Open Ended Lab 01
Modelling Practical Rotational Mechanical System in MATLAB

Objective

To construct a mathematical model of a practical rotational mechanical system in MATLAB.

Introduction

In this open ended lab, a permanent magnet DC motor (PMDC motor) is chosen as an example of a practical rotational mechanical system to be modelled using MATLAB.

Like all DC motors, a PMDC motor (shown in Figure 1) also operates due to a current carrying conductor experiencing a mechanical force when placed in a magnetic field. In the case of the DC motor, the conductor is by the armature circuit, which provides a path for an armature current I_A to flow in response to an applied DC armature voltage V_A.

This current generates a magnetic field which, in turn, will interact with the magnetic field of the motor's permanent magnet to generate a mechanical force based on Fleming's Left Hand Rule. As the armature is usually mounted on a shaft that can rotate about a cylindrical axis, the mechanical force manifests in the form of a torque τ which causes the motor's shaft to rotate, thereby converting electrical energy to mechanical energy.

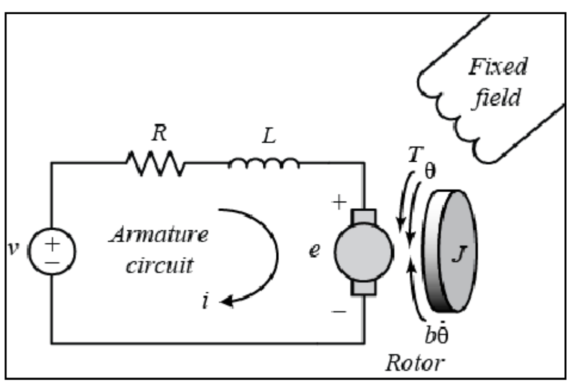


Figure 1: A simple diagram of a DC Motor

PMDC Motor as a Control System

A PMDC motor comprises of two distinct, but interrelated, control systems:

- A mechanical control system that deals with purely rotational kinematic parameters such as the rotor's total moment of inertia J, the applied and shaft torques τ, the shaft's angular speed of rotation ω, and the shaft's angular displacement θ.
- An electrical control system that is concerned primarily with the effects of an applied armature voltage V_A on the current i_A through the resistive and inductive components in the armature circuit.

The two systems are related by the armature current i_A, and a mathematical model of such a motor must be combine both, either in the form of a transfer function or in the form of a system of differential equations.

State Variables

There are two independent energy storage elements in the entire PMDC motor, namely the armature inductance L_A and the mass of the system itself, often represented in rotational dynamics by the moment of inertia J. The differential equations for both energy storage elements are as follows, and the differentiated variable in each case is a state variable. As there are two independent storage elements, there will also be two state variables and equations.

<u>Moment of Inertia</u>

$$\tau_{EQ} = J_{EQ}\frac{d^2\theta}{dt^2} \Rightarrow \tau_{EQ} = \boldsymbol{J_{EQ}\frac{d\omega}{dt}}$$

Therefore, the first state variable is the angular speed of the motor's shaft $x_1(t) = \omega(t)$

<u>Inductor</u>

$$\boldsymbol{V_L = L\frac{di_A}{dt}}$$

Therefore, the second state variable is the armature current $x_2(t) = i_A(t)$.

State Equations

First, consider the mechanical subsystem by equating the acting and opposing torques in the motor's shaft.

$$\tau = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \because \frac{d\theta}{dt} = \omega$$

$$\Rightarrow \tau = J\frac{d\omega}{dt} + \ B\omega \ \Rightarrow \frac{d\omega}{dt} = \frac{\tau}{J} - \frac{B}{J}\omega \because \tau = K_T i_A$$

State Equation (1)

$$\Rightarrow \boldsymbol{\frac{d\omega}{dt} = -\frac{B}{J}\omega \ + \ \frac{K_e}{J} i_A}$$

Next, consider the electrical subsystem and apply Kirchhoff's Voltage Law to the armature circuit

$$V = v_R(t) + v_L(t) + v_B(t)$$

Using Ohm's Law, the inductor equation, and the relationship between back EMF and angular rotation

$$V = i_A(t)R_A + L_A\frac{di_A}{dt} + K_E\omega \ \Rightarrow V = i_A(t)R_A + L_A\frac{di_A}{dt} + K_E\omega$$

