LU decomposition

Cholesky Decomposition of Brownian Motions

Given two stochastic processes that are correlated with a correlation of ρ, it is often beneficial to be able to decompose them and express them as independent Brownian motions. Consider two correlated SDEs

$$dX_t = \mu_X X_t dt + \sigma_X X_t dW_t^X$$
$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y,$$

where $dW_t^X dW_t^Y = \rho\, dt$. We can decompose the two correlated Brownian processes as follow:

$$dW_t^X = \alpha_{11} dZ_t^{(1)}$$
$$dW_t^Y = \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)},$$

where $dZ_t^{(1)} dZ_t^{(2)} = 0$, and

$$\alpha_{11} = 1, \quad \alpha_{12} = \rho, \quad \alpha_{22} = \sqrt{1-\alpha_{12}^2}.$$

Cholesky Decomposition of Brownian Motions

This is because of moment matching. First, the variance relationship is

$$V\left[dW_t^X\right] = V\left[\alpha_{11}dZ_t^{(1)}\right]$$
$$dt = \alpha_{11}^2 dt \quad \Rightarrow \quad \alpha_{11} = 1.$$

Next, the covariance relationship is

$$\text{Cov}\left[dW_t^X,\ dW_t^Y\right] = \text{Cov}\left[\alpha_{11}dZ_t^{(1)},\ \alpha_{12}dZ_t^{(1)} + \alpha_{22}dZ_t^{(2)}\right]$$

$$\rho\ dt = \text{Cov}\left[\alpha_{11}dZ_t^{(1)},\ \alpha_{12}dZ_t^{(1)}\right] + \cancelto{0}{\text{Cov}\left[\alpha_{11}dZ_t^{(1)}, \alpha_{22}dZ_t^{(2)}\right]}$$
$$= \alpha_{11}\alpha_{12}\ dt \quad \Rightarrow \quad \alpha_{12} = \rho.$$

And finally,

$$V\left[dW_t^Y\right] = V\left[\alpha_{12}dZ_t^{(1)} + \alpha_{22}dZ_t^{(2)}\right]$$
$$dt = \alpha_{12}^2\ dt + \alpha_{22}^2\ dt \quad \Rightarrow \quad \alpha_{22} = \sqrt{1-\rho^2}.$$

Forward Exchange Rate Process

Earlier, we mentioned that from the domestic investor's perspective, the spot exchange rate follows

$$dX_t = (r^D - r^F)X_t dt + \sigma_X X_t dW_t^D,$$

while from the foreign investor's perspective, the spot FX rate follows

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt + \sigma_X \frac{1}{X_t} dW_t^F.$$

We have also covered that the forward exchange rate can be written as

$$\begin{aligned}\mathbb{E}^D[X_T] &= \mathbb{E}^D\left[X_t e^{\left(r^D - r^F - \frac{\sigma_X^2}{2}\right)(T-t) + \sigma_X (W_T^D - W_t^D)}\right] \\ &= X_t e^{(r^D - r^F)(T-t)}.\end{aligned}$$

Let $F_t = X_t e^{(r^D - r^F)(T-t)}$ denote the forward exchange rate process (maturity at T), we can use Itô's formula to show that

$$dF_t = \sigma_X F_t dW_t^D.$$

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^*$$

$$F_t = S_t e^{r(T-t)} = f(t, S_t)$$

$$dF_t = \sigma F_t\,dW_t^*$$

Forward Exchange Rate Process

Let $D^D(t,T)$ denote the LIBOR discount factor in the domestic economy, and $D^F(t,T)$ denote the LIBOR discount factor in the foreign economy. Let us express the forward exchange rate F_t (maturing at T) as:

$$F_t = X_t \cdot \frac{D^F(t,T)}{D^D(t,T)}$$

By the same argument, we also have

$$\mathbb{E}^F\left[\frac{1}{X_T}\right] = \frac{1}{X_t} e^{(r^F - r^D)(T-t)}$$

and the forward exchange rate (maturity at T) from the foreign investor's perspective as

