Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 7: Differential Equations II (Ch. 6)

Second-Order Differential Equations (Ch. 6)

In an important area of dynamic optimization called the **calculus of variations**, the first-order condition for optimality involves a second-order differential equation.

The typical second-order differential equation takes the form

$$\ddot{x} = F(t, x, \dot{x}) \quad (*)$$

where F is a given fixed function, $x = x(t)$ is the unknown function, and $\dot{x} = dx/dt$.

The new feature here is the presence of the second derivative $\ddot{x} = d^2x/dt^2$.

A **solution** of $(*)$ on an interval I is a twice differentiable function that satisfies the equation.

Example: $\ddot{x} = k$ (k is a constant)

Taking integration on the equation, we obtain

$$\dot{x} = \int k dt = kt + A,$$

where A is some constant. Taking further integration on the equation above, we obtain

$$\int (kt + A) dt = \frac{k}{2} t^2 + At + B,$$

where B is some constant.

Differential Equations where x or t is Missing

Case 1: $\ddot{x} = F(t, \dot{x})$

In this case, x is missing. We introduce the new variable $u = \dot{x}$. Then, Case 1 becomes $\dot{u} = F(t, u)$, which is a first-order differential equation.

Case 2: $\ddot{x} = F(x, \dot{x})$

In this case, t is not explicitly present in the equation and the equation is called **autonomous**.

Example: $\ddot{x} = \dot{x} + t$.

Define $u = \dot{x}$. Then, the equation is transformed to $\dot{u} = u + t$.

This first-order differential equation has the general solution

$$u = Ae^t - t - 1,$$

where A is a constant. This is equivalent to

$$\dot{x} = Ae^t - t - 1.$$

Integrating this equation, we obtain

$$x = \int (Ae^t - t - 1)dt = Ae^t - \frac{1}{2}t^2 - t + B,$$

where B is a constant.

Assume that $x(0) = 1$ and $\dot{x}(0) = 2$. First,

$$\dot{x}(0) = A - 1 = 2 \Rightarrow A = 3.$$

Second,

$$x(0) = A + B = 1 \underbrace{\Rightarrow}_{A=3} B = -2.$$

Then,

$$x = 3e^t - \frac{1}{2}t^2 - t - 2.$$

Detour: Complex Numbers (Ch. B.3)

Simple quadratic equations like $x^2 + 1 = 0$ and $x^2 + 4x + 8 = 0$ have no solution within the real number system.

The standard formula for solving the equation $x^2 + 4x + 8 = 0$ yields $x = -2 \pm \sqrt{-4} = -2 \pm 2\sqrt{-1}$.

By pretending that $\sqrt{-1}$ is a number i whose square is -1, we make i a solution of the equation $i^2 = -1$.

Mathematical formalism regard complex numbers as 2-vectors (a, b).

We usually write this complex number as $a + bi$, where a and b are real numbers.

The operations of addition, subtraction, and multiplication are defined by

$$
\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(a+bi)-(c+di) &= (a-c)+(b-d)i \\
(a+bi)(c+di) &= (ac-bd)+(ad+bc)i,
\end{aligned}
$$

respectively. The division of two complex numbers is

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Trigonometric Form of Complex Numbers

Each complex number $z = x + yi = (x, y)$ can be represented by a point in the plane.

We could use **polar coordinates**. Let θ be the angle (measured in radians) between the positive real axis and the vector from the origin to the point (x, y), and let r be the distance from the origin to the same point.

Then $x = r\cos\theta$ and $y = r\sin\theta$, so

$$z = x + yi = r(\cos\theta + i\sin\theta).$$

The distance from the origin to the point (x, y) is $r = \sqrt{x^2 + y^2}$. This is called the **modulus** of the complex number, denoted by $|z|$.

If $z = x + iy$, then the **complex conjugate** of z is defined as $\bar{z} = x - iy$. We see that $\bar{z}z = x^2 + y^2 = |z|^2$, where $|z|$ is the modulus of z.

