LIBOR-IN-ARREARS SWAPS

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Abstract: LIBOR-in-arrears swaps differ from regular swaps in that for the former the LIBOR rate is paid at the beginning of the payment period while for the latter the LIBOR rate is paid at the end of the payment period. This mismatch in cashflow timing causes the present value of all the floating payments in a LIBOR-in-arrears swap not equal to the present value of all the forwards. The difference is termed as the convexity adjustment and depends on the volatilities of interest rates. We have derived exact and approximate solutions for the convexity adjustment that uses only the LIBOR yield and volatility curves, both of which are readily observable in the interest rate swap and cap markets. Numerical examples are also provided.

1. INTRODUCTION

The unusually steep yield curve that we had seen recently in the U.S. in 1992 - 1993 promoted a variety of new structures in the market. One of the more popular ones is the LIBOR-in-arrears structure. In standard swap structures, on the floating leg, the rate is set at the beginning of the period and the payment is at the end of the period. In a LIBOR-in arrears structure, the rate paid at the end of the period is the then prevailing rate for the next period. In essence, by utilizing the forward part of an upward sloping curve, the expected coupon levels on the floating side are increased. By definition, as an at-market swap is a zero market value instrument, the fixed coupon also has to be increased for a given maturity of the instrument. In addition, if the investor also has the view that the forward rates reflected in the yield curve are overstated relative to his/her expectation of the future spot rates, the swap also provides an attractive vehicle to capitalize on such a view.

In pricing such a structure, it is tempting to use the discounted value of all the relevant forward rates (with the payment dates moved to a period earlier, compared to a regular swap) to calculate the fixed rate. However, the effect of the convexity of the forward price/rate is ignored in such a calculation. This simplification may not create large errors when the maturities of the structures are short and volatilities in the market are low. However, as maturities lengthen and volatilities move higher, the errors may be significant. Such errors penalize the fixed rate receiver by underpricing the structure, and it is therefore important to know the extent of mispricing inherent in ignoring convexity for various maturities.

Like most interest rate derivatives, LIBOR-in-arrears swaps can be priced in a term structure model of interest rates, such as Black Derman and Toy (1990) and Heath, Jarrow, and Morton (1992). This approach is less intuitive and not easily assessable to swap traders.

In this paper, we derive a simple pricing formula for LIBOR-in-arrears swaps that requires only the LIBOR yield and volatility curves, both of which are readily observable in the interest rate swap and cap markets. In addition to the theoretical underpinnings in such a calculation, we also show the convexity corrections for 5- and 10-year structures.

2. PRICING A LIBOR-IN-ARREARS SWAP

Like a regular swap, a LIBOR-in-arrears swap consists of a series of forward contracts where LIBOR-in-arrears payments are exchanged with fixed payments. Since fixed payments can be easily valued we will focus on the valuation of LIBOR-in-arrears payments.

Consider a LIBOR-in-arrears swap with payment periods $[t_{i-1}, t_i]$, $i = 1, \ldots, n$. For convenience, assume $t_0 = 0$. Let $\Delta t_i = t_i - t_{i-1}$, the length of period *i.* At payment date t_i, the LIBOR in arrears, $l(t_i, t_{i+1})$, which is the LIBOR applicable for period $[t_i, t_{i+1}]$ is paid to the floating receiver. The present value of this payment is

$$PV_{arr}(t_i) = \tilde{E}_{t_i}[l(t_i, t_{i+1})\Delta t_i exp(-\int_0^{t_i} r(t)dt)], \tag{1}$$

where $\tilde{E}[\ ]$ is the expectation operator under the risk-neutral measure $\tilde{Q}$ and $r(t)$ is the instantaneous interest rate at time t. Since we are not interested in using term structure models of interest rates to price LIBOR-in-arrears swaps, we apply change of measure to eliminate the need of $r(t)$. Following Ritchken and Sankarasubramanian (1995) there exists a martingale measure, namely the forward-risk adjusted (FRA) measure $\hat{Q}_{t_i}$, such that Equation (1) can be re-written as

