سید محمود سجادی
402448040

تمرین 6 :

$$\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2 \right)^{1/2} =$$

$$\psi = A e^{ik_0 t} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

نرمالیزیسون : $A = \left(\frac{1}{a^2 2\pi} \right)^{1/4}$

از رابطه احتمال حضور بدست آمده .

1)

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* \hat{x} \psi \, dx = A^2 \int_{-\infty}^{\infty} x \, e^{-\frac{(x-x_0)^2}{2a^2}} dx$$

$$\begin{cases} \frac{x - x_0}{a} = \eta \rightarrow x = a\eta + x_0 = a\left(\eta + \frac{x_0}{a}\right) = a(\eta + \eta_0) \\ dx = a \, d\eta \end{cases}$$

$$\xrightarrow{\text{ابتدا}} = A^2 \int_{-\infty}^{\infty} a(\eta + \eta_0) e^{-\eta^2/2} a \, d\eta = A^2 a^2 \int_{-\infty} \Big(\underbrace{\eta e^{-\eta^2/2}}_{(I)} + \underbrace{\eta_0 e^{\eta^2/2}}_{(II)} \Big) d\eta$$

$$(I) \rightarrow \int_{-\infty}^{\infty} \eta e^{-\eta^2/2} d\eta = \int_{-\infty}^{\infty} \cancel{\eta} e^{-u} \frac{du}{\cancel{\eta}} = \int_{-\infty}^{\infty} e^{-u} du$$

$$\begin{cases} \eta^2/2 = u \\ \eta \, d\eta = du \end{cases}$$

$$= \frac{1}{-1} e^{-u} \Big|_{-\infty}^{\infty}$$

$$= -e^{-\eta^2/2} \Big|_{-\infty}^{\infty} = 0 \; \boxed{\checkmark}$$

$$(\mathrm{II}) \rightarrow \int_{-\infty}^{\infty} \eta_0 \, e^{-\eta^2/2} \, d\eta = \eta_0 \times \sqrt{2\pi} \quad \boxed{\checkmark}$$

$$* \int_{-\infty}^{\infty} e^{-\alpha x} = \sqrt{\pi/\alpha}$$ گوسی !!

$$\xrightarrow{\text{ادامه}} = A^2 a^2 \times \left(0 + \eta_0 \sqrt{2\pi} \right) = \frac{1}{\cancel{a}\cancel{\sqrt{2\pi}}} \cancel{a^2} \times \frac{x_0}{\cancel{a}} \cancel{\sqrt{2\pi}} = x_0 \quad \boxed{\checkmark}$$

2)

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-x_0)^2}{2a^2}} dx$$

$$\begin{cases} \dfrac{x-x_0}{a} = \eta \rightarrow x = a\eta + x_0 = a\left(\eta + \dfrac{x_0}{a}\right) = a(\eta + \eta_0) \\ dx = a\,d\eta \end{cases}$$

$$= A^2 a^2 \int_{-\infty}^{\infty} (\eta + \eta_0)^2 e^{-\eta^2/2} \, a \, d\eta$$

$$= A^2 a^3 \int_{-\infty}^{\infty} \Big(\underbrace{\eta^2 e^{-\eta^2/2}}_{(\mathrm{I})} + \underbrace{\eta_0^2 e^{-\eta^2/2}}_{(\mathrm{II})} + \underbrace{2\eta\eta_0 e^{-\eta^2/2}}_{(\mathrm{III})} \Big) d\eta$$

$$(\mathrm{III}) \rightarrow 2\eta_0 \int_{-\infty}^{\infty} \eta \, e^{-\eta^2/2} = 2\eta_0 \int_{-\infty}^{\infty} \eta \, e^{-u} \frac{du}{\eta} = 2\eta_0 \times \frac{1}{-1} e^{-\eta^2/2} \Big|_{-\infty}^{\infty}$$

$$\begin{cases} \eta^2/2 = u \\ \eta \, d\eta = du \end{cases} \qquad = 0 \quad \boxed{\checkmark}$$

(II) → $\int_{-\infty}^{\infty} \eta_0^2 e^{-\eta^2/2} d\eta = \eta_0^2 \times \sqrt{2\pi}$ ☑

$* \int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\pi/\alpha}$

(I) → $\int_{-\infty}^{\infty} \eta^2 e^{-\eta^2/2} d\eta = -\eta e^{-\eta^2/2}\Big|_{-\infty}^{\infty} + \underline{\int_{-\infty}^{\infty} e^{-\eta^2/2} \times 1\, d\eta}$

