ECON 696
Mathematical Methods for Economic Dynamics
SMU School of Economics; Fall 2024
Answer Key to Homework Assignments

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You are supposed to upload your work (in pdf file) under "Assignments" under our class website (on eLearn) by 17:00 on the date specified below.

1 Homework 1 (Due Date: September 10 (Tue), 2024): Submission is Required and will be Graded

Question 1.1 (15 points) *Determine sup and inf for each of these sets. You need to prove why what you obtain is the sup and inf for each set.*

1. $A = (-3, 7]$.

   $\sup A = 7$ and $\inf A = -3$. Clearly, $x \leq 7$ for all $x \in A$ and $x \geq -3$ for all $x \in A$. Fix $\varepsilon > 0$ arbitrarily small. Then, one can guarantee that $7 - \varepsilon/2 \in A$ such that $7 - \varepsilon/2 > 7 - \varepsilon$. Similarly, one can guarantee that $-3 + \varepsilon/2 \in A$ such that $-3 + \varepsilon/2 < -3 + \varepsilon$.

2. $B = \{1/n | \ n \in \mathbb{N}\}$.

   $\sup B = 1$ and $\inf B = 0$. Clearly, $x \leq 1$ for all $x \in B$ and $x \geq 0$ for all $x \in B$. Fix $\varepsilon > 0$ arbitrarily small. Then, there exists $n \in \mathbb{N}$ small enough so that $1/n > 1 - \varepsilon$ where $1/n \in B$. Then, similarly, there exists $n \in \mathbb{N}$ large enough so that $1/n < \varepsilon$ where $1/n \in B$.

3. $C = \{x | \ x > 0 \text{ and } x^2 > 3\}$.

   $\sup C = \infty$ and $\inf C = \sqrt{3}$. We first claim $\sup C = \infty$. Suppose, on the contrary, that there exists $M \in (0, \infty)$ such that $x \leq M$ for all $x \in C$. Fix $\varepsilon > 0$. By definition of C, $(M + \varepsilon) \in C$, which implies $M + \varepsilon > M$. This is a contradiction.

We next claim that $\inf C = \sqrt{3}$. Suppose, by way of contradiction, there exists $\varepsilon \in (0, \sqrt{3})$ such that $x \geq \sqrt{3} - \varepsilon > 0$ for all $x \in C$. Take $x = \sqrt{3} - \varepsilon$. Then, we have

$$x^2 = (\sqrt{3} - \varepsilon)^2 = 3 + \varepsilon^2 - 2\sqrt{3}\varepsilon = 3 + \varepsilon(\varepsilon - 2\sqrt{3}).$$

Then, we have $x^2 < 3 - 3 = 0$, which is a contradiction.

Question 1.2 (35 points) *Consider the following sets:*

$$\begin{aligned} A &= \left\{ (x,y) \in \mathbb{R}^2 \middle| \; y = 1, \; x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\ B &= \left\{ (x,y) \in \mathbb{R}^2 \middle| \; y \in (0,1), \; x \in \bigcup_{n=1}^{\infty} (2n, 2n+1) \right\}; \\ C &= \left\{ (x,y) \in \mathbb{R}^2 \middle| \; y = 1, \; x \in \bigcup_{n=1}^{\infty} [2n, 2n+1] \right\}. \end{aligned}$$

Answer the following questions.

1. Formally determine whether the set A is open, closed, or neither.

   We claim that A is not open. Fix $(x^0, y^0) = (2.5, 1)$. It is easy to see that $(x^0, y^0) \in A$ because $2.5 \in (2,3) \subseteq \bigcup_{n=1}^{\infty}(2n, 2n+1)$. Fix $\varepsilon \in (0, 0.5)$ and define $B_\varepsilon(x^0, y^0)$ as the open ball around (x^0, y^0) with radius ε. This implies that

   $$B_\varepsilon(x^0, y^0) = \{(x,y) \in \mathbb{R}^2 | \; y \in (1-\varepsilon, 1+\varepsilon), \; x \in (2.5-\varepsilon, 2.5+\varepsilon)\},$$

   which is not a subset of A. Thus, A is not open.

   We claim that A is not closed. Consider a sequence $\{(x^k, y^k)\}_{k=1}^{\infty} \in \mathbb{R}^2$ such that $x^k = 2 + 1/k$ and $y^k = 1$ for each $k \in \mathbb{N}$. Since $x^k \in (2,3)$ for each $k \in \mathbb{N}$, we have that $\{(x^k, y^k)\}_{k=1}^{\infty} \in A$. Furthermore, as $x^k \to 2$ as $k \to \infty$, the sequence $\{(x^k, y^k)\}_{k=1}^{\infty}$ is a convergent sequence in A. However, since $(2,1) \notin (2,3) \subseteq \bigcup_{n=1}^{\infty}(2n, 2n+1)$, $(2,1) \notin A$. Thus, A is not closed.

2. Formally determine whether the set B is open, closed, or neither.

   We claim that B is open. Fix $(x,y) \in B$. This implies that $y \in (0,1)$ and there exists $k \in \mathbb{N}$ such that $x \in (2k, 2k+1)$. Define

   $$\delta = \min\{y, 1-y, x-2k, 2k+1-x\}.$$

   By construction, we have $\delta > 0$. Fix $\varepsilon \in (0, \delta)$. We define $B_\varepsilon(x,y)$ as the open ball around (x,y) with radius ε. By construction, we have

   $$B_\varepsilon(x,y) \subseteq \{(x,y) \in \mathbb{R}^2 | \; y \in (0,1), \; x \in (2k, 2k+1)\}.$$

   This further implies that $B_\varepsilon(x,y) \subseteq B$. Thus, B is open.

3. Formally determine whether the set C is open, closed, or neither

We claim that C is not open. Fix $(x^0, y^0) = (2.5, 1)$. It is easy to see that $(x^0, y^0) \in B$ because $2.5 \in [2, 3] \subseteq \bigcup_{n=1}^{\infty} [2n, 2n+1]$. Fix $\varepsilon \in (0, 0.5)$ and define $B_\varepsilon(x^0, y^0)$ as the open ball around (x^0, y^0) with radius ε. This implies that

$$B_\varepsilon(x^0, y^0) = \{(x, y) \in \mathbb{R}^2 | \ y \in (1-\varepsilon, 1+\varepsilon), \ x \in (2.5-\varepsilon, 2.5+\varepsilon)\},$$

which is not a subset of C. Thus, C is not open.

We claim that C is closed. We take the complement of C:

$$C^c = D_1 \cup D_2 \cup \bigcup_{n=1}^{\infty} D_n,$$

where

$$\begin{aligned} D_1 &\equiv \left\{(x, y) \in \mathbb{R}^2 | \ y \neq 1\right\}; \\ D_2 &\equiv \left\{(x, y) \in \mathbb{R}^2 | \ y = 1, \ x < 2\right\}; \\ D_n &\equiv \left\{(x, y) \in \mathbb{R}^2 | \ y = 1, \ x \in (2n+1, 2n+2)\right\} \end{aligned}$$

for each $n \in \mathbb{N}$. Fix $(x, y) \in C^c$. There are the following cases to consider:

Case 1: $(x, y) \in D_1$

We assume without loss of generality that $y > 1$. Fix $\varepsilon \in (0, y-1)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{y} > 1$, which further implies that $B_\varepsilon(x, y) \subseteq D_1 \subseteq C^c$.

Case 2: $(x, y) \in D_2$

Fix $\varepsilon \in (0, 2-x)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{x} < 2$. Thus,

$$B_\varepsilon(x, y) \subseteq D_1 \cup D_2 \subseteq C^c.$$

Case 3: There exists $k \in \mathbb{N}$ such that $(x, y) \in D_k$

Define

$$\delta = \min\{x - 2k + 1, 2k + 2 - x\}.$$

By our hypothesis, we have $\delta > 0$. So, fix $\varepsilon \in (0, \delta)$. Define $B_\varepsilon(x, y)$ as the open ball around (x, y) with radius ε. Then, by construction, for any $(\tilde{x}, \tilde{y}) \in B_\varepsilon(x, y)$, we have $\tilde{x} \in (2k+1, 2k+2)$. Thus,

$$B_\varepsilon(x, y) \subseteq D_1 \cup D_k \subseteq C^c.$$

Considering all the three cases above, we conclude that C^c is open, which further implies that C is closed.

Question 1.3 (30 points) *Prove the following properties for closed sets:*

1. *The whole space $\mathbb{R}^n$ and the empty set $\emptyset$ are both closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *The union of finitely many closed sets is closed.*

(Hint: Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of sets and Λ denotes the index set with λ as a generic element. You can use the following relationships: $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$ and $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$. This is called de Morgan's law.)

1. $(\mathbb{R}^n)^c$ is the empty set, which is open. So, $\mathbb{R}^n$ is closed. $(\emptyset)^c$ is the entire space $\mathbb{R}^n$, which is open. So, $\emptyset$ is closed.

2. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets. Set $F_* = \bigcap_{\lambda\in\Lambda} F_\lambda$. What we want to show is that F_* is closed, which is equivalent to showing that $(F_*)^c$ is open. By de Morgan's law, we obtain

$$(F_*)^c = \bigcup_{\lambda\in\Lambda} (F_\lambda)^c.$$

Since each F_λ is closed, $(F_\lambda)^c$ is open. Since we know that the arbitrary union of open sets is open, we conclude that $(F_*)^c$ is open, that is, F_* is closed.

3. Let $\{F_k\}_{k=1}^K$ be a collection of finite number of closed sets. Set $F^* = \bigcup_{k=1}^K F_k$. What we want to show is that F^* is closed, which is equivalent to showing that $(F^*)^c$ is open. By de Morgan's law, we obtain

$$(F^*)^c = \bigcap_{k=1}^{K} (F_\lambda)^c.$$

Since each F_λ is closed, $(F_\lambda)^c$ is open. Besides, since we know that the finite intersection of open sets is open, we conclude that $(F_*)^c$ is open, that is, F_* is closed.

Question 1.4 (20 points) *Let $S = \{x \in \mathbb{R}^n |\ g_j(x) \leq 0, j = 1, \ldots, m\}$. Prove that S is closed if each $g_j(\cdot)$ is continuous.*

Proof: Let $\{x^k\}$ be a sequence such that $x^k \in S$ for each k. Assume that $x^k \to x \in \mathbb{R}^n$ as $k \to \infty$. What we want to show is that $x \in S$. Since $x^k \in S$ for each k, we have that for each $j = 1, \ldots, m$,

$$g_j(x^k) \leq 0 \ \ \forall k.$$

Suppose by way of contradiction that $g_j(x) > 0$. Since each $g_j(\cdot)$ is continuous, $g_j(x^k) \to g_j(x)$ so that we have $g_j(x^k) > 0$ for each k large enough. However, this contradicts our hypothesis that $g_j(x^k) \leq 0$ for all k. We therefore must have $g_j(x) \leq 0$. This implies that $x \in S$. ■

2 Homework 2 (Due Date: September 17 (Tue), 2024): Submission is Required but will not be Graded

Question 2.1 *Determine the ranks of the following matrices:*

1. $$\begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix}$$

   We can easily see that
   $$\begin{pmatrix} 2 \\ 16 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 8 \end{pmatrix}.$$
   So, the rank of this matrix is 1.

2. $$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$$

   It is relatively easy to see
   $$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$
   In addition, we can see that $(3, 0)$ cannot be expressed as a scalar multiplication of $(1, 2)$. Therefore, the rank of this matrix is 2.

3. $$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$$

   First, we easily see that
   $$\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$
   So, the vector $(2, 4-2)$ is redundant. Eliminating this vector, we form the following truncated 3×3 matrix:
   $$\begin{pmatrix} 1 & -1 & 3 \\ 2 & -4 & 7 \\ -1 & -1 & -2 \end{pmatrix}$$
   We now claim that $(2, -4, 7)$ can be expressed as a linear combination of $(1, -1, 3)$ and $(-1, -1, -2)$. Let us write the following:
   $$(2, -4, 7) = \alpha(1, -1, 3) + \beta(-1, -1, -2),$$

for some $\alpha, \beta \in \mathbb{R}$. Indeed, we obtain $(\alpha, \beta) = (3, 1)$ as the solution to the above equation. Eliminating the vector $(2, -4, 7)$, we obtain the following truncated 2×3 matrix:

$$\begin{pmatrix} 1 & -1 & 3 \\ -1 & -1 & -2 \end{pmatrix}.$$

Once again, we can easily see that

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

So, eliminating this vector, we obtain the following truncated 2×2 matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Clearly, we cannot express $(1, -1)$ as a scalar multiplication of $(-1, -1)$. Therefore, the rank of the matrix is 2.

Question 2.2 *Let* $\mathbf{a}, \mathbf{b}$*, and* $\mathbf{c}$ *be linearly independent vectors in* $\mathbb{R}^n$*. Answer the following questions*

1. Show that three vectors $\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}$, and $\mathbf{a} + \mathbf{c}$ are linearly independent.

   Suppose, by way of contradiction, that three vectors $\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}$, and $\mathbf{a} + \mathbf{c}$ are linearly dependent. Then, there exists a nonzero vector $(c_1, c_2) \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ such that
   $$\mathbf{a} + \mathbf{b} = c_1(\mathbf{b} + \mathbf{c}) + c_2(\mathbf{a} + \mathbf{c}).$$
   This is equivalent to
   $$(c_2 - 1)\mathbf{a} + (c_1 - 1)\mathbf{b} + (c_1 + c_2)\mathbf{c} = \mathbf{0}.$$
   Since $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent, we must have $c_1 = c_2 = 1$ and $c_1 + c_2 = 0$, which are simply impossible. This is the desired contradiction.

2. Show whether three vectors $\mathbf{a} - \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, and $\mathbf{a} + \mathbf{c}$ are linearly independent or linearly dependent.

   We claim these vectors are linearly dependent, since $\mathbf{a} - \mathbf{b}$ can be expressed as a linear combination of $\mathbf{b} + \mathbf{c}$, and $\mathbf{a} + \mathbf{c}$ as follows:
   $$\mathbf{a} - \mathbf{b} = -(\mathbf{b} + \mathbf{c}) + (\mathbf{a} + \mathbf{c}).$$

Question 2.3 *Consider the following quadratic form subject to two linear equalities:*

$$Q(x, y, z) = x^2 + 2xy + y^2 + z^2 \ \text{ subject to } \ \begin{cases} x + 2y + z = 0; \\ 2x - y - 3z = 0. \end{cases}$$

Show that the associated quadratic form is positive definite.

We denote a 3×3 matrix A by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

If we set

$$\begin{aligned} Q(x, y, z) &= (xyz)A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= a_{11}x^2 + (a_{12} + a_{21})xy + (a_{13} + a_{31})xz + a_{22}y^2 + (a_{23} + a_{32})yz + a_{33}z^2, \end{aligned}$$

then, we can set $a_{11} = a_{22} = a_{33} = 1$; $a_{12} = a_{21} = 1$; and $a_{13} = a_{31} = a_{23} = a_{32} = 0$. Thus, we obtain

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as a symmetric matrix. We also denote a 2×3 matrix B by

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}.$$

If we set

$$B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y + b_{13}z \\ b_{21}x + b_{22}y + b_{23}z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ 2x - y - 3z \end{pmatrix},$$

then, we can set

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \end{pmatrix}$$

Now we define the bordered matrix as follows:

$$\begin{pmatrix} \mathbf{0}_{2\times 2} & B \\ B^T & A \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ b_{11} & b_{21} & a_{11} & a_{12} & a_{13} \\ b_{12} & b_{22} & a_{21} & a_{22} & a_{23} \\ b_{13} & b_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}$$

We compute B_3 as follows:

$$\begin{aligned}
B_3 &= \begin{vmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{vmatrix} \\
&= 1\cdot(-1)^{3+1}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ -1 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{vmatrix} + 2(-1)^{4+1}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ 2 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{vmatrix} \\
&\quad +1(-1)^{5+1}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ 2 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{vmatrix} \\
&= 1(5) - 2(-10) + 1(0) = 25,
\end{aligned}$$

where

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ -1 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{vmatrix} &= (-1)(-1)^{3+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 0 & 0 & 1 \end{vmatrix} + (-3)(-1)^{4+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & 1 & 0 \end{vmatrix} \\
&= -(-5) + 3(0) = 5
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ 2 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{vmatrix} &= 2(-1)^{3+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 0 & 0 & 1 \end{vmatrix} + (-3)(-1)^{4+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & 1 & 0 \end{vmatrix} \\
&= 2(-5) + 3(0) = -10
\end{aligned}$$

and

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & -1 & -3 \\ 2 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{vmatrix} &= 2(-1)^{3+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & 1 & 0 \end{vmatrix} + (-1)(-1)^{4+1}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & 1 & 0 \end{vmatrix} \\
&= 4(0) + (0) = 0.
\end{aligned}$$

Thus, we have

$$(-1)^2 B_3 = 25 > 0,$$

which implies that the quadratic form with two linear constraints is positive definite.

