Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 2: Compactness, Continuity and Linear Algebra

Closedness in terms of Sequences on $\mathbb{R}^n$ (Ch. 13.2)

Theorem: A set $S \subseteq \mathbb{R}^n$ is closed if and only if every convergent sequence of points in S has its limit in S, i.e.,
$(\forall \{x^k\} \in S)((\exists x \in \mathbb{R}^n)(x^k \to x)) \Rightarrow x \in S$.

Proof: We skip the proof. $\blacksquare$

Boundedness in terms of Sequences on $\mathbb{R}^n$ (Ch. 13.2)

Definition: A set S in $\mathbb{R}^n$ is **bounded** if there exists a number $M \in \mathbb{R}_+$ such that $\|x\| \leq M$ for all $x \in S$. A set that is not bounded is called **unbounded**. Here $\|x\| = d(x, \mathbf{0}) = \sqrt{x_1^2 + \cdots + x_n^2}$, called the **Euclidean norm**.

Similarly, a sequence $\{x^k\}$ in $\mathbb{R}^n$ is **bounded** if the set $\{x^k | k \in \mathbb{N}\}$ is bounded.

Lemma: Any convergent sequence $\{x^k\}$ in $\mathbb{R}^n$ is bounded.

Proof: We skip the proof. ■

On the other hand, a bounded sequence $\{x^k\}$ in $\mathbb{R}^n$ is "not" necessarily convergent. This is the same as sequences in $\mathbb{R}$.

Hence, we obtain

Theorem: A subset S of $\mathbb{R}^n$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Proof: We skip the proof. $\blacksquare$

Compactness

Compactness (Ch. 13.2)

Bolzano-Weierstrass Theorem: A subset S of $\mathbb{R}^n$ is closed and bounded if and only if every sequence of points in S has a subsequence that converges to a point in S.

Proof of Bolzano-Weierstrass's Theorem: We skip this. ■

Definition: A set S in $\mathbb{R}^n$ is **compact** if it is closed and bounded.

Continuity

Continuous Functions (Ch. 13.3)

Roughly speaking, $f(\cdot)$ is continuous if small changes in the independent variables cause only small changes in the function value.

In what follows, let S be a nonempty subset of $\mathbb{R}^n$.

Theorem: A function f from S into $\mathbb{R}$ is **continuous** at a point x^0 in S if and only if $(\forall \{x^k\} \in S)(x^k \to x^0 \Rightarrow f(x^k) \to f(x^0))$. If $f(\cdot)$ is continuous at every point in a set S, $f(\cdot)$ is continuous on S.

Proof: We skip the proof. ■

Example: Let $f : \mathbb{R}_+ \to \mathbb{R}$ be given below.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

Let $x^0 = 1$.

If x converges to 1 "from below," we have $\lim_{x \to 1^-} f(x) = 0$.

And, if x converges to 1 "from above," we have $\lim_{x \to 1^+} f(x) = 1$.

Therefore, $f(\cdot)$ is **not** a continuous function at $x^0 = 1$.

Continuity of Vector-Valued Functions (Ch. 13.3)

Let $f = (f^{(1)}, \ldots, f^{(m)})$ be a function from a subset S to $\mathbb{R}^m$.

Theorem: A function $f = (f^{(1)}, \ldots, f^{(m)})$ from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is continuous at a point x^0 in S if and only if each component function $f^{(j)} : S \to \mathbb{R}$, $j = 1, \ldots, m$, is continuous at x^0.

Proof: We omit the proof. ■

Two Preliminary Results for "Weierstrass Theorem" (Ch. 3.1)

Theorem: Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be continuous. Then $f(K) = \{f(x) | x \in K\}$ is compact for every compact subset K of S.

Proof: We omit the proof. ■

Theorem: Let S be a compact set in $\mathbb{R}$ and let x_* be the greatest lower bound of S and x^* be the least upper bound of S. Then, $x_* \in S$ and $x^* \in S$.

Proof: We skip the proof. ■

Weierstrass (Extreme Value) Theorem (Ch. 3.1 and 13.3)

Weierstrass Theorem: Let $f : S \to \mathbb{R}$ be a continuous real-valued mapping where S is a nonempty compact subset of $\mathbb{R}^n$. Then there exist two vectors $x^*, x_* \in S$ such that for all $x \in S$,

$$f(x_*) \leq f(x) \leq f(x^*).$$

Proof: The first theorem shows that $f(S)$ is a nonempty compact set. The second theorem shows that there exist $y^*, y_* \in f(S)$ such that $y_* \leq y \leq y^*$ for any $y \in f(S)$. The rest of the proof is completed by finding $x_*, x^* \in S$ such that $y_* = f(x_*)$ and $y^* = f(x^*)$. ■

Why are the compactness and the continuity needed

(1) Let $S = [0, \infty)$ and $f(x) = x$. Then $f(\cdot)$ cannot attain a maximum because S is not bounded from above. But S is closed and $f(\cdot)$ is continuous.