$$d\frac{1}{F_t} = \sigma_X \frac{1}{F_t} dW_t^F.$$

Therefore, we express it as

$$\frac{1}{F_t} = \frac{1}{X_t} \cdot \frac{D^D(t,T)}{D^F(t,T)}$$

Pricing Quanto LIBOR

In a quanto LIBOR contract, a foreign LIBOR rate L_i^F is observed at T_i and is paid in domestic denomination at time T_{i+1}. Suppose the Brownian motions W_t^{i+1} and W_t^F have a correlation of ρ.

To price this contract, we apply our multi-currency convexity correction formula

$$
\begin{aligned}
\mathbb{E}^{i+1,D}\left[L_i^F(T)\right] &= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{d\mathbb{Q}^{i+1,D}}{d\mathbb{Q}^{i+1,F}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{D_{i+1}^D(T_{i+1})}{D_{i+1}^D(0)}}{\frac{X_T D_{i+1}^F(T_{i+1})}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{D_{i+1}^D(T_{i+1})}{X_T D_{i+1}^F(T_{i+1})}}{\frac{D_{i+1}^D(0)}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}}\right]
\end{aligned}
$$

$$\mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{1}{F_T}}{\frac{1}{F_0}} \right]$$

$$= \mathbb{E}^{i+1,F}\left[L_i^F(0)\, e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1,F}} \cdot \frac{\cancel{\frac{1}{F_0}}\, e^{-\frac{\sigma_x^2 T}{2} + \sigma_x W_T^F}}{\cancel{\frac{1}{F_0}}} \right]$$

$$= L_i^F(0)\, e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}}\, \mathbb{E}^{i+1,F}\left[e^{\sigma_i W_T^{i+1,F} + \sigma_x W_T^F} \right]$$

$$= L_i^F(0)\, e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}}\, \mathbb{E}^{i+1,F}\left[e^{\sigma_i Z_T^{(1)} + \sigma_x\left(\rho Z_T^{(1)} + \sqrt{1-\rho^2}\, Z_T^{(2)}\right)} \right]$$

$$= L_i^F(0)\, e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}}\, \mathbb{E}^{i+1,F}\left[e^{(\sigma_i + \sigma_x \rho) Z_T^{(1)}} \right] \mathbb{E}^{i+1,F}\left[e^{\sigma_x \sqrt{1-\rho^2}\, Z_T^{(2)}} \right]$$

$$= L_i^F(0)\, e^{-\frac{\sigma_i^2 T}{2}}\, e^{-\frac{\sigma_x^2 T}{2}}\, e^{(\sigma_i + \sigma_x \rho)^2 \cdot \frac{T}{2}}\, e^{\sigma_x^2 (1-\rho^2)\cdot \frac{T}{2}}$$

$$= L_i^F(0)\, e^{-\cancel{\frac{\sigma_i^2 T}{2}}}\, e^{-\cancel{\frac{\sigma_x^2 T}{2}}}\, e^{\cancel{\frac{\sigma_i^2 T}{2}} + \sigma_i \sigma_x \rho T + \cancel{\frac{\sigma_x^2 \rho^2 T}{2}}}\, e^{\cancel{\frac{\sigma_x^2 T}{2}} - \cancel{\frac{\sigma_x^2 \rho^2 T}{2}}}$$

$$\tilde{L}_i^F(0) = L_i^F(0)\, e^{\sigma_i \sigma_x \rho T}$$

buy high sell low

$\rho > 0$:

$L_i^F \uparrow \quad \frac{1}{F} \uparrow$ (foreign currency depreciate)

$L_i^F \downarrow \quad \frac{1}{F} \downarrow$ (foreign currency appreciate)

Pricing Quanto LIBOR

Substituting for $L_i^F(T)$ and the forward exchange rate, the expectation to evaluate becomes

$$\mathbb{E}^{i+1,F}\left[L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W^{i+1}}\cdot\frac{1}{F_0}e^{-\frac{\sigma_X^2 T}{2}+\sigma_X W^F}\right]\times F_0.$$

Next, we apply Cholesky decomposition:

$$\begin{aligned}W^{i+1} &:\longrightarrow Z_1\\ W^F &:\longrightarrow \rho Z_1+\sqrt{1-\rho^2}Z_2\end{aligned}$$

where $Z_1 \perp Z_2$.

