SINGAPORE MANAGEMENT UNIVERSITY

Session 1: Bond Market and Bond Risk Management

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QF605 Fixed Income Securities

Remember that time is money.

— Benjamin Franklin (1706–1790)

Gentlemen prefer bonds.

— Andrew Mellon (1855-1937)

Compounding

The convention used in the market is to treat all quoted interests as annualized, i.e. the total amount of interest paid for a whole year.

Simple/Linear Compounding Investing \$1 at the rate R for a period Δ yields

$$\text{Return} = 1 + \Delta \cdot R$$

$\leq 1y$

money mkt

Discrete Compounding Interest is compounded at discrete frequency:

$$\text{Return} = \left(1 + \frac{R}{m}\right)^{m \times n}$$

where m is the number of payments per year and n is the total number of years.

Continuous Compounding Interest is continuously compounded:

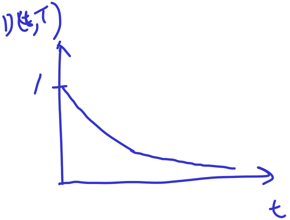
$$\text{Return} = e^{R \cdot T}$$

where T is the number of years. Continuous compounding is often easier to deal with mathematically.

Compounding

We can also turn this question around and ask how much money we need today to compound to \$1 at the maturity date.

This brings in the concept of **discount factor**: $D(t,T)$, denoting the discount factor that discounts from T to back t, in other words, how much money we need at t to compound to \$1 at T.



Depending on the compounding convention, we have

$$D(0,\Delta) = \frac{1}{1+\Delta\cdot R}$$

$$D(0,n) = \frac{1}{\left(1+\frac{R}{m}\right)^{m\times n}}$$

$$D(0,T) = e^{-R\cdot T}$$

$\Rightarrow$ Interest rate is generally positive $(R>0)$, so discount factors are generally less than 1.

$\Rightarrow$ **Liquidity Preference**: people generally prefer to receiver money earlier rather than later, hence $D(0,T)$ should be decreasing in T.

Compounding Frequency

Compound interest is the interest on interest, hence the frequency at which the interest is being compounded is important.

Suppose we invest for one year at an interest rate of 8%. Different compounding frequency leads to different investment outcome.

Compound. freq.	$\frac{r}{m}$	$m \times N$	FV of \$100
Annual	$^{8\%}/_{1}$	1×1	$100 \times (1 + 0.08)^1 = 108$
Semi-annual	$^{8\%}/_{2}$	2×1	$100 \times (1 + 0.04)^2 = 108.16$
Quarterly	$^{8\%}/_{4}$	4×1	$100 \times (1 + 0.02)^4 = 108.24$
Monthly	$^{8\%}/_{12}$	12×1	$100 \times (1 + 0.00667)^{12} = 108.3$
Continuous	8%	1	$100 \times e^{0.08 \times 1} = 108.3287$

Money makes money. And the money that money makes, makes money.

— Benjamin Franklin (1706–1790)

Compound interest is the eighth wonder of the world.

— Attributed to Albert Einstein (1879–1955)

Effective Annual Rate

$1 + r_{EAR} = \left(1 + \frac{r_s}{m}\right)^m$

There is a distinction between stated annual interest rate and **effective annual rate (EAR)** — the interest actually earned over a year.

$$r_{EAR} = \left(1 + \frac{r_S}{m}\right)^m - 1.$$

In our example, \$1 investment that earns 8.16% compounded annually gives the same FV as a \$1 investment that earns 8% compounded semiannually.

For an 8% stated annual interest rate with semi-annual compounding, the EAR is 8.16%

Special Case: **Bond Equivalent Yield** is conventionally used in the U.S. fixed-income market, and restates the yield in terms of semi-annual basis.

$$r_{BEY} = \left[\left(1 + \frac{r_S}{m}\right)^{\frac{m}{2}} - 1\right] \times 2$$

Example The effective annual yield on a fixed-income instrument is 9%. What is the yield on a bond equivalent basis? (ans. 8.81%)

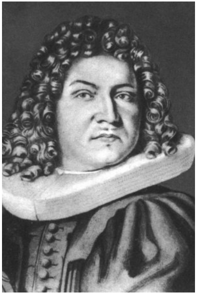
$$1 + r_{EAR} = \left(1 + \frac{r_s}{m}\right)^m = \left(1 + \frac{r_{BEY}}{2}\right)^2$$

$$\left(1 + \frac{r_s}{m}\right)^{\frac{m}{2}} = 1 + \frac{r_{BEY}}{2}$$

$$\left[\left(1 + \frac{r_s}{m}\right)^{\frac{m}{2}} - 1\right] \times 2 = r_{BEY}$$

Side Note: Exponential Constant

An account starts with $1 and pays 100% per year. If the interest is credited once, at the end of the year, the value of the account at year end will be $2. What happens if the interest is computed and credited more frequently during the year?