State Equation (2)

$$\boldsymbol{\frac{di_A}{dt} = -\frac{K_E}{L_A}\omega - \frac{R_A}{L_A} i_A(t) + \frac{V}{L_A}}$$

State Space Representation

Combining the state equations into a system of linear equations and expressing them in matrix form yield the following state space representation for the system.

$$\boldsymbol{\begin{bmatrix} \frac{d\omega}{dt} \\ \frac{di_A}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{B}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \omega \\ i_A \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V}$$

The output state equation can similarly be expressed as

$$\boldsymbol{Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ I_A \end{bmatrix}}$$

MATLAB Implementation

Code Listings 1 and 2 show a MATLAB implementation of the PMDC motor's state space model as derived in the previous section. Code 1, `pmdc_test.m` that uses Code 2, `update_state_vars.m`, with MATLAB's built-in `ode45` solver to obtain a numerical solution to the system of differential state equations. The two state variables, rotational speed ω and armature current i_A, are then plotted on a MATLAB figure to visualize the system's response for a given set of parameters.

Code 01 - `pmdc_test.m`

Code 01 shows the program that is used to calculate and plot the system's response.

Code 1: `pmdc_test.m`

```
%%  FCS Open Ended Lab 01 - Saad Mashkoor Siddiqui, EE-16163, Section D
%   pmdc_test.m - uses a differential equation-based mathematical model of
%   a PMDC motor to plot armature current and shaft speed as state vars

clear all; close all; clc;
%% ODE Solver for d_omega/dt and dI_A/dt
X_0 = [0 0];                     % initially, both omega and I_A are 0

% sys_params are the electromechanical constants of the motor -
% [R L V_A B J K] - passed as array for streamlined testing
sys_params = ones(6);
t_span = [0, 5];                 % the entire system is observed for 5 seconds

% t stores time and X stores omega/current approximated by ODE solver
[t, X] = ode45(@(time, outputs) update_state_vars(time, outputs, ...
    sys_params), t_span, X_0);

%% Plot data
subplot(2, 1, 1); plot(t, X(:, 1), 'k'); grid;
title('PMDC Response - Angular Speed against Time');
xlabel('Time (\it{t/s})'); ylabel('Angular Speed \it({\omega/RPS})');

subplot(2, 1, 2); plot(t, X(:, 2), 'k'); grid;
title('PMDC Response - Armature Current against Time');
xlabel('Time (\it{t/s})'); ylabel('Armature Current (\it{i_A/A})');
```

The program first defines the initial values for both state variables in `X_0` as well as the values of the system's electromechanical parameters, namely armature resistance, armature inductance, armature voltage, rotational damping coefficient, moment of inertia, and arbitrary rotational constant/back EMF constant. As the model was made under the assumption that all initial conditions are 0, both the initial angular speed and initial armature current are set to 0. Similarly, as this model will serve as a baseline for further experimentation to determine system dependency on each dynamic variable, all system parameters were initialized to 1 through `sys_params = ones(7)`.

The program explores the system's response for a duration of 5 seconds as specified by `t_span`. It then passes the system parameters, initial condition, and a reference to the second program as arguments to MATLAB's built-in ODE45 solver, which calculates a numerical solution for the system of differential state equations, storing the results in `t` and `X`. `X` is a two-column matrix with the first and second columns representing values of angular speed and armature current respectively.

The program then plots both parameters on a MATLAB figure to visualize the PMDC motor's response.

Code 2 - `update_state_vars.m`

Code 2 is a simple MATLAB function which defines how the ODE45 solver should attempt to solve differential equations defined in the previous section. It calculates each differential using the previously calculated values of the corresponding state variables, and then appends these terms into a single updated state differential matrix `dX_dt` which is then returned to the caller.

