ECON 696
Mathematical Methods for Economic Dynamics
SMU School of Economics; Fall 2024
Homework Assignments

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You are supposed to upload your work (in pdf file) under "Assignments" under our class website (on eLearn) by 17:00 on the date specified below.

1 Homework 1 (Due Date: September 10 (Tue), 2024): Submission is Required and will be Graded

Question 1.1 (15 points) *Determine sup and inf for each of these sets. You need to prove why what you obtain is the sup and inf for each set.*

1. $A = (-3, 7]$.
2. $B = \{1/n |\ n \in \mathbb{N}\}$.
3. $C = \{x |\ x > 0 \text{ and } x^2 > 3\}$.

Question 1.2 (35 points) *Consider the following sets:*

$$\begin{aligned}
A &= \left\{ (x, y) \in \mathbb{R}^2 \middle|\ y = 1,\ x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\
B &= \left\{ (x, y) \in \mathbb{R}^2 \middle|\ y \in (0, 1),\ x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\
C &= \left\{ (x, y) \in \mathbb{R}^2 \middle|\ y = 1,\ x \in \bigcup_{n=1}^{\infty} [2n, 2n+1] \right\}.
\end{aligned}$$

Answer the following questions.

1. Formally determine whether the set A is open, closed, or neither.
2. Formally determine whether the set B is open, closed, or neither.

3. Formally determine whether the set C is open, closed, or neither

Question 1.3 (30 points) *Prove the following properties for closed sets:*

1. *The whole space $\mathbb{R}^n$ and the empty set $\emptyset$ are both closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *The union of finitely many closed sets is closed.*

(Hint: Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of sets and Λ denotes the index set with λ as a generic element. You can use the following relationships: $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$ and $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$. This is called de Morgan's law.)

Question 1.4 (20 points) *Let $S = \{x \in \mathbb{R}^n |\ g_j(x) \leq 0, j = 1, \ldots, m\}$. Prove that S is closed if each $g_j(\cdot)$ is continuous.*

2 Homework 2 (Due Date: September 17 (Tue), 2024): Submission is Required but will not be Graded

Question 2.1 *Determine the ranks of the following matrices:*

1.
$$\begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$$

3.
$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$$

Question 2.2 *Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be linearly independent vectors in $\mathbb{R}^n$. Answer the following questions*

1. Show that three vectors $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly independent.
2. Show whether three vectors $\mathbf{a}-\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, and $\mathbf{a}+\mathbf{c}$ are linearly independent or linearly dependent.

Question 2.3 *Consider the following quadratic form subject to two linear equalities:*

$$Q(x,y,z) = x^2 + 2xy + y^2 + z^2 \ \textit{subject to} \ \begin{cases} x + 2y + z = 0; \\ 2x - y - 3z = 0. \end{cases}$$

Show that the associated quadratic form is positive definite.

3 Homework 3 (Due Date: September 24 (Tue), 2023): Submission is Required and will be Graded

Question 3.1 (30 points) *Examine the (strict) convexity/concavity of the following functions:*

1. $z = x + y - e^x - e^{x+y}$
2. $z = e^{x+y} + e^{x-y} - y/2$
3. $w = (x + 2y + 3z)^2$

Question 3.2 (30 points) *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x, y) = (x - 1)^2(y - 1)^2,$$

for all $(x, y) \in \mathbb{R}^2$*. Answer the following questions.*

1. Show that the function f is "not" quasiconcave.
2. Derive the bordered Hessian matrix of f, which we denote by $B(x, y)$.
3. Confirm that the "necessary" condition for quasiconcavity of f via bordered Hessian matrix determinant.

Question 3.3 (40 points) *Consider the following constrained optimimization (maximization or minimization) problem:*

$$\max_{(x,y,z)\in\mathbb{R}^3} \left(or \min_{(x,y,z)\in\mathbb{R}^3} \right) x + y + z \text{ subject to } x^2 + y^2 + z^2 = 1 \text{ and } x - y - z = 1.$$

Answer the following questions.

1. Find all the solution candidates of this constrained optimization problem using the Lagrangian method.
2. Classify each solution candidate as either a local maximum point or local minimum point. You are required to leave all the details of your computation.

4 Homework 4: Submission is not Required

Question 4.1 *Consider the following minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} x^2 + y^2 \text{ subject to } (x - 1)^3 - y^2 = 0.$$

Answer the following questions.

