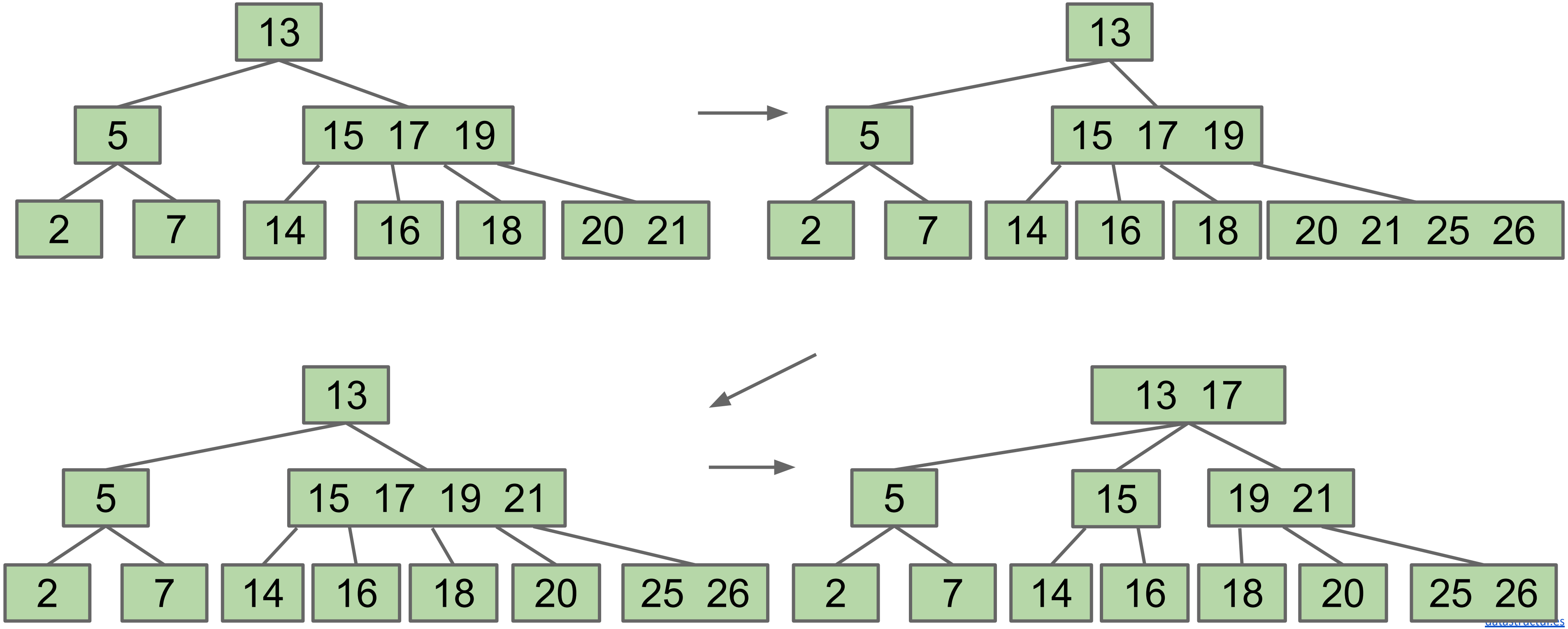


- Q: If our cap is at most L=3 items per node, draw post-split tree:

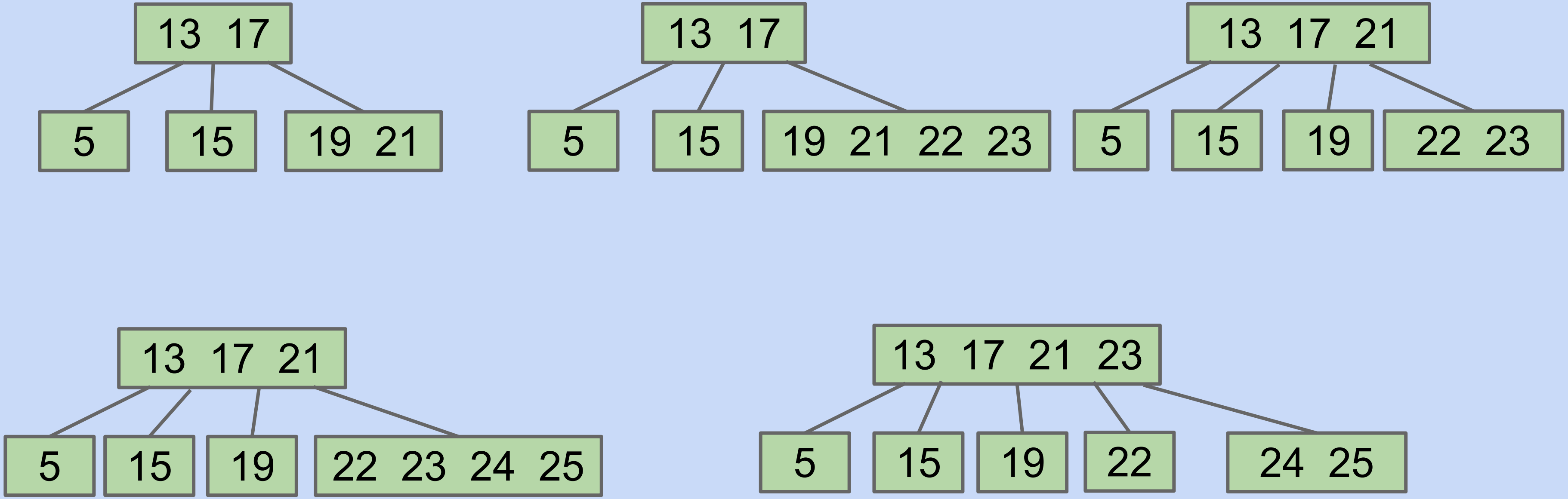


add: Chain Reaction

- Suppose we add 25, 26:

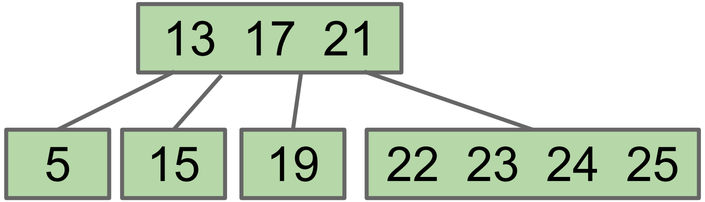


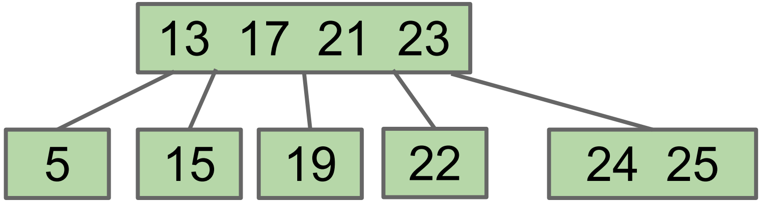
What Happens If The Root Is Too Full?



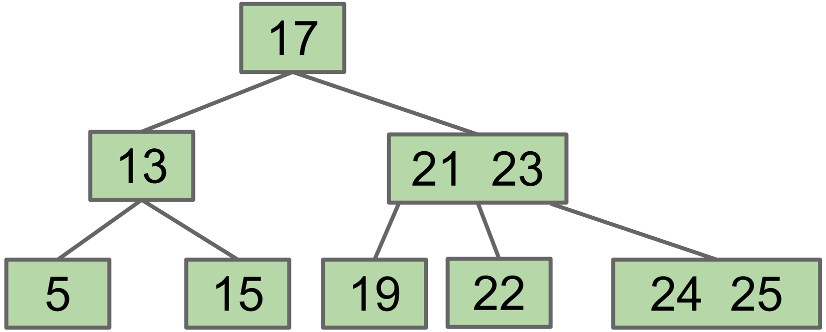
Challenge: Draw the tree after the root is split.

What Happens If The Root Is Too Full?





Challenge: Draw the tree after the root is split.



