ECON 696
Mathematical Methods for Economic Dynamics
Answer Key to the Midterm Examination
Fall 2024, SMU School of Economics

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Oct 2 (Wed), 2024; 8:15 - 9:45 (90 minutes)
at SOE/SCIS2 Building, Seminar Room 3-4

Question 1 (Univariate Calculus (30 points)) *Answer the following questions.*

1. (3 points) Formally state Weierstrass Theorem (or Extreme Value Theorem).

   Let $f : S \to \mathbb{R}$ be a continuous real-valued mapping where S is a nonempty compact subset of $\mathbb{R}^n$. Then there exist two vectors $x^*, x_* \in S$ such that for all $x \in S$,

   $$f(x_*) \leq f(x) \leq f(x^*).$$

2. (7 points) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and continuously differentiable (C^1) on (a, b). Show that if $f(a) = f(b) = 0$, then there is a point $c \in (a, b)$ such that $f'(c) = 0$. This is called Rolle's Theorem.

   If f is constant on $[a, b]$, then $f'(c) = 0$ for all $c \in (a, b)$. In this case, we are done. We next assume that f is not constant on $[a, b]$. Then, we further assume without loss of generality that f is sometimes positive on (a, b). Since $[a, b]$ is a compact subset of $\mathbb{R}$ and f is continuous on $[a, b]$, by Weierstrass Theorem, f achieves both the global maximum and minimum points over $[a, b]$. By our assumption that f is sometimes positive on (a, b), there exists a global maximum point $c \in (a, b)$ such that $f(c) > 0$. Since c is an interior point in $[a, b]$, by the necessity of extreme points for optimization problem, c is a stationary point of f so that $f^{'}(c) = 0$.

3. (10 points) Let $I \subseteq \mathbb{R}$ be an interval on the real line and $f : I \to \mathbb{R}$ be a twice continuously differentiable (or C^2) function. For each $t \in I$, define

$$g_1(t) \equiv f(t) - \left[f(a) + f^{'}(a)(t-a)\right] - M_1(t-a)^2,$$

where

$$M_1 = \frac{1}{h^2}\left[f(a+h) - f(a) - f^{'}(a)h\right].$$

Show that there exists $c_1 \in (a, a+h)$ so that $g_1^{'}(c_1) = 0$.

We compute the following:

$$\begin{aligned} g_1(a) &= f(a) - \left[f(a) + f^{'}(a)(a-a)\right] - M_1(a-a)^2 = f(a) - f(a) = 0. \\ g_1(a+h) &= f(a+h) - \left[f(a) + f^{'}(a)h\right] - M_1h^2 \\ &= f(a+h) - \left[f(a) + f^{'}(a)h\right] - \left[f(a+h) - f(a) - f^{'}(a)h\right] \\ &= 0. \end{aligned}$$

Since $g_1(t)$ is a continuous function, by Rolle's Theorem, there exists $c_1 \in (a, a+h)$ such that $g_1^{'}(c_1) = 0$.

4. (10 points) Show that, for any points $a, a+h \in I$, there exists a point c_2 between a and $a+h$ such that

$$f(a+h) - f(a) = f^{'}(a)h + \frac{1}{2}f^{''}(c_2)h^2.$$

(HINT: Use $g_1^{'}(c_1) = 0$.)

We compute the following:

$$g_1^{'}(t) = f^{'}(t) - f^{'}(a) - 2M_1(t-a).$$

We next confirm the following:

$$g_1^{'}(a) = f^{'}(a) - f^{'}(a) - 2M_1(a-a) = 0$$

It follows from the previous part of the question that $g_1^{'}(c_1) = 0$. Since $g_1^{'}(t)$ is a continuous function, by Rolle's Theorem, there exists $c_2 \in (a, c_1)$ such that $g_1^{''}(c_2) = 0$. We compute the following:

$$g_1^{''}(t) = f^{''}(t) - 2M_1.$$

When $t = c_2$, we have

$$g^{''}(c_2) = f^{''}(c_2) - 2M_1 = 0.$$

Plugging $M_1 = [f(a+h) - f(a) - f^{'}(a)h]/h^2$ into the above equation, we have

$$f^{''}(c_2) - \frac{2}{h^2}\left[f(a+h) - f(a) - f^{'}(a)h\right] = 0.$$

This further implies

$$f(a+h) - f(a) = f^{'}(a)h + \frac{f^{''}(c_2)}{2}h^2.$$

Question 2 (Optimization with Inequality Constraints (40 points)) *Consider the following constrained maximization problem:*

$$\max_{(x,y)\in\mathbb{R}^2} -2x^2-2y^2+2xy+9y \ \textit{ subject to } \ 4x+3y \leq 10,\ y-4x^2 \geq -2,\ x \geq 0,\ y \geq 0.$$

Answer the following questions.

