Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 5: Static Optimization II (Ch. 3) and Differential Equations (Ch. 5, 6)

Concave Programming Problems (Ch. 3.9)

The constrained maximization problem is said to be a **concave programming program** when the objective function $f(\cdot)$ is concave and each constraint $g^j(\cdot)$ is a convex function.

$$\max_{x \in S} f(x) \text{ subject to } g(x) \leq \mathbf{0}.$$

One mild condition is needed for the theorem below:

Definition: The constrained maximization problem satisfies the **Slater qualification** if there exists a vector $z \in S$ such that $g^j(z) < 0$ for all $j = 1, \ldots, m$.

Theorem (Necessary and Sufficient Condition for Concave Programming): Let $f : S \to \mathbb{R}$ be a concave C^1 objective function, where $S \subseteq \mathbb{R}^n$ is open and convex. For each $j = 1, \dots, m$, let $g^j : S \to \mathbb{R}$ be a convex C^1 constraint function. Suppose that the constrained maximization problem satisfies the Slater constraint qualification. Then, $x^* \in S$ is a solution to the constrained maximization problem if and only if there exists $\lambda \in \mathbb{R}^m$ that satisfies:

[KT-1] $\nabla f(x^*) - \sum_{j=1}^m \lambda_j \nabla g^j(x^*) = \mathbf{0}$; and

[KT-2] $\lambda_j \geq 0$, $g^j(x^*) \leq 0$, and $\lambda_j g^j(x^*) = 0$ for each $j = 1, \dots, m$.

Proof: We skip the proof. ■

Example:

$$\max_{x,y,z} f(x,y,z) = \frac{1}{2}x - y \text{ subject to } \begin{cases} g^1(x,y,z) = x + e^{-x} + z^2 \leq y \\ g^2(x,y,z) = -x \leq 0 \end{cases}$$

We setup the Lagrangian:

$$\mathcal{L}(x,y,z) = \frac{1}{2}x - y - \lambda_1(x + e^{-x} - y + z^2) - \lambda_2(-x).$$

The K-T conditions are:

(i) $\quad \mathcal{L}_x^{'} = \frac{1}{2} - \lambda_1(1 - e^{-x}) + \lambda_2 = 0$

(ii) $\quad \mathcal{L}_y^{'} = -1 + \lambda_1 = 0$

(iii) $\quad \mathcal{L}_z^{'} = -2\lambda_1 z = 0$

(iv) $\quad \lambda_1 \geq 0,\ x + e^{-x} + z^2 \leq y,$ and $\lambda_1(x + e^{-x} + z^2 - y) = 0$

(v) $\quad \lambda_2 \geq 0, x \geq 0,$ and $\lambda_2(-x) = 0.$

From (ii), we have $\lambda_1 = 1$. Plugging this into (iii), we obtain $z = 0$.

$\lambda_1 = 1$, $z = 0$, and (iv) together imply $x + e^{-x} = y$.

Plugging $\lambda_1 = 1$ into (i), we obtain

$$e^{-x} = \frac{1}{2} - \lambda_2 \leq \frac{1}{2} \Rightarrow x \geq \ln 2 > 0.$$

Then, $x \geq \ln 2 > 0$ and (v) imply that $\lambda_2 = 0$.

Thus, the only candidate for the solution would be $(x, y, z) = (x, x + e^{-x}, z) = (\ln 2, \ln 2 + 1/2, 0)$.

We confirm that the Slater constraint qualification holds. We evaluate the constraints at $(x, y, z) = (1, 2 + e^{-1}, 0)$. Then,

$$\begin{aligned} &g^1 : 1 + e^{-1} - (2 + e^{-1}) = -1 < 0; \\ &g^2 : -1 < 0. \end{aligned}$$

We next check the concavity of $f(\cdot)$ via the associated Hessian matrix:

$$D^2 f(x, y, z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is trivially NSD, which implies that $f(\cdot)$ is concave.

We check the convexity of g^1 via the associated Hessian matrix:

$$D^2 g(x, y, z) = \begin{pmatrix} e^{-x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $e^{-x} > 0$ for any x, we can conclude that $D^2 g^2(x, y, z)$ is PSD, which implies that $g^1(\cdot)$ is convex.