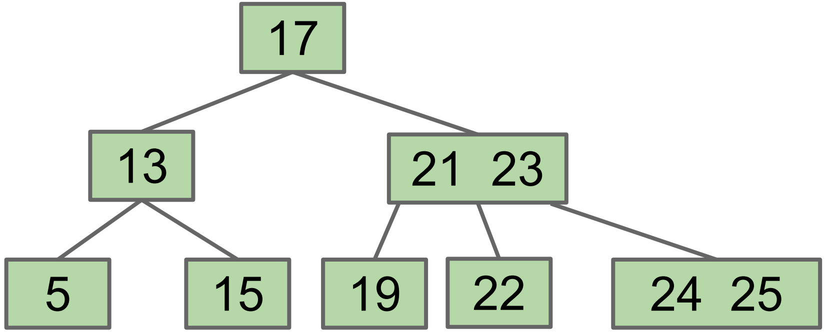
Perfect Balance

Observation: Splitting-trees have perfect balance.

- If we split the root, every node gets pushed down by exactly one level.
- If we split a leaf node or internal node, the height doesn't change.

We will soon prove: All operations have guaranteed O(log N) time.

- More details soon.



The Real Name for Splitting Trees is “B Trees”

Splitting tree is a better name, but I didn’t invent them, so we’re stuck with their real name: **B-trees**.

- B-trees of order L=3 (like we used today) are also called a 2-3-4 tree or a 2-4 tree.
    - “2-3-4” refers to the number of children that a node can have, e.g. a 2-3-4 tree node may have 2, 3, or 4 children.
- B-trees of order L=2 are also called a 2-3 tree.

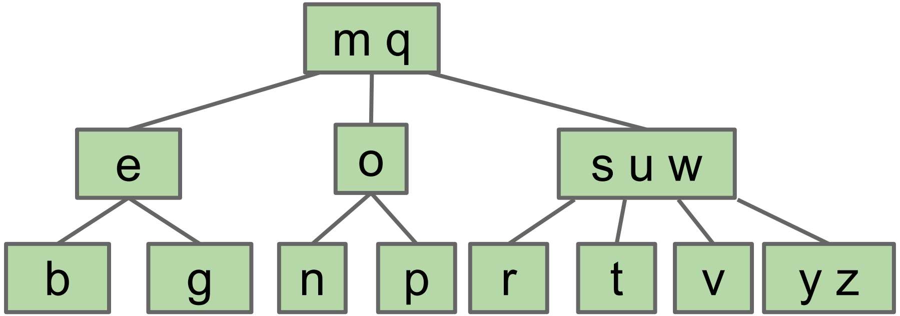
The origin of "B-tree" has never been explained by the authors. As we shall see, "balanced," "broad," or "bushy" might apply. Others suggest that the "B" stands for Boeing. Because of his contributions, however, it seems appropriate to think of B-trees as "Bayer"-trees.

- Douglas Corner (The Ubiquitous B-Tree)

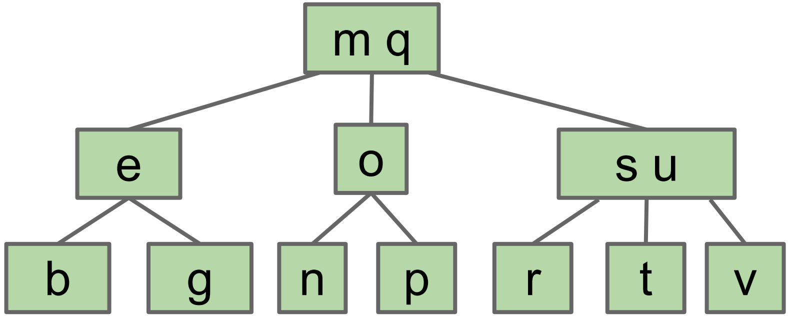
A note on Terminology

B-Trees are most popular in two specific contexts:

- Small L (L=2 or L=3):
  - Used as a conceptually simple balanced search tree (as today).
- L is very large (say thousands).
  - Used in practice for databases and filesystems (i.e. systems with very large records).



2-3-4 a.k.a. 2-4 Tree (L=3):
Max 3 items per node.
Max 4 non-null children per node.



2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.

B-Tree Bushiness Invariants

Exercise

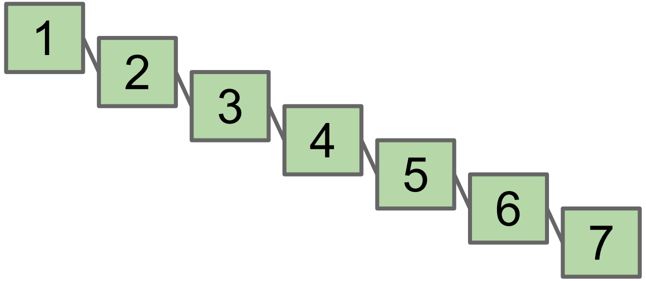
add the numbers 1, 2, 3, 4, 5, 6, then 7 (in that order) into a regular BST.

Then try adding 1, 2, 3, 4, 5, 6, then 7 (in that order) into a 2-3 tree (L=2).

- For L=2, pass the middle item up.

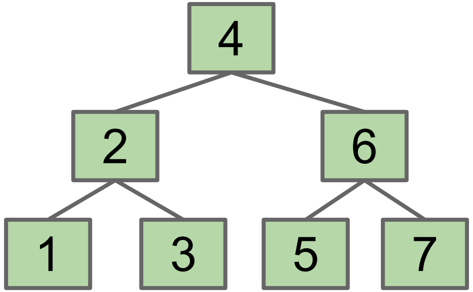
Exercise

add the numbers 1, 2, 3, 4, 5, 6, then 7 (in that order) into a regular BST.



Then try adding 1, 2, 3, 4, 5, 6, then 7 (in that order) into a 2-3 tree.

- Interactive demo: https://tinyurl.com/balanceYD or [this link](#).
- In this demo "max-degree" means the maximum number of children, i.e. 3.



All leaves are at depth 2.

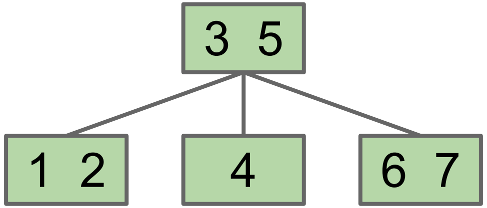
Exercise 2

Find an order such that if you add the items 1, 2, 3, 4, 5, 6, and 7 in that order, the resulting 2-3 tree has height 1.

Exercise 2

Find an order such that if you add the items 1, 2, 3, 4, 5, 6, and 7 in that order, the resulting 2-3 tree has height 1.

- One possible answer: 2, 3, 4, 5, 6, 1, 7



All leaves are at depth 1.

Not sure why? Make sure to see https://tinyurl.com/balanceYD.

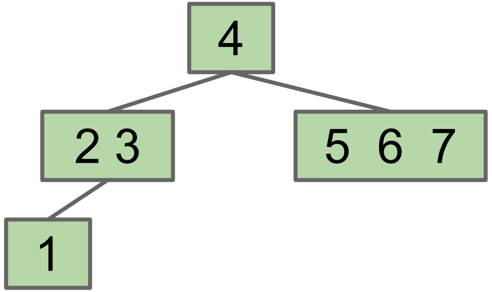
No matter the insertion order you choose, resulting B-Tree is always bushy!

- May vary in height a little bit, but overall guaranteed to be bushy.

B-Tree Invariants

Because of the way B-Trees are constructed, we get two nice invariants:

- All leaves must be the same distance from the source.
- A non-leaf node with k items must have exactly k+1 children.
- Example: The tree given below is impossible.
    - Leaves ([1] and [5 6 7]) are a different distance from the source.
    - Non-leaf node [2 3] has two items but only only one child. Should have three children.



We have not proven these invariants rigorously, but try thinking them through.

B-Tree Invariants

Because of the way B-Trees are constructed, we get two nice invariants:

- All leaves must be the same distance from the source.
- A non-leaf node with k items must have exactly k+1 children.

These invariants guarantee that our trees will be bushy.

