Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 9: Control Theory II (Ch. 9)

The Maximum Principle and the Calculus of Variations (Ch. 9.5)

Consider the standard variational problem:

$$\max \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))dt, \;\; x(t_0) = x_0, \quad \begin{cases} \text{(a)} & x(t_1) = x_1 \\ \text{(b)} & x(t_1) \geq x_1 \; (*) \\ \text{(c)} & x(t_1) \text{ free,} \end{cases}$$

where one of the alternative conditions (a), (b), and (c) is imposed.

To transform this to a control problem, simply use $u(t) = \dot{x}(t)$ as a control variable.

Because there are no restrictions on $\dot{x}(t)$ in the calculus of variations problem, $U = \mathbb{R}$.

The control problem has the particularly simple differential equation $\dot{x}(t) = u(t)$. The Hamiltonian is $H(t, x, u, p) = p_0 F(t, x, u) + pu$.

The maximum principle states that if $u^*(t)$ solves the problem, then H as a function of u must be maximized at $u = u^*(t)$.

Because $U = \mathbb{R}$, a necessary condition for this maximum is

$$H_u^{'}(t, x^*(t), u^*(t), p(t)) = p_0 F_u^{'}(t, x^*(t), u^*(t)) + p(t) = 0 \quad (*)$$

Since $(p_0, p(t)) \neq (0, 0)$, $(*)$ implies that $p_0 \neq 0$ so that $p_0 = 1$.

The differential equation for $p(t)$ is

$$\dot{p}(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t)) = -F_x^{'}(t, x^*(t), u^*(t)) \quad (**)$$

Differentiating $(*)$ w.r.t. t yields

$$\frac{d}{dt}\left(F_u^{'}(t, x^*(t), u^*(t))\right) + \dot{p}(t) = 0 \qquad (***)$$

Since $u^* = \dot{x}^*$, it follows from $(**)$ and $(***)$ that

$$F_x^{'}(t, x^*, \dot{x}^*) - \frac{d}{dt}\left(F_{\dot{x}}^{'}(t, x^*, \dot{x}^*)\right) = 0,$$

which is the Euler equation.

Moreover, $(*)$ implies that

$$p(t) = -F'_{\dot{x}}(t, x^*, \dot{x}^*).$$

Thus,

Terminal Conditions	**Transversality Conditions (Control Theory)**	**Transversality Conditions (Calculus of Variations)**
$x(t_1) = x_1$	$p(t_1)$ no condition	$F'_{\dot{x}}(t_1, x^*, \dot{x}^*)$ no condition
$x(t_1) \geq x_1$	$p(t_1) \geq 0$ with $p(t_1) = 0$ if $x^*(t_1) > x_1$	$F'_{\dot{x}}(t_1, x^*, \dot{x}^*) \leq 0$ with $F'_{\dot{x}}(t_1, x^*, \dot{x}^*) = 0$ if $x^*(t_1) > x_1$
$x(t_1)$ free	$p(t_1) = 0$	$F'_{\dot{x}}(t_1, x^*, \dot{x}^*) = 0$

Note that concavity of the Hamiltonian with respect to (x, u) is equivalent to concavity of $F(t, x, \dot{x})$ with respect to $(x, \dot{x})$.

Adjoint Variables as Shadow Prices
(Ch. 9.6)

We consider the standard end-constrained problem.

Suppose it has a unique solution $(x^*(t), u^*(t))$ with a unique corresponding adjoint function $p(t)$. The corresponding value of the objective function will depend on x_0, x_1, t_0, and t_1, so it is denoted by

$$V(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))dt.$$

We call V the **value function**. (When $x(t_1)$ is free, x_1 is not an argument of V.) At any point where V is differentiable,

$$\frac{\partial V(x_0, x_1, t_0, t_1)}{\partial x_0} = p(t_0).$$

The number $p(t_0)$ therefore measures the marginal change in the value function as x_0 increases.