Finally, the convexity corrected foreign LIBOR rate paid in domestic denomination is given by

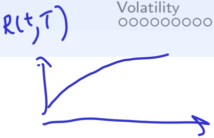
$$\tilde{L}_i^F(T)=L_i^F(0)e^{\rho\sigma_X\sigma_i T}.$$

Session 7
Short Rate Models and Term Structure

Models

Tee Chyng Wen

QF605 Fixed Income Securities



Term Structure Models

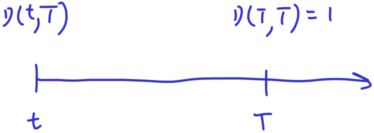
The **Market Models** and static replication method can handle the pricing of derivatives with **European payoffs**, such as caps, floors and European swaptions.

However, they are not able to handle derivatives with **path-dependent payoffs**, e.g. Bermudan or American option.

To value path-dependent products, we need a model of how the whole **term structure** (not just a single forward rate or bond) evolves.

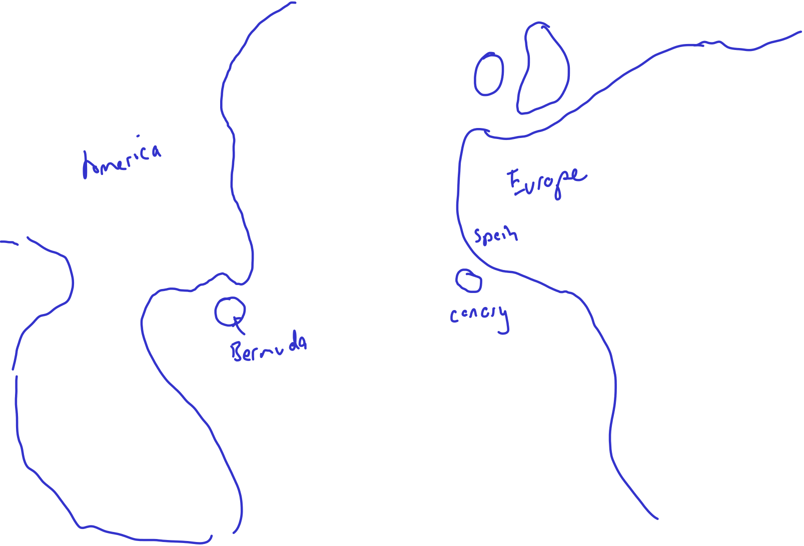
One set of models specifies dynamics for the short rate under the risk-neutral measure. This then determines prices of zero coupon bonds, and hence, the entire term structure:

$$\mathbb{E}_t^*\left[e^{-\int_t^T r_u\,du}\right] = D(t,T) = e^{-R(t,T)(T-t)}$$



$$\mathbb{Q}^* : \quad \frac{D(t,T)}{B_t} = \mathbb{E}^*\left[\frac{\cancel{D(T,T)}}{B_T}\right]$$

$$D(t,T) = \mathbb{E}^*\left[\frac{B_t}{B_t\, e^{\int_t^T r_u\,du}}\right] = \mathbb{E}^*\left[e^{-\int_t^T r_u\,du}\right]$$


America
Bermuda
Europe
Spain

Term Structure Models

A typical **short rate model** will take the following form:

$$dr_t = \mu_t \, dt + \sigma_t \, dW_t^*$$

We begin by considering how two different features of the short rate model affect the spot curve that you obtain from the model:

1. the drift in the short rate (under $\mathbb{Q}^*$)
2. the volatility of the short rate (under $\mathbb{Q}^*$)