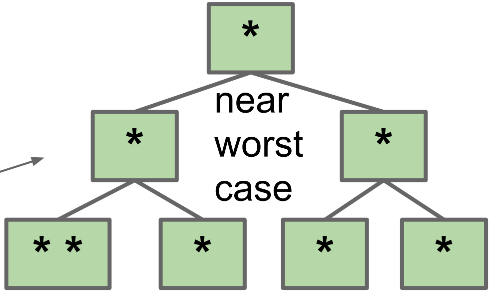
B-Tree Runtime Analysis

Height of a B-Tree with Limit L

L: Max number of items per node.

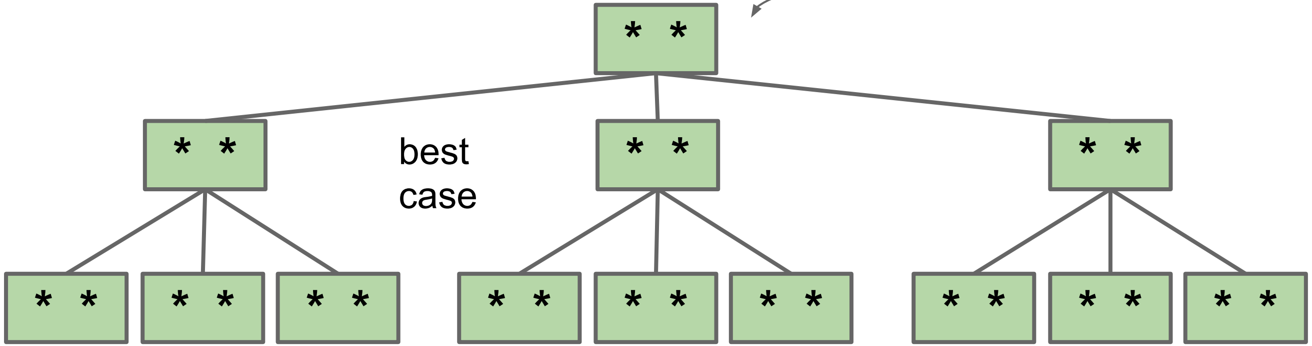
Height: Between ~$\log_{L+1}(N)$ and ~$\log_2(N)$

- Largest possible height is all non-leaf nodes have 1 item.
- Smallest possible height is all nodes have L items.
- Overall height is therefore $\Theta(\log N)$.



N: 8 items
L: 2 max per node
H: 2

Height grows with $\log_2(N)$



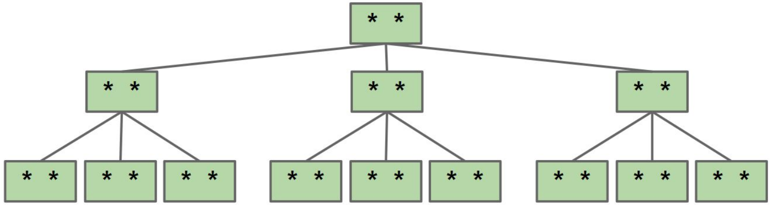
N: 26 items
L: 2 max per node
H: 2

Height grows with $\log_3(N)$

Runtime for `contains`

Runtime for contains:

- Worst case number of nodes to inspect: H + 1
- Worst case number of items to inspect per node: L
- Overall runtime: O(HL)



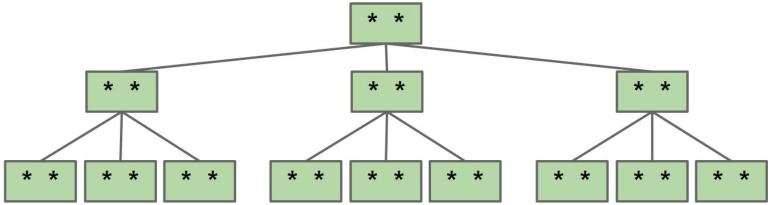
Since H = Θ(log N), overall runtime is O(L log N).

- Since L is a constant, runtime is therefore O(log N).

Runtime for add

Runtime for add:

- Worst case number of nodes to inspect: H + 1
- Worst case number of items to inspect per node: L
- Worst case number of split operations: H + 1
- Overall runtime: O(HL) = O(L)



Since H = Θ(log N), overall runtime is O(L log N).

- Since L is a constant, runtime is therefore O(log N).

Bottom line: contains and add are both O(log N).

Summary

Summary

BSTs have best case height $\Theta(\log N)$, and worst case height $\Theta(N)$.

- Big O is not the same thing as worst case!

B-Trees are a modification of the binary search tree that avoids $\Theta(N)$ worst case.

- Nodes may contain between 1 and L items.
- `contains` works almost exactly like a normal BST.
- add works by adding items to existing leaf nodes.
  - If nodes are too full, they split.
- Resulting tree has perfect balance. Runtime for operations is $O(\log N)$.
- Have not discussed deletion. See extra slides if you're curious.
- Have not discussed how splitting works if $L > 3$ (see some other class).
- B-trees are more complex, but they can efficiently handle ANY insertion order.

Binary Search Tree Deletion
A Quick History (Extra)

This will not be covered in any homework, lab, or exam.

Randomized Search Trees Including Deletion

Earlier, we saw that BSTs have:

- Worst case Θ(N) height.
- Best case Θ(log N) height.
- Θ(log N) height if constructed via random inserts.

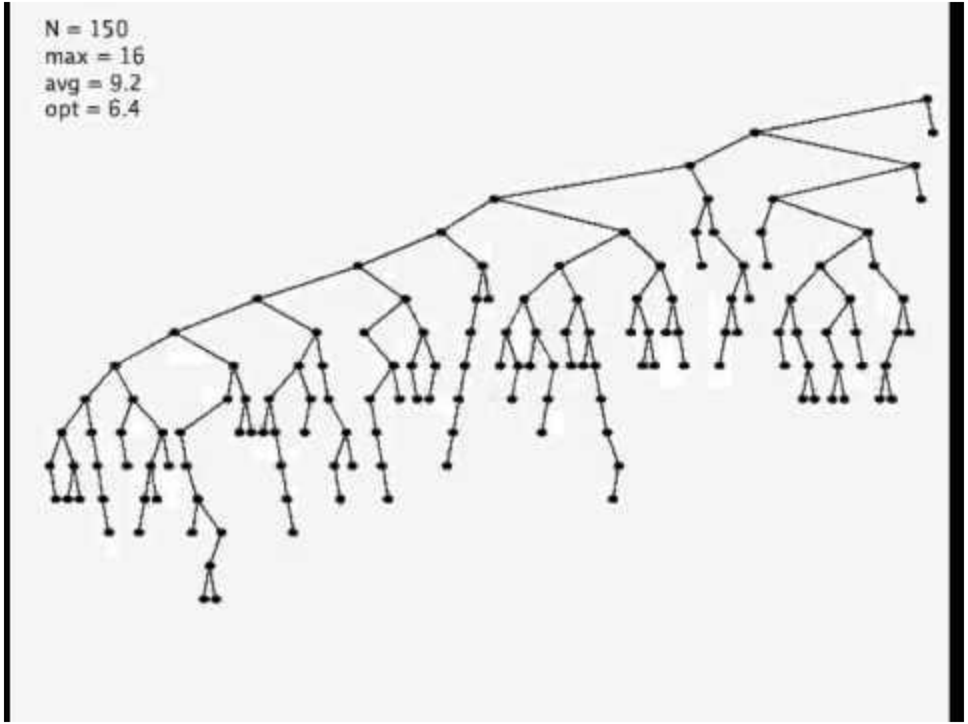
Can also simulate a sequence of random insertions and deletions.

Randomized Search Trees Including Deletion

In real world applications, items are added **and removed** from sets (or maps) all the time! Let's try out a simulation that includes deletion.

max: height + 1

avg: average depth + 1



Video courtesy of Kevin Wayne (Princeton University)

Important Question: What about Real World BSTs?

BSTs have:

- Worst case Θ(N) height.
- Best case Θ(log N) height.
- Θ(log N) height if constructed via random inserts.
- Θ(sqrt N) height after random insertion and deletion.
  - Assumes you always pick the successor when deleting!
  - There's an interesting story how this Θ(sqrt N) bound was found...