3 Homework 3 (Due Date: September 24 (Tue), 2023): Submission is Required and will be Graded

Question 3.1 (30 points) *Examine the (strict) convexity/concavity of the following functions:*

1. $z = x + y - e^x - e^{x+y}$

   We compute the following:

$$\begin{aligned} \partial z/\partial x &= 1 - e^x - e^{x+y} \\ \partial z/\partial y &= 1 - e^{x+y} \\ \partial^2 z/\partial x^2 &= -e^x - e^{x+y} \\ \partial^2 z/\partial y^2 &= -e^{x+y} \\ \partial^2 z/\partial x\partial y = \partial^2/\partial y\partial x &= -e^{x+y}. \end{aligned}$$

   We form the associated Hessian matrix H:

$$H = \begin{pmatrix} -e^x - e^{x+y} & -e^{x+y} \\ -e^{x+y} & -e^{x+y} \end{pmatrix}.$$

   We also compute $|H|$:

$$|H| = e^{x+y}(e^x + e^{x+y}) - e^{2x+2y} = e^{2x+y} > 0.$$

   Since $-e^x - e^{x+y} < 0$, we conclude that H is negative definite. So, the function is strictly concave.

2. $z = e^{x+y} + e^{x-y} - y/2$

   We compute the following:

$$\begin{aligned} \partial z/\partial x &= e^{x+y} + e^{x-y} \\ \partial z/\partial y &= e^{x+y} - e^{x-y} - 1/2 \\ \partial^2 z/\partial x^2 &= e^{x+y} + e^{x-y} \\ \partial^2 z/\partial y^2 &= e^{x+y} + e^{x-y} \\ \partial^2 z/\partial x\partial y = \partial^2/\partial y\partial x &= e^{x+y} - e^{x-y}. \end{aligned}$$

   We form the associated Hessian matrix H:

$$H = \begin{pmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{pmatrix}$$

   We next compute $|H|$:

$$|H| = (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2 = 4e^{2x} > 0.$$

   Since $e^{x+y} + e^{x-y} > 0$, we conclude that H is positive definite. So, the function is strictly convex.

3. $w = (x + 2y + 3z)^2$

   We first compute the following:

$$\begin{aligned}
\partial w/\partial x &= 2(x+2y+3z)\\
\partial w/\partial y &= 4(x+2y+3z)\\
\partial w/\partial z &= 6(x+2y+3z)\\
\partial^2 w/\partial x^2 &= 2\\
\partial^2 w/\partial y^2 &= 8\\
\partial^2 w/\partial z^2 &= 18\\
\partial^2 w/\partial x\partial y = \partial^2 w/\partial y\partial x &= 4\\
\partial^2 w/\partial y\partial z = \partial^2 w/\partial z\partial y &= 12\\
\partial^2 w/\partial z\partial x = \partial^2 w/\partial x\partial z &= 6
\end{aligned}$$

   We next form the associated Hessian matrix H:

$$H = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{pmatrix}.$$

   We compute all the principal minors of H:

$$\begin{aligned}
\Delta_1^2 &= 2, 8, 18\\
\Delta_2^2 &= \begin{vmatrix} 8 & 12 \\ 12 & 18 \end{vmatrix} = 144 - 144 = 0, \begin{vmatrix} 2 & 6 \\ 6 & 18 \end{vmatrix} = 0, \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 16 - 16 = 0\\
\Delta_3^2 &= |H| = \frac{1}{3}\begin{vmatrix} 3\cdot 2 & 4 & 6 \\ 3\cdot 4 & 8 & 12 \\ 3\cdot 6 & 12 & 18 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 6 & 4 & 6 \\ 12 & 6 & 12 \\ 18 & 12 & 18 \end{vmatrix} = 0.
\end{aligned}$$

   Therefore, we have $\Delta_r^2 \geq 0$ for each $r = 1, 2, 3$. That is, the Hessian matrix is positive semidefinite. So, the function is convex.

Question 3.2 (30 points) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x, y) = (x-1)^2(y-1)^2,$$

for all $(x, y) \in \mathbb{R}^2$. Answer the following questions.

1. Show that the function f is "not" quasiconcave.

   We know that $f(0,0) = f(2,2) = 1$. Observe that $(1,1)$ is considered a convex combination of $(0,0)$ and $(2,2)$:

$$(1,1) = \frac{1}{2}(0,0) + \frac{1}{2}(2,2).$$

Then, we have $f(1,1) = 0$. This implies that

$$f\left(\frac{1}{2}(0,0) + \frac{1}{2}(2,2)\right) = f(1,1) = 0 < 1 = f(0,0) = f(2,2) = \min\{f(0,0), f(2,2)\}.$$

Setting $x = (0,0), x^{'} = (2,2)$ and $\lambda = 1/2$ in the definition of quasiconcavity shows that the function f is not quasiconcave.

2. Derive the bordered Hessian matrix of f, which we denote by $B(x,y)$.

   We compute the first-order and second-order derivatives of f:

$$\begin{aligned} f_1^{'}(x,y) &= 2(x-1)(y-1)^2 \\ f_2^{'}(x,y) &= 2(x-1)^2(y-1) \\ f_{11}^{''}(x,y) &= 2(y-1)^2 \\ f_{22}^{''}(x,y) &= 2(x-1)^2 \\ f_{12}^{''}(x,y) = f_{21}^{''}(x,y) &= 4(x-1)(y-1). \end{aligned}$$

   So,

$$B(x,y) = \begin{pmatrix} 0 & 2(x-1)(y-1)^2 & 2(x-1)^2(y-1) \\ 2(x-1)(y-1)^2 & 2(y-1)^2 & 4(x-1)(y-1) \\ 2(x-1)^2(y-1) & 4(x-1)(y-1) & 2(x-1)^2 \end{pmatrix}$$

3. Confirm that the "necessary" condition for quasiconcavity of f via bordered Hessian matrix determinant.

$$\begin{aligned} |B(x,y)| &= 2(x-1)(y-1)^2 \cdot (-1)^{2+1} \begin{vmatrix} 2(x-1)(y-1)^2 & 2(x-1)^2(y-1) \\ 4(x-1)(y-1) & 2(x-1)^2 \end{vmatrix} \\ &\quad +2(x-1)^2(y-1) \cdot (-1)^{3+1} \begin{vmatrix} 2(x-1)(y-1)^2 & 2(x-1)^2(y-1) \\ 2(y-1)^2 & 4(x-1)(y-1) \end{vmatrix} \\ &= -2(x-1)(y-1)^2 \left\{4(x-1)^3(y-1)^2 - 8(x-1)^3(y-1)^2\right\} \\ &\quad +2(x-1)^2(y-1) \left\{8(x-1)^2(y-1)^3 - 4(x-1)^2(y-1)^3\right\} \\ &= 16(x-1)^4(y-1)^4 \geq 0, \end{aligned}$$

   for all $(x,y) \in S$. This is because $(x-1)^4 \geq 0$ for any $x \in \mathbb{R}$ and $(y-1)^4 \geq 0$ for any $y \in \mathbb{R}$.

Question 3.3 (40 points) *Consider the following constrained optimimization (maximization or minimization) problem:*

$$\max_{(x,y,z)\in\mathbb{R}^3} \left(\textit{or} \min_{(x,y,z)\in\mathbb{R}^3} \right) x + y + z \ \ \textit{subject to} \ x^2 + y^2 + z^2 = 1 \ \textit{and} \ x - y - z = 1.$$

Answer the following questions.

1. Find all the solution candidates of this constrained optimization problem using the Lagrangian method.

   We set up the Lagrangian:

   $$\mathcal{L}(x, y, z, \lambda_1, \lambda_2) = x + y + z - \lambda_1(x^2 + y^2 + z^2 - 1) - \lambda_2(x - y - z - 1).$$

   The first-order conditions of the Lagrangian and the two equality constraints are provided as follows:

   $$\begin{array}{ll} (1) & \mathcal{L}_1' = 1 - 2\lambda_1 x - \lambda_2 = 0; \\ (2) & \mathcal{L}_2' = 1 - 2\lambda_1 y + \lambda_2 = 0; \\ (3) & \mathcal{L}_3' = 1 - 2\lambda_1 z + \lambda_2 = 0; \\ (4) & x^2 + y^2 + z^2 - 1 = 0; \\ (5) & x - y - z - 1 = 0. \end{array}$$

   (2) - (3) leads us to

   $$-2\lambda_1(y - z) = 0.$$

   There are two cases to consider: Case 1: $\lambda_1 = 0$ and Case 2: $y = z$.

   Case 1: $\lambda_1 = 0$

   Plugging $\lambda_1 = 0$ into (1), we have $\lambda_2 = 1$. Next, plugging $\lambda_1 = 0$ and $\lambda_2 = 1$ into (2), we obtain $2 = 0$, which is a contradiction. So, we ignore this case.

   Case 2: $y = z$

   Taking $y = z$ into account in (5), we have $x = 2y + 1$. Plugging $x = 2y + 1$ and $z = y$ into (4), we have

   $$(2y + 1)^2 + y^2 + y^2 - 1 = 0 \Rightarrow 6y^2 + 4y = 6y(y + 2/3) = 0.$$

   There are two cases to consider: Case A: $y = 0$ and Case B: $y = -2/3$.

   Case A: $y = 0$

   In this case, we thus obtain $(x, y, z) = (1, 0, 0)$. Plugging $y = 0$ into (2), we obtain $\lambda_2 = -1$. Plugging $x = 2$ and $\lambda_2 = -1$ into (1), we obtain $\lambda_1 = 1$. Thus, the Lagrange multipliers associated with this case are $(\lambda_1, \lambda_2) = (1, -1)$.

   Case B: $y = -2/3$

   In this case, we have $(x, y, z) = (-1/3, -2/3, -2/3)$. Plugging $x = -1/3$ into (1) and plugging $y = -2/3$ into (2), we obtain

   $$\begin{array}{ll} (1') & 1 + \dfrac{2}{3}\lambda_1 - \lambda_2 = 0; \\ (2') & 1 + \dfrac{4}{3}\lambda_1 + \lambda_2 = 0. \end{array}$$

$(1') + (2')$ allows us to obtain $\lambda_1 = -1$. Plugging $\lambda_1 = -1$ into $(1')$, we obtain $\lambda_2 = 1/3$. Thus, the Lagrangian multipliers associated with this case are $(\lambda_1, \lambda_2) = (-1, 1/3)$. Therefore, we obtain two solution candidates with the associated Lagrange multipliers:

$$(x, y, z, \lambda_1, \lambda_2) = \begin{cases} (1, 0, 0, 1, -1) \\ (-1/3, -2/3, -2/3, -1, 1/3) \end{cases}$$

2. Classify each solution candidate as either a local maximum point or local minimum point You are required to leave all the details of your computation.

   Let $g^1(x, y, z) = x^2 + y^2 + z^2 - 1$ and $g^2(x, y, z) = x - y - z - 1$. We then compute

$$\begin{aligned} \nabla g^1(x, y, z) &= (\partial g^1/\partial x, \partial g^1/\partial y, \partial g^1/\partial z) = (2x, 2y, 2z); \\ \nabla g^2(x, y, z) &= (\partial g^2/\partial x, \partial g^2/\partial y, \partial g^2/\partial z) = (1, -1, -1). \end{aligned}$$

   We also compute

$$\begin{aligned} \mathcal{L}''_{11} &= -2\lambda_1; \\ \mathcal{L}''_{12} = \mathcal{L}''_{21} &= 0; \\ \mathcal{L}''_{13} = \mathcal{L}''_{31} &= 0; \\ \mathcal{L}''_{22} &= -2\lambda_1; \\ \mathcal{L}''_{23} = \mathcal{L}''_{32} &= 0; \\ \mathcal{L}''_{33} &= -2\lambda_1. \end{aligned}$$

   We form the Bordered Hessian matrix associated with this problem:

$$\begin{aligned} B(x, y, z) &= \begin{pmatrix} 0 & 0 & \partial g^1/\partial x & \partial g^1/\partial y & \partial g^1/\partial z \\ 0 & 0 & \partial g^2/\partial x & \partial g^2/\partial y & \partial g^2/\partial z \\ \partial g^1/\partial x & \partial g^2/\partial x & \mathcal{L}''_{11} & \mathcal{L}''_{12} & \mathcal{L}''_{13} \\ \partial g^1/\partial y & \partial g^2/\partial y & \mathcal{L}''_{21} & \mathcal{L}''_{22} & \mathcal{L}''_{23} \\ \partial g^1/\partial z & \partial g^2/\partial z & \mathcal{L}''_{31} & \mathcal{L}''_{32} & \mathcal{L}''_{33} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 1 & -1 & -1 \\ 2x & 1 & -2\lambda_1 & 0 & 0 \\ 2y & -1 & 0 & -2\lambda_1 & 0 \\ 2z & -1 & 0 & 0 & -2\lambda_1 \end{pmatrix} \end{aligned}$$

   We evaluate the determinant of the bordered Hessian matrix at each solution candidate.

   Case I: $(x, y, z, \lambda_1, \lambda_2) = (1, 0, 0, 1, -1)$

We compute the following:

$$|B(1,0,0)| = \begin{vmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & -2 \end{vmatrix} = 2\cdot(-1)^{1+3} \begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{vmatrix} = -16,$$

where

$$\begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{vmatrix} = 2\cdot(-1)^{2+1} \begin{vmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{vmatrix} + 1\cdot(-2)^{2+2} \underbrace{\begin{vmatrix} 0 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}}_{=0},$$

where

$$\begin{aligned} \begin{vmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{vmatrix} &= (-1)\cdot(-1)^{1+2} \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} + (-1)\cdot(-1)^{1+3} \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} \\ &= (2-0)-(0-2) = 4. \end{aligned}$$

Hence, $(1,0,0)$ is a local maximum point.

Case II: $(x,y,z,\lambda_1,\lambda_2) = (-1/3,-2/3,-2/3,-1,1/3)$

We compute the following:

$$\begin{aligned} |B(-1/3,-2/3,-2/3)| &= \begin{vmatrix} 0 & 0 & -2/3 & -4/3 & -4/3 \\ 0 & 0 & 1 & -1 & -1 \\ -2/3 & 1 & 2 & 0 & 0 \\ -4/3 & -1 & 0 & 2 & 0 \\ -4/3 & -1 & 0 & 0 & 2 \end{vmatrix} \\ &= 1\cdot(-1)^{2+3} \begin{vmatrix} 0 & 0 & -4/3 & -4/3 \\ -2/3 & 1 & 0 & 0 \\ -4/3 & -1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} \\ &\quad +(-1)\cdot(-1)^{2+4} \begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} \\ &\quad +(-1)\cdot(-1)^{2+5} \begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \\ -4/3 & -1 & 0 & 0 \end{vmatrix} \\ &= -\left(-\frac{32}{3}\right) - \left(-\frac{8}{3}\right) + \frac{8}{3} = \frac{48}{3} = 16, \end{aligned}$$

where

$$\begin{aligned}
\begin{vmatrix} 0 & 0 & -4/3 & -4/3 \\ -2/3 & 1 & 0 & 0 \\ -4/3 & -1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} &= -\frac{4}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix} \\
&\quad -\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix} \\
&= -\frac{4}{3}\cdot 4 + \frac{4}{3}\cdot(-4) = -\frac{32}{3} \\
\begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 0 \\ -4/3 & -1 & 0 & 2 \end{vmatrix} &= -\frac{2}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix} \\
&\quad -\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix} \\
&= -\frac{2}{3}\cdot 4 + \frac{4}{3}\cdot 0 = -\frac{8}{3} \\
\begin{vmatrix} 0 & 0 & -2/3 & -4/3 \\ -2/3 & 1 & 2 & 0 \\ -4/3 & -1 & 0 & 2 \\ -4/3 & -1 & 0 & 0 \end{vmatrix} &= -\frac{2}{3}(-1)^{1+3}\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix} \\
&\quad -\frac{4}{3}(-1)^{1+4}\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix} \\
&= -\frac{2}{3}\cdot(-4) + \frac{4}{3}\cdot 0 = \frac{8}{3},
\end{aligned}$$

where

$$\begin{aligned}
\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 2 \end{vmatrix} &= 2\cdot(-1)^{3+3}\begin{vmatrix} -2/3 & 1 \\ -4/3 & -1 \end{vmatrix} = 2(2/3+4/3) = 4 \\
\begin{vmatrix} -2/3 & 1 & 0 \\ -4/3 & -1 & 2 \\ -4/3 & -1 & 0 \end{vmatrix} &= 2\cdot(-1)^{2+3}\begin{vmatrix} -2/3 & 1 \\ -4/3 & -1 \end{vmatrix} = -2\cdot 2 = -4 \\
\begin{vmatrix} -2/3 & 1 & 2 \\ -4/3 & -1 & 0 \\ -4/3 & -1 & 0 \end{vmatrix} &= 2\cdot(-1)^{1+3}\begin{vmatrix} -4/3 & -1 \\ -4/3 & -1 \end{vmatrix} = 0.
\end{aligned}$$

4 Homework 4: Submission is not Required

Question 4.1 *Consider the following minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} x^2 + y^2 \text{ subject to } (x-1)^3 - y^2 = 0.$$

Answer the following questions.

