QF605 Fixed Income (AY2021/2022)

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Part I – Bootstrapping

OIS Discount Factor

Equations Used (Example for Year 2):

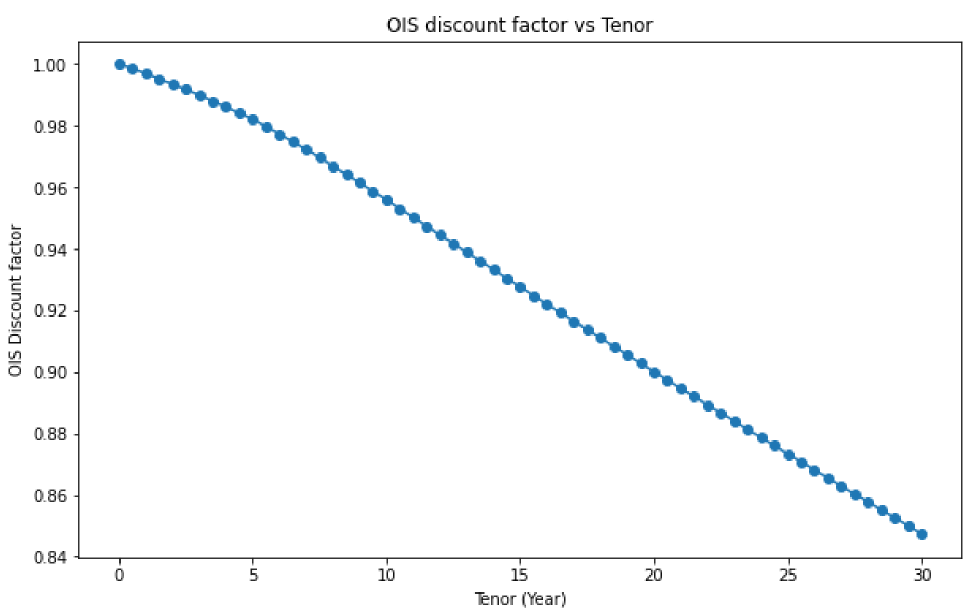
$$D_o(0, T_i) = \frac{1}{1+\frac{f_i}{360}}^{360*T_i}$$

$$PV_{fix}^{2y\ \text{OIS}} = PV_{flt}^{2y\ \text{OIS}}$$

$$\left[D_o(0,1y) + D_o(0,2y)\right] \times 0.325\% = D_o(0,1y) \times \left[\left(1+\frac{f_0}{360}\right)^{180}\left(1+\frac{f_1}{360}\right)^{180} - 1\right] + D_o(0,2y) \times \left[\left(1+\frac{f_2}{360}\right)^{360} - 1\right]$$

Using the above equations, linear interpolation for timeframes without corresponding overnight interest swaps (OIS) and Brent's method as our root finder to solve for $f_{i,}$ we derived the OIS discount factor table as shown below:

Tenor	OIS DF	Tenor	OIS DF	Tenor	OIS DF	Tenor	OIS DF	Tenor	OIS DF	Tenor	OIS DF
0.5	0.99875	5.5	0.97973	10.5	0.95311	15.5	0.92483	20.5	0.89737	25.5	0.87073
1	0.99701	6	0.97728	11	0.95024	16	0.92204	21	0.89467	26	0.86811
1.5	0.99527	6.5	0.97484	11.5	0.94738	16.5	0.91927	21.5	0.89198	26.5	0.86550
2	0.99353	7	0.97241	12	0.94453	17	0.91650	22	0.88929	27	0.86289
2.5	0.99177	7.5	0.96965	12.5	0.94169	17.5	0.91375	22.5	0.88662	27.5	0.86030
3	0.99002	8	0.96690	13	0.93886	18	0.91099	23	0.88395	28	0.85770
3.5	0.98807	8.5	0.96416	13.5	0.93604	18.5	0.90826	23.5	0.88129	28.5	0.85513
4	0.98612	9	0.96142	14	0.93322	19	0.90552	24	0.87864	29	0.85255
4.5	0.98415	9.5	0.95870	14.5	0.93042	19.5	0.90280	24.5	0.87600	29.5	0.84999
5	0.98218	10	0.95598	15	0.92761	20	0.90008	25	0.87336	30	0.84742



