QF605 Fixed-Income Securities
Solutions to Assignment 3

1. (a) The LIBOR market model is given by

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1},$$

where $L_i(t) = L(t; T_i, T_{i+1})$, and W_t^{i+1} is a Brownian motion under the risk-neutral measure associated to the numeraire $D_{i+1}(t) = D(t, T_{i+1})$. $\triangleleft$

(b) The solution to the LIBOR market model is given by

$$L_i(T) = L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}}.$$

Let V_t denote the value of the financial contract at time t. Under the martingale measure, we have

$$\begin{aligned}
\frac{V_0}{D_{i+1}(0)} &= \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right] \\
V_0 &= D(0, T_{i+1}) \mathbb{E}^{i+1}\left[\Delta_i L_i(T)^{\frac{1}{2}}\right] \\
&= D(0, T_{i+1}) \Delta_i \mathbb{E}^{i+1}\left[L_i(0)^{\frac{1}{2}} e^{-\frac{\sigma_i^2 T}{4} + \frac{\sigma_i}{2} W_T^{i+1}}\right] \\
&= D(0, T_{i+1}) \Delta_i L_i(0)^{\frac{1}{2}} e^{-\frac{\sigma_i^2 T}{4} + \frac{\sigma_i^2 T}{8}} \\
&= D(0, T_{i+1}) \Delta_i L_i(0)^{\frac{1}{2}} e^{-\frac{\sigma_i^2 T}{8}} \quad \triangleleft
\end{aligned}$$

(c) Given the solution to the LIBOR market model, we note that this contract pays when

$$K_1 \leq L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i \sqrt{T} x} \leq K_2$$

So the range of values x take when the payoff is nonzero is given by the inequalities

$$\begin{aligned}
& K_1 \leq L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i \sqrt{T} x} \\
\Rightarrow \quad x_l^* = & \frac{\log \frac{K_1}{L_i(0)} + \frac{\sigma_i^2 T}{2}}{\sigma_i \sqrt{T}} \leq x
\end{aligned}$$

and

$$\begin{aligned}
L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i \sqrt{T} x} &\leq K_2 \\
\Rightarrow \quad x &\leq \frac{\log \frac{K_2}{L_i(0)} + \frac{\sigma_i^2 T}{2}}{\sigma_i \sqrt{T}} = x_h^*
\end{aligned}$$

Let V_0 denote the value of this contract today. We can proceed to evaluate the expectation as follows:

$$\begin{aligned}
V_0 &= D(0,T_{i+1})\mathbb{E}^{i+1}\left[\mathbb{1}_{K_1\le L_i(T)\le K_2}\right] \\
&= D(0,T_{i+1})\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbb{1}_{K_1\le L_i(T)\le K_2}\, e^{-\frac{x^2}{2}}\,dx \\
&= D(0,T_{i+1})\frac{1}{\sqrt{2\pi}}\int_{x_l^*}^{x_h^*} e^{-\frac{x^2}{2}}\,dx \\
&= D(0,T_{i+1})\left[\Phi\left(-\frac{\log\frac{L_i(0)}{K_2}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)-\Phi\left(-\frac{\log\frac{L_i(0)}{K_1}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right] \quad \triangleleft
\end{aligned}$$

2. (a) The risk-neutral measure $\mathbb{Q}^{n+1,N}$ is associated with the numeraire asset $P_{n+1,N}$, which is the PVBP, defined as

$$P_{n+1,N}(t) = \sum_{i=n+1}^{N}\Delta_{i-1}D_i(t).$$

Under this probability measure, the forward swap rate $S_{n,N}(t)$ is a martingale. $\triangleleft$

(b) Let V_t denote the value of this contract at time t. For a floating-leg-or-nothing digital option, the payoff on maturity at time T is

$$\begin{aligned}
P_{n+1,N}(T)S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K} &= P_{n+1,N}(T)\cdot\frac{D(T,T_n)-D(T,T_N)}{P_{n+1,N}(T)}\cdot\mathbb{1}_{S_{n,N}(T)>K} \\
&= \Big(D(T,T_n)-D(T,T_N)\Big)\cdot\mathbb{1}_{S_{n,N}(T)>K} \\
&= \sum_{i=n+1}^{N}\Delta_{i-1}D_i(T)L_{i-1}(T)\cdot\mathbb{1}_{S_{n,N}(T)>K}
\end{aligned}$$

In words, the payoff is a floating leg of LIBOR payments if $S_{n,N}(T)>K$, and 0 otherwise. Taking the expectation under the martingale measure, we have

$$\begin{aligned}
\frac{V_0}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N}\left[\frac{V_T}{P_{n+1,N}(T)}\right] \\
&= \mathbb{E}^{n+1,N}\left[\frac{P_{n+1,N}(T)S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K}}{P_{n+1,N}(T)}\right] \\
&= \mathbb{E}^{n+1,N}\left[S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K}\right] \\
\Rightarrow V_0 &= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}S_{n,N}(0)e^{-\frac{\sigma_{n,N}^2T}{2}+\sigma_{n,N}\sqrt{T}x}\,e^{-\frac{x^2}{2}}\,dx \\
&= P_{n+1,N}(0)S_{n,N}(0)\Phi\left(-x^*+\sigma_{n,N}\sqrt{T}\right) \\
&= P_{n+1,N}(0)S_{n,N}(0)\Phi\left(\frac{\log\frac{S_{n,N}(0)}{K}+\frac{\sigma_{n,N}^2T}{2}}{\sigma_{n,N}\sqrt{T}}\right) \quad \triangleleft
\end{aligned}$$

(c) Let V_t denote the value of this contract at time t. In this contract, we observe and pay the swap rate $S_{n,N}$ on maturity date T in a single payment. Under martingale valuation framework, we write

$$\begin{aligned}
\frac{V_0}{D(0,T)} &= \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right] \\
\Rightarrow \quad V_0 &= D(0,T)\mathbb{E}^T\left[S_{n,N}(T)\right] \\
&= D(0,T)\mathbb{E}^T\left[S_{n,N}(0)e^{-\frac{\sigma_{n,N}^2 T}{2}+\sigma_{n,N}W^{n+1,N}(T)}\right]
\end{aligned}$$

Since $W^{n+1,N}(T)$ is not a standard Brownian motion under measure $\mathbb{Q}^T$, we cannot evaluate this expectation without convexity correction. $\triangleleft$