Example: Revisit the problem

$$\max \int_0^T \left[1 - tx(t) - u(t)^2\right] dt,\ \dot{x}(t) = u(t),\ x(0) = x_0,\ x(T) \text{ free},\ u \in \mathbb{R}$$

where x_0 and T are given positive constants. The solution was

$$\begin{aligned} u^*(t) &= -\frac{1}{4}(T^2 - t^2), \\ x^*(t) &= x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3, \\ p(t) &= -\frac{1}{2}(T^2 - t^2). \end{aligned}$$

So, the value function is

$$\begin{aligned} V(x_0, T) &= \int_0^T [1 - tx^*(t) - (u^*(t))^2] dt \\ &= \int_0^T \left[1 - x_0 t + \frac{1}{4}T^2 t^2 - \frac{1}{12}t^4 - \frac{1}{16}(T^2 - t^2)^2\right] dt \end{aligned}$$

By Leibniz' formula, we obtain

$$\frac{\partial V(x_0, T)}{\partial x_0} = \int_0^T (-t)dt = -\frac{1}{2}T^2 = p(0).$$

Define

$$H^*(t) = H(t, x^*(t), u^*(t), p(t)).$$

Provided V is differentiable, we have

$$\frac{\partial V}{\partial x_0} = p(t_0), \; \frac{\partial V}{\partial x_1} = -p(t_1), \; \frac{\partial V}{\partial t_0} = -H^*(t_0), \; \frac{\partial V}{\partial t_1} = H^*(t_1).$$

Leibniz's Formula: General Case

Suppose $f(x,t)$ and $\partial f(x,t)/\partial x$ are continuous over the rectangle by $a \leq x \leq b$ and $c \leq t \leq d$. Assume further that $u(\cdot)$ and $v(\cdot)$ are C^1 functions over $[a,b]$ and

$$\begin{aligned} u([a,b]) &= \{y \in \mathbb{R} | \ \exists x \in [a,b] \text{ s.t. } u(x) = y\} \subseteq [c,d], \\ v([a,b]) &= \{y \in \mathbb{R} | \ \exists x \in [a,b] \text{ s.t. } v(x) = y\} \subseteq [c,d]. \end{aligned}$$

Then,

$$\begin{aligned} F(x) &= \int_{u(x)}^{v(x)} f(x,t)dt \\ \Rightarrow \quad & F^{'}(x) = f(x,v(x))v^{'}(x) - f(x,u(x))u^{'}(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt. \end{aligned}$$

Sketch of Proof: Let $H : [a,b]\times[c,d]\times[c,d] \to \mathbb{R}$ be the following function.

$$H(x,u,v) = \int_u^v f(x,t)dt.$$

Since we can set $F(x) = H(x, u(x), v(x))$, by the chain rule,

$$\begin{aligned} F'(x) &= H'_x + H'_u u'(x) + H'_v v'(x) \\ &= \int_u^v \frac{\partial f(x,t)}{\partial x} dt + f(x,v)v'(x) - f(x,u)u'(x). \end{aligned}$$

This completes the proof. ■

Example: Revisit the problem

$$\max \int_0^T \left[1 - tx(t) - u(t)^2\right] dt,\ \dot{x}(t) = u(t),\ x(0) = x_0,\ x(T) \text{ free},\ u \in \mathbb{R}$$

where x_0 and T are given positive constants. We already obtain

$$V(x_0, T) = \int_0^T \left[1 - x_0 t + \frac{1}{4}T^2 t^2 - \frac{1}{12}t^4 - \frac{1}{16}(T^2 - t^2)^2\right] dt.$$

By Leibniz' rule, we have

$$\begin{aligned}\frac{\partial V}{\partial T} &= \left(1 - x_0 T + \frac{1}{4}T^4 - \frac{1}{12}T^4\right) \cdot 1 + \int_0^T \left[\frac{1}{2}t^2 T - \frac{1}{8}(T^2 - t^2)2T\right] dt \\ &= 1 - x_0 T + \frac{1}{6}T^4.\end{aligned}$$

Since $u^*(T) = 0$ and $x^*(T) = x_0 - T^3/6$, we have

$$H^*(T) = 1 - Tx^*(T) - (u^*(T))^2 + p(T)u^*(T) = 1 - x_0 T + \frac{1}{6}T^4 = \frac{\partial V}{\partial T}.$$

An Elaborated Example of Economic Growth Model

Consider the following economic growth model:

$$\max \int_0^T (1 - s(t)) e^{\rho t} f(k(t)) e^{-\delta t} dt$$

$$\text{subject to} \quad \dot{k}(t) = s(t) e^{\rho t} f(k(t)) - \lambda k(t), \; k(0) = k_0,$$

$$k(T) \geq k_T > k_0, \; s(t) \in [0, 1],$$

where $k(t)$ is the capital stock (a state variable), $s(t)$ is the savings rate (a control variable), and $f(k)$ is a production function.

