ECON 696
Mathematical Methods for Economic Dynamics
SMU School of Economics; Fall 2022
Answer Key to Homework Assignments

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You are supposed to upload your work (in pdf file) at "Assignments" at our class website (on eLearn) by 5:00pm on the date specified below.

1 Homework 1 (Due Date: September 8 (Thu), 2022): Submission is Required and will be Graded

Question 1.1 (32 points) *Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, we define $(x-1/n, x+1/n)$ as an interval over $\mathbb{R}$. Answer the following questions.*

Hint: Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of sets and Λ denotes the index set with λ as a generic element. You can use the following relationships: $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$ and $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$. This is called *de Morgan's law*.)

1. Show that $(x-1/n, x+1/n)$ is an open set for each $n \in \mathbb{N}$.

   Fix $n \in \mathbb{N}$. And fix $x^0 \in (x-1/n, x+1/n)$ accordingly. We choose $\varepsilon > 0$ small enough so that $x - 1/n < x^0 - \varepsilon$ and $x^0 + \varepsilon < x + 1/n$. By construction, we have $B_\varepsilon(x^0) \subseteq (x-1/n, x+1/n)$. This implies that $(x-1/n, x+1/n)$ is an open set.

2. Confirm that
$$\bigcap_{n=1}^{\infty}\left(x-\frac{1}{n}, x+\frac{1}{n}\right) = \{x\}.$$

   As n tends to infinity, $x - 1/n \to x$ and $x + 1/n \to x$.

3. Show that a singleton set $\{x\}$ is "not" open.

   Fix $\varepsilon > 0$. Then, $B_\varepsilon(x)$ always contains points other than $\{x\}$, no matter how small $\varepsilon > 0$ can be made. This implies that $\{x\}$ is not an open set.

4. Using de Morgan's law, show that the union of an infinite number of closed sets may not be closed.

   For each $n \in \mathbb{N}$, we have

$$\left(x - \frac{1}{n}, x + \frac{1}{n}\right)^c = (-\infty, x - 1/n] \cup [1 + 1/n, \infty).$$

   Since $(x - 1/n, x + 1/n)$ is shown to be open, $(-\infty, x - 1/n] \cup [x + 1/n, \infty)$ is closed. By de Morgan's law, we have

$$\begin{aligned} \left(\bigcup_{n=1}^{\infty} (-\infty, x - 1/n] \cup [x + 1/n, \infty)\right)^c &= \bigcap_{n=1}^{\infty} ((-\infty, x - 1/n] \cup [x + 1/n, \infty))^c \\ &= \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) = \{x\}. \end{aligned}$$

   Since $\{x\}$ is shown to be non-open, we are done.

Question 1.2 (20 points) *Sketch the set*

$$S = \{(x, y) \in \mathbb{R}^2 |\ x > 0,\ y \geq 1/x\}$$

in the plane. Is S closed? Provide the formal argument.

Proof: We claim that S is closed. Let $\{(x^k, y^k)\}_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^2$ such that $(x^k, y^k) \in S$ for each $k \in \mathbb{N}$ and $(x^k, y^k) \to (x^*, y^*)$ as $k \to \infty$ for some $(x^*, y^*) \in \mathbb{R}^2$. It thus suffices to prove that $(x^*, y^*) \in S$. Suppose, on the contrary, that $(x^*, y^*) \notin S$. We complete the rest of the proof by checking the following cases.

Case 1: $x^* = 0$

In this case, we have $x^k > 0$ for each $k \in \mathbb{N}$ and $x^k \to 0$ as $k \to \infty$. Since $(x^k, y^k) \in S$ for each $k \in \mathbb{N}$, which further implies that $y^k \geq 1/x^k$ for each $k \in \mathbb{N}$, we conclude that $y^k \to \infty$ as $k \to \infty$. However, this contradicts the hypothesis that $\{(x^k, y^k)\}$ is a convergent sequence.

Case 2: $x^* < 0$

Since $x^k \to x^*$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $x^k < 0$ for each $k > K$. This contradicts the hypothesis that $(x^k, y^k) \in S$ for each $k \in \mathbb{N}$.

Case 3: $x^* > 0$ and $y* < 1/x^*$

In this case, we have $(x^*)(y^*) < 1$. Since $(x^k, y^k) \to (x^*, y^*)$ as $k \to \infty$ and $g(x, y) = xy$ is a continuous function, there exists $K \in \mathbb{N}$ such that $(x^k)(y^k) < 1$ for each $k > K$. This implies that $y^k < 1/x^k$ for each $k > K$. Once again, this contradicts the hypothesis that $(x^k, y^k) \in S$ for each $k \in \mathbb{N}$.

After checking all the three cases, we conclude that $x^* > 0$ and $y^* \geq 1/x^*$. But this implies that $(x^*, y^*) \in S$. This completes the proof. ■

Question 1.3 (18 points) *Prove the following properties for closed sets:*

1. *The whole space $\mathbb{R}^n$ and the empty set $\emptyset$ are both closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *The union of finitely many closed sets is closed.*

1. $(\mathbb{R}^n)^c$ is the empty set, which is open. So, $\mathbb{R}^n$ is closed. $(\emptyset)^c$ is the entire space $\mathbb{R}^n$, which is open. So, $\emptyset$ is closed.

2. Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Set $F_* = \bigcap_{\lambda \in \Lambda} F_\lambda$. What we want to show is that F_* is closed, which is equivalent to showing that $(F_*)^c$ is open. By de Morgan's law, we obtain

$$(F_*)^c = \bigcup_{\lambda \in \Lambda} (F_\lambda)^c.$$

Since each F_λ is closed, $(F_\lambda)^c$ is open. Since we know that the arbitrary union of open sets is open, we conclude that $(F_*)^c$ is open, that is, F_* is closed.

3. Let $\{F_k\}_{k=1}^K$ be a collection of finite number of closed sets. Set $F^* = \bigcup_{k=1}^K F_k$. What we want to show is that F^* is closed, which is equivalent to showing that $(F^*)^c$ is open. By de Morgan's law, we obtain

$$(F^*)^c = \bigcap_{k=1}^{K} (F_\lambda)^c.$$

Since each F_λ is closed, $(F_\lambda)^c$ is open. Besides, since we know that the finite intersection of open sets is open, we conclude that $(F_*)^c$ is open, that is, F_* is closed.

Question 1.4 (30 points) *We say that a sequence $\{x^k\}$ in $\mathbb{R}^n$ is a* ***Cauchy sequence*** *if, for any $\varepsilon > 0$, there is a natural number $N \in \mathbb{N}$ such that $d(x^m, x^n) < \varepsilon$ for each $m, n \geq N$. Answer the following questions.*

1. Show that if a sequence in $\mathbb{R}^n$ is convergent, it is also a Cauchy sequence.

   Let $\{x^k\}$ be a sequence in $\mathbb{R}^n$ and assume that $\{x^k\}$ converges to $x \in \mathbb{R}^n$. Fix $\varepsilon > 0$. Then, there exists a natural number $N \in \mathbb{N}$ such that $d(x^k, x) < \varepsilon/2$ for each $k \geq N$. Fix $m, n \geq N$. By the triangle inequality, we have

   $$d(x^m, x^n) \leq d(x^m, x) + d(x, x^n) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

   This means that $\{x^k\}$ is a Cauchy sequence.

2. Show that any Cauchy sequence in $\mathbb{R}^n$ is bounded.

   Let $\{x^k\}$ be a Cauchy sequence in $\mathbb{R}^n$. By the Cauchy property, there exists a natural number $K \in \mathbb{N}$ such that $d(x^k, x^K) < 1$ for each $k \geq K$. Therefore, $\|x^k\| \leq \|x^K\| + 1$ for each $k \geq K$ so that $\{x^k\}_{k=K}^\infty$ is bounded.

It is also easy to see that $\{x^1, \dots, x^{K-1}\}$ is bounded because, for every $x \in \{x^1, \dots, x^{K-1}\}$, we have $\|x\| \leq M^*$ where $M^* = \max\{\|x^1\|, \dots, \|x^{K-1}\|\}$. Since $\|x^k\| \leq \max\{\|x^K\| + 1, M^*\}$ for each k, the whole sequence $\{x^k\}$ is a bounded sequence.

