Q: Prove the following algebraic expression.

$$\nabla \times (\nabla \times V) = \nabla(\nabla . V) - \nabla^2 V$$

Sol: We know that curl is defined as follows:

$$\nabla \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla \times (\nabla \times V) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} & \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right)$$

$$+ \hat{y}\left(\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right)$$

$$\nabla \times (\nabla \times V) = \hat{x}\left(\left(\frac{\partial\partial v_y}{\partial y \partial x} - \frac{\partial^2 v_x}{\partial y^2}\right) - \left(\frac{\partial^2 v_x}{\partial z^2} - \frac{\partial\partial v_z}{\partial z \partial x}\right)\right)$$

$$+ \hat{y}\left(\left(\frac{\partial\partial v_z}{\partial z \partial y} - \frac{\partial^2 v_y}{\partial z^2}\right) - \left(\frac{\partial^2 v_y}{\partial x^2} - \frac{\partial\partial v_x}{\partial x \partial y}\right)\right)$$

$$+ \hat{z}\left(\left(\frac{\partial\partial v_x}{\partial x \partial z} - \frac{\partial^2 v_z}{\partial x^2}\right) - \left(\frac{\partial^2 v_z}{\partial y^2} - \frac{\partial\partial v_y}{\partial y \partial z}\right)\right)$$

$$\nabla \times (\nabla \times V) = \hat{x}\left(\frac{\partial\partial v_y}{\partial y \partial x} + \frac{\partial\partial v_z}{\partial z \partial x}\right) - \hat{x}\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \hat{y}\left(\frac{\partial\partial v_z}{\partial z \partial y} + \frac{\partial\partial v_x}{\partial x \partial y}\right)$$
$$- \hat{y}\left(\frac{\partial^2 v_y}{\partial z^2} + \frac{\partial^2 v_y}{\partial x^2}\right) + \hat{z}\left(\frac{\partial\partial v_x}{\partial x \partial z} + \frac{\partial\partial v_y}{\partial y \partial z}\right) - \hat{z}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right)$$

Proof of the left side of Eq

$$
\begin{aligned}
&\nabla(\nabla.V)-\nabla^2V\\
&\quad=\nabla\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\\
&\quad-\left[\hat{x}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\hat{y}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)\right.\\
&\quad\left.+\hat{z}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\right]\\
&\quad=\hat{x}\left(\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right)+\hat{y}\left(\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right)\\
&\quad+\hat{z}\left(\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right)\\
&\quad-\left[\hat{x}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\hat{y}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)\right.\\
&\quad\left.+\hat{z}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\right]\\
&\quad=\hat{x}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial\partial v_y}{\partial x\partial y}+\frac{\partial\partial v_z}{\partial x\partial z}\right)-\hat{x}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)\\
&\quad+\hat{y}\left(\frac{\partial\partial v_x}{\partial y\partial x}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial\partial v_z}{\partial y\partial z}\right)-\hat{y}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)\\
&\quad+\hat{z}\left(\frac{\partial\partial v_x}{\partial z\partial x}+\frac{\partial\partial v_y}{\partial z\partial y}+\frac{\partial^2 v_z}{\partial z^2}\right)-\hat{z}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\\
&\quad=\hat{x}\left(\frac{\partial\partial v_y}{\partial x\partial y}+\frac{\partial\partial v_z}{\partial x\partial z}\right)-\hat{x}\left(\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\hat{y}\left(\frac{\partial\partial v_x}{\partial y\partial x}+\frac{\partial\partial v_z}{\partial y\partial z}\right)\\
&\quad-\hat{y}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\hat{z}\left(\frac{\partial\partial v_x}{\partial z\partial x}+\frac{\partial\partial v_y}{\partial z\partial y}\right)-\hat{z}\left(\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned}
$$

Proof of the right side of Eq

As it is known, both sides of the equation were equal. 😊