Multiplication of complex numbers have a neat geometric interpretation:

$$r_1(\cos\theta_1+i\sin\theta_1)r_2(\cos\theta_2+i\sin\theta_2) = r_1 r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

Division of complex numbers have a neat geometric interpretation:

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

Second-Order Linear Differential Equations (Ch. 6.2)

The general second-order linear differential equation is

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t) \quad (*)$$

where $a(t), b(t)$, and $f(t)$ are all continuous functions of t on some interval I.

Let us begin with the **homogeneous** equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0 \quad (**)$$

Assume that $u_1 = u_1(t)$ and $u_2 = u_2(t)$ both satisfy $(**)$. Define $x = Au_1 + Bu_2$ where A and B are constants. Then,

$$\begin{aligned}\dot{x} &= A\dot{u}_1 + B\dot{u}_2 \\ \ddot{x} &= A\ddot{u}_1 + B\ddot{u}_2\end{aligned}$$

Substituting these into $(**)$, we obtain

$$\begin{aligned}\ddot{x} + a(t)\dot{x} + b(t)x &= A\ddot{u}_1 + B\ddot{u}_2 + a(t)(A\dot{u}_1 + B\dot{u}_2) + b(t)(Au_1 + Bu_2) \\ &= A[\ddot{u}_1 + a(t)\dot{u}_1 + b(t)u_1] + B[\ddot{u}_2 + a(t)\dot{u}_2 + b(t)u_2] \\ &= 0.\end{aligned}$$

This is true for all choices of A and B.

Equation $(*)$ is called a **nonhomogeneous equation**, and $(**)$ is the homogeneous equation associated with it.

Suppose we are able to find **some particular solution** $u^* = u^*(t)$ of $(*)$.

Assume further that $x(t)$ is an arbitrary solution to $(*)$. Then, define $v = v(t) = x(t) - u^*(t)$. Then,

$$\begin{aligned} \dot{v} &= \dot{x} - \dot{u}^* \\ \ddot{v} &= \ddot{x} - \ddot{u}^*. \end{aligned}$$

$$\begin{aligned}\ddot{v}+a(t)\dot{v}+b(t)v &= \ddot{x}-\ddot{u}^*+a(t)(\dot{x}-\dot{u}^*)+b(t)(x-u^*)\\ &= [\ddot{x}+a(t)\dot{x}+b(t)x]-[\ddot{u}^*+a(t)\dot{u}^*+b(t)u^*]\\ &= f(t)-f(t)=0.\end{aligned}$$

Thus, $x(t)-u^*(t)$ is a solution to the homogeneous equation $(**)$.

Since we have argued that the solution to $(**)$ is of the form $Au_1(t)+Bu_2(t)$,

$$x(t)-u^*(t)=Au_1(t)+Bu_2(t),$$

where $u_1(t)$ and $u_2(t)$ are two nonproportional solutions to $(**)$, and A and B are arbitrary constants.

Theorem: (a) The **general solution** of the homogeneous differential equation $(**)$ is

$$x = Au_1(t) + Bu_2(t),$$

where $u_1(t)$ and $u_2(t)$ are any two solutions that are not proportional, and A and B are arbitrary constants.

(b) The **general solution** of the nonhomogeneous differential equation $(*)$ is

$$x = Au_1(t) + Bu_2(t) + u^*(t),$$

where $Au_1(t) + Bu_2(t)$ is the general solution of the associated homogeneous equation, and $u^*(t)$ is any **particular solution** of $(*)$.

Constant Coefficients (Ch. 6.3)

Consider

$$\ddot{x} + a\dot{x} + bx = 0, \quad (**)$$

where a and b are arbitrary constants, and $x = x(t)$ is the unknown function.

It **seems** a good idea to try possible solutions x with the property that $x, \dot{x}$, and $\ddot{x}$ are all constant multiples of each other.

