Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 1: Logic, Sets, Functions, Sequences, Compactness

Style of the Course

- We place a big emphasis on the formality of the mathematical argument.

- Although proving a theorem is crucial in mathematics, we often skip the proof in the class.

- Proofs omitted in the class should be found in the lecture notes or textbook.

Logic

Notation

$\neg p$: the **negation** of a statement p.

$p \wedge q$: the **conjunction** of two statements p and q.

$p \vee q$: the **disjunction** of p and q.

The **universal** quantifier $(\forall x)$: "for every x,", and

The **existential** quantifier $(\exists x)$: "there exists x."

$(\exists! x)$: "there is exactly one x."

Thus, the sentence:

For every x, there is a y such that $x < y$

is symbolized:

$$(\forall x)(\exists y)(x < y).$$

The sentence:

For every x, there is exactly one y such that $x + y = 0$

is symbolized:

$$(\forall x)(\exists! y)(x + y = 0).$$

Necessity and Sufficiency (Ch. 1.2)

Consider any two statements, p and q.

When "q is necessary for p," q must be true for p to be true.

I might say that "q is true **if** p is true," or simply that "p **implies** q."

I denote this statement by $p \Rightarrow q$.

Suppose we know that "$p \Rightarrow q$" is a true statement.

What if q is not true? Because q is necessary for p, when q is not true, then p cannot be true, either. But doesn't this just say that "not-q" (denoted by $\neg q$) implies "not-p." (denoted by $\neg p$)

This latter form of the original statement is called the **contra-positive** form. Contraposition of the arguments in the statement **reverses** the direction of implication for a true statement.

When "q is sufficient for p," "q is true **only if** p is true," or that "p is implied by q" ($p \Leftarrow q$).

"$p \Leftarrow q$" and "$p \Rightarrow q$," can both be true. When this is so, "p is necessary and sufficient for q," or "p is true **if and only if** q is true," or "p iff q."

When "p is necessary and sufficient for q," p and q are **equivalent** ("$p \Leftrightarrow q$.")

Set Theory

Sets (Ch. A.1)

A **set** = any collection of elements.

Sets of objects: $A, S, T, \ldots$

their members of a set: $a, s, t, \ldots$.

We mean by $x \in S$ "x is an element of S."

S is a **subset** of T if every element of S is also an element of T: $S \subseteq T \Leftrightarrow (\forall x \in S)(x \in T)$.

S is a **proper subset** of T if $S \subseteq T$ and $S \neq T$; we write $S \subsetneq T$ in this case.

$S = T$: $S \subseteq T$ and $S \supseteq T \Leftrightarrow (\forall x)((x \in S \Rightarrow x \in T) \wedge (x \in T \Rightarrow x \in S))$.

$|S|$: the number of elements in a set S and it is called the **cardinality** of S.

S is **empty** or is an **empty set** if it contains no elements at all. It is a subset of "every" set.

Example: Let $A = \{x \in (-\infty, \infty)|\ x^2 = 0 \text{ and } x > 1\}$. Then A is empty.

We denote the empty set by $\emptyset$.

$S \backslash T$: all elements in the set S that are not elements of T.

$S \cup T \equiv \{x|\ x \in S$ or $x \in T\}$: the **union** of S and T.

$S \cap T \equiv \{x|\ x \in S$ and $x \in T\}$: the **intersection** of S and T.

Vehn Diagrams are often used to depict $S \cup T$, $S \cap T, S \backslash T$ or many other.

S^c: the **complement** of a set S in a universal set U if it is the set of all elements in U that are not in S.

$S, T \subseteq U \Rightarrow S^c = U \backslash S$.

Clearly, what S^c crucially depends on what the universal set U is.

Example: Let $S = \{x | 5 \leq x \leq 10\}$. If $U = (-\infty, \infty)$,

$$S^c = \{x \in U |\ x < 5 \text{ or } x > 10\}.$$

On the other hand, if $U = \{x \in (-\infty, \infty) |\ x \geq 0\}$,

$$S^c = \{x \in U |\ 0 \leq x < 5 \text{ or } x > 10\}.$$

So far, the order of the elements in a set does not matter. In particular, $\{a, b\} = \{b, a\}$.

