Quantum Mechanics

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Lesson V
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LAST CLASS WE SAW

$$E\psi = \hbar\omega\psi = -j\hbar\frac{\partial}{\partial t}\psi \qquad p_x\psi = \hbar k\psi = j\hbar\frac{\partial}{\partial x}\psi$$

$$E = \frac{p^2}{2m} \quad \text{(free-particle)}$$

$$\text{Notation} \Rightarrow i = j$$

$$\boxed{-j\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \quad \text{(free-particle)}}$$

..The Free-Particle Schrodinger Wave Equation !

Erwin Schrödinger (1887–1961)
Image in the Public Domain

probability

$$P(x) = |\psi|^2 dx$$

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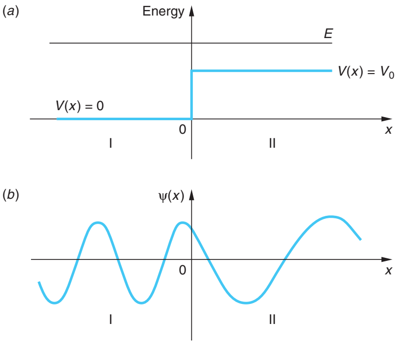
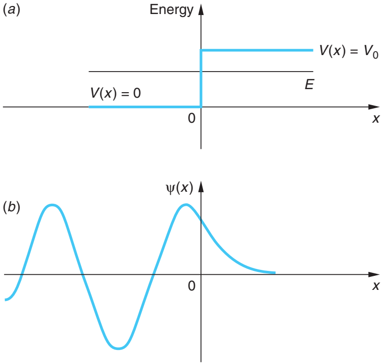
The Schrodinger Wave Equation

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x)$$

Quantum Intuition

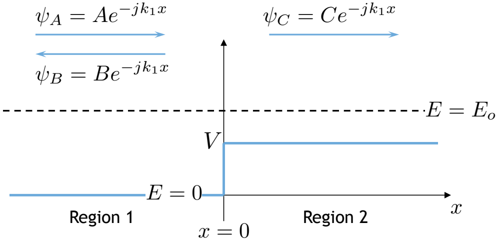
$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x \geq 0 \end{cases} \tag{1}$$





A Simple Potential Step

CASE I : $E_o > V$



In Region 1:

$$E_o\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \quad \Longrightarrow \quad k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2:

$$(E_o - V)\,\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \quad \Longrightarrow \quad k_2^2 = \frac{2m\,(E_o - V)}{\hbar^2}$$

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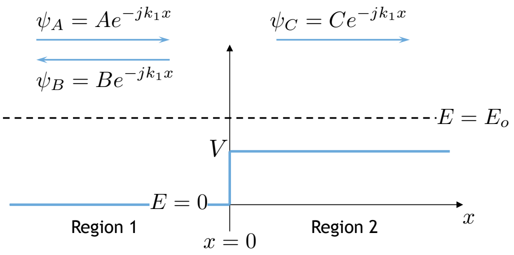
A Simple Potential Step

CASE I : $E_o > V$



$$\psi_1 = Ae^{-jk_1x} + Be^{jk_1x} \qquad \psi_2 = Ce^{-jk_2x}$$

ψ is continuous: $\psi_1(0) = \psi_2(0) \Rightarrow A + B = C$

$\frac{\partial \psi}{\partial x}$ is continuous: $\frac{\partial}{\partial x}\psi(0) = \frac{\partial}{\partial x}\psi_2(0) \Rightarrow A - B = \frac{k_2}{k_1}C$

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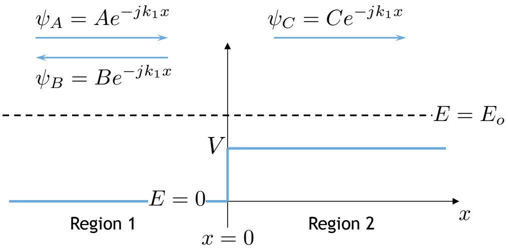
A Simple Potential Step