1. Drift: Suppose a (simplistic) short rate model specifies

$$dr_t = \mu \, dt,$$

where μ is a constant. The short rate grows linearly over time, and is deterministic. We could also write this as

$$\mu = \frac{dr_t}{dt}.$$

Drift in Short Rate Models

Example We consider a discrete approximation of positive μ. Suppose the initial $3m$ rate (with continuous compounding) is 5%. The next $3m$ rates will be 5.1%, 5.2%, 5.3%, $\cdots$ and so on.

$$D(0,3m) = e^{-0.05 \cdot 0.25} \quad \Rightarrow \quad R(0,3m) = 0.05$$

$$D(0,6m) = e^{-(0.05+0.051)\cdot 0.25} \quad \Rightarrow \quad R(0,6m) = 0.0505$$

$$D(0,9m) = e^{-(0.05+0.051+0.052)\cdot 0.25} \quad \Rightarrow \quad R(0,9m) = 0.051$$

$$D(0,12m) = e^{-(0.05+0.051+0.052+0.053)\cdot 0.25} \quad \Rightarrow \quad R(0,12m) = 0.0515.$$

Based on the calculation, we conclude that the term structure is upward sloping.

If μ is negative, then the term structure will be downward sloping.

Drift in Short Rate Models

Mathematically, we proceed as follows:

- First, we integrate the short rate SDE from 0 to t to obtain an expression for the short rate process:

$$r_t = r_0 + \mu t.$$

- Next, we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T-t) + \frac{1}{2}\mu(T^2 - t^2) = r_t(T-t) + \frac{1}{2}\mu(T-t)^2$$

- We can now reconstruct the discount factor as

$$e^{-R(t,T)(T-t)} = D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u \, du}\right] = e^{-r_t(T-t) - \frac{1}{2}\mu(T-t)^2}.$$

- Therefore, the spot curve in this stylized (simplified) model is given by

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = \frac{1}{2}\mu(T-t) + r_t.$$

Clearly, if $\mu > 0$, the spot curve is upward sloping, and if $\mu < 0$, the spot curve is downward sloping.

Model : $dr_t = \mu\, dt$

① $\int_0^t dr_u = \int_0^t \mu\, du$

$r_t - r_0 = \mu t$

$\therefore r_t = r_0 + \mu t$

② $\int_t^T r_u\, du = \int_t^T r_0\, du + \int_t^T \mu u\, du$

$= r_0 \cdot (T - t) + \mu \cdot \left[\frac{u^2}{2} \right]_t^T$

$= r_0 \cdot (T - t) + \mu \cdot \frac{T^2 - t^2}{2}$

$$= r_0 \cdot (T-t) + \mu \cdot \frac{T^2 - t^2 - 2Tt + 2Tt + t^2 - t^2}{2}$$

$$= r_0 \cdot (T-t) + \mu \cdot \frac{\overbrace{2Tt - 2t^2}}{2} + \mu \cdot \frac{\overbrace{(T-t)^2}}{2}$$

$$= r_0 \cdot (T-t) + \mu t (T-t) + \mu \cdot \frac{(T-t)^2}{2}$$

$$= (r_0 + \mu t)(T-t) + \mu \cdot \frac{(T-t)^2}{2}$$

$$= r_t (T-t) + \mu \cdot \frac{(T-t)^2}{2}$$

Volatility in Short Rate Models

2. Volatility: Suppose a (simplistic) short rate model specifies

$$dr_t = \sigma dW_t^* \qquad \Rightarrow \quad \Delta r_t = \sigma \Delta W_t^*$$

where σ is a constant, and W_t^* is a Brownian motion under $\mathbb{Q}^*$.

The short rate follows a random walk without drift under $\mathbb{Q}$, where σ affects the variance of the "error term".

