Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 10: Control Theory III (Ch. 9)

Scrap Values (Ch. 9.10)

Consider the following problem

$$\max_{u(t)\in U} \left\{ \int_{t_0}^{t_1} f(t, x(t), u(t))dt + S(x(t_1)) \right\},$$
$$\text{subject to } \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0\ (*)$$

The function $S(x)$ is called a **scrap value function**, and we assume that it is C^1.

Suppose that $(x^*(t), u^*(t))$ solves this problem (with no additional condition on $x(t_1)$).

Then, $(x^*(t), u^*(t))$ is indeed a solution to the corresponding problem with fixed terminal point $(t_1, x^*(t_1))$.

For all admissible pairs in this new problem, the scrap value function $S(x^*(t_1))$ is constant.

But then $(x^*(t), u^*(t))$ must satisfy all the conditions in the maximum principle, except the transversality conditions.

The Transversality Condition for the Problem with Scrap Value

Lemma: The transversality condition for problem $(*)$ is

$$p(t_1) = S^{'}(x^*(t_1)) \ (**).$$

Proof: We skip the proof. ∎

If $S(x) \equiv 0$, then $(**)$ reduces to $p(t_1) = 0$, which is precisely as expected in a problem with no restrictions on $x(t_1)$.

If $x(t)$ denotes the capital stock of a firm, then according to $(**)$, the shadow price of capital at the end of the planning period is equal to the marginal scrap value of the terminal stock.

Sufficient Conditions with Scrap Value

Theorem: Suppose $(x^*(t), u^*(t))$ is an admissible pair for the scrap value problem $(*)$ and suppose there exists a continuous $p(t)$ such that, for all $t \in [t_0, t_1]$,

(A) $u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ w.r.t. $u \in U$

(B) $\dot{p}(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t))$, $p(t_1) = S^{'}(x^*(t_1))$

(C) $H(t, x, u, p(t))$ is concave in (x, u) and $S(x)$ is concave.

Then, $(x^*(t), u^*(t))$ solves the problem.

Proof: We omit the proof. ■

Example:

$$\max_{u \in \mathbb{R}} \left\{ \int_0^1 -\frac{1}{2}u^2 dt + \sqrt{x(1)} \right\} \text{ subject to } \dot{x} = x+u,\ x(0) = 0,\ x(1) \text{ free.}$$

We set

$$\begin{aligned} H(t,x,u,p) &= -u^2/2 + p(x+u) \\ S(x) &= \sqrt{x} = x^{1/2}. \end{aligned}$$

Since $u \in \mathbb{R}$, the maximum principle, requires that $H_u^{'} = -u+p = 0$ so that $u = p$.

By the maximum principle,

$$\dot{p} = -H_x^{'} \Rightarrow \dot{p} = -p \Rightarrow p(t) = Ae^{-t},$$

where A is a constant.

Since $\dot{x} = x + u$, $u = p$, and $p(t) = Ae^{-t}$, we have

$$\dot{x} - x = Ae^{-t}.$$

Using the formula of first-order linear equations, we have

$$x(t) = Be^t + e^t \int e^{-t} Ae^{-t} dt = Be^t + Ae^t \int e^{-2t} dt = Be^t - \frac{A}{2} e^{-t}.$$

Since $x(0) = 0$, we have

$$0 = B - \frac{A}{2} \Rightarrow B = \frac{A}{2}.$$

Therefore,

$$x(t) = \frac{A}{2}(e^t - e^{-t}).$$

Since $S'(x) = \frac{1}{2}x^{-1/2}$ and $p(t) = Ae^{-t}$, transversality condition reduces to

$$Ae^{-1} = \frac{1}{2}(x(1))^{-1/2} \Rightarrow Ae^{-1} = \frac{1}{2}\left\{\frac{A}{2}(e - e^{-1})\right\}^{-1/2}.$$

