Lab Session 04 – Mathematical Modelling of Electrical Systems

Exercise 01

For each of the electrical systems shown below

- **Plot the voltage across the capacitor $v_C(t)$ and the current through the inductor $i_L(t)$ in separate graphs using the `subplot` command.**
- **Change the values of each parameter, such as**
  - **Voltage source (e)**
  - **Resistance R**
  - **Inductance L**

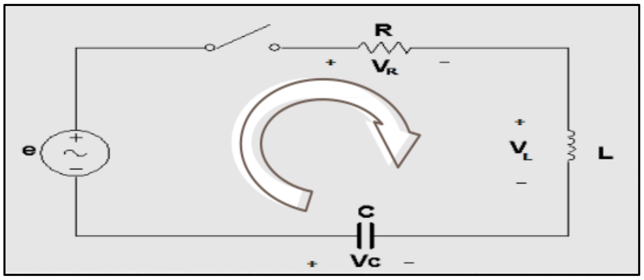


Figure 1(a): System (a) for Exercise 1

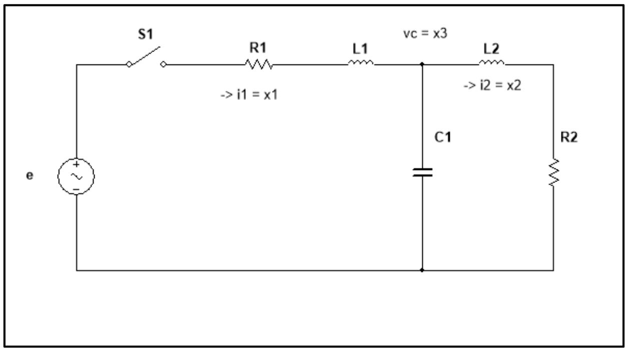


Figure 1(b): System (b) for Exercise 1

System A

Initially assumed $e = 60$ V, $R = 10\ \Omega$, $L = 1$ H, and $C = 10\ F$.

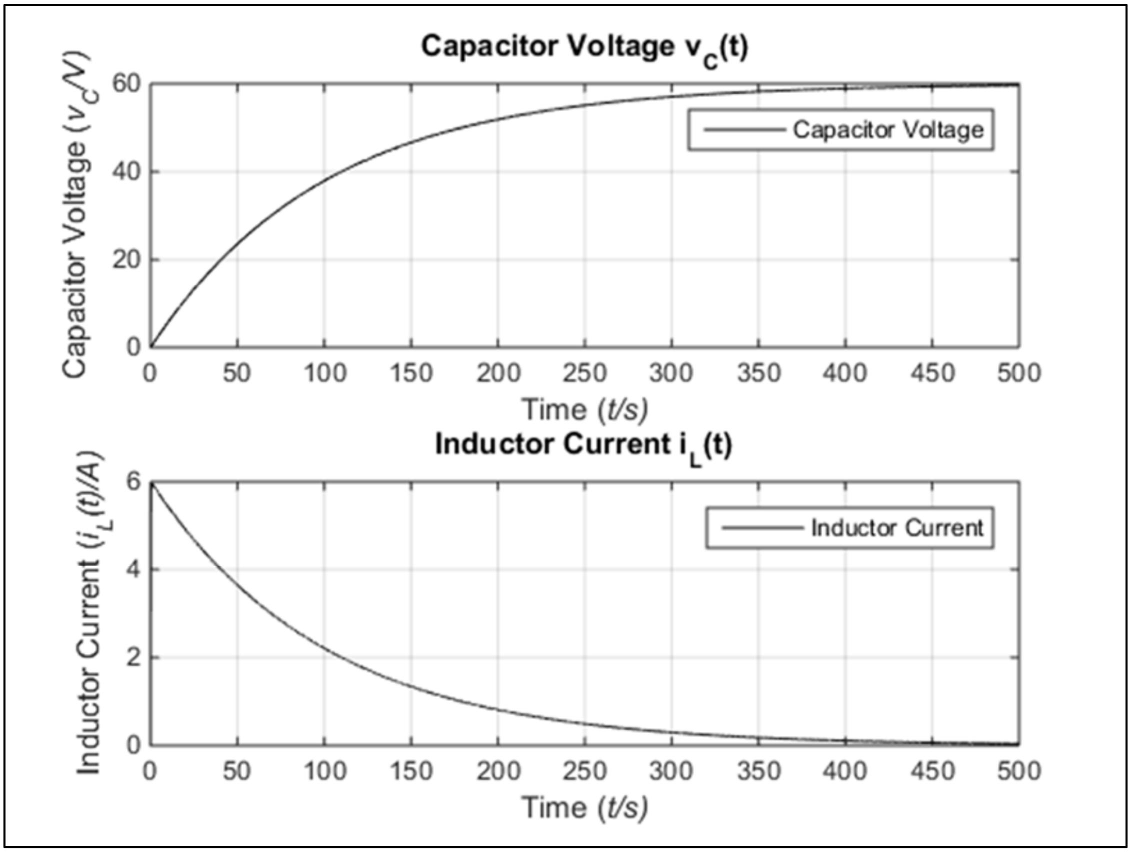


Figure 2: Response for System (a) with default parameters.

Changed the value of e, R, L, C and investigated the changes in the capacitor voltage and inductor current. The values investigated for each parameter are tabulated below.