We set $\rho = 0; f(k) = ak;\ a > 0;\ \delta = 0;$ and $\lambda = 0.$ Assume further that $T > 1/a$ and $k_0 e^{aT} > k_T$.

We rewrite the optimization problem:

$$\max \int_0^T (1 - s(t))ak(t)dt$$
$$\text{subject to}\quad \dot{k}(t) = as(t)k(t),\ k(0) = k_0,\ k(T) \geq k_T > k_0,\ s(t) \in [0, 1],$$

where $a > 0,\ T > 1/a$, and $k_0 e^{aT} > k_T$.

The Hamiltonian is

$$H(t, k, s, p) = (1 - s)ak + pask.$$

Claim: $p(\cdot)$ is strictly decreasing.

Proof: First, we obtain

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1, \\ [0,1] & \text{if } p(t) = 1, \\ 0 & \text{if } p(t) < 1. \end{cases}$$

Second, using the above property of $s^*(t)$, we obtain

$$\begin{aligned} \dot{p} &= -(1 - s^*(t))a - p(t)s^*(t)a \\ &= -a + s^*(t)a(1 - p(t)) \\ &= \begin{cases} -ap(t) < 0 & \text{if } p(t) > 1 \\ -a < 0 & \text{if } p(t) \leq 1. \end{cases} \end{aligned}$$

As $a > 0$, we have $\dot{p}(t) < 0$ for each $t \in [0, T]$, which implies that $p(\cdot)$ is strictly decreasing. ■

Claim: $p(0) > 1$.

Proof: Suppose not, i.e., $p(0) \leq 1$. Since $p(t)$ is strictly decreasing, we have $p(t) < 1$ for any $t \in (0, T]$. Since

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ [0,1] & \text{if } p(t) = 1 \\ 0 & \text{if } p(t) < 1, \end{cases}$$

we have $s^*(t) = 0$ for any $t \in [0, T]$.

Since $\dot{k}^*(t) = as^*(t)k^*(t)$ and $k^*(0) = k_0$, we have $k^*(t) = k_0$ for any $t \in [0, T]$. In particular,

$$k^*(T) = k_0 \qquad \underbrace{\Rightarrow\Leftarrow}_{\text{Contradiction!}} \qquad k^*(T) = k_T > k_0. \blacksquare$$

Case I: $p(T) = 0$

Since $p(t)$ is continuous and strictly decreasing such that $p(0) > 1$ and $p(T) = 0$, there is a unique $t^* \in (0, T)$ such that

$$p(t) \begin{cases} > 1 & \text{if } t \in [0, t^*) \\ = 1 & \text{if } t = t^* \\ < 1 & \text{if } t \in (t^*, T]. \end{cases}$$

Then,

$$s^*(t) = \begin{cases} 1 & t \in [0, t^*) \\ [0, 1] & t = t^* \\ 0 & t \in (t^*, T]. \end{cases}$$

From the preceding claim, we have

$$\dot{p} = \begin{cases} -ap(t) < 0 & \text{if } p(t) > 1, \\ -a < 0 & \text{if } p(t) \leq 1. \end{cases}$$

So,

$$\dot{p}(t) = \begin{cases} -ap(t) & \text{if } t \in [0, t^*), \\ -a & \text{if } t \in [t^*, T]. \end{cases}$$

On the interval $[t^*, T]$, $\dot{p} = -a \Rightarrow p(t) = -at + A$, where A is a constant. $p(T) = 0 \Rightarrow p(t) = a(T - t)$.

Since $p(t^*) = 1$, we have $a(T - t^*) = 1 \Rightarrow t^* = T - 1/a > 0$ by our assumption.

We can also easily see that $t^* < T$, as $t^* = T - 1/a$ and $a > 0$. Hence, $t^* = T - 1/a$ is well-defined.

On the interval $[0, t^*]$, $\dot{p} = -ap \Rightarrow p(t) = Be^{-at}$, where B is a constant.

Since $p(t^*) = 1$, we have $p(t) = e^{-a(t-t^*)}$. Since $t^* = T - 1/a > 0$, we obtain $p(t) = e^{a(T-t)-1}$.

