QF605 Fixed-Income Securities
Solutions to Assignment 4

1. We proceed as follows:

$$
\begin{aligned}
\int_0^\infty h(K)\frac{\partial^2 V^p(K)}{\partial K^2}\,dK &= \int_{K_1}^{K_2} h_1(K)\frac{\partial^2 V^p(K)}{\partial K^2}\,dK + \int_{K_2}^{\infty} h_2(K)\frac{\partial^2 V^p(K)}{\partial K^2}\,dK \\
&= \left[h_1(K)\frac{\partial V^p(K)}{\partial K}\right]_{K_1}^{K_2} - \int_{K_1}^{K_2} h_1'(K)\frac{\partial V^p(K)}{\partial K}\,dK \\
&\quad + \left[h_2(K)\frac{\partial V^p(K)}{\partial K}\right]_{K_2}^{\infty} - \int_{K_2}^{\infty} h_2'(K)\frac{\partial V^p(K)}{\partial K}\,dK \\
&= h_1(K_2)\frac{\partial V^p(K_2)}{\partial K} - h_1(K_1)\frac{\partial V^p(K_1)}{\partial K} - \Big[h_1'(K)V^p(K)\Big]_{K_1}^{K_2} \\
&\quad + \int_{K_1}^{K_2} h_1''(K)V^p(K)\,dK \\
&\quad + \cancelto{0}{h_2(\infty)\frac{\partial V^p(\infty)}{\partial K}} - h_2(K_2)\frac{\partial V^p(K_2)}{\partial K} - \Big[h_2'(K)V^p(K)\Big]_{K_2}^{\infty} \\
&\quad + \int_{K_2}^{\infty} h_2''(K)V^p(K)\,dK \\
&= h_1(K_2)\frac{\partial V^p(K_2)}{\partial K} - h_1(K_1)\frac{\partial V^p(K_1)}{\partial K} - h_1'(K_2)V^p(K_2) + h_1'(K_1)V^p(K_1) \\
&\quad + \int_{K_1}^{K_2} h_1''(K)V^p(K)\,dK \\
&\quad - h_2(K_2)\frac{\partial V^p(K_2)}{\partial K} - \cancelto{0}{h_2'(\infty)V^p(\infty)} + h_2'(K_2)V^p(K_2) \\
&\quad + \int_{K_2}^{\infty} h_2''(K)V^p(K)\,dK
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(K) &= \frac{K-K_1}{\text{IRR}(K)} \\
h_1'(K) &= \frac{\text{IRR}(K) - (K-K_1)\text{IRR}'(K)}{\text{IRR}(K)^2} \\
h_1''(K) &= \frac{-\text{IRR}''(K)(K-K_1) - 2\cdot\text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2(K-K_1)}{\text{IRR}(K)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(K) &= \frac{K_2-K_1}{\text{IRR}(K)} \\
h_2'(K) &= -\frac{(K_2-K_1)\text{IRR}'(K)}{\text{IRR}(K)^2} \\
h_2''(K) &= -\frac{\text{IRR}''(K)(K_2-K_1)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2(K_2-K_1)}{\text{IRR}(K)^3}
\end{aligned}
$$

Substituting h_1, h_1', h_2, and h_2', the static-replication formula can be simplified to

$$\frac{1}{\text{IRR}(K_1)}V^p(K_1) - \frac{1}{\text{IRR}(K_2)}V^p(K_2) + \int_{K_1}^{K_2} h_1''(K)V^p(K)\,dK + \int_{K_2}^{\infty} h_2''(K)V^p(K)\,dK \quad \triangleleft$$

2. Starting with the stochastic differential equation of the Ho-Lee model:

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

we integrate both sides from 0 to t to obtain

$$\begin{aligned}
\int_0^t dr_s &= \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^* \\
\Rightarrow \quad r_t &= r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*
\end{aligned}$$

Next, integrating r_t from 0 to T, we obtain

$$\begin{aligned}
\int_0^T r_u\,du &= r_0T + \int_0^T\int_0^u \theta(s)\,ds\,du + \int_0^T\int_0^u \sigma\,dW_s^*\,du \\
&= r_0T + \int_0^T\int_s^T \theta(s)\,du\,ds + \int_0^T\int_s^T \sigma\,du\,dW_s^* \\
&= r_0T + \int_0^T \theta(s)(T-s)\,ds + \int_0^T \sigma(T-s)\,dW_s^*
\end{aligned}$$

Hence, the mean of the short rate integral is

$$\mathbb{E}\left[\int_0^T r_u\,du\right] = r_0T + \int_0^T \theta(s)(T-s)\,ds \quad \triangleleft$$

while the variance of the short rate integral is

$$\begin{aligned}
V\left[\int_0^T r_u\,du\right] &= V\left[\int_0^T \sigma(T-s)\,dW_s^*\right] \\
&= \int_0^T \sigma^2(T-s)^2\,ds \\
&= \frac{\sigma^2T^3}{3} \quad \triangleleft
\end{aligned}$$

3. Suppose the length of each period is $1y$, based on the discount factor $D(0,1y) = 0.9656$, we can work out that the short rate starts at 3.5%. The 3-period binomial tree for the short-rate under Ho-Lee model is as follows:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & r = 4.5\% + \theta_0 + \theta_1 \\
 & r = 4\% + \theta_0 \arrow[ur] \arrow[dr] & \\
r = 3.5\% \arrow[ur] \arrow[dr] & & r = 3.5\% + \theta_0 + \theta_1 \\
 & r = 3\% + \theta_0 \arrow[ur] \arrow[dr] & \\
 & & r = 2.5\% + \theta_0 + \theta_1
\end{tikzcd}
$$

To match the $2y$ discount factor, we need

$$
\begin{aligned}
D(0,2y) &= D(0,1y) \times \left[0.5 \times e^{-0.04-\theta_0} + 0.5 \times e^{-0.03-\theta_0}\right] \\
0.9224 &= 0.9656 \times \left[0.5 \times e^{-0.04} + 0.5 \times e^{-0.03}\right] \times e^{-\theta_0} \\
\Rightarrow \quad \theta_0 &= 0.01078 \quad \triangleleft
\end{aligned}
$$

Next, we work out

$$
\begin{aligned}
D_u(1y,3y) &= e^{-0.04-\theta_0} \times \left[0.5 \times e^{-0.045-\theta_0-\theta_1} + 0.5 \times e^{-0.035-\theta_0-\theta_1}\right] \approx 0.90343e^{-\theta_1} \\
D_d(1y,3y) &= e^{-0.03-\theta_0} \times \left[0.5 \times e^{-0.035-\theta_0-\theta_1} + 0.5 \times e^{-0.025-\theta_0-\theta_1}\right] \approx 0.92168e^{-\theta_1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
D(0,3y) &= D(0,1y) \times \left[0.5 \times 0.90343e^{-\theta_1} + 0.5 \times 0.92168^{-\theta_1}\right] \\
0.8903 &= 0.9656 \times \left[0.5 \times 0.90343 + 0.5 \times 0.92168\right] \times e^{-\theta_1} \\
\Rightarrow \quad \theta_1 &= -0.01032 \quad \triangleleft
\end{aligned}
$$