ECON 696
Mathematical Methods for Economic Dynamics
Answer Key to the Midterm Examination
Fall 2022, SMU School of Economics

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Sep 30 (Fri), 2022; 12:00 - 13:40 (100 minutes)
at SOE/SCIS2 Building, Seminar Room 3-3

Question 1 (Linear Independence (15 points)) *Let* $\mathbf{a}, \mathbf{b}$*, and* $\mathbf{c}$ *be linearly independent vectors in* $\mathbb{R}^n$*. Answer the following questions*

1. (8 points) Show that three vectors $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly independent.

   Suppose, by way of contradiction, that three vectors $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly dependent. Then, there exists a nonzero vector $(c_1, c_2) \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ such that

   $$\mathbf{a}+\mathbf{b} = c_1(\mathbf{b}+\mathbf{c}) + c_2(\mathbf{a}+\mathbf{c}).$$

   This is equivalent to

   $$(c_2 - 1)\mathbf{a} + (c_1 - 1)\mathbf{b} + (c_1 + c_2)\mathbf{c} = \mathbf{0}.$$

   Since $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent, we must have $c_1 = c_2 = 1$ and $c_1 + c_2 = 0$, which are simply impossible. This is the desired contradiction.

2. (7 points) Show whether three vectors $\mathbf{a}-\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly independent or linearly dependent.

   We claim these vectors are linearly dependent, since $\mathbf{a}-\mathbf{b}$ can be expressed as a linear combination of $\mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ as follows:

   $$\mathbf{a}-\mathbf{b} = -(\mathbf{b}+\mathbf{c}) + (\mathbf{a}+\mathbf{c}).$$

Question 2 (Optimization with Equality Constraints (30 points)) *Consider the following constrained optimimization (maximization or minimization) problem:*

$$\max_{(x,y,z)\in\mathbb{R}^3} \left(\text{or} \min_{(x,y,z)\in\mathbb{R}^3} \right) x+y+z \ \textit{subject to } x^2+y^2+z^2=1 \textit{ and } x-y-z=1.$$

Answer the following questions.

1. (15 points) Find all the solution candidates of this constrained optimization problem using the Lagrangian method.

   We set up the Lagrangian:

   $\mathcal{L}(x,y,z,\lambda_1,\lambda_2) = x+y+z-\lambda_1(x^2+y^2+z^2-1)-\lambda_2(x-y-z-1).$

   The first-order conditions of the Lagrangian and the two equality constraints are provided as follows:

$$\begin{aligned}
&(1) \quad \mathcal{L}_1' = 1-2\lambda_1 x-\lambda_2 = 0;\\
&(2) \quad \mathcal{L}_2' = 1-2\lambda_1 y+\lambda_2 = 0;\\
&(3) \quad \mathcal{L}_3' = 1-2\lambda_1 z+\lambda_2 = 0;\\
&(4) \quad x^2+y^2+z^2-1 = 0;\\
&(5) \quad x-y-z-1 = 0.
\end{aligned}$$

   (2) - (3) leads us to
$$-2\lambda_1(y-z) = 0.$$

   There are two cases to consider: Case 1: $\lambda_1 = 0$ and Case 2: $y = z$.

   Case 1: $\lambda_1 = 0$

   Plugging $\lambda_1 = 0$ into (1), we have $\lambda_2 = 1$. Next, plugging $\lambda_1 = 0$ and $\lambda_2 = 1$ into (2), we obtain $2 = 0$, which is a contradiction. So, we ignore this case.

   Case 2: $y = z$

   Taking $y = z$ into account in (5), we have $x = 2y+1$. Plugging $x = 2y+1$ and $z = y$ into (4), we have

$$(2y+1)^2+y^2+y^2-1 = 0 \Rightarrow 6y^2+4y = 6y(y+2/3) = 0.$$

   There are two cases to consider: Case A: $y = 0$ and Case B: $y = -2/3$.

Case A: $y = 0$

In this case, we thus obtain $(x, y, z) = (1, 0, 0)$. Plugging $y = 0$ into (2), we obtain $\lambda_2 = -1$. Plugging $x = 2$ and $\lambda_2 = -1$ into (1), we obtain $\lambda_1 = 1$. Thus, the Lagrange multipliers associated with this case are $(\lambda_1, \lambda_2) = (1, -1)$.