CASE I : E_o > V



$$\begin{cases} A + B = C \\ A - B = \dfrac{k_2}{k_1} C \end{cases}$$

$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$\frac{C}{A} = \frac{2}{1 + k_2/k_1} = \frac{2k_1}{k_1 + k_2}$$

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Quantum Electron Currents

Given an electron of mass m

that is located in space with charge density $\rho = q\left|\psi(x)\right|^2$

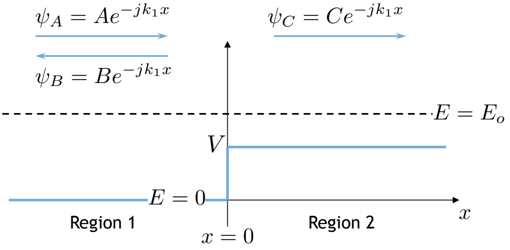
and moving with momentum $<p>$ corresponding to $<v>= \hbar k/m$

... then the current density for a *single electron* is given by

$$\boxed{J = \rho v = q\left|\psi\right|^2(\hbar k/m)}$$

A Simple Potential Step

CASE I : $E_o > V$



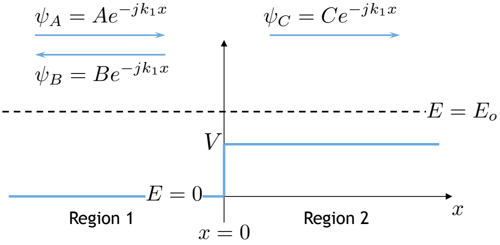
$$\text{Reflection} = R = \frac{J_{reflected}}{J_{incident}} = \frac{J_B}{J_A} = \frac{|\psi_B|^2(\hbar k_1/m)}{|\psi_A|^2(\hbar k_1/m)} = \left|\frac{B}{A}\right|^2$$

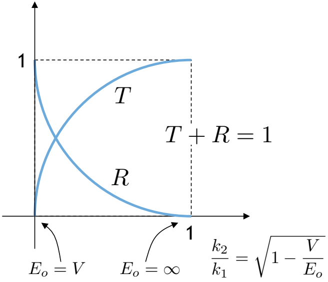
$$\text{Transmission} = T = \frac{J_{transmitted}}{J_{incident}} = \frac{J_C}{J_A} = \frac{|\psi_C|^2(\hbar k_2/m)}{|\psi_A|^2(\hbar k_1/m)} = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}$$

$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} \qquad \frac{C}{A} = \frac{2}{1 + k_2/k_1}$$

A Simple Potential Step

CASE I : $E_o > V$





$$\text{Reflection} = R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

$$\text{Transmission} = T = 1 - R$$

$$= \frac{4k_1k_2}{|k_1 + k_2|^2}$$

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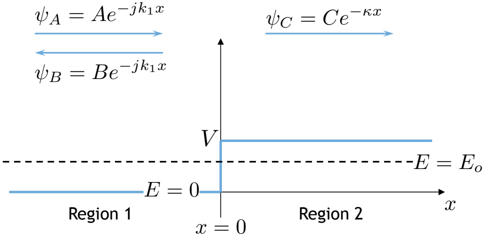
A Simple Potential Step

CASE II : $E_o < V$



In Region 1: $$E_o\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2: $$(E_o - V)\,\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow \kappa^2 = \frac{2m\,(E_o - V)}{\hbar^2}$$

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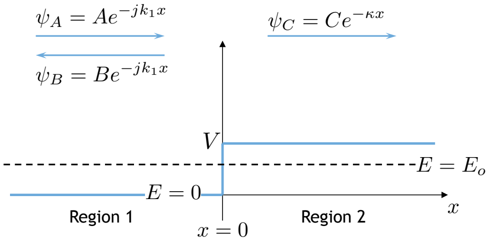
A Simple Potential Step