Code 2: `update_state_vars.m`

```
%%  FCS Open Ended Lab 01 - Saad Mashkoor Siddiqui, EE-16163, Section D
%   update_state_vars.m - passed as a function argument to ODE45 solver in
%   pmdc_test.m along with values of PMDC electromechanical constants.
%   Used to update numerical solution of differential state equations

function dX_dt = update_state_vars(t, X, sys_params)
% dX/dt = [d_omega/dt, d_Ia/dt] and X = [omega, I_a]

R = sys_params(1);              % armature resistance
L = sys_params(2);              % armature inductance
V_A = sys_params(3);            % applied armature voltage
B = sys_params(4);              % rotational damping
J = sys_params(5);              % moment of inertia
K = sys_params(6);              % arbitrary rotational constant
```

```
dw_dt = (K/J) * X(2) - (B/J) * X(1);
dI_dt = (-R/L) * X(2) + (-K/L) * X(1) + V_A/L;
dX_dt = [dw_dt; dI_dt];
```

System Response

Figure 2 shows the PMDC motor's response generated by Codes 01 and 02 in terms of its angular speed and armature current for the baseline case of all electromechanical parameters being set to unity.

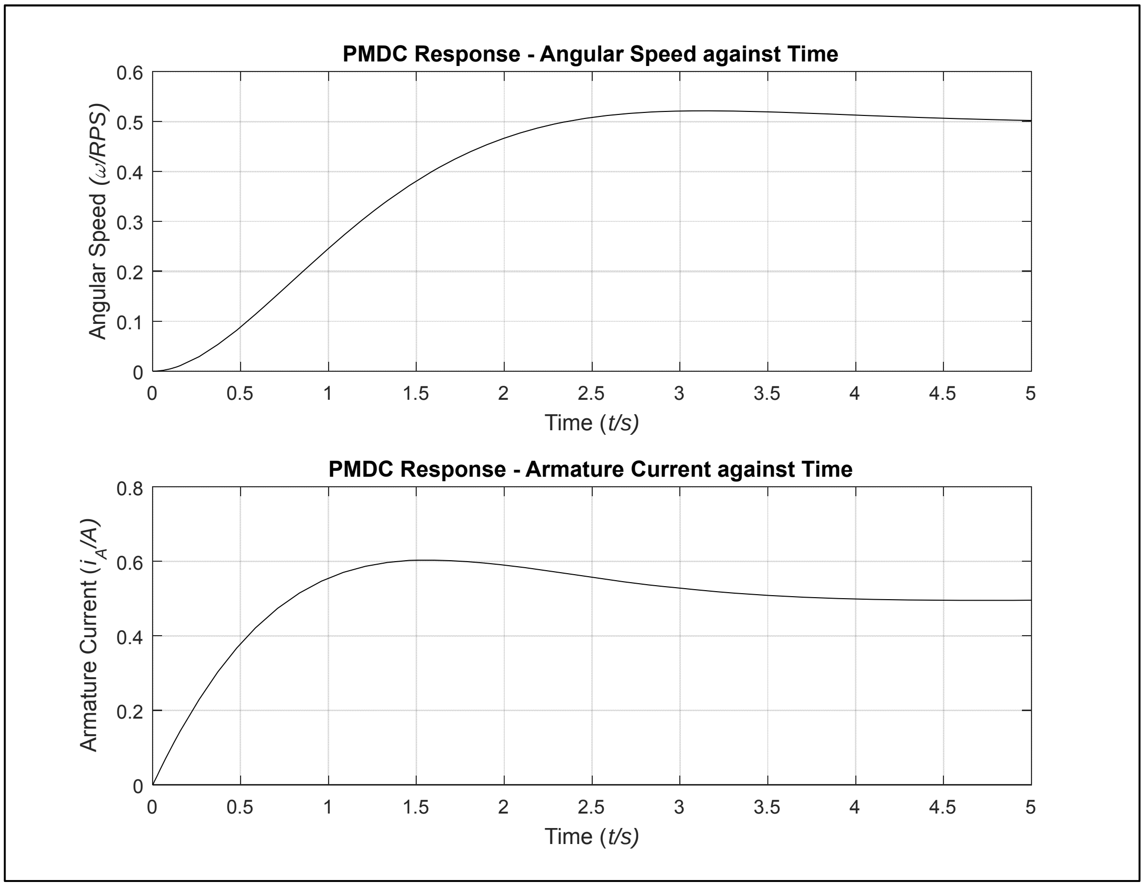


Figure 2: PMDC System Response (baseline case)

Analysis of Results

- Both the armature current and angular speed gradually increase from 0 to a maximum or peak value before settling at a more constant value towards by 5 seconds.

