Dirac Notation: Complex amplitudes, Real Probabilities and Normalization

1. Defining the notation:

Dirac developed a powerful notation which is super-useful in quantum mechanics, hence quantum computation.

$\langle\rangle$ - This is a *bracket*! The right part of the *bracket* is used to denote a *ket.* This represents a vector. Let us look at our familiar examples

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{1}$$

We define an operator called dagger which is complex conjugate transpose (Please see Appendix for a quick refresher of what a complex conjugate or what a transpose is).

This means that if α, β are complex numbers

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix}^\dagger = \left(\begin{pmatrix}\alpha\\\beta\end{pmatrix}^*\right)^T = \begin{pmatrix}\alpha^*\\\beta^*\end{pmatrix}^T = (\alpha^* \quad \beta^*) \tag{2}$$

Empowered with this, we can define the left part of the *bracket* to denote a *bra.*

$$\langle 0| = |0\rangle^\dagger = \begin{pmatrix}1\\0\end{pmatrix}^\dagger = (1 \quad 0) \tag{3}$$

Exercise 1:

Please write down $\langle 1|$. (Ans: $(0 \quad 1)$)

2. Dot product

Let us recall that the dot product between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as:

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^\dagger \boldsymbol{v} \tag{4}$$

Hence in Dirac notation, if we represented vectors $\boldsymbol{u}$ as $|u\rangle$ and $\boldsymbol{v}$ as $|v\rangle$, then the dot product is defined as:

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^\dagger \boldsymbol{v} = \langle u|v\rangle \tag{5}$$

Please note that a dot product is a scalar, hence $\langle u|v\rangle$ is a scalar!

Let us calculate quick one:

$$\langle 0|1\rangle = |0\rangle^\dagger|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}^\dagger \begin{pmatrix}0\\1\end{pmatrix} = (1 \quad 0)\begin{pmatrix}0\\1\end{pmatrix} = 0 + 0 = 0 \tag{6}$$

Exercise 2:

Please calculate $\langle 1|0\rangle$.

Please show or convince yourself that $\langle v|u\rangle = \langle u|v\rangle^*$

3. Complex Amplitudes, Real Probabilities

In Quantum computation, when we write down a qubit, a superposition:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{7}$$

the coefficients α and β are complex numbers. They are called *probability amplitudes.* (NOT probability! Probability has to be a real number between 0 and 1.)

A physicist Max Born provided what is known as Born rule, a foundation of quantum mechanics: For a state ψ, the probability density is given as $|\psi|^2$. (Please do not worry about what probability density is, we will see it soon enough, moral here is that the probability is related to $|\psi|^2$ which is real, not complex.)

For our purposes, we know that the total probability has to be 1 (An axiom of probability). This is expressed in Dirac notation as:

$$\langle\psi|\psi\rangle = 1 \tag{8}$$

Let us calculate this quantity to see what condition we get on the coefficients α and β. We start from the beginning in Eq.(7)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$\langle\psi| = |\psi\rangle^\dagger = (\alpha^* \quad \beta^*)$$

$$\langle\psi|\psi\rangle = (\alpha^* \quad \beta^*)\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2 \tag{9}$$

An alternate way to calculate this could be, starting again from Eq.(7)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

This implies (please check with definitions, following equation works and this is why Dirac notation is awesome! We just switch the bracket shapes around and take complex conjugate of all coefficients.)

$$\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1| \tag{10}$$

$$\begin{aligned}\langle\psi|\psi\rangle &= (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle) \qquad (11)\\ &= \alpha^*\alpha\langle 0|0\rangle + \alpha^*\beta\langle 0|1\rangle + \beta^*\alpha\langle 1|0\rangle + \beta^*\beta\langle 1|1\rangle\\ &= |\alpha|^2 + |\beta|^2\end{aligned}$$

Where we used in the last line that $\langle 0|1\rangle = \langle 1|0\rangle = 0$, and $\langle 0|0\rangle = \langle 1|1\rangle = 1$ (Please just write down the vector definitions and show all this, also see Eq.(6))

Therefore, the probability axiom would imply

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1 \tag{10}$$

4. Normalization

Like with vectors, sometimes quantum states are casually written without normalization. For example someone may write a qubit as

$$|\psi\rangle = |0\rangle + (1 + 2i)|1\rangle$$

Here $\alpha = 1, \beta = 1 + 2i$. Hence $|\alpha|^2 + |\beta|^2 = 1^2 + (1^2 + 2^2) = 6$. Thus this $|\psi\rangle$ does not satisfy the condition of total probability =1 (Probability axiom) as expressed for qubits in Eq.(10). Therefore, they will not give us correct probabilities in calculations! Hence we need to *normalize* these states to get correct probabilities.

Let us imagine a state which is not normalized

$$|\psi\rangle = A|0\rangle + B|1\rangle \tag{11}$$

where A and B are complex numbers.

to normalize them, we need to multiple them with some constant c, such that Eq.(10), the total probability $\langle\psi|\psi\rangle = 1$ is satisfied. So let us have

$$|\psi\rangle = cA|0\rangle + cB|1\rangle \tag{12}$$

Then

$$\langle\psi|\psi\rangle = c^2|A|^2 + c^2|B|^2 = 1$$

$$c = \frac{1}{\sqrt{|A|^2+|B|^2}} \tag{13}$$

Hence the normalized $|\psi\rangle$ is

$$|\psi\rangle = \frac{A}{\sqrt{|A|^2 + |B|^2}}|0\rangle + \frac{B}{\sqrt{|A|^2 + |B|^2}}|1\rangle$$

(Note that now, $\langle\psi|\psi\rangle = 1$. We will get correct probabilities from this $|\psi\rangle$!)

Appendix

A. Complex Conjugate

For a complex number $z = a + ib$, complex conjugate $z^* = a - ib$

Examples:

a) $z = 2, z^* = 2$;

b) $z = 5i, z^* = -5i$;

c) $z = 2 + 3i, z^* = 2-3\mathrm{i}$

B. Transpose

For a matrix M with elements a_{ij}, transpose of the matrix is M^T with elements $a'_{ij} = a_{ji}$.

Examples:

a) $1^T = 1$ (scalar transpose is scalar :))

b) $(1 \quad -1)^T = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ i \end{pmatrix}^T = (1 \quad i)^T$

d) $\begin{pmatrix} 1 & 2 \\ -3 & i \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 2 & i \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 \\ -2 & 5 \\ 3+i & 4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3+i \\ 2 & 5 & 4 \end{pmatrix}$

C. Modulus

For a complex number $z = a + ib$, (with $i^2 = -1$)

$$zz^* = (a + ib)(a - ib) = a^2 - \cancel{iab} + \cancel{iba} - i^2b^2 = a^2 + b^2 = |z|^2$$