***Q:*** According to the Q-factor formula, which is as follows:

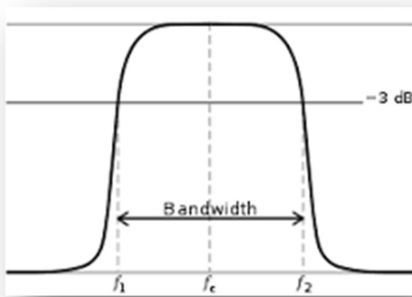
$$Q = -\frac{\omega E_{cavity}}{\frac{dE_{cavity}}{dt}}$$

Prove that this formula can also be defined as:

$$Q = \frac{\lambda}{\delta\lambda}$$

***Sol:***

I will say an introduction that will be an interesting summary of this issue, we used the energy relationship to define the Q-factor in a way that we stated that the input energy becomes the Q-factor; Now, exactly the same problem can be expressed for frequency or wavelength, pay attention as follows:




Imagine that we sent a light with a frequency $f_0$ and we expected to have a sharp vertical line at the output, but it didn't, but it was shown with a bandwidth at the output, which is the beginning of our next definition.

Now let's analyze ourselves step by step:

**Q-factor in Terms of Energy**

The Q-factor is a measure of how efficiently a resonator stores energy compared to how much energy it loses. It can be expressed as:

$$Q = -\frac{\omega E_{\text{cavity}}}{\frac{dE_{\text{cavity}}}{dt}}$$

- $\omega$ is the angular frequency of the resonant mode.

- $E_{\text{cavity}}$ is the energy stored in the resonator.

- $\frac{dE_{\text{cavity}}}{dt}$ is the rate at which energy is lost from the resonator.

**Q-factor and Frequency Bandwidth**

Another common way to express the Q-factor is in terms of the resonant frequency and the bandwidth of the resonance:

$$Q = \frac{f_0}{\Delta f}$$

- $f_0$ is the resonant frequency.

- $\Delta f$ is the full width at half maximum (FWHM) of the resonance, representing the bandwidth over which the energy drops to half its peak value.

**Deriving the Relationship**

To relate these two expressions, we need to understand how energy decay translates to bandwidth. In a resonator, the stored energy $E_{\text{cavity}}$ decays exponentially over time as:

$$E_{\text{cavity}}(t) = E_{\text{cavity}}(0)e^{-\frac{\omega_0 t}{2Q}}$$

where $\omega_0 = 2\pi f_0$ is the resonant angular frequency.

The energy decay rate $\frac{dE_{\text{cavity}}}{dt}$ can be expressed as:

$$\frac{dE_{\text{cavity}}}{dt} = -\frac{\omega_0}{Q}E_{\text{cavity}}$$

Thus, substituting this into the Q-factor definition, we get:

$$Q = -\frac{\omega_0 E_{\text{cavity}}}{-\frac{\omega_0}{Q}E_{\text{cavity}}} = Q$$

This confirms the consistency of the definition. Now, let's connect this to the frequency domain.

**Frequency Domain Perspective**

In the frequency domain, the Q-factor is defined as the ratio of the resonant frequency $f_0$ to the bandwidth $\Delta f$:

$$Q = \frac{f_0}{\Delta f}$$

This comes from the relationship between energy storage and loss in the frequency domain. The bandwidth $\Delta f$ represents the range of frequencies over which the resonator effectively stores energy.

**Wavelength Perspective**

For light or electromagnetic waves, we often use wavelength ($\lambda$) instead of frequency. The relationship between frequency $f$ and wavelength $\lambda$ is:

$$f = \frac{c}{\lambda}$$

where $c$ is the speed of light. The bandwidth in terms of wavelength $\delta\lambda$ can be related to the bandwidth in terms of frequency $\Delta f$:

$$\Delta f = \left|\frac{df}{d\lambda}\right| \delta\lambda = \frac{c}{\lambda^2} \delta\lambda$$

Therefore, the Q-factor in terms of wavelength is:

$$Q = \frac{f_0}{\Delta f} = \frac{\frac{c}{\lambda}}{\frac{c}{\lambda^2} \delta\lambda} = \frac{\lambda}{\delta\lambda}$$

By understanding that:

- The Q-factor is a measure of energy stored versus energy lost.

- In the frequency domain, it is the ratio of the resonant frequency to the bandwidth.

- The relationship between frequency and wavelength allows us to express the Q-factor in terms of wavelength and its bandwidth.

Therefore, the Q-factor can equivalently be expressed as:

$$Q = \frac{\lambda}{\delta\lambda}$$

This shows the direct relationship between the energy decay perspective and the frequency (or wavelength) bandwidth perspective of the Q-factor.

***The concept was very interesting to me*** 😊