History of Asymmetric Hibbard Deletion (AHD) Analysis

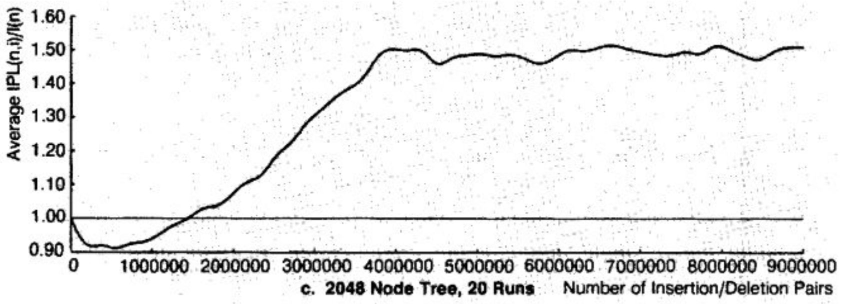
In 1975, Gary Knott wrote his Ph.D. thesis “Deletion in Binary Storage Trees."

- In this thesis, Knott ran simulations and conjectured that random insertion and deletion using AHD improved the average depth of trees.

In 1983, Jeffrey Eppinger of Carnegie Mellon wrote an article overturning this conjecture.

- Showed that average depth did improve with random insertions and deletions, but only for a while. Eventually, it got worse!

Here: Average IPL means average depth. The values are normalized to the height of a tree built only by random insertion.



History of Asymmetric Hibbard Deletion (AHD) Analysis

In 1983, Jeffrey Eppinger of Carnegie Mellon [an article overturning this conjecture].

- Showed that average depth did improve with random insertions and deletions, but only for a while. Eventually, it got worse!
- Conjectured that average depth is $\Theta(\log^2 N)$ rather than $\Theta(\log N)$.
  - $\log(1000000) = 6$, $\log^2(1000000) = 36$

In 1987, Culberson and Munro overturned Eppinger's conjecture.

- Did a more rigorous empirical study, and found that average depth appeared to be $\Theta(\text{sqrt } N)$, not $\Theta(\log^2 N)$.
- This $\Theta(\text{sqrt } N)$ bound has not been mathematically proven!

Symmetric Hibbard Deletion

If you randomly pick between successor and predecessor, you get Θ(log N) height.

- See Culberson and Munro, 1989 or Knuth's TAOCP Volume 3 - 6.2 for more if you're curious (the history is pretty interesting).

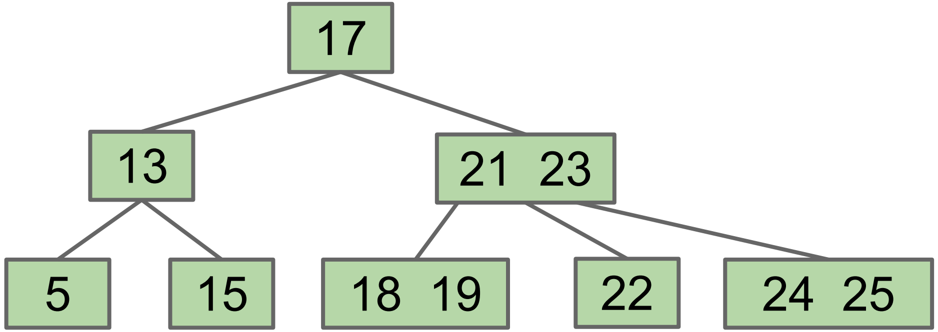
2-3 Tree Deletion (Extra)

This will not be covered in any homework, lab, or exam.

Deletion Operations

As with regular Binary Search Trees, deletion is a more complicated operation.

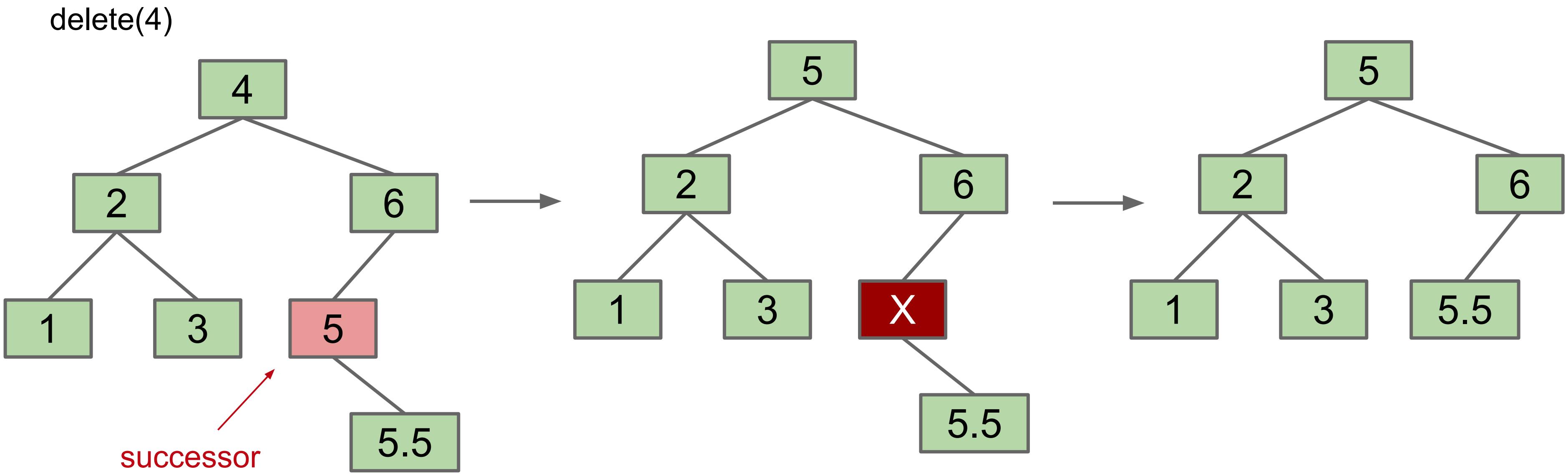
- Many possible deletion algorithms.
- We'll develop a deletion algorithm together.



Deletion from a Regular BST (Review)

In a regular BST, when we delete a value α with 2 children, we:

- Copy the value of the successor into α.
- Then we delete the successor.



Deletion from a 2-3 Tree

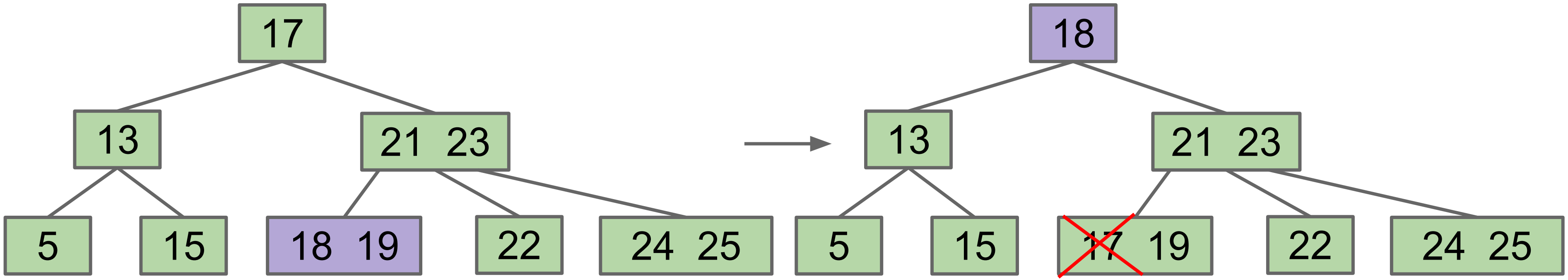
In a 2-3 Tree, when we delete α from a node with 2 or more children, we:

- Swap the value of the successor with α.
- Then we delete the successor value.

Note: Successor will always be in a leaf node!

