Volatility in Short Rate Models

Mathematically, we proceed as follows:

- First, integrate the SDE from 0 to t to obtain the short rate process:

$$r_t = r_0 + \sigma W_t^*, \qquad \text{where } r_t \sim N(r_0, \sigma^2 t)$$

- Next we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T-t) + \sigma \int_t^T W_u^* \, du = r_t(T-t) + \sigma \int_t^T (W_u^* - W_t^*) \, du.$$

Recall that in the previous term, we have demonstrated that by applying Itô's formula to the function $X_t = f(t, W_t) = tW_t$, we can write

$\int dX_t = \int W_t \, dt + \int t \, dW_t$

$$\int_0^T W_u \, du = \int_0^T (T-u) \, dW_u,$$

so that this integral is normally distributed, with mean and variance:

$$\mathbb{E}\left[\int_0^T W_u \, du\right] = 0, \qquad V\left[\int_0^T W_u \, du\right] = \frac{T^3}{3}.$$

Model : $dr_t = \sigma \, dW_t^*$

① $\int_0^t dr_u = \int_0^t \sigma \, dW_u^*$

$$r_t = r_0 + \sigma W_t^*$$

② $\int_t^T r_u \, du = r_0 \cdot (T-t) + \int_t^T \sigma W_u^* \, du$

$$= r_0 \cdot (T-t) + \sigma W_t^* \cdot (T-t)$$

$$- \sigma W_t^* (T-t) + \int_t^T \sigma W_u^* \, du$$

$$= \left(r_0 + \sigma W_t^*\right)(T - t)$$

$$- \sigma W_t^* \int_t^T 1 \, du \qquad + \int_t^T \sigma W_u^* \, du$$

$$= r_t \cdot (T - t) + \sigma \int_t^T \left(W_u^* - W_t^*\right) du$$

$$V\left[\sigma\int_0^T (W_u^* - W_0^*)\,du\right] = \frac{\sigma^2 T^3}{3}$$

Volatility in Short Rate Models

- Applying this results to our integrated short rate process, we note that

$$\mathbb{E}\left[\int_t^T r_u\, du\right] = r_t(T-t)$$

$$V\left[\int_t^T r_u\, du\right] = V\left[\sigma\int_t^T (W_u^* - W_t^*)\, du\right] = \frac{\sigma^2(T-t)^3}{3},$$

and hence

$$\int_t^T r_u\, du \sim N\left(r_t(T-t), \frac{\sigma^2}{3}(T-t)^3\right).$$

- We can now reconstruct the discount factor as

$$D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right].$$

Volatility in Short Rate Models

- We know how to evaluate the expectation of a lognormal random variable. If $X \sim N(\mu, \sigma^2)$, then $\theta = -1$

$r_t(T-t)$

$\frac{\sigma^2(T-t)^3}{3}$

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

- Using this, we have

$$e^{-R(t,T)(T-t)} \equiv \quad D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}.$$

- Finally, we can express the zero rate $R(t,T)$ as follows:

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = r_t - \frac{\sigma^2}{6}(T-t)^2.$$

- The further we look ahead (larger $T-t$), the larger the accumulated uncertainty, and hence the lower the corresponding spot rate. Also, the higher σ, the lower all spot rates.

$dr_t = \mu_t dt + \sigma_t dW_t^*$

Vasicek Model

The Vasicek model for interest rate is a classic short rate model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*$$

Here, κ is the mean reversion coefficient, θ is the long run mean of the short rate, and σ is the volatility of the short rate. Vasicek model is mean reverting.

Applying Itô's formula to $f(r_t, t) = r_t e^{\kappa t}$, we can show that

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-T)} dW_u^*$$

We conclude that r_t is normally distributed, with a mean of

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)$$

and a variance of

$$V[r_t] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

Vasicek Model

Once again, we can now write the integrated short rate process under Vasicek model as

$$\int_t^T r_u\,du \sim N\left(\mathbb{E}\left[\int_t^T r_u\,du\right], V\left[\int_t^T r_u\,du\right]\right).$$