Case B: $y = -2/3$

In this case, we have $(x, y, z) = (-1/3, -2/3, -2/3)$. Plugging $x = -1/3$ into (1) and plugging $y = -2/3$ into (2), we obtain

$$(1') \quad 1 + \frac{2}{3}\lambda_1 - \lambda_2 = 0;$$
$$(2') \quad 1 + \frac{4}{3}\lambda_1 + \lambda_2 = 0.$$

$(1') + (2')$ allows us to obtain $\lambda_1 = -1$. Plugging $\lambda_1 = -1$ into $(1')$, we obtain $\lambda_2 = 1/3$. Thus, the Lagrangian multipliers associated with this case are $(\lambda_1, \lambda_2) = (-1, 1/3)$. Therefore, we obtain two solution candidates with the associated Lagrange multipliers:

$$(x, y, z, \lambda_1, \lambda_2) = \begin{cases} (1, 0, 0, 1, -1) \\ (-1/3, -2/3, -2/3, -1, 1/3) \end{cases}$$

2. (15 points) Classify each solution candidate as either a local maximum point or local minimum point (Hint: Leave all the details of your computation).

   Let $g^1(x, y, z) = x^2 + y^2 + z^2 - 1$ and $g^2(x, y, z) = x - y - z - 1$. We then compute

$$\begin{aligned} \nabla g^1(x, y, z) &= (\partial g^1/\partial x, \partial g^1/\partial y, \partial g^1/\partial z) = (2x, 2y, 2z); \\ \nabla g^2(x, y, z) &= (\partial g^2/\partial x, \partial g^2/\partial y, \partial g^2/\partial z) = (1, -1, -1). \end{aligned}$$

   We also compute

$$\begin{aligned} \mathcal{L}''_{11} &= -2\lambda_1; \\ \mathcal{L}''_{12} = \mathcal{L}''_{21} &= 0; \\ \mathcal{L}''_{13} = \mathcal{L}''_{31} &= 0; \\ \mathcal{L}''_{22} &= -2\lambda_1; \\ \mathcal{L}''_{23} = \mathcal{L}''_{32} &= 0; \\ \mathcal{L}''_{33} &= -2\lambda_1. \end{aligned}$$

We form the Bordered Hessian matrix associated with this problem:

$$
\begin{aligned}
B(x,y,z) &= \begin{pmatrix} 0 & 0 & \partial g^1/\partial x & \partial g^1/\partial y & \partial g^1/\partial z \\ 0 & 0 & \partial g^2/\partial x & \partial g^2/\partial y & \partial g^2/\partial z \\ \partial g^1/\partial x & \partial g^2/\partial x & \mathcal{L}''_{11} & \mathcal{L}''_{12} & \mathcal{L}''_{13} \\ \partial g^1/\partial y & \partial g^2/\partial y & \mathcal{L}''_{21} & \mathcal{L}''_{22} & \mathcal{L}''_{23} \\ \partial g^1/\partial z & \partial g^2/\partial z & \mathcal{L}''_{31} & \mathcal{L}''_{32} & \mathcal{L}''_{33} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 1 & -1 & -1 \\ 2x & 1 & -2\lambda_1 & 0 & 0 \\ 2y & -1 & 0 & -2\lambda_1 & 0 \\ 2z & -1 & 0 & 0 & -2\lambda_1 \end{pmatrix}
\end{aligned}
$$

We evaluate the determinant of the bordered Hessian matrix at each solution candidate.

Case I: $(x, y, z, \lambda_1, \lambda_2) = (1, 0, 0, 1, -1)$

We compute the following:

$$
|B(1,0,0)| = \begin{vmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & -2 \end{vmatrix} = 2 \cdot (-1)^{1+3} \begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{vmatrix} = -16,
$$

where

$$
\begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{vmatrix} = 2\cdot(-1)^{2+1} \begin{vmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{vmatrix} + 1\cdot(-2)^{2+2} \underbrace{\begin{vmatrix} 0 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}}_{=0},
$$

where

$$
\begin{aligned}
\begin{vmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{vmatrix} &= (-1)\cdot(-1)^{1+2} \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} + (-1)\cdot(-1)^{1+3} \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} \\
&= (2-0) - (0-2) = 4.
\end{aligned}
$$

Hence, $(1, 0, 0)$ is a local maximum point.