Jacob Bernoulli (1655–1705)

As the payment frequency increases, the interest paid out needs to be divided by the number of payment, but the interest earned is multiplied by the same amount. Taking the limit yields:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Day Count Conventions

Calculating Δ is important because it tells us how much interest we have accrued over a given period.

Three very common ways of calculating this are

- **Act/Act**: Read "actual/actual", simply count the number of calendar days.
- **30/360**: Read "thirty three-sixty", assumes that there are 30 days in a month and 360 days in a year.

US

- **Act/360**: Read "actual three-sixty", each months has the right number of days, but there are only 360 days in a year.

Asian → Act/365

Business Day Convention If payment date falls on a non-business day, then we need to move the date using business day convention:

- **Actual**: on the actual day, regardless.
- **Following(/Previous)**: rolled onto the following business day.
- **Modified Following(/Previous)**: rolled only the following business day, except if it is the next calendar month, in this case roll backwards.

Day Count Conventions

Example A simple interest rate contract whereby you are paid $\Delta \times 4.5\%$ (Δ is the day count fraction) gives you the optionality to choose one of the following day count convention:

- 30/360
- Act/360
- Act/365
- Act/Act

The 4.5% rate accrues from 1-Jan-2025 to 30-June-2025. Which would you choose and why?

Day Count Conventions

Example Everything else being equal, should a $1m$ expiry call option on a non-dividend paying stock be more valuable on

(A) 15 Jan

(B) 15 Feb

Basic Instruments in the Bond Market

- **Zero-coupon Bonds**: These are pure discount instruments – making a single payment on the maturity date T.
  - $\Rightarrow$ The final payment is commonly referred to as **principal, face value, or notional**.
  - $\Rightarrow$ They tend to have very short maturities (e.g. treasury bills or commercial papers).

- **Coupon Bonds** In addition to the final notional, these also pay a periodic fixed coupon over the life of the bond.
  - $\Rightarrow$ Tends to have longer maturity.
  - $\Rightarrow$ US convention: zero-coupon bonds are called bills, bonds with maturity 2-10 years are called notes, bonds with maturities longer than 10 year are called bonds.

- **Floating Rate Bonds** Coupon payments are not fixed but depend on some benchmark interest rate (floating).
  - $\Rightarrow$ E.g. quarterly compounded $3m$ benchmark rate plus 10 basis points.

Zero-coupon Bonds

The importance of the zero-coupon bonds is that we can write any deterministic set of cashflows as a linear multiple of such bonds.

For instance, suppose we agreed to lend a company a principal N at $T = 1$ and the company is to pay a fixed six-monthly annualized rate of 10% for 5 years, and then at $T = 6$ return the principal.

We can write this transaction in terms of zero-coupon bonds:

$$PV = -N \cdot D(0,1) + \sum_{i=1}^{10} 0.05 \cdot N \cdot D(0, 1 + 0.5i) + N \cdot D(0,6)$$

The convenient property is that—upon substituting the values of the zero-coupon bonds—we have a **present value** for the entire transaction.

If this number is 0, the loan is at fair value. This technique for valuing trades is very standard and is called **present-valuing (or PV)**.

Zero Rates

Zero rate (or spot rate) is the yield-to-maturity of a zero-coupon bond. From the price $D(t,T)$, we can calculate the continuously compounded **spot rate** $R(t,T)$ that is set at t and pays at T.

Continuously compounded zero rate

$$D(t,T) = e^{-R(t,T)(T-t)} \quad \Rightarrow \quad R(t,T) = -\frac{\log D(t,T)}{T-t}.$$

By no-arbitrage, the zero rate should be the interest rate earned on any investment at time t and pays at T $\Rightarrow$ useful for discounting.

Discretely compounded zero rate

$$D(t,T) = \frac{1}{\left(1+\frac{R(t,T)}{m}\right)^{(T-t)\times m}} \quad \Rightarrow \quad R(t,T) = m\left[D(t,T)^{-\frac{1}{m(T-t)}} - 1\right]$$

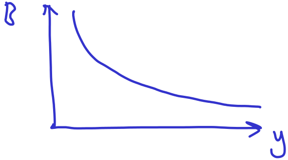
$\Rightarrow$ If $R(t,T)$ is specified for all values of T, then we have a zero rate curve.