- The variation in both state variables is characteristic of a typical second order system's response: there is a rise time, a peak time, a maximum overshoot, as well as a settling time and delay time.
- Angular speed seems to have a lower overshoot than armature current, but has a higher rise time and peak time, suggesting the mechanical system may be more damped than the electrical one.
- Armature current response seems similar to an underdamped response, while the angular speed seems similar to an overdamped response.
- As the electrical system approaches its steady state, transient current has dissipated entirely. This means all steady state current is purely determined by the 1 Ω armature resistance.
- However, even though the armature resistance is 1 Ω and in series with a 1 V DC supply, the actual DC armature current is not 1/1 = 1 A. Instead, it is 0.5 A.
- This is most likely a consequence of back EMF reducing the effective DC voltage in the armature circuit. Concretely, the back EMF can be approximated to be 1 - 0.5 = 0.5 V.

Dynamic Element Dependency Investigation

Resistance

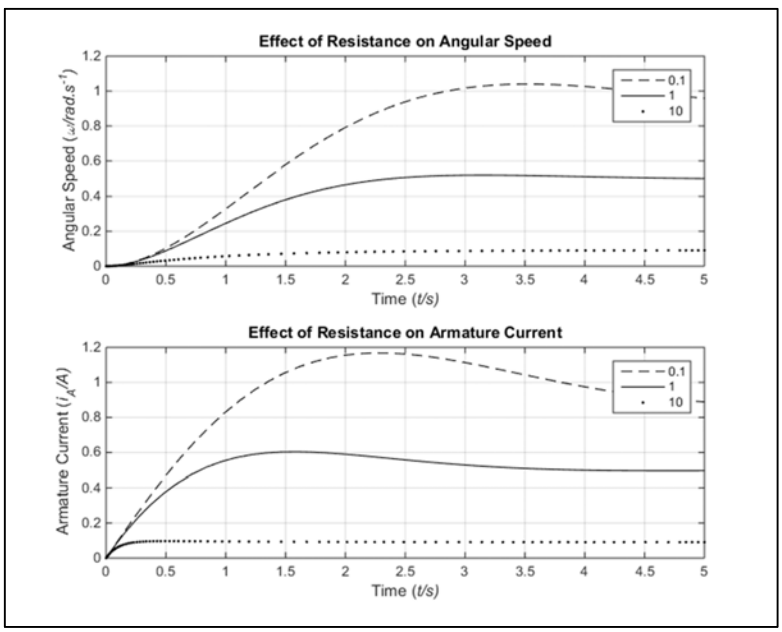


Figure 3: Effect of Resistance on PMDC Motor Response

Increasing the resistance from 1 Ohms to 10 Ohms decreases the peak and steady state values of the armature current, as the increased resistance offers more obstruction to current flow for the same applied voltage. It also decreases the rise time and peak time of the armature current, which makes sense because the time constant of the RL armature circuit is $\frac{L}{R}$, so the higher the resistance the smaller the time constant and the quicker the current's rise to its steady state value. The reverse is also true: decreasing the armature current by a factor of 10 leads to a higher peak and steady state armature current but also increases the rise time and peak time of the armature current response.

A similar trend can be observed in the angular speed. As the resistance increases, the armature current decreases and the rotational torque decreases, which therefore leads to a decrease in the speed. However, the angular speed also reaches its steady state value earlier than in the case of a smaller resistance, possibly because the effect of transient currents on the applied torque and thus the speed dissipates earlier.