We solve this for A as follows:

$$\begin{aligned}
& 4A^2e^{-2} = \left\{\frac{A}{2}(e - e^{-1})\right\}^{-1} \\
\Rightarrow\ & 4A^2e^{-2}\left\{\frac{A}{2}(e - e^{-1})\right\} = 1 \\
\Rightarrow\ & 2A^3(e^{-1} - e^{-3}) = 1 \\
\Rightarrow\ & A = e[2(e^2 - 1)]^{-1/3}.
\end{aligned}$$

Thus, we obtain the following solution candidate:

$$\begin{aligned} u^*(t) &= Ae^{-t} \\ p(t) &= Ae^{-t} \\ x^*(t) &= \frac{A}{2}(e^t - e^{-t}), \end{aligned}$$

where $A = e[2(e^2 - 1)]^{-1/3}$.

Since the Hamiltonian is concave in (x, u) and the scrap function $S(x)$ is strictly concave in x, the solution candidate we have obtained is indeed the solution.

Current Value Formulation

Many control problems in economics have the following structure:

$$\max_{u \in U \subseteq \mathbb{R}} \left\{ \int_{t_0}^{t_1} f(t,x,u) e^{-rt} dt + S(x(t_1)) e^{-rt_1} \right\}, \ \dot{x} = g(t,x,u), \ x(t_0) = x_0 \ (*)$$

$$(a)\ x(t_1) = x_1, \ (b)\ x(t_1) \geq x_1, \text{ or } (c)\ x(t_1) \text{ free } (**)$$

The current value Hamiltonian for the problem is

$$H^c(t,x,u,\lambda) = \lambda_0 f(t,x,u) + \lambda g(t,x,u).$$

Current Value Maximum Principle: Scrap Values

Theorem: Suppose that the admissible pair $(x^*(t), u^*(t))$ solves problem $(*)$ and $(**)$. Then, there exist a continuous function $\lambda(t)$ and a number λ_0, either 0 or 1, such that, for all $t \in [t_0, t_1]$, we have $(\lambda_0, \lambda(t)) \neq (0, 0)$, and:

(A) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ for $u \in U$

(B) $\dot{\lambda}(t) - r\lambda(t) = -\partial H^c(t, x^*(t), u^*(t), \lambda(t))/\partial x$ whenever $u^*(t)$ is continuous

(C) The transversality conditions are :

$(a^{'})$ $\lambda(t_1)$ no condition

$$(b^{'}) \quad \lambda(t_1) \begin{cases} = \lambda_0 S^{'}(x^*(t_1)) & \text{if } x^*(t_1) > x_1 \\ \geq \lambda_0 S^{'}(x^*(t_1)) & \text{otherwise} \end{cases}$$

$(c^{'})$ $\lambda(t_1) = \lambda_0 S^{'}(x^*(t_1))$.

Theorem: The current value maximum principle with scrap values such that $\lambda_0 = 1$ is sufficient if U is convex, $H^c(t, x, u, \lambda(t))$ is concave in (x, u), and $S(x)$ is concave in x.

Proof We omit the proof. ■

Example:

$$\max_{u \in \mathbb{R}} \left\{ \int_0^T (x - u^2) e^{-0.1t} dt + a x(T) e^{-0.1T} \right\}$$

subject to $\dot{x} = -0.4x + u,\ x(0) = 1,\ x(T)$ free,

where a is a positive constant.

We formulate the current value Hamiltonian with $\lambda_0 = 1$:

$$H^c(t, x, u, \lambda) = x - u^2 + \lambda(-0.4x + u).$$

H^c is concave in (x, u). Moreover, $S(x) = ax$ is linear so that it is concave in x.

Therefore, the conditions in the maximum principle are sufficient.

Because H^c is concave in u and $u \in \mathbb{R}$,

$$(1) \quad \frac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial u} = -2u^*(t) + \lambda(t) = 0.$$

By the maximum principle, we also have

$$\begin{aligned} & \dot{\lambda}(t) - 0.1\lambda(t) = -\partial H^c / \partial x = -1 + 0.4\lambda(t) \\ \Rightarrow \quad & \dot{\lambda} - 0.5\lambda = -1. \end{aligned}$$

By the formula of first-order linear differential equations, we obtain

$$(2) \quad \lambda(t) = Ce^{0.5t} + \frac{-1}{-0.5} = Ce^{0.5t} + 2,$$

where C is a constant.

