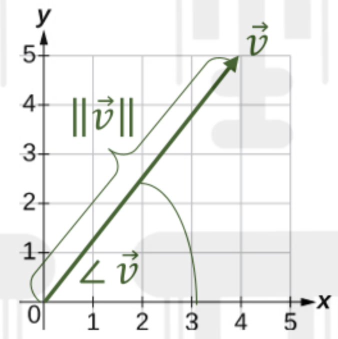
Vectors and Matrices Cheat Sheet

Vectors represent a quantity that has both **magnitude** and **direction**.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \angle\,\vec{v} = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$



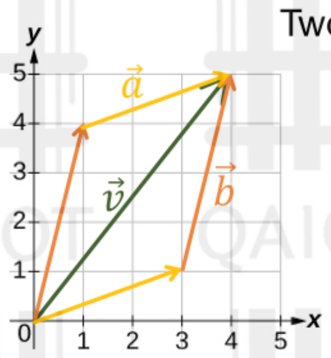
$$\vec{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\|\vec{v}\| = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$$

$$\angle\,\vec{v} = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{5}{4}\right) = 0.896\ radians$$

Two vectors can be added together. Any vector can be multiplied by a **scalar**.

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

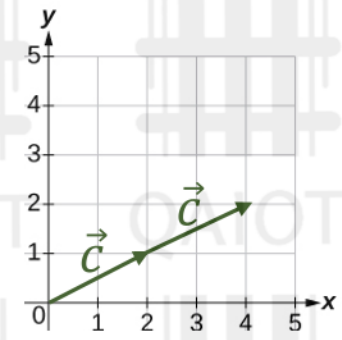


$$\vec{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\vec{a} + \vec{b} = \begin{pmatrix} 3+1 \\ 1+4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \vec{v}$$

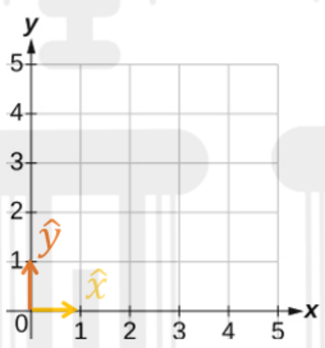
$$\vec{b} + \vec{a} = \begin{pmatrix} 1+3 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \vec{v}$$

$$c * \vec{v} = \begin{pmatrix} c * v_1 \\ c * v_2 \\ \vdots \\ c * v_n \end{pmatrix}$$



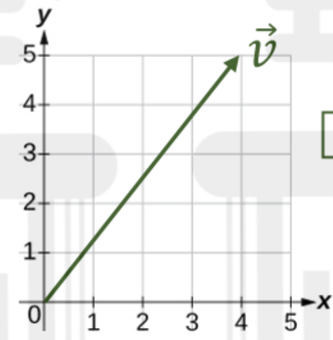
$$\vec{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad 2 * \vec{c} = \begin{pmatrix} 2*2 \\ 2*1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

A vector of magnitude 1 is called a **unit** vector. A vector can be **normalized** to obtain a unit vector in the same direction.



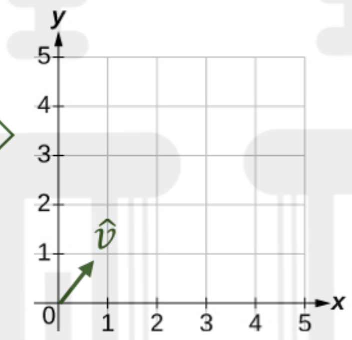
$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\hat{x}$ and $\hat{y}$ are known as the **standard basis** vectors



Normalize

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

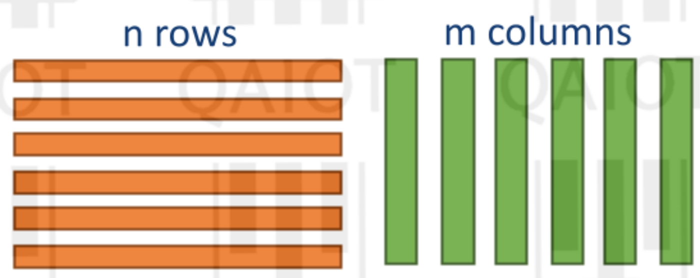


$$\vec{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \hat{v} = \frac{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}{\sqrt{41}}$$

A matrix is a **rectangular array** of numbers organized into **rows** and **columns**. Vectors are special cases of matrices.

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

X is an $(n \times m)$ matrix



$$A = \begin{pmatrix} 10 & -1 \\ 12 & 6 \end{pmatrix} \quad \mathrm{B} = \begin{pmatrix} 3 & -i \\ 1 & 4 \end{pmatrix} \quad \mathrm{C} = \begin{pmatrix} 0 & 12 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}$$

Two matrices can be added together. Matrices can be multiplied by scalars, and by other matrices.