Example: delete(17):

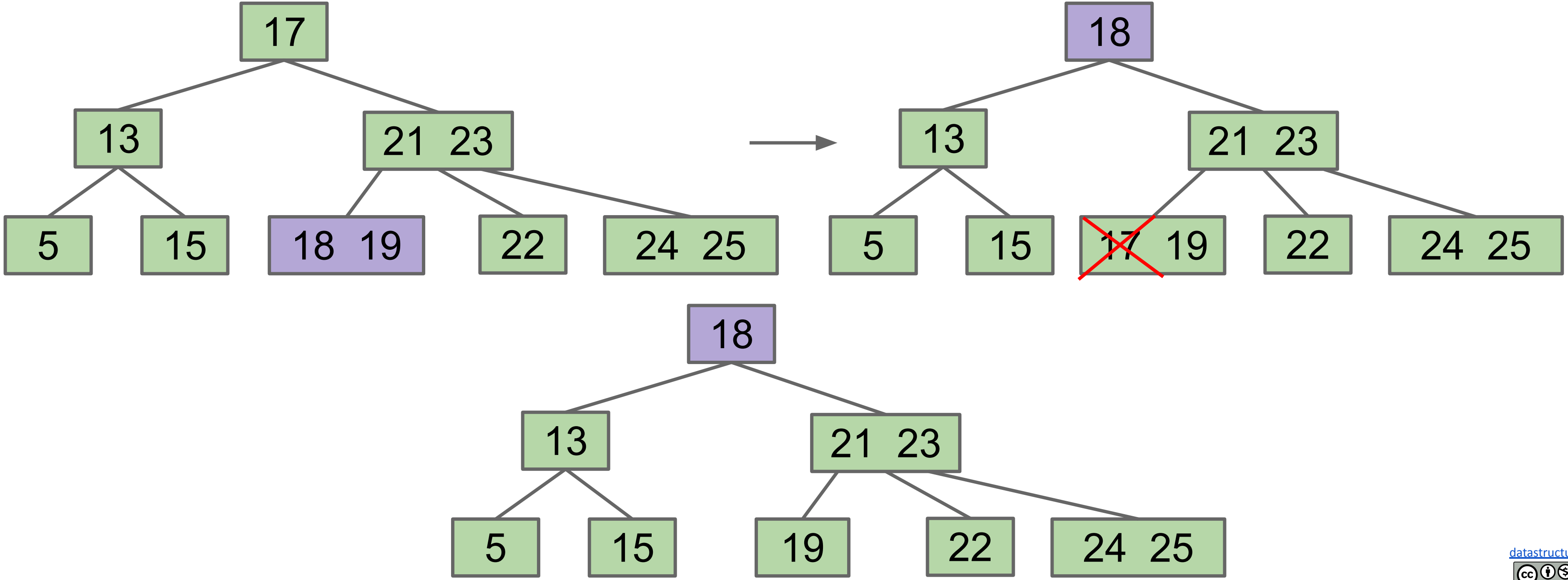
- Swap 17 with its successor 18, then delete 17.



Multi-Key Leaves vs. Single-Key Leaves

If deleting from a leaf with multiple keys, the deletion is trivial. We simply remove the item from the leaf, and we are done.

delete(17):



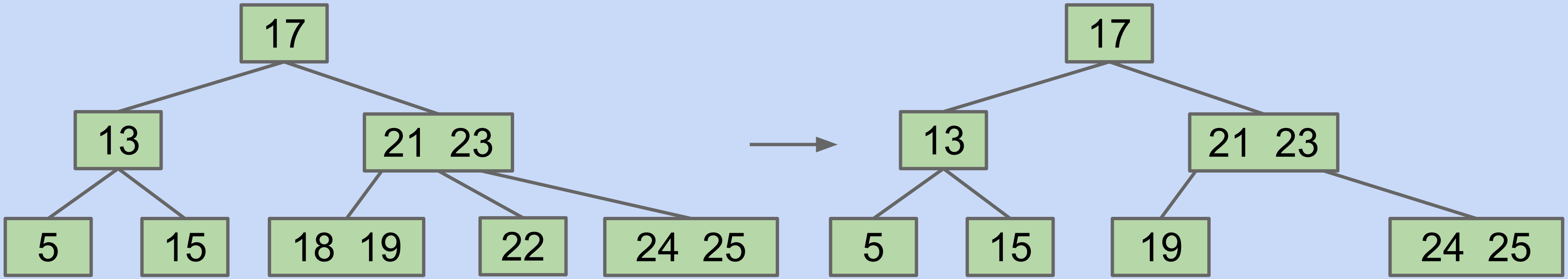
Multi-Key Leaves vs. Single-Key Leaves

If our leaf has multiple keys, the deletion is trivial. We simply remove the item.

If our leaf has a single key, we cannot simply remove the node entirely.

- Why?

delete(22):



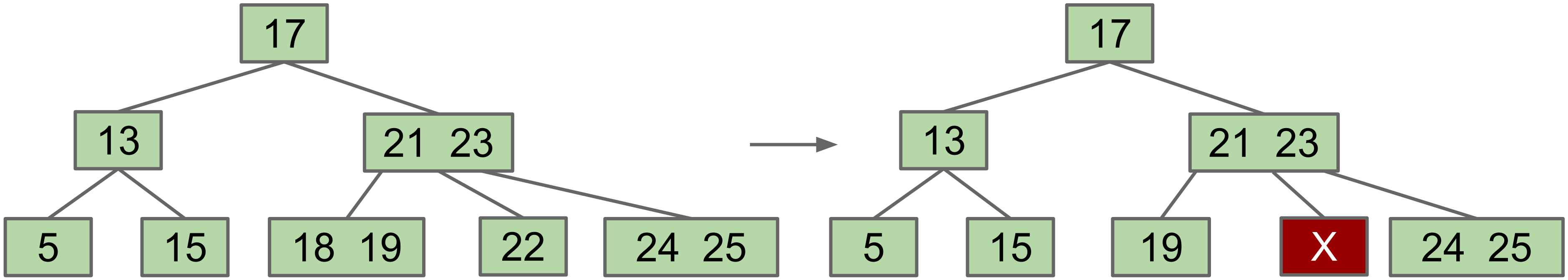
Multi-Key Leaves vs. Single-Key Leaves

If our leaf has multiple keys, the deletion is trivial. We simply remove the item.

If our leaf has a single key, we cannot simply remove the node entirely.

- Any node with k items must have k + 1 children!
- Instead, we'll leave an empty node, which must be filled.
- Filling the empty node is complex and has many cases (coming soon).

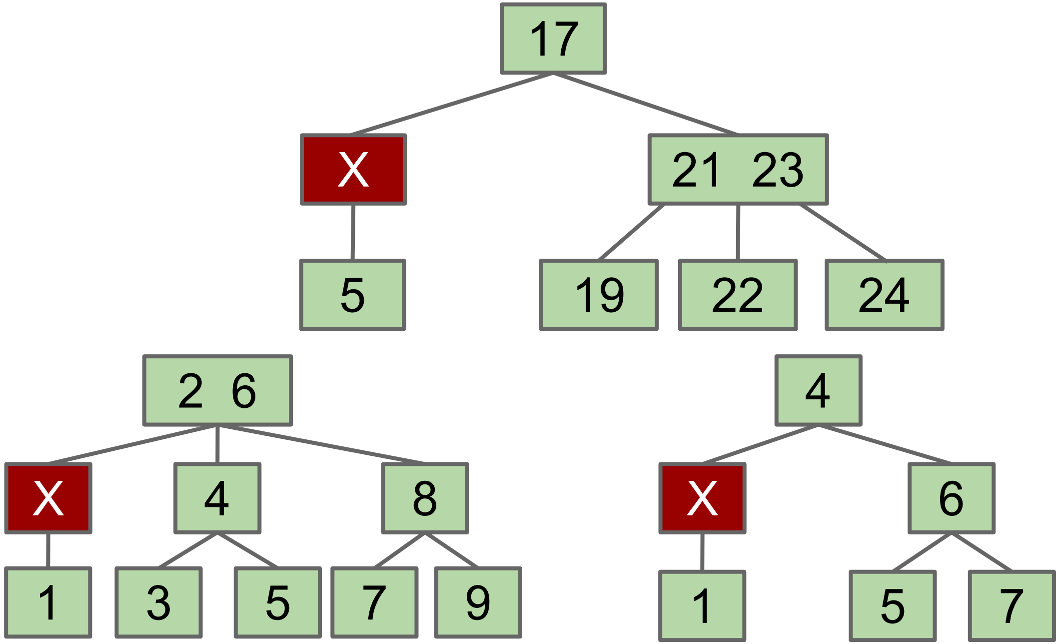
delete(22):



Filling in Empty Nodes (FIEN)

The hard part about deletion is filling empty nodes.