Varied Quantity	Values			Constant 1		Constant 2		Constant 3	
	1	2	3	Qty	Value	Qty	Value	Qty	Value
e (Volts)	6	60	600	R	10	L	1	C	10
R (Ohms)	1	10	100	e	60	L	1	C	10
L (Henries)	0.1	1	10	e	60	R	10	C	10
C (Farads)	1	10	100	e	60	R	10	L	1

Effect of Varying Supply Voltage e

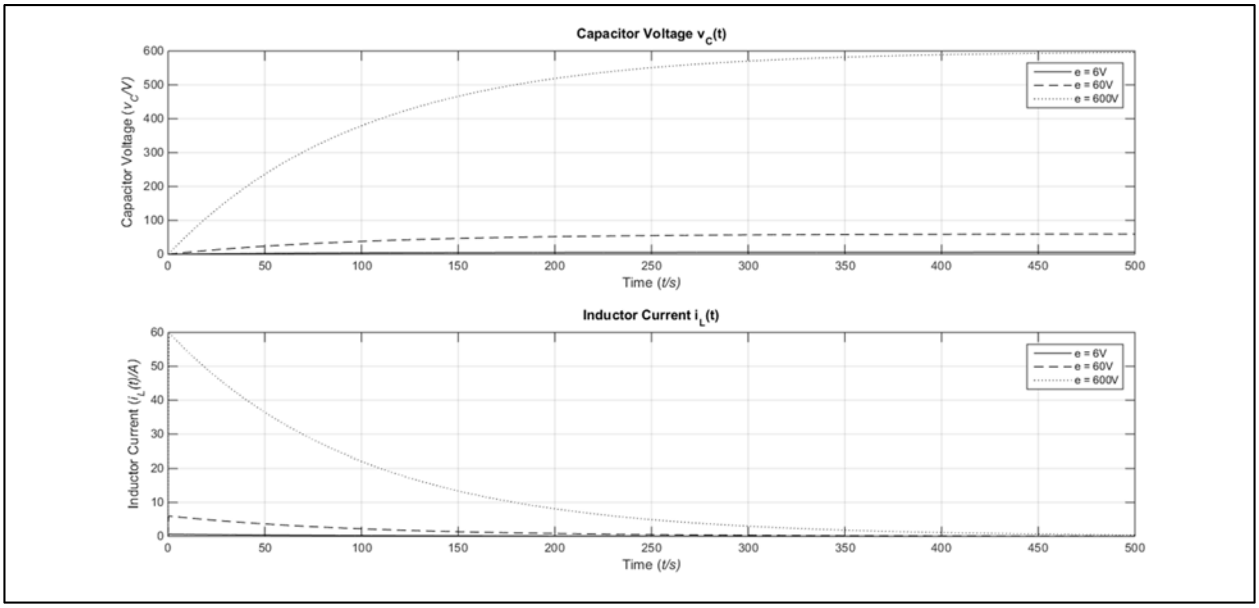


Figure 3: Effect of e on System (a) Response

