Case 1: $|\Omega| < \infty$,

- $Q(x,w)$ recover in finite number of iterations
- $\mathbb{E}\{Q(x,\tilde{w})\}$ recover in finite number of iterations

$$= \sum_{w \in \Omega} p(w) \cdot Q(x,w)$$

if $\{p(w)\} = P$

(a) $\max_{P \in \mathbb{P}} \mathbb{E}\{Q(x,\tilde{w})\}$ recover in finite iterations

if

(i) $\mathbb{P}$ has finite number of extremal distributions.

- $\mathbb{P}$ is finite
- $\mathbb{P}$ is a polytope. ← moment matching set Riis and Andersen

(b) If (i) is not true, then can we

$\max_{P \in \mathbb{P}} \mathbb{E}\{Q(x,\tilde{w})\}$ recover eventually?

If we have $\{x^k\} \to \bar{x}$ then

$$\max_{P \in \mathbb{P}} \mathbb{E}\{Q(x^k,\tilde{w})\} \to \max_{P \in \mathbb{P}} \mathbb{E}\{Q(\bar{x},\tilde{w})\}?$$

This is trivially true.

Case 1: $|\Omega| < \infty$.

(a) $\mathbb{P}$ is a moment matching set. (Riis & Anderson)

→ Distribution separation problem is a linear program.

(b) $\mathbb{P}$ is a polytope (Bansal et. al)

→ Distribution separation problem is a LP.

(c) $\mathbb{P}$ ~~has~~ finite number of extremal distributions.

→

(d) $\mathbb{P}$ does not have finite number of extreme points, but is $\mathbb{P}$ is compact.

→ Analogous to Ω be compact in SD.

→ Check relationship between compactness and weak convergence of distribution (as in Riis & Anderson Prop. 2.1)

Case 2: Ω is a compact set, continuous.

(a) $\mathbb{P}$ is a polytope

(b) $\mathbb{P}$ has finite number of extremal distributions.

(c) $\mathbb{P}$ does not have finite number of extremal distributions, but is compact.

(A) $\max_{P \in \mathbb{P}} \mathbb{E}_P\{Q(x, \tilde{\omega})\} = Q(x)$ → convex ~~piecewise linear~~

$\min_{x \in X} Q(x) = v^*$ with solution x^*

(B) $\max_{P \in \mathbb{P}} \mathbb{E}_P\{Q(x, \tilde{\omega})\} \sim \hat{Q}(x)$ → convex

$\min_{x \in X} \hat{Q}(x) = \hat{v}$ with solution $\hat{x}$.

How does $\hat{v}$ relate to v? $(v^* - \hat{v}) \gtrless 0$ $\hat{v} \leq v$

Note: $Q(x)$ is a max over possible infinite number of convex piecewise linear functions. Hence it is convex, but not necessarily piecewise linear.

Decomposition methods for convex programs (Kelley's).

Note: For (B), work of Zakeri & Philpott as well as Au, Higle & Sen.

When solving a approximate/sample problem, can we use effective scenarios to reduce computation. Solve the distribution separation problem with reduced sample space (only the effective scenarios). This can also be applied to "argmax" procedure. (Rahimian et. al)

Under Case 1 (a), (b), (c)

Riis & Andersen provide the theoretical backing to the algorithm.
But questions on sample size selection and solving SAA to ϵ-optimality still linger.

Under Case 1 (d)

For a given sample size, DR-L method will converge following Kelley's arguments.
The same results from Riis & Andersen provide asymptotic convergence (as sample size goes to ∞)

Under Case 2 (a) & (b)

(a) $v_n \to v^*$ $\quad n \to \infty$

(a') $Q^n(x^n) \to Q(x^*)$ for every $\{x^n\} \to x^*$.

$Q^n(x) \to Q(x)$ pointwise $\forall x \in X$

$$\min_{x\in X} \; \max_{P\in \mathbb{P}} \; \underbrace{c^T x + \mathbb{E}_P\{Q(x,\tilde{\omega})\}}_{f(x)}$$

Let us define

$$Q(x) = \max_{P\in\mathbb{P}} \mathbb{E}_P\{Q(x,\tilde{\omega})\}.$$

Let $\{\Pi^k\}$ be the sequence of dual vertices such that $\Pi^k \subset \Pi^{k+1} \subset \ldots \Pi$. With this we define

(Lemma 1 Higle & Sen 1991)

$$Q^k(x,\omega) = \max\left\{\pi^T[r(\omega) - T(\omega)x] \;/\; \pi\in\Pi^k\right\}.$$

and

$$Q^k(x) = \max_{P\in\mathbb{P}^k} \mathbb{E}_P\{Q^k(x,\tilde{\omega})\}.$$

$(\Omega^k, \mathcal{F}^k)$

- Approximation of recourse function, i.e., only a subset $\Pi^k \subseteq \Pi$ is being used.
- Only a subset of $\mathbb{P}^k \subseteq \mathbb{P}$ is being used.