$* \rightarrow u.v\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v.du$

$$\begin{cases} u = \eta \\ dv = \eta e^{-\eta^2/2} d\eta \end{cases} \Rightarrow \begin{cases} du = 1 \times d\eta \\ v = \eta \times \frac{1}{-\eta} e^{-\eta^2/2} = -e^{-\eta^2/2} \end{cases}$$

ادامه → $= 0 + \sqrt{2\pi} = \sqrt{2\pi}$ ☑

نتیجه نهایی ⇒ $\langle x^2 \rangle = A^2 a^3 \times \left(\sqrt{2\pi} + \eta_0^2 \sqrt{2\pi} + 0\right)$

$$= \frac{1}{\cancel{a}\cancel{\sqrt{2\pi}}} a^{\cancel{3}\,2} \times \cancel{\sqrt{2\pi}} \left(1 + \frac{x_0^2}{a^2}\right)$$

$= a^2 + x_0^2$ ☑

$$* \; \Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2} = \left(a^2 + x_0^2 - x_0^2\right)^{1/2} = a$$

$$\Delta P = \left(\langle P^2 \rangle - \langle P \rangle^2 \right)^{1/2} = ?$$

$$* \; P = -i\hbar \frac{d}{dx}$$

$$\psi = A e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

$$\psi^* = A e^{-i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{-i\omega t}$$

$$\frac{d\psi}{dx} = A\, i k_0\, e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t} - A \times 2 \times \frac{1}{4a^2} \times (x-x_0) \times e^{i k_0 x} \times e^{-\frac{(x-x_0)^2}{4a^2}} \times e^{i\omega t}$$

$$\frac{d^2\psi}{dx^2} = A \Big[-k_0^2 e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t} - i k_0 \times 2 \times \frac{1}{4a^2} \times (x-x_0) \times e^{i k_0 x} \times e^{-\frac{(x-x_0)^2}{4a^2}} \times e^{i\omega t}$$

$$- 2 \times \frac{1}{4a^2} \times (x-x_0) \times i k_0 e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

$$- 2 \times \frac{1}{4a^2}\, e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

$$+ \left(2 \times \frac{1}{(4a^2)} \right)^2 (x-x_0)^2 \times e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t} \Big]$$

$$= A \Big[-k_0^2 e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t} - i k_0 \frac{x-x_0}{a^2} e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

$$+ \left(\frac{x-x_0}{2a^2} \right)^2 e^{i k_0 x} e^{-\frac{(x-x_0)^2}{4a^2}} e^{i\omega t}$$

$$- \frac{1}{2a^2} e^{i k_0 x} e^{\frac{(x-x_0)^2}{4a^2}} e^{i\omega t} \Big]$$

1)

$$\langle P\rangle = \int_{-\infty}^{\infty} \psi^* \hat{P} \psi \, dx = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\delta}{\delta x}\right) \psi \, dx$$

$$= -i\hbar A^2 \int_{-\infty}^{\infty} \Big(\underbrace{ik_0 \, e^{-\frac{(x-x_0)^2}{2a^2}}}_{(\mathrm{I})} - \underbrace{\frac{x-x_0}{2a^2} \, e^{-\frac{(x-x_0)^2}{2a^2}}}_{(\mathrm{II})} \Big) dx$$

$$(\mathrm{I}) \rightarrow \int_{-\infty}^{\infty} ik_0 \, e^{-\frac{(x-x_0)^2}{2a^2}} \, dx$$

$$\begin{cases} \dfrac{x-x_0}{a} = \eta \\ dx = a \, d\eta \end{cases}$$

$$\rightarrow ik_0 \int_{-\infty}^{\infty} e^{-\eta^2/2} \, a \, d\eta = ik_0 a \sqrt{2\pi}$$

$$* \int_{-\infty}^{\infty} e^{-\alpha x} = \sqrt{\pi/\alpha}$$

$$(\mathrm{II}) \rightarrow \int_{-\infty}^{\infty} -\frac{(x-x_0)}{2a^2} \, e^{-\frac{(x-x_0)^2}{2a^2}} \, dx$$

$$\begin{cases} \dfrac{x-x_0}{a} = \eta \\ dx = a \, d\eta \end{cases}$$

$$\rightarrow -\int_{-\infty}^{\infty} \eta/2 \; e^{-\eta^2/2} \, a \, d\eta = 0$$

نتیجه گیری $\Rightarrow$ $$\langle P\rangle = -i\hbar A^2 \times \left(i k_0 a\sqrt{2\pi} + 0 \right)$$

$$= \hbar \frac{1}{\cancel{a\sqrt{2\pi}}} \times k_0 \cancel{a\sqrt{2\pi}} = k_0\hbar = P_0 \quad \boxed{\checkmark}$$