1. Explicitly show that the Lagrangian method does not work.

   We set up the Lagrangian function for this problem:

   $$\mathcal{L}(x,y) = x^2 + y^2 - \lambda[(x-1)^3 - y^2].$$

   The first-order conditions are obtained as follows:

   $$\begin{aligned} &(1) \quad \mathcal{L}_x' = 2x - 3\lambda(x-1)^2 = 0 \\ &(2) \quad \mathcal{L}_y' = 2y + 2\lambda y = 0 \\ &(3) \quad (x-1)^3 - y^2 = 0. \end{aligned}$$

   From (2), we have two cases: Case 1: $y = 0$ and Case 2: $y \neq 0$.

   Case 1: $y = 0$

   From (3), we must have $x = 1$. However, plugging $x = 1$ into (1), we obtain

   $$2 = 0.$$

   This is simply impossible.

   Case 2: $y \neq 0$

   From (2), we obtain $\lambda = -1$. Plugging $\lambda = -1$ into (1), we obtain

   $$3x + 3(x-1)^2 = 0 \Leftrightarrow 3x^2 - 4x + 3 = 0.$$

   However, we confirm

   $$3x^2 - 4x + 3 = 3\left(x - \frac{2}{3}\right)^2 + \frac{5}{3} > 0.$$

   Once again, this is simply impossible. Therefore, we obtain no solutions from the Lagrangian method.

2. Find the solution to the minimization problem. When answering this question, you are required to clarify your argument for this.

From the equality constraint, we have

$$(x-1)^3 = y^2.$$

Since $y^2 \geq 0$, the above equality ensures that we must have $x-1 \geq 0$. That is, we have $x \geq 1$. To minimize x^2+y^2 subject to $(x-1)^3 - y^2 = 0$, we propose $(x^*, y^*) = (1, 0)$ as the solution to the corresponding minimization problem with an equality constraint.

Fix any alternative vector (x, y) such that (1) $x > 1$ and (2) $(x-1)^3 = y^2$. This means that we have $x^2 + y^2 > 1$. Therefore, $(x^*, y^*) = (1, 0)$ is the solution to the constrained minimization problem.

3. Explain why the Lagrangian method does not work.

   Let $g(x, y) = (x-1)^3 - y^2$. Then, we have $\nabla g(x, y) = (\partial g(x, y)/\partial x, \partial g(x, y)/\partial y) = (3(x-1)^2, -2y)$. We therefore have $\nabla g(x^*, y^*) = \nabla g(1, 0) = (0, 0)$. This implies that the constraint qualification does not hold. This is the reason why the Lagrangian method does not work here.

Question 4.2 *Consider the following problem:*

$$\max_{(x,y)\in\mathbb{R}^2} f(x, y) = 1 - (x-2)^2 - y^2 \text{ subject to } x^2 + y^2 \leq a, \; x - y \leq 0,$$

where a is a positive constant. Answer the following questions.

1. Write down the Kuhn-Tucker conditions for this constrained maximization problem.

   We set up the Lagrangian:

   $$\mathcal{L}(x, y, \lambda) = 1 - (x-2)^2 - y^2 - \lambda_1(x^2 + y^2 - a) - \lambda_2(x - y),$$

   where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. The Kuhn-Tucker conditions are

   $$\begin{aligned}
   &(1) \quad \mathcal{L}_x' = -2(x-2) - 2\lambda_1 x - \lambda_2 = 0,\\
   &(2) \quad \mathcal{L}_y' = -2y - 2\lambda_1 y + \lambda_2 = 0,\\
   &(3) \quad \lambda_1 \geq 0;\; x^2 + y^2 - a \leq 0; \;\text{ and }\; \lambda_1(x^2 + y^2 - a) = 0,\\
   &(4) \quad \lambda_2 \geq 0;\; x - y \leq 0; \;\text{ and }\; \lambda_2(x - y) = 0.
   \end{aligned}$$

2. Find the solution of this constrained maximization problem for all positive values of a.

   We will find all the solution candidates satisfying the Kuhn-Tucker condition by considering the following four cases:

   Case 1: $\lambda_1 > 0$ and $\lambda_2 > 0$

By (4), we have $x = y$. Plugging $y = x$ into (1) and (2), we obtain

$$\begin{aligned} &(1') \quad -2(x-2) - 2\lambda_1 x - \lambda_2 = 0 \\ &(2') \quad -2x - 2\lambda_1 x + \lambda_2 = 0. \end{aligned}$$

$(1') - (2')$ leads us to obtain $\lambda_2 = 2$. Plugging $\lambda_2 = 2$ into $(2')$, we obtain

$$x = \frac{1}{1+\lambda_1}$$

By (3), we know $\lambda_1 \geq 0$ so that $x > 0$. Plugging $y = x$ into $x^2 + y^2 = a$ together with $x > 0$, we obtain $(x, y, \lambda_1, \lambda_2) = (\sqrt{a/2}, \sqrt{a/2}, \sqrt{2/a} - 1, 2)$ as a solution candidate. Moreover, from our hypothesis, we must have $\lambda_2 > 0$, which implies that $0 < a < 2$. So, we have a solution candidate $(x, y, \lambda_1, \lambda_2) = (\sqrt{a/2}, \sqrt{a/2}, \sqrt{2/a} - 1, 2)$ for all $a \in (0, 2)$.

Case 2: $\lambda_1 > 0$ **and** $\lambda_2 = 0$

Plugging $\lambda_2 = 0$ into (2), we obtain

$$-2y - 2\lambda_1 y = 0 \Rightarrow (1 + \lambda_1)y = 0.$$

Since $\lambda_1 > 0$ is assumed, we have $y = 0$. Plugging $\lambda_2 = 0$ into (1), we obtain

$$-2x + 4 - 2\lambda_1 x = 0 \Rightarrow x = \frac{2}{1+\lambda_1}.$$

Since we assume $\lambda_1 > 0$, we have $x > 0$. On the other hand, by (4), we have $x - y \leq 0$. Plugging $y = 0$ into $x - y \leq 0$, we obtain $x \leq 0$. This is a contradiction. So, we have no solution candidates in this case.

Case 3: $\lambda_1 = 0$ **and** $\lambda_2 > 0$

Since $\lambda_2 > 0$, by (4), we have $x = y$. As we have in Case 1, we have two equations, $(1')$ and $(2')$. Plugging $\lambda_1 = 0$ into $(1')$ and $(2')$ respectively, we obtain

$$\begin{aligned} &(1'') \quad -2(x-2) - \lambda_2 = 0 \\ &(2'') \quad -2x + \lambda_2 = 0. \end{aligned}$$

From $(2'')$, we have $x = \lambda_2/2$. Plugging $x = \lambda_2/2$ into $(1'')$, we have $\lambda_2 = 2$. Thus, we also obtain $x = 1$. In this case, we have $(x, y) = (1, 1)$. However, by (3), we must have

$$x^2 + y^2 \leq a \underbrace{\Rightarrow}_{x=y=1} 2 \leq a.$$

Hence, in this case, we have a unique solution candidate $(x, y, \lambda_1, \lambda_2) = (1, 1, 0, 2)$ for all $a \geq 2$.

Case 4: $\lambda_1 = 0$ **and** $\lambda_2 = 0$

Plugging $\lambda_1 = \lambda_2 = 0$ into (1), we obtain $x = 2$. Plugging $\lambda_1 = \lambda_2 = 0$ into (2), we obtain $y = 0$. However, $(x, y) = (2, 0)$ implies $x - y = 2 - 0 > 0$, which contradicts $x - y \leq 0$ in (4). So, we have no solution candidates in this case.

By the Kuhn-Tucker conditions, we have the unique solution candidate: for each $a \in (0, \infty)$,

$$(x^*(a), y^*(a), \lambda_1(a), \lambda_2(a)) = \begin{cases} (\sqrt{a/2}, \sqrt{a/2}, \sqrt{2/a} - 1, 2) & \text{if } 0 < a < 2 \\ (1, 1, 0, 2) & \text{if } a \geq 2 \end{cases}$$

We compute the second-order derivatives of the Lagrangian:

$$\begin{aligned} \mathcal{L}''_{xx} &= -2 - 2\lambda_1 \\ \mathcal{L}''_{yy} &= -2 - 2\lambda_1 \\ \mathcal{L}''_{xy} &= 0 \\ \mathcal{L}''_{yx} &= 0. \end{aligned}$$

So, we have the Hessian matrix for the Lagrangian:

$$D^2\mathcal{L}(x, y, \lambda_1, \lambda_2) = \begin{pmatrix} -2 - 2\lambda_1 & 0 \\ 0 & -2 - 2\lambda_1 \end{pmatrix}.$$

We investigate this Hessian matrix by considering the following two cases:

Case I: $0 < a < 2$

We evaluate the Hessian matrix at $(\lambda_1, \lambda_2) = (\sqrt{2/a} - 1, 2)$:

$$D^2\mathcal{L}(x, y, \sqrt{2/a} - 1, 2) = \begin{pmatrix} -2 - 2(\sqrt{2/a} - 1) & 0 \\ 0 & -2 - 2(\sqrt{2/a} - 1) \end{pmatrix}.$$

We check the leading principal minor of $D^2\mathcal{L}(x, y, \sqrt{2/a} - 1, 2)$:

$$\begin{aligned} D^2_{(1)}\mathcal{L}(x, y, \sqrt{2/a} - 1, 2) = -2 - 2(\sqrt{2/a} - 1) = -2\sqrt{2/a} < 0, \\ D^2_{(2)}\mathcal{L}(x, y, \sqrt{a/2} - 1, 2) = 4 \cdot \frac{2}{a} = \frac{8}{a} > 0. \end{aligned}$$

So, $D^2\mathcal{L}(x, y, \sqrt{a/2} - 1, 2)$ is negative definite in all (x, y). This implies that $\mathcal{L}(x, y, \sqrt{2/a} - 1, 2)$ is strictly concave in all (x, y). Thus, the solution candidate we found for this case is indeed a solution to the constrained maximization problem.

Case II: $a \geq 2$

We evaluate the Hessian matrix at $(\lambda_1, \lambda_2) = (0, 2)$:

$$D^2\mathcal{L}(x, y, 0, 2) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

We check the leading principal minor of $D^2\mathcal{L}(x, y, 0, 2)$:

$$\begin{aligned} D^2_{(1)}\mathcal{L}(x, y, 0, 2) &= -2 < 0, \\ D^2_{(2)}\mathcal{L}(x, y, 0, 2) &= 4 > 0. \end{aligned}$$

So, $D^2\mathcal{L}(x, y, 0, 2)$ is negative definite in all (x, y). This implies that $\mathcal{L}(x, y, 0, 2)$ is strictly concave in all (x, y). Thus, the solution candidate we found for this case as well is indeed a solution to the constrained maximization problem.

3. For each positive value of a, let $(x^*(a), y^*(a))$ denote the solution to the constrained maximization problem. For each positive value of a, define $f^*(a) = f(x^*(a), y^*(a))$. Then, find $f^*(a)$ and $df^*(a)/da$.

   As we analyzed in the previous question, we divide our argument into the following two cases:

   Case A: $0 < a < 2$

   In this case, we have $(x^*(a), y^*(a)) = (\sqrt{a/2}, \sqrt{a/2})$. So,

   $$f^*(a) = 1 - (\sqrt{a/2} - 2)^2 - (\sqrt{a/2})^2 = -3 - a + 4\sqrt{a/2}.$$

   We also compute

   $$\frac{df^*(a)}{da} = -1 + 4 \cdot \frac{1}{2}(a/2)^{-1/2} \cdot \frac{1}{2} = (a/2)^{-1/2} - 1.$$

   Case B: $a \geq 2$

   In this case, we have $(x^*(a), y^*(a)) = (1, 1)$. So,

   $$f^*(a) = 1 - (1 - 2)^2 - 1^2 = -1.$$

   We also compute

   $$\frac{df^*(a)}{da} = 0.$$

Question 4.3 *Consider the following constrained minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} x^2 - 2y \;\; \textit{subject to} \;\; x^2 + y^2 \leq 1, \;\; x \geq 0, \;\; y \geq 0.$$

Answer the following questions.

1. Write down the Kuhn-Tucker conditions for this constrained optimization problem.

   We first translate the original minimization problem into the equivalent maximization problem:

   $$\max_{(x,y)\in\mathbb{R}^2} -x^2 + 2y \text{ subject to } x^2 + y^2 - 1 \leq 0,\ -x \leq 0,\ -y \leq 0.$$

   We next set up the Lagrangian:

   $$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3) = -x^2 + 2y - \lambda_1(x^2 + y^2 - 1) - \lambda_2(-x) - \lambda_3(-y),$$

   where λ_1, λ_2, and λ_3 are the Lagrange multipliers. Finally, we provide the Kuhn-Tucker conditions as follows:

   $$\begin{aligned}
   &(1) \quad \mathcal{L}_1^{'} = -2x - 2\lambda_1 x + \lambda_2 = 0;\\
   &(2) \quad \mathcal{L}_2^{'} = 2 - 2\lambda_1 y + \lambda_3 = 0;\\
   &(3) \quad \lambda_1 \geq 0,\ x^2 + y^2 \leq 1,\ \lambda_1(x^2 + y^2 - 1) = 0;\\
   &(4) \quad \lambda_2 \geq 0,\ x \geq 0,\ \lambda_2(-x) = 0;\\
   &(5) \quad \lambda_3 \geq 0,\ y \geq 0,\ \lambda_3(-y) = 0.
   \end{aligned}$$

2. Exhaust all the solution candidates using the Kuhn-Tucker approach.

   We search for the solution candidates by considering the following four cases:

   Case 1: $x > 0$ and $y > 0$

   By (4) and (5), we have $\lambda_2 = \lambda_3 = 0$. Plugging $\lambda_2 = 0$ into (1), we obtain

   $$2x(\lambda_1 + 1) = 0 \underbrace{\Rightarrow}_{x>0} \lambda_1 = -1,$$

   which contradicts the hypothesis that $\lambda_1 \geq 0$ in (3). Hence, there are no solution candidates in this case.

   Case 2: $x > 0$ and $y = 0$

   By (4), we have $\lambda_2 = 0$. Plugging $\lambda_2 = 0$ into (1), we obtain

   $$2x(\lambda_1 + 1) = 0 \underbrace{\Rightarrow}_{x>0} \lambda_1 = -1,$$

   which contradicts the hypothesis that $\lambda_1 \geq 0$ in (3). Hence, there are no solution candidates in this case.

   Case 3: $x = 0$ and $y > 0$

By (5), we have $\lambda_3 = 0$. Plugging $\lambda_3 = 0$ into (2), we obtain

$$2 - 2\lambda_1 y = 0 \Rightarrow \lambda_1 = 1/y \underbrace{\Rightarrow}_{y>0} \lambda_1 > 0.$$

Plugging $x = 0$ into (1), we obtain $\lambda_2 = 0$. Since $\lambda_1 > 0$, by (3), we have $x^2 + y^2 = 1$. Furthermore, since $x = 0$, this equation implies $y = \pm 1$. Since we assume $y > 0$, we have $y = 1$. It then follows from $\lambda_1 = 1/y$ that $\lambda_1 = 1$. Thus, we obtain the unique solution candidate: $(x, y, \lambda_1, \lambda_2, \lambda_3) = (0, 1, 1, 0, 0)$.

Case 4: $x = 0$ and $y = 0$

Plugging $y = 0$ into (2), we obtain

$$2 + \lambda_3 = 0 \Rightarrow \lambda_3 = -2,$$

which contradicts the hypothesis that $\lambda_3 \geq 0$ in (5). Hence, there are no solution candidates in this case.