(2) Let $S = (0, 1)$ and $f(x) = x$. Then $f(\cdot)$ cannot attain a maximum or minimum because S is not closed. But S is bounded and $f(\cdot)$ is continuous.

(3) Let $S = [0,1]$. Define $f : S \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 - x & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

This $f(\cdot)$ fails to attain a maximum because $f(\cdot)$ is not continuous at $x = 0$. But S is compact.

Linear Algebra (Ch. 1)

Basic Concepts of Matrix Algebra in $\mathbb{R}^n$ (Ch. 1.1)

An $m \times n$ **matrix** is a rectangular array with m rows and n columns:

$$A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Here a_{ij} denotes the elements in the ith row and the jth column.

A matrix can be considered a generalization of numbers. We want to understand to what extent we can treat matrices like numbers.

Sum and Subtraction of Matrices

If $A = (a_{ij})_{m\times n},\ B = (b_{ij})_{m\times n}$, and $\alpha \in \mathbb{R}$ is a scalar, I define

- $A + B = (a_{ij} + b_{ij})_{m\times n}$,
- $\alpha A = (\alpha a_{ij})_{m\times n}$,
- $A - B = A + (-1)B = (a_{ij} - b_{ij})_{m\times n}$.

The **zero matrix** $\mathbf{0}_{m\times n}$ is defined as a matrix where all entries are zero:

$$\mathbf{0}_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Multiplications of Matrices

A: $m \times n$ matrix; and B: $n \times p$ matrix.

$C = AB$: $m \times p$ matrix $C = (c_{ij})_{m\times p}$ such that for each (i, j),

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj}}_{n \text{ terms}}$$

AB is well-defined only if the number of columns in A is equal to the number of rows in B.

Example of Multiplication of Matrices:

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 2 + 3 \cdot 6 & 1 \cdot 3 + 0 \cdot 5 + 3 \cdot 2 \\ 2 \cdot 1 + 1 \cdot 2 + 5 \cdot 6 & 2 \cdot 3 + 1 \cdot 5 + 5 \cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 9 \\ 34 & 21 \end{pmatrix}.$$

Theorem: If A, B, and C are matrices such that the given operations are well-defined, then

- $(AB)C = A(BC)$ (**associative law**)

- $A(B + C) = AB + AC$ (**left distributive law**)

- $(A + B)C = AC + BC$ (**right distributive law**)

Proof: We omit the proof. ■

However, matrix multiplication is **not** commutative. In fact,

(1) $AB \neq BA$, except in special cases:

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

(2) $AB = \mathbf{0}$ does not imply that A or B is $\mathbf{0}$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

but

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(3) $AB = AC$ and $A \neq \mathbf{0}$ do not imply that $B = C$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \; B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; \text{ and } C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$AB = AC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A matrix is **square** if it has an equal number of rows and columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the **principal diagonal** of the matrix A.

The **identity matrix** of order n, denoted by $\mathbf{I}_n$, is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ (\textbf{identity matrix})}$$

If A is any $m \times n$ matrix, then $A\mathbf{I}_n = A = \mathbf{I}_m A$. In particular,

$$A\mathbf{I}_n = \mathbf{I}_n A = A \text{ for every } n \times n \text{ matrix } A$$

Let $A = (a_{ij})_{m \times n}$. The **transpose** of A is defined as $A^T = (a_{ji})_{n \times m}$ where the subscripts i and j are interchanged.

A square matrix is said to be **symmetric** if $A = A^T$.

Theorem: The following rules apply to matrix transposition:

1. $(A^T)^T = A$

2. $(A + B)^T = A^T + B^T$

3. $(\alpha A)^T = \alpha A^T$

4. $(AB)^T = B^T A^T$

Proof: We omit the proof. ■

Determinants (Ch. 1.1)

Let A be an $n \times n$ matrix.

The **determinant** of A generates a real number that summarizes some information about what matrix A is.

This is very convenient because we are usually not good at dealing with matrices but we should be OK to handle a single number.

The determinant of such A is denoted $|A|$. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we have $|A| = a_{11}a_{22} - a_{12}a_{21}$.