$\Rightarrow$ $R(t,T)$ is the interest rate implicit in a discount factor/zero-coupon bond $D(t,T)$.

Yield-to-Maturity

For coupon bonds, **yield-to-maturity (YtM)** is a more convenient interest rate measure. The YtM is defined as the interest rate such that if all the different cash flows are discounted at this rate (y), then the resulting net present value is equal to the current price of the bond. Assuming continuous compounding:

$$B = \sum_{i=1}^{N} c_i e^{-y \cdot T_i}$$



Here c_i is the cash flow paid at time T_i.

The price of the bond can be regarded as a function of the yield. Note that the price of the bond $B(y)$ is a decreasing function of the yield y:

$$\frac{\partial B}{\partial y} = B'(y) = -\sum_{i=1}^{N} T_i c_i e^{-yT_i} < 0.$$

Therefore, if the yield of the bond goes up, then the price of the bond goes down, and vice versa.

Yield-to-Maturity

YtM IRR

The **yield-to-maturity** or **internal rate of return** or **redemption yield** is the discount rate that makes the market price of a bond equal to the discounted value of its future cashflows.

Although continuous compounding is elegant, in practice we use discrete compounding to compute bond prices.

Compare the formula with zero rates and formula with the yield-to-maturity:

$$B = \sum_{i=1}^{N} \frac{c_i}{\left(1+\frac{y}{m}\right)^{m\times T_i}} = \sum_{i=1}^{N} \frac{c_i}{\left(1+\frac{R_i}{m}\right)^{m\times T_i}}$$

Notice that the yield is a blend or a kind of average of the different zero rates associated with the cashflows.

In other words, the yield must be between the highest and lowest zero rates.

Yield-to-Maturity

Consider a $1.5y$ semi-annual coupon bond with a coupon of 8.5%. Compare the two formulas for the bond:

$$1.043066 = \frac{0.0425}{\left(1+\frac{0.0554}{2}\right)^1} + \frac{0.0425}{\left(1+\frac{0.0545}{2}\right)^2} + \frac{1.0425}{\left(1+\frac{0.0547}{2}\right)^3}$$
$$1.043066 = \frac{0.0425}{\left(1+\frac{0.054704}{2}\right)^1} + \frac{0.0425}{\left(1+\frac{0.054704}{2}\right)^2} + \frac{1.0425}{\left(1+\frac{0.054704}{2}\right)^3}$$

$\Rightarrow$ Using the zero rates 5.54%, 5.45%, and 5.47%, the bond price is 1.043066 per dollar par value.

$\Rightarrow$ The implied yield of 5.4704% is a kind of average of the discount rates 5.54%, 5.45%, and 5.47%.

$\Rightarrow$ The YtM is useful because it provides a way of converting a price into something resembling an interest rate. This sometimes makes it easier to <u>compare how cheap or expensive</u> a bond is.

Yield-to-Maturity

Par Yield is the coupon rate that makes the value of the bond equal to its **face value**. In other words, par yield is the value of the coupon rate such that

$$B = \text{Face Value}$$

Bonds are typically issued at par.

Under discrete compounding, a bond will trade at par whenever its yield-to-maturity is equal to the coupon rate.

Example Consider the following annual coupon bond:

$$B = \sum_{i=1}^{N} \frac{c_i}{(1+y)^i}.$$

Show that when the yield-to-maturity y is equal to its coupon rate ($c = 100y$), then $B = 100$.

$$B = \sum_{i=1}^{N} \frac{c_i}{(1+y)^i}$$

$$= \frac{c}{1+y} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c}{(1+y)^N} + \frac{100}{(1+y)^N}$$

$$= \frac{c}{1+y}\left[1 + \frac{1}{1+y} + \frac{1}{(1+y)^2} + \cdots + \frac{1}{(1+y)^{N-1}}\right] + \frac{100}{(1+y)^N}$$

$$= \frac{c}{1+y}\left[\frac{1 - \frac{1}{(1+y)^N}}{1 - \frac{1}{1+y}}\right] + \frac{100}{(1+y)^N}$$

$$= \frac{c}{\cancel{1+y}}\left[\frac{1 - \frac{1}{(1+y)^N}}{\frac{\cancel{1}+y-\cancel{1}}{\cancel{1+y}}}\right] + \frac{100}{(1+y)^N}$$

$$= \frac{c}{y}\left[1 - \frac{1}{(1+y)^N}\right] + \frac{100}{(1+y)^N}$$

$$= \frac{100\cancel{y}}{\cancel{y}}\left[1 - \frac{1}{(1+y)^N}\right] + \frac{100}{(1+y)^N}$$

$$= 100$$

Bootstrapping a Bond Curve

Example Suppose we have 3 coupon bonds (annual coupon):