Considering all four cases, we obtain the unique solution candidate: $(x, y) = (0, 1)$ associated with the Lagrange multipliers $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$.

3. Find the solution to this constrained optimization problem.

From the previous question, we obtain $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$. Plugging these values into the Lagrangian, we obtain

$$\mathcal{L}(x, y, 1, 0, 0) = -x^2 + 2y - (x^2 + y^2 - 1).$$

We compute the second-order derivatives of the Lagrangian:

$$\begin{aligned} \mathcal{L}''_{11} &= -4; \\ \mathcal{L}''_{12} = \mathcal{L}''_{21} &= 0; \\ \mathcal{L}''_{22} &= -2. \end{aligned}$$

Then, we form the Hessian matrix associated with the Lagrangian:

$$D^2\mathcal{L}(x, y, 1, 0, 0) = \begin{pmatrix} \mathcal{L}''_{11} & \mathcal{L}''_{12} \\ \mathcal{L}''_{21} & \mathcal{L}''_{22} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus, we have the following leading principal minors:

$$\begin{aligned} D_1 &= -4 < 0; \\ D_2 &= 8 > 0. \end{aligned}$$

This implies that the associated Hessian matrix is negative definite, which further implies that the Lagrangian function is strictly concave in (x, y), given $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$. Thus, by the sufficiency of the Kuhn-Tucker conditions, we conclude that the unique solution candidate we have obtained in the previous step is indeed the solution to the original constrained optimization problem.

Question 4.4 *Consider the following constrained minimization problem:*

$$\min_{(x,y)\in\mathbb{R}^2} 2x^2 + 2y^2 - 2xy - 9y \text{ subject to } 4x + 3y \leq 10,\ y - 4x^2 \geq -2,\ x \geq 0,\ y \geq 0.$$

Answer the following questions.

1. Set up the Lagrangian function and obtain the Kuhn-Tucker condition for this optimization problem.

   We first rewrite this constrained minimization problem into the constrained maximization one:

   $$\max_{(x,y)\in\mathbb{R}^2} -2x^2-2y^2+2xy+9y \text{ subject to } 4x+3y-10 \leq 0,\ -y+4x^2-2 \leq 0,\ -x \leq 0,\ -y \leq 0.$$

   We set up the Lagrangian $\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as follows:

   $$\begin{aligned}\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= -2x^2 - 2y^2 + 2xy + 9y - \lambda_1(4x + 3y - 10) \\ &\quad -\lambda_2(-y + 4x^2 - 2) - \lambda_3(-x) - \lambda_4(-y).\end{aligned}$$

   The Kuhn-Tucker condition is given below:

   (1) $\mathcal{L}_x' = -4x + 2y - 4\lambda_1 - 8\lambda_2 x + \lambda_3 = 0$
   (2) $\mathcal{L}_y' = -4y + 2x + 9 - 3\lambda_1 + \lambda_2 + \lambda_4 = 0$
   (3) $\lambda_1 \geq 0$; $4x + 3y - 10 \leq 0$; $\lambda_1(4x + 3y - 10) = 0$;
   (4) $\lambda_2 \geq 0$; $-y + 4x^2 - 2 \leq 0$; $\lambda_2(-y + 4x^2 - 2) = 0$;
   (5) $\lambda_3 \geq 0$; $x \geq 0$; $\lambda_3(-x) = 0$;
   (6) $\lambda_4 \geq 0$; $y \geq 0$; $\lambda_4(-y) = 0$.

2. Examine each of the following three cases: (i) $x = 0$ and $y = 0$; (ii) $x > 0$ and $y = 0$; and $x = 0$ and $y > 0$, separately and show that each case leads to a violation of the Kuhn-Tucker condition.

   Case (i) $x = 0$ **and** $y = 0$

   Plugging $x = y = 0$ into the inequalities in (3) and (4), we obtain

   $$\begin{aligned}4x + 3y - 10 &= -10 < 0 \\ -y + 4x^2 - 2 &= -2 < 0,\end{aligned}$$

   Since these two inequality constraints are not binding, by (3) and (4), we have $\lambda_1 = \lambda_2 = 0$. Next, plugging $x = y = 0$ and $\lambda_1 = \lambda_2 = 0$ into $(2')$, we obtain

   $$9 + \lambda_4 = 0 \Leftrightarrow \lambda_4 = -9,$$

which contradicts the requirement that $\lambda_4 \geq 0$ in (6). Thus, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

Case (ii) $x > 0$ **and** $y = 0$

Since $x > 0$, by (5), we have $\lambda_3 = 0$. Plugging $y = 0$ and $\lambda_3 = 0$ into (1), we obtain

$$-4x - 4\lambda_1 - 8\lambda_2 x = 0 \Leftrightarrow \lambda_1 = -(2\lambda_2 + 1)x.$$

Since $\lambda_2 \geq 0$ by (2) and we assume $x > 0$, we have $\lambda_1 < 0$, which contradicts the requirement that $\lambda_1 \geq 0$ in (1). Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

Case (iii) $x = 0$ **and** $y > 0$

Plugging $x = 0$ into (1), we obtain

$$2y - 4\lambda_1 + \lambda_3 = 0 \Leftrightarrow \lambda_3 = 4\lambda_1 - 2y.$$

Since $y > 0$ and $\lambda_3 \geq 0$ from (5), we have $\lambda_1 > 0$. Then, by (3), we have

$$4x + 3y - 10 = 0 \underbrace{\Rightarrow}_{\because\ x=0} y = 10/3.$$

Plugging $(x, y) = (0, 10/3)$ into the inequality in (4), we obtain

$$-y + 4x^2 - 2 = -10/3 - 2 < 0.$$

Thus, by (4), we have $\lambda_2 = 0$. Since $y > 0$, by (6), we have $\lambda_4 = 0$. Plugging $(x, y) = (0, 10/3)$ and $\lambda_2 = \lambda_4 = 0$ into (2), we obtain

$$-4y + 9 - 3\lambda_1 = 0 \Leftrightarrow 3\lambda_1 = -4(10/3) + 9 = -13/3,$$

which contradicts the requirement that $\lambda_1 \geq 0$. Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition in this case.

3. Find the solution candidate satisfying the Kuhn-Tucker condition.

   By the previous part of the question, we have that $x > 0$ and $y > 0$. By (5) and (6), we conclude that $\lambda_3 = \lambda_4 = 0$. Taking into account that $\lambda_3 = \lambda_4 = 0$, we simplify the Kuhn-Tucker condition as follows:

   (1$'$) $\mathcal{L}'_x = -4x + 2y - 4\lambda_1 - 8\lambda_2 x = 0$
   (2$'$) $\mathcal{L}'_y = -4y + 2x + 9 - 3\lambda_1 + \lambda_2 = 0$
   (3) $\lambda_1 \geq 0$; $4x + 3y - 10 \leq 0$; $\lambda_1(4x + 3y - 10) = 0$;
   (4) $\lambda_2 \geq 0$; $-y + 4x^2 - 2 \leq 0$; $\lambda_2(-y + 4x^2 - 2) = 0$;

Case 1: $\lambda_1 > 0$ **and** $\lambda_2 > 0$

Since $\lambda_1 > 0$, by (3), we have

$$y = \frac{1}{3}(-4x + 10).$$

Since $\lambda_2 > 0$, by (4), we have

$$-y + 4x^2 - 2 = 0.$$

Combining the above two equations, we obtain

$$12x^2 + 4x - 16 = 0 \Leftrightarrow (3x + 4)(x - 1) = 0.$$

Thus, $x = -4/3$ or 1. However, by the previous part of this question, we conclude that $x > 0$ so that we must have $x = 1$. This implies that $y = 2$. Plugging $x = 1$ and $y = 2$ into $(1^{'})$ and $(2^{'})$, we obtain

$(1^{''})$ $\mathcal{L}_x^{'} = -4 + 4 - 4\lambda_1 - 8\lambda_2 = 0$
$(2^{''})$ $\mathcal{L}_y^{'} = -8 + 2 + 9 - 3\lambda_1 + \lambda_2 = 0$

Solving the above two equations, we obtain $\lambda_2 = -1/2$, which contradicts (4). So, there is no solution candidate in this case.

Case 2: $\lambda_1 > 0$ **and** $\lambda_2 = 0$

Since $\lambda_1 > 0$, by (3), we have

$$y = \frac{1}{3}(-4x + 10).$$

Plugging the above equation and $\lambda_2 = 0$ into $(1^{'})$ and $(2^{'})$, we obtain

$(1^{''})$ $\mathcal{L}_x^{'} = -4x + \frac{2}{3}(-4x + 10) - 4\lambda_1 = 0 \Leftrightarrow -5x + 5 - 3\lambda_1 = 0$
$(2^{''})$ $\mathcal{L}_y^{'} = -\frac{4}{3}(-4x + 10) + 2x + 9 - 3\lambda_1 = 0 \Leftrightarrow 22x - 13 - 9\lambda_1 = 0$

Solving the system of the above two equations, we obtain $x = 28/37$ and $\lambda_1 = 15/37 > 0$. Plugging $x = 28/37$ into $y = (-4x + 10)/3$, we obtain $y = 86/37 > 0$. Plugging $x = 28/37$ and $y = 86/37$ into $-y + 4x^2 - 2$, we obtain

$$-y + 4x^2 - 2 = -\frac{86}{37} + 4 \times \frac{28^2}{37^2} - 2 < -2 + 4 - 2 = 0,$$

where we take into account that $-86/37 < -2$ and $(28^2)/(37^2) < 1$. Therefore, the inequality in (4) is satisfied. In conclusion, $(x, y) = (28/37, 86/37)$ is a solution candidate satisfying the Kuhn-Tucker condition.

Case 3: $\lambda_1 = 0$ **and** $\lambda_2 > 0$

Plugging $\lambda_1 = 0$ into $(1^{'})$ and $(2^{'})$, we obtain

$(1'')$ $\mathcal{L}'_x = -4x + 2y - 8\lambda_2 x = 0$
$(2'')$ $\mathcal{L}'_y = -4y + 2x + 9 + \lambda_2 = 0$

Since $\lambda_2 > 0$, by (4), we have

$$-y + 4x^2 - 2 = 0.$$

Plugging $y = 4x^2 - 2$ into $(2'')$, we obtain

$$\lambda_2 = 4(4x^2 - 2) - 2x - 9 = 16x^2 - 2x - 17.$$

Since we must have $x > 0$ and $\lambda_2 \geq 0$, we have

$$x \geq \frac{1 + \sqrt{263}}{16} > \frac{1 + 16}{16} > 1.$$

Hence, $x > 1$. Taking into account $y = 4x^2 - 2$, we compute the following:

$$4x+3y-10 = 4x+3(4x^2-2)-10 = 12x^2+4x-16 = 4(3x^2+x-4) = 4(3x+4)(x-1).$$

Since we must have $4x + 3y - 10 \leq 0$ by (3), we have $-4/3 \leq x \leq 1$. However, this contradicts the previous conclusion that $x > 1$. So, there are no solution candidates in this case.

Case 4: $\lambda_1 = 0$ **and** $\lambda_2 = 0$

Plugging $\lambda_1 = \lambda_2 = 0$ into $(1')$ and $(2')$, we obtain

$$\begin{aligned} -4x + 2y &= 0 \\ -4y + 2x + 9 &= 0. \end{aligned}$$

We obtain $(x, y) = (3/2, 3)$ as the unique solution to the above system of linear equations. Plugging $(x, y) = (3/2, 3)$ into $4x + 3y - 10$, we obtain

$$4x + 3y - 10 = 4(3/2) + 3 \cdot 3 - 10 = 5,$$

which contradicts the inequality $4x + 3y - 10 \leq 0$ in (3). Therefore, there are no solution candidates satisfying the Kuhn-Tucker condition.

Considering all the four cases above, we have

$$(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (28/37, 86/37, 15/37, 0, 0, 0)$$

as the unique solution candidate satisfying the Kuhn Tucker condition.

5 Homework 5 (Due Date: October 22 (Tue), 2024): Submission is Required and will be Graded

Question 5.1 (20 points) *Solve the following differential equations.*

1. $\dot{x} = t^3 - t$
$$x(t) = \frac{t^4}{4} - \frac{t^2}{2} + C,$$
where C is a constant.

2. $\dot{x} = te^t - t$
$$x(t) = te^t - e^t - \frac{t^2}{2} + C,$$
where C is a constant.

3. $\dot{x}e^x = t + 1$
$$x(t) = ln\left(\frac{t^2}{2} + t + C\right),$$
where C is a constant.

4. $x^2\dot{x} = t + 1$
$$x(t) = \left(\frac{3}{2}t^2 + 3t + C\right)^{1/3},$$
where C is a constant.

Question 5.2 (20 points) *Find the general solutions of the following differential equations. Find also the integral curves through the indicated points.*

1. $t\dot{x} = x(1-t),\ (t_0, x_0) = (1, 1/e)$

Note that $x = 0$ is a trivial solution. Now, we consider any other solutions. We obtain
$$\frac{dx}{x} = \frac{1-t}{t}dt \Rightarrow \int \frac{1}{x}dx = \int \left(\frac{1}{t} - 1\right) dt \Rightarrow \ln|x| = \ln|t| - t + C,$$
where C is a constant. We manipulate the expression of $\ln|x| = \ln|t| - t + C$ as follows:
$$t - C = \ln\frac{|t|}{|x|} \Rightarrow e^{t-C} = \frac{|t|}{|x|} \Rightarrow |x| = e^{-C}|t|e^{-t}.$$
Setting $C_1 = e^{-C} > 0$, we have $x = C_1te^{-t}$ as the general solution.

Plugging $(t_0, x_0) = (1, 1/e)$ into $x = C_1te^{-t}$, we obtain
$$e^{-1} = Ce^{-1} \Rightarrow C_1 = 1.$$
Therefore, we have $x = te^{-t}$ as the integral curve through $(t_0, x_0) = (1, 1/e)$.

2. $(1+t^3)\dot{x} = t^2 x,\ (t_0, x_0) = (0, 2)$

   Note that $x = 0$ is a trivial solution. Now, we consider any other solutions. We obtain

   $$\frac{dx}{x} = \frac{t^2}{1+t^3}dt \Rightarrow \int \frac{1}{x}dx = \int \frac{t^2}{1+t^3}dt \Rightarrow \ln|x| = \frac{1}{3}\ln|1+t^3| + C,$$

   where C is a constant. Setting $C_2 = e^C$, we obtain

   $$\ln \frac{|x|}{|1+t^3|^{1/3}} = C \Rightarrow \frac{x}{(1+t^3)^{1/3}} = C_2 \Rightarrow x = C_2(1+t^3)^{1/3}.$$

   So, $x = C_2(1+t^3)^{1/3}$ is the general solution. Plugging $(t_0, x_0) = (0, 2)$ into the general solution, we have $C_2 = 2$. Therefore, we have $x = 2(1+t^3)^{1/3}$ as the integral curve through $(t_0, x_0) = (0, 2)$.

3. $x\dot{x} = t,\ (t_0, x_0) = (\sqrt{2}, 1)$

   Note that $x = 0$ is not a solution. So, we always have $x \neq 0$. We obtain

   $$xdx = tdt \Rightarrow \int xdx = \int tdt \Rightarrow \frac{x^2}{2} = \frac{t^2}{2} + C/2,$$

   where C is a constant. Since $x^2 > 0$, we must have $t^2 + C > 0$. Thus, we have $x = \pm\sqrt{t^2 + C}$ as the general solution. Plugging $(t_0, x_0) = (\sqrt{2}, 1)$ into the general solution, we obtain $x > 0$ and $C = -1$. So, we have $x = \sqrt{t^2 - 1}$ as the integral curve through $(t_0, x_0) = (\sqrt{2}, 1)$.

4. $e^{2t}\dot{x} - x^2 - 2x = 1,\ (t_0, x_0) = (0, 0)$

   We obtain

   $$\frac{dx}{(1+x)^2} = e^{-2t}dt \Rightarrow \int \frac{dx}{(1+x)^2} = \int e^{-2t}dt \Rightarrow -\frac{1}{1+x} = -\frac{1}{2}e^{-2t} + C,$$

   where C is a constant. Multiplying $2(1+x)$ to both hand sides of the derived equation, we have

   $$-2 = -\frac{1+x}{2}e^{-2t} + 2C(1+x) \Rightarrow x = \frac{2 - e^{-2t} + C}{e^{-2t} - C}$$

   So, we have $x = (2 - e^{-2t} + C)/(e^{-2t} - C)$ as the general solution. Plugging $(t_0, x_0) = (0, 0)$ into the general solution, we obtain $C = -1$. Therefore,

   $$x = \frac{1 - e^{-2t}}{1 + e^{-2t}}$$

   is the integral curve through $(t_0, x_0) = (0, 0)$.