Finally, we check the convexity of g^2 via the associated Hessian matrix:

$$D^2g^2(x,y,z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is trivially PSD, which further implies that $g^2(\cdot)$ is convex.

Therefore, by the concave programming result, $(x,y,z) = (\ln 2, \ln 2 + 1/2, 0)$ is the solution to the constrained maximization problem.

Quasiconcave Programming (Ch. 3.6)

Theorem (Sufficiency for Quasiconcave Programming): Let $f : S \to \mathbb{R}$ be a quasiconcave C^1 function and $S \subseteq \mathbb{R}^n$ be open and convex. Assume that there exist numbers $\lambda_1, \ldots, \lambda_m$ and a vector $x^* \in S$ such that

(1) x^* is feasible and satisfies [KT-1] and [KT-2];

(2) $\lambda_j g_j(\mathbf{x})$ is quasiconvex for each $j = 1, \ldots, m$; and

(3) $\nabla f(x^*) \neq \mathbf{0}$ or $f(\cdot)$ is concave.

Then, x^* is a solution to the constrained maximization problem.

Proof: We first prove that for all x,

$$f(x) > f(x^*) \Rightarrow \nabla f(x^*) \cdot (x - x^*) > 0. \quad (*)$$

Assume $f(x) > f(x^*)$. Due to the continuity of $f(\cdot)$, we choose $\alpha > 0$ small enough so that $f(x - \alpha \nabla f(x^*)) \geq f(x^*)$.

Using the characterization of quasiconcave functions via first-order derivatives, we have

$$\nabla f(x^*) \cdot (x - \alpha \nabla f(x^*) - x^*) \geq 0.$$

This is equivalent to

$$\nabla f(x^*) \cdot (x - x^*) \geq \alpha \|\nabla f(x^*)\|^2 > 0 \ (\because \nabla f(x^*) \neq \mathbf{0}).$$

This establishes $(*)$.

Let x be any feasible vector, i.e., $g^j(x) \leq 0$ for each $j = 1, \ldots, m$.

Let $J = \{j \in \{1, \ldots, m\} | \; g^j(x^*) = 0\}$.

$$j \in J \Rightarrow \lambda_j g^j(x) \leq \lambda_j g^j(x^*) \Leftrightarrow -\lambda_j g^j(x) \geq -\lambda_j g^j(x^*).$$

$$j \notin J \Rightarrow \lambda_j = 0 \Rightarrow -\lambda_j g^j(x) \geq -\lambda_j g^j(x^*).$$

Since each $-\lambda_j g^j(x)$ is quasiconcave (because $\lambda_j g^j(x)$ is quasi-convex), we use the characterization of quasiconcave functions via first derivatives,

$$-\lambda_j \nabla g^j(x^*) \cdot (x - x^*) \geq 0 \Leftrightarrow \lambda_j \nabla g^j(x^*) \cdot (x - x^*) \leq 0$$

for each $j \in \{1, \ldots, m\}$.

So,

$$0 \geq \sum_{j=1}^{m} \lambda_j \nabla g^j(x^*) \cdot (x - x^*) \underbrace{=}_{KT-1} \nabla f(x^*) \cdot (x - x^*)$$

By $(*)$, we conclude $f(x) \leq f(x^*)$. Hence, x^* is the solution. ■

Differential Equations

First-Order Ordinary Differential Equations (Ch. 5.1)

An **ordinally** differential equation is one for which the unknown is a function of only one variable.

Partial differential equations are equations where the unknown is a function of two or more variables, and one or more of the partial derivatives of the function are included.

For the moment, we restrict attention to **first-order** ordinary differential equations where only the first-order derivatives of the unknown functions of one variable are included.

Examples of First-Order Differential Equations (Ch. 5.1)

$$
\begin{aligned}
&(a) \quad \dot{x} = x + t; \\
&(b) \quad \dot{K} = \alpha\sigma K + H_0 e^{\mu t}; \text{ and} \\
&(c) \quad \dot{k} = s f(k) - \lambda k,
\end{aligned}
$$

where we often use $\dot{x} = dx/dt$.