1. (5 points) Set up the Lagrangian function and obtain the Kuhn-Tucker condition for this optimization problem.

   We first rewrite this constrained maximization problem:

   $$\max_{(x,y)\in\mathbb{R}^2} -2x^2-2y^2+2xy+9y \ \text{ subject to } \ 4x+3y-10 \leq 0,\ -y+4x^2-2 \leq 0,\ -x \leq 0,\ -y \leq 0.$$

   We set up the Lagrangian $\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as follows:

   $$\begin{aligned}\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= -2x^2 - 2y^2 + 2xy + 9y - \lambda_1(4x + 3y - 10) \\ &\quad -\lambda_2(-y + 4x^2 - 2) - \lambda_3(-x) - \lambda_4(-y).\end{aligned}$$

   The Kuhn-Tucker condition is given below:

   (1) $\mathcal{L}^{'}_x = -4x + 2y - 4\lambda_1 - 8\lambda_2 x + \lambda_3 = 0$

   (2) $\mathcal{L}^{'}_y = -4y + 2x + 9 - 3\lambda_1 + \lambda_2 + \lambda_4 = 0$

   (3) $\lambda_1 \geq 0$; $4x + 3y - 10 \leq 0$; $\lambda_1(4x + 3y - 10) = 0$;

   (4) $\lambda_2 \geq 0$; $-y + 4x^2 - 2 \leq 0$; $\lambda_2(-y + 4x^2 - 2) = 0$;

   (5) $\lambda_3 \geq 0$; $x \geq 0$; $\lambda_3(-x) = 0$;

   (6) $\lambda_4 \geq 0$; $y \geq 0$; $\lambda_4(-y) = 0$.

2. (10 points) Examine each of the following three cases: (i) $x = 0$ and $y = 0$; (ii) $x > 0$ and $y = 0$; and $x = 0$ and $y > 0$, separately and show that each case leads to a violation of the Kuhn-Tucker condition.

Case (i) $x = 0$ **and** $y = 0$

Plugging $x = y = 0$ into the inequalities in (3) and (4), we obtain

$$\begin{aligned} 4x + 3y - 10 &= -10 < 0 \\ -y + 4x^2 - 2 &= -2 < 0, \end{aligned}$$

Since these two inequality constraints are not binding, by (3) and (4), we have $\lambda_1 = \lambda_2 = 0$. Next, plugging $x = y = 0$ and $\lambda_1 = \lambda_2 = 0$ into $(2')$, we obtain

$$9 + \lambda_4 = 0 \Leftrightarrow \lambda_4 = -9,$$

which contradicts the requirement that $\lambda_4 \geq 0$ in (6). Thus, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

Case (ii) $x > 0$ **and** $y = 0$

Since $x > 0$, by (5), we have $\lambda_3 = 0$. Plugging $y = 0$ and $\lambda_3 = 0$ into (1), we obtain

$$-4x - 4\lambda_1 - 8\lambda_2 x = 0 \Leftrightarrow \lambda_1 = -(2\lambda_2 + 1)x.$$

Since $\lambda_2 \geq 0$ by (2) and we assume $x > 0$, we have $\lambda_1 < 0$, which contradicts the requirement that $\lambda_1 \geq 0$ in (1). Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