The exponential function $x = e^{rt}$ has this property because $\dot{x} = re^{rt} = rx$ and $\ddot{x} = r^2 e^{rt} = r^2 x$.

So, we try adjusting the constant r in order that $x = e^{rt}$ satisfies $(**)$. This requires us to arrange that $r^2 e^{rt} + are^{rt} + be^{rt} = 0$. Therefore, e^{rt} satisfies $(**)$ if and only if r satisfies

$$r^2 + ar + b = 0.$$

This is the **characteristic equation** of the differential equation $(**)$.

If $a^2 - 4b \geq 0$, the characteristic equation has two real roots:

$$\begin{aligned} r_1 &= -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b} \\ r_2 &= -\frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b} \end{aligned}$$

Theorem: The general solution of $\ddot{x} + a\dot{x} + bx = 0$ depends on the roots of the characteristic equation $r^2 + ar + b = 0$ as follows:

(I) If $a^2 - 4b > 0$, when there are two distinct real roots, then

$$x = Ae^{r_1 t} + Be^{r_2 t}, \quad \text{where } r_{1,2} = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b}.$$

(II) If $a^2 - 4b = 0$, when there is a double real root, then

$$x = (A + Bt)e^{rt}, \quad \text{where } r = -\frac{1}{2}a.$$

(III) If $a^2 - 4b < 0$, when there are two complex roots, then

$$x = e^{\alpha t}(A\cos\beta t + B\sin\beta t), \quad \text{where } \alpha = -\frac{1}{2}a, \ \beta = \sqrt{b - \frac{1}{4}a^2}.$$

Example: $\ddot{x} - 3x = 0$

The characteristic equation $r^2 - 3 = 0$ has two real roots: $r_1 = -\sqrt{3}$ and $r_2 = \sqrt{3}$.

Then, the general solution is

$$x = Ae^{-\sqrt{3}t} + Be^{\sqrt{3}t}.$$

Example: $\ddot{x} - 4\dot{x} + 4x = 0$

The characteristic equation $r^2 - 4r + 4 = 0$ has a double real root: $r = 2$.

Hence, the general solution is

$$x = (A + Bt)e^{2t}.$$

Example: $\ddot{x} - 6\dot{x} + 13x = 0$

The characteristic equation $r^2 - 6r + 13 = 0$ has two complex roots because $(r-3)^2 + 4 = 0$.

Then, we compute

$$\begin{aligned} \alpha &= -\frac{1}{2}a = 3 \\ \beta &= \sqrt{13 - \frac{1}{4}(-6)^2} = \sqrt{13-9} = 2. \end{aligned}$$

So, the general solution is

$$x = e^{3t}(A\cos 2t + B\sin 2t).$$

The Nonhomogeneous Equation (Ch. 6.3)

Consider the nonhomogeneous eqaution

$$\ddot{x} + a\dot{x} + bx = f(t), \quad (*)$$

where $f(t)$ is an arbitrary continuous function.

If $b = 0$ in $(*)$, then the term in x is missing and the substitution $u = \dot{x}$ transforms the equation into a linear equation of first order.

So, we may assume $b \neq 0$.

Case (A): $f(t) = A$ (constant)

We check to see if $(*)$ has a solution that is constant, $u^* = c$.

Then, $\dot{u}^* = \ddot{u}^* = 0$. So, the equation reduces to $bc = A$. Hence, $c = A/b$.

For $b \neq 0$:

$\ddot{x} + a\dot{x} + bx = A$ has a particular solution $u^* = A/b$.

Case (B): $f(t)$ is polynominal

Suppose $f(t)$ is a polynomial of degree n. Then, a **reasonable** guess is that $(*)$ has a particular solution that is also a polynomial of degree n, of the form $u^* = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$.

We determine the undetermined coefficients $A_n, A_{n-1}, \ldots, A_0$ by requiring u^* to satisfy $(*)$.