This in turn allows us to reconstruct the discount factor as follows:

$$D(t,T) = \mathbb{E}\left[e^{-\int_t^T r_u\,du}\right].$$

Vasicek Model

Let $R(t,T)$ denote the zero rate covering the period $[t,T]$, so that

$$D(t,T) = e^{-R(t,T)(T-t)}.$$

After some algebra (see Session 7 Additional Examples Q2), we find that we can write

$$D(t,T) = e^{A(t,T)-B(t,T)r_t},$$

or (equivalently)

$$R(t,T) = \frac{1}{T-t}\Big[-A(t,T) + B(t,T)r_t\Big]$$

where

$$B(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

$$A(t,T) = \frac{[B(t,T)-(T-t)](\kappa^2\theta - \frac{1}{2}\sigma^2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}$$

Cox-Ingersoll-Ross Model

FYI (no exam)

binomial tree CRR Cox Ross Rubinstein

In any model in which the short rate is normally distributed (including the Vasicek model), there is always a non-zero probability that the short rate is negative.

An alternative model will be the Cox-Ingersoll-Ross (CIR) model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^*$$

However, r_t is non-centrally χ^2-distributed in the CIR model.

SINGAPORE MANAGEMENT UNIVERSITY

Session 8
Ho-Lee & Hull-White Models

Tee Chyng Wen

QF605 Fixed Income Securities

Note: Integrating W_t wrt t

Consider the following integral:

$$
\begin{aligned}
\int_0^T W_t \, dt &= \int_0^T \int_0^t dW_u \, dt \\
&= \int_0^T \int_u^T dt \, dW_u \\
&= \int_0^T (T-u) \, dW_u \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} (T-t_i)(W_{t_{i+1}} - W_{t_i})
\end{aligned}
$$

As a deterministic function $(T-t_i)$ weighted sum of independent Brownian increment, this integral must be normally distributed.

$\Rightarrow$ It remains to determine the mean and variance of the integral.

Note: Integrating W_t wrt t

The mean is given by

$$\mathbb{E}\left[\int_0^T (T-u)\, dW_u\right] = 0$$

since all stochastic integral has zero-mean, and

$$\begin{aligned} V\left[\int_0^T (T-u)\, dW_u\right] &= \mathbb{E}\left[\int_0^T (T-u)^2\, du\right] \\ &= \int_0^T \left(T^2 - 2uT + u^2\right) du \\ &= \left[T^2u - u^2T + \frac{u^3}{3}\right]_0^T = \frac{T^3}{3} \end{aligned}$$

where we have used Itô's Isometry.

Equilibrium Affine Models

Definition It can be shown that in any **equilibrium short rate model** (e.g. Vasicek, CIR), the zero coupon bond prices can be reconstructed as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)}$$

for some deterministic functions $A(t,T)$ and $B(t,T)$ of t and T only.

This implies that the **spot curve or zero rate curve** can be written as

$$R(t,T) = \frac{1}{T-t}\Big(- A(t,T) + r_t B(t,T) \Big)$$

for this class of model.

In this class of model, spot rates are affine functions of the short rate, and so this class is referred to as the class of **affine term structure models**.

Affine function is composed of a linear function plus a constant (translation).

Affine : $f(x) = a + bx$

Linear : $f(x) = bx$ $\quad (f(0) = 0)$

$$R = a + b \cdot r_t$$

Equilibrium

$$dr_t = \mu\, dt + \sigma\, dW_t^* \quad ; \quad \mu, \sigma$$

Vasicek : $$dr_t = k(\theta - r_t)\, dt + \sigma\, dW_t^* \quad ; \quad k, \theta, \sigma$$

CIR : $$dr_t = k(\theta - r_t)\, dt + \sigma \sqrt{r_t}\, dW_t^* \quad ; \quad k, \theta, \sigma$$

" parsimonious "

No-Arbitrage Affine Models

However, equilibrium models only have a few model parameters—there is no guarantee that we will be able to fit to the observed term structure.

Although it is possible to perform a least square optimization to match the observed discount factors as closely as possible, to prevent arbitrage, we must be able to fit exactly to liquid discount instruments.

Ho-Lee, and subsequently **Hull-White**, proposed to address this problem by letting the model parameters be deterministic function of time – this way, we can match any observed spot curve $R(t,T)$.