$$PV_{arr}(t_i) = \ddot{E}_{t_i}[l(t_i, t_{i+1})\Delta t_i]P(0, t_i), \tag{2}$$

where $\ddot{E}_{t_i}[\ ]$ is the expectation operator under the $\hat{Q}_{t_i}$, and $P(0, t_i)$ is the present value of a discount bond maturing at time t_i. Note that the $\hat{Q}_{t_i}$ depends on the LIBOR reset date t_i. Since the LIBOR rate $l(t_i, t_{i+1})$ is not a traded asset, its expectation under $\ddot{Q}_{t_i}$ does not equal the forward LIBOR rate $l_0(t_i, t_{i+1}) = \frac{1}{\Delta t_i}\left[\frac{P(0, t_i)}{P(0, t_{i+1})} - 1\right]$. Define the discounted LIBOR rate as

$$\xi(t_i, t_{i+1}) = \frac{l(t_i, t_{i+1})}{1 + l(t_i, t_{i+1})\Delta t_{i+1}}, \tag{3}$$

and the corresponding forward discounted LIBOR as

$$\xi_0(t_i, t_{i+1}) = \frac{l_0(t_i, t_{i+1})}{1 + l_0(t_i, t_{i+1})\Delta t_{i+1}}. \tag{4}$$

Then, as shown in the appendix

$$\ddot{E}_{t_i}[\xi(t_i, t_{i+1})] = \xi_0(t_i, t_{i+1}). \tag{5}$$

Assume that the LIBOR rate $l(t_i, t_{i+1})$ is lognormally distributed under $\hat{Q}_{t_i}$, i.e.

$$l(t_i, t_{i+1}) = \eta l_0(t_i, t_{i+1}) exp[-\sigma^2 t_i/2 + \sigma\sqrt{t_i}z], \tag{6}$$

where z is a standard normal variable under $\hat{Q}_{t_i}$ and η is some constant to be determined later. The assumption of lognormality is consistent with the pricing of interest caps and floors in practice where the Black (1976) model are used. Thus LIBOR cap volatilities can be used in Equation (6).

The only unknown parameter in Equation (6) is η. Using Jensen's Inequality we can show that $\ddot{E}_{t_i}[l(t_i, t_{i+1})] \geq l_0(t_i, t_{i+1})$ because $l(t_i, t_{i+1})$ is convex function of $\xi(t_i, t_{i+1})$. This and Equation (6) imply that $\eta \geq 1$. Therefore the expectation of the LIBOR rate is always higher than the forward LIBOR rate under $\ddot{Q}_{t_i}$ due to randomness of the LIBOR rate. In other words, receiving LIBOR-in-arrears is a long volatility position.

The exact solution for η can be obtained numerically through the following equation:

$$\hat{E}_{t_i}\left[\frac{1}{[\eta l_0(t_i, t_{i+1}) exp(-\sigma^2 t_i/2 + \sigma\sqrt{t_i}z)]^{-1} + \Delta t_{i+1}}\right] = \frac{1}{[l_0(t_i, t_{i+1})^{-1}] + \Delta t_{i+1}}, \tag{7}$$

which is a result of Equations (3) - (6). In practice, we recommend the following closed-form approximation for η:

$$\eta \approx 1 + l_0(t_i, t_{i+1})\Delta t_i[exp(\sigma^2 t_i) - 1] \tag{8}$$

Substituting Equation (6) and Equation (8) into Equation (2) yields.

$$PV_{arr}(t_i) \approx l_0(t_i, t_{i+1})\Delta t_i[1 + l_0(t_i, t_{i+1})\Delta t_i\{exp[\sigma^2 t_i] - 1\}]P(0, t_i). \tag{9}$$

This equation essentially decomposes the present value of the LIBOR-in-arrears payment into its intrinsic value and a volatility-related correction term. As we will show in the next section, the approximation is satisfactory.

3. NUMERICAL EXAMPLES

Consider a 10-year semi-annual floating-rate note with principal of $1,000,000. It pays a coupon in the amount of 6-month LIBOR-in-arrears. Assume the yield curve is 5% flat (semi-annual compounding) and the 6-month LIBOR volatility is 20%.The flat yield curve assumption does not invalidate the implication of this example because the correction term in Equation (9) depends only on the level of forward LIBOR rate, not the shape of the yield curve.