3. Show that any Cauchy sequence in $\mathbb{R}^n$ is convergent.

   Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_j}\}_j$ of it such that $\{x^{k_j}\}$ is convergent. Let $x \in \mathbb{R}^n$ be the limit point of $\{x^{k_j}\}$. Because $\{x^k\}$ is a Cauchy sequence, for every $\varepsilon > 0$, there is a natural number N such that $d(x^m, x^n) < \varepsilon/2$ for all $m, n > N$. If we take J sufficiently large, we have $d(x^{k_j}, x) < \varepsilon/2$ for all $j > J$. Then for $k > N$ and $j > \max\{N, J\}$, due to the triangle inequality, we obtain

   $$d(x^k, x) \leq d(x^k, x^{k_j}) + d(x^{k_j}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

   Therefore, $\{x^k\}$ is convergent.

2 Homework 2 (Due Date: September 15 (Thu), 2022): Submission is Required but will not be Graded

Question 2.1 *Let $d(x, y)$ be the Euclidean distance (or metric) between any two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$. Then, show that $d(x, y)$ satisfies the following triangle inequality: for any $x, y, z \in \mathbb{R}^n$,*

$$d(x, z) \leq d(x, y) + d(y, z).$$

This is equivalent to checking $(d(x, z))^2 \leq (d(x, y) + d(y, z))^2$. We show this by the following:

$$\begin{aligned}
(d(x,z))^2 &= \sum_{i=1}^n |x_i - z_i|^2 = \sum_{i=1}^n |x_i - y_i + y_i - z_i|^2 \\
&\leq \sum_{i=1}^n (|(x_i - y_i| + |y_i - z_i|)^2 \;\; (\because \;\; |x + y| \leq |x| + |y|) \\
&= \sum_{i=1}^n |x_i - y_i|^2 + 2\sum_{i=1}^n |x_i - y_i||y_i - z_i| + \sum_{i=1}^n |y_i - z_i|^2 \\
&= \|\mathbf{a}\|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + \|\mathbf{b}\|^2 \;\text{ where we define } \mathbf{a} \equiv |x - y| \text{ and } \mathbf{b} \equiv |y - z| \\
&\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 \;\text{ (by Cauchy-Schwartz inequality)} \\
&= \sum_{i=1}^n |x_i - y_i|^2 + 2\sqrt{\sum_{i=1}^n |x_i - y_i|^2}\sqrt{\sum_{i=1}^n |y_i - z_i|^2} + \sum_{i=1}^n |y_i - z_i|^2 \\
&= d(x,y)^2 + 2d(x,y)d(y,z) + d(y,z)^2 \\
&= (d(x,y) + d(y,z))^2 .
\end{aligned}$$

Question 2.2 *Let S be a nonempty subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}^m$ and $g : S \to \mathbb{R}^m$ be two continuous functions. Let $h : S \to \mathbb{R}^m$ be defined by $h(x) = f(x) + g(x)$ for each $x \in S$. Show that $h : S \to \mathbb{R}^m$ is a continuous function. (Hint: Use Minkowski's inequality.)*

Fix $x \in S$ and consider an arbitrarily sequence $\{x^k\}_{k=1}^{\infty} \in S$ such that $x^k \to x$ as $k \to \infty$. By definition of $h(\cdot)$, for each $k \in \mathbb{N}$, we have

$$h(x) - h(x^k) = (f(x) - f(x^k)) + (g(x) - g(x^k)).$$

For each $k \in \mathbb{N}$, let $X^k = f(x) - f(x^k)$ and $Y^k = g(x) - g(x^k)$. Then, by Minkowski's inequality, for each $k \in \mathbb{N}$, we have

$$d(h(x), h(x^k)) = \|X^k + Y^k\| \leq \|X^k\| + \|Y^k\| = d(f(x), f(x^k)) + d(g(x), g(x^k)).$$

Since each $f(\cdot)$ and $g(\cdot)$ are continuous, we have $d(f(x), f(x^k)) \to 0$ and $d(g(x), g(x^k)) \to 0$ as $k \to \infty$. This implies that $d(h(x), h(x^k)) \to 0$ as $k \to \infty$. Since the choice of $\{x^k\}$ is arbitrarily as long as $x^k \to x$, we show that $h(\cdot)$ is continuous.

Question 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows:*

$$f(x) = \frac{1}{1 + x^2}.$$

Answer the following questions.

1. Show that $f(\cdot)$ is a continuous function (Hint: Use the fact that a differentiable function is a continuous function.).

   We compute the first derivative of f:

   $$f^{'}(x) = -\frac{2x}{(1 + x^2)^2}$$

   So, $f(\cdot)$ is differentiable at any point $x \in \mathbb{R}$, it is a continuous function.

2. Let $K = [0, \infty) \subseteq \mathbb{R}$. Show that K is a closed set

   Observe that $K^c = (-\infty, 0)$. Fix $x \in K^c$. Since $x < 0$, we have $|x| > 0$. Set $\varepsilon \in (0, |x|)$. Then, we conclude that $B_\varepsilon(x) \subseteq K^c = (-\infty, 0)$. This implies that K^c is an open set, which further implies that K is a closed set. ■

3. Show that $f(\cdot)$ is strictly decreasing over $K = [0, \infty)$

   Since $f^{'}(x) < 0$ for any $x > 0$, $f(\cdot)$ is strictly decreasing.

4. Let $f(K) = \{y \in \mathbb{R} | y = f(x) \text{ for some } x \in K\}$. Is $f(K)$ a closed set? If yes, prove it. If not, argue why.

We compute the following: $f(0) = 1$ and $\lim_{x\to+\infty} f(x) = 0$. We note that 0 is not included in $f(K)$. Since f is strictly decreasing over K, we obtain $f(K) = (0, 1]$. We claim that $f(K)$ is "not" a closed set. To show this, consider a sequence $\{1/k\}_{k=1}^{\infty}$. Clearly, $\{1/k\} \in f(K)$. We also note that $1/k \to 0$ as $k \to \infty$. For $f(K)$ to be closed, we need to have $0 \in K$, which is not the case.

Question 2.4 *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a 3×3 matrix where a_{ij} is a constant for each $i, j = 1, 2, 3$. Let $|A|$ denote the determinant of a matrix A and A_{ij} denote the cofactor corresponding to (i, j) component of the matrix A. Assume $|A| \neq 0$. A system of 3 linear equations and 3 unknowns is given as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \qquad (*) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

where b_j is a constant for each $j = 1, 2, 3$. We begin by assuming that (c_1, c_2, c_3) is a solution to $(*)$ so that

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + a_{13}c_3 &= b_1, \\ a_{21}c_1 + a_{22}c_2 + a_{23}c_3 &= b_2, \qquad (**) \\ a_{31}c_1 + a_{32}c_2 + a_{33}c_3 &= b_3. \end{aligned}$$

Answer the following questions

1. Show that for each $j, k = 1, 2, 3$ with $j \neq k$,

$$a_{1k}A_{1j} + a_{2k}A_{2j} + a_{3k}A_{3j} = 0.$$

We prove this by dividing our argument into the following cases:

Case 1: $(j, k) = (1, 2)$

What we want to show is

$$a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the first column of A with its second column:

$$\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

Case 2: $(j, k) = (1, 3)$

What we want to show is

$$a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the first column of A with its third column column:

$$\begin{vmatrix} a_{13} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

Case 3: $(j, k) = (2, 1)$

What we want to show is

$$a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the second column of A with its first column column:

$$\begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

Case 4: $(j, k) = (2, 3)$

What we want to show is

$$a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the second column of A with its third column column:

$$\begin{vmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

Case 5: $(j, k) = (3, 1)$

What we want to show is

$$a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the third column of A with its first column column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{11} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

Case 6: $(j, k) = (3, 2)$

What we want to show is

$$a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} = 0.$$

The left hand side of the above equation is obtained by the determinant of a matrix which is the same as A except that we replace the third column of A with its second column column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}.$$

Since this determinant contains two identical columns in it, it is 0. This completes the argument for this case.

The analysis in all the cases above completes the argument.