Case (iii) $x = 0$ **and** $y > 0$

Plugging $x = 0$ into (1), we obtain

$$2y - 4\lambda_1 + \lambda_3 = 0 \Leftrightarrow \lambda_3 = 4\lambda_1 - 2y.$$

Since $y > 0$ and $\lambda_3 \geq 0$ from (5), we have $\lambda_1 > 0$. Then, by (3), we have

$$4x + 3y - 10 = 0 \underbrace{\Rightarrow}_{\because\, x=0} y = 10/3.$$

Plugging $(x, y) = (0, 10/3)$ into the inequality in (4), we obtain

$$-y + 4x^2 - 2 = -10/3 - 2 < 0.$$

Thus, by (4), we have $\lambda_2 = 0$. Since $y > 0$, by (6), we have $\lambda_4 = 0$. Plugging $(x, y) = (0, 10/3)$ and $\lambda_2 = \lambda_4 = 0$ into (2), we obtain

$$-4y + 9 - 3\lambda_1 = 0 \Leftrightarrow 3\lambda_1 = -4(10/3) + 9 = -13/3,$$

which contradicts the requirement that $\lambda_1 \geq 0$. Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

3. (15 points) Find the solution candidate satisfying the Kuhn-Tucker condition.

   By the previous part of the question, we have that $x > 0$ and $y > 0$. By (5) and (6), we conclude that $\lambda_3 = \lambda_4 = 0$. Taking into account that $\lambda_3 = \lambda_4 = 0$, we simplify the Kuhn-Tucker condition as follows:

   $(1^{'})$ $\mathcal{L}_x^{'} = -4x + 2y - 4\lambda_1 - 8\lambda_2 x = 0$
   $(2^{'})$ $\mathcal{L}_y^{'} = -4y + 2x + 9 - 3\lambda_1 + \lambda_2 = 0$
   (3) $\lambda_1 \geq 0$; $4x + 3y - 10 \leq 0$; $\lambda_1(4x + 3y - 10) = 0$;
   (4) $\lambda_2 \geq 0$; $-y + 4x^2 - 2 \leq 0$; $\lambda_2(-y + 4x^2 - 2) = 0$;

   Case 1: $\lambda_1 > 0$ **and** $\lambda_2 > 0$

   Since $\lambda_1 > 0$, by (3), we have

   $$y = \frac{1}{3}(-4x + 10).$$

   Since $\lambda_2 > 0$, by (4), we have

   $$-y + 4x^2 - 2 = 0.$$

   Combining the above two equations, we obtain

   $$12x^2 + 4x - 16 = 0 \Leftrightarrow (3x + 4)(x - 1) = 0.$$

   Thus, $x = -4/3$ or 1. However, by the previous part of this question, we conclude that $x > 0$ so that we must have $x = 1$. This implies that $y = 2$. Plugging $x = 1$ and $y = 2$ into $(1^{'})$ and $(2^{'})$, we obtain

   $(1^{''})$ $\mathcal{L}_x^{'} = -4 + 4 - 4\lambda_1 - 8\lambda_2 = 0$
   $(2^{''})$ $\mathcal{L}_y^{'} = -8 + 2 + 9 - 3\lambda_1 + \lambda_2 = 0$

Solving the above two equations, we obtain $\lambda_2 = -1/2$, which contradicts (4). So, there is no solution candidate in this case.

Case 2: $\lambda_1 > 0$ **and** $\lambda_2 = 0$

Since $\lambda_1 > 0$, by (3), we have

$$y = \frac{1}{3}(-4x + 10).$$

Plugging the above equation and $\lambda_2 = 0$ into $(1^{'})$ and $(2^{'})$, we obtain

$(1^{''})$ $\mathcal{L}_x^{'} = -4x + \frac{2}{3}(-4x + 10) - 4\lambda_1 = 0 \Leftrightarrow -5x + 5 - 3\lambda_1 = 0$
$(2^{''})$ $\mathcal{L}_y^{'} = -\frac{4}{3}(-4x + 10) + 2x + 9 - 3\lambda_1 = 0 \Leftrightarrow 22x - 13 - 9\lambda_1 = 0$

Solving the system of the above two equations, we obtain $x = 28/37$ and $\lambda_1 = 15/37 > 0$. Plugging $x = 28/37$ into $y = (-4x + 10)/3$, we obtain $y = 86/37 > 0$. Plugging $x = 28/37$ and $y = 86/37$ into $-y + 4x^2 - 2$, we obtain

$$-y + 4x^2 - 2 = -\frac{86}{37} + 4 \times \frac{28^2}{37^2} - 2 < -2 + 4 - 2 = 0,$$

where we take into account that $-86/37 < -2$ and $(28^2)/(37^2) < 1$. Therefore, the inequality in (4) is satisfied. In conclusion, $(x, y) = (28/37, 86/37)$ is a solution candidate satisfying the Kuhn-Tucker condition.