Question 5.3 (30 points) *Find the general solutions of the following differential equations.*

1. $$t\dot{x} + 2x + t = 0 \ (t \neq 0)$$

   Since $t \neq 0$, we can rewrite this equation as

   $$\dot{x} + \frac{2}{t}x = -1.$$

   Define $A(t) = \int (2/t)dt = 2\ln t + K = \ln t^2 + K$, where K is a constant. Here we choose $K = 0$ so that $A(t) = \ln t^2$ because we obtain the same general solution by choosing another value for K. Then, using the formula, we have

   $$\begin{aligned} x &= e^{-A(t)}\left(C - \int e^{A(t)}dt\right) \\ &= e^{-\ln t^2}\left(C - \int e^{\ln t^2}dt\right) \\ &= \frac{1}{t^2}\left(C - \int t^2 dt\right) \\ &= \frac{1}{t^2}\left(C - \frac{t^3}{3}\right) \\ &= \frac{C}{t^2} - \frac{t}{3}. \end{aligned}$$

   where C is a constant.

2. $$\dot{x} - \frac{1}{t}x = t \ (t > 0)$$

   Define $A(t) = -\int \frac{1}{t}dt = -\ln t + K = \ln t^{-1} + K$, where K is a constant. Here we choose $K = 0$ so that $A(t) = \ln t^{-1}$ because we obtain the same general solution by choosing another value for K. Then, using the formula, we have

   $$\begin{aligned} x &= e^{-A(t)}\left(C + \int te^{A(t)}dt\right) \\ &= e^{-\ln t^{-1}}\left(C + \int te^{\ln t^{-1}}dt\right) \\ &= t\left(C + \int dt\right) \\ &= t(C + t) = t^2 + Ct. \end{aligned}$$

   where C is a constant.

3.

$$\dot{x} - \frac{t}{t^2-1}x = t \ (t > 1)$$

Define

$$A(t) = -\int \frac{t}{t^2-1}dt = -\frac{1}{2}\ln(t^2-1) + K = \ln(t^2-1)^{-1/2} + K,$$

where K is a constant. Here we choose $K = 0$ so that $A(t) = \ln(t^2-1)^{-1/2}$ because we obtain the same general solution by choosing another value for K. Then, using the formula, we have

$$\begin{aligned} x &= e^{-A(t)}\left(C + \int te^{A(t)}dt\right) \\ &= e^{-\ln(t^2-1)^{-1/2}}\left(C + \int te^{\ln(t^2-1)^{-1/2}}dt\right) \\ &= (t^2-1)^{1/2}\left(C + \int t(t^2-1)^{-1/2}dt\right) \\ &= (t^2-1)^{1/2}\left(C + (t^2-1)^{1/2}\right) \\ &= C(t^2-1)^{1/2} + t^2 - 1. \end{aligned}$$

4.

$$\dot{x} - \frac{2}{t}x + \frac{2a^2}{t^2} = 0 \ (t > 0)$$

We rewrite this equation as follows:

$$\dot{x} - \frac{2}{t}x = -\frac{2a^2}{t^2}.$$

Define

$$A(t) = -\int \frac{2}{t}dt = -2\ln t + K = \ln t^{-2} + K,$$

where K is a constant. Here we choose $K = 0$ so that $A(t) = \ln t^{-2}$ because we obtain the same general solution by choosing another value for K. Then, using the formula, we have

$$\begin{aligned} x &= e^{-A(t)}\left(C - \int \frac{2a^2}{t^2}e^{A(t)}dt\right) \\ &= e^{-\ln t^{-2}}\left(C - \int \frac{2a^2}{t^2}e^{\ln t^{-2}}dt\right) \\ &= t^2\left(C - \int \frac{2a^2}{t^4}dt\right) \\ &= t^2\left(C + \frac{2a^2}{3t^3}\right) \\ &= Ct^2 + \frac{2a^2}{3}t^{-1}, \end{aligned}$$

where C is a constant.

Question 5.4 (30 points) *Consider the following differential equation:*

$$\ddot{Y} + (\alpha s + \beta)\dot{Y} + \alpha\beta(s+m)Y = -\alpha\beta t - \frac{\alpha s + \beta}{s+m} \quad (*)$$

where $\alpha, \beta, s,$ and m are positive constants. Answer the following questions.

1. Find a particular solution of $(*)$.

   We assume that a particular solution u^* is the form of $u^* = At + B$, where A and B are constants. Then, $\dot{u}^* = A$ and $\ddot{u}^* = 0$. Plugging these into $(*)$, we obtain

   $$\begin{aligned} & A(\alpha s + \beta) + \alpha\beta(s+m)(At+B) = -\alpha\beta t - \frac{\alpha s+\beta}{s+m} \\ \Leftrightarrow \quad & \alpha\beta(s+m)At + [A(\alpha s+\beta) + \alpha\beta(s+m)B] = -\alpha\beta t - \frac{\alpha s+\beta}{s+m} \end{aligned}$$

   This implies that $A = -1/(s+m)$ and $B = 0$ so that $u^* = -t/(s+m)$.

2. Discuss conditions that ensure that the characteristic equation associated with $(*)$ has two complex roots.

   We consider the characteristic equation of the homogeneous part of equation $(*)$:

   $$r^2 + (\alpha s + \beta)r + \alpha\beta(s+m) = 0.$$

   A sufficient condition for $(*)$ to exhibits oscillations is that this characteristic equation has two complex roots. Hence, the corresponding condition is

   $$(\alpha s + \beta)^2 - 4\alpha\beta(s+m) < 0$$

3. For the rest of this question, we set $\alpha = 1/4,\ \beta = 3/4,\ s = 1$, and $m = 17/3$. Then, find the general solution to $(*)$.

   In this case, we have $u^* = -3t/20$. Equation $(*)$ becomes

   $$\ddot{Y} + \dot{Y} + \frac{5}{4}Y = -\frac{3}{16}t - \frac{3}{20}.$$

   We consider the homogeneous equation $\ddot{Y} + \dot{Y} + 5Y/4 = 0$. The characteristic equation of this homogeneous equation becomes

   $$r^2 + r + \frac{5}{4} = 0 \Leftrightarrow \left(r + \frac{1}{2}\right)^2 + 1 = 0.$$

   So, this equation has two complex roots. Using the formula, we obtain

   $$Y = e^{-t/2}(A\cos t + B\sin t),$$

where A and B are constants. Then, the general solution to $(*)$ is

$$Y = e^{-t/2}(A\cos t + B\sin t) - \frac{3}{20}t.$$

4. What can you say about the behavior of the solutions as $t \to \infty$?

   Since $e^{-t/2} \to 0$ as $t \to \infty$, we know that Y becomes approximately $-3t/20$, which is a linear function of t.

6 Homework 6 (Due Date: Oct 29 (Tue), 2024): Submission is Required but will not be Graded

Question 6.1 *Consider the following problem*

$$\max \int_0^T U(\bar{c} - \dot{x}e^{rt})dt, \;\; x(0) = x_0, \;\; x(T) = 0,$$

where $x = x(t)$ is the unknown function, $T, \bar{c}, r$, and x_0 are positive constants, and U is a given C^2 function of one variable. Answer the following questions.

1. Write down the Euler equation associated with this problem.

   Let $F(t, x, \dot{x}) = U(\bar{c} - \dot{x}e^{rt})$. We first compute the following:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= U^{'}(\bar{c} - \dot{x}e^{rt})\frac{\partial}{\partial x}(\bar{c} - \dot{x}e^{rt}) = 0 \\
\frac{\partial F}{\partial \dot{x}} &= U^{'}(\bar{c} - \dot{x}e^{rt}) \cdot (-e^{rt}) = -U^{'}(\bar{c} - \dot{x}e^{rt})e^{rt}.
\end{aligned}$$

   Next, we compute

$$\begin{aligned}
\frac{d}{dt}(-U^{'}(\bar{c} - \dot{x}e^{rt})e^{rt}) &= -U^{''}(\bar{c} - \dot{x}e^{rt}) \cdot (-\ddot{x}e^{rt} - r\dot{x}e^{rt})e^{rt} - rU^{'}(\bar{c} - \dot{x}e^{rt})e^{rt} \\
&= e^{2rt}(\ddot{x} + r\dot{x})U^{''}(\bar{c} - \dot{x}e^{rt}) - rU^{'}(\bar{c} - \dot{x}e^{rt})e^{rt}.
\end{aligned}$$

   Then, the Euler equation is obtained as

$$\begin{aligned}
& \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \\
\Leftrightarrow \;\; & -\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \\
\Leftrightarrow \;\; & rU^{'}(\bar{c} - \dot{x}e^{rt})e^{rt} - e^{2rt}(\ddot{x} + r\dot{x})U^{''}(\bar{c} - \dot{x}e^{rt}) = 0. \\
\Leftrightarrow \;\; & rU^{'}(\bar{c} - \dot{x}e^{rt}) - e^{rt}(\ddot{x} + r\dot{x})U^{''}(\bar{c} - \dot{x}e^{rt}) = 0.
\end{aligned}$$

2. In what follows, we let $U(c) = -e^{-vc}/v$, where v is a positive constant. Write down and solve the Euler equation in this case.

Since
$$\frac{d}{dt}(-U^{'}(\bar{c} - \dot{x}e^{rt})e^{rt}) = 0,$$
we have $U^{'}(\bar{c} - \dot{x}e^{rt})e^{rt} = K$, where K is a constant. We therefore obtain $U^{'}(\bar{c} - \dot{x}e^{rt}) = Ke^{-rt}$. Now, we know that $U(c) = -e^{-vc}/v$ so that we obtain $U^{'}(c) = e^{-vc}$. Therefore, we obtain the following equation:

$$Ke^{-rt} = e^{-v(\bar{c} - \dot{x}e^{rt})}.$$

Taking the natural logarithm on both hand sides of the above equation, we obtain
$$\ln K - rt = -v(\bar{c} - \dot{x}e^{rt})$$
We summarize this into
$$\dot{x} = (\bar{C} - rt/v)e^{-rt},$$
where $\bar{C} = \ln K/v + \bar{c}$. Taking integration on both hand sides of the above equation, we obtain
$$x = \int \left(\bar{C} - \frac{rt}{v}\right) e^{-rt} dt.$$
By integration by parts, we obtain

$$\begin{aligned} x &= (\bar{C} - rt/v)\left(\frac{e^{-rt}}{-r}\right) - \int \frac{-r}{v}\left(\frac{e^{-rt}}{-r}\right) dt = \left(-\frac{\bar{C}}{r} + \frac{t}{v}\right)e^{-rt} + \frac{e^{-rt}}{rv} + A \\ &= \left(-\frac{\bar{C}}{r} + \frac{t}{v} + \frac{1}{rv}\right)e^{-rt} + A \\ &= \left(B + \frac{t}{v}\right)e^{-rt} + A, \end{aligned}$$

where A is a constant and $B = -\bar{C}/r + 1/(rv)$, which is a constant. Since $x(0) = x_0$ and $x(T) = 0$, we have

$$\begin{aligned} x(0) &= A + B = x_0 \\ x(T) &= \left(B + \frac{T}{v}\right)e^{-rT} + A = 0. \end{aligned}$$

Solving this system of equations, we obtain

$$\begin{aligned} A &= \frac{x_0 + T/v}{1 - e^{rT}} \\ B &= \frac{e^{rT}x_0 + T/v}{e^{rT} - 1}. \end{aligned}$$

3. Explain why we also solve the problem in this case.

   If $U(c) = -e^{-vc}/v$, we have $U''(c) = -ve^{-vc} < 0$. This implies that U is strictly concave. Therefore, the Euler equation is sufficient for the optimality so that we solve the problem.

Question 6.2 *Solve the following problem:*

$$\min \int_0^1 (x^2 + (\dot{x})^2)dt \text{ subject to } x(0) = 1 \text{ and } x(1) \text{ free.}$$

Let $F(t, x, \dot{x}) = x^2 + (\dot{x})^2$. Then, we obtain

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2x, \\ \frac{\partial F}{\partial \dot{x}} &= 2\dot{x}. \end{aligned}$$

We also obtain

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2\ddot{x}.$$

So, the Euler equation is given as the following homogeneous differential equation:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \Rightarrow \ddot{x} - x = 0.$$

Its characteristic equation is $r^2 - 1 = 0$ so that we obtain $r = \pm 1$. This implies that $x(t)$ satisfying the Euler equation is expressed as

$$x(t) = Ae^t + Be^{-t},$$

where A and B are arbitrary constants. Since we have $x(0) = 1$, we have

$$x(0) = A + B = 1 \Rightarrow B = 1 - A.$$

It follows from the terminal condition that $x(1)$ free that we have the following transversality condition:

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=1} = 0 \Rightarrow \dot{x}(1) = 0.$$

From $x(t) = Ae^t + Be^{-t}$, we compute the following:

$$\dot{x}(t) = Ae^t - Be^{-t}.$$

Taking $\dot{x}(1) = 0$ and $B = 1 - A$ into account, we obtain

$$\dot{x}(1) = Ae - Be^{-1} = 0 \Rightarrow B = Ae^2 \Rightarrow 1 - A = Ae^2 \Rightarrow A = \frac{1}{1 + e^2}.$$

So, we also obtain

$$B = \frac{e^2}{1+e^2}.$$

Therefore, $x^*(t)$ satisfying the Euler equation is fully pinned down as follows:

$$x^*(t) = \frac{1}{1+e^2}e^t + \frac{e^2}{1+e^2}e^{-t}.$$

It only remains to show that $F(t,x,\dot{x})$ is convex in $(x,\dot{x})$. Since $F(t,x,\dot{x}) = x^2+(\dot{x})^2$ is a sum of two convex functions, x^2 and $(\dot{x})^2$, respectively, $F(t,x,\dot{x})$ is convex in $(x,\dot{x})$. Therefore, $x^*(t)$ specified above is indeed the solution to this minimization problem.

Question 6.3 *Solve the following problem:*

$$\min \int_0^1 (x^2 + (\dot{x})^2)dt \ \textit{ subject to } x(0) = 0 \textit{ and } x(1) \geq 1.$$

From the previous question, we obtain $x(t)$ as the one satisfying the Euler equation:

$$x(t) = Ae^t + Be^{-t}.$$

Since we have $x(0) = 0$, we obtain

$$x(0) = A + B = 0 \Rightarrow B = -A.$$

Then, we can write $x(t)$ satisfying the Euler equation as follows:

$$x(t) = A(e^t - e^{-t}).$$

Since $x(1) \geq 1$, we have

$$x(1) = A(e - e^{-1}) \geq 1.$$

Therefore, we also have $A > 0$. Assume $x(1) > 1$. Then, the following transversality condition holds:

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=1} = 0 \Rightarrow \dot{x}(1) = 0.$$

Since we have $\dot{x}(t) = A(e^t + e^{-t})$, we obtain

$$\dot{x}(1) = A(e + e^{-1}) = 0 \Rightarrow A = 0.$$

However, this contradicts the obtained property that $A > 0$. Therefore, we must satisfy $x(1) = 1$. Then,

$$x(1) = A(e - e^{-1}) = 1 \Rightarrow A = \frac{1}{e - e^{-1}} = \frac{e}{e^2 - 1}.$$

Since we have shown that $F(t,x,\dot{x})$ is convex in $(x,\dot{x})$ in the previous question, we obtain the following as the solution to the minimization problem.

$$x^*(t) = \frac{e}{e^2-1}(e^t - e^{-t}).$$

Question 6.4 *Consider a monopolist that produces a single commodity with a quadratic total cost function $C(Q)$:*

$$C(Q) = \alpha Q^2 + \beta Q + \gamma,$$

where $\alpha, \beta,$ and γ are positive constants. The quantity demanded is assumed to depend not only on its price $P(t)$, but also on the rate of change of its price $\dot{P}(t)$:

$$Q(t) = a - bP(t) + h\dot{P}(t),$$

where $a, b,$ and h are constants such that $a, b > 0$ and $h \neq 0$. Answer the following questions.