Inductance

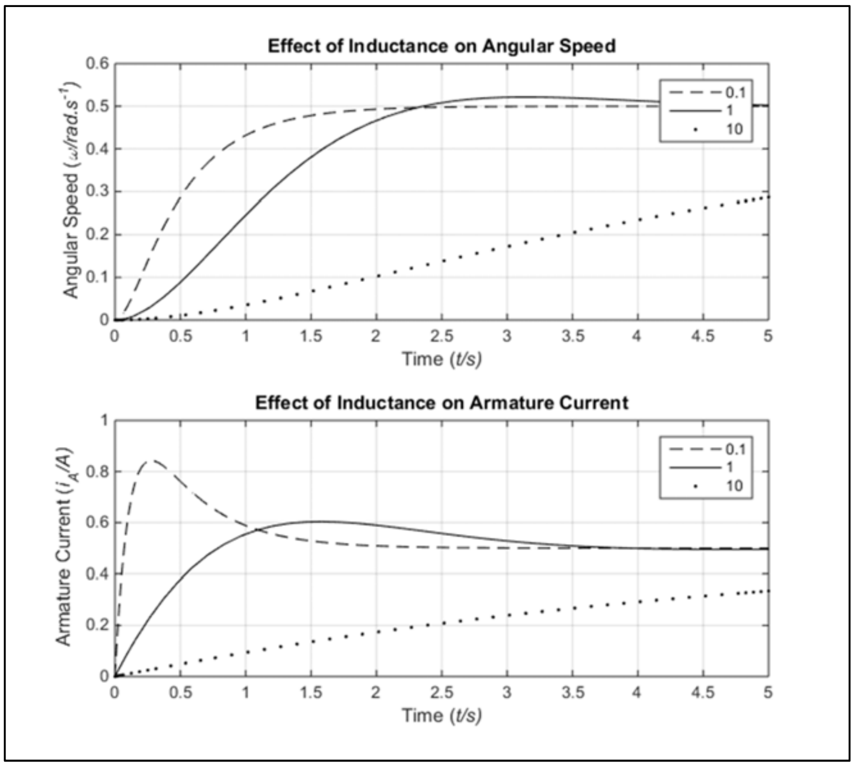


Figure 4: Effect of Inductance on PMDC Motor Response

Variation of armature inductance has a more pronounced and interesting effect on the system's armature current and angular speed responses. For both responses, a lower armature inductance results in a faster response that has lower rise time, delay time, and settling time, thus reaching its steady state value quickly. This is undoubtedly related to the time constant of the armature circuit. As the value of the inductance increases, the armature current and speed take longer to reach their steady state values. However, the steady state value of both state variables is the same for all values of inductance, suggesting that the inductance affects only the delay and not the magnitude in both responses.

However, the nature of the armature current and angular speed responses is different for the same value of inductance. For instance, a 0.1 H inductance leads to a very large maximum overshoot in case of the armature current that is characteristic of an underdamped response.

On the other hand, the angular speed shows virtually no overshoot for a 0.1 H inductance and rises smoothly to its steady state value, which is indicative of a critically damped response.

