Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 8: Calculus of Variations II (Ch. 8) and Control Theory (Ch. 9)

More General Terminal Conditions (Ch. 8.5)

In economic applications, the initial point is usually fixed, while in many models, the terminal value of the unknown function can be free, or subject to more general restrictions.

The problems we study are formulated as

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x}) dt, \;\; x(t_0) = x_0, \;\; \text{(a) } x(t_1) \text{ free or (b) } x(t_1) \geq x_1 \;\; (*)$$

Theorem (Transversality Conditions): If $x^*(t)$ solves problem $(*)$ with either (a) or (b) as the terminal condition, then $x^*(t)$ must satisfy the Euler equation. With the terminal condition (a), the **transversality condition** is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} = 0,$$

where $F^* \equiv F(t, x^*, \dot{x}^*)$. With the terminal condition (b), the **transversality condition** is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} \begin{cases} = 0 & \text{if } x^*(t_1) > x_1 \\ \leq 0 & \text{otherwise} \end{cases}$$

Proof: We skip the proof. ■

We can interpret $(\partial F^*/\partial \dot{x})_{t=t_1}$ as the marginal value of investment at $t = t_1$.

Example: Consider the following problem:

max $\int_0^1 (1-x^2-\dot{x}^2)dt$, $x(0) = 1$, with (a) $x(1)$ free or (b) $x(1) \geq 2$

Let $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$. The Euler equation is

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \Leftrightarrow -2x + 2\ddot{x} = 0 \Leftrightarrow \ddot{x} - x = 0.$$

The characteristic equation of this differential equation is $r^2 - 1 = 0$. So, we have $r = 1, -1$. The general solution is

$$x(t) = Ae^t + Be^{-t}.$$

$$x(0) = 1 \Rightarrow A + B = 1 \Rightarrow B = 1 - A$$

Thus, an optimal solution of either problem must be of the form

$$x^*(t) = Ae^t + (1 - A)e^{-t}.$$

With (a) as the terminal condition, the transversality condition requires

$$\left.\frac{\partial F^*}{\partial \dot{x}}\right|_{t=t_1} = 0 \Rightarrow -2\dot{x}^*(1) = 0 \Rightarrow \dot{x}^*(1) = 0.$$

Since $\dot{x}^*(t) = Ae^t - (1 - A)e^{-t}$,

$$\dot{x}^*(1) = Ae - (1 - A)e^{-1} = 0 \Rightarrow A = \frac{1}{e^2 + 1}.$$

Hence,

$$x^*(t) = \frac{1}{e^2 + 1}\left(e^t + e^2e^{-t}\right).$$

Because $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$ is concave in $(x, \dot{x})$ (why?), the solution has been found.

With (b) as the terminal condition, we require

$$x^*(1) = Ae + (1 - A)e^{-1} \geq 2 \Rightarrow A \geq \frac{2e - 1}{e^2 - 1}$$

Suppose $x^*(1) > 2$. Then, as in (a), the transversality condition gives $A = 1/(e^2 + 1)$. But this violates the inequality $A \geq (2e - 1)/(e^2 - 1)$ because $2e - 1 > 1$. So,

$$x^*(1) = 2 \Rightarrow A = \frac{2e - 1}{e^2 - 1}$$

Then,

$$\dot{x}^*(1) = Ae - (1 - A)e^{-1} = \frac{2(e^2 - e + 1)}{e^2 - 1} > 0.$$

This implies

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=1} = -2\dot{x}^*(1) \leq 0.$$

Hence, the transversality condition holds. Then, the only solution candidate is

$$x^*(t) = \frac{1}{e^2 - 1}\left\{(2e-1)e^t + (e^2 - 2e)e^{-t}\right\}.$$

Because $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$ is concave in $(x, \dot{x})$ (why?), the solution has been found.

Example: Let $A(t)$ denote DM's wealth at time t, and let w be the (constant) income per unit of time.

DM can borrow and save at the same constant rate of interest r.

