Mathematical Methods for Economic Dynamics (ECON 696)

Lecture 7: Differential Equations III (Ch. 6) and Calculus of Variations (Ch. 8)

Stability for Linear Equations (Ch. 6.4)

Question: Will small changes in the initial conditions have any effect on the long-run behavior of the solution to a given system of differential equations or will the effect "die out" as $t \to \infty$?

In the latter case, the system is called **asymptocially stable**.

On the other hand, if small changes in the initial conditions might lead to significant differences in the behavior of the solution in the long run, then the system is **unstable**.

Consider the second-order nonhomogeneous differential equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t). \quad (*)$$

Recall that the general solution of $(*)$ is $x = Au_1(t) + Bu_2(t) + u^*(t)$, where $Au_1(t) + Bu_2(t)$ is the general solution of the associated homogeneous equation (with $f(t)$ replaced by zero), and $u^*(t)$ is a particular solution of the nonhomogeneous equation $(*)$.

Definition: $(*)$ is called **globally asymptotically stable** if every solution $Au_1(t)+Bu_2(t)$ of the associated homogeneous equation tends to 0 as $t \to \infty$ for all values of A and B. Then, the effect of the initial conditions "dies out" as $t \to \infty$.

Examples:

(1) $\ddot{x} + 2\dot{x} + 5x = e^t$.

The corresponding characteristic equation is $r^2 + 2r + 5 = 0$, with complex roots $r_1 = -1 + 2i, r_2 = -1 - 2i$, so $u_1 = e^{-t}\cos 2t$ and $u_2 = e^{-t}\sin 2t$ are linearly independent solutions of the homogeneous equation.

Since $\cos 2t$ and $\sin 2t$ are both less than or equal to 1 in absolute value and $e^{-t} \to 0$ as $t \to \infty$, u_1 and u_2 tend to 0 as $t \to \infty$.

So, the equation is globally asymptotically stable.

(2) $\ddot{x} + \dot{x} - 2x = 3t^2 + 2$.

The corresponding characteristic equation is $r^2 + r - 2 = 0$, with two real roots $r_1 = 1, r_2 = -2$, so $u_1 = e^t$ and $u_2 = e^{-2t}$ are linearly independent solutions of the homogeneous equation.

Since $u_1 = e^t$ does not tend to 0 as $t \to \infty$, the equation is **not** globally asymptotically stable.

Theorem: The equation $\ddot{x}+a\dot{x}+bx=f(t)$ is globally asymptotically stable if and only if both roots of the characteristic equation $r^2+ar+b=0$ have negative real parts.

Proof: We prove this by considering the following three cases:

Case I: $\frac{1}{4}a^2-b>0$

In this case, we have $x=Ae^{r_1 t}+Be^{r_2 t}$, where $r_1,r_2=-\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2-b}$. Then, $Ae^{r_1 t}+Be^{r_2 t}\to 0$ as $t\to\infty$ for all values of A and B if and only if $e^{r_1 t}\to 0$ and $e^{r_2 t}\to 0$, which is equivalent to $r_1<0$ and $r_2<0$.

Case II: $\frac{1}{4}a^2 - b = 0$

In this case, we have $x = (A + Bt)e^{rt}$, where $r = -\frac{1}{2}a$. Then, $(A + Bt)e^{rt} \to 0$ as $t \to \infty$ for all values of A and B if and only if $te^{rt} \to 0$ as $t \to \infty$, which is equivalent to $r < 0$.

Case III: $\frac{1}{4}a^2 - b < 0$

In this case, we have $r_1, r_2 = \alpha \pm i\beta$ so that $x = e^{\alpha t}(A \cos \beta t + B \sin \beta t)$, where $\alpha = -\frac{1}{2}a, \beta = \sqrt{b - \frac{1}{4}a^2}$.

Since $\cos \beta t$ and $\sin \beta t$ are both less than or equal to 1 in absolute value, $x \to 0$ as $t \to \infty$ for all values of A and B if and only if $e^{\alpha t} \to 0$ as $t \to \infty$, which is equivalent to $\alpha < 0$. ■

Corollary: $\ddot{x} + a\dot{x} + bx = f(t)$ is globally asymptotically stable if and only if $a > 0$ and $b > 0$.