2. Show that

$$\begin{aligned} & (a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})c_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})c_2 \\ + \quad & (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})c_3 = b_1A_{11} + b_2A_{21} + b_3A_{31}. \end{aligned}$$

We multiply the first equation in $(**)$ by the cofactor A_{11}; multiply the second equation in $(**)$ by A_{21}; and finally multiply the third equation in $(**)$ by A_{31}. Then, we add all the equations so obtained. The result is

$$\begin{aligned} & (a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})c_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})c_2 \\ + \quad & (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})c_3 = b_1A_{11} + b_2A_{21} + b_3A_{31}. \end{aligned}$$

3. Show that $c_1 = A_1/|A|$, where

$$A_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$$

We can simplify the equation obtained above into the following:

$$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})c_1 = b_1A_{11} + b_2A_{21} + b_3A_{31},$$

because $a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} = 0$ and $a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31} = 0$ by the previous question. Thus, we obtain

$$c_1 = \frac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}} = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} = \frac{A_1}{|A|}.$$

4. Show that $c_2 = A_2/|A|$ and $c_3 = A_3/|A|$, where

$$A_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \text{ and } A_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

In doing so, you are required to provide the whole derivation process which justifies your answer.

We consider each case in turn:

Case I: $c_2 = A_2/|A|$

We multiply the first equation in $(**)$ by the cofactor A_{12}; multiply the second equation in $(**)$ by A_{22}; and finally multiply the third equation in $(**)$ by A_{32}. Then, we add all the equations so obtained. The result is

$$\begin{aligned} & (a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32})c_1 + (a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32})c_2 \\ + \quad & (a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32})c_3 = b_1A_{12} + b_2A_{22} + b_3A_{32}. \end{aligned}$$

We can simplify the above equation into the following:

$$(a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32})c_2 = b_1A_{12} + b_2A_{22} + b_3A_{32},$$

because $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0$ and $a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 0$ by the first question. Thus, we obtain

$$c_2 = \frac{b_1A_{21} + b_2A_{22} + b_3A_{23}}{a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} = \frac{A_2}{|A|}.$$

Case II: $c_3 = A_3/|A|$

We multiply the first equation in ($**$) by the cofactor A_{13}; multiply the second equation in ($**$) by A_{23}; and finally multiply the third equation in ($**$) by A_{33}. Then, we add all the equations so obtained. The result is

$$\begin{aligned} & (a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33})c_1 + (a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33})c_2 \\ + \quad & (a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33})c_3 = b_1A_{13} + b_2A_{23} + b_3A_{33}. \end{aligned}$$

We can simplify the above equation into the following:

$$(a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33})c_3 = b_1A_{13} + b_2A_{23} + b_3A_{33},$$

because $a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0$ and $a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} = 0$ by the first question. Thus, we obtain

$$c_3 = \frac{b_1A_{13} + b_2A_{23} + b_3A_{33}}{a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} = \frac{A_3}{|A|}.$$

3 Homework 3 (Due Date: September 22 (Thu), 2022): Submission is Required and will be Graded

Question 3.1 (20 points) *Examine the definiteness of the following quadratic forms subject to the given linear constraint.*

1. $Q(x_1, x_2) = 2x_1^2 - 4x_1x_2 + x_2^2$ subject to $3x_1 + 4x_2 = 0$

   We first set $x_1 = -4x_2/3$. Plugging this into $Q(x_1, x_2)$, we obtain

   $$\begin{aligned} Q(x_1, x_2) &= 2\left(-\frac{4x_2}{3}\right)^2 - 4\left(-\frac{4x_2}{3}\right)x_2 + x_2^2 \\ &= \frac{89}{9}x_2^2 > 0, \ \ \forall x_2 \neq 0. \end{aligned}$$

   So, $Q(x_1, x_2)$ is positive definite subject to the linear constraint $3x_1 + 4x_2 = 0$.

2. $Q(x_1, x_2) = -x_1^2 + x_1x_2 - x_2^2$ subject to $5x_1 - 2x_2 = 0$

   We first set $x_1 = 2x_2/5$. Plugging this into $Q(x_1, x_2)$, we obtain

   $$\begin{aligned} Q(x_1, x_2) &= -\left(\frac{2x_2}{5}\right)^2 + \left(\frac{2x_2}{5}\right)x_2 - x_2^2 \\ &= -\frac{19}{25}x_2^2 < 0, \ \ \forall x_2 \neq 0. \end{aligned}$$

   Therefore, $Q(x_1, x_2)$ is negative definite subject to the linear constraint $5x_1 - 2x_2 = 0$.

Question 3.2 (20 points) *Consider the following system of linear equations:*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -x_1 + ax_2 - 21x_3 &= 2 \ (*) \\ 3x_1 + 7x_2 + ax_3 &= b, \end{aligned}$$

where $a, b \in \mathbb{R}$. Answer the following questions.

1. Identify the set of values of a and b for which the system $(*)$ has a unique solution.

   Let
   $$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & a & -21 \\ 3 & 7 & a \end{pmatrix}.$$

   The necessary and sufficient condition for the system $(*)$ to have a unique solution is $|A| \neq 0$. So, we compute $|A|$:

   $$\begin{aligned} |A| &= 1 \cdot (-1)^{1+1} \begin{vmatrix} a & -21 \\ 7 & a \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} -1 & -21 \\ 3 & a \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} -1 & a \\ 3 & 7 \end{vmatrix} \\ &= a^2 + 147 + 2a - 126 - 21 - 9a = a^2 - 7a \\ &= a(a-7). \end{aligned}$$

   So, as long as $a \neq 0$ and $a \neq 7$ are satisfied, $|A| \neq 0$. Thus, the corresponding set of values of a and b are

   $$\{(a, b) \in \mathbb{R}^2 |\ a \neq 0,\ a \neq 7\}.$$

2. Identify the set of values of a and b for which the system $(*)$ has multiple (at least two) solutions.

   Since $(*)$ has a unique solution if and only if $|A| \neq 0$, a necessary condition for $(*)$ to have multiple solutions is $|A| = 0$. Therefore, we have $a = 0$ or $a = 7$. $|A| = 0$ means that there is a linear dependence among column vectors in A. If I eliminate the third column from A, we have the following two column vectors:

   $$\begin{pmatrix} 1 & 2 \\ -1 & a \\ 3 & 7 \end{pmatrix}.$$

   It is relatively easy to see that these two vectors are linearly independent regardless of the value of a. So, $\text{rank}(A) = 2$.

   Define A_b as follows:

   $$A_b = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{pmatrix}.$$

We know that the system $(*)$ has a solution if and only if $\text{rank}(A) = \text{rank}(A_b)$. This implies that there exists $(\alpha, \beta) \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ such that

$$\begin{pmatrix} 1 \\ 2 \\ b \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ a \\ 7 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -\alpha + \beta a \\ 3\alpha + 7\beta \end{pmatrix}.$$

Solving the above equation, we obtain

$$\begin{aligned} \alpha &= \frac{a-4}{a+2}, \\ \beta &= \frac{3}{2+a}. \end{aligned}$$

These α and β are well-defined because a is either 0 or 7. Then, we obtain

$$b = 3\alpha + 7\beta = \frac{3a+9}{a+2}.$$

Since a is either 0 or 7, we have

$$b = \begin{cases} 9/2 & \text{if } a = 0 \\ 10/3 & \text{if } a = 7 \end{cases}$$

So, the corresponding set of values of a and b are

$$\left\{(a,b) \in \mathbb{R}^2 \middle| \ (a,b) = (0, 9/2) \ \text{ or } \ (a,b) = (7, 10/3)\right\}$$

3. Identify the set of values of a and b for which the system $(*)$ has no solutions.

We know that the system $(*)$ has no solutions if and only if $\text{rank}(A_b) = \text{rank}(A) + 1$ where $\text{rank}(A) = 2$. We eliminate the third column of A from the matrix A_b so that the remaining matrix B is considered below:

$$B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & a & 2 \\ 3 & 7 & b \end{pmatrix}.$$

Then, the system $(*)$ has no solutions if and only if $|B| \neq 0$. Then, we compute $|B|$ as follows:

$$\begin{aligned} |B| &= 1 \cdot (-1)^{1+1} \begin{vmatrix} a & 2 \\ 7 & b \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} -1 & 2 \\ 3 & b \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} -1 & a \\ 3 & 7 \end{vmatrix} \\ &= (ab - 14) - 2(-b-6) + (-7-3a) \\ &= ab - 3a + 2b - 9. \end{aligned}$$

Since a is either 0 or 7, we have

$$|B| = \begin{cases} 2b - 9 & \text{if } a = 0 \\ 3b - 10 & \text{if } a = 7 \end{cases}$$

Therefore, the corresponding set of values of a and b is

$$\left\{(a,b) \in \mathbb{R}^2 \middle| \ a = 0 \ \text{ or } \ a = 7\right\} \backslash \{(a,b) \in \mathbb{R}^2 | \ (a,b) = (0, 2/9) \ \text{ and } \ (a,b) = (7, 10/3)\}.$$

Question 3.3 (20 points) *The CES (Constant Elasticity of Substitution) function $f(\cdot)$ defined for $K > 0,\ L > 0$ by*

$$f(K, L) = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho}$$

where $A > 0,\ \rho \neq 0$, and $0 \leq \delta \leq 1$. Show that $f(\cdot)$ is concave if $\rho \geq -1$ and convex if $\rho \leq -1$. (Hint: You might find it useful to define $X = \delta K^{-\rho} + (1-\delta)L^{-\rho}$.)