Therefore, the solution candidate is given as follows:

$$s^*(t) = \begin{cases} 1 & \text{if } t \in [0, T - 1/a] \\ 0 & \text{if } t \in (T - 1/a, T], \end{cases}$$

$$k^*(t) = \begin{cases} k_0 e^{at} & \text{if } t \in [0, T - 1/a] \\ k_0 e^{aT-1} & \text{if } t \in (T - 1/a, T], \end{cases}$$

$$p(t) = \begin{cases} e^{a(T-t)-1} & \text{if } t \in [0, T - 1/a] \\ -a(t - T) & \text{if } t \in (T - 1/a, T]. \end{cases}$$

It remains to verify the terminal condition $k(T) \geq k_T$.

This reduces to $k_0 e^{aT-1} \geq k_T$, i.e., $k_0 e^{at^*} \geq k_T$. This inequality holds if and only if

$$t^* = T - \frac{1}{a} \geq \frac{1}{a} \ln \left(\frac{k_T}{k_0} \right).$$

Case II: $p(T) > 0$

Claim: $p(T) > 0$ implies $p(T) < 1$.

Proof: Suppose, on the contrary, that $p(T) \geq 1$. Since $p(t)$ is strictly decreasing, $p(t) > 1$ for any $t \in [0, T)$. This implies that $s^*(t) = 1$ for any $t \in [0, T]$.

So, we have $\dot{k}^*(t) = ak^*(t)$. Solving this differential equation together with $k^*(0) = k_0$, we obtain $k^*(t) = k_0 e^{at}$.

Thus, we have $k^*(T) = k_0 e^{aT} > k_T$ by our assumption. By the transversality condition, we have $p(T) = 0$, which contradicts the hypothesis that $p(T) > 0$. ■

So, $p(T) < 1$.

As we did for Case I: $p(T) = 0$, there is a unique $t^* \in (0, T)$ such that

$$p(t) \begin{cases} > 1 & \text{if } t \in [0, t^*) \\ = 1 & \text{if } t = t^* \\ < 1 & \text{if } t \in (t^*, T]. \end{cases}$$

Then,

$$s^*(t) = \begin{cases} 1 & t \in [0, t^*) \\ [0, 1] & t = t^* \\ 0 & t \in (t^*, T]. \end{cases}$$

Taking into account

$$\dot{p}(t) = \begin{cases} -ap(t) & \text{if } t \in [0, t^*), \\ -a & \text{if } t \in [t^*, T], \end{cases}$$

we obtain the solution candidate:

$$\begin{aligned} s^*(t) &= \begin{cases} 1 & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in (t^*, T] \end{cases} \\ k^*(t) &= \begin{cases} k_0 e^{at} & \text{if } t \in [0, t^*] \\ k_0 e^{at^*} & \text{if } t \in (t^*, T] \end{cases} \\ p(t) &= \begin{cases} e^{a(t^*-t)} & \text{if } t \in [0, t^*] \\ 1 - a(t - t^*) & \text{if } t \in (t^*, T] \end{cases} \end{aligned}$$

It remains to identify t^*.

From $k^*(T) = k_T$, it follows that $e^{at^*} = k_T/k_0$. So,

$$t^* = \frac{1}{a} \ln\left(\frac{k_T}{k_0}\right).$$

For t^* to be well-defined, we must have

$$\frac{1}{a} \ln\left(\frac{k_T}{k_0}\right) \leq T \Rightarrow k_T \leq k_0 e^{aT}.$$

This inequality is guaranteed by our assumption that $k_0 e^{aT} > k_T$.

$p(T) > 0 \Rightarrow 1 - a(T - t^*) > 0 \Rightarrow$

$$T - \frac{1}{a} < t^* = \frac{1}{a} \ln\left(\frac{k_T}{k_0}\right).$$

Considering Cases I and II together, there is only one solution candidate such that

$$s^*(t) = \begin{cases} 1 & \text{if } t \in [0, \bar{t}], \\ 0 & \text{if } t \in (\bar{t}, T], \end{cases}$$

where $\bar{t} = \max\{T - 1/a, (1/a)\ln(k_T/k_0)\}$.

