

Mathematical Methods for Economic Dynamics (ECON 696)

**Lecture 3: Linear Algebra II (Ch. 1) and
Multivariate Calculus (Ch. 2)**

Quadratic Forms (Ch. 1.7, 1.8)

A **quadratic form** in n variables is a function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form:

$$\begin{aligned} Q(x_1, \dots, x_n) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{ij}x_ix_j + \cdots + a_{nn}x_n^2. \end{aligned}$$

where the a_{ij} are constants.

Let $x = (x_1, \dots, x_n)^T$ and $A = (a_{ij})$. Then, it follows that

$$Q(x_1, \dots, x_n) = Q(x) = x^T A x = x \cdot A x.$$

Definition: A quadratic form $Q(x) = x^T Ax = x \cdot Ax$, as well as its associated matrix A , are said to be **positive definite**, **positive semidefinite**, **negative definite**, or **negative semidefinite** according as

$$Q(x) > 0, \quad Q(x) \geq 0, \quad Q(x) < 0, \quad Q(x) \leq 0,$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. The quadratic form $Q(x)$ is **indefinite** if there exist vectors x^* and y^* in $\mathbb{R}^n \setminus \{0\}$ such that $Q(x^*) < 0$ and $Q(y^*) > 0$.

Let $A = (a_{ij})$ be any $n \times n$ matrix.

An arbitrary **principal minor** of A of **order** r is the determinant of the matrix obtained by deleting all but r rows and r columns such that if the i -th row (column) is selected, then so is the i -th column (row).

In particular, a principal minor of order r always includes exactly r elements of the main (principal) diagonal.

The determinant $|A|$ itself is called a principal minor (no rows and columns are deleted).

Let Δ_r denote an arbitrary principal minor of A of order r .

A principal minor is said to be a **leading principal minor** of order r ($1 \leq r \leq n$), if it consists of the first “leading” rows and columns of $|A|$. The leading principal minors of A of order r is

$$D_r = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{vmatrix}.$$

where D_r denotes the leading principal minor of A of order r

Theorem: Consider the quadratic form $Q = x^T A x$ with the associated symmetric matrix $A = (a_{ij})_{n \times n}$. Then

1. Q is positive definite $\Leftrightarrow D_r > 0$ for $r = 1, \dots, n$
2. Q is positive semidefinite $\Leftrightarrow \Delta_r \geq 0$ for all Δ_r and $r = 1, \dots, n$.
3. Q is negative definite $\Leftrightarrow (-1)^r D_r > 0$ for $r = 1, \dots, n$
4. Q is negative semidefinite $\Leftrightarrow (-1)^r \Delta_r \geq 0$ for all Δ_r and $r = 1, \dots, n$.

Proof: We only prove this for $n = 2$ and focus on PD and ND. Then, the quadratic form is

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

After some manipulation through **perfect square**, we obtain

$$Q(x_1, x_2) = a_{11} \underbrace{\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2}_{\geq 0} + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right) \underbrace{x_2^2}_{\geq 0}$$

Thus, we obtain

$$Q(x_1, x_2) > 0 \quad \forall (x_1, x_2) \neq \mathbf{0} \iff a_{11} > 0 \text{ and } a_{11}a_{22} - a_{12}^2 > 0.$$

When $a_{11} < 0$, we observe the following:

$$a_{11}a_{22} - a_{12}^2 > 0 \Leftrightarrow a_{22} - \frac{a_{12}^2}{a_{11}} < 0.$$

Therefore, we obtain

$$Q(x_1, x_2) < 0 \ \forall (x_1, x_2) \neq \mathbf{0} \iff a_{11} < 0 \text{ and } a_{11}a_{22} - a_{12}^2 > 0. \blacksquare$$

Quadratic Forms with Linear Constraints (Ch. 1.8)

In constrained optimization theory, the second-order conditions involve the signs of quadratic forms subject to homogeneous linear constraints.