Standard terminology for these models is that these are **no-arbitrage short rate models**:

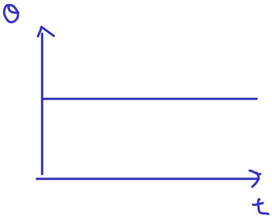
(Extended Vasicek)

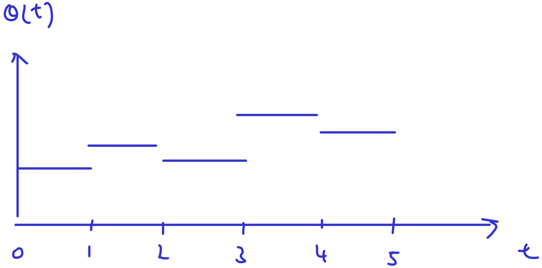
$$\begin{aligned} \text{Ho-Lee:} \quad & dr_t = \theta(t)dt + \sigma dW_t^*. \\ \text{Hull-White:} \quad & dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t^*. \end{aligned}$$

The simplest no-arbitrage model is the Ho-Lee model: where we choose the deterministic function $\theta(t)$ to match the observed spot curve.

$\theta \quad : \quad \longrightarrow \quad \theta(t)$

constant piecewise constant





Ho-Lee Model

In the Ho-Lee interest rate model, the short rate follows:

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a Brownian motion under the measure $\mathbb{Q}^*$. To **fit the initial term structure**, we require that

$$\theta(T) = -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.$$

To show this, first write out the interest rate process by integrating both sides:

$$r_t = r_0 + \int_0^t \theta(s)\ ds + \int_0^t \sigma\ dW_s^*.$$

Next, integrate again to obtain an expression for the integrated rate:

$$\begin{aligned}\int_0^T r_u\ du &= \int_0^T r_0\ du + \int_0^T \int_0^u \theta(s)\ ds\ du + \int_0^T \int_0^u \sigma\ dW_s^*\ du \\ &= r_0 T + \int_0^T \theta(s)(T-s)\ ds + \int_0^T \sigma(T-s)\ dW_s^*.\end{aligned}$$

Model : $dr_t = \theta(t)\, dt + \sigma\, dW_t^*$

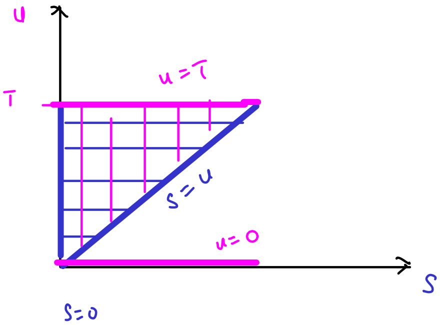
① $$\int_0^t dr_s = \int_0^t \theta(s)\, ds + \int_0^t \sigma\, dW_s^*$$

$$r_t = r_0 + \text{''} + \text{''}$$

② $$\int_0^T r_u\, du = r_0 \cdot T + \int_0^T \int_0^u \theta(s)\, ds\, du + \int_0^T \int_0^u \sigma\, dW_s^*\, du$$

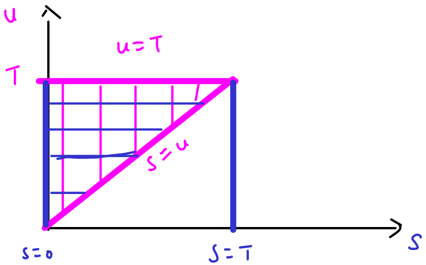
$$= r_0 T + \int_0^T \int_s^T \theta(s)\, du\, ds + \int_0^T \int_s^T \sigma\, du\, dW_s^*$$

$$= r_0 T + \int_0^T \theta(s) \cdot (T-s)\, ds + \int_0^T \sigma \cdot (T-s)\, dW_s^*$$



$$\int_{u=0}^{u=T} \int_{s=0}^{s=u} ds \quad du$$

Fubini's Theorem



$$\int_{s=0}^{s=T} \int_{u=s}^{u=T} du \quad ds$$

Ho-Lee Model

The mean of this stochastic integral is given by

$$\mathbb{E}\left[\int_0^T r_u \, du\right] = r_0 T + \int_0^T \theta(s)(T-s)\, ds,$$

and the variance is given by

$$V\left[\int_0^T r_u \, du\right] = \int_0^T \sigma^2 (T-s)^2 \, ds = \frac{1}{3}\sigma^2 T^3,$$

where we have used **Itô Isometry**.