The **cofactor** A_{ij} is the determinant of $(n-1)\times(n-1)$ matrices given by deleting ith row and jth columns from the matrix A:

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Definition: Let A be an $n \times n$ matrix. The **determinant** of A is computed by

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

for each $j = 1, \ldots, n$. This is called the **expansion of the determinant** $|A|$ **with respect to the** j**-th column**.

Naturally, we can write a similar formula for any **row** of the determinant $|A|$. For example, for the i-th row,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{ij}A_{ij} + \cdots + a_{in}A_{in}.$$

The role of determinants in solving a system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Eliminating one of the unknowns in the usual way, one can easily obtain the formulas:

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \text{ and } x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

assuming that $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

The numerators and denominators of the ratio can be represented by:

$$
\begin{aligned}
a_{11}a_{22} - a_{12}a_{21} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \\
b_1 a_{22} - b_2 a_{12} &= \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \\
a_{11} b_2 - a_{21} b_1 &= \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.
\end{aligned}
$$

A square matrix A is **nonsingular** if $|A| \neq 0$ and **singular** if $|A| = 0$.

Some Rules for Manipulating Determinants

(1) If two rows (or two columns) of A are interchanged, the determinant changes sign but its absolute value remains unchanged.

(2) If all the elements in a single row (or column) of A are multiplied by a number c, the determinant is multiplied by c.

(3) If two of the rows (or columns) of A are proportional, then $|A| = 0$.

(4) The value of $|A|$ remains unchanged if a multiple of one row (or one column) is added to another row (or column).

Cramer's Rule (Ch. 1.1)

A linear system of n equations and n unknowns is given:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \quad (*) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array}$$

Theorem (Cramer's Rule): $(*)$ has a unique solution if and only if $|A| \neq 0$. The solution is then

$$x_j = \frac{|A_j|}{|A|},\ j = 1, \ldots, n$$

where the determinant $|A_j|$ is defined as:

$$|A_j| = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & \boxed{b_1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & \boxed{b_2} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & \boxed{b_n} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}.$$

Note that $|A_j|$ is obtained by replacing the jth column of $|A|$ by the column whose components are $b_1, b_2, \ldots, b_n$.

Proof: We omit the proof. ■

A linear system of n equations and n unknowns $Ax = b$ is given:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \quad (*) \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array}$$

If $b_1 = \cdots = b_n = 0$, i.e., $Ax = \mathbf{0}$, the system $(*)$ is called **homogeneous**.

A homogeneous system always has the **trivial** solution: $x_1 = \cdots = x_n = 0$.

Matrix Inverse (Ch. 1.1)

Let A be an $n \times n$ matrix.

When can we find an $n \times n$ matrix B such that $BA = AB = I$?

If such B exists, it is called the **inverse** of matrix A and is denoted by A^{-1}.

If A^{-1} exists, matrix A itself is said to be **invertible** (**nonsingular**).

Theorem: Every nonsingular matrix A has a unique inverse matrix B such that

$$AB = BA = I.$$

If $A = (a_{ij})_{n\times n}$ and $|A| \neq 0$, the unique inverse of A is given by

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A), \text{ where } \mathrm{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

with A_{ij}, the **cofactor** of the element a_{ij}. Note carefully the order of the indices in adj(A) with the column number preceding the row number.

Proof: We omit the proof. ■

We provide the results on matrix inverse without proofs.

Lemma (Rules for Matrix Inverse):

- $(A^{-1})^{-1} = A$,

- $(AB)^{-1} = B^{-1}A^{-1}$,

- $(A^T)^{-1} = (A^{-1})^T$,

- $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$, where $\alpha \in \mathbb{R}$.

Proposition: If $|A| \neq 0$, then $|A^{-1}| = 1/|A|$.

Vectors (Ch. 1.1)

An n-**vector** is an ordered n-tuple of (real) numbers.

An n-vector can be understood either as a $1 \times n$ matrix $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ (a **row vector**) or as an $n \times 1$ matrix (a **column** vector)

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The operations of addition, subtraction and "scalar" multiplication of vectors are defined in the obvious way.

Inner Product (Dot Product)

We introduce the multiplication of two vectors:

The **dot product** (or **inner product**) of the n-vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

Proposition (Properties of the Inner Product): If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are n-vectors and α is a scalar, then

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,

2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,

3. $(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$.

4. $\mathbf{a} \cdot \mathbf{a} = 0 \Rightarrow \mathbf{a} = \mathbf{0}$

5. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b} \cdot \mathbf{b}$.