However, the coordinates of a point in xy-plane are given as an **ordered pair** (a, b) of real numbers: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

The **product** of S and T is

$$S \times T \equiv \{(s, t) | \ s \in S, \ t \in T\}.$$

$\mathbb{R} \equiv \{x| \ -\infty < x < \infty\}$: the set of real numbers

$$\mathbb{R}^n \equiv \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \equiv \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, \ i = 1, \ldots, n\}.$$

$\mathbb{R}^n$ is called n-**dimensional Euclidean space** and any element of it is considered a "point" in $\mathbb{R}^n$.

$$\begin{aligned} \mathbb{R}^n_+ &\equiv \{(x_1, \ldots, x_n) \mid x_i \geq 0, \ i = 1, \ldots, n\} \subseteq \mathbb{R}^n \\ \mathbb{R}^n_{++} &\equiv \{(x_1, \ldots, x_n) \mid x_i > 0, \ i = 1, \ldots, n\} \subseteq \mathbb{R}^n_+. \end{aligned}$$

Functions

Functions (Ch. A.1)

A **function** associates each element of one set with a single, unique element of another set.

$f : D \to R$: a **mapping, map**, or **transformation** from one set D to another set R, where D denotes the **domain** and R denotes the **range** of the mapping.

$y = f(x)$: y is the point in the range mapped into by the point x in the domain.

Examples

(1) $f(x) = 1/(x+3)$: For $x = -3$, the formula reduces to the meaningless expression "1/0." For all other values of x, the formula makes $f(x)$ a well-defined number. Thus, the domain of f consists of all (real) numbers $x \neq -3$.

(2) $g(x) = \sqrt{2x+4}$: The expression $\sqrt{2x+4}$ is uniquely defined for all x such that $2x+4$ is nonnegative. Solving the inequality $2x+4 \geq 0$ for x gives $x \geq -2$. The domain of g is therefore the interval $[-2, \infty)$.

The **image** of f is the set of points in the range into which some point in the domain is mapped, i.e.,

$$f(D) \equiv \{y \in R \mid y = f(x) \text{ for some } x \in D\} \subseteq R.$$

The **inverse image** of a set of points $S \subseteq f(D)$ is defined as

$$f^{-1}(S) \equiv \{x \in D \mid f(x) \in S\}.$$

Example: Let $f(x) = x^2$. Then,

If $D = \mathbb{R}$, the image of f is $f(D) = \mathbb{R}_+$.

Let $S = [0, 1]$. Then, the inverse image of $S = [0, 1]$ is $f^{-1}([0, 1]) = [-1, 1]$.

A function $f : D \to R$ is **one-to-one** (or **injective**) if, $(\forall x, x^{'} \in D)(f(x) = f(x^{'}) \Rightarrow x = x^{'})$.

If $f(D) = R$, the function f is said to be **onto** (or **surjective**).

If a function is one-to-one and onto (or **bijective**), then an **inverse function** $f^{-1} : R \to D$ exists and f^{-1} is also one-to-one and onto.

Example: $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is **not** a one-to-one mapping. the same $f(\cdot)$ is **not** onto either. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) = x^2$. In this case, $f(\cdot)$ is bijective.

Examples of Inverse Functions

(1) $y = 4x - 3$. Solving the equation for x,

$$y = 4x - 3 \Leftrightarrow 4x = y + 3 \Leftrightarrow x = \frac{1}{4}y + \frac{3}{4}.$$

So, $g(y) = y/4 + 3/4$ is the inverse function of $f(x) = 4x - 3$.

(2) $y = (3x - 1)/(x + 4)$. Multiplying both sides of the equation by $(x + 4)$, we obtain

$$y(x + 4) = 3x - 1 \Leftrightarrow x(3 - y) = 4y + 1.$$

Hence,

$$x = \frac{4y + 1}{3 - y}.$$

Note that we exclude $x = -4$ and $y = 3$ to make both functions well-defined.