Case 3: $\lambda_1 = 0$ **and** $\lambda_2 > 0$

Plugging $\lambda_1 = 0$ into $(1^{'})$ and $(2^{'})$, we obtain

$(1^{''})$ $\mathcal{L}_x^{'} = -4x + 2y - 8\lambda_2 x = 0$
$(2^{''})$ $\mathcal{L}_y^{'} = -4y + 2x + 9 + \lambda_2 = 0$

Since $\lambda_2 > 0$, by (4), we have

$$-y + 4x^2 - 2 = 0.$$

Plugging $y = 4x^2 - 2$ into $(2^{''})$, we obtain

$$\lambda_2 = 4(4x^2 - 2) - 2x - 9 = 16x^2 - 2x - 17.$$

Since we must have $x > 0$ and $\lambda_2 \geq 0$, we have

$$x \geq \frac{1+\sqrt{263}}{16} > \frac{1+16}{16} > 1.$$

Hence, $x > 1$. Taking into account $y = 4x^2 - 2$, we compute the following:

$$4x+3y-10 = 4x+3(4x^2-2)-10 = 12x^2+4x-16 = 4(3x^2+x-4) = 4(3x+4)(x-1).$$

Since we must have $4x + 3y - 10 \leq 0$ by (3), we have $-4/3 \leq x \leq 1$. However, this contradicts the previous conclusion that $x > 1$. So, there are no solution candidates in this case.

Case 4: $\lambda_1 = 0$ **and** $\lambda_2 = 0$

Plugging $\lambda_1 = \lambda_2 = 0$ into $(1')$ and $(2')$, we obtain

$$\begin{aligned} -4x + 2y &= 0 \\ -4y + 2x + 9 &= 0. \end{aligned}$$

We obtain $(x, y) = (3/2, 3)$ as the unique solution to the above system of linear equations. Plugging $(x, y) = (3/2, 3)$ into $4x + 3y - 10$, we obtain

$$4x + 3y - 10 = 4(3/2) + 3 \cdot 3 - 10 = 5,$$

which contradicts the inequality $4x+3y-10 \leq 0$ in (3). Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition.

Considering all the four cases above, we have

$$(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (28/37, 86/37, 15/37, 0, 0, 0)$$

as the unique solution candidate satisfying the Kuhn Tucker condition.

4. (10 points) Find the solution to this constrained maximization problem.

We obtain the Lagrangian associated with the Lagrange multipliers:

$$\mathcal{L}(x, y, z, \lambda_1, 0, 0, 0) = -2x^2 - 2y^2 + 2xy + 9y - \lambda_1(4x + 3y - 10),$$

where $\lambda_1 = 15/37$. We compute the Hessian matrix associated with this Lagrangian:

$$H_{\mathcal{L}}(x, y, z, \lambda_1) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}.$$

Since $h_{11} = -4 < 0$ and $|H_{\mathcal{L}}(x, y, z, \lambda_1)| = 16 - 4 = 12 > 0$, the Lagrangian function $\mathcal{L}(x, y, z, \lambda_1 = 15/37)$ is concave in (x, y, z). Thus, the solution candidate we have obtained in the previous question is indeed a solution.

Question 3 (The System of Linear Equations (30 points)) *Consider the following system of linear equations:*

$$\begin{aligned} 2x + 5y - z + 5u &= 8 \\ -x - 3y + z - 2u + 2v &= -4 \\ -3y + 3z + 7u + 4v &= 4 \\ x + 2y + 3u + 2v &= 4. \end{aligned}$$

Let

$$A = \begin{pmatrix} 2 & 5 & -1 & 5 & 0 \\ -1 & -3 & 1 & -2 & 2 \\ 0 & -3 & 3 & 7 & 4 \\ 1 & 2 & 0 & 3 & 2 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 8 \\ -4 \\ 4 \\ 4 \end{pmatrix}.$$

Then, we can rewrite the system as $A\mathbf{x} = \mathbf{b}$. Answer the following questions.