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

$$A + B = \begin{pmatrix} 10 & -1 \\ 12 & 6 \end{pmatrix} + \begin{pmatrix} 3 & -i \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 10+3 & -1-i \\ 12+1 & 6+4 \end{pmatrix} = \begin{pmatrix} 13 & -1-i \\ 13 & 10 \end{pmatrix}$$

$$c * A = c * \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} c*a_{11} & c*a_{12} & \cdots & c*a_{1m} \\ c*a_{21} & c*a_{22} & \cdots & c*a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c*a_{n1} & c*a_{n2} & \cdots & c*a_{nm} \end{pmatrix}$$

$$3 * \mathrm{B} = 3 * \begin{pmatrix} 3 & -i \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3*3 & 3*-i \\ 3*1 & 3*4 \end{pmatrix} = \begin{pmatrix} 9 & -3i \\ 3 & 12 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, \vec{b}_1 \rangle & \langle \vec{a}_1, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_1, \vec{b}_k \rangle \\ \langle \vec{a}_2, \vec{b}_1 \rangle & \langle \vec{a}_2, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_2, \vec{b}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{a}_n, \vec{b}_1 \rangle & \langle \vec{a}_n, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_n, \vec{b}_k \rangle \end{pmatrix}$$

$$\mathrm{AB} = \begin{pmatrix} 10 & -1 \\ 12 & 6 \end{pmatrix} * \begin{pmatrix} 3 & -i \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 10*3+(-1)*1 & 10*-i+(-1)*4 \\ 12*3+6*1 & 12*-i+6*4 \end{pmatrix} = \begin{pmatrix} 29 & -4-10i \\ 42 & 24-12i \end{pmatrix}$$

The **transpose** is an operation that **flips** the shape of a matrix.The **conjugate transpose** additionally replaces each entry with its **conjugate**.

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \quad X^T = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m} & x_{2m} & \cdots & x_{nm} \end{pmatrix} \quad X^\dagger = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{n1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m}^* & x_{2m}^* & \cdots & x_{nm}^* \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & -1 \\ 12 & 6 \end{pmatrix} \quad \mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 10 & 12 \\ -1 & 6 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} 3 & -i \\ 1 & 4 \end{pmatrix} \quad B^\dagger = \begin{pmatrix} 3 & 1 \\ i & 4 \end{pmatrix}$$

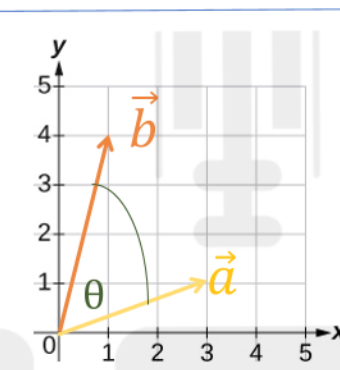
The **inner product** is an important operation on two vectors. It can be used to find the **angle between** two vectors.

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^\dagger \vec{w} = v_1^* w_1 + \cdots + v_n^* w_n = \sum_{i=1}^{n} v_i^* w_i$$

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos(\theta) \qquad \theta = \cos^{-1}\left(\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|}\right)$$

$$\vec{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \langle \vec{a}, \vec{b} \rangle = \vec{a}^\dagger \vec{b} = 3 * 1 + 1 * 4 = 12$$

$$\theta = \cos^{-1}\left(\frac{12}{\sqrt{10}\sqrt{17}}\right) = 0.402 \text{ radians}$$



The **identity matrix** has 1s along it's diagonals and 0s elsewhere. Matrix multiplication by the identity is analogous to scalar multiplication by 1. We define the **inverse** of a matrix using the identity matrix.

$$X\,\mathbb{I} = \mathbb{I}\,X = X \qquad \vec{x}\,\mathbb{I} = \mathbb{I}\,\vec{x} = \vec{x}$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

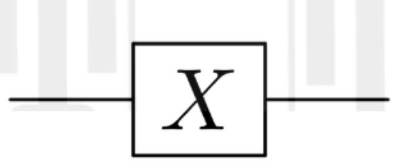
$$XX^{-1} = X^{-1}X = \mathbb{I}$$

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow X^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & -1 \\ 12 & 6 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{72}\begin{pmatrix} 6 & 1 \\ -12 & 10 \end{pmatrix}$$

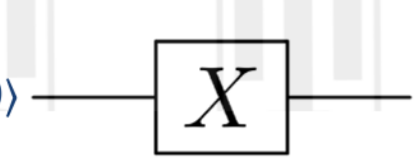
Why is all this important!? Well it turns out that vectors and matrices are the language we use to talk about quantum computing. Quantum states are represented by vectors, quantum gates are represented by matrices and the application of a gate to a state is represented by matrix-vector multiplication.

$$|0\rangle \Leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$|0\rangle$



$$|1\rangle \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$