Let $X = \delta K^{-\rho} + (1-\delta)L^{-\rho}$. So, we have

$$\frac{\partial X}{\partial K} = -\delta\rho K^{-(1+\rho)} \text{ and } \frac{\partial X}{\partial L} = -(1-\delta)\rho L^{-(1+\rho)}.$$

We compute the following:

$$\begin{aligned}
f_K(K,L) &= \frac{-1}{\rho} A X^{-(1+1/\rho)} \frac{\partial X}{\partial K} = \delta A K^{-(1+\rho)} X^{-(1+1/\rho)} \\
f_L(K,L) &= \frac{-1}{\rho} A X^{-(1+1/\rho)} \frac{\partial X}{\partial L} = (1-\delta) A L^{-(1+\rho)} X^{-(1+1/\rho)} \\
f_{KK}(K,L) &= -\delta(1+\rho)AK^{-(2+\rho)}X^{-(1+1/\rho)} - \delta(1+1/\rho)AK^{-(1+\rho)}X^{-(2+1/\rho)}\frac{\partial X}{\partial K} \\
&= -\delta(1+\rho)AK^{-(2+\rho)}X^{-(1+1/\rho)} + \delta^2(1+\rho)AK^{-2(1+\rho)}X^{-(2+1/\rho)} \\
&= -\delta(1+\rho)AK^{-(2+\rho)}X^{-(2+1/\rho)}\left[X - \delta K^{-\rho}\right] \\
&= -\delta(1-\delta)(1+\rho)AK^{-(2+\rho)}X^{-(2+1/\rho)}L^{-\rho}
\end{aligned}$$

This means that $f_{KK}(K,L) \leq 0$ if and only if $\rho \geq -1$ and $f_{KK}(K,L) \geq 0$ if and only if $\rho \leq -1$.

$$\begin{aligned}
f_{LL}(K,L) &= -(1-\delta)(1+\rho)AL^{-(2+\rho)}X^{-(1+1/\rho)} - (1-\delta)(1+1/\rho)AL^{-(1+\rho)}X^{-(2+1/\rho)}\frac{\partial X}{\partial L} \\
&= -(1-\delta)(1+\rho)AL^{-(2+\rho)}X^{-(1+1/\rho)} + (1-\delta)^2(1+\rho)AL^{-(2+2\rho)}X^{-(2+1/\rho)} \\
&= -(1-\delta)(1+\rho)AL^{-(2+\rho)}X^{-(2+1/\rho)}\left[X - (1-\delta)L^{-\rho}\right] \\
&= -\delta(1-\delta)(1+\rho)AL^{-(2+\rho)}X^{-(2+1/\rho)}K^{-\rho}
\end{aligned}$$

This means that $f_{LL}(K,) \leq 0$ if and only if $\rho \geq -1$ and $f_{LL}(K,L) \geq 0$ if and only if $\rho \leq -1$.

$$\begin{aligned}
f_{KL}(K,L) = f_{LK}(K,L) &= -\delta AK^{-(1+\rho)}(1+1/\rho)X^{-(2+1/\rho)}\frac{\partial X}{\partial L} \\
&= \delta(1-\delta)(1+\rho)AK^{-(1+\rho)}L^{-(1+\rho)}X^{-(2+1/\rho)}.
\end{aligned}$$

We also obtain the following:

$$f_{KK}(K,L)f_{LL}(K,L) - f_{KL}(K,L)f_{LK}(K,L) = 0.$$

We thus conclude that $f(\cdot)$ is concave if $\rho \geq -1$ and it is convex if $\rho \leq -1$.

Question 3.4 (20 points) *Let* $f(x_1,x_2,x_3) = 100-2x_1^2-x_2^2-3x_3-x_1x_2-e^{x_1+x_2+x_3}$ *be a function from* $\mathbb{R}^3 \to \mathbb{R}$. *Show that* $f(\cdot)$ *is strictly concave. (Hint: Set* $u = x_1+x_2+x_3$*).*

$$\begin{aligned}
f_1(x_1,x_2,x_3) &= -4x_1 - x_2 - e^{x_1+x_2+x_3} \\
f_2(x_1,x_2,x_3) &= -2x_2 - x_1 - e^{x_1+x_2+x_3} \\
f_3(x_1,x_2,x_3) &= -3 - e^{x_1+x_2+x_3} \\
f_{11}(x_1,x_2,x_3) &= -4 - e^{x_1+x_2+x_3} \\
f_{12}(x_1,x_2,x_3) = f_{21}(x_1,x_2,x_3) &= -1 - e^{x_1+x_2+x_3} \\
f_{13}(x_1,x_2,x_3) = f_{31}(x_1,x_2,x_3) &= -e^{x_1+x_2+x_3} \\
f_{22}(x_1,x_2,x_3) &= -2 - e^{x_1+x_2+x_3} \\
f_{23}(x_1,x_2,x_3) = f_{32}(x_1,x_2,x_3) &= -e^{x_1+x_2+x_3} \\
f_{33}(x_1,x_2,x_3) = -e^{x_1+x_2+x_3}
\end{aligned}$$

Then, the Hessian matrix is

$$D^2 f(x_1,x_2,x_3) = \begin{pmatrix} -4-e^u & -1-e^u & -e^u \\ -1-e^u & -2-e^u & -e^u \\ -e^u & -e^u & -e^u \end{pmatrix}.$$

where $u = x_1+x_2+x_3$. By definition, we have $e^u > 0$ for any u. We next compute the leading principal minors:

$$\begin{aligned}
D^2_{(1)} f(x_1,x_2,x_3) &= -4-e^u < 0 \\
D^2_{(2)} f(x_1,x_2,x_3) &= \begin{vmatrix} -4-e^u & -1-e^u \\ -1-e^u & -2-e^u \end{vmatrix} = 7+4e^u > 0 \\
D^2_{(3)} f(x_1,x_2,x_3) &= (-1)^{1+3}(-e^u)\begin{vmatrix} -1-e^u & -2-e^u \\ -e^u & -e^u \end{vmatrix} + (-1)^{2+3}(-e^u)\begin{vmatrix} -4-e^u & -1-e^u \\ -e^u & -e^u \end{vmatrix} \\
&\quad +(-1)^{3+3}(-e^u)\begin{vmatrix} -4-e^u & -1-e^u \\ -1-e^u & -2-e^u \end{vmatrix} \\
&= -e^u\left(\begin{vmatrix} -1-e^u & -2-e^u \\ -e^u & -e^u \end{vmatrix} - \begin{vmatrix} -4-e^u & -1-e^u \\ -e^u & -e^u \end{vmatrix} + \begin{vmatrix} -4-e^u & -1-e^u \\ -1-e^u & -2-e^u \end{vmatrix}\right) \\
&= -e^u(-e^u - 3e^u + 7 + 4e^u) = -7e^u < 0.
\end{aligned}$$

This implies that $(-1)^r D^2_{(r)} f(x_1,x_2,x_3) > 0$ for each $r = 1,2,3$. Thus, $f(\cdot)$ is strictly concave.