Effect of Varying Resistance R

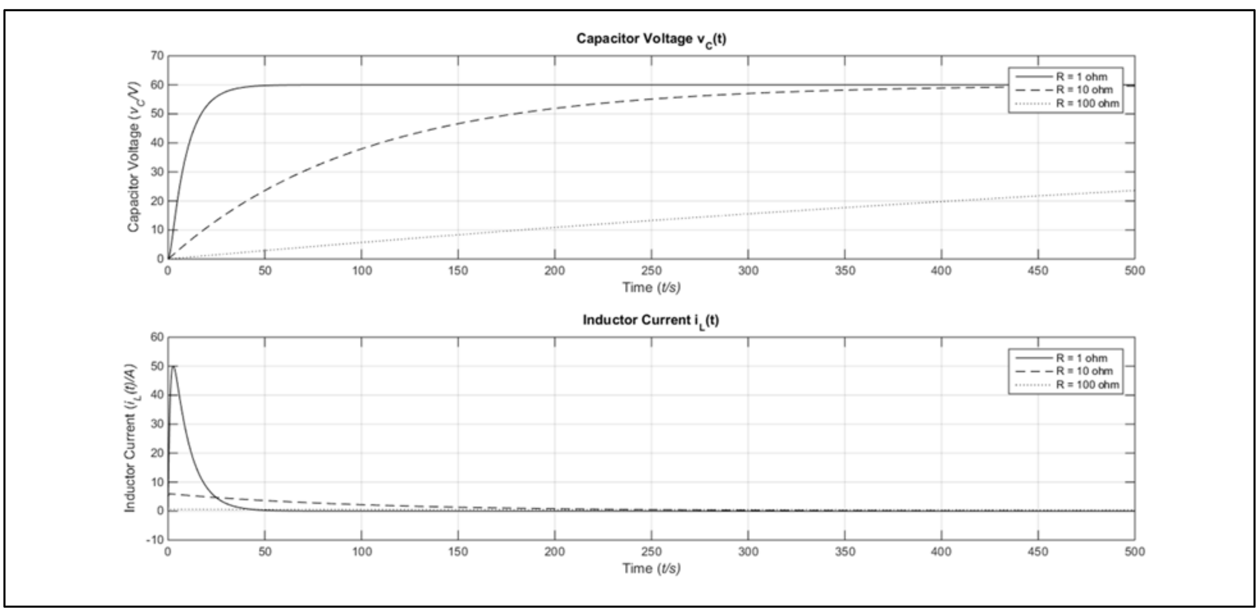


Figure 4: Effect of R on System (a) Response

Effect of Varying Inductance L

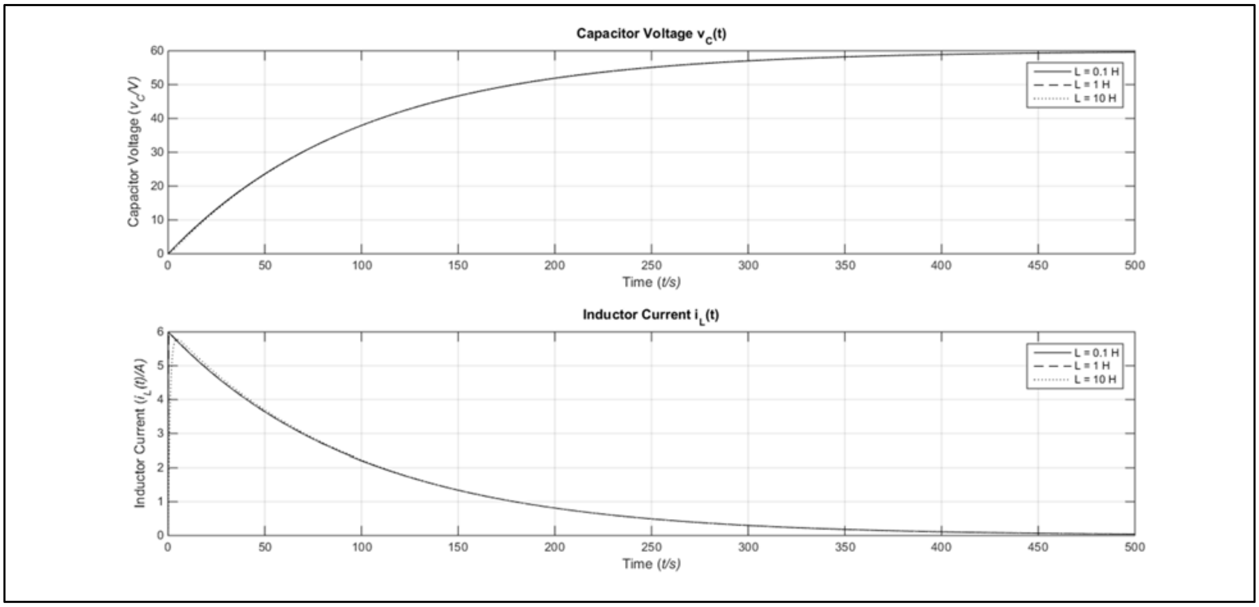


Figure 5: Effect of L on System (a) Response

Effect of Varying Capacitance C

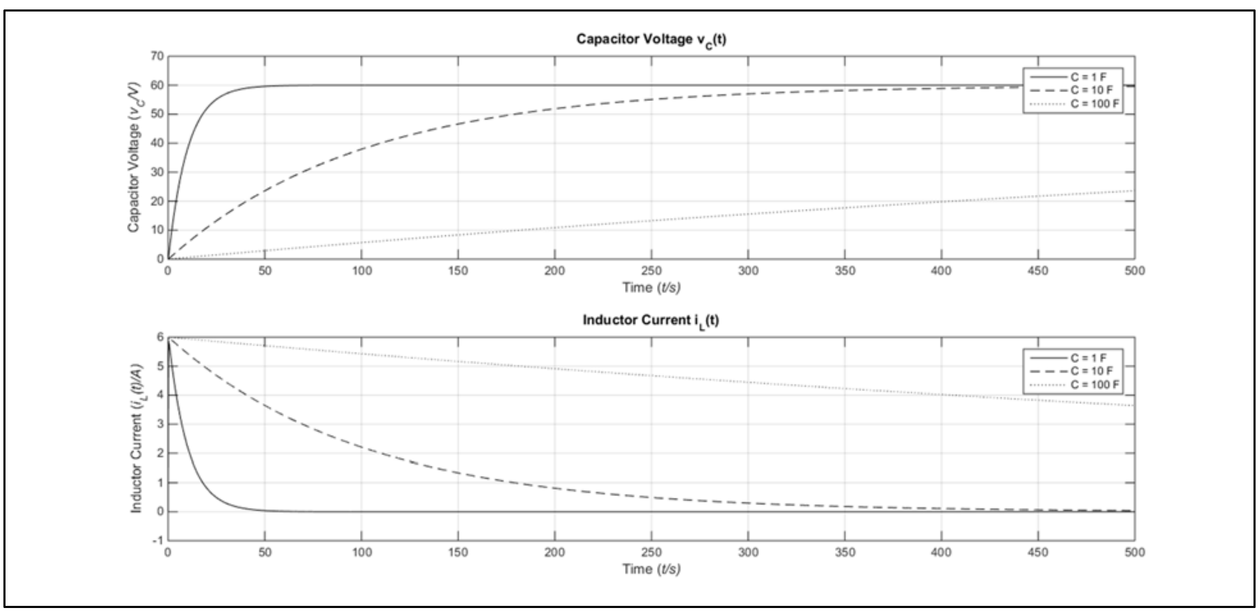
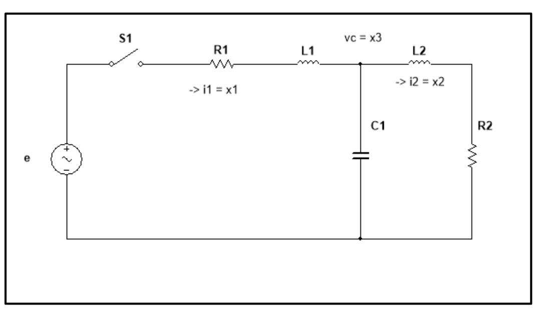


