CS 70 Fall 2024

Note 1

Discrete Mathematics and Probability Theory Rao, Hug

DIS 0B

1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

- (a) There is a real number which is not rational.
- (b) All integers are natural numbers or are negative, but not both.
- (c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.
- (d) $(\forall x \in \mathbb{Z}) (x \in \mathbb{Q})$
- (e) $(\forall x \in \mathbb{Z}) (((2 \mid x) \lor (3 \mid x)) \Longrightarrow (6 \mid x))$
- (f) $(\forall x \in \mathbb{N}) ((x > 7) \implies ((\exists a, b \in \mathbb{N}) (a + b = x)))$

Solution:

- (a) $(\exists x \in \mathbb{R}) \ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R}) \ \neg (x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.
- (b) $(\forall x \in \mathbb{Z}) (((x \in \mathbb{N}) \lor (x < 0)) \land \neg ((x \in \mathbb{N}) \land (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.
- (c) $(\forall x \in \mathbb{N})$ $((6 \mid x) \implies ((2 \mid x) \lor (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.
- (d) All integers are rational numbers. This is true, since any integer number n can be written as n/1.
- (e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.
- (f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take a = x and b = 0.
 - (Aside: this is a reference to the very weak Goldback Conjecture (https://xkcd.com/1310/).)

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2 Truth Tables

Note 1

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a)
$$P \wedge (Q \vee P) \equiv P \wedge Q$$

(b)
$$(P \lor Q) \land R \equiv (P \land R) \lor (Q \land R)$$

(c)
$$(P \land Q) \lor R \equiv (P \lor R) \land (Q \lor R)$$

Solution:

(a) Not equivalent.

-		
Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F
	T F T	T T T T T T T T F

(b) Equivalent.

1						
P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$		
T	T	T	T	T		
T	T	F	F	F		
T	F	T	T	Т		
T	F	F	F	F		
F	T	T	T	Т		
F	T	F	F	F		
F	F	T	F	F		
F	F	F	F	F		

(c) Equivalent.

P	Q	R	$(P \wedge Q) \vee R$	$(P \lor R) \land (Q \lor R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

3 Implication

Note 0 Note 1 Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement P(x, y) that would make the implication false).

- (a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.
- (b) $\forall x \exists y P(x, y) \implies \exists y \forall x P(x, y)$.
- (c) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

- (a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.
- (b) False. Let P(x, y) be x < y, and the universe for x and y be the integers. Or let P(x, y) be x = y and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.
- (c) True. The first statement says that there is an x, say x' where for every y, P(x,y) is true. Thus, one can choose x = x' for the second statement and that statement will be true again for every y.

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