Real Numbers

Natural Numbers, Integers, and Rationals

$\mathbb{N} \equiv \{1, 2, 3, \ldots\}$: the set of all **natural numbers** .

$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$: the set of all **nonnegative integers**.

Construct the set $\mathbb{N}_- = \{-1, -2, \ldots\}$ of all **negative integers**.

$\mathbb{Z} = \mathbb{Z}_+ \cup \mathbb{N}_-$: the set of all **integers**.

For instance, consider an equation like $2x = 1$, which makes sense in $\mathbb{Z}$. However, it cannot possibly be solved in $\mathbb{Z}$. Hence, we need to extend $\mathbb{Z}$ to the set $\mathbb{Q}$ of all **rational numbers**:

$$\mathbb{Q} = \{m/n |\ m, n \in \mathbb{Z},\ n \neq 0\}\,.$$

Sets of Real Numbers

There are certainly two rational numbers p and q such that $p^2 > 2 > q^2$, but now we know that there is no $r \in \mathbb{Q}$ with $r^2 = 2$. It is as if there were a "hole" in the set of rational numbers.

So, we wish to **complete** $\mathbb{Q}$ by filling up its holes with "new" numbers. And doing this leads us to the set $\mathbb{R}$ of **real numbers**.

Note that any member of the set $\mathbb{R}\backslash\mathbb{Q}$ is said to be an **irrational number**.

Properties of Real Numbers (Ch. A.2)

A set $S \subseteq \mathbb{R}$ is **bounded above** if $(\exists b \in \mathbb{R})(\forall x \in S)(b \geq x)$, where b is called an **upper bound** for S.

Definition: b^* is a **least upper bound** for the set S if it is an upper bound for S with the property that $b^* \leq b$ for every upper bound b.

A set S in $\mathbb{R}$ can have at most one least upper bound, because if b_1^* and b_2^* are both least upper bounds for S, then $b_1^* \leq b_2^*$ and $b_2^* \leq b_1^*$, which thus implies that $b_1^* = b_2^*$.

Notation: $b^* = \sup S$ or $b^* = \sup_{x \in S} x$, where "sup" stands for **supremum**.

Example: The set $S = (0, 5) = \{x \in \mathbb{R} | 0 < x < 5\}$ has many upper bounds, some of which are 100, 6.73, and 5.

Clearly no number smaller than 5 can be an upper bound, so 5 is the least upper bound. Thus, $\sup S = 5$.

A set S in $\mathbb{R}$ is **bounded below** if $(\exists a \in \mathbb{R})(\forall x \in S)(x \geq a)$, where a is called a lower bound for S.

Similarly, a set S in $\mathbb{R}$ that is bounded below has a greatest lower bound a^*.

Notation: $a^* = \inf S$ or $a^* = \inf_{x \in S} x$, where "inf" stands for **infimum**.

Axiom of Real Numbers (Ch. A.2)

Any nonempty set of real numbers that is bounded above has a least upper bound.

Theorem: Let $S \subseteq \mathbb{R}$ and $b^* \in \mathbb{R}$. Then, $\sup S = b^*$ if and only if the following two conditions are satisfied:

1. $(\forall x \in S)(x \leq b^*)$.

2. $(\forall \varepsilon > 0)(\exists x \in S)(x > b^* - \varepsilon)$.

Proof: We omit the proof. ■

Topology

Topology in the Euclidean Space

Topology is a concept that allows us to determine how close two objects are.

Key Concepts to be Developed: openness, closedness, boundedness, and compactness of sets and continuity of functions.

Sequences on $\mathbb{R}$ (Ch. A.3)

For $x \in \mathbb{R}$, the **absolute value** of x is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

A **sequence** is a function $k \mapsto x(k)$ whose domain is $\mathbb{N} = \{1, 2, 3, \ldots\}$ and whose range is $\mathbb{R}$.

$x(1), x(2), \ldots, x(k), \ldots$ of the sequence $\Leftrightarrow$ $x^1, x^2, \ldots, x^k, \ldots$.