LIBOR Discount Factor

Equations Used (Example for Year 2):

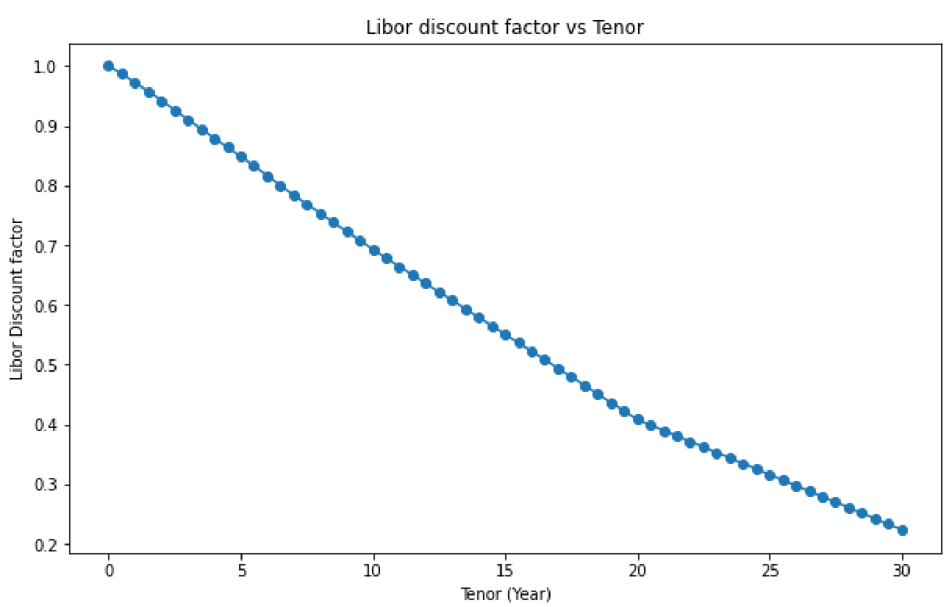
$$L(T_i, T_{i+1}) = \frac{D_L(0,T_{i-1}) - D_L(0,T_i)}{Daycount * D_L(0,T_i)}$$

$$PV_{fix}^{2yLibor} = PV_{flt}^{2yLibor}$$

$$0.5 * IRS * \sum_{i=0.5}^{2} D_o(0, T_i) = 0.5 * \sum_{i=0.5}^{2} D_o(0, T_i) * L(T_{i-0.5}, T_i)$$

Using the above equations, linear interpolation for timeframes without corresponding Interest Rate Swaps (IRS) and Brent's method as our root finder to solve for $L(T_i, T_{i+1})$, we derived the Libor discount factor table as shown below:

Tenor	Libor DF	Tenor	Libor DF	Tenor	Libor DF	Tenor	Libor DF	Tenor	Libor DF	Tenor	Libor DF
0.5	0.98765	5.5	0.83280	10.5	0.67855	15.5	0.53679	20.5	0.39899	25.5	0.30671
1	0.97258	6	0.81660	11	0.66438	16	0.52251	21	0.38976	26	0.29748
1.5	0.95738	6.5	0.80041	11.5	0.65022	16.5	0.50822	21.5	0.38053	26.5	0.28825
2	0.94218	7	0.78422	12	0.63606	17	0.49393	22	0.37130	27	0.27902
2.5	0.92633	7.5	0.76897	12.5	0.62189	17.5	0.47965	22.5	0.36208	27.5	0.26980
3	0.91048	8	0.75371	13	0.60773	18	0.46536	23	0.35285	28	0.26057
3.5	0.89473	8.5	0.73846	13.5	0.59357	18.5	0.45107	23.5	0.34362	28.5	0.25134
4	0.87898	9	0.72321	14	0.57941	19	0.43679	24	0.33439	29	0.24211
4.5	0.86398	9.5	0.70796	14.5	0.56524	19.5	0.42250	24.5	0.32516	29.5	0.23288
5	0.84899	10	0.69271	15	0.55108	20	0.40822	25	0.31594	30	0.22366