Proof: The two roots (real or complex) r_1 and r_2 of the quadratic characteristic equation $r^2 + ar + b = 0$ have the property that $r^2 + ar + b = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$.

Hence, $a = -r_1 - r_2$ and $b = r_1 r_2$. In Cases (I) and (II) in the previous theorem, the system is globally asymptotically stable if and only if $r_1 < 0$ and $r_2 < 0$, which is equivalent to $a > 0$ and $b > 0$.

In Case (III) in the previous theorem, we have $r_1, r_2 = \alpha \pm i\beta$. Then, the system is globally asymptotically stable if and only if $\alpha < 0$. Then, $a = -(r_1 + r_2) = -2\alpha > 0$ and $b = r_1 r_2 = \alpha^2 + \beta^2 > 0$. $\blacksquare$

Example:

$$\ddot{\nu} + \left(\mu - \frac{\lambda}{a}\right)\dot{\nu} + \lambda\gamma\nu = -\frac{\lambda}{a}\dot{b}(t),$$

where μ, λ, γ, and a are constants, and $\dot{b}(t)$ is a fixed function.

By the previous corollary, the equation is globally asymptotically stable if and only if $\mu > \frac{\lambda}{a}$ and $\lambda\gamma > 0$.

Introduction: How Much Should a Nation Save?

Example: Consider an economy evolving over time where $K = K(t)$ denotes the capital stock, $C = C(t)$ consumption, and $Y = Y(t)$ net national product at time t. Suppose that

$$Y = f(K), \quad \text{where } f^{'}(K) > 0 \text{ and } f^{''}(K) \leq 0.$$

For each t, assume that

$$f(K(t)) = C(t) + \dot{K}(t),$$

which means that output, $Y(t) = f(K(t))$, is divided between consumption, $C(t)$, and investment, $\dot{K}(t)$.

Let $K(0) = K_0$ be a historically given capital stock existing "today" at $t = 0$ and suppose that there is a fixed planning period $[0, T]$.

For each choice of investment function $\dot{K}(t)$ on the interval $[0, T]$, capital is fully determined by

$$K(t) = K_0 + \int_0^t \dot{K}(\tau) d\tau,$$

and in turn, determines $C(t)$.

Assume that the society has a utility function U, where $U(C)$ is the utility (flow) the country enjoys when the total consumption is C. Suppose also that

$$U^{'}(C) > 0 \text{ and } U^{''}(C) < 0.$$

For each $t \geq 0$, we multiply $U(C(t))$ by the discount factor e^{-rt}.

Frank Ramsey (1928) argues that r must be zero.

The goal of investment policy is to find the path of capital $K = K(t)$, with $K(0) = K_0$, that maximizes

$$\int_0^T U(C(t))e^{-rt}dt = \int_0^T U(f(K(t)) - \dot{K}(t))e^{-rt}dt.$$

Usually, some **terminal condition** on $K(t)$ is imposed. For example, $K(T) = K_T$ where K_T is given.

One possibility is $K_T = 0$, with no capital left for times after T.

The Euler Equation (Ch. 8.2)

More generally, we consider the following problem:

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x}) dt \ \text{ subject to } x(t_0) = x_0 \text{ and } x(t_1) = x_1 \quad (*)$$

Here F is a given C^2 function of three variables, whereas t_0, t_1, x_0, and x_1 are given numbers.

Leonhard Euler (1744) proved that a function $x(t)$ can only solve problem $(*)$ if $x(t)$ satisfies the differential equation:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \quad (**)$$

where $\partial F / \partial x = F_2'(t, x, \dot{x})$ and $\partial F / \partial \dot{x} = F_3'(t, x, \dot{x})$.

Equation $(**)$ is called the **Euler equation**.