Let $Q = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ and assume that the variables are subject to the homogeneous linear constraint $b_1x_1 + b_2x_2 = 0$, where $b_1 \neq 0$.

$$b_1 \neq 0 \Rightarrow x_1 = -b_2x_2/b_1.$$

Plugging $x_1 = -b_2x_2/b_1$ into Q , we obtain

$$\begin{aligned} Q &= a_{11} \left(-\frac{b_2x_2}{b_1} \right)^2 + 2a_{12} \left(-\frac{b_2x_2}{b_1} \right) x_2 + a_{22}x_2^2 \\ &= \frac{1}{b_1^2} (a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2) x_2^2. \end{aligned}$$

It is easy to see

$$a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2 = - \begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix}.$$

Therefore,

$$Q \text{ is PD subject to } b_1x_1 + b_2x_2 = 0 \Leftrightarrow \begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix} < 0.$$

$$Q \text{ is ND subject to } b_1x_1 + b_2x_2 = 0 \Leftrightarrow \begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix} > 0.$$

This is also valid when $b_1 = 0$ but $b_2 \neq 0$.

General Case of Q when there are the linear constraints

$$Q(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \quad (a_{ij} = a_{ji})$$

subject to m linear homogeneous constraints:

$$\begin{aligned} b_{11}x_1 + \cdots + b_{1n}x_n &= 0 \\ b_{21}x_1 + \cdots + b_{2n}x_n &= 0 \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots &= \vdots \Leftrightarrow \mathbf{B}\mathbf{x} = \mathbf{0} (*) \\ b_{m1}x_1 + \cdots + b_{mn}x_n &= 0, \end{aligned}$$

where \mathbf{B} is an $m \times n$ matrix.

PD/ND when there are the linear constraints

Definition:

(1) Q is PD subject to the linear constraints $(*)$ if $Q(x) > 0$ for all $x = (x_1, \dots, x_n) \neq 0$ that satisfy $(*)$

(2) Q is ND subject to the linear constraints $(*)$ if $Q(x) < 0$ for all $x = (x_1, \dots, x_n) \neq 0$ that satisfy $(*)$

Define the symmetric determinants

$$B_r = \begin{vmatrix} 0 & \cdots & 0 & b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{m1} & \cdots & b_{mr} \\ b_{11} & \cdots & b_{m1} & a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{1r} & \cdots & b_{mr} & a_{r1} & \cdots & a_{rr} \end{vmatrix}$$

The determinant B_r is the $(m+r)$ th leading principal minor of the $(m+n) \times (m+n)$ bordered matrix

$$\begin{pmatrix} \mathbf{0}_{m \times m} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{pmatrix}.$$

Theorem: Assume that $\text{rank}(B) = m$. Then, a necessary and sufficient condition for the quadratic form Q to be positive definite (PD) subject to the linear constraints (*) is

$$(-1)^m B_r > 0, \quad \forall r = m + 1, \dots, n.$$

The corresponding necessary and sufficient condition for Q to be negative definite (ND) subject to the linear constraints (*) is

$$(-1)^r B_r > 0, \quad \forall r = m + 1, \dots, n.$$

Proof: We omit the proof. ■

Multivariate Calculus

Real-Valued Functions of Several Variables

$f : D \rightarrow \mathbb{R}$ is a **real-valued** function if D is any nonempty set in \mathbb{R}^n and $\mathbb{R} \subseteq \mathbb{R}$.

Rather than having a single slope, a function of n variables can be thought to have n **partial slopes**, each giving only the rate at which y would change if one x_i alone were to change.

Each of these partial slopes is called the **partial derivative**.

Definition: Let $y = f(x_1, \dots, x_n)$. The **partial** derivative of f with respect to x_i is defined as

$$\frac{\partial f(x)}{\partial x_i} \equiv \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

$\partial y / \partial x_i$ or $f'_i(x)$ are used to denote partial derivatives.

Gradients (Ch. 2.1)

If $z = F(x, y)$ and C is any number, we call the graph of the equation $F(x, y) = C$ a **level curve** for F .

The slope of the level curve $F(x, y) = C$ at a point (x, y) is given by the formula:

$$F(x, y) = C \Rightarrow y' = \frac{dy}{dx} = -\frac{\partial F(x, y)/\partial x}{\partial F(x, y)/\partial y} = -\frac{F'_1(x, y)}{F'_2(x, y)},$$

where $\partial F(x, y)/\partial y \neq 0$.

