Discussion 14B

Markov Chains Intro II

Note 22

Recall that a Markov chain is defined with the following: the state space $\mathscr{X}$, the transition matrix P, and the initial distribution π_0. This implicitly defines a sequence of random variables X_n with distribution π_n, which denote the state of the Markov chain at timestep n. This sequence of random variables also obey the Markov property: the transition probabilities only depend on the current state, and not any prior states.

The **stationary distribution** (or the **invariant distribution**) of a Markov chain is the row vector π such that $\pi P = \pi$. (That is, transitioning does not change the distribution of states.)

Irreducibility: A Markov chain is *irreducible* if one can reach any state from any other state in a finite number of steps.

Periodicity: In an irreducible Markov chain, we define the *period* of a state i as

$$d(i) = \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}.$$

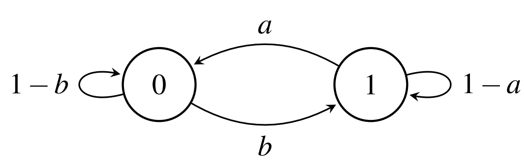
If $d(i) = 1$ for all i, then a Markov chain is *aperiodic*. Otherwise, we say that the Markov chain is *periodic*.

Fundamental Theorem of Markov Chains: If a Markov chain is irreducible and aperiodic, then for any initial distribution π_0, we have that $\pi_n \to \pi$ as $n \to \infty$, and π is the unique invariant distribution for the Markov chain.

1 Markov Chain Terminology

Note 22

In this question, we will walk you through terms related to Markov chains. Consider the following Markov chain.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

(a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one of a and b is 0.

(b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2,4,6,\ldots\} = 2.$$

This is because if we start at a state X then we can get back to it after taking an even number of steps only (2, 4, 6, 8, etc.), not by taking an odd number of steps (1, 3, 5, 7, etc.). Thus, the chain is periodic with period 2.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1,2,3,\ldots\} = 1.$$

Thus, the chain is aperiodic. Notice that the self-loops allow us to stay at the same node, thereby letting us get to any other node in an odd *or* even number of steps.

(d) The transition matrix is:

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e) To solve for the stationary distribution, we need to solve for π in $\pi = \pi P$. This gives us the following system of equations:

$$\begin{aligned} \pi(0) &= (1-b)\pi(0) + a\pi(1), \\ \pi(1) &= b\pi(0) + (1-a)\pi(1). \end{aligned}$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

$$\pi = \frac{1}{a+b}\begin{bmatrix} a & b \end{bmatrix}.$$

2 Allen's Umbrella Setup

Note 22

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring exactly one umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix. (*Hint*: You should have 3 states. Keep in mind that our goal is to construct a Markov chain to solve part (c).)

(b) Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution.

(c) In the long term, what is the probability that Allen walks through rain with no umbrella?

Solution:

(a) Let state i represent the situation that Allen has i umbrellas at his current location, for $i = 0$, 1, or 2.

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

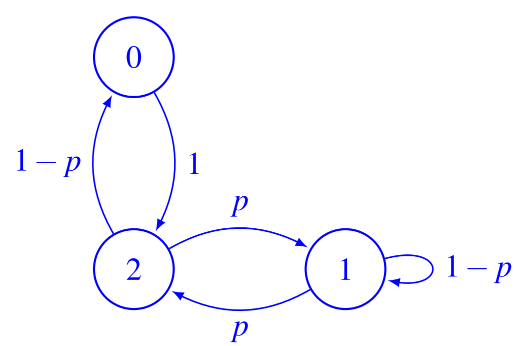
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \qquad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \qquad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1-p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

(b) Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution.

To solve for the invariant distribution, we set $\pi P = \pi$, or $\pi(P-I) = 0$. This yields the balance equations

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & p\\ 1-p & p & -1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0)+\pi(1)+\pi(2) = 1$.

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & 1\\ 1-p & p & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{3-p}\begin{bmatrix}1-p & 1 & 1\end{bmatrix}$$

(c) Allen walks through rain with no umbrella if and only if it is raining when we take the transition from state 0 to 2 (i.e. Allen had no umbrellas, and moved to a location with 2 umbrellas). Note that given that we are in state 0, we must always take this transition with probability 1, so it suffices to compute the probability that it rains *and* we are in state 0.

Since the invariant distribution has $\pi(0) = \frac{1-p}{3-p}$, and it rains with probability p, the probability of walking through rain with no umbrella in the long term is

$$\mathbb{P}[\text{rain} \wedge \text{no umbrella}] = p \cdot \frac{1-p}{3-p} = \frac{p(1-p)}{3-p}.$$

3 Three Tails

Note 22 You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

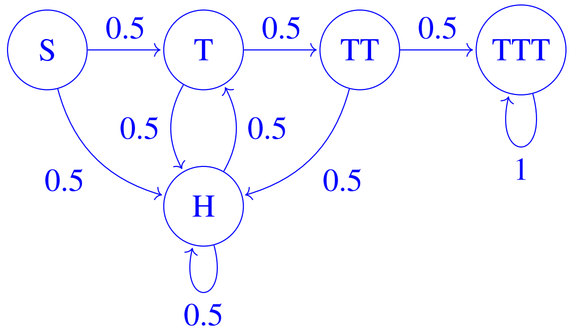
Hint: It can help to start by thinking about how to compute the number of *coins* flipped until getting TTT, and then slightly modifying your equations to solve the original question.

Solution:

We can model this problem as a Markov chain with the following states:

- S: Start state, which we are only in before flipping any coins.
- H: We see a head, which means no streak of tails currently exists.
- T: We've seen exactly one tail in a row so far.
- TT: We've seen exactly two tails in a row so far.

- TTT: We've accomplished our goal of seeing three tails in a row and stop flipping.



We can write the first step equations and solve for $\beta(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\beta(S) = 0.5\beta(T) + 0.5\beta(H) \tag{1}$$
$$\beta(H) = 1 + 0.5\beta(H) + 0.5\beta(T) \tag{2}$$
$$\beta(T) = 0.5\beta(TT) + 0.5\beta(H) \tag{3}$$
$$\beta(TT) = 0.5\beta(H) + 0.5\beta(TTT) \tag{4}$$
$$\beta(TTT) = 0 \tag{5}$$

From (2), we see that

$$0.5\beta(H) = 1 + 0.5\beta(T)$$

and can substitute that into (3) to get

$$0.5\beta(T) = 0.5\beta(TT) + 1.$$

Substituting this into (4), we can deduce that $\beta(TT) = 4$. This allows us to conclude that $\beta(T) = 6$, $\beta(H) = 8$, and $\beta(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.