

Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 6: Differential Equations (Ch. 5)

First-Order Ordinary Differential Equations (Ch. 5.1)

An **ordinally** differential equation is one for which the unknown is a function of only one variable.

Partial differential equations are equations where the unknown is a function of two or more variables, and one or more of the partial derivatives of the function are included.

For the moment, we restrict attention to **first-order** ordinary differential equations where only the first-order derivatives of the unknown functions of one variable are included.

Examples of First-Order Differential Equations (Ch. 5.1)

$$(a) \quad \dot{x} = x + t;$$

$$(b) \quad \dot{K} = \alpha\sigma K + H_0 e^{\mu t}; \text{ and}$$

$$(c) \quad \dot{k} = sf(k) - \lambda k,$$

where we often use $\dot{x} = dx/dt$.

Solving equation (a) means finding all functions $x(t)$ such that, for every value of t , the derivative $\dot{x}(t)$ is equal to $x(t) + t$.

In equation (b), $K(t)$ is the unknown function, whereas α, σ, H_0 , and μ are constants.

In equation (c), $f(k)$ is a given function, whereas s and λ are constants. The unknown function is $k = k(t)$.

Solutions

A first-order differential equation is written as:

$$\dot{x} = F(t, x) \quad (*)$$

where $F(\cdot)$ is a given function of two variables and $x = x(t)$ is the unknown function.

A **solution** of $(*)$ in an interval $I \subseteq \mathbb{R}$ is any differentiable function $\varphi : I \rightarrow \mathbb{R}$ such that $x = \varphi(t)$ satisfies $(*)$, that is, $\dot{\varphi}(t) = F(t, \varphi(t))$ for all $t \in I$.

The graph of a solution is called a **solution curve** or **integral curve**.

The equations (a), (b), and (c) are all of the form $(*)$.

A Differential Equation has Infinitely Many Solutions

Example: Let $x = Ce^t - t - 1$ where C is a constant. Then,

$$\dot{x} = Ce^t - 1 = Ce^t - t - 1 + t = x + t.$$

Therefore, $x = Ce^t - t - 1$ is a solution to $\dot{x} = x + t$.

In particular, $x = -t - 1$ and $x = e^t - t - 1$ are also solutions to $\dot{x} = x + t$.

The set of all solutions of a differential equation is called its **general solution**, while any specific function that satisfies the equation is called a **particular solution**.

Initial Value Problem

Many models specify both that a function satisfies a differential equation and that the value of the function, or the values of the derivatives of the function, take certain values for some values of the variable.

Example: $\dot{x}(t) = x(t) + t$ and $x(0) = 1$ (*)

If $t = 0$ denotes the initial time, then $x(0) = 1$ is called an **initial condition** and we call (*) an **initial value problem**.

We have $x(t) = 2e^t - t - 1$ as a unique solution to (*).

Separable Equations (Ch. 5.3)

The differential equation $\dot{x} = F(t, x)$ is **separable** if $F(t, x) = f(t)g(x)$.

Step 1: Write

$$\frac{dx}{dt} = f(t)g(x).$$

Step 2: Separate the variables:

$$\frac{dx}{g(x)} = f(t)dt.$$

Step 3: Integrate each side:

$$\int \frac{dx}{g(x)} = \int f(t)dt.$$

An Example of Separable Equations

$$\frac{dx}{dt} = -2tx^2.$$

$$\text{Separate: } -\frac{dx}{x^2} = 2t dt$$

$$\text{Integrate: } -\int \frac{dx}{x^2} = \int 2t dt$$

$$\text{Evaluate: } \frac{1}{x} = t^2 + C$$

So, the general solution is $x(t) = 1/(t^2 + C)$. If $x(1) = -1$, we have $C = -2$. So, we obtain

$$x(t) = \frac{1}{t^2 - 2}.$$

An Example: The Solow Model

There is a representative consumer who produces final output $Y(t)$ using only two inputs, physical capital $K(t)$ and labor $L(t)$ at each period t .

The production function $F(\cdot)$ takes the form

$$Y(t) = F(K(t), L(t)).$$

Output $Y(t)$ is a homogeneous good that can be consumed (i.e., $C(t)$) or invested (i.e., $I(t)$), to create new units of physical capital.