CASE II : E_o < V



$$\psi_1 = Ae^{-jk_1x} + Be^{jk_1x} \qquad \psi_2 = Ce^{-\kappa x}$$

ψ is continuous: $\psi_1(0) = \psi_2(0) \Rightarrow A + B = C$

$\dfrac{\partial \psi}{\partial x}$ is continuous: $\dfrac{\partial}{\partial x}\psi(0) = \dfrac{\partial}{\partial x}\psi_2(0) \Rightarrow A - B = -j\dfrac{\kappa}{k_1}C$

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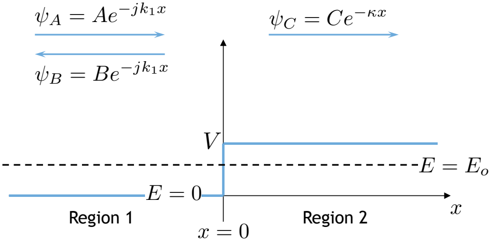
A Simple Potential Step



CASE II : $E_o < V$

$$\frac{B}{A} = \frac{1 + j\kappa/k_1}{1 - j\kappa/k_1} \qquad \frac{C}{A} = \frac{2}{1 - j\kappa/k_1} \Longleftarrow \begin{cases} A + B = C \\ A - B = -j\dfrac{\kappa}{k_1}C \end{cases}$$

$$\boxed{R = \left|\frac{B}{A}\right|^2 = 1 \qquad\qquad T = 0}$$

Total reflection → Transmission must be zero

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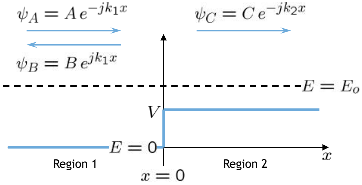
KEY TAKEAWAYS

A Simple Potential Step

CASE I : $E_o > V$



$$Reflection = R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

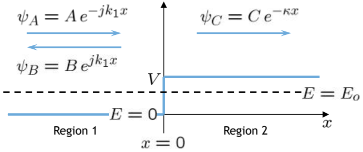
$$Transmission = T = 1 - R = \frac{4\, k_1 k_2}{|k_1 + k_2|^2}$$

PARTIAL REFLECTION

$$\psi_1 = A\, e^{-jk_1x} + B\, e^{jk_1x} \qquad \psi_2 = C\, e^{-jk_2x}$$

$$k_1^2 = \frac{2m\, E_o}{\hbar^2} \qquad k_2^2 = \frac{2m\,(E_o - V)}{\hbar^2}$$

CASE II : $E_o < V$



$$R = |\frac{B}{A}|^2 = 1$$

$$T = 0$$

TOTAL REFLECTION

$$\psi_1 = A\, e^{-jk_1x} + B\, e^{jk_1x} \qquad \psi_2 = C\, e^{-\kappa x}$$

$$k_1^2 = \frac{2m\, E_o}{\hbar^2} \qquad \kappa^2 = \frac{2m\,(V - E_o)}{\hbar^2}$$

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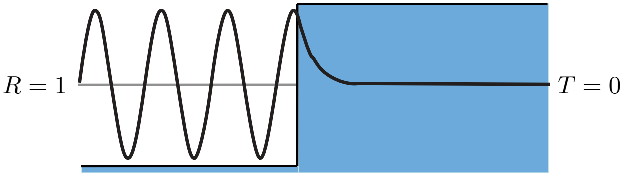
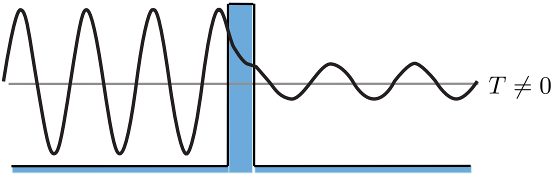
Quantum Tunneling Through a Thin Potential Barrier

Total Reflection at Boundary



Frustrated Total Reflection (Tunneling)



