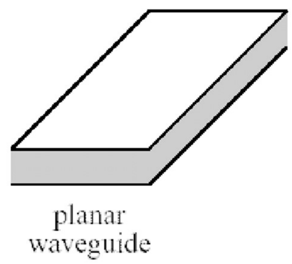
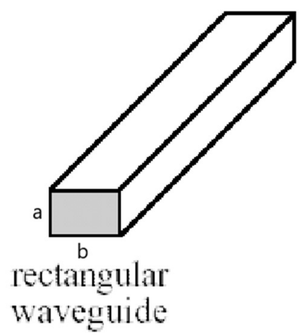
Q:

a) A flat metal waveguide is assumed, find its modes for that waveform.



b) A metal waveguide channel with a rectangular cross-section of length a and width b is assumed. Find its modes for that waveform.

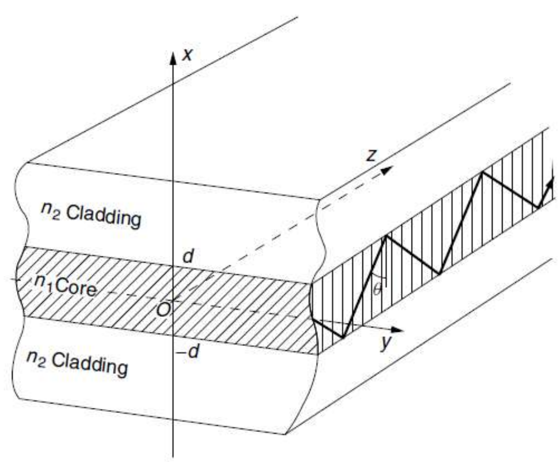


Sol:

a) In TM mode, we check the release:

$$E = E_0(x,y)e^{i(\beta z - \quad)}$$

$$H = H_0(x,y)e^{i(\beta z - \omega\ )}$$



Because we are in TM mode:

$$H_z = H_x = 0, E_y = 0$$

Based on the Helmholtz equation:

$$\nabla^2 H + \left(n_{1,2}k\right)^2 H = 0 \rightarrow \nabla^2 H + \left(n_{1,2}^2 k^2 - \beta^2\right) H = 0$$

$$n_1 k > \beta > n_2 k$$

$$n_1^2 - \beta^2 = K^2 \qquad 0 < x < d$$

Let's solve the above equation only inside the core:

$$H_y = A\cos(Kx) + B\sin(Kx)$$

$$H_y(x) = 0 \quad x = 0, x = d \rightarrow \boldsymbol{H_y(x) = B\sin\left(\frac{m\pi}{d}x\right)}$$

$$\boldsymbol{H_y(x,y,z,t) = B\sin\left(\frac{m\pi}{d}x\right) e^{i(\beta z - \omega t)}}$$

Now, based on Maxwell's equations, we can get the rest of the equations:

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = -\epsilon * i\omega(\boldsymbol{E_x}\hat{x} + \boldsymbol{E_z}\hat{z}) \rightarrow \begin{cases} -\frac{\partial H_y}{\partial z} = -\epsilon * i\omega \boldsymbol{E_x} \\ \frac{\partial H_y}{\partial x} = -\epsilon * i\omega \boldsymbol{E_z} \end{cases}$$

$$\begin{cases} \boldsymbol{E_x(x,y,z,t)} = \frac{i\beta H_y}{\epsilon * i\omega} = \frac{\beta}{\epsilon * \omega} H_y = \boldsymbol{\frac{\beta}{\epsilon * \omega} \sin\left(\frac{m\pi}{d}x\right) e^{i(\beta z - \omega t)}} \\ \boldsymbol{E_z(x,y,z,t)} = \frac{i}{\epsilon * \omega}\frac{\partial H_y}{\partial x} = \boldsymbol{\frac{i}{\epsilon * \omega}\frac{m\pi}{d} \cos\left(\frac{m\pi}{d}x\right) e^{i(\beta z - \omega t)}} \end{cases}$$

b) In TM mode, we check the release:

$$E = E_0(x,y)e^{i(\beta z-\omega t)}$$

$$H = H_0(x,y)e^{i(\beta z-\omega t)}$$

$$\boldsymbol{H(x,y) = H(x)H(y)}$$

Because we are in TM mode, Of course, it should be noted that since the two states x and y are independent of each other, we must solve two equations:

