Ho-Lee Model

The mean of this stochastic integral is given by

$$\mathbb{E}\left[\int_0^T r_u \, du\right] = r_0 T + \int_0^T \theta(s)(T-s)\, ds,$$

and the variance is given by

$$V\left[\int_0^T r_u \, du\right] = \int_0^T \sigma^2 (T-s)^2 \, ds = \frac{1}{3}\sigma^2 T^3,$$

where we have used **Itô Isometry**.

Therefore, the **zero-coupon discount bond can be reconstructed** as

$$e^{-R(0,T)\cdot(T-0)} = D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u \, du}\right] = \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right].$$

Since we can express $D(0,T)$ in the form of $e^{A(0,T) - r_0 B(0,T)}$, we see that <u>Ho-Lee is an affine model</u>.

Ho-Lee Model

$$C(K) = e^{-rT} \int_K^{\infty} (s-K) f(s)\, ds$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} f(K)$$

Fitting the initial term structure

From here we can work out that

$$
\begin{aligned}
\log D(0,T) &= -r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3 \\
\frac{\partial}{\partial T}\log D(0,T) &= -r_0 - \int_0^T \theta(s)\, ds + \frac{1}{2}\sigma^2 T^2 \\
\frac{\partial^2}{\partial T^2}\log D(0,T) &= -\theta(T) + \sigma^2 T \\
\Rightarrow \quad \theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.
\end{aligned}
$$

This allows Ho-Lee model to fit the initial term structure $D(0,T)$ observed in the market.

$$\log D(t,T) = -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3$$

$$\frac{\partial}{\partial T} \log D(t,T) = -r_0 - \Big[\theta(T)\cancelto{0}{(T-T)} \cdot \cancelto{1}{\frac{dT}{dT}} - \theta(0)(T-0) \cdot \cancelto{0}{\frac{d0}{dT}}$$

$$+ \int_0^T \frac{\partial}{\partial T}\Big(\theta(s)(T-s)\Big)\,ds \Big] + \frac{1}{2}\sigma^2 T^2$$

$$= -r_0 - \int_0^T \theta(s)\,ds + \frac{1}{2}\sigma^2 T^2$$

$$\frac{\partial^2}{\partial T^2} \log D(t,T) = -0 - \Big[\theta(T) \cdot \frac{dT}{dT} - \theta(0) \cdot \cancelto{0}{\frac{d0}{dT}}$$

$$+ \int_0^T \cancelto{0}{\frac{\partial}{\partial T}\theta(s)}\,ds \Big] + \sigma^2 T$$

$dr_t = \theta(t)\,dt + \sigma\, dW_t^*$

Ho-Lee Model

$dD(t,T) = \quad ?$

We have shown that Ho-Lee model allows us to reconstruct the discount factor

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2 (T-t)^3}{6},$$
$$B(t,T) = T - t.$$

What does Ho-Lee model tell us about the evolution of discount factors over time?

⇒ Note that the reconstructed discount factor is given as a function of time and short rate, i.e. $D(t,T) = f(t, r_t)$.

This means that we can use **Itô's formula** to derive the stochastic differential equation describing the evolution of the discount factors over time.

Ho-Lee Model

First, we work out the partial derivatives

$$\begin{aligned}
f(t,x) &= e^{A(t,T)-xB(t,T)} \\
f_t(t,x) &= e^{A(t,T)-xB(t,T)}\left[\frac{\partial A(t,T)}{\partial t} - x\cdot\frac{\partial B(t,T)}{\partial t}\right] \\
f_x(t,x) &= e^{A(t,T)-xB(t,T)}\Big[-B(t,T)\Big] \\
f_{xx}(t,x) &= e^{A(t,T)-xB(t,T)}\Big[B(t,T)^2\Big],
\end{aligned}$$

where an application of **Leibniz's rule** yields

$$\begin{aligned}
A(t,T) &= -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6} \\
\frac{\partial A(t,T)}{\partial t} &= \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}.
\end{aligned}$$