- There are three interesting cases to consider.
- For reasons that will become clear later, we will talk about how to fill empty boxes ANYWHERE in the tree, not just in the leaves.

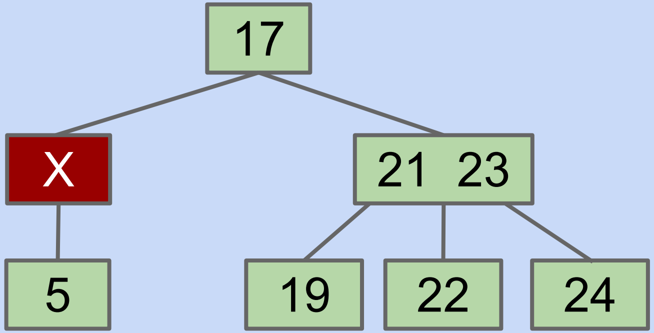


Since empty nodes contain k=0 items, non-leaf empty nodes have 1 child!

FIEN Case 1A: Multi-Key Sibling

In case 1A, the empty node's adjacent sibling has multiple keys.

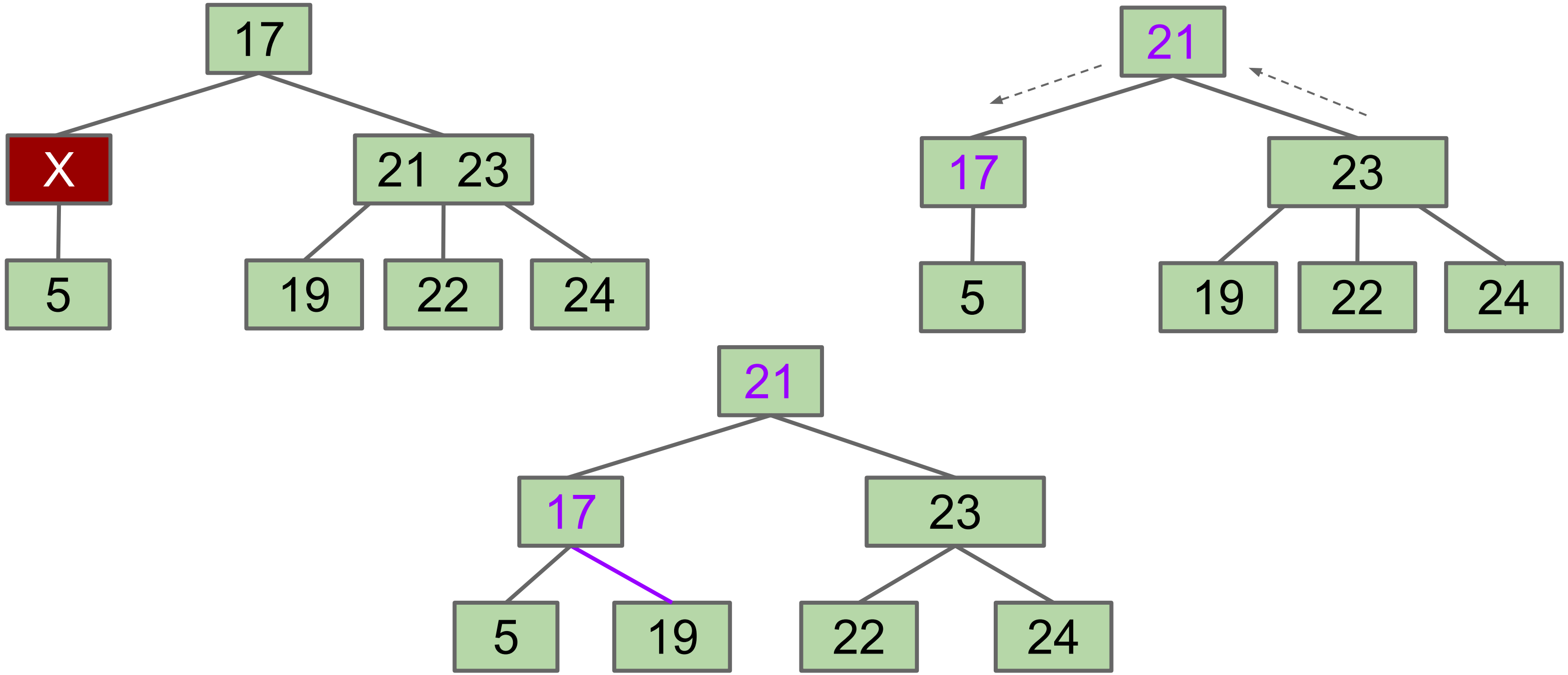
- Very hard optional challenge: Try to fill in X in the diagram below so that the result is a valid 2-3 tree.



FIEN Case 1A: Multi-Key Sibling

In case 1A, the empty node's adjacent sibling has multiple keys.

- X steals parent's item. Parent steals sibling's item.
- If X was not a leaf node, X steals one of sibling's subtrees.



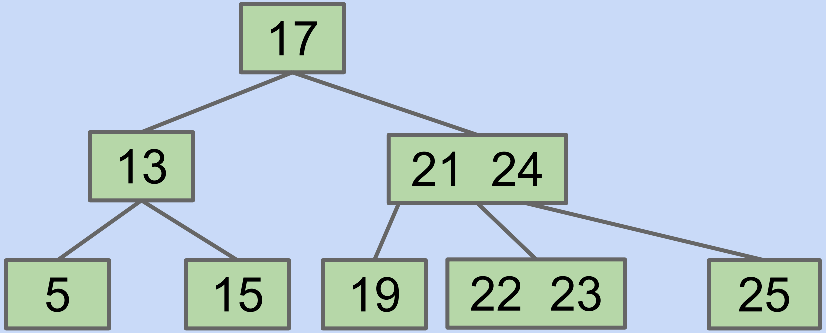
FIEN Case 1A: Multi-Key Sibling

In case 1A, the empty node's adjacent sibling has multiple keys.

- X steals parent's item. Parent steals sibling's item.
- If X was not a leaf node, X steals one of sibling's subtrees.

Try to delete 17 from the tree below.

- Hint: You'll end up in FIEN Case 1.



FIEN Case 1A: Multi-Key Sibling

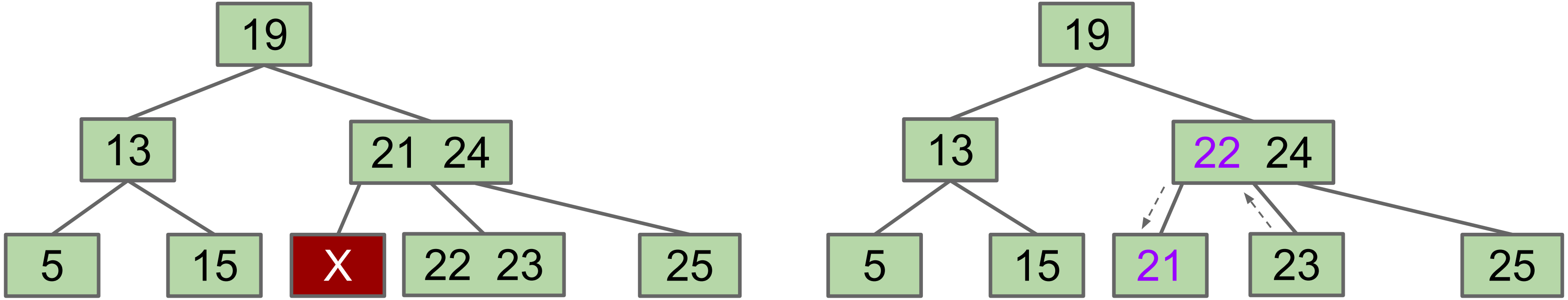
In case 1A, the empty node's adjacent sibling has multiple keys.