Consumption per unit of time at time t: $C(t) = rA(t) + w - \dot{A}(t)$.

$$\max_{\{C(t)\}_{t \in [0,T]}} \int_0^T U(C(t)) e^{-\rho t} dt$$

$$\text{s.t. } \; C(t) = rA(t) + w - \dot{A}(t), \; \forall t \in [0, T], \; A(0) = A_0, \; A(T) \geq A_T,$$

where $U'(\cdot) > 0$, $U''(\cdot) < 0$, $\rho > 0$, $A_0 > 0$, and $A_T > 0$.

The objective function: $F(t, A, \dot{A}) = U(rA + w - \dot{A})e^{-\rho t}$.

$$\begin{aligned}
\frac{\partial F}{\partial A} &= rU^{'}(rA + w - \dot{A})e^{-\rho t}, \\
\frac{\partial F}{\partial \dot{A}} &= -U^{'}(rA + w - \dot{A})e^{-\rho t}.
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}}\right) = -U^{''}(rA + w - \dot{A})(r\dot{A} - \ddot{A})e^{-\rho t} + \rho U^{'}(rA + w - \dot{A})e^{-\rho t}$$

The Euler equation:

$$\begin{aligned}
& \frac{\partial F}{\partial A} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}}\right) = 0 \\
\Leftrightarrow \quad & rU^{'}e^{-\rho t} + U^{''}(r\dot{A} - \ddot{A})e^{-\rho t} - \rho U^{'}e^{-\rho t} = 0 \\
\Leftrightarrow \quad & (r - \rho)U^{'} + U^{''}(r\dot{A} - \ddot{A}) = 0 \\
\Leftrightarrow \quad & \ddot{A} - r\dot{A} + (\rho - r)\frac{U^{'}}{U^{''}} = 0.
\end{aligned}$$

$A^*(t)$: an admissible function that satisfies the Euler equation and the corresponding transversality condition.

Since we have $A(T) \geq A_T$ as the terminal condition, the following transversality condition holds:

$$\left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} \leq 0 \text{ and } A^*(T) > A_T \Rightarrow \left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} = 0.$$

Taking its contrapositive form, we have

$$\left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} < 0 \Rightarrow A^*(T) = A_T$$

Because $U^{'} > 0$, we have

$$\left(\frac{\partial F^*}{\partial \dot{A}}\right)_{t=T} = -U^{'}(rA(T) + w - \dot{A}(T))e^{-\rho T} < 0.$$

This implies that $A^*(T) = A_T$.

The second derivatives of F:

$$
\begin{aligned}
\frac{\partial^2 F}{\partial A^2} &= r^2 U^{''} e^{-\rho t} < 0 \\
\frac{\partial^2 F}{\partial \dot{A} \partial A} = \frac{\partial^2 F}{\partial A \partial \dot{A}} &= -r U^{''} e^{-\rho t} > 0 \\
\frac{\partial^2 F}{\partial \dot{A}^2} &= U^{''} e^{-\rho t} < 0
\end{aligned}
$$

$$
\mathcal{H}(A, \dot{A}) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} r^2 U^{''} e^{-\rho t} & -r U^{''} e^{-\rho t} \\ -r U^{''} e^{-\rho t} & U^{''} e^{-\rho t} \end{pmatrix}.
$$

Since $U^{''} < 0$, $h_{11} < 0$, $h_{22} < 0$, and $h_{11}h_{22} - h_{12}h_{21} = 0$. Thus, $\mathcal{H}(A, \dot{A})$ is NSD $\Rightarrow F(t, A, \dot{A})$ is concave in $(A, \dot{A})$.

By the sufficiency of the Euler equation, we conclude that $A^*(t)$ solves the problem.

Control Theory: Basic Technique (Ch. 9.1, 9.2)

We consider a control problem with no restrictions on the control variable and no restrictions on the terminal state.

Given the fixed time t_0 and t_1, our problem is

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \ u(t) \in (-\infty, \infty) \ \ (*)$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t)), \ \ x(t_0) = x_0, \ x_0 \text{ fixed}, \ x(t_1) \text{ free. } (**)$$

A pair $(x(t), u(t))$ that satisfies $(**)$ is called an **admissible pair**.

