Homework 5 Solutions

Question 1

- The pdf of r, the radial distance one can go before encountering the first neighbour is given by,

$$f(r) = \frac{d}{dr}(1 - e^{-\lambda\pi r^2}) = 2\lambda\pi r e^{-\lambda\pi r^2}$$

- Integration by parts:

$$\int u\frac{dv}{dr}.dr = uv - \int v\frac{du}{dr}.dr$$

- Gaussian integral:

$$\begin{aligned} \int_{\mathbb{R}} e^{-r^2}.dr &= \sqrt{\pi} \\ \int_{\mathbb{R}} e^{-a(r+b)^2}.dr &= \sqrt{\frac{\pi}{a}} \end{aligned}$$

- Expected value of r,

$$\begin{aligned} E(r) &= \int_{\mathbb{R}+} rf(r).dr \\ &= \int_{\mathbb{R}+} 2\lambda\pi r^2 e^{-\lambda\pi r^2}.dr \\ \text{Let} \quad & u = r, \quad v = -e^{-\lambda\pi r^2} \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \therefore E(r) &= -re^{-\lambda\pi r^2}\Big|_0^\infty + \int_{\mathbb{R}+} e^{-\lambda\pi r^2}.dr \\ &= 0 + \frac{1}{2}\cdot\frac{\sqrt{\pi}}{\sqrt{\lambda\pi}} \quad \text{where } a = \lambda\pi \text{ and support } [0,\infty) \\ &= \frac{1}{2\sqrt{\lambda}} \end{aligned}$$

- Second moment of r,

$$E(r^2) = \int_{\mathbb{R}+} 2\lambda\pi r^3 e^{-\lambda\pi r^2}.dr$$

$$\begin{aligned}
\text{Let} \quad u = r^2, &\quad v = -e^{-\lambda\pi r^2} \\
= &\quad 0 + \int_{\mathbb{R}^+} 2re^{-\lambda\pi r^2}.dr \\
= &\quad \left.\frac{-e^{-\lambda\pi r^2}}{\lambda\pi}\right|_0^\infty \\
= &\quad \frac{1}{\lambda\pi}
\end{aligned}$$

- Variance of r,

$$\begin{aligned}
Var(r) &= E(r^2) - E^2(r) \\
&= \frac{1}{\lambda\pi} - \frac{1}{4\lambda} \\
&= \frac{4-\pi}{4\lambda\pi}
\end{aligned}$$

- Standard error of $\bar{r}_N$,

$$\begin{aligned}
s.e.(\bar{r}_N) &= \frac{\sigma_r}{\sqrt{N}} \\
&= \sqrt{\frac{4-\pi}{4\lambda\pi N}} \\
&\approx \frac{0.26136}{\sqrt{N\lambda}}
\end{aligned}$$