Forward Swap Rate (FSR)

Using the calculated OIS discount factors and the Libor forwards (calculated from the Libor discount factors), we can calculate the FSR as shown below:

Tenor	FSR
1 x 1	0.032007
1 x 2	0.033259
1 x 3	0.034011
1 x 5	0.035255
1 x 1	0.038428
5 x 1	0.039274
5 x 2	0.040075
5 x 3	0.040072
5 x 5	0.041093
5 x 10	0.043634
10 x 1	0.042190
10 x 2	0.043116
10 x 3	0.044097
10 x 5	0.046249
10 x 10	0.053458

Part II – Model Calibration

Displaced Diffusion Model

Sigma					
Expiry\Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.307542	0.345404	0.341398	0.289518	0.259990
5Y	0.302838	0.312472	0.308905	0.272008	0.247628
10Y	0.296200	0.297524	0.294184	0.267308	0.243554
Beta					
Expiry\Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.012154	0.015845	0.031462	0.111442	0.295188
5Y	0.052560	0.107758	0.144572	0.249040	0.367633
10Y	0.121313	0.180841	0.146493	0.286947	0.386948

SABR Model

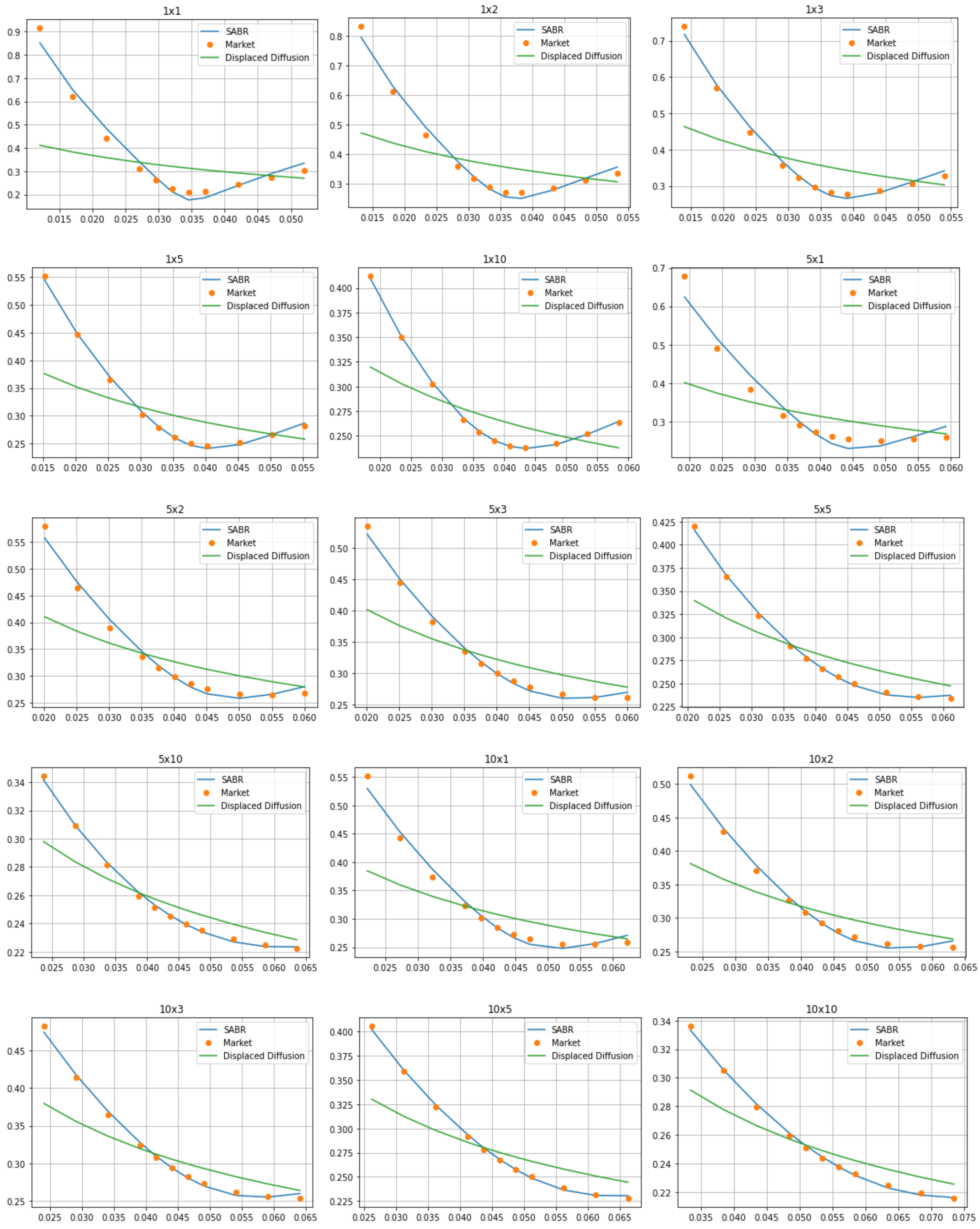
Alpha					
Expiry\Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.139070	0.184646	0.196851	0.178052	0.171237
5Y	0.166509	0.199497	0.210346	0.191011	0.177441
10Y	0.177551	0.195043	0.207212	0.201519	0.180061
Rho					
Expiry\Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	-0.63322	-0.52512	-0.48284	-0.41443	-0.26565
5Y	-0.58516	-0.54687	-0.54976	-0.51106	-0.44079
10Y	-0.54586	-0.54376	-0.55087	-0.56245	-0.50698
Nu					
Expiry\Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	2.049482	1.677437	1.438138	1.064877	0.776535
5Y	1.339619	1.061909	0.936718	0.671724	0.493849
10Y	1.007291	0.924810	0.869132	0.719875	0.579936