In discrete term, we have

$$r_{t+\Delta t} \approx r_t + \Delta r_t = r_t + \sigma \Delta W_t^*$$

Volatility in Short Rate Models

Example We consider a discrete approximation of this short rate model with small σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are describe by a tree, where each period the short rate can move up or down by 0.5%. The risk-neutral probability of an up/down move is always 0.5.

A 2-period tree looks as follows:

$$
\begin{array}{ccc}
 & & \begin{array}{c} r{=}5.5\% \\ D(1y{,}2y){=}0.94649 \end{array} = e^{-0.055\cdot 1} \\
 & \nearrow & \\
\begin{array}{c} r{=}5\% \\ D(0{,}2y){=}0.90485 \end{array} & & \\
 & \searrow & \\
 & & \begin{array}{c} r{=}4.5\% \\ D(1y{,}2y){=}0.956 \end{array} = e^{-0.045\cdot 1}
\end{array}
$$

$$D(0,2y) = \mathbb{E}^*\Big[D(0,1y)\, D(1y,2y) \Big]$$

$$= D(0,1y)\ \mathbb{E}^*\big[D(1y,2y) \big]$$

$$= e^{-0.05\cdot 1} \times \left(\frac{1}{2} \times 0.94649 + \frac{1}{2} \times 0.956 \right)$$

$$= 0.90485$$

$D(1,3) = D(1,2)\, \mathbb{E}^*\left[D(2,3)\right]$

$= e^{-0.055} \cdot \left(\frac{1}{2} \cdot 0.94176 + \frac{1}{2} \times 0.95123\right)$

Volatility in Short Rate Models

A 3-period tree looks as follows:

$r{=}6\%$

$D(2y{,}3y){=}0.94176$

$r{=}5.5\%$

$D(1y{,}3y){=}0.89585$

$r{=}5\%$

$D(0{,}3y){=}0.86076$

$r{=}5\%$

$D(2y{,}3y){=}0.95123$

$r{=}4.5\%$

$D(1y{,}3y){=}0.91394$

$r{=}4\%$

$D(2y{,}3y){=}0.96079$

From the zero coupon bond prices, we work out the spot rates:

$$R(0,1y) = 5\%,\ D(0,2y) = 0.90485 \quad \Rightarrow \quad R(0,2y) = 4.9994\%$$

$$D(0,3y) = 0.86076 \quad \Rightarrow \quad R(0,3y) = 4.9979\%$$

Volatility in Short Rate Models

Example We now consider a discrete approximation of the short rate model with large σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are described by a tree, where each period the short rate can move up or down by 2%, and the risk-neutral probability of an up/down move is always $\frac{1}{2}$.

A 2-period tree looks as follows:

$$
\begin{array}{ccc}
 & & \begin{array}{c} r{=}7\% \\ D(1y{,}2y){=}0.93239 \end{array} \\
 & \nearrow & \\
\begin{array}{c} r{=}5\% \\ D(0{,}2y){=}0.90502 \end{array} & & \\
 & \searrow & \\
 & & \begin{array}{c} r{=}3\% \\ D(1y{,}2y){=}0.97045 \end{array}
\end{array}
$$

Volatility in Short Rate Models

A 3-period tree looks as follows:

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c} r{=}9\% \\ D(2y{,}3y){=}0.91393 \end{array} \\
 & \begin{array}{c} r{=}7\% \\ D(1y{,}3y){=}0.86953 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} r{=}5\% \\ D(0{,}3y){=}0.86157 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} r{=}5\% \\ D(2y{,}3y){=}0.95123 \end{array} \\
 & \begin{array}{c} r{=}3\% \\ D(1y{,}3y){=}0.94195 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} r{=}1\% \\ D(2y{,}3y){=}0.99005 \end{array}
\end{tikzcd}
$$

From the zero coupon bond prices, we work out the spot rates:

$$
\begin{aligned}
R(0,1y) = 5\%,\ D(0,2y) &= 0.90502 \quad \Rightarrow \quad R(0,2y) = 4.99\% \\
D(0,3y) &= 0.86157 \quad \Rightarrow \quad R(0,3y) = 4.9667\%
\end{aligned}
$$