- X steals parent's item. Parent steals sibling's item.
- If X was not a leaf node, X steals one of sibling's subtrees.

Wasn't necessary since X was not a leaf node.

Try to delete 17 from the tree below.

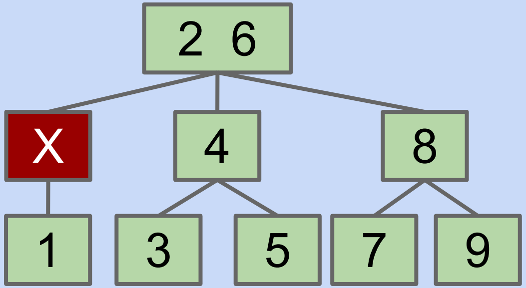
- Swap 17 with 19, then delete 17.



FIEN Case 2A: Multi-Key Parent

In case 2A, siblings on the right all have one key, but parent has two.

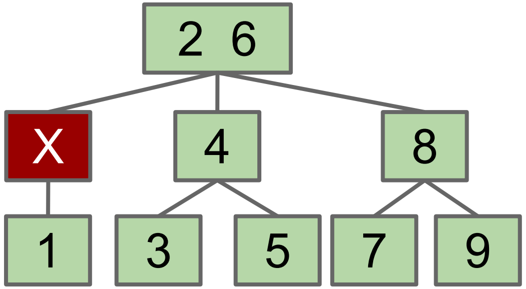
- Very hard optional challenge: Try to fill in X in the diagram below so that the result is a valid 2-3 tree.

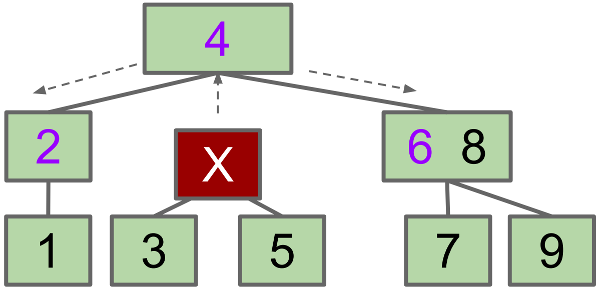


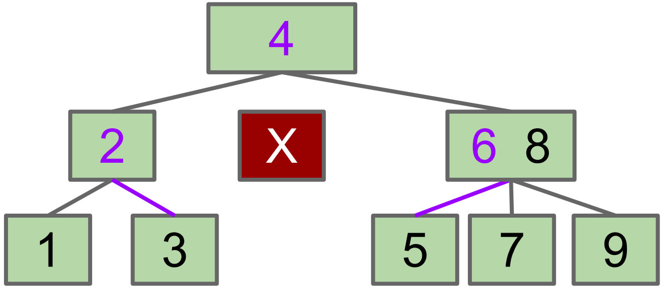
FIEN Case 2A: Multi-Key Parent

In case 2A, siblings on the right all have one key, but parent has two.

- X and right sibling steal parent's keys. Middle sibling moves into the parent.
- Subtrees are passed along so that every node has the correct children.





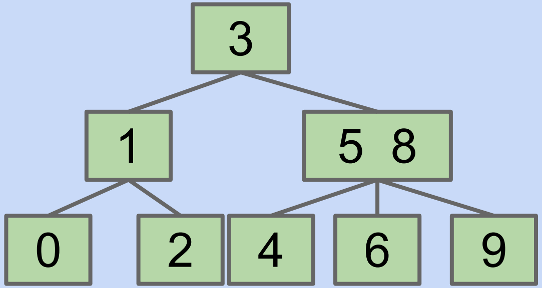


FIEN Case 2A Exercise

In case 2A, siblings on the right all have one key, but parent has two.

- X and right sibling steal parent's keys. Middle sibling moves into the parent.
- Subtrees are passed along so that every node has the correct children.

Try to delete 3 from the tree below.

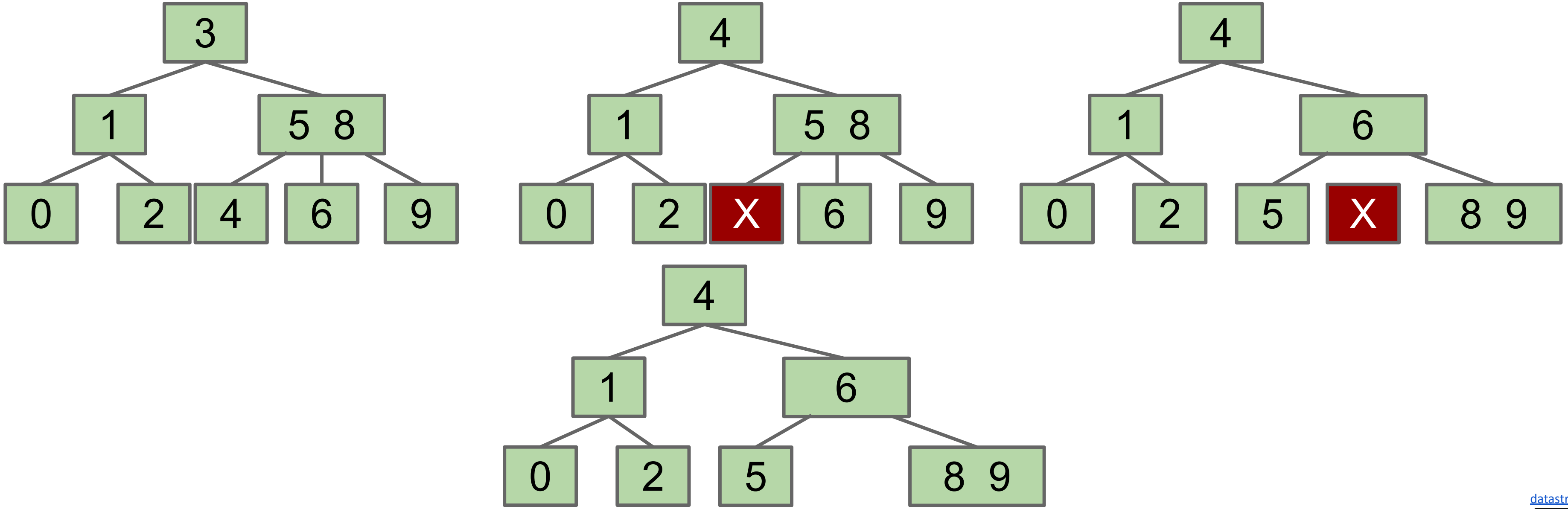


FIEN Case 2A Exercise

In case 2A, siblings on the right all have one key, but parent has two.

- X and right sibling steal parent's keys. Middle sibling moves into the parent.
- Subtrees are passed along so that every node has the correct children.

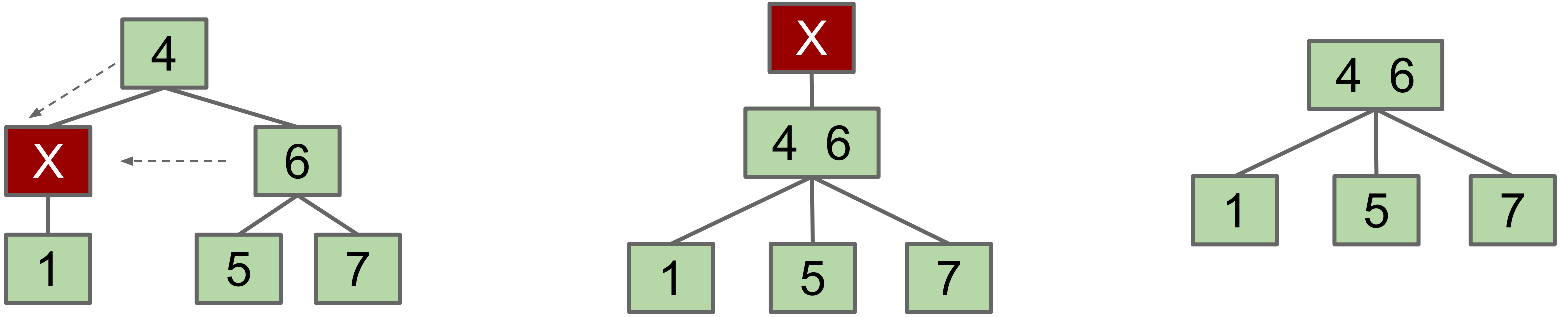
delete(3)



FIEN Case 3: Single-Key Parent and Sibling

In FIEN Case 3: The parent and all siblings have only one item.