For simplicity, we assume that the representative consumer saves a constant fraction of the income, denoted by $s \in [0, 1]$.

For simplicity, the population grows at a constant, exogenous rate $n = \dot{L}/L \geq 0$:

$$L(t) = L_0 e^{nt}.$$

where $L_0 > 0$ is constant.

For all $K > 0$ and $L > 0$, $F(\cdot)$ is **neoclassical** in the following sense:

$$\frac{\partial F}{\partial K} > 0; \quad \frac{\partial F}{\partial L} > 0; \quad \frac{\partial^2 F}{\partial K^2} < 0; \quad \text{and} \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

In addition, $F(\cdot)$ exhibits **constant returns to scale**: $\forall \lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

Assume that $F(K, L) = AK^{1-\alpha}L^\alpha$ with $\alpha \in (0, 1)$. Then, it can be easily checked that the Cobb-Douglass function satisfies the neoclassical properties and it exhibits constant returns to scale.

In a closed economy, outputs equals income, and the amount invested equals the amount saved. So,

$$\dot{K} = \frac{dK}{dt} = sY = sAK^{1-\alpha}L^\alpha = sAL_0^\alpha e^{\alpha nt} K^{1-\alpha}.$$

Clearly, this is a separable differential equation.

Separate: $K^{\alpha-1}dK = sAL_0^\alpha e^{\alpha nt}dt$

Integrate: $\int K^{\alpha-1}dK = sAL_0^\alpha \int e^{\alpha nt}dt$

Evaluate: $\frac{1}{\alpha}K^\alpha = \frac{sAL_0^\alpha}{\alpha n}e^{\alpha nt} + C.$

Setting $C_1 = \alpha C$, we obtain $K^\alpha = (sA/n)L_0^\alpha e^{\alpha nt} + C_1$.

Furthermore, if $K(0) = K_0 > 0$ (initial condition), we obtain $C_1 = K_0^\alpha - (sA/n)L_0^\alpha$. Therefore, the solution is

$$K = \left[K_0^\alpha + (sA/n)L_0^\alpha(e^{\alpha nt} - 1) \right]^{1/\alpha}.$$

First-Order Linear Equations (Ch. 5.4)

A **first-order linear differential equation** is one that can be written in the form

$$\dot{x} + a(t)x = b(t) \quad (*)$$

where $a(t)$ and $b(t)$ denote continuous functions of t in a certain interval, and $x = x(t)$ is the unknown function.

Equation $(*)$ is called **linear** because the LHS is a linear function of x and \dot{x} .

Examples of First-Order Linear Equations

1. $\dot{x} + x = t;$

2. $\dot{x} + 2tx = 4t;$

3. $(t^2 + 1)\dot{x} + e^t x = t \ln t.$

The last equation can be rearranged into:

$$\dot{x} + \frac{e^t}{t^2 + 1}x = \frac{t \ln t}{t^2 + 1}.$$

The Simplest Case of First-Order Linear Equations

Consider the following equation with a and b as constants, where $a \neq 0$:

$$\dot{x} + ax = b.$$

Multiplying this equation by e^{at} , we have

$$\dot{x}e^{at} + axe^{at} = be^{at}.$$

This is equivalent to

$$\frac{d}{dt}(xe^{at}) = be^{at}.$$

Taking the integration on both hand sides, we obtain

$$xe^{at} = \int be^{at} dt = \frac{b}{a}e^{at} + C$$

Therefore, we obtain

$$\dot{x} + ax = b \Leftrightarrow x = Ce^{-at} + \frac{b}{a},$$

where C is a constant.

If $C = 0$, we obtain the constant function $x(t) = b/a$. We say that $x = b/a$ is an **equilibrium state** or **steady state**.

If $a > 0$, the solution $x = Ce^{-at} + b/a$ converges to b/a as $t \rightarrow \infty$. In this case, the equation is said to be **stable**, because every solution of the equation converges to an equilibrium as $t \rightarrow \infty$.