1. Explicitly show that the Lagrangian method does not work.

2. Find the solution to the minimization problem. When answering this question, you are required to clarify your argument for this.

3. Explain why the Lagrangian method does not work.

Question 4.2 *Consider the following problem:*

$$\max_{(x,y)\in\mathbb{R}^2} f(x,y) = 1-(x-2)^2-y^2 \ \textit{subject to}\ x^2+y^2 \le a,\ x-y\le 0,$$

where a is a positive constant. Answer the following questions.

1. Write down the Kuhn-Tucker conditions for this constrained maximization problem.

2. Find the solution of this constrained maximization problem for all positive values of a.

3. For each positive value of a, let $(x^*(a), y^*(a))$ denote the solution to the constrained maximization problem. For each positive value of a, define $f^*(a) = f(x^*(a), y^*(a))$. Then, find $f^*(a)$ and $df^*(a)/da$.

Question 4.3 *Consider the following constrained minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} x^2-2y \ \textit{subject to}\ x^2+y^2\le 1,\ x\ge 0,\ y\ge 0.$$

Answer the following questions.

1. Write down the Kuhn-Tucker conditions for this constrained optimization problem.

2. Exhaust all the solution candidates using the Kuhn-Tucker approach.

3. Find the solution to this constrained optimization problem.

Question 4.4 *Consider the following constrained minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} 2x^2+2y^2-2xy-9y \ \textit{subject to}\ 4x+3y\le 10,\ y-4x^2\ge -2,\ x\ge 0,\ y\ge 0.$$

Answer the following questions.

1. Set up the Lagrangian function and obtain the Kuhn-Tucker condition for this optimization problem.

2. Examine each of the following three cases: (i) $x=0$ and $y=0$; (ii) $x>0$ and $y=0$; and $x=0$ and $y>0$, separately and show that each case leads to a violation of the Kuhn-Tucker condition.

3. Find the solution candidate satisfying the Kuhn-Tucker condition.

5 Homework 5 (Due Date: October 22 (Tue), 2024): Submission is Required and will be Graded

Question 5.1 (20 points) *Solve the following differential equations.*

1. $\dot{x} = t^3 - t$
2. $\dot{x} = te^t - t$
3. $\dot{x}e^x = t + 1$
4. $x^2\dot{x} = t + 1$

Question 5.2 (20 points) *Find the general solutions of the following differential equations. Find also the integral curves through the indicated points.*

1. $t\dot{x} = x(1-t),\ (t_0, x_0) = (1, 1/e)$
2. $(1+t^3)\dot{x} = t^2 x,\ (t_0, x_0) = (0, 2)$
3. $x\dot{x} = t,\ (t_0, x_0) = (\sqrt{2}, 1)$
4. $e^{2t}\dot{x} - x^2 - 2x = 1,\ (t_0, x_0) = (0, 0)$

Question 5.3 (30 points) *Find the general solutions of the following differential equations.*

1. $$t\dot{x} + 2x + t = 0\ (t \neq 0)$$
2. $$\dot{x} - \frac{1}{t}x = t\ (t > 0)$$
3. $$\dot{x} - \frac{t}{t^2 - 1}x = t\ (t > 1)$$
4. $$\dot{x} - \frac{2}{t}x + \frac{2a^2}{t^2} = 0\ (t > 0)$$

Question 5.4 (30 points) *Consider the following differential equation:*

$$\ddot{Y} + (\alpha s + \beta)\dot{Y} + \alpha\beta(s+m)Y = -\alpha\beta t - \frac{\alpha s + \beta}{s + m} \quad (*)$$

where $\alpha, \beta, s,$ and m are positive constants. Answer the following questions.

1. Find a particular solution of $(*)$.

2. Discuss conditions that ensure that the characteristic equation associated with $(*)$ has two complex roots.

3. For the rest of this question, we set $\alpha = 1/4$, $\beta = 3/4$, $s = 1$, and $m = 17/3$. Then, find the general solution to $(*)$.

4. What can you say about the behavior of the solutions as $t \to \infty$?

6 Homework 6 (Due Date: Oct 29 (Tue), 2024): Submission is Required but will not be Graded

Question 6.1 *Consider the following problem*

$$\max \int_0^T U(\bar{c} - \dot{x}e^{rt})dt, \quad x(0) = x_0, \quad x(T) = 0,$$

where $x = x(t)$ is the unknown function, $T, \bar{c}, r,$ and x_0 are positive constants, and U is a given C^2 function of one variable. Answer the following questions.