Armature Voltage

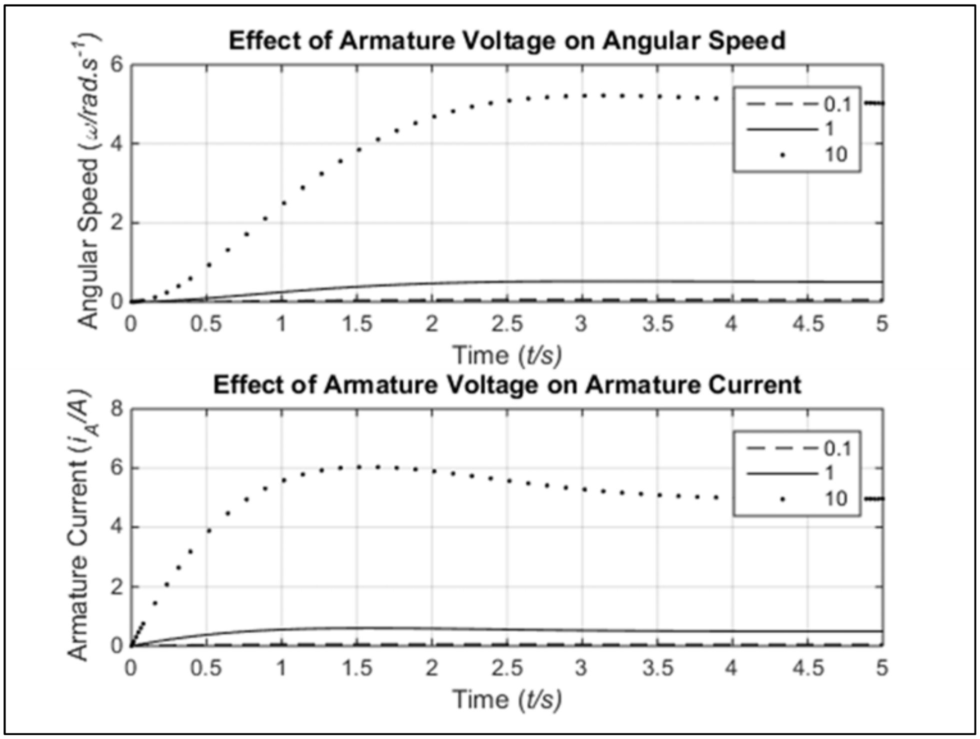


Figure 5: Effect of Armature Voltage on PMDC Motor Response

Armature voltage has a pronounced effect on the magnitudes of the armature current and angular speed. The higher the armature voltage, the higher the magnitude of the steady state armature current that flows through the circuit and thus the angular speed.

The increase in angular speed and armature current corresponding to a tenfold increase in armature voltage is much more substantial than the change corresponding to a tenfold decrease, which indicates a non-linear relationship between these parameters. The observation about back EMF reducing the effective armature voltage still applies and can be seen in all three cases – the steady state current caused by a 0.1/1/10V DC source powering a 1 Ohm resistor is always less than the magnitude of the DC voltage itself.

Moment of Inertia

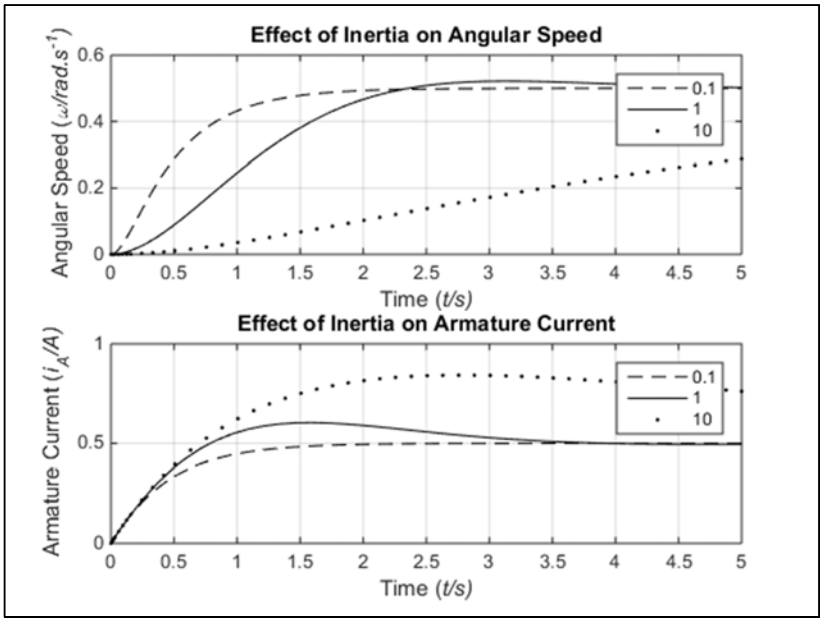


Figure 6: Effect of Inertia on PMDC Motor Response

The higher the moment of inertia, the greater the reluctance of the motor to change its state of rest or motion, and thus the lower the rate of hange in the motor's angular speed. This is why the lowest moment of inertia leads to the fastest angular speed response which gradually rises from 0 to reach its steady state value by 2 s. When the moment of inertia is 10 times the base value, the motor still hasn't reached its steady state speed by the end of 5s. However, the final angular speed for all three cases is the same, suggesting that moment of inertia only affects the speed of the response and not its steady state value.