Figure 6: Effect of C on System (a) Response

Summary of Results

- Increasing supply voltage causes the capacitor voltage's steady state value to increase. However, the time taken for the capacitor to charge to its steady state voltage also increases.
- Initially, the inductor current is 0 in all investigations. This is because the inductor is in series with the power source and other components, and as a result cannot allow the system's current to change instantaneously, regardless of the system's other components.
- The inductor current is always initially 0, and then rises very quickly to its maximum value before eventually decaying to its steady state value.
- Increasing supply voltage increases the maximum value of the inductor current. However, in all cases the steady state inductor current is always 0 A.
- The smaller the value of R, the quicker the rate at which the capacitor charges to its steady state voltage.
- Smaller value of R also corresponds to a larger maximum value of the inductor current, as well as the rate at which this current decays to its steady state value of 0 A. This is because a smaller R means a smaller overall circuit impedance (hence larger maximum current) and a smaller time constant (hence faster charging and discharging of the capacitor).
- Changing inductance does not seem to have a marked effect on the capacitor voltage, but has a slightly more significant effect on the inductor current. Smaller inductance corresponds to marginally higher current.
- A smaller capacitance corresponds to a shorter time duration over which the capacitor charges to its steady state voltage. It also leads to faster decay of the inductor current to its steady state value.
- The steady state capacitor voltage is dependent only on the supply voltage, and the inductor current's maximum value depends on both the supply voltage and the resistance (with some evidence to support its dependence on the inductance and capacitance).

System B



In this system, there are three independent energy storage elements, namely inductors L_1 and L_2 along with capacitor C. Writing differential equations describing I-V relationships for all three components

$$v_{L1}(t) = L_1 \frac{di_1(t)}{dt} \ldots (1)$$
$$v_{L2}(t) = L_2 \frac{di_2(t)}{dt} \ldots (2)$$
$$i_C(t) = C \frac{dv_3(t)}{dt} \ldots (3)$$

The differentiated variables in (1), (2), and (3) are $i_1(t)$, $i_2(t)$, and $v_3(t)$ are the state variables for this system.

Expressing the LHS of (1), (2), (3) in terms of state variables and the input voltage $e(t)$.

<u>Mesh Analysis for Mesh Containing L_1 and R_1</u>

$$-e(t) + v_{R1}(t) + v_{L1}(t) + v_C(t) = 0$$
$$e(t) = v_{R1}(t) + v_{L1}(t) + v_C(t)$$
$$v_{L1}(t) = e(t) - v_{R1}(t) - v_C(t)$$

As $v_{R1}(t) = i_1(t)R_1$ and $v_C(t) = v_3(t)$

$$\boldsymbol{v_{L1}(t) = e(t) - v_{R1}(t) - v_C(t) \ldots (4)}$$

<u>Mesh Analysis for Mesh Containing $L_2, C, and\ R_2$</u>

$$0 = -v_C(t) + v_{L2}(t) + v_{R2}(t)$$
$$v_{L2}(t) = v_C(t) - v_{R2}(t)$$

As $v_C(t) = v_3(t)$ and $v_{R2}(t) = i_2(t)R_2$

$$\boldsymbol{v_{L2}(t) = v_3(t) - i_2(t)R_2 \ldots (5)}$$

<u>Nodal Analysis for Node 3</u>

$$\boldsymbol{i_3(t) = i_1(t) - i_2(t) \ldots (6)}$$

Substituting (4), (5), (6) back into (1), (2), and (3) respectively

$$\boldsymbol{\frac{di_1(t)}{dt} = \frac{e(t)}{L_1} - \frac{R_1}{L_1} i_1(t) - \frac{v_3(t)}{L_1} \ldots (7)}$$

$$\boldsymbol{\frac{di_2(t)}{dt} = \frac{v_3(t)}{L_2} - \frac{R_2}{L_2} i_2(t) \ldots (8)}$$

$$\boldsymbol{\frac{dv_3(t)}{dt} = \frac{i_1(t)}{C} - \frac{i_2(t)}{C} \ldots (9)}$$

Expressing these state equations in matrix form $\dot{x} = Ax + Bu$

$$\boldsymbol{\begin{bmatrix} \frac{di_1(t)}{dt} \\ \frac{di_2(t)}{dt} \\ \frac{dv_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \\ v_3(t) \end{bmatrix} + e(t) \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}}$$

This system of differential equations was solved using MATLAB's ODE45 solver using a function `RLC_sys_b.m` which was called by a test script `RLC_sys_b_test.m`. Assumed the following parameters $R_1 = 10\ \Omega$, $R_2 = 10\ \Omega$, $L_1 = 1$ H, $L_2 = 1$ H, and $C = 10\ F$.

Exercise 01(b), Code 01 - RLC_sys_b.m

```
function dXdt = RLC_sys_b(t, X)
e = 60;                 % supply voltage in volts
R1 = 10;                % in ohms
R2 = 10;
L1 = 1;                 % in Henries
L2 = 1;                 % in Henries
C = 10;                 % in Farads

% dXdt = [di1/dt; di2/dt; dv3/dt]
% X = [i1; i2; v3] = [L1 current; L2 current; C voltage]
dXdt(1, 1) = e/L1 - X(1) * (R1 / L1) - X(3) / L1;
dXdt(2, 1) = X(3)/L2 - X(2) * (R2 / L2);
dXdt(3, 1) = X(1)/C - X(2) / C;
```

Exercise 01(b), Code 01 - RLC_sys_b.m

```
clear all; close all; clc;

%% ODE Solver for i1, i2, v3
X0 = [0 0 0];           % init values - i1, i2 v3
[t, X] = ode45('RLC_sys_b', [0, 500], X0);

%% Plot L1 Current i1
subplot(3, 1, 1); plot(t, X(:, 1), 'k'); xlabel('Time (\it{t/s})');
ylabel('L1 Current (\it{i_{L1}(t)/A})'); title('Current i_1(t)'); grid;

%% Plot L2 Current i2
subplot(3, 1, 2); plot(t, X(:, 2), 'k'); xlabel('Time (\it{t/s})');
ylabel('L2 Current (\it{i_{L2}(t)/A})'); title('Current i_2(t)'); grid;

%% Plot C Voltage v3
subplot(3, 1, 3); plot(t, X(:, 3), 'k'); xlabel('Time (\it{t/s})');
ylabel('C Voltage (\it{v_{C}(t)/V})'); title('Voltage v_3(t)'); grid;
```

The output generated by these programs is shown in the figure below.

