Introduction to Spatial Econometrics

ECON6027 7a, Solution A

1.

$$\begin{aligned}
\hat{\beta}_{OLS} &= \left(X^{*'}X^{*}\right)^{-1}X^{*'}Y^{*} \\
&= \left(X^{'}P^{-1'}P^{-1}X\right)^{-1}X^{'}P^{-1'}P^{-1}Y \\
&= \left(X^{'}\Omega^{-1}X\right)^{-1}X^{'}\Omega^{-1}Y \\
&= \hat{\beta}_{GLS}
\end{aligned}$$

2. The likelihood function with respect to ϵ_n^*

$$L(\beta,\sigma_\epsilon^2) = \frac{1}{(2\pi\sigma_\epsilon^2)^{\frac{n}{2}}}|\Omega|^{-1/2}\exp\left\{-\frac{(Y_n - X_n\beta)'\,\Omega^{-1}\,(Y_n - X_n\beta)}{2\sigma_\epsilon^2}\right\}$$

The transformation from ϵ_i^* to Y_i is via $\epsilon_i^* = Y_n^* - X_n^*\beta = P_n^{-1}(Y_n - X_n\beta) = \Omega^{-1/2}(Y_i - X_i\beta)$, thus, the Jacobian of the transformation for each observation is, $|\partial\epsilon_i/\partial Y_i| = |\Omega|^{-1/2}$ which appears in the likelihood function. The log-likelihood function, $\ln(L(\theta))$ of the parameters can be written as,

$$\ell(\beta,\sigma_\epsilon^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma_\epsilon^2 - \frac{1}{2}\ln|\Omega| - \frac{1}{2\sigma_\epsilon^2}(Y_n - X_n\beta)'\,\Omega^{-1}\,(Y_n - X_n\beta)$$

Maximising the log-likelihood function w.r.t. β gives the F.O.C.,

$$\frac{1}{2\sigma_\epsilon^2}X_n^{'}\Omega^{-1}(Y_n - X_n\beta) = 0$$

and the maximum likelihood estimator of β as,

$$\begin{aligned}
\hat{\beta}_{ML} &= \left(X_n^{'}\Omega^{-1}X_n\right)^{-1}X_n^{'}\Omega^{-1}Y_n \\
&= \hat{\beta}_{GLS}
\end{aligned}$$

3.

$$\begin{aligned}
\hat{\sigma}_\epsilon^2 &= \frac{\epsilon^{*'}\epsilon^{*}}{n-k} \\
&= \frac{1}{n-k}(Y_n - X_n\beta)'\,\Omega^{-1}\,(Y_n - X_n\beta)
\end{aligned}$$