Assuming that $x = x(t)$ is C^2, we find that

$$\frac{d}{dt}\left(\frac{\partial F(t,x,\dot{x})}{\partial \dot{x}}\right) = \frac{\partial^2 F}{\partial t \partial \dot{x}} \cdot 1 + \frac{\partial^2 F}{\partial x \partial \dot{x}} \cdot \dot{x} + \frac{\partial^2 F}{\partial \dot{x} \partial \dot{x}} \cdot \ddot{x}$$

Inserting this into $(**)$, we obtain

$$\frac{\partial^2 F}{\partial \dot{x} \partial \dot{x}} \cdot \ddot{x} + \frac{\partial^2 F}{\partial x \partial \dot{x}} \cdot \dot{x} + \frac{\partial^2 F}{\partial t \partial \dot{x}} - \frac{\partial F}{\partial x} = 0.$$

This equation can be written as

$$F_{33}^{''}\ddot{x} + F_{32}^{''}\dot{x} + F_{31}^{''} - F_{2}^{'} = 0.$$

So, the Euler equation is a differential equation of the second-order (if $F_{33}^{''} \neq 0$).

Example: Consider

$$\max \int_0^2 (4 - 3x^2 - 16\dot{x} - 4(\dot{x})^2)e^{-t}dt, \;\; x(0) = -8/3, \;\; x(2) = 1/3.$$

Set $F(t, x, \dot{x}) = (4 - 3x^2 - 16\dot{x} - 4(\dot{x})^2)e^{-t}$. So,

$$\begin{aligned} \frac{\partial F}{\partial x} &= -6xe^{-t} \\ \frac{\partial F}{\partial \dot{x}} &= (-16 - 8\dot{x})e^{-t}. \end{aligned}$$

Next, compute

$$\frac{d}{dt}\left[(-16 - 8\dot{x})e^{-t}\right] = 16e^{-t} + 8\dot{x}e^{-t} - 8\ddot{x}e^{-t}.$$

By the Euler equation, we have

$$-6xe^{-t} - 16e^{-t} + 8\ddot{x}e^{-t} - 8\dot{x}e^{-t} = 0 \Rightarrow \ddot{x} - \dot{x} - \frac{3}{4}x = 2.$$

This is a second-order linear differential equation with constant coefficients. The characteristic equation is

$$r^2 - r - \frac{3}{4} = 0 \Leftrightarrow \left(r + \frac{1}{2}\right)\left(r - \frac{3}{2}\right) = 0.$$

The nonhomogeneous equation has a particular solution, A/b (in the formula) $= 2/(-3/4) = -8/3$. Thus, the general solution is

$$x = Ae^{-\frac{1}{2}t} + Be^{\frac{3}{2}t} - \frac{8}{3},$$

where A and B are arbitrary constants.

The boundary conditions $x(0) = -8/3$ and $x(2) = 1/3$ imply

$$\begin{aligned} 0 &= A + B \\ Ae^{-1} + Be^{3} &= 3 \end{aligned}$$

Then, we obtain $A = -3/(e^3 - e^{-1})$ and $B = -A$ so that

$$x = x(t) = -\frac{3}{e^3 - e^{-1}} e^{-\frac{1}{2}t} + \frac{3}{e^3 - e^{-1}} e^{\frac{3}{2}t} - \frac{8}{3}.$$

This is the only solution of the Euler equation that satisfies the given boundary conditions.

Why the Euler Equation is Necessary (Ch. 8.3)

Leibniz's Formula: Simple Case

$$F(x) = \int_c^d f(x,t)dt \Rightarrow F^{'}(x) = \int_c^d \frac{\partial f(x,t)}{\partial x} dt$$

Sketch of Proof:

$$\begin{aligned} F^{'}(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \int_c^d \frac{f(x+h,t) - f(x,t)}{h} dt \\ &= \int_c^d \lim_{h \to 0} \frac{f(x+h,t) - f(x,t)}{h} dt \\ &\quad (\because \text{ we can change the order of lim and } \textstyle\int.) \\ &= \int_c^d \frac{\partial f(x,t)}{\partial x} dt. \ \blacksquare \end{aligned}$$

Example: Discounted Present Value of an Asset

Let an asset generate a value $f(t) \in \mathbb{R}$ in period $t \in [0, T]$ and T be the terminal period.

Assume that r is the interest rate for the safe asset.