Bond	Maturity	Coupon	Price
A	$1y$	5	101
B	$2y$	6.5	102
C	$3y$	7	102.5

What can we say about $D(0,1)$, $D(0,2)$, and $D(0,3)$ and the continuously compounded zero rates $R(0,1)$, $R(0,2)$, $R(0,3)$?

ans.: $D(0,1) = 0.9619$, $D(0,2) = 0.899$, $D(0,3) = 0.8362$

$$101 = D(0,1) \times (5 + 100)$$

$$D(0,1) = \frac{101}{105} = \_\_\_\_$$

$$102 = D(0,1) \times 6.5 + D(0,2) \times 106.5$$

$$102.5 = D(0,1) \times 7 + D(0,2) \times 7 + D(0,3) \times 107$$

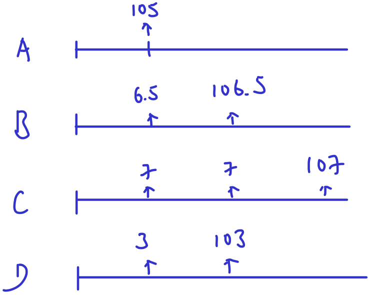
Bootstrapping a Bond Curve

Example Following the previous question, suppose we have another coupon bond D that pays a coupon of 3 annually, with 2 years left to maturity.

1. What should be the no-arbitrage price for this bond?
2. Suppose this bond trades at \$94. Form an arbitrage.

ans.: $B_D = 95.483$. $w_A = 0.0313$, $w_B = -0.967$.

1 $D(0,1) \times 3 + D(0,2) \times 103 = 95.483$



$$\text{Portfolio} = B_D + \omega_A \cdot B_A + \omega_B \cdot B_B$$

$$t=1: \quad 0 = 3 + \omega_A \cdot 105 + \omega_B \cdot 6.5$$

$$t=2: \quad 0 = 103 + \omega_A \cdot 0 + \omega_B \cdot 106.5$$

Bootstrapping a Bond Curve

Example Consider the following bonds from the same issuer:

Maturity (years)	Coupon	Price
0.25	0	97.5
0.50	0	94.9
1.00	0	90.0
1.50	8 (semi-annual)	96.0
2.00	12 (semi-annual)	101.6

$D(0,0.25) = 0.975$

$D(0,.5) = 0.949$

$D(0,1) = 0.9$

Bootstrap the continuously compounded zero rate curve.

ans.: $D(0,1.5) = 0.85196,\ D(0,2) = 0.8056$

$$96 = D(0,0.5)\times 4 + D(0,1)\times 4 + D(0,1.5)\times 104$$

$$101.6 = D(0,0.5)\times 6 + D(0,1)\times 6 + D(0,1.5)\times 6 + D(0,2)\times 106$$

Case Study: On-the-run vs. Off-the-run

On-the-run treasuries are the most recently issued US Treasury bonds of a particular maturity. Because on-the-run issues are the most liquid, they typically trade at a slight premium and therefore yield a little less than their **off-the-run** counterparts.

Long-Term Capital Management (LTCM) successfully exploited this price differential through an arbitrage strategy that involves selling on-the-run treasuries and buying off-the-run treasuries between 1994–1998.

The idea is that this disparity in pricing should not persist. At the same time, there was an economic justification underpinning the disparity.

Convergence Arbitrage

- Trade made between two different securities that tend to become more similar over time. Because they differ from one another, there is daily mark-to-market risk.
- But over the longer term the prices will tend to move closer and closer to each other because it will become increasingly obvious that the two securities are similar.

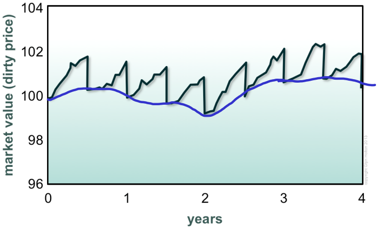
Clean Price *vs.* Dirty Price

The price of bonds quoted in the market are **clean prices**. That is, they are quoted without any accrued interest. The accrued interest is the amount of interest that has built up since the last coupon payment.

trade

The actual payment is called the **dirty price** and is the sum of the quoted clean price and the accrued interest.

The cash price of a coupon bond falls by the amount of the coupon just after the coupon has been paid, this means that the cash price have a "saw" pattern.



