

QF605 Additional Examples

Session 7: Short-Rate Models and Term Structure

1 Questions

1. Consider a stylized interest rate model

$$dr_t = \mu dt + \sigma dW_t^*,$$

where W_t^* is a standard Brownian motion under the risk-neutral measure \mathbb{Q}^* .

- (a) Determine the distribution, mean, and variance of the integral

$$\int_t^T r_u du.$$

- (b) Identify the expressions $A(t, T)$ and $B(t, T)$ in the following expectation:

$$D(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} \right] = e^{A(t, T) - r_t B(t, T)}.$$

- (c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

Determine the mean and variance of the integral

$$\int_0^T r_u du,$$

and use this to evaluate the expectation

$$D(0, T) = \mathbb{E}^* \left[e^{-\int_0^T r_u du} \right].$$

2 Suggested Solutions

1. (a) First we integrate the stochastic differential equation from t to s to obtain:

$$r_s = r_t + \mu(s - t) + \int_t^s \sigma dW_u^*.$$

Next we integrate r_s from t to T to obtain

$$\begin{aligned} \int_t^T r_s ds &= r_t(T - t) + \mu \int_t^T (s - t) ds + \int_t^T \int_t^s \sigma dW_u^* ds \\ &= r_t(T - t) + \mu \left[\frac{s^2}{2} - ts \right]_t^T + \int_t^T \int_u^T \sigma ds dW_u^* \\ &= r_t(T - t) + \frac{\mu}{2}(T - t)^2 + \int_t^T \sigma(T - u) dW_u^* \end{aligned}$$

Hence the mean is given by

$$\mathbb{E}^* \left[\int_t^T r_s ds \right] = r_t(T - t) + \frac{\mu}{2}(T - t)^2, \quad \triangleleft$$

and the variance is given by

$$\begin{aligned} V \left[\int_t^T r_s ds \right] &= \int_t^T \sigma^2(T - u)^2 du \\ &= \frac{\sigma^2(T - t)^3}{3} \quad \triangleleft \end{aligned}$$

- (b) Having identified the mean and variance of the short rate integral, we have

$$\begin{aligned} D(t, T) &= \mathbb{E}^* \left[e^{-\int_t^T r_u du} \right] \\ &= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2} \frac{\sigma^2(T-t)^3}{3}} \end{aligned}$$

Comparing this against

$$D(t, T) = e^{A(t, T) - r_t B(t, T)},$$

we note that

$$\begin{aligned} A(t, T) &= -\frac{\mu}{2}(T - t)^2 + \frac{\sigma^2(T - t)^3}{6} \quad \triangleleft \\ B(t, T) &= (T - t) \quad \triangleleft \end{aligned}$$

- (c) For affine interest rate model, the zero coupon bond prices can be written as

$$D(t, T) = e^{A(t, T) - r_t B(t, T)},$$

for some deterministic functions of $A(t, T)$ and $B(t, T)$ of t and T only. This implies that

$$R(t, T) = \frac{1}{T - t} \left(-A(t, T) + r_t B(t, T) \right),$$

i.e. the zero (spot) rates are affine functions of the short rate. \triangleleft

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

We can solve this stochastic differential equation by applying Itô's formula to the function $f(r_t, t) = e^{\kappa t} r_t$, and the solution is given by

$$r_t = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u^*$$

Integrating both sides from 0 to T , we have

$$\int_0^T r_t dt = \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \underbrace{\int_0^T \int_0^t \sigma e^{\kappa(u-t)} dW_u^* dt}_{\text{double integral}}.$$

On the right hand side, the first and second integrals can be carried out directly. The double integral can be simplified by exchanging the order of integration (Fubini's Theorem):

$$\begin{array}{ll} \text{Inner Integral } u : & 0 \leq u \leq T \\ \text{Outer Integral } t : & 0 \leq t \leq T \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{Inner Integral } t : & u \leq t \leq T \\ \text{Outer Integral } u : & 0 \leq u \leq T \end{array}$$

So we have

$$\begin{aligned} \int_0^T \int_0^t \sigma e^{\kappa(u-t)} dW_u^* dt &= \int_0^T \int_u^T \sigma e^{\kappa(u-t)} dt dW_u^* \\ &= \int_0^T \left[-\frac{\sigma}{\kappa} e^{\kappa(u-t)} \right]_u^T dW_u^* \\ &= \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \end{aligned}$$

So we can write the overall integral as:

$$\int_0^T r_t dt = \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^*$$

Taking expectation on both sides gives us the mean of this integral

$$\begin{aligned} \mathbb{E}^* \left[\int_0^T r_t dt \right] &= \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt \\ &= \frac{r_0}{\kappa} (1 - e^{-\kappa T}) + \theta T - \frac{\theta}{\kappa} (1 - e^{-\kappa T}). \end{aligned}$$

Taking the variance, we obtain

$$\begin{aligned} V \left[\int_0^T r_t dt \right] &= V \left[\int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \right] \\ &= V \left[\frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \right] \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T (1 - e^{\kappa(u-T)})^2 du \quad \because \text{Itô's Isometry} \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T (1 - 2e^{\kappa(u-T)} + e^{2\kappa(u-T)}) du \\ &= \frac{\sigma^2}{\kappa^2} \left[T - \frac{2}{\kappa} (1 - e^{-\kappa T}) + \frac{1}{2\kappa} (1 - e^{-2\kappa T}) \right] \end{aligned}$$

Finally, we can express the discount factor as

$$\begin{aligned}
D(0, T) &= \mathbb{E}^* \left[e^{-\int_0^T r_t dt} \right] \\
&= \exp \left(\underbrace{-\frac{r_0}{\kappa} (1 - e^{-\kappa T}) - \theta T + \frac{\theta}{\kappa} (1 - e^{-\kappa T})}_{\text{mean}} + \frac{1}{2} \cdot \underbrace{\frac{\sigma^2}{\kappa^2} \left[T - \frac{2}{\kappa} (1 - e^{-\kappa T}) + \frac{1}{2\kappa} (1 - e^{-2\kappa T}) \right]}_{\text{variance}} \right)
\end{aligned}$$