Solving equation (a) means finding all functions $x(t)$ such that, for every value of t, the derivative $\dot{x}(t)$ is equal to $x(t) + t$.

In equation (b), $K(t)$ is the unknown function, whereas α, σ, H_0, and μ are constants.

In equation (c), $f(k)$ is a given function, whereas s and λ are constants. The unknown function is $k = k(t)$.

Solutions

A first-order differential equation is written as:

$$\dot{x} = F(t, x) \quad (*)$$

where $F(\cdot)$ is a given function of two variables and $x = x(t)$ is the unknown function.

A **solution** of $(*)$ in an interval $I \subseteq \mathbb{R}$ is any differentiable function $\varphi : I \to \mathbb{R}$ such that $x = \varphi(t)$ satisfies $(*)$, that is, $\dot{\varphi}(t) = F(t, \varphi(t))$ for all $t \in I$.

The graph of a solution is called a **solution curve** or **integral curve**.

The equations $(a), (b)$, and (c) are all of the form $(*)$.

A Differential Equation has Infinitely Many Solutions

Example: Let $x = Ce^t - t - 1$ where C is a constant. Then,

$$\dot{x} = Ce^t - 1 = Ce^t - t - 1 + t = x + t.$$

Therefore, $x = Ce^t - t - 1$ is a solution to $\dot{x} = x + t$.

In particular, $x = -t - 1$ and $x = e^t - t - 1$ are also solutions to $\dot{x} = x + t$.

The set of all solutions of a differential equation is called its **general solution**, while any specific function that satisfies the equation is called a **particular solution**.

Initial Value Problem

Many models specify both that a function satisfies a differential equation and that the value of the function, or the values of the derivatives of the function, take certain values for some values of the variable.

Example: $\dot{x}(t) = x(t) + t$ and $x(0) = 1 \qquad (*)$

If $t = 0$ denotes the initial time, then $x(0) = 1$ is called an **initial condition** and we call $(*)$ an **initial value problem**.

We have $x(t) = 2e^t - t - 1$ as a unique solution to $(*)$.

Separable Equations (Ch. 5.3)

The differential equation $\dot{x} = F(t, x)$ is **separable** if $F(t, x) = f(t)g(x)$.

Step 1: Write

$$\frac{dx}{dt} = f(t)g(x).$$

Step 2: Separate the variables:

$$\frac{dx}{g(x)} = f(t)dt.$$

Step 3: Integrate each side:

$$\int \frac{dx}{g(x)} = \int f(t)dt.$$

An Example of Separable Equations

$$\frac{dx}{dt} = -2tx^2.$$

$$\text{Separate:} \quad -\frac{dx}{x^2} = 2tdt$$
$$\text{Integrate:} \quad -\int \frac{dx}{x^2} = \int 2tdt$$
$$\text{Evaluete:} \quad \frac{1}{x} = t^2 + C$$

So, the general solution is $x(t) = 1/(t^2 + C)$. If $x(1) = -1$, we have $C = -2$. So, we obtain

$$x(t) = \frac{1}{t^2 - 2}.$$

An Example: The Solow Model

There is a representative consumer who produces final output $Y(t)$ using only two inputs, physical capital $K(t)$ and labor $L(t)$ at each period t.

The production function $F(\cdot)$ takes the form

$$Y(t) = F(K(t), L(t)).$$

Output $Y(t)$ is a homogeneous good that can be consumed (i.e., $C(t)$) or invested (i.e., $I(t)$), to create new units of physical capital.

For simplicity, we assume that the representative consumer saves a constant fraction of the income, denoted by $s \in [0,1]$.

For simplicity, the population grows at a constant, exogenous rate $n = \dot{L}/L \geq 0$:

$$L(t) = L_0 e^{nt}.$$

where $L_0 > 0$ is constant.