Proof: We skip the proof. ■

The **Euclidean norm** or **length** of the vector $\mathbf{a}$ is defined:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

NOTE: $\|\alpha \mathbf{a}\| = |\alpha| \|\mathbf{a}\|$ for all scalars and vectors.

Lemma: The following useful inequalities hold.

1. $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$ (**Cauchy-Schwartz inequality**)

2. $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ (**Minkowski inequality**)

Proof: We omit the proof. ■

Remark: You are referred to Ch.B.1 for trigonometric functions.

Cauchy-Schwartz inequality implies that, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \leq 1.$$

Thus, the **angle** θ between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\cos\theta = \frac{\mathbf{a}}{\|\mathbf{a}\|} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}, \;\; \theta \in [0, \pi]$$

This definition reveals that $\cos\theta = 0$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. Then $\theta = \pi/2 = 90^\circ$. In symbols,

$$\mathbf{a} \perp \mathbf{b} \Longleftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

Linear Independence (Ch. 1.2)

Definition: The n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ of an Euclidean space are **linearly dependent** if there exist numbers $c_1, c_2, \ldots, c_n \in \mathbb{R}$, not all zero, such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

If this equation holds only when $c_1 = c_2 = \cdots = c_n = 0$, then the vectors are **linearly independent**.

The next result is a characterization of linear dependence in a linear space.

Proposition: A set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ of an Euclidean space is linearly dependent if and only if at least one of them can be written as a linear combination of the others.

Proof: Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly dependent. Then the equation $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = 0$ holds with at least **one** of the coefficients c_i differently from 0. We can, without loss of generality, assume that $c_1 \neq 0$. Solving the equation for $\mathbf{a}_1$ yields

$$\mathbf{a}_1 = -\frac{c_2}{c_1}\mathbf{a}_2 - \cdots - \frac{c_n}{c_1}\mathbf{a}_n.$$

Thus, $\mathbf{a}_1$ is a linear combination of the other vectors. ■

The relation between the determinant and linear dependence:

Theorem: Let A be an $n \times n$ matrix. Then, $|A| = 0$ if and only if there is a linear dependence between its columns of A.

Proof: We omit the proof. $\blacksquare$

The Rank of a Matrix (Ch. 1.3)

Definition: The **rank** of a matrix A, written rank(A), is the maximum number of linearly independent column vectors in A. If A is the $\mathbf{0}$ matrix, we put rank(A) $= 0$.

$A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$: $n \times n$ matrix

rank(A) $= n \Rightarrow |A| \neq 0$;

rank(A) $< n \Rightarrow |A| = 0$.

Theorem: Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be an $n \times n$ matrix. Then, $|A| \neq 0$ if and only if $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent.

Proof: We skip the proof. ■

Example

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

We can easily see that the third and fourth columns are parallel to each other:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}.$$

So, we now consider the following truncated matrix:

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{pmatrix}$$

We compute the determinant of the truncated matrix:

$$\begin{aligned}\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} &= 1 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} \\ &= 4 - 8 = -4 \neq 0.\end{aligned}$$

Since $|B| \neq 0$, all the columns vectors in B are linearly independent.

Thus, rank$(A) = 3$.

Main Results on Linear Systems (Ch. 1.4)

Consider the general **system of linear equations**:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & . & . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & . & . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & . & . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \qquad (*)$$

can be written as $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Let

$$A_b = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

be the **augmented** matrix of the system $(*)$.

It turns out that the relationship between the ranks of A and A_b is crucial in determining whether system $(*)$ has a solution.

Because all the columns in A occur in A_b, the rank of A_b is certainly greater than or equal to the rank of A. Moreover, because A_b contains only one more column than A, rank$(A_b) \leq$ rank$(A) + 1$.

Theorem: $Ax = b$ has at least one solution if and only if rank$(A) =$ rank(A_b).

Proof: We omit the proof. ■

Remark: When we say $Ax = b$ has at least one solution, there may well be multiple solutions to $Ax = b$.

Theorem: Suppose that system $(*)$ has solutions with $\text{rank}(A) = \text{rank}(A_b) = k$.

1. If $k < m$, then $m - k$ equations are **superfluous** in the sense that if we choose any subsystem of equations corresponding to k linearly independent rows, then any solution of these k equations also satisfies the remaining $m - k$ equations.

2. If $k < n$, there exist $n - k$ variables that can be chosen freely, whereas the remaining k variables are uniquely determined by the choice of these $n - k$ free variables. The system then has $n - k$ **degrees of freedom**.

Proof: We omit the proof. ■