1. Show that the firm's profit function $\pi(P, \dot{P})$ is obtained a function of P and $\dot{P}$ as follows:

$$\begin{aligned} \pi(P, \dot{P}) &= -b(1+\alpha b)P^2 + (a + 2\alpha ab + \beta b)P - \alpha h^2(\dot{P})^2 \\ &\quad -h(2\alpha a + \beta)\dot{P} + h(1 + 2\alpha b)P\dot{P} - (\alpha a^2 + \beta a + \gamma). \end{aligned}$$

The firm's profit is

$$\begin{aligned} \pi &= PQ - C \\ &= P(a - bP + h\dot{P}) - \alpha(a - bP + h\dot{P})^2 - \beta(a - bP + h\dot{P}) - \gamma \\ &= -bP^2 + (a + \beta b)P + hP\dot{P} - \beta h\dot{P} - (\beta a + \gamma) \\ &\quad -\alpha(a^2 + b^2P^2 + h^2(\dot{P})^2 - 2abP - 2bhP\dot{P} + 2ah\dot{P}) \\ &= -(b + \alpha b^2)P^2 + (a + \beta b + 2\alpha ab)P + (h + 2\alpha bh)P\dot{P} - (\beta h + 2\alpha ah)\dot{P} \\ &\quad -(\alpha a^2 + \beta a + \gamma) \\ &= -b(1 + \alpha b)P^2 + (a + 2\alpha ab + \beta b)P - \alpha h^2(\dot{P})^2 - h(2\alpha a + \beta)\dot{P} \\ &\quad +h(1 + 2\alpha b)P\dot{P} - (\alpha a^2 + \beta\alpha + \gamma). \end{aligned}$$

2. The objective of the firm is to find an optimal path of price P that maximizes the total profit over a finite time period $[0, T]$. Assume that the initial price P_0 and the terminal price P_T are fixed. Therefore, the objective of the monopolist is summarized as follows:

$$\max \int_0^T \pi(P, \dot{P})dt \text{ s.t. } P(0) = P_0 \text{ and } P(T) = P_T.$$

Obtain the solution to the associated Euler equation.

We first compute the following:

$$\begin{aligned} \frac{\partial \pi}{\partial P} &= -2b(1 + \alpha b)P + (a + 2\alpha ab + \beta b) + h(1 + 2\alpha b)\dot{P} \\ \frac{\partial \pi}{\partial \dot{P}} &= -2\alpha h^2\dot{P} - h(2\alpha a + \beta) + h(1 + 2\alpha b)P. \end{aligned}$$

We further compute

$$\frac{d}{dt}\left(\frac{\partial \pi}{\partial \dot{P}}\right) = -2\alpha h^2 \ddot{P} + h(1+2\alpha b)\dot{P}.$$

By the Euler equation, we obtain

$$\begin{aligned}
& \frac{\partial \pi}{\partial P} - \frac{d}{dt}\left(\frac{\partial \pi}{\partial \dot{P}}\right) = 0 \\
\Leftrightarrow \quad & -2b(1+\alpha b)P + (a + 2\alpha ab + \beta b) + h(1+2\alpha b)\dot{P} - (-2\alpha h^2\ddot{P} + h(1+2\alpha b)\dot{P}) = 0 \\
\Leftrightarrow \quad & 2\alpha h^2 \ddot{P} - 2b(1+\alpha b)P = -(a + 2\alpha ab + \beta b) \\
\Leftrightarrow \quad & \ddot{P} - \frac{b(1+\alpha b)}{\alpha h^2}P = -\frac{a+2\alpha ab+\beta b}{2\alpha h^2}. \quad (*)
\end{aligned}$$

The characteristic equation of the homogeneous equation of $(*)$ is

$$r^2 - \frac{b(1+\alpha b)}{\alpha h^2} = 0.$$

So, we have

$$\begin{aligned}
r_1 &= \sqrt{\frac{b(1+\alpha b)}{\alpha h^2}} \\
r_2 &= -\sqrt{\frac{b(1+\alpha b)}{\alpha h^2}}.
\end{aligned}$$

Let $P(t) = C$ be a particular solution to $(*)$ where C is some constant. Then, we have

$$\begin{aligned}
-\frac{b(1+\alpha b)}{\alpha h^2}C &= -\frac{a+2\alpha ab+\beta b}{2\alpha h^2} \\
C &= \frac{a+2\alpha ab+\beta b}{2b(1+\alpha b)}.
\end{aligned}$$

Hence, the general solution to $(*)$ is

$$P(t) = Ae^{r_1 t} + Be^{r_2 t} + P^*,$$

where A and B are arbitrary constants and

$$P^* = \frac{a+2\alpha ab+\beta b}{2b(1+\alpha b)}.$$

Since $P(0) = P_0$ and $P(T) = P_T$, we have

$$\begin{aligned}
P_0 &= A + B + P^* \\
P_T &= Ae^{r_1 T} + Be^{r_2 T} + P^*.
\end{aligned}$$

Solving this system of equations, we can exactly pin down A and B as follows:

$$\begin{aligned}
A &= \frac{(P^* - P_0)e^{r_2 T} + (P_T - P^*)}{e^{r_1 T} - e^{r_2 T}} \\
B &= -\frac{(P^* - P_0)e^{r_1 T} + (P_T - P^*)}{e^{r_1 T} - e^{r_2 T}}
\end{aligned}$$

3. Do we know whether the solution to the Euler equation is a solution to the original maximization problem formulated in Part 2 of the question? Justify your answer.

   In what follows, we check whether or not the solution we obtained using the Euler equation is indeed a solution. This boils down to checking if $\pi(\cdot)$ is concave in $(P, \dot{P})$. We first compute the following:

$$\begin{aligned}
\frac{\partial^2 \pi}{\partial P^2} &= -2b(1+\alpha b) \\
\frac{\partial^2 \pi}{\partial (\dot{P})^2} &= -2\alpha h^2 \\
\frac{\partial^2 \pi}{\partial P \partial \dot{P}} = \frac{\partial^2 \pi}{\partial \dot{P} \partial P} &= h(1+2\alpha b).
\end{aligned}$$

   We form the Hessian matrix associated with $\pi(P, \dot{P})$:

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} -2b(1+\alpha b) & h(1+2\alpha b) \\ h(1+2\alpha b) & -2\alpha h^2 \end{pmatrix}.$$

   Since a, b, α, and β are positive and $h \neq 0$,

$$\begin{aligned}
h_{11} &= -2b(1+\alpha b) < 0, \\
|H| &= 4\alpha h^2 b(1+\alpha b) - h^2(1+2\alpha b)^2 = -h^2 < 0.
\end{aligned}$$

   This implies that H is neither negative semidefinite nor positive semidefinite. So, we do not know if the solution to the Euler equation is a solution to the original maximization problem.

7 Homework 7 (Due Date: Nov 5 (Tue), 2024): Submission is Required and will be graded

Question 7.1 (30 points) *Solve the following control problem:*

$$\max \int_0^1 -u^2 dt \ \ s.t. \ \dot{x} = x + u, \ x(0) = 1, \ x(1) \geq 3, \ and \ u \in \mathbb{R}.$$

Set up the Hamiltonian:

$$H = -u^2 + p(x+u).$$

Since $\partial^2 H / \partial u^2 = -2$, H is a concave function of u. Since u is unconstrained (i.e., $u \in \mathbb{R}$), $u^*(t)$ can be obtained via the first-order condition:

$$u^*(t) \in \arg\max_{u \in \mathbb{R}} H(t, x^*(t), u, p(t)) \Leftrightarrow \frac{\partial H(t, x^*(t), u^*(t), p(t))}{\partial u} = 0.$$

So,

$$\frac{\partial H}{\partial u} = -2u + p = 0 \Rightarrow u(t) = \frac{p(t)}{2}.$$

By the maximum principle, we have

$$\dot{p}(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t)) \Rightarrow \dot{p} = -p.$$

We solve this differential equation:

$$\begin{aligned}
& \frac{dp}{dt} = -p \\
\Leftrightarrow \quad & \frac{dp}{p} = -dt \\
\Leftrightarrow \quad & \int \frac{dp}{p} = -\int dt \\
\Leftrightarrow \quad & \ln p = -t + C_1 \\
\Leftrightarrow \quad & p = e^{C_1} e^{-t} \\
\Leftrightarrow \quad & p = k_1 e^{-t},
\end{aligned}$$

where C_1 is a constant and we set $k_1 = e^{C_1}$. If $x(1) > 3$, by the transversality condition, we have $p(1) = 0$. Thus,

$$k_1 e^{-1} = 0 \Rightarrow k_1 = 0.$$

So, we have $p(t) = 0$ for all $t \in [0, 1]$. This further implies that $u^*(t) = 0$ for all $t \in [0, 1]$. From the difference equation $\dot{x} = x + u$, we have $\dot{x} = x$. We solve this differential equation:

$$\begin{aligned}
& \frac{dx}{dt} = x \\
\Leftrightarrow \quad & \frac{dx}{x} = dt \\
\Leftrightarrow \quad & \int \frac{dx}{x} = \int dt \\
\Leftrightarrow \quad & \ln x = t + C_2 \\
\Leftrightarrow \quad & x = e^{C_2} e^{t} \\
\Leftrightarrow \quad & x = k_2 e^{t}
\end{aligned}$$

where C_2 is a constant and we set $k_2 = e^{C_2}$. Since $x(0) = 1$, we have $k_2 = 1$ so that $x = e^t$. However, this contradicts our hypothesis that $x(1) > 3$ because $x(1) = e < 3$. So, we exclude this case.

In what follows, we assume $x(1) = 3$. From $p(t) = k_1 e^{-t}$, we have

$$u(t) = \frac{k_1}{2} e^{-t}.$$

Plugging this into the difference equation $\dot{x} = x + u$, we obtain

$$\dot{x} = x + \frac{k_1}{2}e^{-t} \Rightarrow \dot{x} - x = \frac{k_1}{2}e^{-t}.$$

Multiplying the equation by e^{-t}, we have

$$\begin{aligned} & e^{-t}\dot{x} - e^{-t}x = \frac{k_1}{2}e^{-2t} \\ \Leftrightarrow\ & \frac{d}{dx}(e^{-t}x) = \frac{k_1}{2}e^{-2t} \\ \Leftrightarrow\ & e^{-t}x = \frac{k_1}{2}\int e^{-2t}dt \\ \Leftrightarrow\ & e^{-t}x = -\frac{k_1}{4}e^{-2t} + C_3 \\ \Leftrightarrow\ & x = C_3e^{t} - \frac{k_1}{4}e^{-t}, \end{aligned}$$

where C_3 is a constant. Since we have $x(0) = 1$ and we assume $x(1) = 3$, we have the following system of equations:

$$\begin{aligned} C_3 - \frac{k_1}{4} &= 1 \\ C_3e - \frac{k_1}{4}e^{-1} &= 3. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} C_3 &= \frac{3e-1}{e^2-1} \\ k_1 &= \frac{4e(3-e)}{e^2-1} \end{aligned}$$

Hence, using the maximum principle, we obtain the solution candidate as follows:

$$\begin{aligned} x^*(t) &= \frac{3e-1}{e^2-1}e^{t} + \frac{e(3-e)}{e^2-1}e^{-t} \\ p(t) &= \frac{4e(3-e)}{e^2-1}e^{-t} \\ u^*(t) &= \frac{2e(3-e)}{e^2-1}e^{-t}. \end{aligned}$$

Finally, we claim that the solution candidate we obtained using the maximum principle is indeed a solution. This is reduced to checking the Hamiltonian being concave in (u, x). Recall the Hamiltonian: $H = -u^2 + p(x + u)$. We compute the following:

$$\begin{aligned} H_{uu} &= -2 \\ H_{xx} &= 0 \\ H_{ux} = H_{xu} &= 0. \end{aligned}$$

So, we form the Hessian matrix associated with the Hamiltonian:

$$H = \begin{pmatrix} H_{uu} & H_{ux} \\ H_{xu} & H_{xx} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $H_{uu} = -2 < 0$, $H_{xx} = 0$, and $|H| = 0$, the Hessian matrix is associated with the Hamiltonian is negative semidefinite for all (u, x), which implies that H is concave in (u, x). This completes the argument.

Question 7.2 (30 points) *Consider the following problem:*

$$\max \int_0^2 (2x - 3u)dt \ \ s.t. \ \ \dot{x} = x + u, \ x(0) = 4, \ x(2) \textit{ free, and } u \in [0, 2].$$

Using the maximum principle, answer the following questions. Note also that $x(t)$ is assumed to be C^1, in particular, a continuous function.

1. Characterize the optimal control $u^*(t)$ in terms of $p(t)$, which is the adjoint function associated with the Hamiltonian.

   Define the Hamiltonian:

   $$H = 2x - 3u + p(x + u) = (2 + p)x + (p - 3)u.$$

   By the maximum principle, we have

   $$u^*(t) \in \arg \max_{u \in [0,2]} H(t, x^*(t), u, p(t)).$$

   Thus,

   $$u^*(t) = \begin{cases} 2 & \text{if } p(t) > 3, \\ [0, 2] & \text{if } p(t) = 3, \\ 0 & \text{if } p(t) < 3. \end{cases}$$

2. Determine $p(t)$.

   Once again, by the maximum principle, we have

   $$\dot{p}(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t).$$

   So,

   $$\dot{p} = -2 - p.$$

   We can solve this differential equation as follows:

   $$\begin{aligned} & \frac{dp}{dt} = -(p + 2) \\ \Leftrightarrow \quad & \frac{dp}{p + 2} = -dt \\ \Leftrightarrow \quad & \int \frac{dp}{p + 2} = -\int dt \\ \Leftrightarrow \quad & \ln(p + 2) = -t + C \\ \Leftrightarrow \quad & p + 2 = e^C e^{-t} \\ \Leftrightarrow \quad & p = ke^{-t} - 2, \end{aligned}$$

where C is a constant and we define $k = e^C$. By the transversality condition,

$$p(2) = 0 \Leftrightarrow ke^{-2} - 2 = 0 \Rightarrow k = 2e^2.$$

Thus, $p(t) = 2e^2e^{-t} - 2 = 2e^{2-t} - 2 = 2(e^{2-t} - 1)$.

3. Determine the optimal control $u^*(t)$.

   Let τ be such that $p(\tau) = 3$. Then,

   $$p(\tau) = 2(e^{2-\tau}-1) = 3 \Leftrightarrow e^{2-\tau} = \frac{5}{2} \Leftrightarrow 2-\tau = \ln(5/2) \Leftrightarrow \tau = 2-\ln(5/2) \approx 1.084.$$

   So, $\tau \in [0, 2]$. We also confirm that $p(t)$ is strictly decreasing because $\dot{p}(t) = -2e^{2-t} < 0$. Hence, for any $t \in [0, 2]$,

   $$u^*(t) = \begin{cases} 2 & \text{if } t \in [0, \tau) \\ 0 & \text{if } t \in [\tau, 2]. \end{cases}$$

4. Determine the optimal state $x^*(t)$.

   For any $t \in [0, \tau)$, from the given differential equation, we have

   $$\dot{x} = x + u = x + 2.$$

   We solve this differential equation:

   $$\begin{aligned} & \frac{dx}{dt} = x + 2 \\ \Leftrightarrow\quad & \frac{dx}{x+2} = dt \\ \Leftrightarrow\quad & \int \frac{dx}{x+2} = \int dt \\ \Leftrightarrow\quad & \ln(x+2) = t + C \\ \Leftrightarrow\quad & x + 2 = e^C e^t \\ \Leftrightarrow\quad & x = k_1 e^t - 2, \end{aligned}$$

   where C is a constant and we set $k_1 = e^C$. Since $x(0) = 4$, we have $k_1 = 6$. Hence, $x = 6e^t - 2 = 2(3e^t - 1)$.

   For any $t \in [\tau, 2]$, from the given differential equation, we have

   $$\dot{x} = x + u = x.$$

We solve this differential equation:

$$
\begin{aligned}
& \frac{dx}{dt} = x \\
\Leftrightarrow\quad & \frac{dx}{x} = dt \\
\Leftrightarrow\quad & \int \frac{dx}{x} = \int dt \\
\Leftrightarrow\quad & \ln(x) = t + C \\
\Leftrightarrow\quad & x = e^C e^t \\
\Leftrightarrow\quad & x = k_2 e^t,
\end{aligned}
$$

where C is a constant and we set $k_2 = e^C$. Since x is required to be continuous, we have

$$2(3e^\tau - 1) = k_2 e^\tau \Rightarrow k_2 = \frac{2(3e^\tau - 1)}{e^\tau} = 2(3 - e^{-\tau}).$$

Hence,

$$x(t) = 2(3 - e^{-\tau})e^t.$$

In sum, for any $t \in [0, 2]$,

$$x^*(t) = \begin{cases} 2(3e^t - 1) & \text{if } t \in [0, \tau) \\ 2(3 - e^{-\tau})e^t & \text{if } t \in [\tau, 2] \end{cases}$$

where $\tau = 2 - \ln(5/2)$.