For all $K > 0$ and $L > 0$, $F(\cdot)$ is **neoclassical** in the following sense:

$$\frac{\partial F}{\partial K} > 0; \ \ \frac{\partial F}{\partial L} > 0; \ \ \frac{\partial^2 F}{\partial K^2} < 0; \text{ and } \frac{\partial^2 F}{\partial L^2} < 0.$$

In addition, $F(\cdot)$ exhibits **constant returns to scale**: $\forall \lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

Assume that $F(K, L) = AK^{1-\alpha}L^{\alpha}$ with $\alpha \in (0, 1)$. Then, it can be easily checked that the Cobb-Douglass function satisfies the neoclassical properties and it exhibits constant returns to scale.

In a closed economy, outputs equals income, and the amount invested equals the amount saved. So,

$$\dot{K} = \frac{dK}{dt} = sY = sAK^{1-\alpha}L^{\alpha} = sAL_0^{\alpha}e^{\alpha nt}K^{1-\alpha}.$$

Clearly, this is a separable differential equation.

$$\begin{aligned}
\text{Separate:}\quad & K^{\alpha-1}dK = sAL_0^{\alpha}e^{\alpha nt}dt \\
\text{Integrate:}\quad & \int K^{\alpha-1}dK = sAL_0^{\alpha}\int e^{\alpha nt}dt \\
\text{Evaluate:}\quad & \frac{1}{\alpha}K^{\alpha} = \frac{sAL_0^{\alpha}}{\alpha n}e^{\alpha nt} + C.
\end{aligned}$$

Setting $C_1 = \alpha C$, we obtain $K^{\alpha} = (sA/n)L_0^{\alpha}e^{\alpha nt} + C_1$.

Furthermore, if $K(0) = K_0 > 0$ (initial condition), we obtain $C_1 = K_0^{\alpha} - (sA/n)L_0^{\alpha}$. Therefore, the solution is

$$K = \left[K_0^{\alpha} + (sA/n)L_0^{\alpha}(e^{\alpha nt} - 1)\right]^{1/\alpha}.$$

First-Order Linear Equations (Ch. 5.4)

A **first-order linear differential equation** is one that can be written in the form

$$\dot{x} + a(t)x = b(t) \quad (*)$$

where $a(t)$ and $b(t)$ denote continuous functions of t in a certain interval, and $x = x(t)$ is the unknown function.

Equation $(*)$ is called **linear** because the LHS is a linear function of x and $\dot{x}$.

Examples of First-Order Linear Equations

1. $\dot{x} + x = t$;

2. $\dot{x} + 2tx = 4t$;

3. $(t^2 + 1)\dot{x} + e^t x = t \ln t$.

The last equation can be rearranged into:

$$\dot{x} + \frac{e^t}{t^2 + 1} x = \frac{t \ln t}{t^2 + 1}.$$

The Simplest Case of First-Order Linear Equations

Consider the following equation with a and b as constants, where $a \neq 0$:

$$\dot{x} + ax = b.$$

Multiplying this equation by e^{at}, we have

$$\dot{x}e^{at} + axe^{at} = be^{at}.$$

This is equivalent to

$$\frac{d}{dt}(xe^{at}) = be^{at}.$$

Taking the integration on both hand sides, we obtain

$$xe^{at} = \int be^{at}dt = \frac{b}{a}e^{at} + C$$

Therefore, we obtain

$$\dot{x} + ax = b \Leftrightarrow x = Ce^{-at} + \frac{b}{a},$$

where C is a constant.

If $C = 0$, we obtain the constant function $x(t) = b/a$. We say that $x = b/a$ is an **equilibrium state** or **steady state**.

If $a > 0$, the solution $x = Ce^{-at} + b/a$ converges to b/a as $t \to \infty$. In this case, the equation is said to be **stable**, because every solution of the equation converges to an equilibrium as $t \to \infty$.

Variable RHS Case of First-Order Linear Equations (Ch. 5.4)

Consider the following equation:

$$\dot{x} + ax = b(t)$$

Multiplying the equation by e^{at}, we obtain

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at} \Leftrightarrow \frac{d}{dt}(xe^{at}) = b(t)e^{at}.$$

Hence, we obtain

$$xe^{at} = \int b(t)e^{at}dt + C.$$

Multiplying the last equation by e^{-at}, we obtain

$$\dot{x} + ax = b(t) \Leftrightarrow x = Ce^{-at} + e^{-at}\int e^{at}b(t)dt.$$

General Case of First-Order Linear Equations (Ch. 5.4)

Consider

$$\dot{x} + a(t)x = b(t).$$

Let $A(t) = \int a(t)dt$. This means that $\dot{A}(t) = a(t)$.