Question 3.5 (20 points) *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a real-valued function such that*

$$f(x,y) = (x^2-y^2)e^{-(x^2+y^2)}.$$

Find all local extreme points of f *and classify them into either local maximum points, local minimum points, or saddle points.*

We first compute the partial derivatives of f:

$$\begin{aligned} f_x'(x,y) &= 2xe^{-(x^2+y^2)} - 2x(x^2-y^2)e^{-(x^2+y^2)} = 2x(1-x^2+y^2)e^{-(x^2+y^2)} \\ f_y'(x,y) &= -2ye^{-(x^2+y^2)} - 2y(x^2-y^2)e^{-(x^2+y^2)} = -2y(1+x^2-y^2)e^{-(x^2+y^2)} \end{aligned}$$

To find the stationary points of f, we solve the following system of equations: $f_x'(x,y) = 0$ and $f_y'(x,y) = 0$. Then, the stationary points are

$$(x,y) = \begin{cases} (0,0) \\ (0,1) \\ (0,-1) \\ (1,0) \\ (-1,0). \end{cases}$$

We next compute the second-order derivatives of f:

$$\begin{aligned} f_{xx}''(x,y) &= 2(1-x^2+y^2)e^{-(x^2+y^2)} - 4x^2e^{-(x^2+y^2)} - 4x^2(1-x^2+y^2)e^{-(x^2+y^2)} \\ &= 2[(1-2x^2)(1-x^2+y^2) - 2x^2]e^{-(x^2+y^2)} \\ f_{yy}''(x,y) &= -2(1+x^2-y^2)e^{-(x^2+y^2)} + 4y^2e^{-(x^2+y^2)} + 4y^2(1+x^2-y^2)e^{-(x^2+y^2)} \\ &= -2[(1-2y^2)(1+x^2-y^2) - 2y^2]e^{-(x^2+y^2)} \\ f_{xy}''(x,y) &= 4xye^{-(x^2+y^2)} - 4xy(1-x^2+y^2)e^{-(x^2+y^2)} \\ &= 4xy(1-(1-x^2+y^2))e^{-(x^2+y^2)} = 4xy(x^2-y^2)e^{-(x^2+y^2)} \\ &= f_{yx}''(x,y) \end{aligned}$$

Thus, we obtain the Hessian matrix associated with f:

$$D^2f(x,y) = \begin{pmatrix} f_{xx}''(x,y) & f_{xy}''(x,y) \\ f_{yx}''(x,y) & f_{yy}''(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} 2[(1-2x^2)(1-x^2+y^2) - 2x^2]e^{-(x^2+y^2)} & 4xy(x^2-y^2)e^{-(x^2+y^2)} \\ 4xy(x^2-y^2)e^{-(x^2+y^2)} & -2[(1-2y^2)(1+x^2-y^2) - 2y^2]e^{-(x^2+y^2)} \end{pmatrix}.$$

Case 1: $(x,y) = (0,0)$

In this case, we have

$$D^2f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

We can see that $f_{xx}''(0,0) = 2 > 0$ and $|D^2f(0,0)| = -4 \neq 0$. Thus, $(0,0)$ is a saddle point.

Case 2: $(x,y) = (0,1)$

In this case, we have

$$D^2f(0,1) = \begin{pmatrix} 4e^{-1} & 0 \\ 0 & 4e^{-1} \end{pmatrix}.$$

We can see that $f'_{xx}(0,1) = 4e^{-1} > 0$ and $|D^2 f(0,1)| = 16e^{-2} > 0$. Thus, $(0,1)$ is a local minimum point.

Case 3: $(x, y) = (0, -1)$

In this case, we have

$$D^2 f(0,-1) = \begin{pmatrix} 4e^{-1} & 0 \\ 0 & 4e^{-1} \end{pmatrix}.$$

We can see that $f'_{xx}(0,-1) = 4e^{-1} > 0$ and $|D^2 f(0,-1)| = 16e^{-2} > 0$. Thus, $(0,-1)$ is a local minimum point.

Case 4: $(x, y) = (1, 0)$

In this case, we have

$$D^2 f(1,0) = \begin{pmatrix} -4e^{-1} & 0 \\ 0 & -4e^{-1} \end{pmatrix}.$$

We can see that $f''_{xx}(1,0) = -4e^{-1} < 0$ and $|D^2 f(1,0)| = 16e^{-2} > 0$. Thus, $(1,0)$ is a local maximum point.

Case 5: $(x, y) = (-1, 0)$

In this case, we have

$$D^2 f(-1,0) = \begin{pmatrix} -4e^{-1} & 0 \\ 0 & -4e^{-1} \end{pmatrix}.$$

We can see that $f''_{xx}(-11,0) = -4e^{-1} < 0$ and $|D^2 f(-1,0)| = 16e^{-2} > 0$. Thus, $(-1,0)$ is a local maximum point.

4 Homework 4: Submission is not Required

Question 4.1 *Consider the following constrained optimization problem:*

$$\max_{(x,y,z)\in\mathbb{R}^3} x^2 + y^2 + z^2 \text{ subject to } x^2 + y^2 + 4z^2 = 1 \text{ and } x + 3y + 2z = 0.$$

Answer the following questions.

1. Find the solution candidate using the Lagrangian method.

   We first set up the Lagrangian:

   $$\mathcal{L}(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 + 4z^2 - 1) - \lambda_2(x + 3y + 2z).$$

The first-order conditions for the Lagrangian together with two equality constraints are given as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 2x - 2\lambda_1 x_1 - \lambda_2 = 0 && (1)\\
\frac{\partial \mathcal{L}}{\partial y} &= 2y - 2\lambda_1 y - 3\lambda_2 = 0 && (2)\\
\frac{\partial \mathcal{L}}{\partial z} &= 2z - 8\lambda_1 z - 2\lambda_2 = 0 && (3)\\
x^2 + y^2 + 4z^2 &= 1 && (4)\\
x + 3y + 2z &= 0 && (5)
\end{aligned}$$

We rewrite the above conditions as follows:

$$\begin{aligned}
\lambda_2 &= 2x(1-\lambda_1) && (1^{'})\\
3\lambda_2 &= 2y(1-\lambda_1) && (2^{'})\\
\lambda_2 &= z(1-4\lambda_1) && (3^{'}).
\end{aligned}$$

We proceed to the analysis by dividing the following two cases: (i) $\lambda_1 = 1$ and (ii) $\lambda_1 \neq 1$.

(i) $\lambda_1 = 1$

In this case, it follows from $(1^{'})$ that $\lambda_2 = 0$. Plugging $\lambda_1 = 1$ and $\lambda_2 = 0$ into $(3^{'})$, we obtain $z = 0$. Plugging $z = 0$ into (5), we obtain $x = -3y$. Plugging $x = -3y$ and $z = 0$ into (4), we obtain $y = \pm\sqrt{10}/10$. Thus, in this case, we have the following solution candidates:

$$(x, y, z, \lambda_1, \lambda_2) = \begin{cases} (-3\sqrt{10}/10, \sqrt{10}/10, 0, 1, 0), \\ (3\sqrt{10}/10, -\sqrt{10}/10, 0, 1, 0). \end{cases}$$

In this case, we obtain

$$f(x, y, z) = x^2 + y^2 + z^2 = \frac{9 \cdot 10}{100} + \frac{10}{100} = 1,$$

where we denote by $f(\cdot)$ the objective function for this constrained optimization problem.