1. (5 points) Let A be expressed by $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a}_5)$, where

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}; \mathbf{a}_2 = \begin{pmatrix} 5 \\ -3 \\ -3 \\ 2 \end{pmatrix}; \mathbf{a}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 0 \end{pmatrix}; \mathbf{a}_4 = \begin{pmatrix} 5 \\ -2 \\ 7 \\ 3 \end{pmatrix}; \mathbf{a}_5 = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 2 \end{pmatrix}$$

   Show that $\mathbf{a}_3$ can be expressed as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

   Let $x, y \in \mathbb{R}$ be such that

$$\mathbf{a}_3 = x\mathbf{a}_1 + y\mathbf{a}_2 \Leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x + 5y \\ -x - 3y \\ -3y \\ x + 2y \end{pmatrix}$$

   So, $x = 2$ and $y = -1$.

2. (5 points) Show that $\mathbf{a}_5$ can be expressed as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a_4}$.

   Let $x, y, u \in \mathbb{R}$ be such that

$$\mathbf{a}_5 = x\mathbf{a}_1 + y\mathbf{a}_2 + u\mathbf{a}_4 \Leftrightarrow \begin{pmatrix} 0 \\ 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2x + 5y + 5u \\ -x - 3y - 2u \\ -3y + 7u \\ x + 2y + 3u \end{pmatrix}.$$

   So, $x = 20, y = -6$, and $u = -2$.

3. (5 points) Let

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4) = \begin{pmatrix} 2 & 5 & 5 \\ -1 & -3 & -2 \\ 0 & -3 & 7 \\ 1 & 2 & 3 \end{pmatrix}$$

be the truncated matrix by eliminating $\mathbf{a}_3$ and $\mathbf{a}_5$ from A. Show that one row of this truncated matrix can be expressed as a linear combination of two other rows of the same matrix.

This is shown because

$$(-1, -3, -2) = -(2, 5, 5) + (1, 2, 3).$$

4. (5 points) Find the rank of A.

Following all the previous steps, we obtain the following truncated matrx:

$$\begin{pmatrix} 2 & 5 & 5 \\ 0 & -3 & 7 \\ 1 & 2 & 3 \end{pmatrix}$$

We compute the determinant of this matrix:

$$\begin{aligned} \begin{vmatrix} 2 & 5 & 5 \\ 0 & -3 & 7 \\ 1 & 2 & 3 \end{vmatrix} &= 2 \cdot (-1)^{1+1} \begin{vmatrix} -3 & 7 \\ 2 & 3 \end{vmatrix} + 1 \cdot (-1)^{3+1} \begin{vmatrix} 5 & 5 \\ -3 & 7 \end{vmatrix} \\ &= 2(-9 - 14) + (35 + 15) \\ &= -46 + 50 = 4 \neq 0. \end{aligned}$$

Thus, $\text{rank}(A) = 3$.

5. (10 points) Show that the system has at least one solution. (HINT: You may use the following result: $Ax = b$ has at least one solution if and only if $\text{rank}(A) = \text{rank}(A_b)$, where A_b denotes the augmented matrix.)

Let A_b be the augmented matrix such that

$$A_b = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{b}).$$

By the previous steps, we can consider the truncated matrix by eliminating $\mathbf{a}_3$ and $\mathbf{a}_5$ from A_b:

$$\begin{pmatrix} 2 & 5 & 5 & 8 \\ -1 & -3 & -2 & -4 \\ 0 & -3 & 7 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

We claim that the second row of this matrix can be expressed as a linear combination of the first and fourth rows of this matrix:

$$(-1, -3, -2, -4) = -(2, 5, 5, 8) + (1, 2, 3, 4).$$

We thus obtain the following further truncated matrix by eliminating the second row of the above matrix:

$$\begin{pmatrix} 2 & 5 & 5 & 8 \\ 0 & -3 & 7 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

We claim that the fourth column can be expressed as a linear combination of the other three columns of the above matrix:

$$\begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} = -\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}.$$

So, we have the following truncated matrix by eliminating the fourth columns from the above matrix.

$$\begin{pmatrix} 2 & 5 & 5 \\ 0 & -3 & 7 \\ 1 & 2 & 3 \end{pmatrix}.$$

It turns out that this matrix is identical to the one we have obtained in Part 4 of this question. So, we compute the determinant of the above truncated matrix:

$$\begin{vmatrix} 2 & 5 & 5 \\ 0 & -3 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 4 \neq 0.$$

Thus, $\text{rank}(A_b) = 3$. Since $\text{rank}(A) = \text{rank}(A_b) = 3$, the system of equations has at least one solution.