Among all admissible pairs we search for an **optimal pair**, a pair of functions that maximizes the integral in $(*)$.

Analogously to the Lagrange multiplier to each constraint, we associate a number $p(t)$, called the **co-state** variable, with the constraint $(**)$ for each $t \in [t_0, t_1]$.

The resulting function $p(t)$ is called the **adjoint function** associated with the differential equation. We interpret $p(t)$ as the rental price for the use of one unit of capital.

Very much like the Lagrangian method, for each time $t \in [t_0, t_1]$ and each possible triple (x, u, p), of the state, control, and adjoint variables, we define what is called the **Hamiltonian** by

$$H(t, x, u, p) = f(t, x, u) + pg(t, x, u)$$

Theorem (Necessity for the Maximum Principle)

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for the problem $(*)$ subject to the constraints $(**)$. Then, there exists a continuous function $p(t)$ such that, for each $t \in [t_0, t_1]$,

(i) $u^*(t) \in \arg \max_{u \in (-\infty, \infty)} H(t, x^*(t), u, p(t))$

(ii) $\dot{p}(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t))$

(iii) $p(t_1) = 0$ (the transversality condition)

Proof: We rather provide a heuristic argument.

We setup the Lagrangian as follows:

$$\mathcal{L} = \int_{t_0}^{t_1} \{f(t, x(t), u(t)) - p(t)[\dot{x}(t) - g(t, x(t), u(t))]\}\, dt,$$

where $p(t)$ is considered the Lagrange multiplier, which changes over time.

By integration by parts, we obtain

$$\begin{aligned}\int_{t_0}^{t_1} p(t)\dot{x}(t)dt &= [p(t)x(t)]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}(t)x(t)dt \\ &= p(t_1)x(t_1) - p(t_0)x(t_0) - \int_{t_0}^{t_1} \dot{p}(t)x(t)dt.\end{aligned}$$

Plugging this back into the Lagrangian, we obtain

$$\mathcal{L} \;=\; \int_{t_0}^{t_1} \left[f(t,x(t),u(t)) + p(t)g(t,x(t),u(t)) + \dot{p}(t)x(t)\right] dt$$
$$-p(t_1)x(t_1) + p(t_0)x(t_0).$$

Assume that the optimal pair $(x^*(t), u^*(t))$ has been found and the Lagrangian is concave in (x, u) so that it is maximized at $(x^*(t), u^*(t))$.

Then, we compute the differential of the Lagrangian $d\mathcal{L}$. Given our hypothesis, we shall show $d\mathcal{L} \leq 0$. The proof that follows is a very simple version of the proof for $d\mathcal{L} \leq 0$.

Claim: $d\mathcal{L} \leq 0$

A Heuristic Argument: Let $\mathcal{L} : [x_0, x_1] \to \mathbb{R}$ be a concave function and $x^* \in \arg\max_{x \in [x_0, x_1]} \mathcal{L}(x)$. We consider the following cases.

Case 1: $x^* \in (x_0, x_1)$

In this case, $\mathcal{L}^{'}(x^*) = 0$. So, $d\mathcal{L} = \mathcal{L}^{'}(x^*)dx = 0$ regardless of whether $dx > 0$ or $dx < 0$.

Case 2: $x^* = x_0$

In this case, $\mathcal{L}^{'}(x^*) \leq 0$ and $dx > 0$. So, $d\mathcal{L} = \mathcal{L}^{'}(x^*)dx \leq 0$.

Case 3: $x^* = x_1$

In this case, $\mathcal{L}^{'}(x^*) \geq 0$ and $dx < 0$. So, $d\mathcal{L} = \mathcal{L}^{'}(x^*)dx \leq 0$.

