IOM Exercise 4. Taylor Series Expansion.

① $f(x) = \frac{x}{x+1}$

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \cancel{f}\, O((x-x_0)^3)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ $\cancel{f'(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$f'(x) = \frac{(x+1)(1) - (x)(1)}{(x+1)^2}$

$f'(x) = \frac{1}{(x+1)^2}$

$f''(x) = -2(x+1)^{-3} \cdot \frac{d}{dx}(x+1)$

$f''(x) = \frac{-2}{(x+1)^3}$

$\cancel{f(x)}$ $\therefore f(x) = \frac{x_0}{x_0+1} + \frac{(x_0+1)^{-2}(x-x_0)}{1} - \frac{2(x_0+1)^{-3}(x-x_0)^2}{2!}$

~~$f(x) = \frac{x_0}{x_0+1} + (x_0+1)^{-2}(x-x_0)$~~

$f(x) = \frac{x_0}{x_0+1} + \frac{(x-x_0)}{(x_0+1)^2} + \frac{(x-x_0)^2}{(x_0+1)^3} + O(x-x_0)^3$

(2) $f(x) = \frac{2x^2}{x^4} = 2x^{-2}$

$\therefore f'(x) = -4x^{-3}$

$f''(x) = 12x^{-4}$

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + O(x-x_0)^3$

$\therefore f(x) = 2x^{-2} - 4x^{-3}(x-x_0) + \frac{12x^{-4}(x-x_0)^2}{2!} + O(x-x_0)^3$

$\cancel{2x^{-2}} \; \cancel{4x} \; \frac{2}{x^2} - \frac{4}{x^3} + \frac{6}{x^{-4}}$

$= 2x^{-2} - 4x^{-3}(x-x_0) + 6x^{-4}(x-x_0)^2 + O(x-x_0)^3$

$= \frac{2}{x^2} - \frac{4(x-x_0)}{x^3} + \frac{6(x-x_0)^2}{x^4} + O(x-x_0)^3$

③ $f(x) = \ln x^2 = 2\ln x$

$f'(x) = \frac{2}{x} = 2x^{-1}$

$f''(x) = -2x^{-2}$

$\therefore f(x) = 2\ln x_0 + 2x^{-1}(x-x_0) - \frac{2x^{-2}(x-x_0)^2}{2!} + O(x-x_0$

$= \ln x_0^2 + \frac{2(x-x_0)}{x} - \frac{\cancel{2}(x-x_0)^2}{x^2} + O(x-x_0)^3$