Since $x(T)$ is free and $S(x) = ax$, the transversality condition $\lambda(t_1) = \lambda_0 S^{'}(x^*(t_1))$ reduces to

$$(3) \quad \lambda(T) = a.$$

Combining (2) and (3) together, we have

$$\lambda(t) = (a-2)e^{-0.5(T-t)} + 2.$$

From (1), $u^*(t) = \lambda(t)/2$.

Plugging $u^*(t) = \lambda(t)/2$ into $\dot{x} = -0.4x + u$,

$$\dot{x} + 0.4x = \lambda(t)/2$$

By the formula of first-order linear differential equations, we have

$$x^*(t) = Ce^{-0.4t} + e^{-0.4t} \int e^{0.4t} \lambda(t)/2 dt,$$

where C is a constant. Since $\lambda(t) = (a-2)e^{-0.5(T-t)} + 2$,

$$\begin{aligned}
x^*(t) &= Ce^{-0.4t} + \frac{e^{-0.4t}}{2} \int \left\{ (a-2)e^{-0.5T+0.9t} + 2e^{0.4t} \right\} dt \\
&= Ce^{-0.4t} + \frac{(a-2)e^{-0.4t-0.5T}}{2} \int e^{0.9t} dt + e^{-0.4t} \int e^{0.4t} dt \\
&= Ce^{-0.4t} + \frac{(a-2)e^{-0.4t-0.5T}}{2} \cdot \frac{e^{0.9t}}{0.9} + e^{-0.4t} \frac{e^{0.4t}}{0.4} \\
&= Ce^{-0.4t} + \frac{5(a-2)}{9} e^{-0.5(T-t)} + \frac{5}{2}.
\end{aligned}$$

Since $x^*(0) = 0$, we can pin down the value of C:

$$C = -\frac{5(a-2)}{9}e^{-0.5T} - \frac{5}{2}.$$

Thus, the solution to this problem is obtained:

$$\begin{aligned}
\lambda(t) &= (a-2)e^{-0.5(T-t)} + 2 \\
u^*(t) &= \lambda(t)/2 \\
x^*(t) &= Ce^{-0.4t} + \frac{5(a-2)}{9}e^{-0.5(T-t)} + \frac{5}{2},
\end{aligned}$$

where

$$C = -\frac{5(a-2)}{9}e^{-0.5T} - \frac{5}{2}.$$

Infinite Horizon (Ch. 9.11)

A typical infinite horizon optimal control problem in economics takes the following form:

$$\max \int_{t_0}^{\infty} f(t, x(t), u(t)) e^{-rt} dt,$$

$$\text{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \; u(t) \in U \quad (1)$$

Many problems do impose the constraint

$$\lim_{t \to \infty} x(t) \geq x_1 \;\; (x_1 \text{ is a fixed number}) \quad (2)$$

The pair $(x(t), u(t))$ is **admissible** if it satisfies $\dot{x}(t) = g(t, x(t), u(t))$, $x(t_0) = x_0$, $u(t) \in U$, along with (2) when that is imposed.

Theorem (Sufficient Conditions with an Infinite Horizon): Suppose that an admissible pair $(x^*(t), u^*(t))$ for problem (1), with or without terminal condition (2), satisfies the following conditions for some $\lambda(t)$ for all $t \geq t_0$, with $\lambda_0 = 1$:

(a) $u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ with respect to $u \in U$;

(b) $\dot{\lambda}(t) - r\lambda = -\partial H^c(t, x^*(t), u^*(t), \lambda(t))/\partial x$;

(c) $H^c(t, x, u, \lambda(t))$ is concave in (x, u);

(d) $\lim_{t\to\infty} \lambda(t)e^{-rt}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$.