Under $\mathbb{P}^k \subseteq \mathbb{P}^{k+1} \ldots \subseteq \mathbb{P}$, we have

$$Q^k(x) \leq Q^{k+1}(x) \leq Q(x)$$

Lets also define

$$\bar{Q}^k(x) = \max_{P\in\mathbb{P}^k} \mathbb{E}_P\{Q(x,\tilde{\omega})\}.$$

(Proposition 2.1 Rlis & Anderson) $\min_{x \in X} Q^k(x) \to \min_{x \in X} Q(x) \quad k \to \infty$

The above uses two assumptions

(i) $\{\mathbb{P}^k\}$ is such that $\mathbb{P}^k \subseteq \mathbb{P}^{k+1} \ldots \subseteq \mathbb{P}$ and

(ii) for all $P \in \mathbb{P}$ there exists a sequence of $\{P^k\}_{k \geq 1}$ such that $P^k \in \mathbb{P}^k$ and $P^k \to P$ as $k \to \infty$.

(iii) $\mathbb{E}_P\{h(x, \tilde{\omega})\}$ is continuous in $\mathbb{P}$, $\forall x \in X$.

Conjecture 1: Under assumption (i) & (ii) we have

$$Q^k(x) \to Q(x) \quad \text{as } k \to \infty$$

pointwise over $x \in X$.

Conjecture 2: Under assumptions (i), (ii) & (iii)

$$\min_{x \in X} Q^k(x) \to \min_{x \in X} Q(x).$$

<u>Lemma</u>: The sequence of functions $\{Q^k(x, \omega)\}$ converge uniformly to $Q(x, \omega)$.

Lemma 1. of Higle & Sen (1991).

For a given sequence $\{x^k\}_{k \geq 1}$ that has an accumulation point $\bar{x} \in X$.

$$P^{*,k} \in \arg\max_{P \in \mathbb{P}^k} \mathbb{E}_P\{Q^k(x^k, \omega)\}$$

Under (ii) $\{P^{*,k}\}_{k \geq 1} \to P^* \in \mathbb{P}$.

<u>Consequently</u> $\mathbb{E}_{P^{*,k}}\{Q^k(x^k, \omega)\} \to \mathbb{E}_{P^*}\{Q(\bar{x}, \omega)\}. \quad \forall x \in X$

$\exists K$ s.t., for all $k \geq K$ $\exists\, \delta > 0$ s.t.

$$\left| Q_j(x^k, \omega) - Q_j^k(x^k, \omega) \right| < \varepsilon_1$$

$$\left| Q^{k+1}(x^{k+1}, \omega) - Q^k(x^k, \omega) \right| < \varepsilon_2'$$

for all x^{k+1} and x^k s.t. $\| x^{k+1} - x^k \| < \delta > 0$.

$$\left| p^{*,k+1}(\omega) Q^k(x^k, \omega) - p^{*,k}(\omega) Q^k(x^k, \omega) \right| < \varepsilon_3(\omega)$$

$$< \left| Q^k(x^k, \omega) \right| \left| p^{*,k+1}(\omega) - p^*(\omega) \right|$$

$$\left| p^{*,k+1} Q^{k+1}(x^{k+1}, \omega) - p^{*,k+1} Q^k(x^k, \omega) \right| < \varepsilon_2$$

$$\left| p^{*,k+1} Q^{k+1}(x^{k+1}, \omega) - p^{*,k} Q^k(x^k, \omega) \right| < \varepsilon_1 + \varepsilon_2$$

$$\cancel{\left| p^{*,2} Q^2(x^2, \omega) - p^{*,1} Q^1(x^1, \omega) \right| < \varepsilon / M}$$

$$\cancel{\left| p^{*,3} Q^3(x^3, \omega) - p^{*,2} Q^2(x^2, \omega) \right| < \varepsilon / M}$$

$\vdots$

$| p^*$

Theorem. Given a sequence $\{x^k\}_{k\geq 1}$ that converges to $\bar{x}$, i.e., $x^k \to \bar{x}$, and a sequence of approximate ambiguity sets $\{\mathbb{P}^k\}_{k\geq 1}$ that satisfy conditions (i)-(iii) of Proposition 2.1 of Riis and Anderson;

$$\max_{P \in \mathbb{P}^k} \mathbb{E}_{\xi_P}\left[Q^k(x^k, \omega)\right] \longrightarrow \max_{P \in \mathbb{P}} \mathbb{E}_{\xi_P}\left[Q(\bar{x}, \omega)\right]$$

with probability one.

Proof:

$$Q^k(x^k, \omega^j) = \pi_j^k\left[r(\omega^j) - T(\omega^j)\cdot x^k\right]$$

$$\sum_{\omega^j \in \Omega^k} p^k(\omega^j)\cdot Q^k(x^k, \omega^j) = \sum_{\omega^j \in \Omega^k} p^k(\omega^j)\,\pi_j^k\left[r(\omega^j) - T(\omega^j)\,x^k\right]$$

where $\{p^k(\omega^j)\}_{\forall \omega^j}$ is obtained by solving distribution separation problem at x^k over $(\Omega^k, \mathbb{P}^k)$.

Since $\{Q^k(x, \omega)\}$ uniformly converges to $g(x, \omega)$, we have for

$$p^k \in \operatorname*{argmax}_{P \in \mathbb{P}^k} \sum_{\omega^j \in \Omega^k} p(\omega^j)\, Q^k(x^k, \omega^j)$$

Similarly we have

$$\hat{p}^k \in \operatorname*{argmax}_{P \in \mathbb{P}^k} \sum_{\omega \in \Omega^k} p(\omega)\, g(\bar{x}, \omega)$$

$$\sum_{\omega^j \in \Omega^k}\left[p^k(\omega^j)\, Q^k(x^k, \omega^j) - \hat{p}(\omega^j)\, g(\bar{x}, \omega^j)\right] \to 0$$