Summarizing all three cases, we conclude that $d\mathcal{L} \leq 0$. ■

So, by Leibniz's formula, we derive the expression for $d\mathcal{L}$:

$$\begin{aligned} d\mathcal{L} \;=\; & \int_{t_0}^{t_1} \left[\left(f_u'(t, x^*(t), u^*(t)) + p(t) g_u'(t, x^*(t), u^*(t)) \right) du(t) \right] dt \\ & + \int_{t_0}^{t_1} \left[\left(f_x'(t, x^*(t), u^*(t)) + p(t) g_x'(t, x^*(t), u^*(t)) + \dot{p}(t) \right) dx(t) \right] dt \\ & - p(t_1) dx(t_1) + p(t_0) dx(t_0). \end{aligned}$$

Observe that $x(t_0) = x_0$ so that $dx(t_0) = 0$. So, we must have $d\mathcal{L} \leq 0$ for **any** $dx(t), du(t)$, and $dx(t_1)$.

This is equivalent to the following conditions:

$$
\begin{aligned}
f_u^{'}(t, x^*(t), u^*(t)) + p(t) g_u^{'}(t, x^*(t), u^*(t)) &= 0 \\
f_x^{'}(t, x^*(t), u^*(t)) + p(t) g_x^{'}(t, x^*(t), u^*(t)) + \dot{p}(t) &= 0 \\
p(t_1) &= 0.
\end{aligned}
$$

Then, set $H(t, x(t), u(t), p(t)) = f(t, x(t), u(t)) + p(t) g(t, x(t), u(t))$. The above conditions are translated into:

$$
\begin{aligned}
H_u^{'}(t, x^*(t), u^*(t), p(t)) &= 0 \\
H_x^{'}(t, x^*(t), u^*(t), p(t)) &= -\dot{p}(t) \\
p(t_1) &= 0.
\end{aligned}
$$

This completes the Heuristic argument. $\blacksquare$

Theorem (Sufficiency for the Maximum Principle)

Suppose that there exists an admissible pair of $(x^*(t), u^*(t))$ satisfying $(**)$ for which there exists a continuous function $p(t)$ such that, for each $t \in [t_0, t_1]$, conditions (i), (ii), and (iii) hold.

In addition if $H(t, x, u, p(t))$ is concave in (x, u) for each $t \in [t_0, t_1]$, $(x^*(t), u^*(t))$ is optimal.

Proof: We provide the proof later. ■

Example: Consider the problem

$$\max \int_0^T \left[1 - tx(t) - u(t)^2\right] dt,\ \dot{x}(t) = u(t),\ x(0) = x_0,\ x(T) \text{ free},\ u \in \mathbb{R}$$

where x_0 and T are given positive constants.

The Hamiltonian is

$$H(t, x, u, p) = 1 - tx - u^2 + pu.$$

Since $\partial^2 H/\partial u^2 = -2$, H is strictly concave in u. So,

$$u^*(t) \in \arg\max_{u \in \mathbb{R}} H(t, x^*(t), u, p(t)) \Leftrightarrow H_u^{'} = -2u + p(t) = 0.$$

Thus, $u^*(t) = p(t)/2$.

Because $H_x^{'} = -t$, Condition (ii) reduces to $\dot{p}(t) = t$ and Condition (iii) reduces to $p(T) = 0$.

$$\dot{p}(t) = t \Rightarrow p(t) = \frac{1}{2}t^2 + C,$$

where C is a constant. Taking into account $p(T) = T^2/2 + C = 0$, we have

$$p(t) = -\frac{1}{2}(T^2 - t^2) \Rightarrow u^*(t) = -\frac{1}{4}(T^2 - t^2).$$

$$\dot{x}^*(t) = u^*(t) = -\frac{1}{4}(T^2 - t^2) \Rightarrow x^*(t) = -\frac{1}{4}T^2 t + \frac{1}{12}t^3 + K,$$

where K is a constant. Taking into account $x(0) = x_0$, we obtain

$$x^*(t) = x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3.$$

We have therefore found the only possible pair that can solve the problem.

Because $H(t, x, u, p) = 1 - tx - u^2 + pu$ is concave in (x, u) for each fixed t (why?), $(x^*(t), u^*(t))$ is indeed optimal.