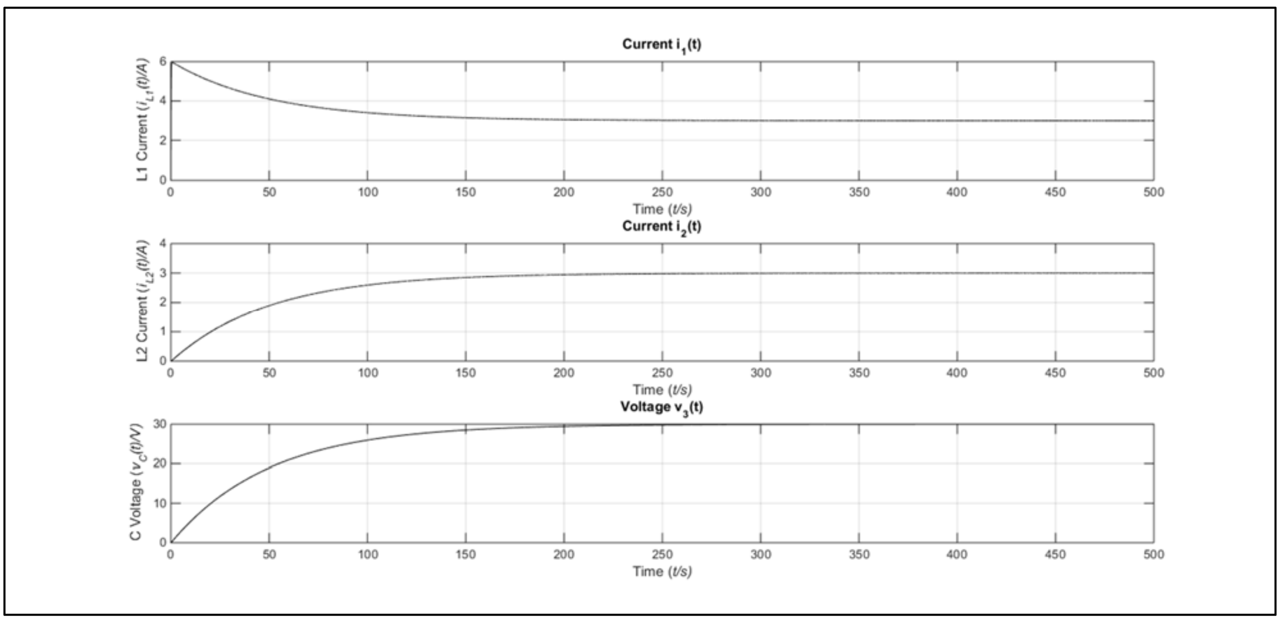


Figure 7: Default Response for System 1(b)

The same programs were used to investigate the effect of varying each of the system's parameters on the system's state variables/response. The values of the parameters used in this investigation is tabulated below.

Effect of Varying Resistances

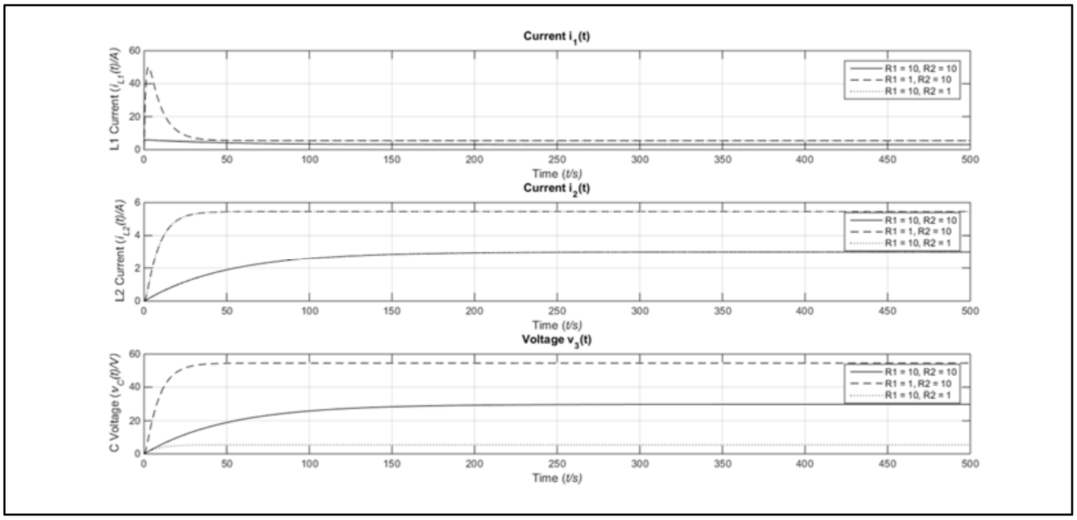


Figure 8: Effect of Resistance on System (b)