(ii) $\lambda_1 \neq 1$

It follows from $(1^{'})$ and $(2^{'})$ that $y = 3x$. Plugging $y = 3x$ into (5), we obtain $z = -5x$. Plugging $y = 3x$ and $z = -5x$ into (4), we obtain $x = \pm\sqrt{110}/110$. Thus, in this case, we have the following solution candidates:

$$(x, y, z, \lambda_1, \lambda_2) = \begin{cases} (\sqrt{110}/110, 3\sqrt{110}/110, -5\sqrt{110}/110, 7/22, 3\sqrt{110}/242) \\ (-\sqrt{110}/110, -3\sqrt{110}/110, 5\sqrt{110}/110, 7/22, 3\sqrt{110}/242) \end{cases}$$

In this case, we obtain

$$f(x,y,z) = x^2 + y^2 + z^2 = \frac{1}{110} + \frac{9}{110} + \frac{25}{110} = \frac{35}{110} = \frac{7}{22}.$$

Therefore, the solution candidates for this problem are

$$(x, y, z, \lambda_1, \lambda_2) = \begin{cases} (-3\sqrt{10}/10, \sqrt{10}/10, 0, 1, 0), \\ (3\sqrt{10}/10, -\sqrt{10}/10, 0, 1, 0). \end{cases}$$

2. Find the solution to the constrained optimization problem.

From the previous part of the question, we have $\lambda_1 = 1$ and $\lambda_2 = 0$ at the solution candidates. So, we have

$$\mathcal{L}(x, y, z, 1, 0) = x^2 + y^2 + z^2 - (x^2 + y^2 + 4z^2 - 1) = -3z^2 + 1.$$

Then, the Hessian matrix associated with the Lagrangian is given as follows:

$$D^2\mathcal{L}(x, y, z, 1, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

We compute all the principal minors of $D^2\mathcal{L}(x, y, z, 1, 0)$:

$$\begin{aligned} \Delta_1^2 &: \quad 0, 0, -6, \\ \Delta_2^2 &: \quad \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0; \begin{vmatrix} 0 & 0 \\ 0 & -6 \end{vmatrix} = 0; \begin{vmatrix} 0 & 0 \\ 0 & -6 \end{vmatrix} = 0, \\ \Delta_3^2 &: |D^2\mathcal{L}(x, y, z, 1, 0)| = 0. \end{aligned}$$

This implies that $D^2\mathcal{L}(x, y, z, 1, 0)$ is negative semidefinite for all (x, y, z) so that $\mathcal{L}(x, y, z, 1, 0)$ is a concave function. Thus, by the sufficiency of the Lagrangian method, the solutions to the constrained optimization problems are

$$(x^*, y^*, z^*) = \begin{cases} (-3\sqrt{10}/10, \sqrt{10}/10, 0), \\ (3\sqrt{10}/10, -\sqrt{10}/10, 0). \end{cases}$$

There is an alternative, easy way of solving this part of the problem. By the equality constraint $x^2+y^2+4z^2 = 1$, we plug $x^2+y^2 = 1-4z^2$ into the objective function. Then, the objective function becomes $1-3z^2$. It is clear that we need to set $z = 0$ to maximize the objective function. Then, we uniquely determine the values of x and y by considering the two equality constraints. The answer turns out to be the same.

There is also another, tedious way of solving this part of the problem. Let $g^1(x, y, z) = x^2 + y^2 + 4z^2 - 1$ and $g^2(x, y, z) = x + 3y + 2z$. Next we compute $\nabla g^1(x, y, z) = (2x, 2y, 8z)$ and $\nabla g^2(x, y, z) = (1, 3, 2)$. We need to check that these vectors are linearly independent for any feasible point (x, y, z). Finally, we appeal to Weierstass Theorem to show the existence of solutions to the constrained optimization problem. Hence, we complete the argument by the necessity of the Lagrangian method.

Question 4.2 *Let f and g be functions on $\mathbb{R}^2$ defined respectively by*

$$\begin{aligned} f(x,y) &= \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x \\ g(x,y) &= x - y. \end{aligned}$$

Consider the problem of maximizing and minimizing $f(x,y)$ subject to $g(x,y) = 0$. Answer the following questions.

1. Set up the Lagrangian for this problem and obtain all the solution candidates together with the Lagrange multiplier by solving the first-order condition of the Lagrangian.

   We set up the Lagrangian:

   $$\mathcal{L}(x,y) = \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x - \lambda(x-y).$$

   The FOCs of the Lagrangian are given:

   $$\begin{aligned} \mathcal{L}_x &= x^2 + 2 - \lambda = 0 \\ \mathcal{L}_y &= -3y + \lambda = 0 \end{aligned}$$

   Taking into account that the equality constraint $x - y = 0$, we obtain two solution candidates: $(x,y,\lambda) = (1,1,3), (2,2,6)$.

2. Confirm that $\nabla g(x,y)$ at each solution candidate (x,y) derived in the previous question is linearly independent.

   We compute the following:

   $$\nabla g(x,y) = (\partial g/\partial x, \partial g/\partial y) = (1,-1).$$

   This implies that $\nabla g(x,y)$ is always a nonzero vector. Therefore, the constraint qualification holds.

3. Check if each solution candidate is a local maximum point, local minimum point, or neither. You are required to use the bordered Hessian matrix of the Lagrangian function in this question.

   We compute the second-order derivatives of the Lagrangian:

   $$\begin{aligned} \mathcal{L}''_{xx} &= 2x \\ \mathcal{L}''_{yy} &= -3 \\ \mathcal{L}''_{xy} = \mathcal{L}''_{yx} &= 0. \end{aligned}$$

   We form the bordered Hessian matrix B:

   $$B(x,y,\lambda) = \begin{pmatrix} 0 & \partial g/\partial x & \partial g/\partial y \\ \partial g/\partial x & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} \\ \partial g/\partial y & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2x & 0 \\ -1 & 0 & -3 \end{pmatrix}.$$

At $(x, y, \lambda) = (1, 1, 3)$, we have

$$B(1,1,3) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -3 \end{pmatrix}.$$

We compute the determinant of $B(x, y, \lambda)$ at $(1, 1, 3)$:

$$|B(1,1,3)| = 1 \cdot (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix} - 1 \cdot (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 1.$$

Therefore, $(1, 1)$ is a local maximum point and the associated local maximum value is $f(1, 1) = 5/6$. At $(x, y, \lambda) = (2, 2, 6)$, on the other hand, we have

$$B(2,2,6) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix}.$$

We compute the determinant of $B(x, y, \lambda)$ at $(2, 2, 6)$:

$$|B(2,2,6)| = 1 \cdot (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix} - 1 \cdot (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} = -1 < 0.$$

Therefore, $(2, 2)$ is a local minimum point and the associated local minimum value is $f(2, 2) = 2/3$.

4. Argue that each solution candidate is neither a global maximum point nor global minimum point.

   Taking into account that $x = y$, we write the objective function:

$$f(x, x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x = x^2 \left(\frac{1}{3}x - \frac{3}{2} + \frac{2}{x} \right).$$

   Since $2/x \to 0$ and $x^2 \to \infty$ as $x \to \pm\infty$, we make the following observation:

$$\begin{aligned} \lim_{x \to \infty} f(x, x) &= +\infty > 5/6 \quad \left(\because \ \frac{1}{3}x \to \infty \right) \\ \lim_{x \to -\infty} f(x, x) &= -\infty < 2/3 \quad \left(\because \ \frac{1}{3}x \to -\infty \right) \end{aligned}$$

   This implies that all the solution candidates we found are neither the global maximum point nor the global minimum point.

Question 4.3 *Consider the following problem:*

$$\min_{(x,y) \in \mathbb{R}^2} 4 \ln(x^2 + 2) + y^2 \ \text{ subject to } \ x^2 + y \geq 2, \ x \geq 1$$

Reformulate this as an equivalent maximization problem with inequality constraints, and find the solution of the constrained maximization problem, assuming that it has a solution.

The equivalent maximization problem is given as follows:

$$\max_{(x,y)\in\mathbb{R}^2} -4\ln(x^2+2)-y^2 \text{ subject to } g^1(x,y)=-x^2-y+2\leq 0,\ g^2(x,y)=1-x\leq 0.$$

Set up the Lagrangian $\mathcal{L}$ as follows:

$$\mathcal{L}=-4\ln(x^2+2)-y^2-\lambda_1(-x^2-y+2)-\lambda_2(1-x).$$

The FOCs of $\mathcal{L}$ are given:

$$\begin{aligned}\partial\mathcal{L}/\partial x &= \frac{-8x}{x^2+2}+2\lambda_1 x+\lambda_2=0\\ \partial\mathcal{L}/\partial y &= -2y+\lambda_1=0.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\lambda_1 &= 2y,\\ \lambda_2 &= \frac{8x}{x^2+2}-4xy.\end{aligned}$$

The complementary slackness conditions are

- $\lambda_1\geq 0$ and $\lambda_1(-x^2-y+2)=0$
- $\lambda_2\geq 0$ and $\lambda_2(1-x)=0$.