For Case I, we obtain the value function:

$$V(k_0, k_T, T) = \int_{T-1/a}^{T} ak_0 e^{aT-1} dt = ak_0 e^{aT-1}[T-(T-1/a)] = k_0 e^{aT-1}.$$

So, we obtain the following relationships:

$$\frac{\partial V}{\partial k_0} = e^{aT-1} = p(0);\ \frac{\partial V}{\partial k_T} = 0 = -p(T);\ \text{and}\ \frac{\partial V}{\partial T} = ak_0 e^{aT-1} = H^*(T),$$

where

$$\begin{aligned} H^*(T) &\equiv H(T, k^*(T), s^*(T), p(T)) \\ &= (1 - s^*(T))ak^*(T) + p(T)as^*(T)k^*(T) \\ &= ak^*(T) = ak_0 e^{aT-1} = \partial V/\partial T. \end{aligned}$$

For Case II, we obtain the value function:

$$V(k_0, k_T, T) = \int_{t^*}^{T} ak_0 e^{at^*} dt = ak_0 e^{at^*}(T - t^*) = ak_T \left(T - \frac{1}{a} \ln k_T + \frac{1}{a} \ln k_0 \right),$$

where $t^* = (1/a)\ln(k_T/k_0)$ and $k_T = k_0 e^{at^*}$. Since $p(0) = e^{at^*} = k_T/k_0$, we obtain

$$\frac{\partial V}{\partial k_0} = \frac{k_T}{k_0} = p(0).$$

Since $p(T) = 1 - a(T - t^*)$, we have

$$\frac{\partial V}{\partial k_T} = a\left(T - \frac{1}{a} \ln k_T + \frac{1}{a} \ln k_0 \right) - 1 = a(T - t^*) - 1 = -P(T).$$

Finally, we obtain

$$\frac{\partial V}{\partial T} = ak_T = H^*(T),$$

where $H^*(T) = ak^*(T) = ak_0 e^{at^*} = ak_0(k_T/k_0) = ak_T$.

Sufficient Conditions (Ch. 9.7)

In many economic models, the Hamiltonian is not concave in (x, u). Arrow has suggested a weakening of this concavity condition. Define

$$\widehat{H}(t, x, p) = \max_{u \in U} H(t, x, u, p),$$

assuming that the maximum value is attained. The function $\widehat{H}(t, x, p)$ is called the **maximized Hamiltonian**.

Theorem (Arrow's Sufficient Conditions): Suppose that $(x^*(t), u^*(t))$ is an admissible pair in the standard end-constrained problem that satisfies all the requirements in the maximum principle, with $p(t)$ as the adjoint function, and with $p_0 = 1$. Suppose further that $\widehat{H}(t, x, p(t))$ is concave in x for every $t \in [t_0, t_1]$.

Then, $(x^*(t), u^*(t))$ solves the problem.

Proof: We skip the proof. ■

Example: Consider the problem

$$\max \int_0^2 (u^2 - x)dt, \ \dot{x} = u, \ x(0) = 0, \ x(2) \text{ free}, \ 0 \le u \le 1.$$

Let $H(t, x, u, p) = u^2 - x + pu$ be the Hamiltonian with $p_0 = 1$. We use the maximum principle to propose the solution candidate.

$$\dot{p} = -\partial H/\partial x = 1 \Rightarrow p(t) = t + A \qquad \underbrace{\Rightarrow}_{x(2) \text{ free} \Rightarrow p(2)=0} \qquad p(t) = t - 2.$$

Now we have

$$H(t, x, u, p) = u^2 - x + u(t - 2).$$

Since $\partial^2 H/\partial u^2 = 2$, $H(t, x, u, p)$ is a strictly convex function of u. Noting $u \in [0, 1]$, we have

$$u^*(t) \in \arg\max H(t, x^*, u, p) \Rightarrow u^*(t) = 0, \text{ or } 1.$$

Since $H(t, x^*, 0, p) = -x^*$ and $H(t, x^*, 1, p) = -x^* + t - 1$, we have

$$u^*(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ 1 & \text{if } t \in (1,2]. \end{cases} \underbrace{\Rightarrow}_{\dot{x}=u} \dot{x}^*(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ 1 & \text{if } t \in (1,2] \end{cases}$$