Effect of Varying Inductance

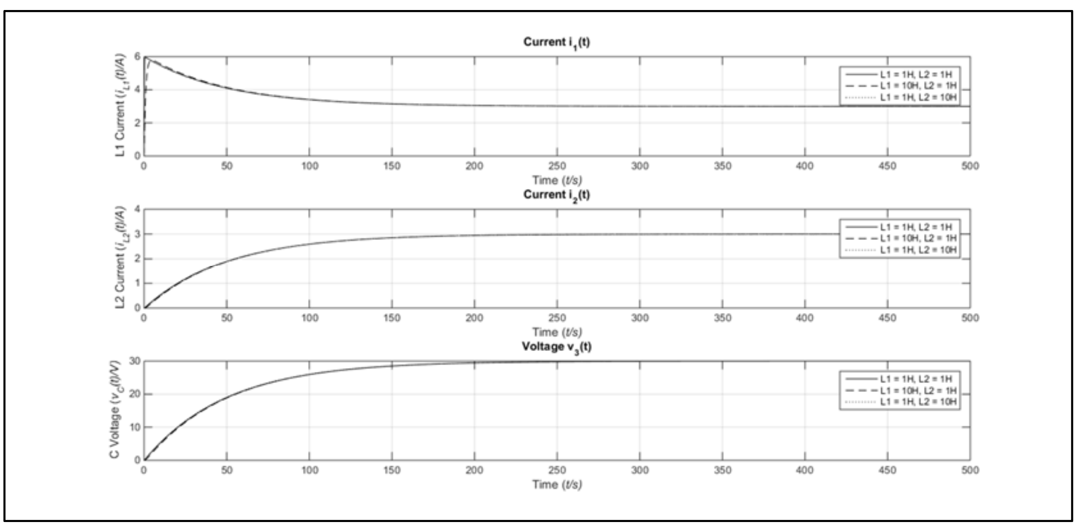


Figure 9: Effect of Inductance on System (b)

Effect of Varying Supply Voltage

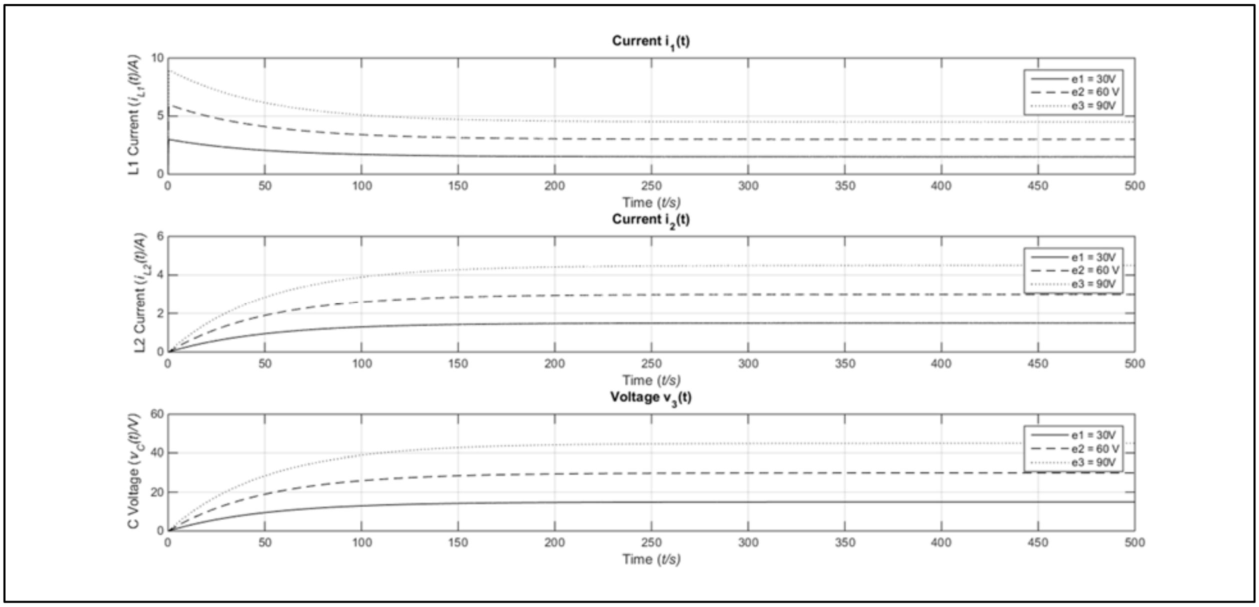


Figure 10: Effect of Supply Voltage on System (b)

Effect of Varying Capacitance

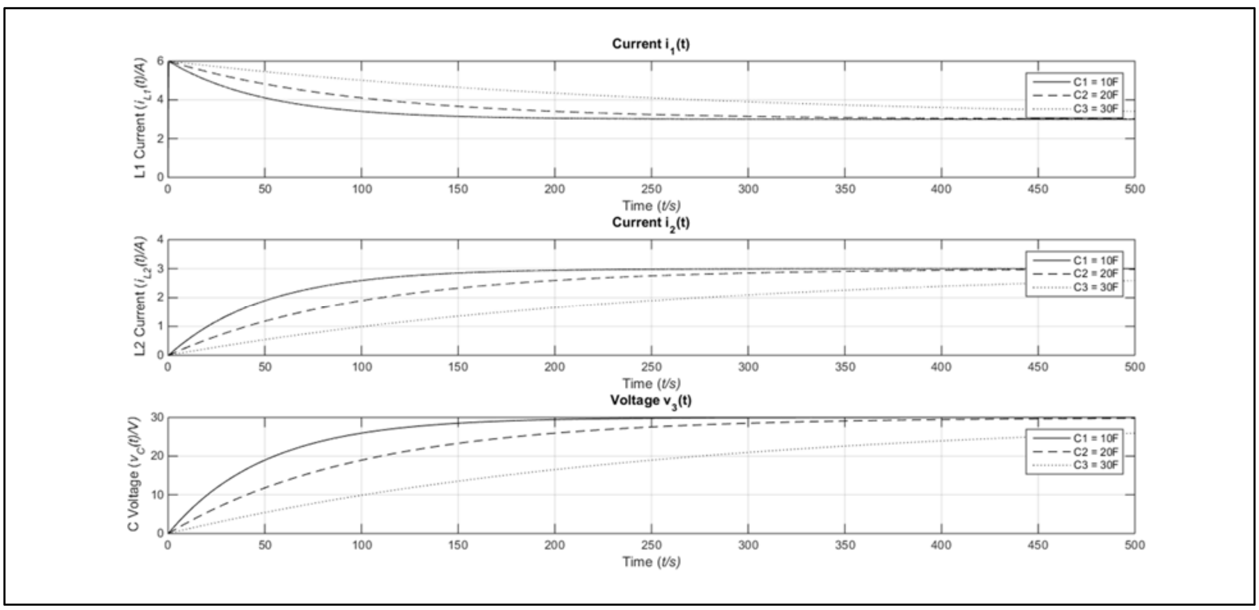


Figure 11: Effect of Supply Voltage on System (b)