If (x_0, y_0) is a particular point on the level curve $F(x, y) = C$, the slope at (x_0, y_0) is $-F'_1(x_0, y_0)/F'_2(x_0, y_0)$.

The equation for the **tangent hyperplane** T is

$$y - y_0 = - \left[F'_1(x_0, y_0) / F'_2(x_0, y_0) \right] (x - x_0)$$

or, rearranging

$$F'_1(x_0, y_0)(x - x_0) + F'_2(x_0, y_0)(y - y_0) = 0.$$

Recalling the inner product, the equation can be written as

$$\left(F'_1(x_0, y_0), F'_2(x_0, y_0) \right) \cdot (x - x_0, y - y_0) = 0.$$

The vector $(F'_1(x_0, y_0), F'_2(x_0, y_0))$ is the **gradient** of F at (x_0, y_0) and often denoted by $\nabla F(x_0, y_0)$ (∇ is pronounced as “nabla”).

Then, the previous equation can be rewritten as

$$\nabla F(x^0, y^0) \cdot dx = 0,$$

where $dx = (x - x^0, y - y^0)$.

The vector $(x - x_0, y - y_0)$ is a vector on the tangent hyperplane T which implies that $\nabla F(x_0, y_0)$ is **orthogonal** to the tangent hyperplane T at (x_0, y_0) .

Examples of Gradients

(i) $F(x, y) = xy$ and $(x^0, y^0) = (1/2, 1)$.

$$\nabla F(x, y) = (\partial F(x, y)/\partial x, \partial F(x, y)/\partial y) = (y, x).$$

So,

$$\nabla F(x^0, y^0) = (1, 1/2).$$

(ii) $F(x, y, z) = xe^{xy} - z^2$ and $(x^0, y^0, z^0) = (0, 0, 1)$.

$$\nabla F(x, y, z) = (F'_x, F'_y, F'_z) = (e^{xy} + xye^{xy}, x^2e^{xy}, -2z) \text{ So,}$$

$$\nabla F(x^0, y^0, z^0) = (1, 0, -2).$$

Example: Indifference Curves

Let $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be the utility function for a consumer.

Fix $x^0 \in \mathbb{R}_+^2$. If we move from x^0 to $x^0 + dx$, the resulting change of utility becomes

$$du = \frac{\partial u(x^0)}{\partial x_1} dx_1 + \frac{\partial u(x^0)}{\partial x_2} dx_2 \text{ (differentials).}$$

If we satisfy $u(x^0 + dx) = u(x^0)$, we must have $du = 0$, i.e.,

$$\underbrace{-\frac{dx_2}{dx_1} \Big|_{x=x^0}}_{\text{Marginal Rate of Substitution}} = \frac{\partial u(x^0)/\partial x_1}{\partial u(x^0)/\partial x_2}.$$

Gradients (Ch. 2.1)

Suppose that $F(x) = F(x_1, \dots, x_n)$ is a function of n variables defined on an open set S in \mathbb{R}^n , and let $x^0 = (x_1^0, \dots, x_n^0)$ be a point in S .

The **gradient** of F at x^0 is the vector

$$\nabla F(x^0) = \left(\frac{\partial F(x^0)}{\partial x_1}, \dots, \frac{\partial F(x^0)}{\partial x_n} \right)$$

of first-order partial derivatives.

T_{x^0} : the tangent hyperplane that passes through $x^0 \in \mathbb{R}^n$. Then,

$$T_{x^0} = \left\{ x \in \mathbb{R}^n \mid \nabla F(x^0) \cdot (x - x^0) = 0 \right\}.$$

Example: Budget Frontier as a Tangent Hyperplane at Optimum

In the standard consumer optimization problem, the optimal choice is determined such that the marginal rate of substitution is equalized to the price ratio. That is,

$$\frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = \frac{p_1}{p_2}.$$

This implies that $\nabla u(x^*)$ and p are parallel to each other at the optimal consumption x^* .

Therefore, there exists some number $\lambda \in \mathbb{R}$ such that $\nabla u(x^*) = \lambda p$.