Pricing Swaptions

2 x 10 Payer Swaption			***8 x 10 Receiver Swaption***		
K	***Displaced Diffusion***	***SABR***	***K***	***Displaced Diffusion***	***SABR***
1%	*0.287825*	*0.290670*	*1%*	*0.010445*	*0.019217*
2%	*0.194357*	*0.199352*	*2%*	*0.024849*	*0.038276*
3%	*0.112743*	*0.116159*	*3%*	*0.049254*	*0.060689*
4%	*0.054014*	*0.052858*	*4%*	*0.085163*	*0.089214*
5%	*0.021025*	*0.021666*	*5%*	*0.132747*	*0.128133*
6%	*0.006680*	*0.010836*	*6%*	*0.191083*	*0.182277*
7%	*0.001762*	*0.006650*	*7%*	*0.258592*	*0.252024*
8%	*0.000394*	*0.004645*	*8%*	*0.333466*	*0.332294*

Based on the result above, we can observe that as strike goes larger, the price of payer becomes lower, but the price of receiver becomes higher, and both models' results are very close to each other. However, since SABR does a better job in fitting market implied volatility, the result from SABR model is more accurate than Displaced Diffusion Model (please refer to next page for visualized comparison).

Model Calibration Plots



Part III – CMS Static Replication

Constant Maturity Swap Valuation

A constant maturity swap (CMS) pays a swap rate rather than a LIBOR rate on its floating leg. CMS products gives us an easy way to gain exposure to fixed-length longer-term interest rates by taking a view on a fixed point on the yield curve. The standard practice in the market is to use the static-replication method to obtain a model-independent convexity correction. By static-replication approach, and choosing the forward swap rate F = Sn,N (0) as our expansion point.

$$
\begin{aligned}
V_0 = {} & D(0,T)g(F) + h'(F)\left[V^{\text{pay}}(F) - V^{rec}(F)\right] \\
& + \int_0^F h''(K)V^{rec}(K)dK + \int_{F'}^{\infty} h''(K)V^{\text{pay}}(K)dK \\
= {} & D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^{\infty} h''(K)V^{pay}(K)dK
\end{aligned}
$$

IRR-settled option pricer (V-Pay or V-Rec) is given by:

$$
V(K) = D(0,T) \cdot \text{IRR}\left(S_{n,N}(0)\right) \cdot \text{Black 76}\left(S_{n,N}(0), K, \sigma_{\text{SABR}}, T\right)
$$

whereby Sigma for Black76 model is derived from the SABR model

From the calibrated SABR parameters in Part II, we can derive the sigma used in the Black76 lognormal model. This enables us to obtain the IRR settled option for both receiver and payer swaption. From there we can get the CMS rate followed by the PV of CMS leg. Obtaining the CMS rate requires the SABR cubic spline interpolated Alpha, Rho & Nu parameters (in Appendix) since these parameters are not linear across time.

CMS Rate is written as:

$$
E^T\left[S_{n,N}(T)\right] = g(F) + \frac{1}{D_0(0,T)}\left[\int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^{\infty} h''(K)V^{\text{pay}}(K)dK\right]
$$

PV of a CMS leg is the sum of the discounted values of the CMS rates, multiplied by the day count fraction:

$$
CMS \text{ leg } PV = \sum_{i=0.5}^{N} D_0(0,T_i) * \text{CMS}\left(S_{n,N}(T_i)\right) * \text{ Date Count}
$$

Hence:

- PV of a leg receiving CMS10y semi-annually over the next 5 years is **0.20209**.
- PV of a leg receiving CMS2y quarterly over the next 10 years is **0.38106**.