Case II: $(x, y, z, \lambda_1, \lambda_2) = (-1/3, -2/3, -2/3, -1, 1/3)$

We compute the following:

$$
\begin{aligned}
|B(-1/3,-2/3,-2/3)| &= \begin{vmatrix} 0 & 0 & -2/3 & -4/3 & -4/3 \\ 0 & 0 & 1 & -1 & -1 \\ -2/3 & 1 & 2 & 0 & 0 \\ -4/3 & -1 & 0 & 2 & 0 \\ -4/3 & -1 & 0 & 0 & 2 \end{vmatrix} \\
&= 1\cdot(-1)^{2+3} \begin{vmatrix} 0 & 0 & -4/3 & -4/3 \\ -2/3 & 1 & 0 & 0 \\ -4/3 & -1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} \\
&\quad +(-1)\cdot(-1)^{2+4} \begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} \\
&\quad +(-1)\cdot(-1)^{2+5} \begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \\ -4/3 & -1 & 0 & 0 \end{vmatrix} \\
&= -\left(-\frac{32}{3}\right) - \left(-\frac{8}{3}\right) + \frac{8}{3} = \frac{48}{3} = 16,
\end{aligned}
$$

where

$$\begin{vmatrix} 0 & 0 & -4/3 & -4/3 \\ -2/3 & 1 & 0 & 0 \\ -4/3 & -1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} = -\frac{4}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix}$$

$$-\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix}$$

$$= -\frac{4}{3}\cdot 4 + \frac{4}{3}\cdot(-4) = -\frac{32}{3}$$

$$\begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} = -\frac{2}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix}$$

$$-\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix}$$

$$= -\frac{2}{3}\cdot 4 + \frac{4}{3}\cdot 0 = -\frac{8}{3}$$

$$\begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \\ -4/3 & -1 & 0 & 0 \end{vmatrix} = -\frac{2}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix}$$

$$-\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix}$$

$$= -\frac{2}{3}\cdot(-4) + \frac{4}{3}\cdot 0 = \frac{8}{3},$$

where

$$\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix} = 2\cdot(-1)^{3+3}\begin{vmatrix} -2/3 & 1 \\ -4/3 & -1 \end{vmatrix} = 2(2/3+4/3) = 4$$

$$\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix} = 2\cdot(-1)^{2+3}\begin{vmatrix} -2/3 & 1 \\ -4/3 & -1 \end{vmatrix} = -2\cdot 2 = -4$$

$$\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix} = 2\cdot(-1)^{1+3}\begin{vmatrix} -4/3 & -1 \\ -4/3 & -1 \end{vmatrix} = 0.$$

Question 3 (Optimization with Inequality Constraints (30 points)) *Consider the following constrained minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} x^2 - 2y \ \textit{ subject to } \ x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0.$$

Answer the following questions.

1. (5 points) Write down the Kuhn-Tucker conditions for this constrained optimization problem.

   We first translate the original minimization problem into the equivalent maximization problem:

   $$\max_{(x,y)\in\mathbb{R}^2} -x^2 + 2y \ \text{ subject to } \ x^2 + y^2 - 1 \leq 0, \ -x \leq 0, \ -y \leq 0.$$

   We next set up the Lagrangian:

   $$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3) = -x^2 + 2y - \lambda_1(x^2 + y^2 - 1) - \lambda_2(-x) - \lambda_3(-y),$$

   where λ_1, λ_2, and λ_3 are the Lagrange multipliers. Finally, we provide the Kuhn-Tucker conditions as follows:

   $$\begin{array}{ll} (1) & \mathcal{L}_1' = -2x - 2\lambda_1 x + \lambda_2 = 0; \\ (2) & \mathcal{L}_2' = 2 - 2\lambda_1 y + \lambda_3 = 0; \\ (3) & \lambda_1 \geq 0, \ x^2 + y^2 \leq 1, \ \lambda_1(x^2 + y^2 - 1) = 0; \\ (4) & \lambda_2 \geq 0, \ x \geq 0, \ \lambda_2(-x) = 0; \\ (5) & \lambda_3 \geq 0, \ y \geq 0, \ \lambda_3(-y) = 0. \end{array}$$

2. (15 points) Exhaust all the solution candidates using the Kuhn-Tucker approach.

   We search for the solution candidates by considering the following four cases:

   Case 1: $x > 0$ and $y > 0$

   By (4) and (5), we have $\lambda_2 = \lambda_3 = 0$. Plugging $\lambda_2 = 0$ into (1), we obtain

   $$2x(\lambda_1 + 1) = 0 \underbrace{\Rightarrow}_{x>0} \lambda_1 = -1,$$

   which contradicts the hypothesis that $\lambda_1 \geq 0$ in (3). Hence, there are no solution candidates in this case.