Definition: A function $f : S \rightarrow \mathbb{R}$ is **continuously differentiable** (or C^1) on an open set $S \subseteq \mathbb{R}^n$ if, for each $i = 1, \dots, n$, $(\partial f / \partial x_i)(x)$ exists for all $x \in S$ and is continuous on S .

Definition: f is **k -times continuously differentiable** or C^k on S if all the derivatives of f of order less than or equal to k (≥ 1) exist and they are continuous on S .

Convex Sets (Ch. 2.2)

Convex sets are basic building blocks in virtually every area of Economics.

Convexity guarantees that the analysis is mathematically tractable and the results are clear-cut and “well-behaved.”

Definition: $S \subseteq \mathbb{R}^n$ is a **convex** set if for all $x, y \in S$, we have

$$\alpha x + (1 - \alpha)y \in S,$$

for all $\alpha \in [0, 1]$.

z is called a **convex combination** of x and y if $z = \alpha x + (1 - \alpha)y$ for some $\alpha \in [0, 1]$.

A set is convex \Leftrightarrow we can connect any two points in the set by a straight line that lies entirely within the set.

NOTE: Intuitively, a convex set must be “connected” without any “holes,” and its boundary must not “bend inwards” at any point.

Example: Suppose that $p_i > 0$ for each $i = 1, \dots, n$ and $w \geq 0$. Let $B(p, w) = \{x \in \mathbb{R}_+^n \mid p \cdot x \leq w\}$ be the budget set of the consumer.

Fix $x, x' \in B(p, w)$. Fix also $\alpha \in [0, 1]$.

Define $x^\alpha = \alpha x + (1 - \alpha)x'$. Since $x, x' \in B(p, w)$, we know

$$p \cdot x \leq w \text{ and } p \cdot x' \leq w.$$

Then,

$$p \cdot x^\alpha = p \cdot (\alpha x + (1 - \alpha)x') = \alpha p \cdot x + (1 - \alpha)p \cdot x' \leq \alpha w + (1 - \alpha)w = w.$$

Thus, $x^\alpha \in B(p, w)$ so that $B(p, w)$ is convex.

Concave/Convex Functions (Ch. 2.3, 2.4)

Concave and Convex Functions in One Variable

Let I be an interval on \mathbb{R} .

A C^2 function $f : I \rightarrow \mathbb{R}$ is said to be **concave (convex)** on the interval I if $f''(x) \leq (\geq) 0$ for all $x \in I$.

A C^2 function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **strictly concave (strictly convex)** if $f''(x) < (>) 0$ for all $x \in I$.

Concave and Convex Functions in Several Variables (Ch. 2.3)

Let S be a convex, open subset of \mathbb{R}^n throughout.

Definition: A function $f : S \rightarrow \mathbb{R}$ is **concave** on S if, $\forall x, x' \in S$, $\forall \lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)x') \geq \lambda f(x) + (1 - \lambda)f(x').$$

Similarly, $f : S \rightarrow \mathbb{R}$ is **convex** if $-f(\cdot)$ is concave.

Definition: A function $f : S \rightarrow \mathbb{R}$ is **strictly concave** on S if, $\forall x, x' \in S$ with $x \neq x'$, $\forall \lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)x') > \lambda f(x) + (1 - \lambda)f(x').$$

Similarly, $f : S \rightarrow \mathbb{R}$ is **strictly convex** if $-f(\cdot)$ is strictly concave.

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$$f(\lambda x + (1 - \lambda)x') > \lambda f(x) + (1 - \lambda)f(x').$$

Similarly, $f : S \rightarrow \mathbb{R}$ is **strictly convex** if $-f(\cdot)$ is strictly concave.