TABLE 1. Prices for an FRN with LIBOR-in-arrears coupon

Coupon Date (yr.)	Intrinsic Value	Theoretical Correction	Approximate Correction	Approximation Error in $	Swap Rate Correction
0.5	$24,390	$12.01	$12.32	$0.31	
1.0	$23,795	$23.66	$24.28	$0.62	.37
1.5	$23,215	$34.95	$35.89	$0.93	
2.0	$22,649	$45.91	$47.16	$1.24	.62
2.5	$22,096	$56.52	$58.09	$1.57	
3.0	$21,557	$66.81	$68.71	$1.90	.87
3.5	$21,032	$76.77	$79.01	$2.24	
4.0	$20,519	$86.42	$89.01	$2.58	1.12
4.5	$20,018	$95.76	$98.70	$2.94	
5.0	$19,530	$104.80	$108.10	$3.30	1.38
5.5	$19,054	$113.55	$117.22	$3.67	
6.0	$18,589	$122.00	$126.06	$4.05	1.64
6.5	$18,136	$130.18	$134.62	$4.44	
7.0	$17,693	$138.08	$142.93	$4.84	1.89
7.5	$17,262	$145.71	$150.98	$5.27	
8.0	$16,841	$153.08	$158.78	$5.70	2.15
8.5	$16,430	$160.19	$166.33	$6.14	
9.0	$16,029	$167.05	$173.65	$6.60	2.42
9.5	$15,638	$173.65	$180.73	$7.08	
10.0	$15,257	$180.02	$187.56	$7.57	2.68
Principal	$610,271	$0.00	$0.00	$0.00	
Total	$1,000,000	$2,087.15	$2,160.15	-$73.06	

Table 1 shows the intrinsic value, full correction, approximate correction, approximation error and swap rate correction in basis points. The data shown here can be used for any swap with maturity not exceeding 10 years. It can be seen that the total correction for a 10-year swap amounts to a 20.8715 basis point differential in up-front price. That translates into a swap rate correction of 2.68 basis points. For a 5-year swap, the impact is relatively small: 6 basis points up-front, or 1.38 basis points in terms of the swap rate. The error due to the approximation is negligible (-$73.06 or -0.7306 basis point up front for a 10 year swap).

Tables 2 and 3 show the sensitivities of 5-year and 10-year LIBOR-in-arrears swap rates (in basis points) to interest rate and volatility levels.

TABLE 2. 5 Year Swap Rate Correction Sensitivities

Rate Level (Flat)	Volatility = 15%	Volatility = 20%	Volatility = 25%
5.0%	0.75	1.38	2.24
7.5%	1.64	3.00	4.86
10.0%	2.83	5.16	8.33

TABLE 3. 10 Year Swap Rate Correction Sensitivities

Rate Level (Flat)	Volatility = 15%	Volatility = 20%	Volatility = 25%
5.0%	1.43	2.68	4.49
7.5%	3.03	5.67	9.42
10.0%	5.08	9.47	15.55

4. CONCLUSION

The convexity correction to a LIBOR-in-arrears swap was shown not to be trivial. The correction depends on the volatility and forward rate levels and not on the shape of the yield curve. It was also demonstrated that the approximation to the correction term can be computed in closed form. In addition, the volatility used in the correction term is the implied interest rate cap volatility which is observable from the market.

APPENDIX A: Proof of Equation (5)

Consider a regular swap that differs from the LIBOR-in-arrears swap only in that at payment date t_i, the LIBOR $l(t_{i-1}, t_i)$ which is applicable to period $[t_{i-1}, t_i]$ is paid to the floating receiver. The present value (time t_0 price) of this floating payment is

$$PV_{reg}(t_i) = \tilde{E}[l(t_{i-1}, t_i)\Delta t_i exp(-\int_0^{t_i} r(u)du) \Big| I(t_0)] \quad \textbf{(10)}$$

where $I(t)$ is the information set at time t. Applying change of measure to Equation (10) yields:

$$PV_{reg}(t_i) = \hat{E}_{t_i}[l(t_{i-1}, t_i)\Delta t_i]P(0, t_i), \tag{11}$$

We also know that the present value of a LIBOR payment in a regular swap equals the present value of the corresponding forward LIBOR, $l_0(t_{i-1}, t_i)$; that is

$$PV_{reg}(t_i) = [l_0(t_{i-1}, t_i)\Delta t_i]P(0, t_i). \tag{12}$$

Equation (11) and Equation (12) confirm the fact that, under the FRA measure $\ddot{Q}_{t_i}$ at time t_i, the forward LIBOR rate $l_0(t_{i-1}, t_i)$ is an unbiased estimate of the future LIBOR rate $l(t_{i-1}, t_i)$.