So,

$$x^*(t) = \begin{cases} A & \text{if } t \in [0,1] \\ t + B & \text{if } t \in (1,2] \end{cases}$$

where A and B are constants. Since $x^*(0) = 0$, we have $A = 0$. By the continuity of x^*, we must have

$$x^*(1) = 0 = 1 + B \Rightarrow B = -1.$$

Thus,

$$x^*(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ t - 1 & \text{if } t \in (1,2] \end{cases}$$

The maximized Hamiltonian is

$$\hat{H}(t,x,p(t)) = \max_{u\in[0,1]} u^2 - x + (t-2)u = \begin{cases} -x & \text{if } t \in [0,1] \\ -x + t - 1 & \text{if } t \in (1,2] \end{cases}$$

For each $t \in [0,2]$, the maximized Haimltonian is linear in x, hence concave in x.

Therefore, by Arrow's sufficient condition, the solution candidate we found using the maximum principle is a solution to the problem.

Variable Final Time (Ch. 9.8)

In the optimal control problems so far, the time interval has been fixed.

Yet, for some control problems in economics, the final time is also a variable to be chosen optimally, along with the function $u(t),\ t \in [t_0, t_1]$.

One such instance is the optimal extraction problem in the example below, where it is natural to choose for how long to extract the resource, as well as how fast.

The variable final time problem can be formulated as follows:

$$\max_{u,t_1} \int_{t_0}^{t_1} f(t,x,u)dt$$

$$\text{subject to} \quad \dot{x}(t) = g(t,x,u),\ x(t_0) = x_0, \begin{cases} (a) & x(t_1) = x_1 \\ (b) & x(t_1) \geq x_1 \\ (c) & x(t_1) \text{ free} \end{cases} \qquad (*)$$

The only difference from the standard end-constrained problem is that t_1 can now be chosen.

Theorem (The Maximum Principle with Variable Final Time)

Let $(x^*(t), u^*(t))$ be an admissible pair defined on $[t_0, t_1^*]$ which solves problem $(*)$ with t_1 free $(t_1 \in (t_0, \infty))$. Then, all the conditions in the maximum principle with standard end constraints are satisfied on $[t_0, t_1^*]$, and, in addition

$$H(t_1^*, x^*(t_1^*), u^*(t_1^*), p(t_1^*)) = 0 \quad (**).$$

An Example of Oil Extraction

Let $x(t)$ denote the amount of oil in a reservoir at time t.

Assume that at $t = 0$ the field contains K barrels of oil, so that $x(0) = K$.

If $u(t)$ is the rate of extraction, we have

$$x(t) = x(0) - \int_0^t u(\tau)d\tau = K - \int_0^t u(\tau)d\tau, \ \forall t \geq 0.$$

Differentiating this equation gives

$$\dot{x}(t) = -u(t),\ x(0) = K \quad (*)$$

Suppose that the market price of oil at time t is $q(t)$, so that the sales revenue per unit of time at t is $q(t)u(t)$.

Assume further that the cost C per unit of time depends on t, x, and u, so that $C = C(t, x, u)$.

The instantaneous profit per unit of time at time t is then

$$\pi(t, x(t), u(t)) = q(t)u(t) - C(t, x(t), u(t)).$$

If the discount rate is r, the total discounted profit over the interval $[0, T]$ is

$$\int_0^T [q(t)u(t) - C(t, x(t), u(t))]\, e^{-rt} dt.$$

It is natural to assume that $u(t) \geq 0$, and that $x(T) \geq 0$.

We further assume that $C = C(t, u)$ is independent of x and convex in u, i.e., $C''_{uu} > 0$. Thus, the problem we consider is

$$\max_{u,T} \int_0^T [q(t)u(t) - C(t, u(t))]\, e^{-rt} dt$$
$$\text{s.t.} \quad \dot{x}(t) = -u(t),\ x(0) = K,\ x(T) \geq 0,\ u(t) \geq 0.$$

Suppose $(x^*(t), u^*(t))$, defined on $[0, T^*]$, solves this problem.

