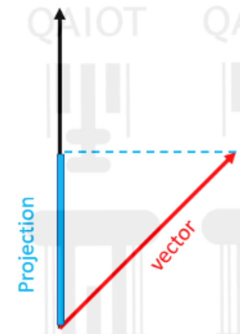
The Math of Measurement: Projections

Projections: The value of a vector quantity along an axis is known as the *projection* of the vector quantity along that axis. Projection along the axis is given as a dot product.

Projection = measurement axis · vector

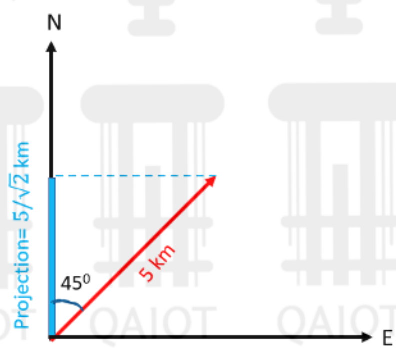


Simple Classical Example: A car has traveled 5 km along North-East (NE). How much it has traveled along North (N)?

Let us say North represents +y-axis and East represents +x-axis, then NE is a direction in first quadrant at 45 degrees from both +x and +y-directions

Vector: 5 km along NE

Answer: Value of vector along N: Projection of vector along N = N · 5 km along NE = $5/\sqrt{2}$ km



Rapid Review of Notation:

Let us quickly recall some basic math about quantum states and Dirac notation first.

Qubit states along z and -z, typically called up and down (on Bloch Sphere) are:

$|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}$

Here are dot products:

$$\langle 0|1\rangle = |0\rangle^{\dagger}|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\dagger}\begin{pmatrix}0\\1\end{pmatrix} = (1 \quad 0)\begin{pmatrix}0\\1\end{pmatrix} = 0 + 0 = 0$$

$$\langle 0|0\rangle = |0\rangle^{\dagger}|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\dagger}\begin{pmatrix}1\\0\end{pmatrix} = (1 \quad 0)\begin{pmatrix}1\\0\end{pmatrix} = 1 + 0 = 1$$

Similarly, please check yourself that $\langle 1|0\rangle = 0$ and $\langle 1|1\rangle = 1$

Quantum Measurement as Projections:

In quantum world, the results are not known deterministically. Projections are related to probability of measurement instead in the following way:

Probability of measurement along measurement axis = |projection of the quantum state along measurement axis|^2 = $|\langle \mathit{measurement\ axis}|\mathit{quantum\ state}\rangle|^2$

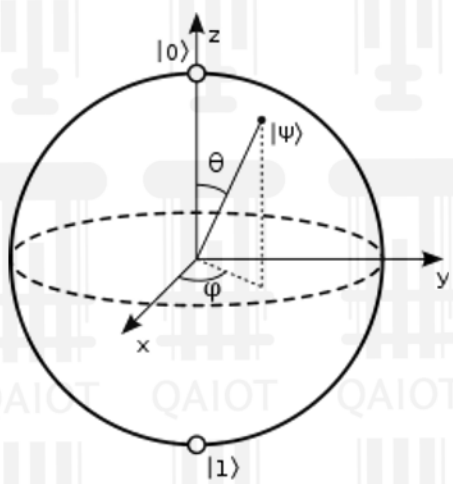
OK, let us now look at Stern-Gerlach apparatuses (just measurement axes for our purposes) oriented along different directions.

Quantum Examples:

Measurement axis +z = $|0\rangle$, measurement axis -z = $|1\rangle$,

measurement axis +x = $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, measurement axis -x = $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Why x-axis has representations as shown? Please check out Bloch Sphere at https://en.wikipedia.org/wiki/Bloch_sphere.



In general, for a qubit at an angle θ and ϕ, the qubit is represented as:

$$\psi(\theta,\phi) = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Measurement axis would follow the same rules as the qubit for determining the direction because a measurement axis is after all, a possible qubit value along that axis.

Also, recall as I said in lab, theory comes after experiments, so people did these measurements first, and then they designed this Bloch sphere theory.

Now let use the definition

Probability of measurement along measurement axis = |projection of the quantum state along measurement axis|^2 = $|\langle \textit{measurement axis}|\textit{quantum state}\rangle|^2$

1) Let us take qubit along +z i.e. $|0\rangle$

a) Probability of measurement of qubit ($|0\rangle$) along along +z ($|0\rangle$):

$$|\langle 0|0\rangle^2| = 1$$

b) Probability of measurement of qubit ($|0\rangle$) along along -z($|1\rangle$):

$$|\langle 1|0\rangle|^2 = 0$$

c) Probability of measurement of qubit ($|0\rangle$) along along +x($|+\rangle$):

$$|\langle +|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|0\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(\langle 0|0\rangle + \langle 1|0\rangle)\right|^2 = \left|\frac{1}{\sqrt{2}}(1+0)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

d) Probability of measurement of qubit ($|0\rangle$) along along +x($|+\rangle$):

$$|\langle -|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)|0\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(\langle 0|0\rangle - \langle 1|0\rangle)\right|^2 = \left|\frac{1}{\sqrt{2}}(1-0)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Quantum Exercise:

What if qubit is along is along +x ($|+\rangle$)?

Please do the math and verify that what we would think intuitively:

a) Probability of measurement of qubit ($|+\rangle$) along along +z ($|0\rangle$): 1/2

b) Probability of measurement of qubit ($|+\rangle$) along along -z($|1\rangle$): 1/2

c) Probability of measurement of qubit ($|+\rangle$) along along +x($|+\rangle$): 1

d) Probability of measurement of qubit ($|+\rangle$) along along +x($|+\rangle$): 0