The Standard Problem (Ch. 9.4)

We consider the **standard end-constrained problem**.

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \ \ u(t) \in U \subseteq \mathbb{R}, \ \ \forall t, \quad (1)$$

$$\dot{x}(t) = g(t, x(t), u(t)), \ \ x(t_0) = x_0 \quad (2)$$

with one of the following conditions imposed

$$(a) \ x(t_1) = x_1, \ (b) \ x(t_1) \geq x_1, \text{ or (c) } x(t_1) \text{ free} \quad (3)$$

where t_0, t_1, x_0, and x_1 are fixed numbers and U is the fixed control region.

NOTE: U may be a closed set so that $u(t)$ takes the values of the boundary of U.

A pair $(x(t), u(t))$ that satisfies (2) and (3) with $u(t) \in U$ is called an **admissible (or feasible) pair**.

Among all admissible pairs we seek an **optimal pair**, a pair of functions that maximizes the integral in (1).

We define the Hamiltonian as

$$H(t, x, u, p) = p_0 f(t, x, u) + pg(t, x, u) \qquad (4)$$

The new feature is the constant number p_0 in front of $f(t, x, u)$. If $p_0 \neq 0$, we can divide by p_0 to get a new Hamiltonian in which $p_0 = 1$, in effect. But if $p_0 = 0$, this normalization is impossible, although we do not consider such a case in this course.

Theorem: The Maximum Principle with Standard End Constraints

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for the standard end-constrained problem (1) - (3). Then, there exists a continuous function $p(t)$ and a number p_0, which is either 0 or 1, such that for all $t \in [t_0, t_1]$, we have $(p_0, p(t)) \neq (0, 0)$ and, moreover,

(A) The control $u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ w.r.t. $u \in U$, i.e.,

$$H(t, x^*(t), u, p(t)) \leq H(t, x^*(t), u^*(t), p(t)) \text{ for all } u \in U$$

(B) $\dot{p}(t) = -H_x'(t, x^*(t), u^*(t), p(t))$

(C) Corresponding to each of the terminal conditions in (3) there is a **transversality condition** on $p(t_1)$:

(a') $p(t_1)$ no condition

(b') $p(t_1) \geq 0$, with $p(t_1) = 0$ if $x^*(t_1) > x_1$

(c') $p(t_1) = 0$.

Proof: We omit the proof. ■

Theorem: Mangasarian's Sufficiency for the Maximum Principle in the Standard End-Constraints Problem (Ch. 9.7)

Suppose that $(x^*(t), u^*(t))$ is an admissible pair with a corresponding adjoint function $p(t)$ such that the conditions (A) - (C) are satisfied with $p_0 = 1$. Suppose further that the control region U is convex and that $H(t, x, u, p(t))$ is concave in (x, u) for every $t \in [t_0, t_1]$. Then, $(x^*(t), u^*(t))$ is an optimal pair.

Remark: Almost all papers in economics literature that use control theory assume that the problem is "normal" in the sense that $p_0 = 1$.

Proof: Suppose that $(x, u) = (x(t), u(t))$ is an arbitrary alternative admissible pair. What we want is

$$D_u = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))dt - \int_{t_0}^{t_1} f(t, x(t), u(t))dt \geq 0.$$

First, simplify notation by writing H^* instead of $H(t, x^*(t), u^*(t), p(t))$ and H instead of $H(t, x(t), u(t), p(t))$.

Then, using the definition of the Hamiltonian and the fact that $\dot{x}^*(t) = g(t, x^*(t), u^*(t))$ and $\dot{x}(t) = g(t, x(t), u(t))$, we have $f^* = H^* - p\dot{x}^*$ and $f = H - p\dot{x}$. Therefore,

$$D_u = \int_{t_0}^{t_1} (H^* - H)dt + \int_{t_0}^{t_1} p(\dot{x} - \dot{x}^*)dt \qquad (*)$$

Because H is concave in (x, u), the first-order characterization of a concave function implies that

$$H - H^* \leq \frac{\partial H^*}{\partial x}(x - x^*) + \frac{\partial H^*}{\partial u}(u - u^*) \qquad (**)$$