Then, the discounted present value of this asset in period 0 is computed as:

$$K = \int_0^T f(t)e^{-rt}dt.$$

By Leibniz's rule,

$$\frac{dK}{dr} = \int_0^T f(t)(-t)e^{-rt}dt = -\int_0^T tf(t)e^{-rt}dt.$$

Theorem: Suppose that F is a C^2 function of three variables. Suppose that $x^*(t)$ maximizes or minimizes

$$J(x) = \int_{t_0}^{t_1} F(t, x, \dot{x})dt,$$

among all **admissible** functions $x(t)$, i.e., all C^1 functions $x(t)$ defined on $[t_0, t_1]$ that satisfy the boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad (x_0 \text{ and } x_1 \text{ given numbers})$$

Then, $x^*(t)$ is a solution of the Euler equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0.$$

If $F(t, x, \dot{x})$ is concave (convex) in $(x, \dot{x})$, an admissible $x^*(t)$ that satisfies the Euler equation solves the maximization (minimization) problem.

Proof: (**Necessity**) Suppose $x^* = x^*(t)$ is an optimal solution to the maximization problem and let $\mu(t)$ be any C^2 function that satisfies $\mu(t_0) = \mu(t_1) = 0$.

For each real number $\alpha \in \mathbb{R}$, define a **perturbed** function $x(t)$ by

$$x(t) = x^*(t) + \alpha\mu(t).$$

Note that if α is small, the function $x(t)$ is near the function $x^*(t)$.

Clearly, $x(t)$ is admissible because $x^*(t)$ and $\mu(t)$ are C^2 and

$$\begin{aligned} x(t_0) &= x^*(t_0) + \alpha\mu(t_0) = x_0 + \alpha \cdot 0 = x_0 \\ x(t_1) &= x^*(t_1) + \alpha\mu(t_1) = x_1 + \alpha \cdot 0 = x_1 \end{aligned}$$

If $\mu(t)$ is a fixed function, then $J(x^* + \alpha\mu)$ is a function $I(\alpha)$ of only the single scalar α, given by

$$I(\alpha) = \int_{t_0}^{t_1} F(t, x^*(t) + \alpha\mu(t), \dot{x}^*(t) + \alpha\dot{\mu}(t))dt \qquad (1)$$

Obviously, $I(0) = J(x^*)$. Also, because of the hypothesis that $x^*(t)$ is optimal,

$$J(x^*) \geq J(x^* + \alpha\mu),\ \forall\alpha \Leftrightarrow I(0) \geq I(\alpha),\ \forall\alpha$$

Because I is a differentiable function and $\alpha = 0$ is an interior point in the domain of I, one must have

$$I^{'}(0) = 0.$$

By Leibniz's formula, we differentiate $I(\alpha)$ with respect to α,

$$I^{'}(\alpha) = \int_{t_0}^{t_1} \frac{\partial}{\partial \alpha} F(t, x^*(t) + \alpha\mu(t), \dot{x}^*(t) + \alpha\dot{\mu}(t))dt.$$

By the chain rule,

$$\frac{\partial}{\partial \alpha} F(t, x^*(t) + \alpha\mu(t), \dot{x}^*(t) + \alpha\dot{\mu}(t)) = F_2^{'} \cdot \mu(t) + F_3^{'} \cdot \dot{\mu}(t),$$

where $F_2^{'}$ and $F_3^{'}$ are evaluated at $(t, x^*(t) + \alpha\mu(t), \dot{x}^*(t) + \alpha\dot{\mu}(t))$.

When $\alpha = 0$, we have

$$I^{'}(0) = \int_{t_0}^{t_1} \left[F_2^{'}(t, x^*(t), \dot{x}^*(t)) \cdot \mu(t) + F_3^{'}(t, x^*(t), \dot{x}^*(t)) \cdot \dot{\mu}(t) \right] dt,$$

or, in more compact notation,

$$I^{'}(0) = \int_{t_0}^{t_1} \left[\frac{\partial F^*}{\partial x} \mu(t) + \frac{\partial F^*}{\partial \dot{x}} \dot{\mu}(t) \right] dt$$

where $*$ indicates that the derivatives are evaluated at $(t, x^*, \dot{x}^*)$.