1. Write down the Euler equation associated with this problem.

2. In what follows, we let $U(c) = -e^{-vc}/v$, where v is a positive constant. Write down and solve the Euler equation in this case.

3. Explain why we also solve the problem in this case.

Question 6.2 *Solve the following problem:*

$$\min \int_0^1 (x^2 + (\dot{x})^2)dt \;\; \textit{subject to } x(0) = 1 \textit{ and } x(1) \textit{ free.}$$

Question 6.3 *Solve the following problem:*

$$\min \int_0^1 (x^2 + (\dot{x})^2)dt \;\; \textit{subject to } x(0) = 0 \textit{ and } x(1) \geq 1.$$

Question 6.4 *Consider a monopolist that produces a single commodity with a quadratic total cost function $C(Q)$:*

$$C(Q) = \alpha Q^2 + \beta Q + \gamma,$$

where $\alpha, \beta,$ and γ are positive constants. The quantity demanded is assumed to depend not only on its price $P(t)$, but also on the rate of change of its price $\dot{P}(t)$:

$$Q(t) = a - bP(t) + h\dot{P}(t),$$

where $a, b,$ and h are constants such that $a, b > 0$ and $h \neq 0$. Answer the following questions.

1. Show that the firm's profit function $\pi(P, \dot{P})$ is obtained a function of P and $\dot{P}$ as follows:

$$\begin{aligned}\pi(P, \dot{P}) &= -b(1+\alpha b)P^2 + (a + 2\alpha ab + \beta b)P - \alpha h^2(\dot{P})^2 \\ &\quad -h(2\alpha a + \beta)\dot{P} + h(1 + 2\alpha b)P\dot{P} - (\alpha a^2 + \beta a + \gamma).\end{aligned}$$

2. The objective of the firm is to find an optimal path of price P that maximizes the total profit over a finite time period $[0, T]$. Assume that the initial price P_0 and the terminal price P_T are fixed. Therefore, the objective of the monopolist is summarized as follows:

$$\max \int_0^T \pi(P, \dot{P})dt \text{ s.t. } P(0) = P_0 \text{ and } P(T) = P_T.$$

Obtain the solution to the associated Euler equation.

3. Do we know whether the solution to the Euler equation is a solution to the original maximization problem formulated in Part 2 of the question? Justify your answer.

7 Homework 7 (Due Date: Nov 5 (Tue), 2024): Submission is Required and will be graded

Question 7.1 (30 points) *Solve the following control problem:*

$$\max \int_0^1 -u^2 dt \;\; s.t. \;\; \dot{x} = x + u, \; x(0) = 1, \; x(1) \geq 3, \text{ and } u \in \mathbb{R}.$$

Question 7.2 (30 points) *Consider the following problem:*

$$\max \int_0^2 (2x - 3u)dt \;\; s.t. \;\; \dot{x} = x + u, \;\; x(0) = 4, \;\; x(2) \text{ free, and } u \in [0, 2].$$

Using the maximum principle, answer the following questions. Note also that $x(t)$ is assumed to be C^1, in particular, a continuous function.

1. Characterize the optimal control $u^*(t)$ in terms of $p(t)$, which is the adjoint function associated with the Hamiltonian.

2. Determine $p(t)$.

3. Determine the optimal control $u^*(t)$.

4. Determine the optimal state $x^*(t)$.

5. Argue whether the maximum principle is strong enough to generate the solution to the original optimization problem.

Question 7.3 (40 points) *Consider the following problem:*

$$\max \int_0^2 (x^2 - 2u)dt, \;\; \textit{subject to } \dot{x} = u, \; x(0) = 1, \; x(2) \textit{ free}, \; u \in [0,1].$$

Answer the following questions.

1. Show that the optimal control $u^*(t)$ satisfies the following condition:
$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 2 \\ 0 & \text{if } p(t) < 2, \end{cases}$$
where $p(t)$ denotes the adjoint function associated with the Hamiltonian.
2. Show that the optimal state variable $x^*(t)$ is nondecreasing.
3. Show that $p(t)$ is strictly decreasing.
4. Show that there exists a unique $t^* \in (0,2)$ such that
$$p(t) \begin{cases} > 2 & \text{if } t \in [0, t^*) \\ = 2 & \text{if } t = t^* \\ < 2 & \text{if } t \in (t^*, 2], \end{cases}$$
and
$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in (t^*, 2]. \end{cases}$$
5. Show that
$$x^*(t) = \begin{cases} t+1 & \text{if } t \in [0, t^*] \\ t^*+1 & \text{if } t \in [t^*, 2]. \end{cases}$$
6. Show that when $t \in [t^*, 2]$,
$$p(t) = 2(1+t^*)(2-t).$$
Show also that $t^* = (1+\sqrt{5})/2$.
7. Derive $p(t)$ for any $t \in [0,2]$.