Concavity/Convexity via Second Derivatives (Ch. 2.3)

Suppose that $f : S \rightarrow \mathbb{R}$ is a C^2 function. The **Hessian** matrix of $f(\cdot)$ at x :

$$D^2 f(x) = \left(f''_{ij}(x) \right)_{n \times n}.$$

For each $r = 1, \dots, n$,

$$D^2_{(r)} f(x) = \begin{vmatrix} f''_{11}(x) & f''_{12}(x) & \cdots & f''_{1r}(x) \\ f''_{21}(x) & f''_{22}(x) & \cdots & f''_{2r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{r1}(x) & f''_{r2}(x) & \cdots & f''_{rr}(x) \end{vmatrix}$$

is the **leading principal minors** of $D^2 f(x)$ of order r .

$$f''_{ij}(x) = \partial^2 f(x) / \partial x_i \partial x_j \text{ for any } i, j = 1, \dots, r.$$

$\Delta_{(r)}^2 f(x)$: a principal minor of $D^2 f(x)$ of order r .

Theorem (Characterization of Concave (Convex) Functions via Second Derivatives): Let $f : S \rightarrow \mathbb{R}$ be a C^2 function. Then

(1) $f(\cdot)$ is convex in $S \iff \forall x \in S, \forall r = 1, \dots, n, \forall \Delta_{(r)}^2 f(x)$, it follows that $\Delta_{(r)}^2 f(x) \geq 0 \iff D^2 f(x)$ is positive semidefinite for any $x \in S$.

(2) $f(\cdot)$ is concave in $S \iff \forall x \in S, \forall r = 1, \dots, n, \forall \Delta_{(r)}^2 f(x)$, it follows that $(-1)^r \Delta_{(r)}^2 f(x) \geq 0 \iff D^2 f(x)$ is negative semidefinite for any $x \in S$.

Proof: We skip the proof. ■

Example of Cobb-Douglas Utility Function: Define $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be the utility function of the consumer: for any $x \in \mathbb{R}_+^2$, $u(x) = x_1^\alpha x_2^\beta$ where α and β are positive real numbers.

$$\begin{aligned} u_1 &= \alpha x_1^{\alpha-1} x_2^\beta \\ u_{11} &= -\alpha(1-\alpha) x_1^{\alpha-2} x_2^\beta \\ u_{12} = u_{21} &= \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ u_2 &= \beta x_1^\alpha x_2^{\beta-1} \\ u_{22} &= -\beta(1-\beta) x_1^\alpha x_2^{\beta-2}. \end{aligned}$$

Then,

$$D^2 u(x) = \begin{pmatrix} -\alpha(1-\alpha) x_1^{\alpha-2} x_2^\beta & \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} & -\beta(1-\beta) x_1^\alpha x_2^{\beta-2} \end{pmatrix}.$$

$$\begin{aligned}
u_{11}u_{22} - u_{12}u_{21} &= \alpha\beta(1-\alpha)(1-\beta)x_1^{2\alpha-2}x_2^{2\beta-2} - \alpha^2\beta^2x_1^{2\alpha-2}x_2^{2\beta-2} \\
&= \alpha\beta x_1^{2\alpha-2}x_2^{2\beta-2} \{(1-\alpha)(1-\beta) - \alpha\beta\} \\
&= \alpha\beta x_1^{2\alpha-2}x_2^{2\beta-2}(1-\alpha-\beta).
\end{aligned}$$

Then, $u(\cdot)$ is concave if and only if $0 < \alpha + \beta \leq 1$.

“Strict” Concavity/Convexity via Second Derivatives (Ch. 2.3)

Theorem: Let $f : S \rightarrow \mathbb{R}$ be a C^2 function. Then

(1) $D^2f(x)$ is positive definite for any $x \in S \iff D_{(r)}^2f(x) > 0$ for all $x \in S$ and all $r = 1, \dots, n \Rightarrow f(\cdot)$ is strictly convex.

(2) $D^2f(x)$ is negative definite for any $x \in S \iff (-1)^r D_{(r)}^2f(x) > 0$ for all $x \in S$ and all $r = 1, \dots, n \Rightarrow f(\cdot)$ is strictly concave.

Proof: We skip the proof. ■

Example of Cobb-Douglas Utility Function Revisited: Define $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ as the utility function of a consumer such that for any $x \in \mathbb{R}_+^2$, $u(x) = x_1^\alpha x_2^\beta$ where α and β are positive real numbers.

Then, $u(\cdot)$ is strictly concave if $0 < \alpha + \beta < 1$.