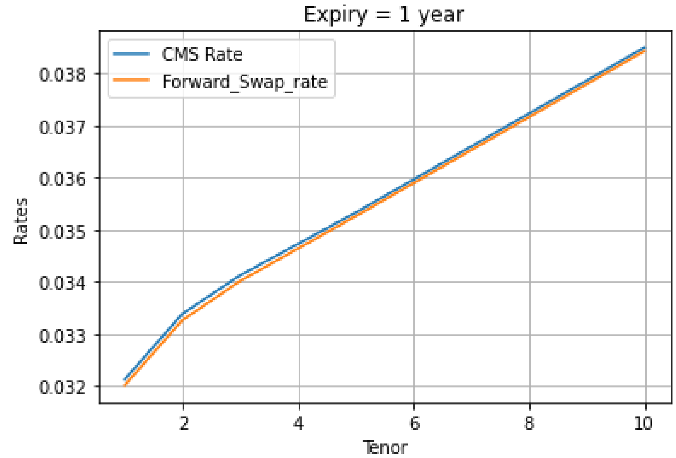
Effects on Convexity Correction

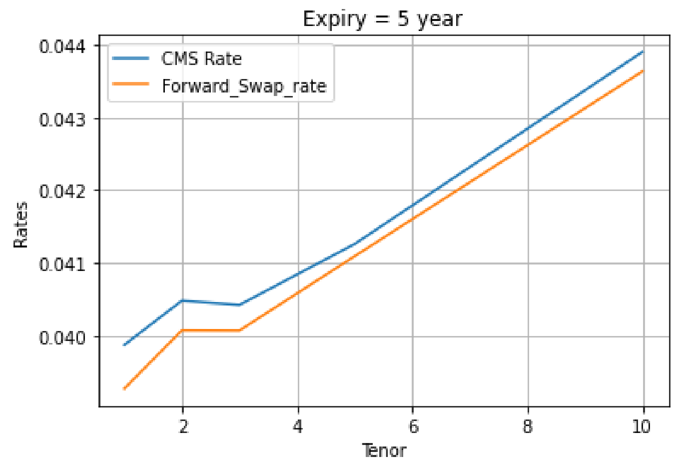
Comparison of CMS Rates & Forward Swap Rates			
Tenor	**CMS Rate**	**FSR Rate**	**Difference**
1 x 1	0.032120	0.032007	0.000113
1 x 2	0.033382	0.033259	0.000122
1 x 3	0.034120	0.034011	0.000109
1 x 5	0.035326	0.035255	0.000070
1 x 10	0.038496	0.038428	0.000068
5 x 1	0.040129	0.039274	0.000855
5 x 2	0.040756	0.040075	0.000681
5 x 3	0.040664	0.040072	0.000591
5 x 5	0.041532	0.041093	0.000439
5 x 10	0.044050	0.043634	0.000416
10 x 1	0.043635	0.042190	0.001446
10 x 2	0.044323	0.043116	0.001206
10 x 3	0.045225	0.044097	0.001128
10 x 5	0.047210	0.046249	0.000961
10 x 10	0.054528	0.053458	0.001069

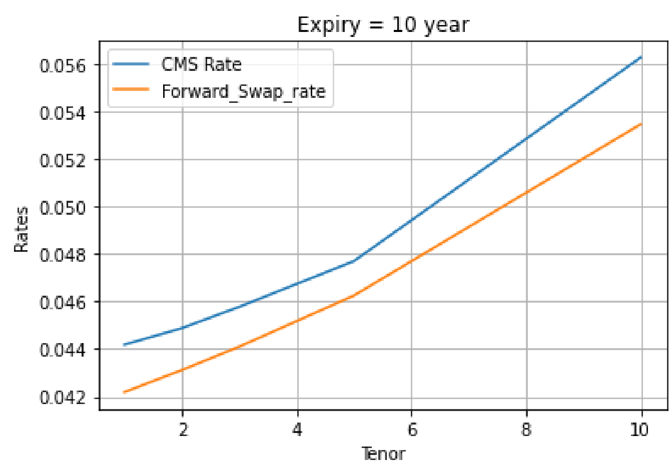
CMS convexity correction is the difference between the expected CMS rate and the implied forward swap rate (under the swap measure).

Correction is needed for CMS because the rates are being paid at a wrong time. Hence, the CMS rate should not be equal to the FSR, this difference should be positive because of the convexity of the function.

From our result, we see that this is true with CMS rates always being higher than FSR rates. The difference between CMS rate and FSR rate is a function of expiry, the longer the maturity the greater the effects of convexity correction.







Part IV – Decompounded Options Static Replication

Decompounded Option Valuation

Payoff: $CMS\ 10Y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$ (Decompounded Option)

The Static Replication equation can be represented using the IRR payer and receiver swaption. The formula is stated below:

$$CMS\ Payoff\ =\ D(0,T)F\ +\int_0^F h''(K)\,V^{rec}(K)\,dK\ +\ \int_F^\infty h''(K)\,V^{pay}(K)\,dK$$

Below is the equation necessary to complete the derivation:

$$g(F) = F^{\frac{1}{4}} - 0.2 \qquad g'(F) = \frac{1}{4}\mathrm{F}^{\frac{-3}{4}} \qquad g''(F) = \frac{-3}{16}\mathrm{F}^{\frac{-7}{4}}$$

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(K)g'(K)}{IRR(K)^2\mathrm{a}} + \frac{2IRR'(K)^2g(K)}{IRR(K)^3}$$

By substituting the 5 x 1 FSR, 5Y OIS DF and SABR parameters, the PV is **0.1773379**.