Variable RHS Case of First-Order Linear Equations (Ch. 5.4)

Consider the following equation:

$$\dot{x} + ax = b(t)$$

Multiplying the equation by e^{at} , we obtain

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at} \Leftrightarrow \frac{d}{dt}(xe^{at}) = b(t)e^{at}.$$

Hence, we obtain

$$xe^{at} = \int b(t)e^{at}dt + C.$$

Multiplying the last equation by e^{-at} , we obtain

$$\dot{x} + ax = b(t) \Leftrightarrow x = Ce^{-at} + e^{-at} \int e^{at}b(t)dt.$$

General Case of First-Order Linear Equations (Ch. 5.4)

Consider

$$\dot{x} + a(t)x = b(t).$$

Let $A(t) = \int a(t)dt$. This means that $\dot{A}(t) = a(t)$.

Multiplying the equation by $e^{A(t)}$, we obtain

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}.$$

This is equivalent to

$$\frac{d}{dt}(xe^{A(t)}) = b(t)e^{A(t)}.$$

Integrating the equation, we obtain

$$xe^{A(t)} = \int b(t)e^{A(t)}dt + C.$$

Multiplying the equation by $e^{-A(t)}$, we have

$$x = Ce^{-A(t)} + e^{-A(t)} \int b(t)e^{A(t)}dt.$$

Summary:

$$\dot{x} + a(t)x = b(t) \Leftrightarrow x = e^{-\int a(t)dt} \left(C + \int e^{\int a(t)dt} b(t)dt \right).$$

Qualitative Theory and Stability (Ch. 5.7)

We are often interested not in the exact form of the solution of a differential equation, but only in the **qualitative** properties of this solution.

Many differential equations in economics can be expressed as:

$$\dot{x} = F(x). \quad (*)$$

This equation is called **autonomous**.

One of the most important properties of a differential equation is whether it has any **equilibrium** or **steady states**.

These corresponds to solutions of the equation that do not change over time.

Stability

In many economic applications, it is also very important to know whether an equilibrium state is **stable**.

Definition: A point a represents an **equilibrium (steady) state** for equation $(*)$ if $F(a) = 0$.

Definition: An equilibrium state a is called **globally asymptotically stable** if for any initial point $x(0) = x_0$, $x(t) \rightarrow a$ as $t \rightarrow \infty$.

Definition: An equilibrium state a is called **locally asymptotically stable** if for any initial point $x(0) = x_0 \in B_\varepsilon(a)$, $x(t) \rightarrow a$ as $t \rightarrow \infty$.

Theorem: Let a be an equilibrium state to $\dot{x} = F(x)$.

(1) $F(a) = 0$ and $F'(a) < 0 \Rightarrow a$ is a locally asymptotically stable equilibrium.

(2) $F(a) = 0$ and $F'(a) > 0 \Rightarrow a$ is an unstable equilibrium.

The Solow Model Revisited

Now assume that capitals depreciate at the constant rate $\delta > 0$.

The net increase in the stock of physical capital at a point in time equals gross investment less depreciation:

$$\dot{K} = I - \delta K = s \cdot F(K, L) - \delta K.$$

Exploiting constant returns to scale technology, we obtain

$$Y = F(K, L) = LF(K/L, 1) = Lf(k),$$

where $k \equiv K/L$ is the capital-labor ratio, $y \equiv Y/L$ is per capita output, and the function $f(k) = F(k, 1)$.

Since $Y = Lf(k)$, we obtain

$$\partial Y / \partial K = f'(k) > 0 \text{ and } \partial^2 Y / \partial K^2 = f''(k) / L < 0.$$

So, we have $f''(k) < 0$.

The production function can now be expressed as $y = f(k)$.

$$\dot{K} = sF(K, L) - \delta K \Rightarrow \frac{\dot{K}}{L} = sf(k) - \delta k.$$

Using this, we compute

$$\dot{k} = \frac{d(K/L)}{dt} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L} = sf(k) - (n + \delta)k.$$

The Steady State of the Solow Model

$\dot{k} = sf(k) - (n + \delta)k$ is the fundamental differential equation of the Solow model.

Since $f(0) = 0$, $f'(k) > 0$, and $f''(k) < 0$ for all $k > 0$, there is a unique steady state $k^* > 0$ in the Solow Model:

$$sf(k^*) = (n + \delta)k^*.$$

If $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow \infty} f'(k) = 0$ (known as the Inada Conditions), the unique steady state k^* is locally asymptotically stable.