Multiplying the equation by $e^{A(t)}$, we obtain

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}.$$

This is equivalent to

$$\frac{d}{dt}(xe^{A(t)}) = b(t)e^{A(t)}.$$

Integrating the equation, we obtain

$$xe^{A(t)} = \int b(t)e^{A(t)}dt + C.$$

Multiplying the equation by $e^{-A(t)}$, we have

$$x = Ce^{-A(t)} + e^{-A(t)} \int b(t)e^{A(t)}dt.$$

Summary:

$$\dot{x} + a(t)x = b(t) \Leftrightarrow x = e^{-\int a(t)dt} \left(C + \int e^{\int a(t)dt} b(t)dt \right).$$

Qualitative Theory and Stability (Ch. 5.7)

We are often interested not in the exact form of the solution of a differential equation, but only in the **qualitative** properties of this solution.

Many differential equations in economics can be expressed as:

$$\dot{x} = F(x). \quad (*)$$

This equation is called **autonomous**.

One of the most important properties of a differential equation is whether it has any **equilibrium** or **steady states**.

These corresponds to solutions of the equation that do not change over time.

Stability

In many economic applications, it is also very important to know whether an equilibrium state is **stable**.

Definition: A point a represents an **equilibrium (steady) state** for equation $(*)$ if $F(a) = 0$.

Definition: An equilibrium state a is called **globally asymptotically stable** if for any initial point $x(0) = x_0$, $x(t) \to a$ as $t \to \infty$.

Definition: An equilibrium state a is called **locally asymptotically stable** if for any initial point $x(0) = x_0 \in B_\varepsilon(a)$, $x(t) \to a$ as $t \to \infty$.

Theorem: Let a be an equilibrium state to $\dot{x} = F(x)$.

(1) $F(a) = 0$ and $F^{'}(a) < 0 \Rightarrow a$ is a locally asymptotically stable equilibrium.

(2) $F(a) = 0$ and $F^{'}(a) > 0 \Rightarrow a$ is an unstable equilibrium.

The Solow Model Revisited

Now assume that capitals depreciate at the constant rate $\delta > 0$.

The net increase in the stock of physical capital at a point in time equals gross investment less depreciation:

$$\dot{K} = I - \delta K = s \cdot F(K, L) - \delta K.$$

Exploiting constant returns to scale technology, we obtain

$$Y = F(K, L) = LF(K/L, 1) = Lf(k),$$

where $k \equiv K/L$ is the capital-labor ratio, $y \equiv Y/L$ is per capita output, and the function $f(k) = F(k, 1)$.

Since $Y = Lf(k)$, we obtain

$$\partial Y/\partial K = f^{'}(k) > 0 \text{ and } \partial^2 Y/\partial K^2 = f^{''}(k)/L < 0.$$

So, we have $f^{''}(k) < 0$.

The production function can now be expressed as $y = f(k)$.

$$\dot{K} = sF(K, L) - \delta K \Rightarrow \frac{\dot{K}}{L} = sf(k) - \delta k.$$

Using this, we compute

$$\dot{k} = \frac{d(K/L)}{dt} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L} = sf(k) - (n + \delta)k.$$

The Steady State of the Solow Model

$\dot{k} = sf(k) - (n + \delta)k$ is the fundamental differential equation of the Solow model.

Since $f(0) = 0, f^{'}(k) > 0$, and $f^{''}(k) < 0$ for all $k > 0$, there is a unique steady state $k^* > 0$ in the Solow Model:

$$sf(k^*) = (n + \delta)k^*.$$

If $\lim_{k \to 0} f^{'}(k) = \infty$ and $\lim_{k \to \infty} f^{'}(k) = 0$ (known as the Inada Conditions), the unique steady state k^* is locally asymptotically stable.