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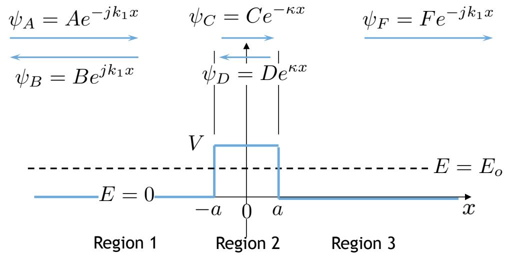
A Rectangular Potential Step



CASE II : $E_o < V$

In Regions 1 and 3: $$E_o\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2: $$(E_o - V)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow \kappa^2 = \frac{2m(V - E_o)}{\hbar^2}$$

for $E_o < V$:

$$\boxed{T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V-E_o)}\sinh^2(2\kappa a)}}$$

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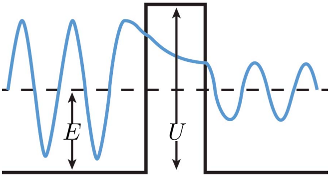
A Rectangular Potential Step



for $E_o < V$:

$$\boxed{T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V - E_o)}\sinh^2(2\kappa a)}}$$

$$\sinh^2(2\kappa a) = \left[e^{2\kappa a} - e^{-2\kappa a}\right]^2 \approx e^{-4\kappa a}$$

$$\boxed{T = \left|\frac{F}{A}\right|^2 \approx \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V - E_o)}} e^{-4\kappa a}}$$

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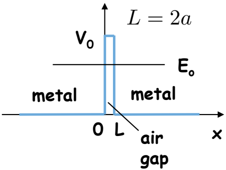
Example: Barrier Tunneling

- Let's consider a tunneling problem:

An electron with a total energy of $E_o = 6$ eV approaches a potential barrier with a height of $V_0 = 12$ eV. If the width of the barrier is L = 0.18 nm, what is the probability that the electron will tunnel through the barrier?



$$T = \left|\frac{F}{A}\right|^2 \approx \frac{16E_o(V - E_o)}{V^2} e^{-2\kappa L}$$

$$\kappa = \sqrt{\frac{2m_e}{\hbar^2}(V - E_o)} = 2\pi\sqrt{\frac{2m_e}{h^2}(V - E_o)} = 2\pi\sqrt{\frac{6\text{eV}}{1.505\text{eV-nm}^2}} \approx 12.6 \text{ nm}^{-1}$$

$$T = 4e^{-2(12.6 \text{ nm}^{-1})(0.18 \text{ nm})} = 4(0.011) = \boxed{4.4\%}$$

Question: What will T be if we double the width of the gap?

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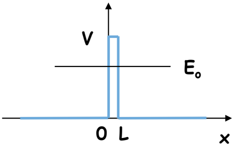
Multiple Choice Questions

Consider a particle tunneling through a barrier:



1. Which of the following will increase the likelihood of tunneling?

a. decrease the height of the barrier
b. decrease the width of the barrier
c. decrease the mass of the particle

2. What is the energy of the particles that have successfully "escaped"?

a. < initial energy
$\boxed{\text{b. = initial energy}}$
c. > initial energy

Although the *amplitude* of the wave is smaller after the barrier, no energy is lost in the tunneling process

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Schrodinger Equations

Key to solving for the wave function of a particle hitting a potential barrier is finding the **Schrodinger equations** which describe the system. First, define the energy potential, $V(x)$, of the system as this:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 < x < a \\ 0, & x > a \end{cases} \tag{1}$$

Writing the wave function of the particle as $\psi_1(x)$ for $x < 0$, $\psi_2(x)$ for $0 < x < a$, and $\psi_3(x)$ for $x > a$, the **Schrodinger equations** for $x < 0, 0 < x < a$, and $x > a$ are respectively:

$$E\psi_1(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1(x) \tag{2}$$

$$E\psi_2(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2(x) + V_0\psi_2(x) \tag{3}$$

$$E\psi_3(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3(x) \tag{4}$$

This can be simplified, considering the wavenumbers, k_1 and k_2, of the wave function for inside and outside the barrier respectively. Since ${k_1}^2 = 2mE/\hbar^2$ and ${k_2}^2 = 2m(E - V_0)/\hbar^2$, this can be said of the wave function of a particle with $E \geq V_0$.