Exercise 02 (From Lab Presentation) – RLC System

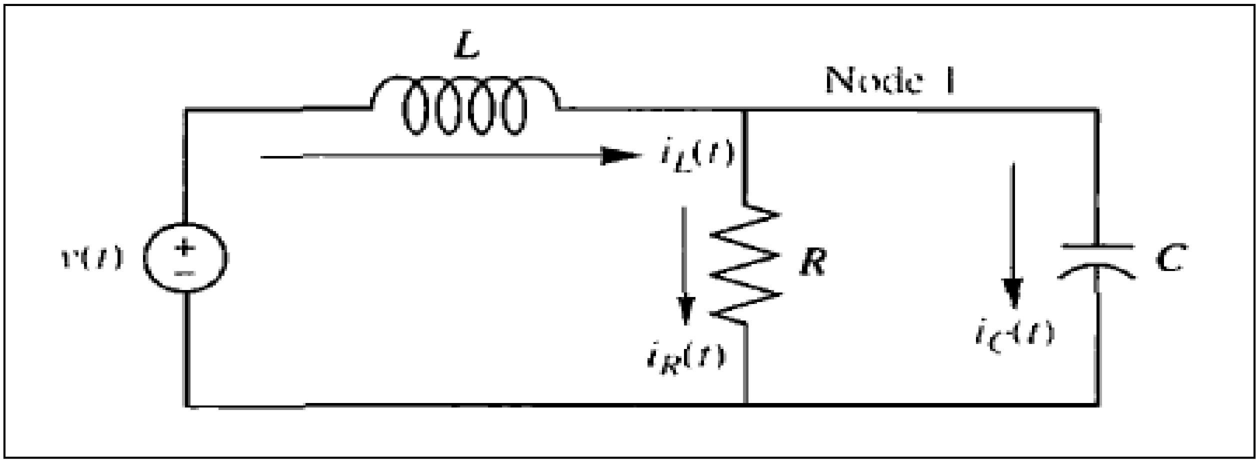


Figure 12: RLC System for Task 02

Write the differential equations for the circuit given in the figure above. Using the inductor current $i_L(t)$ and capacitor voltage $v_C(t)$ as state variables, implement and solve the state equation using MATLAB, and plot v_C and i_L as separate graphs for a circuit with supply voltage e of 50 V, R of 100 Ω, C of 1 F, and L of 10 H.

Writing differential equations for the two independent energy storage elements in the circuit, namely the inductor L and capacitor C.

$$v_L(t) = L\frac{di_L}{dt} \dots (1)$$
$$i_C(t) = C\frac{dv_C}{dt} \dots (2)$$

Assuming the differentiated variables i.e. $i_L(t)$ and $v_C(t)$ as state variables and expressing the RHS of each of (1) and (2) in terms of the state variable and the input voltage $v(t)$.

Using Kirchhoff's current law, $i_C = i_L - i_R$ and $\boldsymbol{i_R = \frac{V_R}{R} = \frac{V_C}{R}} \because V_C = V_R$ as the capacitor and resistor are parallel to one another.

$$\boldsymbol{i_C = i_L - \frac{V_C}{R}}$$

Using Kirchhoff's voltage law in the mesh containing the voltage source $v(t)$, the inductor, and resistor

$$
\begin{aligned}
0 &= -v(t) + v_L(t) + v_R(t) \\
v(t) &= v_L(t) + v_R(t) \\
v_R(t) &= v_C(t) \\
\boldsymbol{v(t)} &= \boldsymbol{v_L(t)} + \boldsymbol{v_C(t)}
\end{aligned}
$$

Substituting these equations back into (1) and (2) to express the differential equation in terms of the state variables.

For the differential equation describing the capacitor

$$
\begin{aligned}
C\frac{dv_C}{dt} &= i_L - i_R \\
\frac{dv_C}{dt} &= \frac{i_L}{C} - \frac{v_C}{RC} \dots (3)
\end{aligned}
$$

For the differential equation describing the inductor

$$
\begin{aligned}
L\frac{di_L}{dt} &= v(t) - v_C(t) \\
\frac{di_L}{dt} &= \frac{v(t)}{L} - \frac{v_C(t)}{L} \dots (4)
\end{aligned}
$$

The output is the current through the resistor R $i_R(t) = i_L(t) - i_C(t)$

$$
\Rightarrow \frac{V_C}{R} = i_L(t) - i_C(t) \dots (5)
$$

Using (3), (4), and (5) to derive state space representation of the system $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}$

$$
\begin{bmatrix} \frac{dv_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)
$$

The state equation is implemented in MATLAB using the function provided in Code 01, with the built-in ODE45 solver used to solve the system for the state variables for given initial conditions.