The smaller the moment of inertia, the less time taken by the armature circuit's current to reach its steady state value. Increasing the moment of inertia increases both the time taken to reach the steady state armature current as well as the overshoot – with a moment of inertia of 10 kg.m.s^2, the current overshoots to almost 1 A before gradually decreasing to (presumably) the same steady state armature current.

Damping Coefficient

The lower the damping coefficient, the higher the steady state speed of the motor. This can be seen in the first graph of Figure (xx), where a damping coefficient of 0.1 leads to a steady state speed of 1 rad.s^-1, with the steady state speed decreasing by approximately 0.5 rad.s^-1 with every tenfold increase in the damping coefficient. This makes sense because a lower damping coefficient means a lower net opposing torque acting on the motor, which means the same armature voltage, armature current, and resulting applied torque can achieve a larger rated speed.

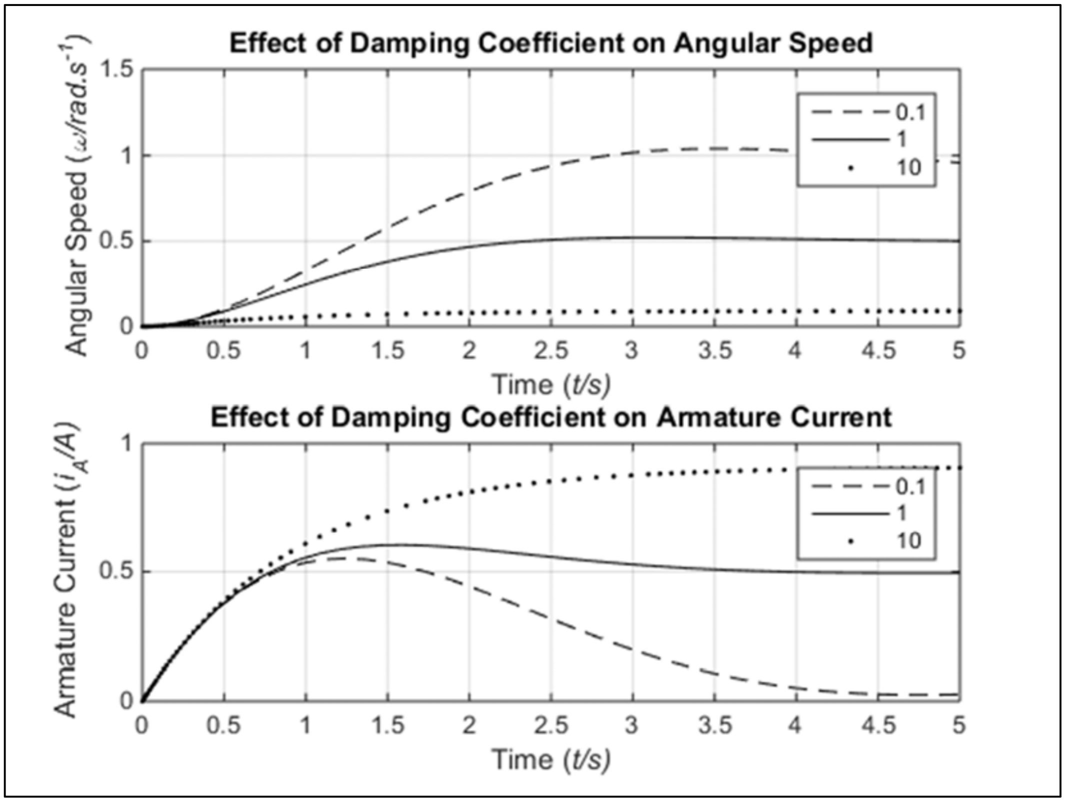
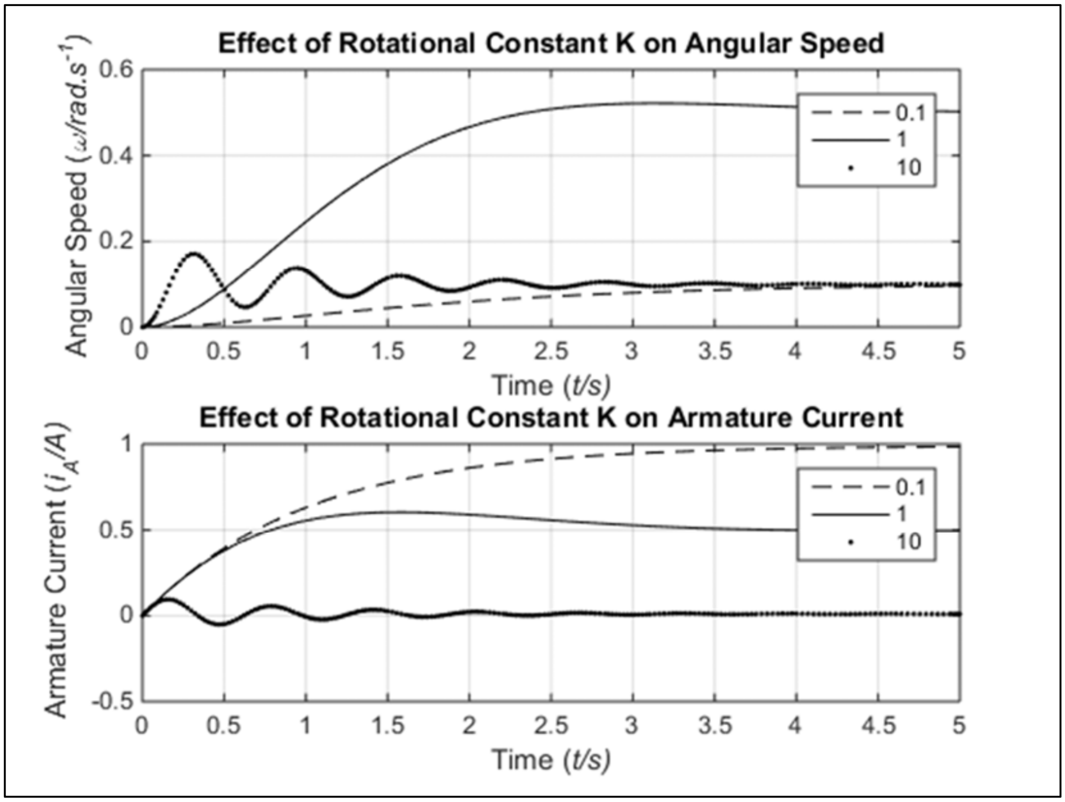