Therefore, the **zero-coupon discount bond can be reconstructed** as

$$e^{-R(0,T)\cdot(T-0)} = D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u \, du}\right] = \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right].$$

Since we can express $D(0,T)$ in the form of $e^{A(0,T) - r_0 B(0,T)}$, we see that <u>Ho-Lee is an affine model</u>.

Ho-Lee Model

Fitting the initial term structure

From here we can work out that

$$\begin{aligned}
\log D(0,T) &= -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3 \\
\frac{\partial}{\partial T}\log D(0,T) &= -r_0 - \int_0^T \theta(s)\,ds + \frac{1}{2}\sigma^2 T^2 \\
\frac{\partial^2}{\partial T^2}\log D(0,T) &= -\theta(T) + \sigma^2 T \\
\Rightarrow \quad \theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.
\end{aligned}$$

This allows Ho-Lee model to fit the initial term structure $D(0,T)$ observed in the market.

Ho-Lee Model

We have shown that Ho-Lee model allows us to reconstruct the discount factor

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where

$$\begin{aligned} A(t,T) &= -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6}, \\ B(t,T) &= T - t. \end{aligned}$$

What does Ho-Lee model tell us about the evolution of discount factors over time?

⇒ Note that the reconstructed discount factor is given as a function of time and short rate, i.e. $D(t,T) = f(t, r_t)$.

This means that we can use **Itô's formula** to derive the stochastic differential equation describing the evolution of the discount factors over time.

Ho-Lee Model

First, we work out the partial derivatives

$$
\begin{aligned}
f(t,x) &= e^{A(t,T)-xB(t,T)} \\
f_t(t,x) &= e^{A(t,T)-xB(t,T)}\left[\frac{\partial A(t,T)}{\partial t} - x\cdot\frac{\partial B(t,T)}{\partial t}\right] \\
f_x(t,x) &= e^{A(t,T)-xB(t,T)}\Big[-B(t,T)\Big] \\
f_{xx}(t,x) &= e^{A(t,T)-xB(t,T)}\Big[B(t,T)^2\Big],
\end{aligned}
$$

where an application of **Leibniz's rule** yields

$$
\begin{aligned}
A(t,T) &= -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6} \\
\frac{\partial A(t,T)}{\partial t} &= \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}.
\end{aligned}
$$

On the other hand, the time derivative for $B(t,T)$ is simply

$$
\frac{\partial B(t,T)}{\partial t} = -1.
$$

Ho-Lee Model

Applying **Itô's formula**, we obtain the following stochastic differential equation:

$$
\begin{aligned}
dD(t,T) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2}f_{xx}(t,r_t)(dr_t)^2 \\
&= D(t,T)\left[\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t}\right]dt \\
&\qquad - D(t,T)(T-t)\Big(\theta(t)dt + \sigma dW_t^*\Big) \\
&\qquad + \frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt \\
&= r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*.
\end{aligned}
$$

$dr_t = \theta(t)\,dt + \sigma\, dW_t^*$, $\int_0^t dr_u = \int_0^t \theta(u)\,du + \int_0^t \sigma\, dW_u^*$

Ho-Lee Binomial Tree

$r_t = r_0 +$ " $+$ "

Integrating the Ho-Lee model from 0 to t, we obtain:

$$r_t = r_0 + \int_0^t \theta(s)\, ds + \sigma W_t^*$$

$X_i = \begin{cases} +1, & p^* \\ -1, & 1-p^* \end{cases}$

$r_t = r_0 + \sum_{i=0}^{n-1} \theta_i \cdot \Delta t + \sum_{i=0}^{n-1} \sigma \sqrt{\Delta t} \cdot X_i$

Example Ho-Lee binomial tree

- Suppose we have an initial 1-year rate of $R = 5\%$ (with continuous compounding).