$$0 = \frac{d^2}{dx^2}\psi_1(x) + {k_1}^2\psi_1(x) \tag{5}$$

$$0 = \frac{d^2}{dx^2}\psi_2(x) + {k_2}^2\psi_2(x) \tag{6}$$

$$0 = \frac{d^2}{dx^2}\psi_3(x) + {k_1}^2\psi_3(x) \tag{7}$$

Notice, however that if $E < V_0$, k_2 is imaginary and thus no longer an observable. By convention therefore, κ, defined by $\kappa^2 = 2m(V_0 - E)/\hbar^2$, is used instead for $E < V_0$. The differential equations defining the wave function of a particle with insufficient energy are thus:

$$0 = \frac{d^2}{dx^2}\psi_1(x) + {k_1}^2\psi_1(x) \tag{8}$$
$$0 = \frac{d^2}{dx^2}\psi_2(x) - \kappa^2\psi_2(x) \tag{9}$$
$$0 = \frac{d^2}{dx^2}\psi_3(x) + {k_1}^2\psi_3(x) \tag{10}$$

If There Is Sufficient Energy

For $E \geq V_0$, to find the wave function of the particle, equations (5), (6), and (7) must be solved. These are homogeneous second-order linear differential equations and have the following general solutions:

$$\psi_1(x) = Ae^{r_A x} + Be^{r_B x} \tag{11}$$
$$\psi_2(x) = Ce^{r_C x} + De^{r_D x} \tag{12}$$
$$\psi_3(x) = Fe^{r_F x} + Ge^{r_G x} \tag{13}$$

where A, B, C, D, F, and G are constants and $r_A = r_F$ and $r_B = r_G$ are the two solutions to the equation $r^2 + {k_1}^2 = 0$ while r_C and r_D are the two solutions to the equation $r^2 + {k_2}^2 = 0$.

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \tag{14}$$
$$\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x} \tag{15}$$
$$\psi_3(x) = Fe^{ik_1x} + Ge^{-ik_1x} \tag{16}$$

tice, considering **Euler's formula**, that Ae^{ik_1x}, Ce^{ik_2x}, and Fe^{ik_1x} represent waves travelling in the positive ection while Be^{-ik_1x}, De^{-ik_2x}, and Ge^{-ik_1x} represent waves travelling in the negative direction. Since reflection by barrier is conceivable, it is possible to have wave components travelling in the negative direction for $x < a$, but ere is no reason to have waves doing so for $x > a$. Thus, $G = 0$.

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \tag{17}$$
$$\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x} \tag{18}$$
$$\psi_3(x) = Fe^{ik_1x} \tag{19}$$

solve for B and F in relation to A, impose these four boundary conditions to ensure that the wave function is a ooth curve as $x \to 0$ and as $x \to a$:

$$\begin{aligned}
\lim_{x\to 0^-} \psi_1(x) &= \lim_{x\to 0^+} \psi_2(x) \\
\lim_{x\to 0^-} \frac{d}{dx}\psi_1(x) &= \lim_{x\to 0^+} \frac{d}{dx}\psi_2(x) \\
\lim_{x\to a^-} \psi_2(x) &= \lim_{x\to a^+} \psi_3(x) \\
\lim_{x\to a^-} \frac{d}{dx}\psi_2(x) &= \lim_{x\to a^+} \frac{d}{dx}\psi_3(x)
\end{aligned}$$

$$A + B = C + D \tag{20}$$
$$ik_1 A - ik_1 B = ik_2 C - ik_2 D \tag{21}$$
$$Ce^{ik_2 a} + De^{-ik_2 a} = Fe^{ik_1 a} \tag{22}$$
$$ik_2 Ce^{ik_2 a} - ik_2 De^{-ik_2 a} = ik_1 Fe^{ik_1 a} \tag{23}$$