Plugging $(**)$ into $(*)$, we obtain

$$\begin{aligned} D_u &\geq -\int_{t_0}^{t_1} \left[\frac{\partial H^*}{\partial x}(x - x^*) + \frac{\partial H^*}{\partial u}(u - u^*)\right] dt + \int_{t_0}^{t_1} p(\dot{x} - \dot{x}^*) dt \\ &= \int_{t_0}^{t_1} \left[\dot{p}(x - x^*) + p(\dot{x} - \dot{x}^*)\right] dt + \int_{t_0}^{t_1} \frac{\partial H^*}{\partial u}(u^* - u) dt, \end{aligned}$$

where the equality follows because $\dot{p} = -\partial H^*/\partial x$.

Assume $U = [u_0, u_1]$. Since the Hamiltonian is concave in u, Condition (A) is equivalent to

$$\frac{\partial H^*}{\partial u} \begin{cases} \leq 0 & \text{if } u^*(t) = u_0 \\ = 0 & \text{if } u^*(t) \in (u_0, u_1) \\ \geq 0 & \text{if } u^*(t) = u_1. \end{cases}$$

This can be further replaced by the equivalent inequality

$$\frac{\partial H^*}{\partial u}(u^*(t) - u) \geq 0 \quad \text{for all } u \in [u_0, u_1]$$

So,

$$
\begin{aligned}
D_u &\geq \int_{t_0}^{t_1} \left[\dot{p}(x - x^*) + p(\dot{x} - \dot{x}^*)\right] dt \\
&= \int_{t_0}^{t_1} \frac{d}{dt} \left[p(x - x^*)\right] dt \\
&= \left[p(t)(x(t) - x^*(t))\right]_{t_0}^{t_1} \\
&= p(t_1)(x(t_1) - x^*(t_1)) - p(t_0)(x(t_0) - x^*(t_0)) \\
&= p(t_1)(x(t_1) - x^*(t_1))
\end{aligned}
$$

where the last equality follows because $x(t_0) = x^*(t_0) = x_0$

It only remains to prove the claim below.

Claim: $p(t_1)(x(t_1) - x^*(t_1)) \geq 0 \quad (***)$

In Case (a), we have $x(t_1) = x^*(t_1) = x_1$ so that $(***)$ holds.

In Case (b), we have $x(t_1) \geq x_1$ and $x^*(t_1) \geq x_1$. If $x^*(t_1) > x_1$, by transversality condition $(b^{'})$, we have $p(t_1) = 0$. So, $(***)$ holds. If $x^*(t_1) = x_1$, by $(b^{'})$, we have $p(t_1) \geq 0$. Since $x(t_1) \geq x_1$ by $(b^{'})$, $(***)$ holds.

In Case (c), $x(t_1)$ is free. By transversality condition $(c^{'})$, we have $p(t_1) = 0$ so that $(***)$ holds. This completes the proof of the theorem. ■

Example

$$\max \int_0^1 x(t)dt$$
$$\text{subject to } \dot{x}(t) = x(t) + u(t),\ x(0) = 0,\ x(1) \text{ free},\ u \in [-1, 1]$$

We want to have $x(t)$ as large as possible all the time, and from the differential equation, this is achieved by having u as large as possible.

So, $u(t) = 1$ for all t and this must be the optimal control.

Let us confirm this by using the maximum principle.

The Hamiltonian function with $p_0 = 1$ is

$$H(t, x, u, p) = x + px + pu,$$

which is linear and hence concave in (x, u). So, the sufficiency for the maximum principle applies. Then,

$$\left[H_x^{'} = -\dot{p} \Leftrightarrow \dot{p} = -1 - p\right], \quad p(1) = 0.$$

We solve $\dot{p} = -1 - p$:

$$\frac{dp}{dt} = -(1 + p) \Leftrightarrow \frac{dp}{1 + p} = -dt \Leftrightarrow \int \frac{dp}{1 + p} = -\int dt \Leftrightarrow \ln(1 + p) = -t + A$$

where A is a constant.