Task 02, Code 01 - RLC.m

```
function dXdt=RLC(t,X)
e=50;                                   % supply voltage in volts
R=100;                                  % resistance in Ohms
L=10;                                   % inductance in Henries
C=1;                                    % capacitance in Farads
dXdt(1,1)=(-1/R*C)*X(1)+(1/C)*X(2);     % computing dvc/dt
dXdt(2,1)=(-1/L)*X(1)+(1/L)*e;          % computing dvl/dt
```

Task 02, Code 02 - RLC_test.m

```
Clear; close; clc;
X0 = [0 0];                             % initial cap voltage and inductor current
[t,X]=ode45('RLC',[0 500],X0);
subplot(2,1,1);  plot(t,X(:,1));  legend('Vc');  grid on;  title('Vc');
subplot(2,1,2);  plot(t,X(:,2));  legend('i');   grid on;  title('i');
```

Graphs of Capacitor Voltage and Inductor Current

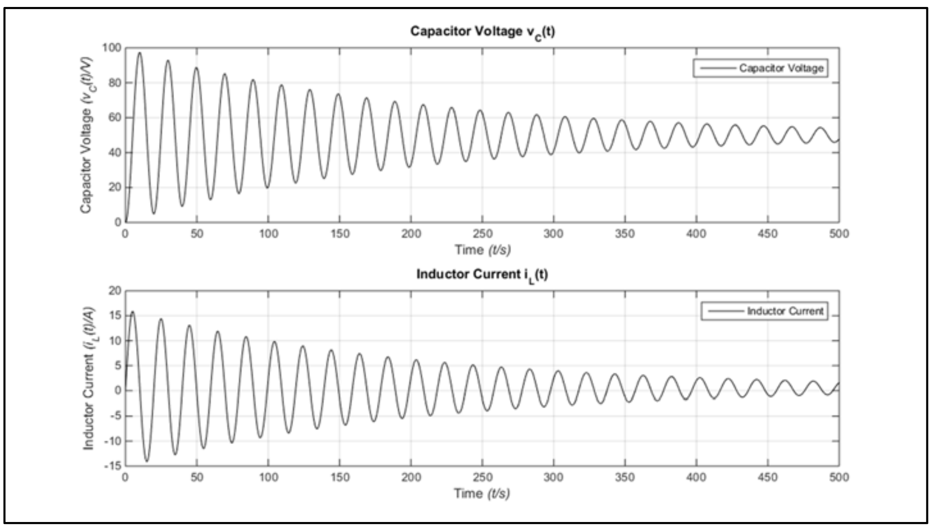


Figure 13: RLC System Response

Analysis of Results

- Both the capacitor voltage and the inductor current experience rapid oscillations whose amplitude decays with time. This shows the system is not particularly stable.

- In 500 seconds, neither capacitor voltage nor inductor current have reached their mean or steady state values. However, both have decayed sufficiently for their steady state values to be approximated from the graph (50 V and 0.49A respectively). This is confirmed by extending simulation period to 2000 s and examining the plots in the figure below.

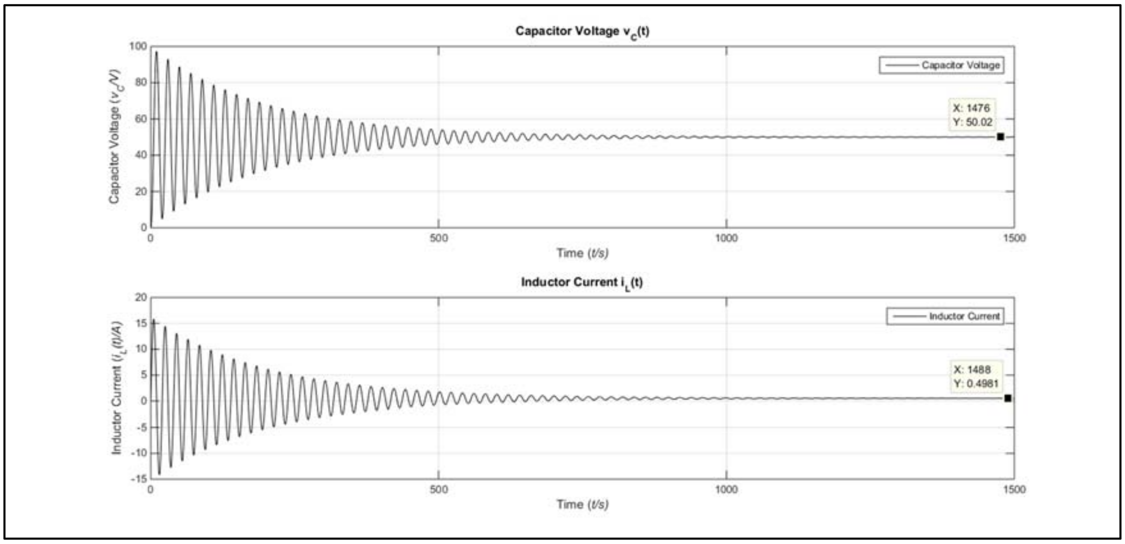


Figure 14: Extended Simulation Results to Investigate Steady State Values

- The oscillations represent the system's transient response as expressed through its state variables. Both capacitor voltage and inductor current show a clear exponential decay.
- In the mean/steady state condition, the capacitor C acts as an open circuit and the inductor L acts as a short circuit.
- This means no current goes through the capacitor branch, which means the current through the inductor is the same as the current through the inductor.
- This is why the theoretical steady state inductor current is simply $\frac{e}{R} = \frac{50}{100} = 0.5$A, which is confirmed by MATLAB's steady state inductor current value of 0.49 A.

This is also why the steady state capacitor voltage is the same as the supply voltage of 50 V – with the inductor acting as a short circuit, 0 V are dropped across it, causing all of the supply voltage to appear across the resistor (and the parallel capacitor