On the other hand, the time derivative for $B(t,T)$ is simply

$B(t,T) = (T-t)$

$$\frac{\partial B(t,T)}{\partial t} = -1.$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6}$$

$$\frac{\partial A(t,T)}{\partial t} = -\Big[\theta(T)\underbrace{\cancel{(T-T)}}_{0}\cdot\frac{dT}{dt} - \theta(t)(T-t)\cdot\frac{dt}{dt}$$

$$+ \int_t^T \frac{\partial}{\partial t}\underbrace{\cancel{\big(\theta(s)(T-s)\big)}}_{0}\,ds\Big] - \frac{\sigma^2(T-t)^2}{2}$$

$$= \theta(t)(T-t)\cdot 1 - \frac{\sigma^2(T-t)^2}{2}$$

Ho-Lee Model

$$dD(t,T) = D(t,T)\left[\cancel{\theta(t)(T-t)} - \cancel{\frac{\sigma^2(T-t)^2}{2}} + r_t\right] dt$$
$$- D(t,T)(T-t)\left(\cancel{\theta(t)}\, dt + \sigma dW_t^*\right)$$
$$+ \frac{1}{2} D(t,T)\cancel{(T-t)^2}\sigma^2 dt$$

Applying **Itô's formula**, we obtain the following stochastic differential equation:

$$\begin{aligned} dD(t,T) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2} f_{xx}(t,r_t)(dr_t)^2 \\ &= D(t,T)\left[\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t}\right] dt \\ &\qquad - D(t,T)(T-t)\Big(\theta(t)dt + \sigma dW_t^*\Big) \\ &\qquad + \frac{1}{2} D(t,T)(T-t)^2\sigma^2 dt \qquad = \\ &= r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*. \end{aligned}$$

$$dr_t = \theta(t)\, dt + \sigma\, dW_t^*$$

$$(dr_t)^2 = \sigma^2 dt$$

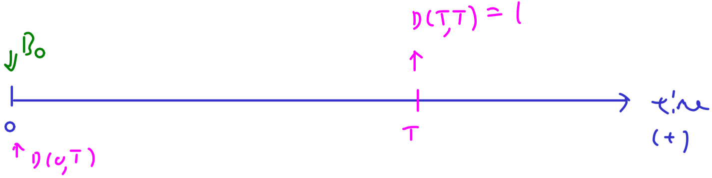
$$dr_t = \theta(t)\,dt + \sigma\, dW_t^*$$

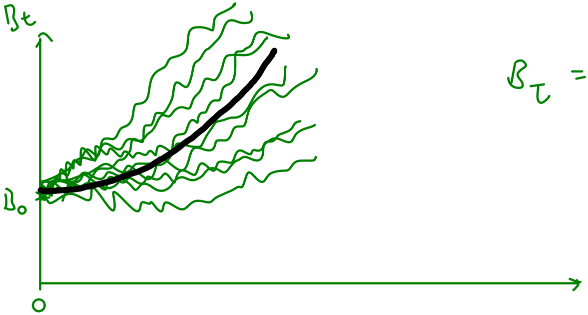
Money-market account : $$dB_t = r_t B_t \, dt$$

$$\Delta B_t \simeq r_t B_t \, \Delta t$$

$$B_{t+\Delta t} \simeq B_t + r_t B_t \, \Delta t$$

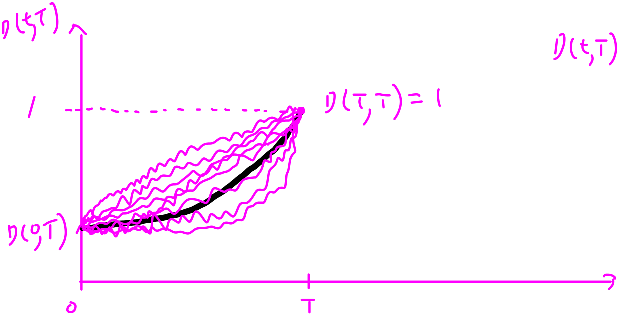
Discount factor (zero-coupon bond) : $$dD(t,T) = r_t D(t,T)\, dt - \sigma \cdot (T-t) D(t,T)\, dW_t^*$$





time (t)

$$B_T = B_0 e^{\int_0^T r_u \, du}$$



time (t)

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u \, du} \right]$$