CMS Caplet Valuation

Payoff: $(CMS\ 10Y^{\frac{1}{4}} - 0.04^{\frac{1}{2}})^+$ (CMS Caplet)

To have a positive payoff there is a condition that needs to be satisfied:

$$\mathrm{F} > 0.2^4$$

The formula for CMS caplet with strike of 0.0016 will be:

$$\begin{aligned} \mathrm{CMS\ Caplet}\ &=\ \mathrm{D(0,T)}\int_{\mathrm{L}}^\infty \mathrm{g(K)f(K)dK} \\ &= \int_{\mathrm{L}}^\infty \mathrm{h(K)}\,\frac{\partial^2 \mathrm{V^{pay}(K)}}{\partial \mathrm{K}^2}\,\mathrm{dK}\ =\mathrm{h'(L)\ V^{pay}(L)}\ +\int_{\mathrm{L}}^\infty \mathrm{h''(K)\ V^{Pay}(K)\,dK} \end{aligned}$$

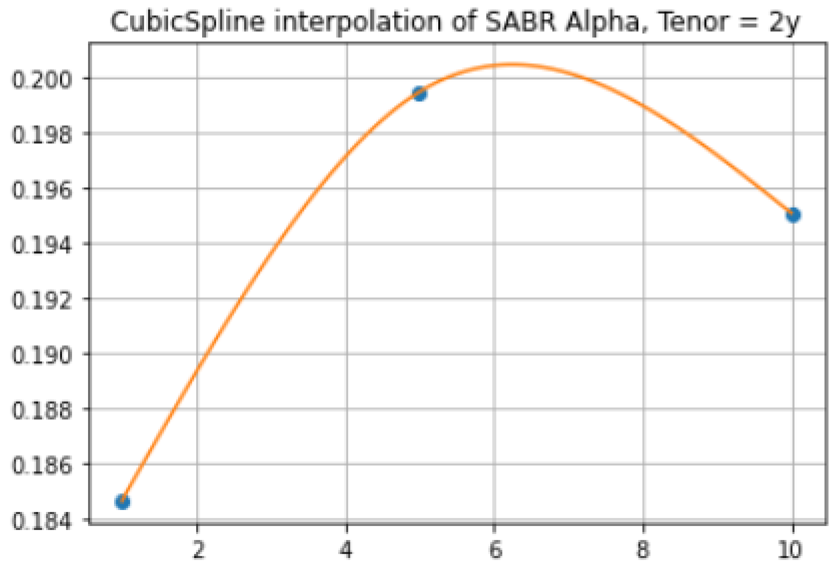
By substituting the FSR and SABR parameters, the PV is **2.6351730**.

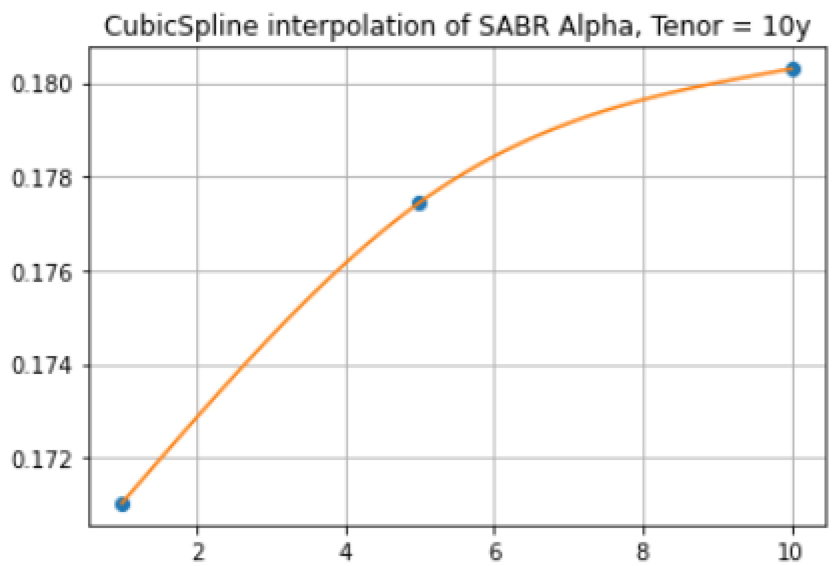
Appendix:

Calibrated SABR Parameters with Cubic Spline Interpolation for CMS Rate

Alpha

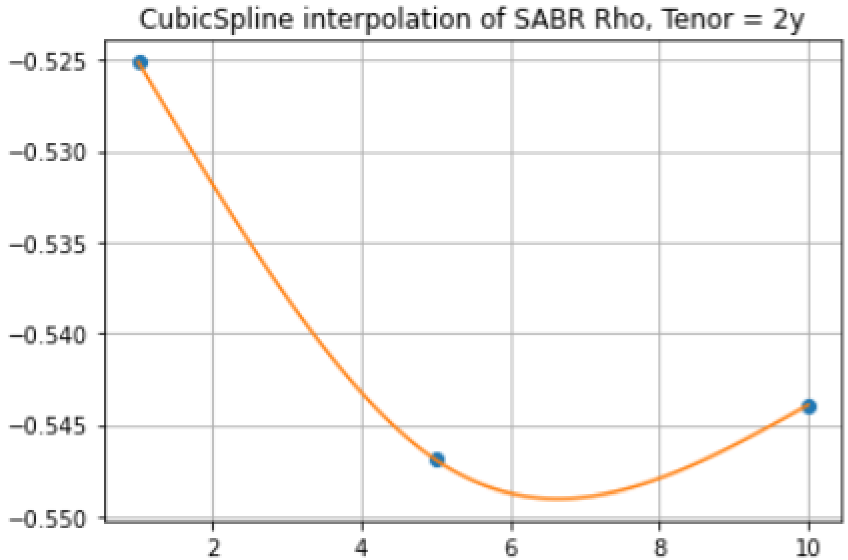
	1	2	3	5	10
1	0.139067	0.184646	0.196851	0.178052	0.171030
5	0.166504	0.199505	0.210363	0.191005	0.177441
10	0.177515	0.195098	0.207194	0.201576	0.180295

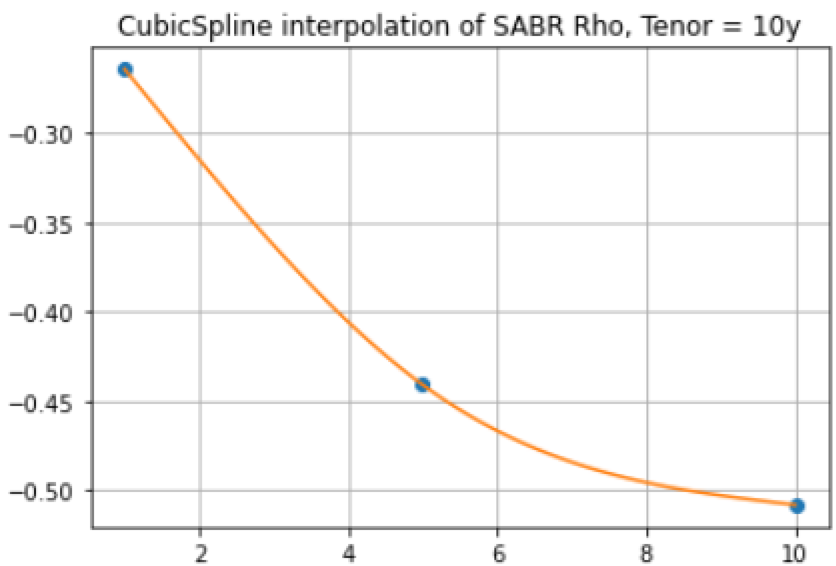




Rho

	1	2	3	5	10
1	-0.633239	-0.525118	-0.482846	-0.414426	-0.264156
5	-0.585021	-0.546913	-0.549830	-0.510985	-0.440790
10	-0.545691	-0.543919	-0.550819	-0.562582	-0.508008





Nu

	1	2	3	5	10
1	2.049533	1.677436	1.438140	1.064877	0.778640
5	1.339381	1.061940	0.936731	0.671750	0.493849
10	1.007122	0.924928	0.869109	0.719742	0.579536