Since $p(1) = 0$, we have $A = 1$. Therefore, $p(t) = e^{1-t} - 1$.

Due to the very expression for H, we must have $u^*(t) = 1$ for all $t \in [0, 1]$.

By the differential equation $\dot{x}(t) = x(t) + u(t)$, $\dot{x}^*(t) = x^*(t) + 1$.

$$\frac{dx^*(t)}{dt} = x^*(t) + 1 \Leftrightarrow \int \frac{dx^*(t)}{x^*(t) + 1} = \int dt \Leftrightarrow \ln(x^*(t) + 1) = t + B,$$

where B is a constant.

Since $x^*(0) = 0$, we have $B = 0$. So, $x^*(t) = e^t - 1$.

We now see that $u^*(t), x^*(t)$, and $p(t)$ satisfy all the requirements in the sufficiency for the maximum principle.

An Example of the Bang-Bang Solution: Consider the following problem.

$\max \int_0^1 (2x - x^2) dt$ subject to $\dot{x} = u,\ x(0) = 0,\ x(1) = 0,\ u \in [-1, 1]$.

We set up the Hamiltonian:

$$H = 2x - x^2 + pu.$$

If we want to maximize the value of H with respect to $u \in [-1, 1]$, we obtain

$$(*)\ u^*(t) = \begin{cases} 1 & \text{if } p(t) > 0 \\ -1 & \text{if } p(t) < 0, \end{cases}$$

where $p(t)$ denotes the adjoint function associated with the Hamiltonian.

By the maximum principle, we have

$$\dot{p}(t) = -H_x^{'} \Rightarrow \dot{p}(t) = -2 + 2x.$$

So,

$$\dot{p}(t) = 2(x^*(t) - 1). \; (**)$$

By the differential equation $\dot{x} = u$,

$$\dot{x}^*(t) = u^*(t) \underbrace{\Rightarrow}_{x^*(0)=0} x^*(t) = \int_0^t u^*(\tau) d\tau \underbrace{\leq}_{u^*(t)\leq 1} \int_0^t d\tau = t.$$

Thus, $x^*(t) \leq t$ for every $t \in [0, 1]$.

This implies that $x^*(t) < 1$ for all $t \in [0, 1)$.

Since $\dot{p}(t) = 2(x^*(t) - 1)$ and $x^*(t) < 1$ for all $t \in [0, 1)$

$$\Rightarrow \quad \dot{p}(t) < 0 \text{ for all } t \in [0, 1)$$

$$\Rightarrow \quad p(t) \text{ is strictly decreasing over } [0, 1]$$

Claim: There is no solution to the problem with $p(1) \geq 0$.

Proof: Suppose not, i.e., there is a solution with $p(1) \geq 0$.

$p(t)$ is strictly decreasing $\Rightarrow p(t) > 0$ for all $t \in [0, 1)$.

By $(*)$, we also have $u^*(t) = 1$ for all $t \in [0, 1]$.

$$u^*(t) = 1, \forall t \in [0,1] \quad \underbrace{\Rightarrow}_{\dot{x}^*(t)=u^*(t)} \quad \dot{x}^*(t) = 1 \quad \underbrace{\Rightarrow}_{x^*(0)=0} \quad x^*(t) = t, \forall t \in [0,1]$$

This implies

$$x^*(1) = 1 \qquad \underbrace{\Rightarrow\Leftarrow}_{\text{Contradiction!}} \qquad x^*(1) = 0. \blacksquare$$

Claim: There is no solution to the problem such that $p(t) < 0$ for all $t \in (0, 1]$.

Proof: Suppose not, i.e., there exists a solution such that $p(t) < 0$ for all $t \in (0, 1]$.

By $(*)$, we have $u^*(t) = -1$ for all $t \in [0, 1]$.

By the differential equation $\dot{x}^*(t) = u^*(t) = -1$, we have

$$\frac{dx^*(t)}{dt} = -1 \Leftrightarrow x^*(t) = -\int dt = -t + B,$$

where B is a constant.