We consider this problem in the following four cases:

Case 1: $\lambda_1>0$ and $\lambda_2>0$

$\lambda_1>0$ implies $y>0$. Since $\lambda_1(-x^2-y+2)=0$, we have $y=2-x^2$. $\lambda_2>0$ implies $\lambda_2(1-x)=0$, which further implies $x=1$. So, $y=1$. Substituting $(x,y)=(1,1)$ for $\lambda_2=8x/(x^2+2)-4xy$, we obtain $\lambda_2=-4/3$, which contradicts to our hypothesis.

Case 2: $\lambda_1=\lambda_2=0$

In this case, we have $(x,y)=(0,0)$. However, this contradicts one of our constraints, $1-x\leq 0$.

Case 3: $\lambda_1=0$ and $\lambda_2>0$

$\lambda_2>0$ implies $x=1$ and $\lambda_1=0$ implies $y=0$. In this case, we have $\lambda_2=8/3$, which is consistent with our hypothesis. However, plugging $(x,y)=(1,0)$ into $g^1(x,y)\leq 0$, we obtain

$$g^1(1,0)=1+0+2=1,$$

which contradicts the hypothesis that $g^1(x,y) \leq 0$. Thus, there are no solution candidates in this case.

Case 4: $\lambda_1 > 0$ and $\lambda_2 = 0$

$\lambda_1 > 0$ implies $\lambda_1(-x^2 - y + 2) = 0$. So, we have $y = 2 - x^2$. In addition, $\lambda_1 > 0$ implies $y > 0$. So, we must have $-\sqrt{2} < x < \sqrt{2}$. Substituting $y = 2 - x^2$ for $\lambda_2 = 8x/(x^2+2) - 4xy = 0$, we obtain $x^2 = \sqrt{2}$. This guarantees that $y = 2 - \sqrt{2} > 0$, which is consistent with our hypothesis $\lambda_1 > 0$. So, we obtain $(x,y) = (2^{1/4}, 2 - \sqrt{2})$. We can also check $2^{1/4} > 1$. So, $g^2(2^{1/4}, 2 - \sqrt{2}) \leq 0$.

Therefore, $(x,y) = (2^{1/4}, 2 - \sqrt{2})$ is the unique solution candidate to the maximization problem. We next check the constraint qualification for this problem. Fix (x,y) be an arbitrary feasible point. We divide our argument into the following cases:

Case 1: No constraints are binding

Constraint qualification trivially holds in this case.

Case 2: g^1 is binding but g^2 is not binding

Since $\nabla g^1(x,y) = (-2x, -1) \neq (0,0)$, constraint qualification holds in this case.

Case 3: g^1 is not binding but g^2 is binding

Since $\nabla g^2(x,y) = (-1,0) \neq (0,1)$, constraint qualification holds in this case.

Case 4: both constraints are binding

Since $\nabla g^1(x,y) = (-2x,-1)$ and $\nabla g^2(x,y) = (-1,0)$ cannot be parallel to each other, $\nabla g^1(x,y)$ and $\nabla g^2(x,y)$ are linearly independent. Thus, constraint qualification holds in this case.

Since the solution is assumed to exist, by the necessity of the Kuhn-Tucker condition, $(x,y) = (2^{1/4}, 2 - \sqrt{2})$ is the solution to the maximization problem.

Question 4.4 *Consider the following problem:*

$$\max_{(x,y)\in\mathbb{R}^2} f(x,y) = 1 - (x-2)^2 - y^2 \;\; \textit{subject to} \;\; x^2 + y^2 \leq a, \;\; x - y \leq 0,$$

where a is a positive constant. Answer the following questions.

1. Write down the Kuhn-Tucker conditions for this constrained maximization problem.

We set up the Lagrangian:

$$\mathcal{L}(x,y,\lambda) = 1-(x-2)^2-y^2-\lambda_1(x^2+y^2-a)-\lambda_2(x-y),$$

where $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$. The Kuhn-Tucker conditions are

$$\begin{array}{ll} (1) & \mathcal{L}_x^{'} = -2(x-2)-2\lambda_1 x-\lambda_2=0, \\ (2) & \mathcal{L}_y^{'} = -2y-2\lambda_1 y+\lambda_2=0, \\ (3) & \lambda_1\geq 0;\; x^2+y^2-a\leq 0;\; \text{ and } \lambda_1(x^2+y^2-a)=0, \\ (4) & \lambda_2\geq 0;\; x-y\leq 0;\; \text{ and } \lambda_2(x-y)=0. \end{array}$$

2. Find the solution of this constrained maximization problem for all positive values of a.

   We will find all the solution candidates satisfying the Kuhn-Tucker condition by considering the following four cases:

   Case 1: $\lambda_1>0$ **and** $\lambda_2>0$

   By (4), we have $x=y$. Plugging $y=x$ into (1) and (2), we obtain

$$\begin{array}{ll} (1^{'}) & -2(x-2)-2\lambda_1 x-\lambda_2=0 \\ (2^{'}) & -2x-2\lambda_1 x+\lambda_2=0. \end{array}$$

   $(1^{'})-(2^{'})$ leads us to obtain $\lambda_2=2$. Plugging $\lambda_2=2$ into $(2^{'})$, we obtain

$$x=\frac{1}{1+\lambda_1}$$

   By (3), we know $\lambda_1\geq 0$ so that $x>0$. Plugging $y=x$ into $x^2+y^2=a$ together with $x>0$, we obtain $(x,y,\lambda_1,\lambda_2)=(\sqrt{a/2},\sqrt{a/2},\sqrt{2/a}-1,2)$ as a solution candidate. Moreover, from our hypothesis, we must have $\lambda_2>0$, which implies that $0<a<2$. So, we have a solution candidate $(x,y,\lambda_1,\lambda_2)=(\sqrt{a/2},\sqrt{a/2},\sqrt{2/a}-1,2)$ for all $a\in(0,2)$.

   Case 2: $\lambda_1>0$ **and** $\lambda_2=0$

   Plugging $\lambda_2=0$ into (2), we obtain

$$-2y-2\lambda_1 y=0\Rightarrow(1+\lambda_1)y=0.$$

   Since $\lambda_1>0$ is assumed, we have $y=0$. Plugging $\lambda_2=0$ into (1), we obtain

$$-2x+4-2\lambda_1 x=0\Rightarrow x=\frac{2}{1+\lambda_1}.$$

   Since we assume $\lambda_1>0$, we have $x>0$. On the other hand, by (4), we have $x-y\leq 0$. Plugging $y=0$ into $x-y\leq 0$, we obtain $x\leq 0$. This is a contradiction. So, we have no solution candidates in this case.

Case 3: $\lambda_1 = 0$ **and** $\lambda_2 > 0$

Since $\lambda_2 > 0$, by (4), we have $x = y$. As we have in Case 1, we have two equations, $(1^{'})$ and $(2^{'})$. Plugging $\lambda_1 = 0$ into $(1^{'})$ and $(2^{'})$ respectively, we obtain

$$\begin{array}{ll} (1^{''}) & -2(x-2) - \lambda_2 = 0 \\ (2^{''}) & -2x + \lambda_2 = 0. \end{array}$$

From $(2^{''})$, we have $x = \lambda_2/2$. Plugging $x = \lambda_2/2$ into $(1^{''})$, we have $\lambda_2 = 2$. Thus, we also obtain $x = 1$. In this case, we have $(x, y) = (1, 1)$. However, by (3), we must have

$$x^2 + y^2 \leq a \underbrace{\Rightarrow}_{x=y=1} 2 \leq a.$$

Hence, in this case, we have a unique solution candidate $(x, y, \lambda_1, \lambda_2) = (1, 1, 0, 2)$ for all $a \geq 2$.

Case 4: $\lambda_1 = 0$ **and** $\lambda_2 = 0$

Plugging $\lambda_1 = \lambda_2 = 0$ into (1), we obtain $x = 2$. Plugging $\lambda_1 = \lambda_2 = 0$ into (2), we obtain $y = 0$. However, $(x, y) = (2, 0)$ implies $x - y = 2 - 0 > 0$, which contradicts $x - y \leq 0$ in (4). So, we have no solution candidates in this case.