Case 2: $x > 0$ and $y = 0$

By (4), we have $\lambda_2 = 0$. Plugging $\lambda_2 = 0$ into (1), we obtain

$$2x(\lambda_1 + 1) = 0 \underbrace{\Rightarrow}_{x>0} \lambda_1 = -1,$$

which contradicts the hypothesis that $\lambda_1 \geq 0$ in (3). Hence, there are no solution candidates in this case.

Case 3: $x = 0$ and $y > 0$

By (5), we have $\lambda_3 = 0$. Plugging $\lambda_3 = 0$ into (2), we obtain

$$2 - 2\lambda_1 y = 0 \Rightarrow \lambda_1 = 1/y \underbrace{\Rightarrow}_{y>0} \lambda_1 > 0.$$

Plugging $x = 0$ into (1), we obtain $\lambda_2 = 0$. Since $\lambda_1 > 0$, by (3), we have $x^2 + y^2 = 1$. Furthermore, since $x = 0$, this equation implies $y = \pm 1$. Since we assume $y > 0$, we have $y = 1$. It then follows from $\lambda_1 = 1/y$ that $\lambda_1 = 1$. Thus, we obtain the unique solution candidate: $(x, y, \lambda_1, \lambda_2, \lambda_3) = (0, 1, 1, 0, 0)$.

Case 4: $x = 0$ and $y = 0$

Plugging $y = 0$ into (2), we obtain

$$2 + \lambda_3 = 0 \Rightarrow \lambda_3 = -2,$$

which contradicts the hypothesis that $\lambda_3 \geq 0$ in (5). Hence, there are no solution candidates in this case.

Considering all four cases, we obtain the unique solution candidate: $(x, y) = (0, 1)$ associated with the Lagrange multipliers $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$.

3. (10 points) Find the solution to this constrained optimization problem.

From the previous question, we obtain $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$. Plugging these values into the Lagrangian, we obtain

$$\mathcal{L}(x, y, 1, 0, 0) = -x^2 + 2y - (x^2 + y^2 - 1).$$

We compute the second-order derivatives of the Lagrangian:

$$\begin{aligned} \mathcal{L}''_{11} &= -4; \\ \mathcal{L}''_{12} = \mathcal{L}''_{21} &= 0; \\ \mathcal{L}''_{22} &= -2. \end{aligned}$$

Then, we form the Hessian matrix associated with the Lagrangian:

$$D^2\mathcal{L}(x, y, 1, 0, 0) = \begin{pmatrix} \mathcal{L}''_{11} & \mathcal{L}''_{12} \\ \mathcal{L}''_{21} & \mathcal{L}''_{22} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus, we have the following leading principal minors:

$$\begin{aligned} D_1 &= -4 < 0; \\ D_2 &= 8 > 0. \end{aligned}$$

This implies that the associated Hessian matrix is negative definite, which further implies that the Lagrangian function is strictly concave in (x, y), given $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$. Thus, by the sufficiency of the Kuhn-Tucker conditions, we conclude that the unique solution candidate we have obtained in the previous step is indeed the solution to the original constrained optimization problem.

Question 4 (Open and Closed Sets (25 points)) *Consider the following sets:*

$$\begin{aligned} A &= \left\{ (x, y) \in \mathbb{R}^2 \middle| \; y = 1, \; x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\ B &= \left\{ (x, y) \in \mathbb{R}^2 \middle| \; y \in (0, 1), \; x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\ C &= \left\{ (x, y) \in \mathbb{R}^2 \middle| \; y = 1, \; x \in \bigcup_{n=1}^{\infty} [2n, 2n+1] \right\}. \end{aligned}$$

Answer the following questions.

1. (8 points) Formally determine whether the set A is open, closed, or neither.

   We claim that A is not open. Fix $(x^0, y^0) = (2.5, 1)$. It is easy to see that $(x^0, y^0) \in A$ because $2.5 \in (2, 3) \subseteq \bigcup_{n=1}^{\infty}(2n, 2n+1)$. Fix $\varepsilon \in (0, 0.5)$ and define $B_\varepsilon(x^0, y^0)$ as the open ball around (x^0, y^0) with radius ε. This implies that

   $$B_\varepsilon(x^0, y^0) = \{(x, y) \in \mathbb{R}^2 | \; y \in (1 - \varepsilon, 1 + \varepsilon), \; x \in (2.5 - \varepsilon, 2.5 + \varepsilon)\},$$

   which is not a subset of A. Thus, A is not open.