By integration by parts,

$$
\begin{aligned}
\int_{t_0}^{t_1} \frac{\partial F^*}{\partial \dot{x}} \dot{\mu}(t) dt &= \left[\left(\frac{\partial F^*}{\partial \dot{x}} \right) \mu(t) \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial F^*}{\partial \dot{x}} \right) \mu(t) dt \\
&= \left(\frac{\partial F^*}{\partial \dot{x}} \right)_{t=t_1} \mu(t_1) - \left(\frac{\partial F^*}{\partial \dot{x}} \right)_{t=t_0} \mu(t_0) \\
&\quad - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial F^*}{\partial \dot{x}} \right) \mu(t) dt \\
&= - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial F^*}{\partial \dot{x}} \right) \mu(t) dt \\
&\quad (\because \ \mu(t_0) = \mu(t_1) = 0)
\end{aligned}
$$

Therefore, $I'(0) = 0$ reduces to

$$\int_{t=t_0}^{t=t_1} \left[\frac{\partial F^*}{\partial x} - \frac{d}{dt} \left(\frac{\partial F^*}{\partial \dot{x}} \right) \right] \mu(t) dt = 0. \qquad (2)$$

So far, $\mu(t)$ was a fixed function. But the above equation (2) must hold for **all** functions $\mu(t)$ that are C^2 on $[t_0, t_1]$ and that are zero at t_0 and t_1.

Then, it seems to be reasonable to conclude that the bracket expression in (2) must be zero for all $t \in [t_0, t_1]$.

(**Sufficiency**) Suppose that $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$. Assume further that $x^* = x^*(t)$ satisfies the Euler equation as well as the boundary conditions $x^*(t_0) = x_0$ and $x^*(t_1) = x_1$.

Let $x = x(t)$ be an arbitrary admissible function in the problem. Since $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$, we use the following characterization: $f : S \to \mathbb{R}$ is concave if and only if, for any $x, x^{'} \in S$,

$$f(x^{'}) - f(x) \leq \nabla f(x) \cdot (x^{'} - x).$$

Setting $x = (x^*, \dot{x}^*), x^{'} = (x, \dot{x})$, and $f(x) = F(t, x, \dot{x})$, we obtain

$$F(t, x, \dot{x}) - F(t, x^*, \dot{x}^*) \leq \frac{\partial F(t, x^*, \dot{x}^*)}{\partial x}(x - x^*) + \frac{\partial F(t, x^*, \dot{x}^*)}{\partial \dot{x}}(\dot{x} - \dot{x}^*).$$

Using the Euler equation, we further obtain

$$\begin{aligned} F^* - F &\geq \frac{\partial F^*}{\partial x}(x^* - x) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x}^* - \dot{x}) \\ &= \left[\frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)\right](x^* - x) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x}^* - \dot{x}) \\ &= \frac{d}{dt}\left[\frac{\partial F^*}{\partial \dot{x}}(x^* - x)\right]. \end{aligned}$$

Since the above inequality holds for all $t \in [t_0, t_1]$, integrating the above yeilds

$$\int_{t_0}^{t_1}(F^* - F)dt \geq \int_{t_0}^{t_1} \frac{d}{dt}\left[\frac{\partial F^*}{\partial \dot{x}}(x^* - x)\right]dt = \left[\frac{\partial F^*}{\partial \dot{x}}(x^* - x)\right]_{t_0}^{t_1} = 0,$$

where the last equality follows because $x^*(t_0) = x(t_0) = x_0$ and $x^*(t_1) = x(t_1) = x_1$.

It follows that

$$\int_{t_0}^{t_1} \left[F(t, x^*, \dot{x}^*) - F(t, x, \dot{x})\right] dt \geq 0,$$

for every admissible function $x = x(t)$. This confirms that $x^*(t)$ solves the maximization problem.

This completes the proof. ■

Theorem: Suppose that F is a C^2 function of three variables. Suppose that $x^*(t)$ maximizes or minimizes

$$J(x) = \int_{t_0}^{t_1} F(t, x, \dot{x}) dt,$$

among all **admissible** functions $x(t)$, i.e., all C^1 functions $x(t)$ defined on $[t_0, t_1]$ that satisfy the boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad (x_0 \text{ and } x_1 \text{ given numbers})$$

Then, $x^*(t)$ is a solution of the Euler equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0.$$

If $F(t, x, \dot{x})$ is concave (convex) in $(x, \dot{x})$, an admissible $x^*(t)$ that satisfies the Euler equation solves the maximization (minimization) problem.