Volatility in Short Rate Models

$\mathbb{E}^*\left[e^{-r_t \Delta t}\right] > e^{-\mathbb{E}^*[r_t]\Delta t}$

$\parallel$

Main Conclusions

$e^{-R(t,t+\Delta t)\cdot\Delta t} \downarrow \quad = \quad D(t,t+\Delta t) \uparrow$

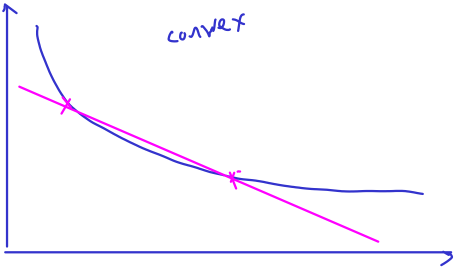
1. Volatility of the short rate by itself produces a slightly downward sloping spot curve.

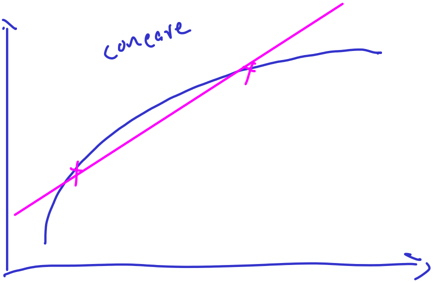
2. The higher the volatility, the more negative the slope of the spot curve.

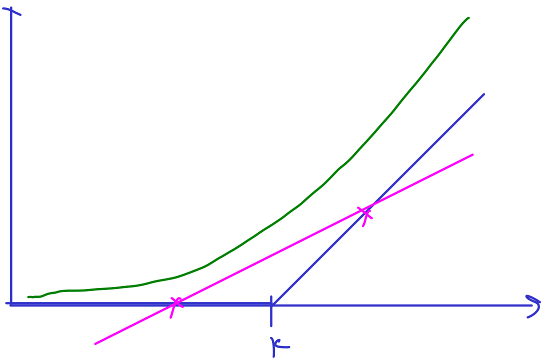
3. This is a consequence of Jensen's inequality and the fact that $f(x) = e^{-x}$ and $f(x) = \frac{1}{1+x}$ are convex in x.

Jensen's inequality states that $\downarrow$

$$\Rightarrow \quad \mathbb{E}[f(X)] \geq f\left(\mathbb{E}[X]\right) \text{ if } f \text{ is convex}$$
$$\mathbb{E}[f(X)] \leq f\left(\mathbb{E}[X]\right) \text{ if } f \text{ is concave}$$


convex


concave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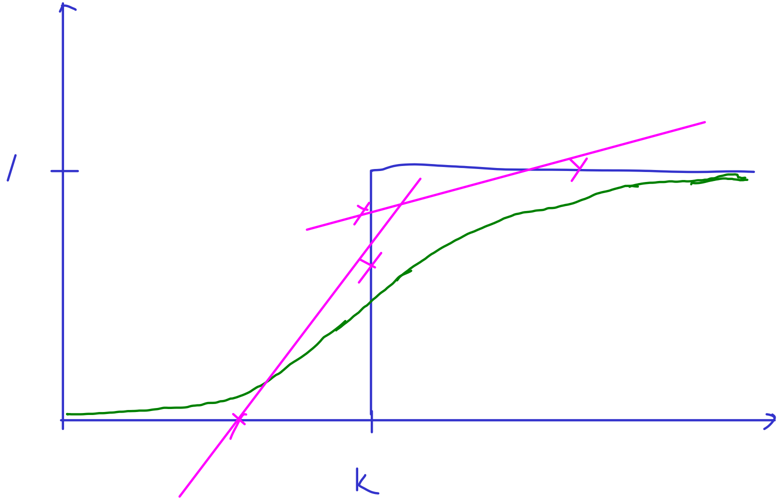

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