   We claim that A is not closed. Consider a sequence $\{(x^k, y^k)\}_{k=1}^{\infty} \in \mathbb{R}^2$ such that $x^k = 2 + 1/k$ and $y^k = 1$ for each $k \in \mathbb{N}$. Since $x^k \in (2, 3)$ for each $k \in \mathbb{N}$, we have that $\{(x^k, y^k)\}_{k=1}^{\infty} \in A$. Furthermore, as $x^k \to 2$ as $k \to \infty$, the sequence $\{(x^k, y^k)\}_{k=1}^{\infty}$ is a convergent sequence in A. However, since $(2, 1) \notin (2, 3) \subseteq \bigcup_{n=1}^{\infty}(2n, 2n+1)$, $(2, 1) \notin A$. Thus, A is not closed.

2. (8 points) Formally determine whether the set B is open, closed, or neither.

   We claim that B is open. Fix $(x, y) \in B$. This implies that $y \in (0, 1)$ and there exists $k \in \mathbb{N}$ such that $x \in (2k, 2k+1)$. Define

   $$\delta = \min\{y, 1-y, x-2k, 2k+1-x\}.$$

   By construction, we have $\delta > 0$. Fix $\varepsilon \in (0, \delta)$. We define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. By construction, we have

   $$B_\varepsilon(x, y) \subseteq \{(x, y) \in \mathbb{R}^2 |\ y \in (0, 1),\ x \in (2k, 2k+1)\}.$$

   This further implies that $B_\varepsilon(x, y) \subseteq B$. Thus, B is open.

3. (9 points) Formally determine whether the set C is open, closed, or neither (Hint: This question can be more challenging than the previous ones. You might find it better to solve the other questions and come back to this one later).

   We claim that C is not open. Fix $(x^0, y^0) = (2.5, 1)$. It is easy to see that $(x^0, y^0) \in B$ because $2.5 \in [2, 3] \subseteq \bigcup_{n=1}^{\infty}[2n, 2n+1]$. Fix $\varepsilon \in (0, 0.5)$ and define $B_\varepsilon(x^0, y^0)$ as the open ball around (x^0, y^0) with radius ε. This implies that

   $$B_\varepsilon(x^0, y^0) = \{(x, y) \in \mathbb{R}^2 |\ y \in (1-\varepsilon, 1+\varepsilon),\ x \in (2.5-\varepsilon, 2.5+\varepsilon)\},$$

   which is not a subset of C. Thus, C is not open.

   We claim that C is closed. We take the complement of C:

   $$C^c = D_1 \cup D_2 \cup \bigcup_{n=1}^{\infty} D_n,$$

   where

   $$\begin{aligned} D_1 &\equiv \left\{(x, y) \in \mathbb{R}^2 |\ y \neq 1\right\}; \\ D_2 &\equiv \left\{(x, y) \in \mathbb{R}^2 |\ y = 1,\ x < 2\right\}; \\ D_n &\equiv \left\{(x, y) \in \mathbb{R}^2 |\ y = 1,\ x \in (2n+1, 2n+2)\right\} \end{aligned}$$

for each $n \in \mathbb{N}$. Fix $(x, y) \in C^c$. There are the following cases to consider:

Case 1: $(x, y) \in D_1$

We assume without loss of generality that $y > 1$. Fix $\varepsilon \in (0, y - 1)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{y} > 1$, which further implies that $B_\varepsilon(x, y) \subseteq D_1 \subseteq C^c$.

Case 2: $(x, y) \in D_2$

Fix $\varepsilon \in (0, 2 - x)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{x} < 2$. Thus,

$$B_\varepsilon(x, y) \subseteq D_1 \cup D_2 \subseteq C^c.$$

Case 3: There exists $k \in \mathbb{N}$ such that $(x, y) \in D_k$

Define

$$\delta = \min\{x - 2k + 1, 2k + 2 - x\}.$$

By our hypothesis, we have $\delta > 0$. So, fix $\varepsilon \in (0, \delta)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{x} \in (2k + 1, 2k + 2)$. Thus,

$$B_\varepsilon(x, y) \subseteq D_1 \cup D_k \subseteq C^c.$$

Considering all the three cases above, we conclude that C^c is open, which further implies that C is closed.