- Combine 1 sibling and parent into a single node that replaces X. Send the blank X up one level.
- If blank ends up as the new root, just delete the blank and we are done.

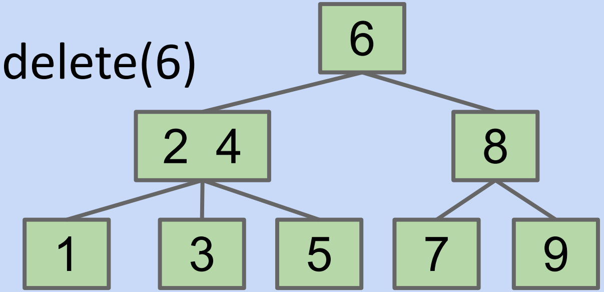


FIEN Case 3 Exercise

In FIEN Case 3: The parent and all siblings have only one item.

- Combine 1 sibling and parent into a single node that replaces X. Send the blank X up one level.
- If blank ends up as the new root, just delete the blank and we are done.

delete(6)

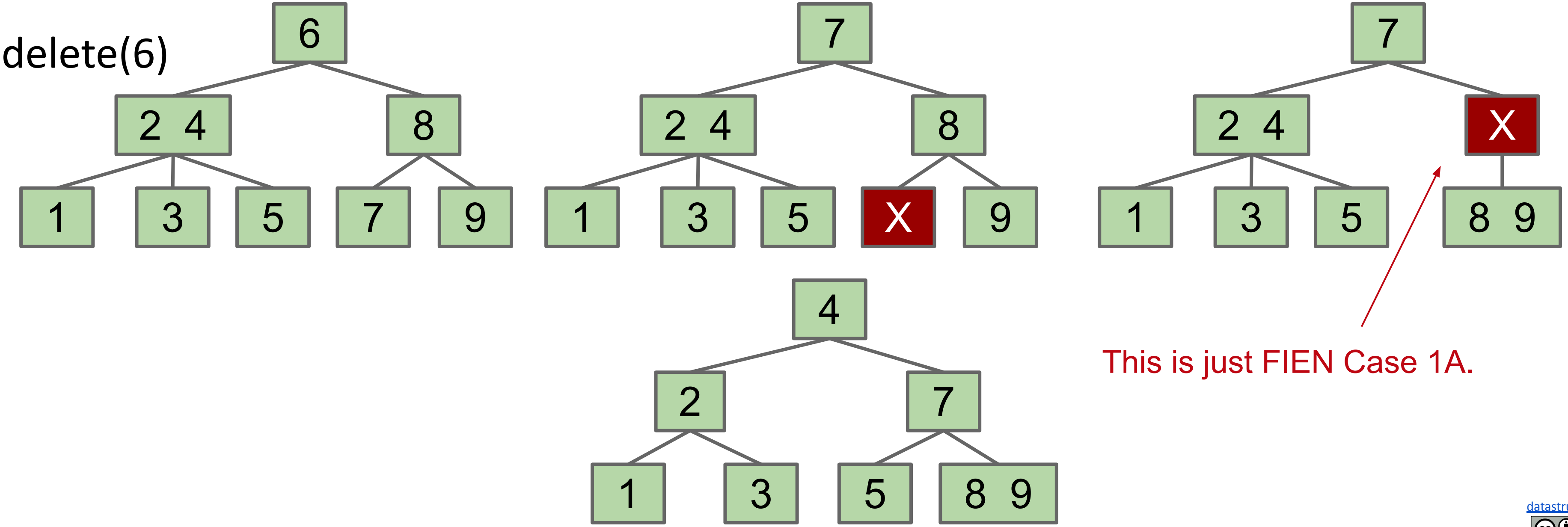


FIEN Case 3 Exercise

In FIEN Case 3: The parent and all siblings have only one item.

- Combine 1 sibling and parent into a single node that replaces X. Send the blank X up one level.
- If blank ends up as the new root, just delete the blank and we are done.

delete(6)

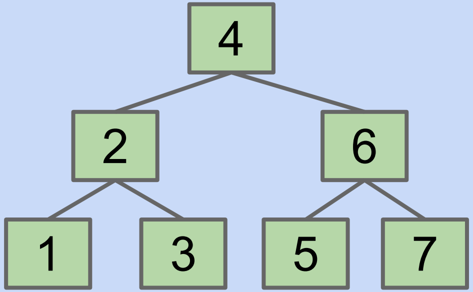


FIEN Case 3 Exercise #2

In FIEN Case 3: The parent and all siblings have only one item.

- Combine 1 sibling and parent into a single node that replaces X. Send the blank X up one level.
- If blank ends up as the new root, just delete the blank and we are done.

delete(4)

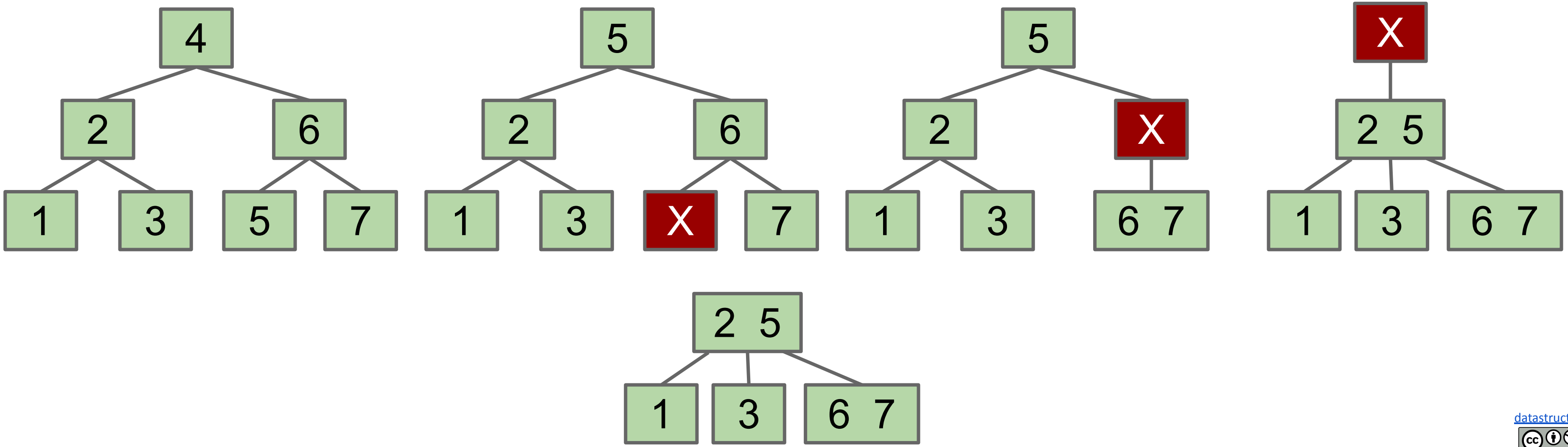


FIEN Case 3 Exercise #2

In FIEN Case 3: The parent and all siblings have only one item.

- Combine 1 sibling and parent into a single node that replaces X. Send the blank X up one level.
- If blank ends up as the new root, just delete the blank and we are done.

delete(4)



Citations

Bee-tree from https://beelore.files.wordpress.com/2010/01/thai_beetree.jpg

Some isometry figures from Algorithms textbook.

Fantasy code for 2-3 Tree courtesy of Kevin Wayne