2)

$$\langle P^2\rangle = \int_{-\infty}^{\infty} \psi^* \hat{P}^2 \psi \, dx = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\delta}{\delta x}\right)\left(-i\hbar\frac{\delta}{\delta x}\right)\psi \, dx$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} \Big[\underbrace{-k_0^2 e^{-\frac{(x-x_0)^2}{2a^2}}}_{(I)} - \underbrace{i k_0 \frac{x-x_0}{a^2} e^{-\frac{(x-x_0)^2}{2a^2}}}_{(II)}$$

$$+ \underbrace{\left(\frac{x-x_0}{2a^2}\right)^2 e^{-\frac{(x-x_0)^2}{2a^2}}}_{(III)} - \underbrace{\frac{1}{2a^2} e^{-(x-x_0)^2/2a^2}}_{(IV)} \Big] dx$$

$$(I) \Rightarrow -k_0^2 \int_{-\infty}^{\infty} e^{-(x-x_0)^2/2a^2} dx = -k_0^2 \int_{-\infty}^{\infty} e^{-\eta^2/2} a\, d\eta = -k_0^2 a\sqrt{2\pi} \quad \boxed{\checkmark}$$

$$\begin{cases} \frac{x-x_0}{a} = \eta \\ dx = a\, d\eta \end{cases}$$

(II) $\Rightarrow$ $$\int_{-\infty}^{\infty} -ik_0 \frac{x-x_0}{a^2} e^{-(x-x_0)^2/2a^2} dx = -\frac{ik_0}{a}\int_{-\infty}^{\infty} \eta\, e^{-\eta^2/2} a\, d\eta$$

$$\begin{cases} \frac{x-x_0}{a} = \eta \\ dx = a\,d\eta \end{cases}$$

$$= 0 \quad \boxed{\checkmark}$$

(III) $\Rightarrow$ $$\int_{-\infty}^{\infty} \left(\frac{x-x_0}{2a^2}\right)^2 e^{-(x-x_0)^2/2a^2} dx = \frac{1}{4a^2}\int_{-\infty}^{\infty} \eta^2 e^{-\eta^2/2} d\eta \times a$$

$$\begin{cases} \frac{x-x_0}{a} = \eta \\ dx = a\,d\eta \end{cases} \qquad \begin{cases} U = \eta \\ dV = \eta e^{-\eta^2/2} d\eta \end{cases} \rightarrow \begin{cases} dU = 1 \\ V = -e^{-\eta^2/2} \end{cases}$$

$$= \frac{1}{4a}\left[U.V\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} V.dU \right]$$

$$= \frac{1}{4a}\left[\underbrace{-\eta e^{-\eta^2/2}\Big|_{-\infty}^{\infty}}_{0} + \int e^{-\eta^2/2} \times 1 \right] = \frac{1}{4a} \times \sqrt{2\pi} \quad \boxed{\checkmark}$$

(IV) $\Rightarrow$ $$\int_{-\infty}^{\infty} -\frac{1}{2a^2} e^{-(x-x_0)^2/2a^2} dx = -\frac{1}{2a^2} \times a \int_{-\infty}^{\infty} e^{-\eta^2/2} d\eta$$

$$\begin{cases} \frac{x-x_0}{a} = \eta \\ dx = a\,d\eta \end{cases}$$

$$= -\frac{1}{2a}\sqrt{2\pi} \quad \boxed{\checkmark}$$

نتیجه می $\Longrightarrow$ $$\langle p^2 \rangle = -\hbar^2 A^2 \times \left(-k_0^2 a \sqrt{\pi} + 0 + \frac{1}{4a}\sqrt{2\pi} - \frac{1}{2a}\sqrt{2\pi} \right)$$

$$= -\hbar^2 \frac{1}{\cancel{a}\,\cancel{\sqrt{2\pi}}}\, \cancel{a} \times \cancel{\sqrt{2\pi}} \left(-k_0^2 + \frac{1}{4a^2} - \frac{1}{2a^2} \right)$$

$$= +k_0^2 \hbar^2 + \frac{\hbar^2}{4a^2} = +p_0^2 + \frac{\hbar^2}{4a^2} \quad \boxed{\checkmark}$$

$$* \; \Delta P = \left(\langle p^2 \rangle - \langle p \rangle^2 \right)^{1/2} = \left(\cancel{p_0^2} - \hbar^2/4a^2 - \cancel{p_0^2} \right)^{1/2}$$

$$= \frac{\hbar}{2a}$$

$$\Delta x \cdot \Delta P = a \cdot \hbar/2a = \hbar/2$$