Then, $(x^*(t), u^*(t))$ is a solution to Problem (1).

Proof: For any admissible pair $(x(t), u(t))$ and for all $t \geq t_0$, define

$$\begin{aligned} D_u(t) &= \int_{t_0}^{t} \underbrace{f(\tau, x^*(\tau), u^*(\tau))}_{=f^*} e^{-r\tau} d\tau - \int_{t_0}^{t} \underbrace{f(\tau, x(\tau), u(\tau))}_{=f} e^{-r\tau} d\tau \\ &= \int_{t_0}^{t} (f^* - f) e^{-r\tau} d\tau \end{aligned}$$

Using the similar simplified notation,

$$\begin{aligned} f^* &= (H^c)^* - \lambda g^* \underbrace{=}_{\dot{x}^* = g^*} (H^c)^* - \lambda \dot{x}^*, \\ f &= H^c - \lambda g \underbrace{=}_{\dot{x} = g} H^c - \lambda \dot{x}. \end{aligned}$$

So,

$$D_u(t) = \int_{t_0}^{t} \left[(H^c)^* - H^c \right] e^{-r\tau} d\tau + \int_{t_0}^{t} \lambda e^{-r\tau} (\dot{x} - \dot{x}^*) d\tau.$$

By concavity of H^c w.r.t. (x, u), one has

$$H^c - (H^c)^* \leq \frac{\partial (H^c)^*}{\partial x}(x - x^*) + \frac{\partial (H^c)^*}{\partial u}(u - u^*).$$

This is equivalent to

$$\begin{aligned} (H^c)^* - H^c &\geq -\frac{\partial (H^c)^*}{\partial x}(x - x^*) + \frac{\partial (H^c)^*}{\partial u}(u^* - u) \\ &= (\dot{\lambda} - r\lambda)(x - x^*) + \frac{\partial (H^c)^*}{\partial u}(u^* - u). \end{aligned}$$

So,

$$D_u(t) \geq \int_{t_0}^{t} e^{-r\tau} \left[(\dot{\lambda} - r\lambda)(x - x^*) + \lambda(\dot{x} - \dot{x}^*) \right] d\tau + \int_{t_0}^{t} \frac{\partial (H^c)^*}{\partial u}(u^* - u) e^{-r\tau} d\tau$$

As we have already shown elsewhere,

$$(a) \Rightarrow \frac{\partial (H^c)^*}{\partial u}(u^* - u) \geq 0 \Rightarrow \int_{t_0}^{t} \frac{\partial (H^c)^*}{\partial u}(u^* - u)e^{-r\tau} d\tau \geq 0.$$

Thus,

$$\begin{aligned} D_u(t) &\geq \int_{t_0}^{t} \frac{d}{d\tau}\left[e^{-r\tau}\lambda(\tau)(x(\tau) - x^*(\tau))\right] d\tau \\ &= \left[e^{-r\tau}\lambda(\tau)(x(\tau) - x^*(\tau))\right]_{t_0}^{t} \\ &= e^{-rt}\lambda(t)(x(t) - x^*(t)) \ (\because \ x(t_0) = x^*(t_0) = x_0) \end{aligned}$$

Then,

$$D_u(\infty) \geq \lim_{t\to\infty} e^{-rt}\lambda(t)(x(t) - x^*(t)) \underbrace{\geq}_{\because\ (d)} 0$$

Noting

$$D_u(\infty) = \int_{t_0}^{\infty}(f^* - f)e^{-r\tau}d\tau \geq 0,$$

we conclude that $(x^*(t), u^*(t))$ is a solution to Problem (1). ■

Lemma (A condition guaranteeing (d)): Let $(x^*(t), u^*(t))$ be an admissible pair for Problem (1) satisfying (a), (b), and (c) in the sufficiency result. Suppose that $\lim_{t\to\infty} x(t) \geq x_1$ for any admissible $x(t)$. Assume further that the following three conditions hold:

(A) $\lim_{t\to\infty} \lambda(t)e^{-rt}(x_1 - x^*(t)) \geq 0$;

(B) $\exists M \in \mathbb{R}_+$ such that $|\lambda(t)e^{-rt}| \leq M$ for all $t \geq t_0$;

(C) $\exists t' \in \mathbb{R}$ such that $\lambda(t) \geq 0$ for all $t \geq t'$.