The Hamiltonian with $p_0 = 1$ is

$$H(t, x, u, p) = [q(t)u(t) - C(t, u(t))]\, e^{-rt} + p(-u).$$

By the maximum principle, there exists a continuous function $p(t)$ such that

$$\begin{aligned}
&(i) \quad u^*(t) \in \arg\max_{u \geq 0} \left\{ [q(t)u(t) - C(t, u(t))]\, e^{-rt} - p(t)u \right\} \\
&(ii) \quad \dot{p}(t) = -\frac{\partial H}{\partial x} = 0,\ \ p(T^*) \geq 0, \text{ with } p(T^*) = 0 \text{ if } x^*(T^*) > 0 \\
&(iii) \quad [q(T^*)u^*(T^*) - C(T^*, u^*(T^*)]\, e^{-rT^*} = p(T^*)u^*(T^*)
\end{aligned}$$

Because $p(t)$ is continuous, it follows from (ii) that there exists a constant $\bar{p}$ such that $p(t) = \bar{p} \geq 0$.

Let

$$g(u) = [q(t)u(t) - C(t, u(t))]\, e^{-rt} - \bar{p}u.$$

Since $C(t, u)$ is convex in u and the other terms in $g(u)$ are linear in u, the function $g(u)$ is concave.

According to (i), $u^*(t)$ maximizes $g(u)$ subject to $u \geq 0$. So, if $u^*(t) = 0$ (i.e., a boundary solution), then $g^{'}(u^*(t)) = g^{'}(0) \leq 0$.

If $u^*(t) > 0$ (i.e., an interior solution), then $g^{'}(u^*(t)) = 0$.

Therefore, (i) implies

$$(iv) \quad \left[q(t) - C_u^{'}(t, u^*(t))\right] e^{-rt} - \bar{p} \leq 0 \; (= 0 \text{ if } u^*(t) > 0)$$

Because $g(\cdot)$ is concave, Condition (iv) is also sufficient for (i) to hold.

At any time t where $u^*(t) > 0$, equation (iv) implies that

$$(v) \quad q(t) - C_u^{'}(t, u^*(t)) = \bar{p}e^{rt}.$$

The left-hand side of equation (v) is the marginal profit from extraction, $\partial\pi/\partial u$.

Putting $t = T^*$ in (v), and using (iii), we deduce that, if $u^*(T^*) > 0$, then

$$(vi) \quad C_u^{'}(T^*, u^*(T^*)) = \frac{C(T^*(, u^*(T^*))}{u^*(T^*)}.$$

This means that we terminate extraction at a time when the marginal cost of extraction is equal to average cost. If the problem has a solution with $u^*(t) > 0$, then (v) and (vi) both hold. If $C(T^*, 0) > 0$, then $u^*(T^*) > 0$, because $u^*(T^*) = 0$ contradicts (iii).

Current Value Formulations (Ch. 9.9)

Consider the standard end-constrained problem:

$$\max \int_{t_0}^{t_1} f(t,x,u)e^{-rt}dt, \;\; u \in U \subseteq \mathbb{R}, \;\; \forall t,$$
$$\dot{x}(t) = g(t, x(t), u(t)), \;\; x(t_0) = x_0$$

with one of the following conditions imposed

$$(a) \; x(t_1) = x_1, \; (b) \; x(t_1) \geq x_1, \text{ or (c) } x(t_1) \text{ free} \quad (1)$$

where t_0, t_1, x_0, and x_1 are fixed numbers and U is the fixed control region.

The ordinary Hamiltonian is $H = p_0 f(t,x,u)e^{-rt} + pg(t,x,u)$. We multiply it by e^{rt} to obtain the **current value Hamiltonian** H^c:

$$H^c = He^{rt} = p_0 f(t,x,u) + e^{rt} pg(t,x,u).$$

Introducing $\lambda = e^{rt}p$ as the **current value shadow price** for the problem, one can write H^c in the form (where we put $p_0 = \lambda_0$)

$$H^c(t, x, u, \lambda) = \lambda_0 f(t, x, u) + \lambda g(t, x, u).$$

$$\lambda = e^{rt}p \Rightarrow \dot{\lambda} = re^{rt}p + e^{rt}\dot{p} = r\lambda + e^{rt}\dot{p} \Rightarrow \dot{p} = e^{-rt}(\dot{\lambda} - r\lambda).$$

$$H^c = He^{rt} \Rightarrow \frac{\partial H^c}{\partial x} = e^{rt}\left(\frac{\partial H}{\partial x}\right)$$

So,

$$\dot{p} = -\frac{\partial H}{\partial x} \Rightarrow \dot{\lambda} - r\lambda = -\frac{\partial H^c}{\partial x}.$$