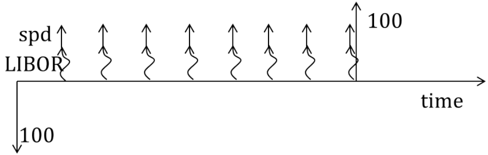
$$\text{Clean Price} = \text{Dirty Price} - \text{AI}$$

$$\text{AI} = c \times \frac{\text{days since last coupon date}}{\text{days between coupon dates}}$$

Floating Rate Notes (FRN)

$\frac{c}{1+y+1\%}$

- **FRN** is a bond that pays a coupon linked to a variable interest rate index – typically LIBOR (or Euribor for EUR).
- FRN eliminates most of the interest rate sensitivity, making it almost a pure credit play – the price action of a FRN is driven mostly by the changes in the market-perceived credit quality of the note issuer.
- While the senior short-term floaters of AA-rated banks pay a coupon close to LIBOR and trade at a price close to par, in the credit markets, many floaters are issued by corporates with much lower credit ratings.
- Many AA-rated banks also issue FRN that are subordinate in the capital structure.

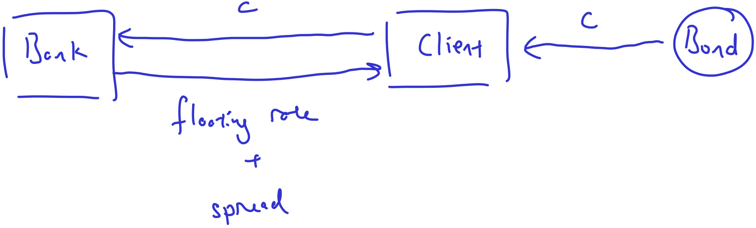


Floating Rate Notes (FRN)

- Either way, investors require a higher yield to compensate them for the increased **credit risk**. At the same time, the coupons of the bonds must be discounted at a higher interest rate than LIBOR to account for this higher credit risk.
- Therefore, in order to issue the note at par, the coupon on the FRN must be set at a fixed spread over LIBOR, this is known as the **par floater spread**.
- Example: 2y FRN with semi-annual payment paying LIBOR + 50bps.
- FRN has a much lower interest rate sensitivity than fixed-rate bonds. If LIBOR moves up, the increment in coupons will be offset by the higher discount rate and vice versa.
- On coupon dates, whether the price of a FRN is above or below par is determined by its par floater spread. If it is above the fixed spread in the coupon, then it will trade below par, and vice versa.

Floating Rate Notes (FRN)

- Between coupon dates, the LIBOR component of the impending coupon payment is fixed and its value is known today.
- However, it is discounted with LIBOR + spread, giving us interest rate exposure for this single cashflow – this is known as **reset risk**. It is usually small and drops to 0 as we approach the payment date.
- A large proportion of the FRN is issued by banks to satisfy their bank capital requirements.
- A large number of corporate and emerging market bonds are issued in FRN format.
- FRN is a way for credit investor to buy bond and take exposure to credit without taking exposure to interest rate movements.
- Most bonds are fixed rate, and so incorporate a significant interest rate sensitivity. We can turn them into pure credit exposure using asset swaps.


Bank
c
floating rate
+
spread
Client
c
Bond

Floating Rate Notes (FRN)

By no-arbitrage, just after an interest payment has been made, the price of a floating rate bond must be equal to par if credit outlook remains constant.

Consider the following trading strategy, which requires an initial amount of cash equal to par:

- ⇒ Invest the cash at the floating rate, until the date of the next coupon payment on the floating rate bond.
- ⇒ When you receive the cash and interest, reinvest the cash amount at the new floating rate until the next coupon payment date.
- ⇒ At maturity of the floating rate bond, you receive the cash equal to par, plus the last interest payment.

Since the payoff of this strategy is equal to the payoff of buying the floating rate bond, the price of the bond must be equal to par.

Just before a coupon is paid, the price of the bond must be equal to par plus the value of the coupon.

$$f(x+\Delta x,\ y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}\cdot\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\Delta x\,\Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right]$$

$$+ \ldots$$

delta vega

$$C(S+\Delta S,\ \sigma+\Delta\sigma) = C(S,\sigma) + \frac{\partial C}{\partial S}\cdot\Delta S + \frac{\partial C}{\partial \sigma}\cdot\Delta\sigma$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + 2\frac{\partial^2 f}{\partial S\partial \sigma}\Delta S\,\Delta\sigma + \frac{\partial^2 f}{\partial \sigma^2}(\Delta\sigma)^2\right]$$

gamma vanna volga

vomma

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \ldots$$

$$B(y+\Delta y) = B(y) + \frac{\partial B}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 B}{\partial y^2}(\Delta y)^2 + \ldots$$

duration

convexity