Then, (d) $\lim_{t\to\infty} \lambda(t)e^{-rt}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$.

Proof: We skip the proof. ■

Example: Consider the problem

$$\max \int_0^\infty -u^2 e^{-rt} dt,\ \dot{x} = u e^{-at},\ x(0) = 0,\ \lim_{t\to\infty} x(t) \geq K,\ u \in \mathbb{R},$$

where r, a, and K are positive constants with $a > r/2$.

Set $H^c = -u^2 + \lambda u e^{-at}$ as the current value Hamiltonian. We compute the Hessian matrix of H^c:

$$H(x,u) = \begin{pmatrix} \partial^2 H^c/\partial x^2 & \partial^2 H^c/\partial u \partial x \\ \partial^2 H^c/\partial x \partial u & \partial^2 H^c/\partial u^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since $H(x,u)$ is negative semidefinite for all (x,u), H^c is concave in (x,u).

We find

$$\begin{aligned}\frac{\partial H^c}{\partial x} &= 0,\\ \frac{\partial H^c}{\partial u} &= -2u + \lambda e^{-at}.\end{aligned}$$

Since every $u \in \mathbb{R}$ is an interior point,

$$u^*(t) \in \arg\max H^c(t, x^*(t), u, \lambda(t)) \Rightarrow u^*(t) = \frac{1}{2}\lambda e^{-at}.$$

Next, since the differential equation below is separable, we obtain

$$\dot{\lambda} - r\lambda = -\partial H^c/\partial x = 0 \Rightarrow \lambda = Ae^{rt},$$

where A is a constant. Thus,

$$u^*(t) = \frac{1}{2}Ae^{(r-a)t}.$$

Plugging $u = (1/2)Ae^{(r-a)t}$ into $\dot{x} = ue^{-at}$, we obtain

$$\dot{x}^* = \frac{1}{2}Ae^{(r-2a)t}.$$

Solving this differential equation, we obtain

$$x^* = C + \int \frac{1}{2}Ae^{(r-2a)t}dt \Rightarrow x^* = C + \frac{1}{2(r-2a)}Ae^{(r-2a)t},$$

where C is a constant. Noting $x^*(0) = 0$, we have $C = -A/2(r-2a)$ so that

$$x^*(t) = \frac{A}{2(2a-r)}\left(1 - e^{(r-2a)t}\right).$$

As $a > r/2$, we have

$$\lim_{t\to\infty} x^*(t) = \frac{A}{2(2a-r)} \underbrace{\Rightarrow}_{\lim_{t\to\infty} x(t) \geq K} A \geq 2K(2a-r) \Rightarrow A > 0.$$

So, (C) in the lemma holds.

We compute

$$\begin{aligned}\lambda(t)e^{-rt}(K - x^*(t)) &= Ae^{rt}e^{-rt}\left[K - \frac{A}{2(2a-r)}\left(1 - e^{(r-2a)t}\right)\right] \\ &\to A\left[K - \frac{A}{2(2a-r)}\right] \text{ as } t \to \infty\end{aligned}$$

Setting $A = 2K(2a - r)$, we obtain

$$\lim_{t\to\infty} \lambda(t)e^{-rt}(K - x^*(t)) = 0 \Rightarrow \text{(A) in the lemma hold.}$$

We also confirm (B) in the lemma holds:

$$|\lambda(t)e^{-rt}| = |Ae^{rt}e^{-rt}| = A = 2K(2a - r) = M, \ \forall t \geq 0.$$

It follows from the lemma (A condition guaranteeing (d)) that (d) holds.

Using the theorem on sufficient conditions with an Infinite Horizon, we have found the solution to Problem (1).