Since $x^*(0) = 0$, we have $B = 0$. Thus, $x^*(t) = -t$.

This implies

$$x^*(1) = -1 \qquad \underbrace{\Rightarrow\Leftarrow}_{\text{Contradiction!}} \qquad x^*(1) = 0. \blacksquare$$

Therefore, we can assume that there exists $t^* \in (0,1)$ such that

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0,t^*] \\ -1 & \text{if } t \in (t^*,1]. \end{cases}$$

Assume also that $x^*(\cdot)$ and $p(\cdot)$ are continuous at $t = t^*$.

Since $u^*(t) = 1$ for any $t \in [0,t^*]$,

$$\dot{x}^*(t) = u^*(t) \underbrace{\Rightarrow}_{x^*(0)=0} x^*(t) = t, \ \forall t \in [0,t^*].$$

Since $u^*(t) = -1$ for any $t \in (t^*, 1]$,

$$\dot{x}^*(t) = u^*(t) \Rightarrow x^*(t) = -t + C, \forall t \in (t^*, 1],$$

where C is a constant.

Since $x^*(\cdot)$ is assumed to be continuous at $t = t^*$, we have

$$C = 2t^* \Rightarrow x^*(t) = -t + 2t^*,\ \forall t \in (t^*, 1] \underbrace{\Rightarrow}_{x^*(1)=0} t^* = 1/2$$

We thus have

$$x^*(t) = -t + 1,\ \forall t \in (1/2, 1].$$

Since $x^*(t) = t$ for any $t \in [0, 1/2]$,

$\dot{p}(t) = 2(x^*(t) - 1), \ \forall t \in [0, 1/2] \Rightarrow \dot{p}(t) = 2(t - 1), \ \forall t \in [0, 1/2]$.

Then,

$$\frac{dp}{dt} = 2(t-1) \quad \underbrace{\Rightarrow}_{\text{integration}} \quad p(t) = t^2 - 2t + D,$$

where D is a constant.

Since

$$(*)\ u^*(t) = \begin{cases} 1 & \text{if } p(t) > 0 \\ -1 & \text{if } p(t) < 0 \end{cases} \quad \text{and } u^*(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2] \\ -1 & \text{if } t \in (1/2, 1] \end{cases}$$

we must have

$$p(1/2) = 0 \Rightarrow D = 3/4 \Rightarrow p(t) = t^2 - 2t + 3/4, \ \forall t \in [0, 1/2].$$

For any $t \in (1/2, 1]$, we have

$$\dot{p}(t) = 2(x^*(t) - 1) \underbrace{\Rightarrow}_{x^*(t) = -t+1} \dot{p}(t) = -2t \Rightarrow p(t) = -t^2 + E,$$

where E is a constant.

Since $p(\cdot)$ is assumed to be continuous at $t = 1/2$,

$$p(1/2) = 0 \Rightarrow -\frac{1}{4} + E = 0 \Rightarrow E = 1/4.$$

Hence, $p(t) = -t^2 + 1/4$.

The solution candidate we obtain is summarized as follows:

$$\begin{aligned} u^*(t) &= \begin{cases} 1 & \text{if } t \in [0, 1/2] \\ -1 & \text{if } t \in (1/2, 1]. \end{cases} \\ x^*(t) &= \begin{cases} t & \text{if } t \in [0, 1/2] \\ -t + 1 & \text{if } t \in (1/2, 1] \end{cases} \\ p(t) &= \begin{cases} t^2 - 2t + 3/4 & \text{if } t \in [0, 1/2] \\ -t^2 + 1/4 & \text{if } t \in (1/2, 1] \end{cases} \end{aligned}$$

Recall that we have $H = 2x - x^2 + pu$ as the Hamiltonian.

$2x - x^2$ and pu are both concave functions in (x, u) and a sum of two concave functions is also a concave function.

Therefore, the Hamiltonian is a concave function in (x, u).

By the sufficiency for the maximum principle, we conclude that the solution candidate we found is indeed a solution to the original problem.