Example Revisited:

$$\max \int_0^2 (4 - 3x^2 - 16\dot{x} - 4(\dot{x})^2)e^{-t}dt,\ x(0) = -8/3,\ x(2) = 1/3.$$

Set $F(t, x, \dot{x}) = (4 - 3x^2 - 16\dot{x} - 4(\dot{x})^2)e^{-t}$. We compute the Hessian matrix of F:

$$H(x, \dot{x}) = \begin{pmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial \dot{x}\partial x \\ \partial^2 F/\partial x \partial \dot{x} & \partial^2 F/\partial \dot{x}^2 \end{pmatrix} = \begin{pmatrix} -6e^{-t} & 0 \\ 0 & -8e^{-t} \end{pmatrix}.$$

This implies that $H(x, \dot{x})$ is negative definite, which further implies that F is strictly concave in x and $\dot{x}$.

Therefore, the solution of the Euler equation that satisfies the given boundary conditions is indeed the solution to the maximization problem.

Optimal Savings (Ch. 8.4)

Consider the Ramsey (optimal growth) problem:

$$\max \int_0^T U(f(K(t)) - \dot{K}(t))e^{-rt}dt, \;\; K(0) = K_0, \;\; K(T) = K_T.$$

Assume $f^{'}(K) > 0, f^{''}(K) \leq 0, U^{'}(C) > 0$, and $U^{''}(C) < 0$.

Let $F(t, K, \dot{K}) = U(C)e^{-rt}$ with $C = f(K) - \dot{K}$. Then, we obtain

$$\begin{aligned} \frac{\partial F}{\partial K} &= U^{'}(C)f^{'}(K)e^{-rt} \\ \frac{\partial F}{\partial \dot{K}} &= -U^{'}(C)e^{-rt}. \end{aligned}$$

This implies that the Euler equation reduces to

$$U^{'}(C)f^{'}(K)e^{-rt} - \frac{d}{dt}\left(-U^{'}(C)e^{-rt}\right) = 0.$$

We compute

$$\frac{d}{dt}\left(U^{'}(C)e^{-rt}\right) = U^{''}(C)\dot{C}e^{-rt} - rU^{'}(C)e^{-rt}$$

Inserting this into the Euler equation, we obtain

$$\begin{aligned}
& U^{'}(C)f^{'}(K)e^{-rt} + U^{''}(C)\dot{C}e^{-rt} - rU^{'}(C)e^{-rt} = 0 \\
\Leftrightarrow \;& \left[U^{'}(C)f^{'}(K) + U^{''}(C)\dot{C} - rU^{'}(C)\right]e^{-rt} = 0 \\
\Leftrightarrow \;& U^{'}(C)f^{'}(K) + U^{''}(C)\dot{C} - rU^{'}(C) = 0 \\
\Leftrightarrow \;& U^{'}(C)(f^{'}(K) - r) + U^{''}(C)\dot{C} = 0.
\end{aligned}$$

This implies

$$\frac{\dot{C}}{C} = \frac{f^{'}(K) - r}{\frac{-CU^{''}(C)}{U^{'}(C)}}$$

Let

$$\sigma(C) = -\frac{CU^{''}(C)}{U^{'}(C)}.$$

Define $1/\sigma(C)$ as the **intertemporal elasticity of substitution** at C. With this additional concept, the Euler equation simplifies to

$$\frac{\dot{C}}{C} = \frac{f^{'}(K) - r}{\sigma(C)}.$$

This means

$$\frac{\dot{C}}{C} > 0 \Leftrightarrow f^{'}(K(t)) > r.$$

Hence, consumption increases if and only if the marginal productivity of capital exceeds the discount rate.

If we use the fact that $\dot{C} = f^{'}(K)\dot{K} - \ddot{K}$ in the Euler equation, we get

$$\ddot{K} - f^{'}(K)\dot{K} + \frac{U^{'}(C)}{U^{''}(C)}(r - f^{'}(K)) = 0. \quad (*)$$

Because f is concave ($f^{''}(K) \leq 0$), it follows that $f(K) - \dot{K}$ is also concave in $(K, \dot{K})$, as it is a sum of two concave functions.

The function U is increasing and concave, so $U(f(K) - \dot{K})e^{-rt}$ is also concave in $(K, \dot{K})$. This can be directly checked via the negative semidefiniteness of the associated Hessian matrix.