Example: $\ddot{x} - 4\dot{x} + 4x = t^2 + 2$

Let $u^* = At^2 + Bt + C$. Then,

$$\begin{aligned} \dot{u}^* &= 2At + B \\ \ddot{u}^* &= 2A. \end{aligned}$$

Plugging these into the LHS of the equation, we obtain

$$2A - 4(2At + B) + 4(At^2 + Bt + C) = 4At^2 + 4(B - 2A)t + (2A - 4B + 4C).$$

Then, we must have $A = 1/4$; $B = 2A = 1/2$; and $1/2 - 2 + 4C = 2$, which implies $4C = 7/2$, which further implies $C = 7/8$. Hence,

$$u^* = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{7}{8}.$$

Case (C): $f(t) = pe^{qt}$

It seems natural to try a particular solution of the form $u^* = Ae^{qt}$. Then,

$$\dot{u}^* = Aqe^{qt} \text{ and } \ddot{u}^* = Aq^2e^{qt}.$$

$$\ddot{x} + a\dot{x} + bx = f(t) \Rightarrow Ae^{qt}(q^2 + aq + b) = pe^{qt}.$$

Hence, if $q^2 + aq + b \neq 0$,

$$u^* = \frac{p}{q^2 + aq + b}e^{qt}$$

is a particular solution to $\ddot{x} + a\dot{x} + bx = f(t)$.

The condition $q^2 + aq + b \neq 0$ means that q is not a solution of the characteristic equation.

Case (D): $f(t) = p \sin rt + q \cos rt$

Let $u^* = A \sin rt + B \cos rt$ and adjust the constants A and B so that the coefficients of $\sin rt$ and $\cos rt$ match.

Example: $\ddot{x} - 4\dot{x} + 4x = 2\cos 2t$.

Let $u^* = A\sin 2t + B\cos 2t$. Then, we have

$$\dot{u}^* = 2A\cos 2t - 2B\sin 2t \text{ and } \ddot{u}^* = -4A\sin 2t - 4B\cos 2t.$$

Therefore,

$$\begin{aligned} & \ddot{x} + a\dot{x} + bx = f(t) \\ \Leftrightarrow\ & -4A\sin 2t - 4B\cos 2t - 4(2A\cos 2t - 2B\sin 2t) \\ & +4(A\sin 2t + B\cos 2t) = 2\cos 2t \\ \Leftrightarrow\ & 8B\sin 2t - 8A\cos 2t = 2\cos 2t \end{aligned}$$

This implies that $A = -1/4$ and $B = 0$. Thus,

$$u^* = -\frac{1}{4}\sin 2t.$$

Stability for Linear Equations (Ch. 6.4)

Question: Will small changes in the initial conditions have any effect on the long-run behavior of the solution to a given system of differential equations or will the effect "die out" as $t \to \infty$?

In the latter case, the system is called **asymptocially stable**.

On the other hand, if small changes in the initial conditions might lead to significant differences in the behavior of the solution in the long run, then the system is **unstable**.

Consider the second-order nonhomogeneous differential equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t). \quad (*)$$

Recall that the general solution of $(*)$ is $x = Au_1(t) + Bu_2(t) + u^*(t)$, where $Au_1(t) + Bu_2(t)$ is the general solution of the associated homogeneous equation (with $f(t)$ replaced by zero), and $u^*(t)$ is a particular solution of the nonhomogeneous equation $(*)$.

Definition: $(*)$ is called **globally asymptotically stable** if every solution $Au_1(t)+Bu_2(t)$ of the associated homogeneous equation tends to 0 as $t \to \infty$ for all values of A and B. Then, the effect of the initial conditions "dies out" as $t \to \infty$.

Examples:

(1) $\ddot{x} + 2\dot{x} + 5x = e^t$.