Now, let us examine an alternative way to price a regular swap. In Equation (10), the LIBOR rate $l(t_{i-1}, t_i)$ is known at time t_{i-1}. Paying LIBOR $l(t_{i-1}, t_i)$ at time t_i is equivalent to paying the "discounted" LIBOR $\xi(t_{i-1}, t_i)$ (defined in Equation (3)) at time t_{i-1}. This is evident through the following steps:

$$\begin{aligned}
PV_{reg}(t_i) &= \tilde{E}\left[l(t_{i-1}, t_i)\Delta t_i \tilde{E}\left[exp\left(-\int_{t_{i-1}}^{t_i} r(u)du\right)\Big| I(t_{i-1})\right] exp\left(-\int_0^{t_{i-1}} r(u)du\right)\Big| I(t_0)\right] \\
&= \tilde{E}[l(t_{i-1}, t_i)\Delta t_i P(t_{i-1}, t_i) exp\left(-\int_0^{t_{i-1}} r(u)du\right)\Big| I(t_0)] \\
&= \tilde{E}\left[\frac{l(t_{i-1}, t_i)\Delta t_i}{1 + l(t_{i-1}, t_i)\Delta t_i} exp\left(-\int_0^{t_{i-1}} r(u)du\right)\Big| I(t_0)\right] \\
&= \tilde{E}[\xi(t_{i-1}, t_i)\Delta t_i exp\left(-\int_0^{t_{i-1}} r(u)du\right)\Big| I(t_0)]
\end{aligned} \tag{13}$$

Rewrite Equation (13) in terms of the FRA measure at time t_{i-1}:

$$PV_{reg}(t_i) = \ddot{E}_{t_{i-1}}[\xi(t_{i-1}, t_i)]P(0, t_{i-1}) \tag{14}$$

Comparing Equation (11) and Equation (13) yields

$$\hat{E}_{t_{i-1}}[\xi(t_{i-1}, t_i)] = l_0(t_{i-1}, t_i)\frac{P(0, t_i)}{P(0, t_{i-1})} = \xi_0(t_{i-1}, t_i) \tag{15}$$

Thus, under the forward-risk adjusted measure at time t_{i-1}, the forward discounted LIBOR rate $\xi_0(t_{i-1}, t_i)$ is an unbiased estimate of the future discounted LIBOR rate $\xi(t_{i-1}, t_i)$.

ENDNOTE

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APPENDIX B. Exact Solution of η is Equation (7).

Substitute Equation (6) into Equation (2) yields

$$PV_{arr}(t_i) = \eta l_0(t_i, t_{i+1})\Delta t_i P(0, t_i) \tag{16}$$

Since the forward LIBOR is usually a biased estimate of $l(t_i, t_{i+1})$, we have $\eta \neq 1$. Using Equation (3), Equation (4) and Equation (5) yields

$$0 = \bar{E}_{t_i}\left[\frac{l(t_i, t_{i+1})}{1 + l(t_i, t_{i+1})\Delta t_{i+1}}\right] - \frac{l_0(t_i, t_{i+1})}{1 + l_0(t_i, t_{i+1})\Delta t_{i+1}} \tag{17}$$