Therefore, any solution to $(*)$ that satisfies the boundary conditions must be a solution to the problem.

Suppose that $f(K) = bK$ and $U(C) = C^{1-v}/(1-v)$.

Assume further that $b > 0, v > 0, v \neq 1$, and $b \neq (b-r)/v$.

In this case, the Euler equation becomes

$$\ddot{K} - \left(b - \frac{r-b}{v}\right)\dot{K} + \frac{b-r}{v} bK = 0 \;\Rightarrow (\lambda - b)\left(\lambda - \frac{b-r}{v}\right) = 0,$$

which is the characteristic function of λ. Because $b \neq (b-r)/v$, this second-order differential equation has the general solution

$$K(t) = Ae^{bt} + Be^{(b-r)t/v}.$$

The constants A and B are determined by

$$\begin{aligned} K_0 &= A + B \\ K_T &= Ae^{bT} + Be^{(b-r)T/v}. \end{aligned}$$

More General Terminal Conditions (Ch. 8.5)

In economic applications, the initial point is usually fixed, while in many models, the terminal value of the unknown function can be free, or subject to more general restrictions.

The problems we study are formulated as

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x}) dt, \;\; x(t_0) = x_0, \;\; \text{(a) } x(t_1) \text{ free or (b) } x(t_1) \geq x_1 \;\; (*)$$

Theorem (Transversality Conditions): If $x^*(t)$ solves problem $(*)$ with either (a) or (b) as the terminal condition, then $x^*(t)$ must satisfy the Euler equation. With the terminal condition (a), the **transversality condition** is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} = 0,$$

where $F^* \equiv F(t, x^*, \dot{x}^*)$. With the terminal condition (b), the **transversality condition** is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} \begin{cases} = 0 & \text{if } x^*(t_1) > x_1 \\ \leq 0 & \text{otherwise} \end{cases}$$

Proof: We skip the proof. ■

We can interpret $(\partial F^*/\partial \dot{x})_{t=t_1}$ as the marginal value of investment at $t = t_1$.

Example: Consider the following problem:

$\max \int_0^1 (1-x^2-\dot{x}^2)dt$, $x(0) = 1$, with (a) $x(1)$ free or (b) $x(1) \geq 2$

Let $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$. The Euler equation is

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \Leftrightarrow -2x + 2\ddot{x} = 0 \Leftrightarrow \ddot{x} - x = 0.$$

The characteristic equation of this differential equation is $r^2-1 = 0$. So, we have $r = 1, -1$. The general solution is

$$x(t) = Ae^t + Be^{-t}.$$

$$x(0) = 1 \Rightarrow A + B = 1 \Rightarrow B = 1 - A$$

Thus, an optimal solution of either problem must be of the form

$$x^*(t) = Ae^t + (1 - A)e^{-t}.$$

With (a) as the terminal condition, the transversality condition requires

$$\left.\frac{\partial F^*}{\partial \dot{x}}\right|_{t=t_1} = 0 \Rightarrow -2\dot{x}^*(1) = 0 \Rightarrow \dot{x}^*(1) = 0.$$

Since $\dot{x}^*(t) = Ae^t - (1 - A)e^{-t}$,

$$\dot{x}^*(1) = Ae - (1 - A)e^{-1} = 0 \Rightarrow A = \frac{1}{e^2 + 1}.$$

Hence,

$$x^*(t) = \frac{1}{e^2 + 1}\left(e^t + e^2 e^{-t}\right).$$

Because $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$ is concave in $(x, \dot{x})$ (why?), the solution has been found.

With (b) as the terminal condition, we require

$$x^*(1) = Ae + (1 - A)e^{-1} \geq 2 \Rightarrow A \geq \frac{2e - 1}{e^2 - 1}$$

Suppose $x^*(1) > 2$. Then, as in (a), the transversality condition gives $A = 1/(e^2 + 1)$. But this violates the inequality $A \geq (2e - 1)/(e^2 - 1)$ because $2e - 1 > 1$. So,

$$x^*(1) = 2 \Rightarrow A = \frac{2e - 1}{e^2 - 1}$$

Then,

$$\dot{x}^*(1) = Ae - (1 - A)e^{-1} = \frac{2(e^2 - e + 1)}{e^2 - 1} > 0.$$

This implies

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=1} = -2\dot{x}^*(1) \leq 0.$$

Hence, the transversality condition holds. Then, the only solution candidate is

$$x^*(t) = \frac{1}{e^2 - 1}\left\{(2e-1)e^t + (e^2 - 2e)e^{-t}\right\}.$$

Because $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$ is concave in $(x, \dot{x})$ (why?), the solution has been found.