$$H_y(x,0) = H_y(x,b) = H_y(0,y) = H_y(a,y) = 0$$

$$H_x(x,0) = H_x(x,b) = H_x(0,y) = H_x(a,y) = 0$$

Based on the Helmholtz equation:

$$\nabla^2 H + (k)^2 H = 0 \rightarrow \nabla^2 H + \left({K_x}^2 - {K_y}^2\right)H = 0$$

$${K_x}^2 - {K_y}^2 = K^2 \qquad 0 < x < a, 0 < y < b$$

Let's solve the above equation only inside the core:

$$K_x = \frac{m\pi}{a}, K_y = \frac{n\pi}{b}$$

$$H_y = A\cos(Kx) + B\sin(Kx)$$

$$H_y(x,y) = 0 \quad x = 0, x = a, y = 0, y = b \rightarrow \boldsymbol{H_y(x,y) = B\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}$$

$$\boldsymbol{H_y(x,y,z,t) = B\sin(k_x x)\sin\left(k_y y\right)e^{i(\beta z-\omega t)}}$$

Now, based on Maxwell's equations, we can get the rest of the equations:

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \mu * i\omega\left(\boldsymbol{H_x}\hat{x} + \boldsymbol{H_y}\hat{y}\right) \rightarrow \begin{cases} \mu * i\omega\boldsymbol{H_x} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \mu * i\omega\boldsymbol{H_y} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ 0 = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{cases}$$

$$\rightarrow \begin{cases} \mu * i\omega\boldsymbol{H_x} = \frac{\partial E_z}{\partial y} - i\beta E_y \\ \mu * i\omega\boldsymbol{H_y} = i\beta E_x - \frac{\partial E_z}{\partial x} \\ 0 = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{cases}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = -\epsilon * i\omega\left(\boldsymbol{E_x}\hat{x} + \boldsymbol{E_y}\hat{y} + \boldsymbol{E_z}\hat{z}\right)$$

$$\rightarrow \begin{cases} -\epsilon * i\omega\boldsymbol{E_x} = -\dfrac{\partial H_y}{\partial z} \\ -\epsilon * i\omega\boldsymbol{E_y} = \dfrac{\partial H_x}{\partial z} \\ -\epsilon * i\omega\boldsymbol{E_z} = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} \end{cases} \rightarrow \begin{cases} -\epsilon * i\omega\boldsymbol{E_x} = -i\beta H_y \\ -\epsilon * i\omega\boldsymbol{E_y} = i\beta H_x \\ -\epsilon * i\omega\boldsymbol{E_z} = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} \end{cases}$$

$$-\epsilon * i\omega\boldsymbol{E_x} = -\frac{\partial H_y}{\partial z} \rightarrow \boldsymbol{E_x(x,y,z,t) = \frac{\beta}{\epsilon * \omega} B\sin(k_x x)\sin(k_y y)e^{i(\beta z-\omega t)}}$$

$$0 = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \rightarrow \boldsymbol{E_y(x,y,z,t)} = \int \frac{\beta}{\epsilon * \omega} k_y B\sin(k_x x)\cos(k_y y)e^{i(\beta z-\omega t)}\,dx$$

$$= \frac{\beta}{\epsilon * \omega} k_y * \frac{-1}{k_x} B\cos(k_x x)\cos(k_y y)e^{i(\beta z-\omega t)}$$

$$= \boldsymbol{-\frac{\beta}{\epsilon * \omega}\frac{k_y}{k_x} B\cos(k_x x)\cos(k_y y)e^{i(\beta z-\omega t)}}$$

$$-\epsilon * i\omega\boldsymbol{E_y} = \frac{\partial H_x}{\partial z} \rightarrow$$

$$\boldsymbol{H_x(x,y,z,t) = \beta\frac{k_y}{k_x} B\cos(k_x x)\cos(k_y y)e^{i(\beta z-\omega t)}}$$

$$-\epsilon * i\omega\boldsymbol{E_z} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \rightarrow$$

$$\boldsymbol{E_z(x,y,z,t)} = \Big(k_x B\cos(k_x x)\sin(k_y y)e^{i(\beta z-\omega t)} + \beta\frac{k_y^2}{k_x} B\cos(k_x x)\sin(k_y y)e^{i(\beta z-\omega t)}\Big)\frac{1}{-\epsilon * i\omega}$$

$$= \boldsymbol{\frac{-B}{\epsilon * i\omega}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i(\beta z-\omega t)}\left(k_x + \beta\frac{k_y^2}{k_x}\right)}$$