$\{x^k\}_{k=1}^{\infty}$, or simply $\{x^k\}$: arbitrary sequence of real numbers.

A sequence $\{x^k\}$ of real numbers is said to be:

1. **nondecreasing** if $x^k \leq x^{k+1}$ for $k = 1, 2, \ldots$

2. **strictly increasing** if $x^k < x^{k+1}$ for $k = 1, 2, \ldots$

3. **nonincreasing** if $x^k \geq x^{k+1}$ for $k = 1, 2, \ldots$

4. **strictly decreasing** if $x^k > x^{k+1}$ for $k = 1, 2, \ldots$

A sequence that is nondecreasing or nonincreasing is called **monotone**.

Example: (1) Let $x^k = 1 - 1/k$ for each $k \in \mathbb{N}$. Then, $\{x^k\}$ is monotone and $x^k \to 1$ as $k \to \infty$.

(2) $y^k = (-1)^k$ for each $k \in \mathbb{N}$. Then, $\{y^k\}$ is clearly neither nonincreasing nor nondecreasing because its terms are $-1, 1, -1, 1, -1, 1, -1, \ldots$.

(3) $z^k = \sqrt{k+1} - \sqrt{k}$ for each $k \in \mathbb{N}$.

$$z^k = \sqrt{k+1} - \sqrt{k} = \frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.$$

So, $\{z^k\}$ is strictly decreasing.

Convergence in $\mathbb{R}$ (Ch. A.3)

Definition: $\{x^k\}$ **converges** to x if, $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x^k - x| < \varepsilon$, $\forall k > N_\varepsilon$. The number x is called the **limit** of the sequence $\{x^k\}$. A **convergent** sequence is one that converges to some number.

Example: (1) $x^k = 1 - 1/k$. the sequence $\{x^k\}$ converges to 1 as $k \to \infty$. Fix $\varepsilon > 0$. Choose $N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon \geq 1/\varepsilon$. Then, for any $k > N_\varepsilon$, which means $1/k < \varepsilon$,

$$|x^k - 1| = |(1 - 1/k) - 1| = |-1/k| < \varepsilon.$$

Let us introduce an extra restriction on sequences.

Definition: A sequence $\{x^k\}$ is **bounded** if $\exists M \in \mathbb{R}$ such that $|x^k| \leq M$, $\forall k \in \mathbb{N}$.

Lemma: Every convergent sequence is bounded.

Proof: We omit the proof. ■

Is every bounded sequence convergent?

The answer is "No." For example, the sequence $\{x^k\} = \{(-1)^k\}$ is bounded but not convergent.

Theorem: Every bounded **monotone** sequence is convergent.

Proof: We omit the proof. ■

Subsequences (Ch. A.3)

Let $\{x^k\}$ be a sequence. Consider a strictly increasing sequence of natural numbers

$$k_1 < k_2 < k_3 < \cdots$$

and form a new sequence $\{y^j\}_{j=1}^{\infty}$, where $y^j = x^{k_j}$ for $j = 1, 2, \ldots$.

The sequence $\{y^j\}_j = \{x^{k_j}\}_j$ is called a **subsequence** of $\{x^k\}$.

Example: Let $x^k = k$ for each k. Define $y^j = 2(j-1)+1$ for each j. Then, $\{y^j\}$ is a subsequence of $\{x^k\}$.

NOTE: Every subsequence of a convergent sequence is itself convergent, and has the same limit as the original sequence.

Not every bounded sequence is convergent, but

Theorem: If the sequence $\{x^k\}$ is bounded, then it contains a convergent subsequence.

Proof: We omit the proof. ■

Example: Let $x^k = (-1)^k$ for each k.

Topology on $\mathbb{R}^n$

Point Set Topology in $\mathbb{R}^n$ (Ch. 13.1)

Consider the n-dimensional Euclidean space $\mathbb{R}^n$, whose elements, or points, are n-vectors $x = (x_1, \ldots, x_n)$.

The **Euclidean distance (or metric)** $d(x, y)$ between any two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is:

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

If $x, y, z \in \mathbb{R}^n$, then

$$d(x, z) \leq d(x, y) + d(y, z) \text{ (\textbf{triangle inequality})}.$$

Let $x^0 \in \mathbb{R}^n$ and $r > 0$.