Example: Let $A(t)$ denote DM's wealth at time t, and let w be the (constant) income per unit of time.

DM can borrow and save at the same constant rate of interest r.

Consumption per unit of time at time t: $C(t) = rA(t) + w - \dot{A}(t)$.

$$\max_{\{C(t)\}_{t \in [0,T]}} \int_0^T U(C(t)) e^{-\rho t} dt$$
$$\text{s.t.} \quad C(t) = rA(t) + w - \dot{A}(t),\ \forall t \in [0,T],\ A(0) = A_0,\ A(T) \geq A_T,$$

where $U'(\cdot) > 0$, $U''(\cdot) < 0$, $\rho > 0$, $A_0 > 0$, and $A_T > 0$.

The objective function: $F(t, A, \dot{A}) = U(rA + w - \dot{A})e^{-\rho t}$.

$$\begin{aligned}
\frac{\partial F}{\partial A} &= rU^{'}(rA + w - \dot{A})e^{-\rho t}, \\
\frac{\partial F}{\partial \dot{A}} &= -U^{'}(rA + w - \dot{A})e^{-\rho t}.
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}}\right) = -U^{''}(rA + w - \dot{A})(r\dot{A} - \ddot{A})e^{-\rho t} + \rho U^{'}(rA + w - \dot{A})e^{-\rho t}$$

The Euler equation:

$$\begin{aligned}
& \frac{\partial F}{\partial A} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{A}}\right) = 0 \\
\Leftrightarrow \quad & rU^{'}e^{-\rho t} + U^{''}(r\dot{A} - \ddot{A})e^{-\rho t} - \rho U^{'}e^{-\rho t} = 0 \\
\Leftrightarrow \quad & (r - \rho)U^{'} + U^{''}(r\dot{A} - \ddot{A}) = 0 \\
\Leftrightarrow \quad & \ddot{A} - r\dot{A} + (\rho - r)\frac{U^{'}}{U^{''}} = 0.
\end{aligned}$$

$A^*(t)$: an admissible function that satisfies the Euler equation and the corresponding transversality condition.

Since we have $A(T) \geq A_T$ as the terminal condition, the following transversality condition holds:

$$\left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} \leq 0 \text{ and } A^*(T) > A_T \Rightarrow \left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} = 0.$$

Taking its contrapositive form, we have

$$\left(\frac{\partial F^*(t)}{\partial \dot{A}}\right)_{t=T} < 0 \Rightarrow A^*(T) = A_T$$

Because $U^{'} > 0$, we have

$$\left(\frac{\partial F^*}{\partial \dot{A}}\right)_{t=T} = -U^{'}(rA(T) + w - \dot{A}(T))e^{-\rho T} < 0.$$

This implies that $A^*(T) = A_T$.

The second derivatives of F:

$$\begin{aligned} \frac{\partial^2 F}{\partial A^2} &= r^2 U'' e^{-\rho t} < 0 \\ \frac{\partial^2 F}{\partial \dot{A} \partial A} = \frac{\partial^2 F}{\partial A \partial \dot{A}} &= -r U'' e^{-\rho t} > 0 \\ \frac{\partial^2 F}{\partial \dot{A}^2} &= U'' e^{-\rho t} < 0 \end{aligned}$$

$$\mathcal{H}(A, \dot{A}) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} r^2 U'' e^{-\rho t} & -r U'' e^{-\rho t} \\ -r U'' e^{-\rho t} & U'' e^{-\rho t} \end{pmatrix}.$$

Since $U'' < 0$, $h_{11} < 0$, $h_{22} < 0$, and $h_{11}h_{22} - h_{12}h_{21} = 0$. Thus, $\mathcal{H}(A, \dot{A})$ is NSD $\Rightarrow F(t, A, \dot{A})$ is concave in $(A, \dot{A})$.

By the sufficiency of the Euler equation, we conclude that $A^*(t)$ solves the problem.