Figure 7: Effect of Damping Coefficient on PMDC Motor Response

Armature current shows a slightly different response. Increasing the damping increases the steady state current, with smaller damping coefficients characterized by a sharp maximum overshoot. For instance, with a damping coefficient of 0.1, the peak armature current is 0.5 A whereas the steady state current is only 0.05 A. This can be explained by the relationship between torque and armature current. The higher the damping coefficient, the greater the torque required to overcome the damping torque, and since torque is proportional to armature current, this therefore leads to a higher armature current being drawn.

The current drawn during the first 0.75s of the observation period is almost the same regardless of the damping coefficient. This could be due to the inrush current drawn by the motor being the same, regardless of damping torque, as the motor tries to overcome rotational inertia by generating a high torque. However, the difference between the steady state and initial torques varies depending on the opposing damping torque, which is why the three curves have different shapes.

Rotational Coefficient/Back EMF Coefficient



The smaller the rotational/back EMF constant, the lower the magnitude of the back EMF that acts on the armature circuit. This, in turn, leads to a smaller decrease in the effective armature voltage and therefore leads to a higher magnitude of steady state armature current. This is evident from the current's variation in the figure shown above: the steady state current approaches the same magnitude as V_A as the back EMF constant approaches 0.1 as the back EMF becomes almost negligible in this case.

However, this constant is also associated with the torque applied on the rotor since $\tau = K_T I_A$ and for SI units $K_T = K_E$. A higher value of this constant therefore means a higher torque and back EMF, but reduced speed. When the back EMF is very large, the net armature voltage becomes smaller, as does the steady state current it generates in the armature circuit. This makes the transients generated by the back EMF considerably more significant in terms of relative magnitude, and thus the armature current they generate shows a clear decaying oscillatory behaviour until the transient voltage dissipates entirely. The transient currents will likewise lead to transient, oscillating torque and thus similarly oscillating speed.