8 Homework 8: Submission is not Needed

Question 8.1 *Consider the following economic growth model:*

$$\max \int_0^T (1 - s(t))e^{\rho t} f(k(t)) e^{-\delta t} dt$$

$$\textit{subject to} \;\; \dot{k}(t) = s(t)e^{\rho t} f(k(t)) - \lambda k(t), \; k(0) = k_0, \; k(T) \geq k_T > k_0, \; s(t) \in [0,1],$$

where $k(t)$ is the capital stock (a state variable), $s(t)$ is the savings rate (a control variable), and $f(k)$ is a production function. Assume that (i) $f(k) > 0$ whenever $k \geq k_0 e^{-\lambda T}$; (ii) $f'(k) > 0$; and (iii) $\rho, \delta, \lambda, T, k_0$, and k_T are all positive constants. We interpret ρ as the rate of technical progress; δ as the discount rate; λ as the capital depreciation rate; T as the end of the planning period; k_0 as the initial level of capital; and k_T as the final level of capital. Answer the following questions.

1. Let $(k^*(t), s^*(t))$ be a solution to the optimization problem. Using the maximum principle, characteirze $(k^*(t), s^*(t), p(t))$ where $p(t)$ is the adjoint function, to the extent possible.
2. In the rest of the question, we set $\rho = 0; f(k) = ak$; $a > 0$; $\delta = 0$; and $\lambda = 0$. Assume further that $T > 1/a$ and $k_0 e^{aT} > k_T$ in the rest of the question. Show that $p(\cdot)$ is strictly decreasing.
3. Show that $p(0) > 1$.
4. Find the solution candidate to the problem when $p(T) = 0$.
5. Show that $p(T) > 0$ implies $p(T) < 1$.
6. Find the solution candidate to the problem when $p(T) > 0$.
7. Show that either $p(T) = 0$ holds or $p(T) > 0$ holds, but not both.
8. Show that the Hamiltonian is "not" concave in (k, s).
9. Show that the solution candidate you obtained is indeed the solution to the simplified version of the optimization problem. For this question, we assume that every possible value of $k(t)$ is nonnegative for any $t \in [0, T]$, i.e., $k(t) \geq 0$ for any $t \in [0, T]$.

Question 8.2 *Consider the following problem:*

$$\max \int_0^5 [10u - (u^2 + 2)]e^{-0.1t} dt, \;\; \dot{x} = -u, \;\; x(0) = 10, \;\; x(5) \geq 0, \;\; u \in [0, \infty).$$

Solve this problem using the current value Hamiltonian.

Question 8.3 *Consider the problem*

$$\max \int_0^\infty x(2-u)e^{-t} dt$$
$$\textit{subject to} \;\; \dot{x} = uxe^{-t}, \;\; x(0) = 1, \;\; \lim_{t\to\infty} x(t) \geq 0, \;\; u \in [0, 1].$$

Answer the following questions.

1. Show that

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda(t)e^{-t} > 1, \\ [0,1] & \text{if } \lambda(t)e^{-t} = 1, \\ 0 & \text{if } \lambda(t)e^{-t} < 1. \end{cases}$$

2. Assume that $\lambda(t)e^{-t} \to 0$ as $t \to \infty$. Then, we have either $\lambda(t)e^{-t} < 1$ for any $t \in [0, \infty)$ or there exists $t^* \geq 0$ such that

$$\lambda(t)e^{-t} \begin{cases} = 1 & \text{if } t = t^*, \\ < 1 & \text{if } t \in (t^*, \infty). \end{cases}$$

   What is $\lambda(t)$ when $t \in [t^*, \infty)$? Moreover, what is t^*?

3. Obtain $u^*(t)$ and $x^*(t)$?

4. Obtain $\lambda(t)$.

5. Show that $\lim_{t\to\infty} \lambda(t)e^{-t}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$.

6. Show that Arrow's sufficient condition holds.