5. Argue whether the maximum principle is strong enough to generate the solution to the original optimization problem.

   Let $F(x, u) = 2x - 3u$. Both $2x$ and $-3u$ are linear functions, the Hessian matrix associated with $F(x, u)$ is the zero matrix, which is trivially negative semidefinite for all (x, u). Hence, $F(x, u) = 2x - 3u$ is concave in (x, u). Therefore, the solution obtained by the maximum principle is a solution to the original optimization problem.

Question 7.3 (40 points) *Consider the following problem:*

$$\max \int_0^2 (x^2 - 2u)dt, \;\; \textit{subject to } \dot{x} = u, \;\; x(0) = 1, \;\; x(2) \textit{ free}, \; u \in [0, 1].$$

Answer the following questions.

1. Show that the optimal control $u^*(t)$ satisfies the following condition:

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 2 \\ 0 & \text{if } p(t) < 2, \end{cases}$$

   where $p(t)$ denotes the adjoint function associated with the Hamiltonian.

We set up the Hamiltonian:

$$H(t, x, p, u) = x^2 - 2u + pu = x^2 + (p - 2)u.$$

Only the term $(p - 2)u$ involves u in H. So, if we want to maximize the value of H with respect to $u \in [0, 1]$, we obtain the answer as it is given.

2. Show that the optimal state variable $x^*(t)$ is nondecreasing.

   By the differential equation $\dot{x} = u$, we have $\dot{x}^*(t) = u \geq 0$. Thus, $\dot{x}^*(t) \geq 0$, which is nondecreasing.

3. Show that $p(t)$ is strictly decreasing.

   By the maximum principle, we have

$$\dot{p}(t) = -H_x^{'}(t, x, p, u) \Rightarrow \dot{p}(t) = -2x^*(t).$$

   Since $x^*(t)$ is nondecreasing, we have

$$\dot{p}(t) = -2x^*(t) \leq -2x^*(0) = -2,$$

   where the last equality follows because $x^*(0) = 1$. This shows that $p(t)$ is strictly decreasing.

4. Show that there exists a unique $t^* \in (0, 2)$ such that

$$p(t) \begin{cases} > 2 & \text{if } t \in [0, t^*) \\ = 2 & \text{if } t = t^* \\ < 2 & \text{if } t \in (t^*, 2], \end{cases}$$

   and

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in (t^*, 2]. \end{cases}$$

   Since $x(2)$ is free, by the transversality condition, we have $p(2) = 0$. Since $p(t)$ is strictly decreasing, we have $p(t) > 0$ for all $t \in [0, 2)$. In the previous question, we obtain

$$\dot{p}(t) \leq -2 \Rightarrow \int_0^2 \dot{p}(t)dt \leq -2\int_0^2 dt \Rightarrow p(2) - p(0) \leq -4 \underbrace{\Rightarrow}_{p(2)=0} p(0) \geq 4.$$

   Once again, due to the strict decreasingness of $p(t)$, there exists a unique $t^* \in (0, 2)$ such that

$$p(t) \begin{cases} > 2 & \text{if } t \in [0, t^*) \\ = 2 & \text{if } t = t^* \\ < 2 & \text{if } t \in (t^*, 2]. \end{cases}$$

As we show in the previous question,

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 2 \\ 0 & \text{if } p(t) < 2. \end{cases}$$

So, we also obtain

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in (t^*, 2] \end{cases}$$

5. Show that

$$x^*(t) = \begin{cases} t+1 & \text{if } t \in [0, t^*] \\ t^*+1 & \text{if } t \in [t^*, 2]. \end{cases}$$

By the given differential equation $\dot{x}^*(t) = u^*(t)$, if we consider the case where $t \leq t^*$, we have

$$\dot{x}^*(t) = u^*(t) = 1 \Rightarrow \int_0^t \dot{x}^*(\tau) d\tau = \int_0^t d\tau \Rightarrow x^*(t) - x^*(0) = t.$$

Since $x(0) = 1$, we obtain $x^*(t) = t + 1$ for any $t \in [0, t^*]$. On the contrary, if we consider the case where $t \geq t^*$, we have

$$\dot{x}^*(t) = u^*(t) = 0 \Rightarrow \int_{t^*}^t \dot{x}^*(t) d\tau = 0 \Rightarrow x^*(t) - x^*(t^*) = 0 \Rightarrow x^*(t) = x^*(t^*).$$

This implies that $x^*(t) = x^*(t^*)$ for any $t \in [t^*, 2]$, By continuity of $x^*(t)$, we have $x^*(t^*) = t^* + 1$.

6. Show that when $t \in [t^*, 2]$,

$$p(t) = 2(1 + t^*)(2 - t).$$

Show also that $t^* = (1 + \sqrt{5})/2$.

When $t \geq t^*$, we have $\dot{p}(t) = -2x^*(t) = -2(1+t^*)$ so that $p(t) = -2(1+t^*)t + C_1$, where C_1 is a constant. By the transversality condition, we know $p(2) = 0$. Therefore,

$$p(2) = -2(1 + t^*)2 + C_1 = 0 \Rightarrow C_1 = 4(1 + t^*).$$

Hence, $p(t) = -2(1 + t^*)t + 4(1 + t^*) = 2(1 + t^*)(2 - t)$. Since $p(t^*) = 2$, we have

$$p(t^*) = -2(1 + t^*)t^* + 4(1 + t^*) = 2 \Rightarrow (t^*)^2 - t^* - 1 = 0 \Rightarrow t^* = \frac{1 \pm \sqrt{5}}{2}$$

Since $t^* \in (0, 2)$, we obtain $t^* = (1 + \sqrt{5})/2$.

7. Derive $p(t)$ for any $t \in [0, 2]$.

   When $t \le t^*$, we have $\dot{p}(t) = -2x^*(t) = -2 - 2t$ so that $p(t) = -2t - t^2 + C_2$, where C_2 is a constant. Since $p(t^*) = 2$, we have

   $$p(t^*) = -2t^* - (t^*)^2 + C_2 = 2 \Rightarrow C_2 = (t^*)^2 + 2t^* + 2.$$

   Plugging $t^* = (1 + \sqrt{5})/2$ into $C_2 = (t^*)^2 + 2t^* + 2$, we obtain

   $$C_2 = \frac{9 + 3\sqrt{5}}{2}.$$

   Therefore,

   $$p(t) = \begin{cases} -2t - t^2 + \frac{9+3\sqrt{5}}{2} & \text{if } t \in [0, (1+\sqrt{5})/2] \\ (3+\sqrt{5})(2-t) & \text{if } t \in [(1+\sqrt{5})/2, 2] \end{cases}$$

8 Homework 8: Submission is not Needed

Question 8.1 *Consider the following economic growth model:*

$$\max \int_0^T (1 - s(t))e^{\rho t} f(k(t))e^{-\delta t} dt$$
$$\textit{subject to} \quad \dot{k}(t) = s(t)e^{\rho t} f(k(t)) - \lambda k(t),\ k(0) = k_0,\ k(T) \ge k_T > k_0,\ s(t) \in [0, 1],$$

where $k(t)$ is the capital stock (a state variable), $s(t)$ is the savings rate (a control variable), and $f(k)$ is a production function. Assume that (i) $f(k) > 0$ whenever $k \ge k_0 e^{-\lambda T}$; (ii) $f^{'}(k) > 0$; and (iii) $\rho, \delta, \lambda, T, k_0$, and k_T are all positive constants. We interpret ρ as the rate of technical progress; δ as the discount rate; λ as the capital depreciation rate; T as the end of the planning period; k_0 as the initial level of capital; and k_T as the final level of capital. Answer the following questions.

1. Let $(k^*(t), s^*(t))$ be a solution to the optimization problem. Using the maximum principle, characteirze $(k^*(t), s^*(t), p(t))$ where $p(t)$ is the adjoint function, to the extent possible.

   We set up the Hamiltonian as follows:

   $$H(t, k, s, p) = (1 - s)e^{\rho t} f(k)e^{-\delta t} + p(se^{\rho t} f(k) - \lambda k).$$

   By the maximum principle, we have

   $$s^*(t) \in \arg\max_{s \in [0,1]} H(t, k^*(t), s, p) \Rightarrow s^*(t) \in \arg\max_{s \in [0,1]} se^{\rho t} f(k) \left[p - e^{-\delta t}\right].$$

   So,

   $$s^*(t) = \begin{cases} 1 & \text{if } p(t) > e^{-\delta t}, \\ [0, 1] & \text{if } p(t) = e^{-\delta t}, \\ 0 & \text{if } p(t) < e^{-\delta t}. \end{cases}$$

By the maximum principle, we also have

$$\dot{p} = -\frac{\partial H(t, k^*(t), s^*(t), p(t)}{\partial k}.$$

Thus, we obtain

$$\dot{p} = -(1 - s^*(t))e^{\rho t} f'(k^*(t))e^{-\delta t} - p(t)s^*(t)e^{\rho t} f'(k^*(t)) + \lambda p(t).$$

By the transversality condition, we satisfy the following condition:

$$p(T) \geq 0 \text{ such that } p(T) = 0 \text{ if } k^*(T) > k_T.$$

2. In the rest of the question, we set $\rho = 0; f(k) = ak$; $a > 0$; $\delta = 0$; and $\lambda = 0$. Assume further that $T > 1/a$ and $k_0 e^{aT} > k_T$ in the rest of the question. Show that $p(\cdot)$ is strictly decreasing.

   Given the specification of the parameters, we rewrite the optimization problem:

$$\begin{aligned} &\max \int_0^T (1 - s(t))ak(t)dt \\ \text{subject to } \quad &\dot{k}(t) = as(t)k(t),\ k(0) = k_0,\ k(T) \geq k_T > k_0,\ s(t) \in [0, 1], \end{aligned}$$

   where $a > 0,\ T > 1/a$, and $k_0 e^{aT} > k_T$. The Hamiltonian can also be rewritten as follows:

$$H(t, k, s, p) = (1 - s)ak + pask.$$

   We can simplify the characterization of the solution $(k^*(t), s^*(t))$ we have obtained in the previous question. First, we obtain

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1, \\ [0, 1] & \text{if } p(t) = 1, \\ 0 & \text{if } p(t) < 1. \end{cases}$$

   Second, using the above property of $s^*(t)$, we obtain

$$\begin{aligned} \dot{p} &= -(1 - s^*(t))a - p(t)s^*(t)a \\ &= -a + s^*(t)a(1 - p(t)) \\ &= \begin{cases} -ap(t) < 0 & \text{if } p(t) > 1 \\ -a < 0 & \text{if } p(t) \leq 1. \end{cases} \end{aligned}$$

   As $a > 0$, we have $\dot{p}(t) < 0$ for each $t \in [0, T]$, which implies that $p(\cdot)$ is strictly decreasing.

3. Show that $p(0) > 1$.

Suppose, on the contrary, that $p(0) \leq 1$. Since $p(t)$ is strictly decreasing, we have $p(t) < 1$ for any $t \in (0, T]$. Since

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ [0,1] & \text{if } p(t) = 1 \\ 0 & \text{if } p(t) < 1, \end{cases}$$

we have $s^*(t) = 0$ for any $t \in [0, T]$. Since $\dot{k}^*(t) = as^*(t)k^*(t)$ and $k^*(0) = k_0$, we have $k^*(t) = k_0$ for any $t \in [0, T]$. In particular, we obtain $k^*(T) = k_0$, which contradicts the restriction that $k^*(T) = k_T > k_0$. Thus, we conclude that $p(0) > 1$.

4. Find the solution candidate to the problem when $p(T) = 0$.

Since $p(t)$ is continuous and strictly decreasing such that $p(0) > 1$ and $p(T) = 0$, there is a unique $t^* \in (0, T)$ such that

$$p(t) \begin{cases} > 1 & \text{if } t \in [0, t^*) \\ = 1 & \text{if } t = t^* \\ < 1 & \text{if } t \in (t^*, T]. \end{cases}$$

Then,

$$s^*(t) = \begin{cases} 1 & t \in [0, t^*) \\ [0,1] & t = t^* \\ 0 & t \in (t^*, T]. \end{cases}$$

From the analysis in the previous question, we have

$$\dot{p}(t) = \begin{cases} -ap(t) & \text{if } t \in [0, t^*) \\ -a & \text{if } t \in [t^*, T] \end{cases}$$

On the interval $[t^*, T]$, by solving the differential equation $\dot{p} = -a$, we have $p(t) = -at + A$, where A is a constant. Taking into account that $p(T) = 0$, we have $p(t) = a(T - t)$. Since $p(t^*) = 1$, we have $a(T - t^*) = 1$, which implies that $t^* = T - 1/a$, which is positive by our assumption. We can also easily see that $t^* < T$, as $t^* = T - 1/a$ and $a > 0$. Hence, $t^* = T - 1/a$ is well-defined.

On the interval $[0, t^*]$, by solving the differential equation $\dot{p} = -ap$, we obtain $p(t) = Be^{-at}$, where B is a constant. Since $p(t^*) = 1$, we have $p(t) = e^{-a(t-t^*)}$. Since $t^* = T - 1/a > 0$, we obtain $p(t) = e^{a(T-t)-1}$. Therefore, the solution candidate is given as follows:

$$\begin{aligned} s^*(t) &= \begin{cases} 1 & \text{if } t \in [0, T - 1/a] \\ 0 & \text{if } t \in (T - 1/a, T], \end{cases} \\ k^*(t) &= \begin{cases} k_0 e^{at} & \text{if } t \in [0, T - 1/a] \\ k_0 e^{aT-1} & \text{if } t \in (T - 1/a, T], \end{cases} \\ p(t) &= \begin{cases} e^{a(T-t)-1} & \text{if } t \in [0, T - 1/a] \\ a(t - T) & \text{if } t \in (T - 1/a, T]. \end{cases} \end{aligned}$$

It remains to verify the terminal condition $k(T) \geq k_T$. This terminal condition reduces to $k_0 e^{aT} \geq k_T$, which is guaranteed by our assumption that $k_0 e^{aT} > k_T$.

5. Show that $p(T) > 0$ implies $p(T) < 1$.

   Suppose, on the contrary, that $p(T) \geq 1$. Since $p(t)$ is strictly decreasing, $p(t) > 1$ for any $t \in [0, T)$. This implies that $s^*(t) = 1$ for any $t \in [0, T]$. So, we have $\dot{k}^*(t) = ak^*(t)$. Solving this differential equation together with $k^*(0) = k_0$, we obtain $k^*(t) = k_0 e^{at}$. Thus, we have $k^*(T) = k_0 e^{aT}$ which is smaller than k_T by our assumption. $p(T) > 0$ together with the transversality condition guarantees that $k^*(T) = k_T$. This is a contradiction.

6. Find the solution candidate to the problem when $p(T) > 0$.

   From the previous question, we know that $p(T) < 1$. As we did for the case where $p(T) = 0$, there is a unique $t^* \in (0, T)$ such that

$$p(t) \begin{cases} > 1 & \text{if } t \in [0, t^*) \\ = 1 & \text{if } t = t^* \\ < 1 & \text{if } t \in (t^*, T]. \end{cases}$$

   Then,

$$s^*(t) = \begin{cases} 1 & t \in [0, t^*) \\ [0, 1] & t = t^* \\ 0 & t \in (t^*, T]. \end{cases}$$

   From the analysis in the previous question, we have

$$\dot{p}(t) = \begin{cases} -ap(t) & \text{if } t \in [0, t^*) \\ -a & \text{if } t \in [t^*, T] \end{cases}$$

   Therefore, the solution candidate is given as follows:

$$\begin{aligned} s^*(t) &= \begin{cases} 1 & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in (t^*, T] \end{cases} \\ k^*(t) &= \begin{cases} k_0 e^{at} & \text{if } t \in [0, t^*] \\ k_0 e^{at^*} & \text{if } t \in (t^*, T] \end{cases} \\ p(t) &= \begin{cases} e^{a(t^* - t)} & \text{if } t \in [0, t^*] \\ 1 - a(t - t^*) & \text{if } t \in (t^*, T] \end{cases} \end{aligned}$$

   It remains to identify t^*. From $k^*(T) = k_T$, it follows that $e^{at^*} = k_T / k_0$. So,

$$t^* = \frac{1}{a} \ln \left(\frac{k_T}{k_0} \right).$$

   For t^* to be well-defined, we must have

$$\frac{1}{a} \ln \left(\frac{k_T}{k_0} \right) \leq T \Rightarrow k_T \leq k_0 e^{aT}.$$

   This inequality is guaranteed by our assumption that $k_0 e^{aT} > k_T$.