$$k_1 A + k_1 B = k_1 C + k_1 D \tag{24}$$
$$k_1 A - k_1 B = k_2 C - k_2 D \tag{25}$$
$$k_2 Ce^{ik_2 a} + k_2 De^{-ik_2 a} = k_2 Fe^{ik_1 a} \tag{26}$$
$$k_2 Ce^{ik_2 a} - k_2 De^{-ik_2 a} = k_1 Fe^{ik_1 a} \tag{27}$$

$$2k_1 A = (k_1 + k_2)C + (k_1 - k_2)D \tag{28}$$
$$2k_1 B = (k_1 - k_2)C + (k_1 + k_2)D \tag{29}$$
$$2k_2 C e^{ik_2 a} = (k_1 + k_2)F e^{ik_1 a} \tag{30}$$
$$2k_2 D e^{-ik_2 a} = (k_2 - k_1)F e^{ik_1 a} \tag{31}$$

To solve for F in relation to A, consider equations (28), (30), and (31).

$$2k_1 A = \frac{(k_1 + k_2)^2}{2k_2} F e^{i(k_1 - k_2)a} - \frac{(k_1 - k_2)^2}{2k_2} F e^{i(k_1 + k_2)a} \tag{32}$$

$$4k_1 k_2 e^{-ik_1 a} A = (k_1 + k_2)^2 F e^{-ik_2 a} - (k_1 - k_2)^2 F e^{ik_2 a} \tag{33}$$

Using **Euler's formula** to expand $e^{-ik_2 a}$ and $e^{ik_2 a}$, the following can be derived:

$$4k_1 k_2 e^{-ik_1 a} A = \left(-2i{k_1}^2 \sin k_2 a + 4k_1 k_2 \cos k_2 a - 2i{k_2}^2 \sin k_2 a\right) F \tag{34}$$

$$F = \frac{2k_1 k_2 e^{-ik_1 a} A}{2k_1 k_2 \cos k_2 a - i\left({k_1}^2 + {k_2}^2\right) \sin k_2 a} \tag{35}$$

To solve for B in relation to A, consider equations (29), (30), and (31).

$$2k_1 B = \frac{{k_1}^2 - {k_2}^2}{2k_2} F e^{i(k_1-k_2)a} - \frac{{k_1}^2 - {k_2}^2}{2k_2} F e^{i(k_1+k_2)a} \tag{36}$$

$$\frac{4k_1 k_2 e^{-ik_1 a} B}{{k_1}^2 - {k_2}^2} = \left(e^{-ik_2 a} - e^{ik_2 a}\right) F \tag{37}$$

Using **Euler's formula** to expand $e^{-ik_2 a}$ and $e^{ik_2 a}$, the following can be derived:

$$\frac{2k_1 k_2 e^{-ik_1 a} B}{-i\left({k_1}^2 - {k_2}^2\right)\sin k_2 a} = F \tag{38}$$

Comparing equations (35) and (38), B is solved for in relation to A.

$$B = \frac{-i\left({k_1}^2 - {k_2}^2\right)\sin(k_2 a)A}{2k_1 k_2 \cos k_2 a - i\left({k_1}^2 + {k_2}^2\right)\sin k_2 a} \tag{39}$$

Considering equations (30) and (31) alongside equation (35), C and D can also be solved for in relation to A, but since only A, B, and F are needed to calculate the reflection and transmission coefficients, the derivations of C and D are omitted here. In order to find the reflection and transmission coefficients, the wave function must be first written in terms of its incident, reflected, and transmitted components, $\psi_i(x)$, $\psi_r(x)$, and $\psi_t(x)$ respectively.

