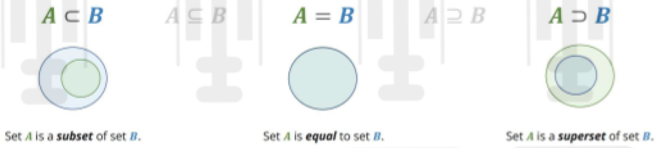
Probability Cheat Sheet

A **set** is a collection of **distinct objects**, which can be considered an object in its own right. The **arrangement** of the objects inside the set **does not matter**!

A = {1, 9, 5, 3, 7} is the set of positive odd numbers less than 10.

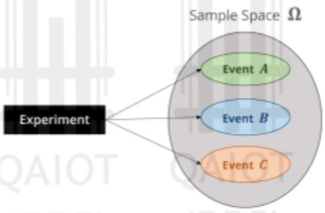
B = {Red, Green, Blue} is the set containing these 3 colors.

C = {A, B} is the set containing the sets A and B!



{Cat, Dog} ⊂ {Cat, Dog, Mouse}

The **sample space (Ω)** is the set of all possible outcomes of an experiment.



$$\Omega = \{A, B, C\}$$

(we denote all the events in the space using set notation)

Rolling a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Taking a pass/fail class: $\Omega = \{\text{Pass}, \text{Fail}\}$

Measuring a qubit: $\Omega = \{|0\rangle, |1\rangle\}$

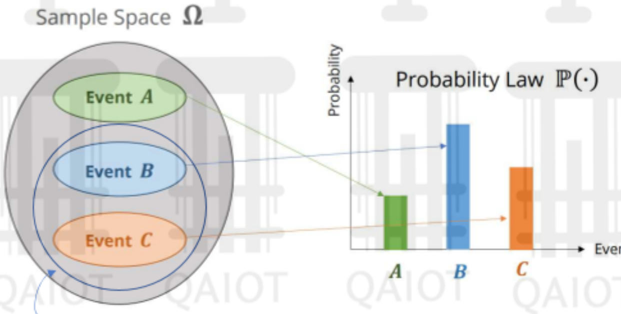
Properties

Collectively exhaustive

Mutually exclusive elements

In simple words, two outcomes cannot occur simultaneously

A **probability law** ($\mathbb{P}$) assigns a probability to each element of the sample space.



Event D

Events can combine multiple possible outcomes

If $B \subseteq D$, then $\mathbb{P}(B) \leq \mathbb{P}(D)$

$\mathbb{P}(\text{empty set}) = \mathbb{P}(\{\}) = P(\emptyset) = 0$

$\mathbb{P}(A^C) = 1 - P(A)$

Nonnegativity: $\mathbb{P}(A) \geq 0$, for every event A

The probability of any event has to be **greater than or equal** to 0.

Addivity: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

If A and B are **disjoint/mutually exclusive events,** then the above is true.

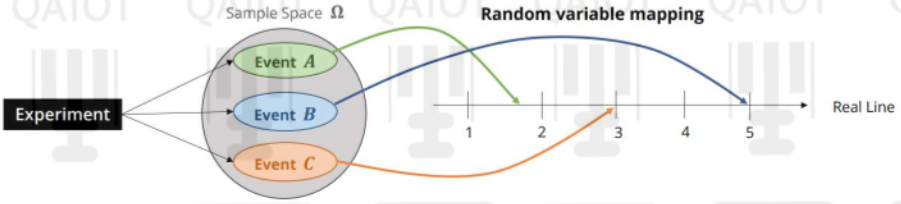
Normalization: $\mathbb{P}(\Omega) = 1$

Since the sample spaceis **collectively exhaustive,** the probablity of the outcome lying inside it is 1.

$\mathbb{P}(A \cap B) = 0$ when A and B are disjoint

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

A **random variable (RV)** maps each possible outcome of an experiment to a real number.
A random variable is just a **function** from the sample space to the real line ($\Omega \to \mathbb{R}$) !



The **Probability Mass Function (PMF)** associates a probability to each possible value of the random variable

HT & TH are disjoint events, they never occur simultaneously!

If we toss 2 **fair** coins, the possible outcomes are HH, HT, TH, HH, all **equally likely** (probability = ¼)

Define the RV as the number of heads. We get the following mapping:
HH -> 2, HT -> 1, TH -> 1, TT -> 0

We obtain the following PMF:
P(2) = P(HH) = ¼
P(1) = P(HT ∪ TH) = P(HT) + P(TH) = ½
P(0) = P(TT) = ¼

The **expectation E[X]** of a random variable X is the **weighted average** of its possible values. The **variance** is the **expectation** of the random variable **(X – E[X])²**.

$$\mathbb{E}[X] = \langle X \rangle = \sum_x x \cdot \mathbb{P}(X = x)$$

The upper case "X" is the random variable while the lower case "x" is the different values "X" can take

$$var[X] = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \left(x - \mathbb{E}(X)\right)^2 \cdot \mathbb{P}(X = x)$$

The variance of a random variable provides a measure of how **dispersed** the data individual values of X are, around the mean.

Even though the random variable might never equal its expected value, it is still a useful property.

Law of Large Numbers: As the number of trials of an experiment is increased, the observed average gets closer and closer to the expected value.

The **Bra-Ket** notation

A **ket** is a column vector

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

A **bra** is the conjugate transpose of a ket (a row vector)

$$\langle v| = |v\rangle^\dagger = (\overline{v_1} \ \ \overline{v_2} \ \cdots \ \overline{v_n})$$

Ground state: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Excited state: $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Superposition state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

α and β here are called **probability amplitudes.**
Measuring $|\Psi\rangle$ will give $|0\rangle$ w.p. $|\alpha|^2$ and $|1\rangle$ w.p. $|\beta|^2$