Theorem: The Maximum Principle: Current Value Formulation

Suppose that the admissible pair $(x^*(t), u^*(t))$ solves Problem (1) and let H^c be the current value Hamiltonian. Then, there exists a continuous function $\lambda(t)$ and a number λ_0, either 0 or 1, such that for all $t \in [t_0, t_1]$, we have $(\lambda_0, \lambda(t)) \neq (0, 0)$, and:

(A) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ for $u \in U$

(B) $\dot{\lambda}(t) - r\lambda(t) = -\frac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$

(C) The transversality conditions are: $(a^{'})$ $\lambda(t_1)$ no condition; $(b^{'})$ $\lambda(t_1) \geq 0$ and $x^*(t_1) > x_1 \Rightarrow \lambda(t_1) = 0$; and $(c^{'})$ $\lambda(t_1) = 0$.

Sufficiency for Current Value Hamiltonian

The conditions in the maximum principle via current value formulation are sufficient for optimality if $\lambda_0 = 1$ and

Mangasarian: $H^c(t, x, u, \lambda(t))$ is concave in (x, u)

or (more generally)

Arrow: $\widehat{H}^c(t, x, \lambda(t)) = \max_{u \in U} H^c(t, x, u, \lambda(t))$ is concave in x.

Example: Consider the problem

$$\max_{u \geq 0} \int_0^{20} (4K - u^2) e^{-0.25t} dt, \ \dot{K} = -0.25K + u, \ K(0) = K_0, \ K(20) \text{ free}$$

$K(t)$ is the value of a firm's capital stock, which depreciates at the constant proportional rate 0.25 per unit of the time, whereas $u(t)$ is gross investment, which costs $u(t)^2$ because the marginal cost of investment increases.

Profits $(4K - u^2)$ are discounted at the constant proportional rate 0.25 per unit of the time.

Let $H^c = 4K - u^2 + \lambda(-0.25K + u)$ with $\lambda_0 = 1$. So,

$$\begin{aligned} \partial H^c/\partial u &= -2u + \lambda \\ \partial H^c/\partial K &= 4 - 0.25\lambda. \end{aligned}$$

Assuming that $u^*(t) > 0$,

$$\partial (H^c)^*/\partial u = 0 \Rightarrow u^*(t) = 0.5\lambda(t).$$

The adjoint function λ satisfies

$$\dot{\lambda} - 0.25\lambda = -\partial (H^c)^*/\partial K = -4 + 0.25\lambda \text{ and } \underbrace{\lambda(20) = 0}_{\because K(20) \text{ free}}.$$

Using the formula for the first-order linear differential equations,

$$\dot{\lambda} - 0.5\lambda = -4 \Rightarrow \lambda = Ce^{0.5t} + \frac{-4}{-0.5} \underbrace{\Rightarrow}_{\lambda(20)=0} \lambda = 8(1 - e^{0.5t-10}).$$

and

$$u^*(t) = 0.5\lambda = 4(1 - e^{0.5t-10}).$$

Plugging $u = 4(1 - e^{0.5t-10})$ into $\dot{K} = -0.25K + u$, we obtain

$$\dot{K} + 0.25K = 4(1 - e^{0.5t-10}).$$

Using the formula for the first-order linear differential equations,

$$\begin{aligned} K^*(t) &= Ce^{-0.25t} + e^{-0.25t} \int e^{0.25t} 4(1 - e^{0.5t-10}) dt \\ &= Ce^{-0.25t} + 16 - \frac{16}{3} e^{0.5t-10} \end{aligned}$$

Noting $K^*(0) = K_0$, we obtain

$$K^*(t) = \left(K_0 - 16 + \frac{16}{3} e^{-10} \right) e^{-0.25t} + 16 - \frac{16}{3} e^{0.5t-10}.$$

To justify the solution we have obtained, it suffices to show that H^c is concave in (K, u). We compute the Hessian matrix of H^c:

$$H(K,u) = \begin{pmatrix} \partial^2 H^c/\partial K^2 & \partial^2 H^c/\partial u \partial K \\ \partial^2 H^c/\partial K \partial u & \partial^2 H^c/\partial u^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since $H(K, u)$ is negative semidefinite for all (K, u), H^c is concave in (K, u).