$$\psi_i(x) = Ae^{ik_1 x} \tag{40}$$

$$\psi_r(x) = Be^{-ik_1 x} \tag{41}$$

$$\psi_t(x) = Fe^{ik_1 x} \tag{42}$$

The reflection and transmission coefficients, R and T respectively, are defined as follows:

$$R = -\frac{j_r}{j_i} \tag{43}$$
$$T = \frac{j_t}{j_i} \tag{44}$$

where j_i, j_r, and j_t are the incident, reflected, and transmitted **probability currents** respectively.

$$R = -\frac{\psi_r(x)\frac{d}{dx}\psi_r{}^*(x) - \psi_r{}^*(x)\frac{d}{dx}\psi_r(x)}{\psi_i(x)\frac{d}{dx}\psi_i{}^*(x) - \psi_i{}^*(x)\frac{d}{dx}\psi_i(x)} \tag{45}$$
$$T = \frac{\psi_t(x)\frac{d}{dx}\psi_t{}^*(x) - \psi_t{}^*(x)\frac{d}{dx}\psi_t(x)}{\psi_i(x)\frac{d}{dx}\psi_i{}^*(x) - \psi_i{}^*(x)\frac{d}{dx}\psi_i(x)} \tag{46}$$

$$R = \frac{|B|^2}{|A|^2} \tag{47}$$
$$T = \frac{|F|^2}{|A|^2} \tag{48}$$

Applying the solutions for B and F found in equations (39) and (35) respectively gives:

$$R = \frac{\left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a}{4{k_1}^2{k_2}^2 \cos^2 k_2 a + \left({k_1}^2 + {k_2}^2\right)^2 \sin^2 k_2 a} \tag{49}$$

$$T = \frac{4{k_1}^2{k_2}^2}{4{k_1}^2{k_2}^2 \cos^2 k_2 a + \left({k_1}^2 + {k_2}^2\right)^2 \sin^2 k_2 a} \tag{50}$$

$$R = \frac{\left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a}{4{k_1}^2{k_2}^2 + \left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a} \tag{51}$$

$$T = \frac{4{k_1}^2{k_2}^2}{4{k_1}^2{k_2}^2 + \left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a} \tag{52}$$

$$R = \left[\frac{4{k_1}^2{k_2}^2}{\left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a} + 1 \right]^{-1} \tag{53}$$

$$T = \left[\frac{\left({k_1}^2 - {k_2}^2\right)^2 \sin^2 k_2 a}{4{k_1}^2{k_2}^2} + 1 \right]^{-1} \tag{54}$$

Interestingly contrary to classical mechanics, quantum mechanics suggests that the particle may actually be reflected by the potential barrier, despite having a total energy of equal or greater value than V_0 .

If There Is Insufficient Energy

For $E < V_0$, equations (8), (9), and (10) must be solved to find $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$. To do this, follow the methodology employed in the previous section, **"If There Is Sufficient Energy"**. The solutions of equations (8), (9), and (10) are identical to those of (5), (6), and (7) respectively save for the use of $i\kappa$ in the place of k_2 .

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \tag{55}$$

$$\psi_2(x) = Ce^{-\kappa x} + De^{\kappa x} \tag{56}$$

$$\psi_3(x) = Fe^{ik_1x} \tag{57}$$

Applying the same boundary conditions as in the previous section and manipulating algebra in the same manner, it can also be found that:

$$\begin{aligned} 2ik_1 A &= (ik_1 - \kappa)C + (ik_1 + \kappa)D && (58) \\ 2ik_1 B &= (ik_1 + \kappa)C + (ik_1 - \kappa)D && (59) \\ 2\kappa C e^{-\kappa a} &= (\kappa - ik_1)F e^{ik_1 a} && (60) \\ 2\kappa D e^{\kappa a} &= (ik_1 + \kappa)F e^{ik_1 a} && (61) \end{aligned}$$

To solve for F in relation to A, consider equations (58), (60), and (61).