The corresponding characteristic equation is $r^2 + 2r + 5 = 0$, with complex roots $r_1 = -1 + 2i, r_2 = -1 - 2i$, so $u_1 = e^{-t}\cos 2t$ and $u_2 = e^{-t}\sin 2t$ are linearly independent solutions of the homogeneous equation.

Since $\cos 2t$ and $\sin 2t$ are both less than or equal to 1 in absolute value and $e^{-t} \to 0$ as $t \to \infty$, u_1 and u_2 tend to 0 as $t \to \infty$.

So, the equation is globally asymptotically stable.

(2) $\ddot{x} + \dot{x} - 2x = 3t^2 + 2$.

The corresponding characteristic equation is $r^2 + r - 2 = 0$, with two real roots $r_1 = 1, r_2 = -2$, so $u_1 = e^t$ and $u_2 = e^{-2t}$ are linearly independent solutions of the homogeneous equation.

Since $u_1 = e^t$ does not tend to 0 as $t \to \infty$, the equation is **not** globally asymptotically stable.

Theorem: The equation $\ddot{x}+a\dot{x}+bx=f(t)$ is globally asymptotically stable if and only if both roots of the characteristic equation $r^2+ar+b=0$ have negative real parts.

Proof: We prove this by considering the following three cases:

Case I: $\frac{1}{4}a^2-b>0$

In this case, we have $x=Ae^{r_1t}+Be^{r_2t}$, where $r_1, r_2=-\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2-b}$. Then, $Ae^{r_1t}+Be^{r_2t}\to 0$ as $t\to\infty$ for all values of A and B if and only if $e^{r_1t}\to 0$ and $e^{r_2t}\to 0$, which is equivalent to $r_1<0$ and $r_2<0$.

Case II: $\frac{1}{4}a^2 - b = 0$

In this case, we have $x = (A + Bt)e^{rt}$, where $r = -\frac{1}{2}a$. Then, $(A + Bt)e^{rt} \to 0$ as $t \to \infty$ for all values of A and B if and only if $te^{rt} \to 0$ as $t \to \infty$, which is equivalent to $r < 0$.

Case III: $\frac{1}{4}a^2 - b < 0$

In this case, we have $r_1, r_2 = \alpha \pm i\beta$ so that $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$, where $\alpha = -\frac{1}{2}a, \beta = \sqrt{b - \frac{1}{4}a^2}$.

Since $\cos\beta t$ and $\sin\beta t$ are both less than or equal to 1 in absolute value, $x \to 0$ as $t \to \infty$ for all values of A and B if and only if $e^{\alpha t} \to 0$ as $t \to \infty$, which is equivalent to $\alpha < 0$. ■

Corollary: $\ddot{x} + a\dot{x} + bx = f(t)$ is globally asymptotically stable if and only if $a > 0$ and $b > 0$.

Proof: The two roots (real or complex) r_1 and r_2 of the quadratic characteristic equation $r^2 + ar + b = 0$ have the property that $r^2 + ar + b = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$.

Hence, $a = -r_1 - r_2$ and $b = r_1 r_2$. In Cases (I) and (II) in the previous theorem, the system is globally asymptotically stable if and only if $r_1 < 0$ and $r_2 < 0$, which is equivalent to $a > 0$ and $b > 0$.

In Case (III) in the previous theorem, we have $r_1, r_2 = \alpha \pm i\beta$. Then, the system is globally asymptotically stable if and only if $\alpha < 0$. Then, $a = -(r_1 + r_2) = -2\alpha > 0$ and $b = r_1 r_2 = \alpha^2 + \beta^2 > 0$. ■

Example:

$$\ddot{\nu} + \left(\mu - \frac{\lambda}{a}\right)\dot{\nu} + \lambda\gamma\nu = -\frac{\lambda}{a}\dot{b}(t),$$

where μ, λ, γ, and a are constants, and $\dot{b}(t)$ is a fixed function.

By the previous corollary, the equation is globally asymptotically stable if and only if $\mu > \frac{\lambda}{a}$ and $\lambda\gamma > 0$.