7. Show that either $p(T) = 0$ holds or $p(T) > 0$ holds, but not both. (Hint: You should check $k^*(t)$ obtained in the case where $p(T) = 0$.)

   When $p(T) = 0$ holds, the previous analysis in this case concludes

$$T - \frac{1}{a} \geq \frac{1}{a} \ln \left(\frac{k_T}{k_0} \right). \; (*)$$

   When $p(T) > 0$ holds, the previous analysis in this case concludes

$$T - \frac{1}{a} < \frac{1}{a} \ln \left(\frac{k_T}{k_0} \right). \; (**)$$

   Clearly, either $(*)$ holds or $(**)$ holds, but not both.

8. Show that the Hamiltonian is "not" concave in (k, s).

   We have $H(t, k, s, p) = (1 - s)ak + pask$ as the Hamiltonian. We compute the following:

$$\begin{aligned} H_k^{'} &= (1-s)a + pas \\ H_s^{'} &= -ak + pak \\ H_{kk}^{''} &= 0 \\ H_{ss}^{''} &= 0 \\ H_{ks}^{''} = H_{sk}^{''} &= -a + pa = -a(1-p). \end{aligned}$$

   So, we define the associated Hessian matrix $\mathcal{H}(k, s)$ as follows:

$$\mathcal{H}(k,s) = \begin{pmatrix} H_{kk}^{''} & H_{ks}^{''} \\ H_{sk}^{''} & H_{ss}^{''} \end{pmatrix} = \begin{pmatrix} 0 & -a(1-p) \\ -a(1-p) & 0 \end{pmatrix}.$$

   Then, we have the determinant of $\mathcal{H}(k, s)$ as follows:

$$|\mathcal{H}(k,s)| = -a^2(1-p)^2 < 0 \text{ if } p \neq 1.$$

   This implies that the Haimiltonian H is not concave if $p \neq 1$. Since we have shown that $p(0) > 1$, it is possible that $p(t) \neq 1$. Thus, this completes the argument.

9. Show that the solution candidate you obtained is indeed the solution to the simplified version of the optimization problem. For this question, we assume that every possible value of $k(t)$ is nonnegative for any $t \in [0, T]$, i.e., $k(t) \geq 0$ for any $t \in [0, T]$.

   We define the maximized Hamiltonian $\hat{H}$:

$$\hat{H}(t, k, p) = ak \max_{s \in [0,1]} [1 + (p-1)s].$$

We define

$$t^* \equiv \begin{cases} T - \frac{1}{a} & \text{if } p(T) = 0, \\ \\ \frac{1}{a}\ln\left(\frac{k_T}{k_0}\right) & \text{if } p(T) > 0. \end{cases}$$

Then, we obtain that, for any $t \in [0, T]$,

$$\hat{H}(t, k, p) = \begin{cases} ak(t)p(t) & \text{if } t \in [0, t^*] \text{ where } p(t) > 1, \\ ak(t) & \text{if } t \in (t^*, T] \text{ where } p(t) \le 1. \end{cases}$$

Since $k(t) \ge 0$ for any $t \in [0, T]$, the maximized Haimltonian $\hat{H}$ is linear in k. This implies that $\hat{H}$ is concave in k. Therefore, Arrow's sufficient condition is satisfied so that the solution candidate we have found is the solution to the optimization problem.

Question 8.2 *Consider the following problem:*

$$\max \int_0^5 [10u - (u^2 + 2)]e^{-0.1t} dt, \ \dot{x} = -u, \ x(0) = 10, \ x(5) \ge 0, \ u \in [0, \infty).$$

Solve this problem using the current value Hamiltonian.

Let the current value Hamiltonian be

$$H^c = 10u - u^2 - 2 - \lambda u.$$

By the maximum principle, we have

$$\dot{\lambda} - 0.1\lambda = -\partial H^c/\partial x = 0.$$

Using the formula of first-order linear differential equation, we have $\lambda = Ae^{0.1t}$ where A is a constant. Assume that $u^*(t) > 0$ for any $t \in [0, 5]$. Then, by the maximum principle, we have

$$u^*(t) = \frac{10 - \lambda}{2} = 5 - \frac{A}{2}e^{0.1t}.$$

Plugging $u^*(t) = 5 - Ae^{0.1t}/2$ into $\dot{x} = -u$, we have

$$\dot{x}^* = \frac{A}{2}e^{0.1t} - 5.$$

So, we obtain

$$x^* = 5Ae^{0.1t} - 5t + B,$$

where B is a constant. Since $x^*(0) = 10$, we have $5A + B = 10$ so that $B = 10 - 5A$. Hence,

$$x^*(t) = 5Ae^{0.1t} - 5t + 10 - 5A.$$

Assume that $x^*(5) > 0$. By the transversality condition, we then have $\lambda(5) = 0$. This implies

$$Ae^{0.5} = 0 \Rightarrow A = 0.$$

This further implies that $\lambda(t) = 0$; $u^*(t) = 5$ and $x^*(t) = -5t + 10$. Thus, $x^*(5) = -15$, which contradicts the hypothesis that $x^*(5) > 0$. Hence, we must assume $x^*(5) = 0$. This implies

$$5Ae^{0.5} - 25 + 10 - 5A = 0 \Rightarrow A = \frac{3}{e^{0.5} - 1}.$$

Plugging this into $u^*(t)$, we have

$$u^*(t) = 5 - \frac{3}{2(e^{0.5} - 1)}e^{0.1t} \Rightarrow u^*(5) = \frac{7e^{0.5} - 10}{2(e^{0.5} - 1)}$$

Noting $e^{0.5} > 1$, we have $e^{0.5} - 1 > 0$. We also check the following:

$$49e \approx 132.79 \;\; (\because e \approx 2.71) \Rightarrow 49e > 100 \Rightarrow 7e^{0.5} > 10.$$

Since $u^*(t) = 5 - 3e^{0.1t}/2(e^{0.5} - 1)$ is strictly decreasing, we conclude that $u^*(t) > 0$ for any $t \in [0, 5]$. Thus, the assumption that $u^*(t) > 0$ for any $t \in [0, 5]$ is justified. In sum, we obtain the following:

$$\begin{aligned} u^*(t) &= 5 - \frac{3}{2(e^{0.5} - 1)}e^{0.1t} \\ x^*(t) &= \frac{15e^{0.1t}}{e^{0.5} - 1} - 5t + \frac{10(e^{0.5} - 1) - 15}{e^{0.5} - 1} \\ \lambda(t) &= \frac{3e^{0.1t}}{e^{0.5} - 1} \end{aligned}$$

We derive the Hessian matrix associated with H^c:

$$\mathcal{H}(x, u) = \begin{pmatrix} \partial^2 H^c/\partial x^2 & \partial^2 H^c/\partial x \partial u \\ \partial^2 H^c/\partial u \partial x & \partial^2 H^c/\partial u^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since the Hessian matrix $\mathcal{H}$ is negative semidefinite for all (x, u), the current value Hamiltonian H^c is concave in (x, u). By Mangasarian's sufficiency result, the solution candidate we obtained via the maximum principle is indeed the solution to the given optimization problem.

Question 8.3 *Consider the problem*

$$\max \int_0^\infty x(2 - u)e^{-t} dt$$
$$\textit{subject to} \quad \dot{x} = uxe^{-t}, \;\; x(0) = 1, \;\; \lim_{t\to\infty} x(t) \geq 0, \;\; u \in [0, 1].$$

Answer the following questions.

1. Show that

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda(t)e^{-t} > 1, \\ [0, 1] & \text{if } \lambda(t)e^{-t} = 1, \\ 0 & \text{if } \lambda(t)e^{-t} < 1. \end{cases}$$

Let $H^c = x(2-u) + \lambda uxe^{-t}$ be the current value Hamiltonian. Let $(x^*(t), u^*(t))$ be an admissible pair satisfying the following conditions:

(a)

$$u^*(t) \in \arg\max_{u\in[0,1]} H^c(t, x^*(t), u, \lambda(t)) = \arg\max_{u\in[0,1]} 2x^*(t)+x^*(t)(e^{-t}\lambda(t)-1)u;$$

(b)

$$\dot{\lambda}(t) - \lambda(t) = -\partial(H^c(t, x, u^*(t), \lambda(t))/\partial x = -(2 - u^*(t)) - \lambda(t)u^*(t)e^{-t}.$$

Since $\dot{x}^*(t) = u^*(t)x^*(t)e^{-t}$ and $x^*(0) = 1$, we have $x^*(t) \geq 1$ for any $t \in [0, \infty)$. Then,

$$\max_{u\in[0,1]} 2x^*(t) + x^*(t)(\lambda(t)e^{-t} - 1)u \Longleftrightarrow \max_{u\in[0,1]} (\lambda(t)e^{-t} - 1)u.$$

This completes the argument.

2. Assume that $\lambda(t)e^{-t} \to 0$ as $t \to \infty$. Then, we have either $\lambda(t)e^{-t} < 1$ for any $t \in [0, \infty)$ or there exists $t^* \geq 0$ such that

$$\lambda(t)e^{-t} \begin{cases} = 1 & \text{if } t = t^*, \\ < 1 & \text{if } t \in (t^*, \infty). \end{cases}$$

What is $\lambda(t)$ when $t \in [t^*, \infty)$? Moreover, what is t^*?

Assume such $t^* \geq 0$ exists. Then, due to the property of $u^*(t)$ outlined above, we have $u^*(t) = 0$ for any $t \in (t^*, \infty)$. Using Condition (b) in the maximum principle, we have that, for any $t \in (t^*, \infty)$,

$$\dot{\lambda}(t) - \lambda(t) = -2.$$

Since this is a linear first-order differential equation, we obtain

$$\lambda(t) = Ce^t + 2,$$

where C is a constant. Since we assume $\lambda(t)e^{-t} \to 0$ as $t \to \infty$, we have

$$\lambda(t)e^{-t} = C + 2e^{-t} \to 0 \text{ as } t \to \infty.$$

So, we have $C = 0$ so that $\lambda(t) = 2$ for any $t \in [t^*, \infty)$. Since $\lambda(t^*)e^{-t^*} = 1$ and $\lambda(t^*) = 2$, we have

$$2e^{-t^*} = 1 \Rightarrow e^{t^*} = 2 \Rightarrow t^* = \ln 2.$$

3. Obtain $u^*(t)$ and $x^*(t)$?

We propose $u^*(t)$ as follows:

$$u^*(t) = \begin{cases} 1 & \text{for any } t \in [0, \ln 2), \\ [0, 1] & \text{for any } t = \ln 2, \\ 0 & \text{for any } t \in (\ln 2, \infty). \end{cases}$$

Since $\dot{x}^*(t) = u^*(t)x^*(t)e^{-t}$ and we use the property of $u^*(t)$ above, we have

$$\dot{x}^*(t) = \begin{cases} x^*(t)e^{-t} & \text{if } t \in [0, \ln 2) \\ 0 & \text{if } t \in [\ln 2, \infty) \end{cases}$$

So,

$$\frac{dx^*(t)}{dt} = x^*(t)e^{-t} \Rightarrow \frac{dx^*(t)}{x^*(t)} = e^{-t}dt \Rightarrow \ln x^*(t) = -e^{-t} + K,$$

where K is a constant. Since $x^*(0) = 1$, we have $K = 1$ so that $\ln x^*(t) = -e^{-t} + 1$. So, $x^*(t) = e^{1-e^{-t}}$. Plugging $\ln 2$ into t in the equation, we obtain

$$x^*(\ln 2) = e^{1-e^{-\ln 2}} = e^{1-(e^{\ln 2})^{-1}} = e^{1-2^{-1}} = e^{1-1/2} = e^{1/2}.$$

Thus,

$$x^*(t) = \begin{cases} e^{1-e^{-t}} & \text{if } t \in [0, \ln 2), \\ e^{1/2} & \text{if } t \in [\ln 2, \infty). \end{cases}$$

4. Obtain $\lambda(t)$.

   Using (b) in the maximum principle and taking into account that $u^*(t) = 1$ for any $t \in [0, \ln 2]$, we obtain

   $$\dot{\lambda} - \lambda = -1 - \lambda(t)e^{-t} \Rightarrow \dot{\lambda} + (e^{-t} - 1)\lambda = -1.$$

   Since this is a linear first-order differential equation, we obtain

   $$\begin{aligned} \lambda(t) &= e^{-\int (e^{-t}-1)dt}\left(C - \int e^{\int (e^{-t}-1)dt}dt\right) = e^{e^{-t}+t}\left(C - \int e^{-e^{-t}-t}dt\right) \\ &= Ce^{e^{-t}+t} - e^{e^{-t}+t}\int e^{-e^{-t}}e^{-t}dt = Ce^{e^{-t}+t} - e^{e^{-t}+t}e^{-e^{-t}} = Ce^{e^{-t}+t} - e^{t}. \end{aligned}$$

   Since $\lambda(\ln 2) = 2$, we have

   $$Ce^{(e^{\ln 2})^{-1}}e^{\ln 2} - e^{\ln 2} = 2 \Rightarrow 2Ce^{1/2} - 2 = 2 \Rightarrow C = 2e^{-1/2}.$$

   Thus,

   $$\lambda(t) = \begin{cases} 2e^{e^{-t}+t-1/2} - e^{t} & \text{if } t \in [0, \ln 2), \\ 2 & \text{if } t \in [\ln 2, \infty). \end{cases}$$

5. Show that (d) $\lim_{t\to\infty} \lambda(t)e^{-t}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$.

   Lemma (A condition guaranteeing (d)): Let $(x^*(t), u^*(t))$ be an admissible pair for the optimal control problem. Suppose that $\lim_{t\to\infty} x(t) \geq x_1$ for any admissible $x(t)$. Assume further that the following three conditions hold:

   (A) $\lim_{t\to\infty} \lambda(t)e^{-rt}(x_1 - x^*(t)) \geq 0$;

   (B) $\exists M \in \mathbb{R}_+$ such that $|\lambda(t)e^{-rt}| \leq M$ for all $t \geq t_0$;

   (C) $\exists t' \in \mathbb{R}$ such that $\lambda(t) \geq 0$ for all $t \geq t'$.

Then, (d) $\lim_{t\to\infty} \lambda(t)e^{-rt}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$.

We first check Condition (A):

$$\lim_{t\to\infty} \lambda(t)e^{-t}(0 - x^*(t)) = \lim_{t\to\infty} 2e^{-t}(0 - e^{1/2}) = \lim_{t\to\infty} -2e^{1/2}e^{-t} = 0,$$

where $e^{-t} \to 0$ as $t \to \infty$. We next check Condition B. We compute the following: for each $t \in [0, \infty)$,

$$\begin{aligned}
\lambda(t)e^{-t} &= \begin{cases} (2e^{e^{-t}+t-1/2} - e^t)e^{-t} & \text{if } t \in [0, \ln 2) \\ 2e^{-t} & \text{if } t \in [\ln 2, \infty) \end{cases} \\
&= \begin{cases} 2e^{e^{-t}-1/2} - 1 & \text{if } t \in [0, \ln 2) \\ 2e^{-t} & \text{if } t \in [\ln 2, \infty) \end{cases} \\
&\leq \begin{cases} 2e^{-1/2} - 1 & \text{if } t \in [0, \ln 2) \\ 2 & \text{if } t \in [\ln 2, \infty), \end{cases}
\end{aligned}$$

where e^x is a strictly increasing function and $e^{-t} \leq 1$ for any $t \in [0, \infty)$. Since $2 > 2e^{-1/2} - 1$, we have

$$|\lambda(t)e^{-t}| \leq 2, \ \forall t \in [0, \infty).$$

So, Condition (B) holds. Finally, we check Condition C. Since $\lambda(t) = 2$ for any $t \in [\ln 2, \infty)$, we show

$$\exists t' \in [0, \infty) \ \text{such that } \lambda(t) \geq 0 \text{ for any } t \geq t'.$$

Hence, Condition (C) holds. Therefore, by the lemma, Condition (d) holds.

6. Show that Arrow's sufficient condition holds.

Recall the current value Hamiltonian:

$$H^c(t, x, u, \lambda) = x(2 - u) + \lambda u x e^{-t}.$$

We define the maximized current value Hamiltonian $\hat{H}^c$: for any $t \in [0, \infty)$,

$$\hat{H}^c(t, x, \lambda) = \max_{u\in[0,1]} H^c(t, x, u, \lambda) = \begin{cases} x + \lambda(t)xe^{-t} & \text{if } t \in [0, \ln 2) \\ 2x & \text{if } t \in [\ln 2, \infty) \end{cases}$$

So, $\hat{H}^c$ is linear in x so that it is concave in x. Therefore, Arrow's sufficient condition holds.