The set of all points $x \in \mathbb{R}^n$ whose distance from x^0 is less than r is called the **open ball** around x^0 with radius r.

$$B_r(x^0) = \{x \in \mathbb{R}^n | d(x^0, x) < r\}$$

Definition: A set $S \subseteq \mathbb{R}^n$ is **open** if, $\forall x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.

On the real line $\mathbb{R}$, the simplest type of open set is an **open interval**.

Let $S \subseteq \mathbb{R}^n$. A point $x^0 \in S$ is called an **interior point** of S if there is some $\varepsilon > 0$ such that $B_\varepsilon(x^0) \subseteq S$.

The set of all interior points of S is called the **interior** of S, and is denoted $\text{Int}(S)$ or S°.

Example: Let $S = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1\}$. Then, $\text{Int}(S) = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 < 1\}$.

Definition: A set S in $\mathbb{R}^n$ is **closed** if its complement, $S^c = \mathbb{R}^n \backslash S$ is open.

Topology and Convergence (Ch. 13.2)

A **sequence** $\{x^k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is a function such that each natural number k yields a corresponding point x^k in $\mathbb{R}^n$.

Definition: A sequence $\{x^k\}$ in $\mathbb{R}^n$ **converges** to a point $x \in \mathbb{R}^n$ if, for each $\varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that $x^k \in B_\varepsilon(x)$ for all $k > N$, or equivalently, $d(x^k, x) < \varepsilon$ for all $k > N$.

Convergence in $\mathbb{R}^n$ (Ch. 13.2)

Theorem: Let $\{x^k\}$ be a sequence in $\mathbb{R}^n$. Then, $\{x^k\}$ converges to the vector $x \in \mathbb{R}^n$ if and only if for each $j = 1, \ldots, n$, the real number sequence $\{x_j^k\}_{k=1}^{\infty}$, consisting of jth component of each vector x^k, converges to $x_j \in \mathbb{R}$, the jth component of x.

Proof: We omit the proof. ■

Closedness in terms of Sequences on $\mathbb{R}^n$ (Ch. 13.2)

Theorem: A set $S \subseteq \mathbb{R}^n$ is closed if and only if every convergent sequence of points in S has its limit in S, i.e., $(\forall \{x^k\} \in S)((\exists x \in \mathbb{R}^n)(x^k \to x) \Rightarrow x \in S)$.

Proof: We skip the proof. $\blacksquare$

Boundedness in terms of Sequences on $\mathbb{R}^n$ (Ch. 13.2)

Definition: A set S in $\mathbb{R}^n$ is **bounded** if there exists a number $M \in \mathbb{R}_+$ such that $\|x\| \leq M$ for all $x \in S$. A set that is not bounded is called **unbounded**. Here $\|x\| = d(x, \mathbf{0}) = \sqrt{x_1^2 + \cdots + x_n^2}$, called the **Euclidean norm**.

Similarly, a sequence $\{x^k\}$ in $\mathbb{R}^n$ is **bounded** if the set $\{x^k | k \in \mathbb{N}\}$ is bounded.

Lemma: Any convergent sequence $\{x^k\}$ in $\mathbb{R}^n$ is bounded.

Proof: We skip the proof. ■

On the other hand, a bounded sequence $\{x^k\}$ in $\mathbb{R}^n$ is "not" necessarily convergent. This is the same as sequences in $\mathbb{R}$.

Hence, we obtain

Theorem: A subset S of $\mathbb{R}^n$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Proof: We skip the proof. ■

Compactness

Compactness (Ch. 13.2)

Bolzano-Weierstrass Theorem: A subset S of $\mathbb{R}^n$ is closed and bounded if and only if every sequence of points in S has a subsequence that converges to a point in S.

Proof of Bolzano-Weierstrass's Theorem: We skip this. $\blacksquare$

Definition: A set S in $\mathbb{R}^n$ is **compact** if it is closed and bounded.