$$2ik_1 A = -\frac{(ik_1 - \kappa)^2}{2\kappa} F e^{(ik_1 + \kappa)a} + \frac{(ik_1 + \kappa)^2}{2\kappa} F e^{(ik_1 - \kappa)a} \quad (62)$$

$$4ik_1 \kappa e^{-ik_1 a} A = -(ik_1 - \kappa)^2 F e^{\kappa a} + (ik_1 + \kappa)^2 F e^{-\kappa a} \quad (63)$$

$$4ik_1 \kappa e^{-ik_1 a} A = \left[\left({k_1}^2 - \kappa^2\right)\left(e^{\kappa a} - e^{-\kappa a}\right) + 2ik_1\kappa\left(e^{\kappa a} + e^{-\kappa a}\right)\right] F \quad (64)$$

$$F = \frac{2ik_1 \kappa e^{-ik_1 a} A}{\left({k_1}^2 - \kappa^2\right)\sinh \kappa a + 2ik_1 \kappa \cosh \kappa a} \quad (65)$$

(In case you are unfamiliar with **hyperbolic functions**, $\sinh u = (e^u - e^{-u})/2$ is the **hyperbolic sine function** and $\cosh u = (e^u + e^{-u})/2$ is the **hyperbolic cosine function**.) To solve for B in relation to A, consider equations (59), (60), and (61).

$$2ik_1 B = \frac{{k_1}^2 + \kappa^2}{2\kappa} F e^{(ik_1+\kappa)a} - \frac{{k_1}^2 + \kappa^2}{2\kappa} F e^{(ik_1-\kappa)a} \tag{66}$$

$$\frac{4ik_1\kappa e^{-ik_1 a} B}{{k_1}^2 + \kappa^2} = F e^{\kappa a} - F e^{-\kappa a} \tag{67}$$

$$\frac{2ik_1\kappa e^{-ik_1 a} B}{\left({k_1}^2 + \kappa^2\right) \sinh \kappa a} = F \tag{68}$$

Comparing equations (65) and (68), B is solved for in relation to A.

$$B = \frac{\left({k_1}^2 + \kappa^2\right)\sinh(\kappa a) A}{\left({k_1}^2 - \kappa^2\right)\sinh\kappa a + 2ik_1\kappa\cosh\kappa a} \tag{69}$$

As in the previous section, the wave function written in terms of its incident, reflected, and transmitted components is:

$$\psi_i(x) = Ae^{ik_1x} \tag{70}$$
$$\psi_r(x) = Be^{-ik_1x} \tag{71}$$
$$\psi_t(x) = Fe^{ik_1x} \tag{72}$$

Furthermore, the reflection and transmission coefficients, derivable using the same method as in the previous section, are again given by:

$$R = \frac{|B|^2}{|A|^2} \tag{73}$$

$$T = \frac{|F|^2}{|A|^2} \tag{74}$$

$$R = \frac{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a}{\left({k_1}^2 - \kappa^2\right)^2 \sinh^2 \kappa a + 4{k_1}^2\kappa^2 \cosh^2 \kappa a} \tag{75}$$

$$T = \frac{4{k_1}^2\kappa^2}{\left({k_1}^2 - \kappa^2\right)^2 \sinh^2 \kappa a + 4{k_1}^2\kappa^2 \cosh^2 \kappa a} \tag{76}$$

$$R = \frac{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a}{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a + 4{k_1}^2\kappa^2} \tag{77}$$

$$T = \frac{4{k_1}^2\kappa^2}{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a + 4{k_1}^2\kappa^2} \tag{78}$$

$$R = \left[\frac{4{k_1}^2\kappa^2}{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a} + 1 \right]^{-1} \tag{79}$$

$$T = \left[\frac{\left({k_1}^2 + \kappa^2\right)^2 \sinh^2 \kappa a}{4{k_1}^2\kappa^2} + 1 \right]^{-1} \tag{80}$$

Contrary to classical expectations which would suggest that the particle has zero probability of travelling beyond $x = 0$, quantum mechanics asserts that the particle has a non-zero probability of tunneling through the rectangular potential barrier, despite having a total energy less than V_0. This phenomenon marks a major difference between quantum and classical mechanics.