Second-Order Differential Equations (Ch. 6)

In an important area of dynamic optimization called the **calculus of variations**, the first-order condition for optimality involves a second-order differential equation.

The typical second-order differential equation takes the form

$$\ddot{x} = F(t, x, \dot{x}) \quad (*)$$

where F is a given fixed function, $x = x(t)$ is the unknown function, and $\dot{x} = dx/dt$.

The new feature here is the presence of the second derivative $\ddot{x} = d^2x/dt^2$.

A **solution** of $(*)$ on an interval I is a twice differentiable function that satisfies the equation.

Example: $\ddot{x} = k$ (k is a constant)

Taking integration on the equation, we obtain

$$\dot{x} = \int k dt = kt + A,$$

where A is some constant. Taking further integration on the equation above, we obtain

$$\int (kt + A) dt = \frac{k}{2} t^2 + At + B,$$

where B is some constant.

Differential Equations where x or t is Missing

Case 1: $\ddot{x} = F(t, \dot{x})$

In this case, x is missing. We introduce the new variable $u = \dot{x}$. Then, Case 1 becomes $\dot{u} = F(t, u)$, which is a first-order differential equation.

Case 2: $\ddot{x} = F(x, \dot{x})$

In this case, t is not explicitly present in the equation and the equation is called **autonomous**.

Example: $\ddot{x} = \dot{x} + t$.

Define $u = \dot{x}$. Then, the equation is transformed to $\dot{u} = u + t$.

This first-order differential equation has the general solution

$$u = Ae^t - t - 1,$$

where A is a constant. This is equivalent to

$$\dot{x} = Ae^t - t - 1.$$

Integrating this equation, we obtain

$$x = \int (Ae^t - t - 1)dt = Ae^t - \frac{1}{2}t^2 - t + B,$$

where B is a constant.

Assume that $x(0) = 1$ and $\dot{x}(0) = 2$. First,

$$\dot{x}(0) = A - 1 = 2 \Rightarrow A = 3.$$

Second,

$$x(0) = A + B = 1 \underbrace{\Rightarrow}_{A=3} B = -2.$$

Then,

$$x = 3e^t - \frac{1}{2}t^2 - t - 2.$$

Detour: Complex Numbers (Ch. B.3)

Simple quadratic equations like $x^2 + 1 = 0$ and $x^2 + 4x + 8 = 0$ have no solution within the real number system.

The standard formula for solving the equation $x^2 + 4x + 8 = 0$ yields $x = -2 \pm \sqrt{-4} = -2 \pm 2\sqrt{-1}$.

By pretending that $\sqrt{-1}$ is a number i whose square is -1, we make i a solution of the equation $i^2 = -1$.

Mathematical formalism regard complex numbers as 2-vectors (a, b).

We usually write this complex number as $a + bi$, where a and b are real numbers.

The operations of addition, subtraction, and multiplication are defined by

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\(a+bi)-(c+di) &= (a-c)+(b-d)i\\(a+bi)(c+di) &= (ac-bd)+(ad+bc)i,\end{aligned}$$

respectively. The division of two complex numbers is

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Trigonometric Form of Complex Numbers

Each complex number $z = x + yi = (x, y)$ can be represented by a point in the plane.

We could use **polar coordinates**. Let θ be the angle (measured in radians) between the positive real axis and the vector from the origin to the point (x, y), and let r be the distance from the origin to the same point.

Then $x = r\cos\theta$ and $y = r\sin\theta$, so

$$z = x + yi = r(\cos\theta + i\sin\theta).$$

The distance from the origin to the point (x, y) is $r = \sqrt{x^2 + y^2}$. This is called the **modulus** of the complex number, denoted by $|z|$.

If $z = x + iy$, then the **complex conjugate** of z is defined as $\bar{z} = x - iy$. We see that $\bar{z}z = x^2 + y^2 = |z|^2$, where $|z|$ is the modulus of z.

Multiplication of complex numbers have a neat geometric interpretation:

$$r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

Division of complex numbers have a neat geometric interpretation:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$