By the Kuhn-Tucker conditions, we have the unique solution candidate: for each $a \in (0, \infty)$,

$$(x^*(a), y^*(a), \lambda_1(a), \lambda_2(a)) = \begin{cases} (\sqrt{a/2}, \sqrt{a/2}, \sqrt{2/a} - 1, 2) & \text{if } 0 < a < 2 \\ (1, 1, 0, 2) & \text{if } a \geq 2 \end{cases}$$

We compute the second-order derivatives of the Lagrangian:

$$\begin{aligned} \mathcal{L}^{''}_{xx} &= -2 - 2\lambda_1 \\ \mathcal{L}^{''}_{yy} &= -2 - 2\lambda_1 \\ \mathcal{L}^{''}_{xy} &= 0 \\ \mathcal{L}^{''}_{yx} &= 0. \end{aligned}$$

So, we have the Hessian matrix for the Lagrangian:

$$D^2\mathcal{L}(x, y, \lambda_1, \lambda_2) = \begin{pmatrix} -2 - 2\lambda_1 & 0 \\ 0 & -2 - 2\lambda_1 \end{pmatrix}.$$

We investigate this Hessian matrix by considering the following two cases:

Case I: $0 < a < 2$

We evaluate the Hessian matrix at $(\lambda_1, \lambda_2) = (\sqrt{2/a} - 1, 2)$:

$$D^2\mathcal{L}(x, y, \sqrt{2/a} - 1, 2) = \begin{pmatrix} -2 - 2(\sqrt{2/a} - 1) & 0 \\ 0 & -2 - 2(\sqrt{2/a} - 1) \end{pmatrix}.$$

We check the leading principal minor of $D^2\mathcal{L}(x, y, \sqrt{2/a} - 1, 2)$:

$$D^2_{(1)}\mathcal{L}(x, y, \sqrt{2/a} - 1, 2) = -2 - 2(\sqrt{2/a} - 1) = -2\sqrt{2/a} < 0,$$
$$D^2_{(2)}\mathcal{L}(x, y, \sqrt{a/2} - 1, 2) = 4 \cdot \frac{2}{a} = \frac{8}{a} > 0.$$

So, $D^2\mathcal{L}(x, y, \sqrt{a/2} - 1, 2)$ is negative definite in all (x, y). This implies that $\mathcal{L}(x, y, \sqrt{2/a} - 1, 2)$ is strictly concave in all (x, y). Thus, the solution candidate we found for this case is indeed a solution to the constrained maximization problem.

Case II: $a \geq 2$

We evaluate the Hessian matrix at $(\lambda_1, \lambda_2) = (0, 2)$:

$$D^2\mathcal{L}(x, y, 0, 2) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

We check the leading principal minor of $D^2\mathcal{L}(x, y, 0, 2)$:

$$D^2_{(1)}\mathcal{L}(x, y, 0, 2) = -2 < 0,$$
$$D^2_{(2)}\mathcal{L}(x, y, 0, 2) = 4 > 0.$$

So, $D^2\mathcal{L}(x, y, 0, 2)$ is negative definite in all (x, y). This implies that $\mathcal{L}(x, y, 0, 2)$ is strictly concave in all (x, y). Thus, the solution candidate we found for this case as well is indeed a solution to the constrained maximization problem.

3. For each positive value of a, let $(x^*(a), y^*(a))$ denote the solution to the constrained maximization problem. For each positive value of a, define $f^*(a) = f(x^*(a), y^*(a))$. Then, find $f^*(a)$ and $df^*(a)/da$.

   As we analyzed in the previous question, we divide our argument into the following two cases:

   Case A: $0 < a < 2$

   In this case, we have $(x^*(a), y^*(a)) = (\sqrt{a/2}, \sqrt{a/2})$. So,

   $$f^*(a) = 1 - (\sqrt{a/2} - 2)^2 - (\sqrt{a/2})^2 = -3 - a + 4\sqrt{a/2}.$$

   We also compute

   $$\frac{df^*(a)}{da} = -1 + 4 \cdot \frac{1}{2}(a/2)^{-1/2} \cdot \frac{1}{2} = (a/2)^{-1/2} - 1.$$

Case B: $a \geq 2$

In this case, we have $(x^*(a), y^*(a)) = (1, 1)$. So,

$$f^*(a) = 1 - (1-2)^2 - 1^2 = -1.$$

We also compute

$$\frac{df^*(a)}{da} = 0.$$

Question 4.5 *Consider the following problem:*

$$\max_{(x,y)\in\mathbb{R}^2} -x^2 - y^2 \text{ subject to } (x-1)^3 - y^2 \geq 0.$$

Answer the following questions.

1. Show that the Slater's constraint qualification is satisfied.

   Set $g(x, y) = -(x-1)^3 + y^2$. We can easily check that $g(2, 0) = -1 < 0$. Thus, the Slater's constraint qualification is satisfied.

2. Show that the Kuhn-Tucker approach fails to work for this constrained maximization problem.

   We set up the Lagrangian:

   $$\mathcal{L}(x, y, \lambda) = -x^2 - y^2 - \lambda(-(x-1)^3 + y^2).$$

   The Kuhn-Tucker conditions are:

   $$\begin{aligned} &(1) \quad \mathcal{L}_x = -2x + 3\lambda(x-1)^2 = 0 \\ &(2) \quad \mathcal{L}_y = -2y - 2\lambda y = 0 \\ &(3) \quad \lambda \geq 0; \ -(x-1)^3 + y^2 \leq 0; \text{ and } \lambda(-(x-1)^3 + y^2) = 0. \end{aligned}$$

   From (2), we have $y(1+\lambda) = 0$. Thus, $y = 0$ because we have $\lambda \geq 0$ by (3). We divide our argument into the following two cases:

   Case 1: $\lambda = 0$

   Plugging $\lambda = 0$ into (1), we obtain $x = 0$. In this case, however, we have $g(0, 0) = 1 > 0$, which contradicts $-(x-1)^3 + y^2 \leq 0$ in (3). So, we have no solutions in this case.

   Case 2: $\lambda > 0$

   By the complementary slackness condition in (3), we have $-(x-1)^3 + y^2 = 0$. Since $y = 0$, $x = 1$. Plugging $(x, y) = (1, 0)$ into (1), we obtain

   $$-2 = 0,$$

   which is simply impossible. So, we have no solutions in this case as well.

   We thus conclude that the Kuhn-Tucker approach does not generate a solution.

3. Find the solution to this constrained maximization problem.

   We have the following inequality constraint: $g(x, y) = -(x-1)^3 + y^2 \leq 0$. So, $(x-1)^3 \geq y^2$. Since $y^2 \geq 0$ for any $y \in \mathbb{R}$, we have $(x-1)^3 \geq 0$, which implies $x \geq 1$. In order to maximize $f(x, y) = -x^2 - y^2$ subject to $-(x-1)^3 + y^2 \leq 0$, we need to make both x and y as close as 0. That is, $(x, y) = (1, 0)$ is the solution.

4. Argue why the Kuhn-Tucker approach does not work.

   We compute

$$\nabla g(x, y) = (\partial g(x, y)/\partial x, \partial g(x, y)/\partial y) = (-3(x-1)^2, 2y).$$

   At the solution $(x, y) = (1, 0)$, we have $\nabla g(1, 0) = (0, 0)$. Since the constraint $g(x, y) \leq 0$ is binding at $(x, y) = (1, 0)$, the constraint qualification fails. This is why the Kuhn-Tucker approach fails for this problem.

5. Explain why the result of Concave Programming does not work.

   We compute the second-order derivatives of $g(x, y)$:

$$\begin{aligned} g_{xx}(x, y) &= -6(x-1) \\ g_{xy}(x, y) &= 0 \\ g_{yx}(x, y) &= 0 \\ g_{yy}(x, y) &= 2. \end{aligned}$$

   We obtain the Hessian matrix of $g(\cdot)$ as follows:

$$D^2 g(x, y) = \begin{pmatrix} -6(x-1) & 0 \\ 0 & 2 \end{pmatrix}.$$

   When $(x, y) = (2, 0)$, we have $\Delta^2_{(1)} f(2, 0) = -6 < 0$. This implies that $D^2 g(x, y)$ is not positive semidefinite. This further implies that $g(x, y)$ is not a convex function. This is why Concave Programming does not work here.

5 Homework 5 (Due Date: October 20 (Thu), 2022): Submission is Required and will be Graded