  $\Delta t = 1$, $r_0 = 0.05$, $D(0,1) = e^{-0.05 \cdot 1}$

- We assume that the probability of a rate increase/decrease is $\frac{1}{2}$. $= p^*$
- At every node, we assume that the rate randomly increases by 1% or decreases by -1% (this is determined by the volatility of the short rate).

  $\sigma \sqrt{\Delta t} = 0.01$

- Start at time $t = 0$. At time $s + 1$, we also add a deterministic amount $\sum_{u=0}^{u=s} \theta_u$ to the rate, in all nodes.
- We then choose the θ_u to ensure that we can match the observed spot rates $R(0, 2)$, $R(0, 3)$, $\cdots$.

Ho-Lee Binomial Tree

The 2-period binomial tree model looks as follows:

$$
\begin{array}{ccc}
 & & r=6\%+\theta_0 \\
 & & D(1,2)=e^{-(6\%+\theta_0)\cdot 1} \\
 & \nearrow & \\
r=5\% & & \\
D(0,2)=\ldots & & \\
 & \searrow & \\
 & & r=4\%+\theta_0 \\
 & & D(1,2)=e^{-(4\%+\theta_0)\cdot 1}
\end{array}
$$

We can choose θ_0 to match the observed spot rate $R(0,2)$.

$r_0 \rightarrow$ match $D(0,1)$

$$D(0,2) = \overline{\underline{\mathbb{E}}}^*\left[D(0,1) \cdot D(1,2) \right]$$

$$= D(0,1)\ \overline{\underline{\mathbb{E}}}^*\left[D(1,2) \right]$$

$$= e^{-0.05} \times \left[\frac{1}{2} \times e^{-0.06-\theta_0} + \frac{1}{2} \times e^{-0.04-\theta_0} \right]$$

Solve for θ_0.

Ho-Lee Binomial Tree

The 3-period binomial tree model looks as follows:

$$
\begin{array}{ccccc}
 & & & & r=7\%+\theta_0+\theta_1 \\
 & & & & D(2,3)=e^{-(7\%+\theta_0+\theta_1)\cdot 1} \\
 & & & \nearrow & \\
 & & r=6\%+\theta_0 & & \\
 & & D(1,3)=\ldots & & \\
 & \nearrow & & \searrow & \\
r=5\% & & & & r=5\%+\theta_0+\theta_1 \\
D(0,3)=\ldots & & & & D(2,3)=e^{-(5\%+\theta_0+\theta_1)\cdot 1} \\
 & \searrow & & \nearrow & \\
 & & r=4\%+\theta_0 & & \\
 & & D(1,3)=\ldots & & \\
 & & & \searrow & \\
 & & & & r=3\%+\theta_0+\theta_1 \\
 & & & & D(2,3)=e^{-(3\%+\theta_0+\theta_1)\cdot 1}
\end{array}
$$

We can then choose θ_1 to match the observed spot rate $R(0,3)$.

$r_0 \to$ match $D(0,1)$, $\theta_0 \to$ match $D(0,2)$

$$D(0,3) = \mathbb{E}^*\left[D(0,1) \cdot D(1,2)\, D(2,3) \right]$$

$$= D(0,1)\, \mathbb{E}^*\left[D(1,2)\, \mathbb{E}_1^*\left[D(2,3) \right] \right]$$

$$= e^{-0.05} \times \left\{ \frac{1}{2} \cdot e^{-0.06-\theta_0} \cdot \left[\frac{1}{2} \cdot e^{-0.07-\theta_0-\theta_1} + \frac{1}{2} \cdot e^{-0.05-\theta_0-\theta_1} \right] \right.$$

$$\left. + \frac{1}{2} e^{-0.04-\theta_0} \cdot \left[\frac{1}{2} e^{-0.05-\theta_0-\theta_1} + \frac{1}{2} e^{-0.03-\theta_0-\theta_1} \right] \right\}$$

Solve for θ_1

Ho-Lee Binomial Tree

Example Consider the same Ho-Lee binomial tree given in the previous example. Suppose we observe the following in the interest rate market:

Instrument	Value
$D(0, 1y)$	0.95123
$D(0, 2y)$	0.90
$D(0, 3y)$	0.86

Determine the no-arbitrage value of θ_0 and θ_1.

ans.: $\theta_0 = 0.00556$, $\theta_1 = -0.01$.