Concavity/Convexity via First Derivatives (Ch. 2.4)

The following result is extremely important in both static and dynamic optimization.

Theorem: Suppose that $f : S \rightarrow \mathbb{R}$ is a C^1 function. Then

(1) $f(\cdot)$ is concave in $S \Leftrightarrow$

$$f(x) - f(x^0) \leq \nabla f(x^0) \cdot (x - x^0) = \sum_{i=1}^n \frac{\partial f(x^0)}{\partial x_i} (x_i - x_i^0)$$

for all $x, x^0 \in S$.

(2) $f(\cdot)$ is strictly concave \Leftrightarrow the above inequality is always strict when $x \neq x^0$.

Remark: Geometrically, this result says that the tangent at any point on the graph will lie above the graph.

Proof: We skip the proof. ■

Quasiconcave and Quasiconvex Functions (Ch. 2.5)

Definition: A function $f : S \rightarrow \mathbb{R}$ is **quasiconcave** if the upper level set $P_\alpha = \{x \in S | f(x) \geq \alpha\}$ is convex for each $\alpha \in \mathbb{R}$. We say that f is **quasiconvex** if $-f$ is quasiconcave. So, f is quasiconvex if the lower level set $P^\alpha = \{x \in S | f(x) \leq \alpha\}$ is convex for each $\alpha \in \mathbb{R}$.

There are equivalent definitions of quasiconcavity.

Theorem: $f : S \rightarrow \mathbb{R}$ is quasiconcave if and only if either of the following conditions is satisfied for all $x, x' \in S$ and all $\lambda \in [0, 1]$,

$$(1) f(\lambda x + (1 - \lambda)x') \geq \min\{f(x), f(x')\}$$

$$(2) f(x') \geq f(x) \Rightarrow f(\lambda x + (1 - \lambda)x') \geq f(x).$$

Proof: We omit the proof. ■

Concavity \Rightarrow Quasiconcavity

Proposition: If $f : S \rightarrow \mathbb{R}$ is concave, then it is quasiconcave. Similarly, if $f(\cdot)$ is convex, then it is quasiconvex.

Proof: We omit the proof. ■

Quasiconcavity is preserved under positive monotone transformation

A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is said to be **strictly increasing** if $F(x) > F(y)$ whenever $x > y$.

Theorem: Let $f : S \rightarrow \mathbb{R}$ and let $F : D \rightarrow \mathbb{R}$ where $f(S) \subseteq D \subseteq \mathbb{R}$. If $f(\cdot)$ is quasiconcave (quasiconvex) and F is strictly increasing, then $F(f(\cdot))$ is quasiconcave (quasiconvex).

Proof: Suppose $f(\cdot)$ is quasiconcave. Using the previous theorem, we must have

$$f\left(\lambda x + (1 - \lambda)x'\right) \geq \min\{f(x), f(x')\}.$$

Since $F(\cdot)$ is strictly increasing,

$$F\left(f\left(\lambda x + (1 - \lambda)x'\right)\right) \geq F\left(\min\{f(x), f(x')\}\right) = \min\{F(f(x)), F(f(x'))\}.$$

It follows that $F \circ f$ is quasiconcave. The argument in the quasiconvex case is entirely similar, replacing \geq with \leq and min with max. ■

Definition: A function $f : S \rightarrow \mathbb{R}$ is **strictly quasiconcave** if

$$f(\lambda x + (1 - \lambda)x') > \min\{f(x), f(x')\}$$

for all $x, x' \in S$ with $x \neq x'$ and all $\lambda \in (0, 1)$.

The function $f(\cdot)$ is **strictly quasiconvex** if $-f(\cdot)$ is strictly quasiconcave.

The Cobb-Douglas Function

Example: Let $f(x_1, \dots, x_n) = Ax_1^{a_1} \cdots x_n^{a_n}$, where $x_1, \dots, x_n > 0$, $A > 0$, and $a_1, \dots, a_n > 0$. Set $a = a_1 + \cdots + a_n$.

- $f(\cdot)$ is quasiconcave for all a_1, \dots, a_n ;
- $f(\cdot)$ is concave for $a \leq 1$;
- $f(\cdot)$ is strictly concave for $a < 1$.