Using Taylor's expansion

$$\begin{aligned}
\bar{E}_{t_i}\left[\frac{l(t_i, t_{i+1})\Delta t_{i+1}}{1 + l(t_i, t_{i+1})\Delta t_{i+1}}\right] &= \bar{E}_{t_i}\left[\frac{l(t_i, t_{i+1})\Delta t_{i+1} I_{\{l(t_i, t_{i+1})\Delta t_{i+1} < 1\}}}{1 + l(t_i, t_{i+1})\Delta t_{i+1}}\right] + \bar{E}_{t_i}\left[\frac{I_{\{l(t_i, t_{i+1})\Delta t_{i+1} > 1\}}}{[l(t_i, t_{i+1})\Delta t_{i+1}]^{-1} + 1}\right] \\
&= \bar{E}_{t_i}\left[\sum_{k=1}^{\infty}(-1)^{k-1}[l(t_i, t_{i+1})\Delta t_{i+1}]^k I_{\{l(t_i, t_{i+1})\Delta t_{i+1} < 1\}}\right] \\
&\quad + \bar{E}_{t_i}\left[\sum_{k=0}^{\infty}(-1)^k[l(t_i, t_{i+1})\Delta t_{i+1}]^{-k} I_{\{l(t_i, t_{i+1})\Delta t_{i+1} > 1\}}\right] \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}[\eta l_0(t_i, t_{i+1})\Delta t_{i+1}]^k exp\left[\frac{k(k-1)\sigma^2 t_i}{2}\right] N(-d_1^{(k)}) \\
&\quad + \sum_{k=0}^{\infty}(-1)^k[\eta l_0(t_i, t_{i+1})\Delta t_{i+1}]^{-k} exp\left[\frac{k(k+1)\sigma^2 t_i}{2}\right] N(-d_2^{(k)})
\end{aligned} \tag{18}$$

where

$$d_1^{(k)} = \frac{k[\log(\eta I_0(t_i, t_{i+1})\Delta t_{i+1})]}{\sigma\sqrt{t_i}} + \frac{(k^2 - k + 1)\sigma\sqrt{t_i}}{2} \tag{19}$$

$$d_2^{(k)} = \frac{-k[\log(\eta I_0(t_i, t_{i+1})\Delta t_{i+1})]}{\sigma\sqrt{t_i}} + \frac{(k^2 + k + 1)\sigma\sqrt{t_i}}{2} \tag{20}$$

Note that

$$0 \le \bar{E}_{t_i}\left[\frac{I_{\{l(t_i, t_{i+1})\Delta t_{i+1} > 1\}}}{l^{-1}(t_i, t_{i+1}) + \Delta t_{i+1}}\right] \le \bar{E}_{t_i}\left[\frac{I_{\{l(t_i, t_{i+1})\Delta t_{i+1} > 1\}}}{2\Delta t_{i+1}}\right] = \frac{Prob[l(t_i, t_{i+1})\Delta t_{i+1} > 1]}{2\Delta t_{i+1}} \tag{21}$$

Using Equation (6) and the fact that $\eta \ge 1$ we have

$$\begin{aligned} Prob[l(t_i, t_{i+1})\Delta t_{i+1} > 1] &\le Prob[\, l_0(t_i, t_{i+1}) exp[-\sigma^2 t_i/2 + \sigma\sqrt{t_i} z]\Delta t_{i+1} > 1] \\ &= N\left[\frac{\log(l_0(t_i, t_{i+1})\Delta t_{i+1}) - \sigma^2 t_i/2}{\sigma\sqrt{t_i}}\right] \end{aligned} \tag{22}$$

One can solve this equation to find the adjustment η. When the volatility is zero, the solution is $\eta = 1$. In this case, the price of an LIBOR-in-arrears payment is simply the present value of the forward LIBOR-in-arrears; that is, $l_0(t_i, t_{i+1})\Delta t_i P(0, t_i)$. We refer this to the intrinsic value of the LIBOR-in-arrears payment.

Combining Equation (16) and Equation (17) yields

$$PV_{arr}(t_i) = l_0(t_i, t_{i+1})\Delta t_i\left[1 - \sum_{k=2}^{\infty}[-l_0(t_i, t_{i+1})\Delta t_{i+1}]^{k-1}\{\eta^k exp[k(k-1)\sigma^2 t_i/2] - 1\}\right]P(0, t_i) \tag{23}$$

This equation essentially decomposes the present value of the LIBOR-in-arrears into its intrinsic value and a volatility-related correction term.

An approximate solution for Equation (18) is

$$PV_{arr}(t_i) = l_0(t_i, t_{i+1})\Delta t_i[1 + l_0(t_i, t_{i+1})\Delta t_i\{exp[\sigma^2 t_i] - 1\}]P(0, t_i)\,. \tag{24}$$

The above approximation was calculated with $k = 2$ and $\eta = 1$, as the actual solution of Equation (17) gives a value of η very close to 1. Accuracy of the approximation also depends on the coupon level.