Supplementary Note (No Exam)
Advanced Short Rate Models

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QF605 Fixed Income Securities

Instantaneous Forward Rate and Discount Factors

Let $F(t;T,T+\delta)$ denote the forward interest rate observed at time t for the period between T and $T+\delta$. Note that $t<T$ and $\delta>0$. No-arbitrage gives us

$$D(t,T) = D(t,T+\delta)e^{\delta F(t;T,T+\delta)}$$
$$\Rightarrow \quad F(t;T,T+\delta) = -\frac{\log D(t,T+\delta) - \log D(t,T)}{\delta}.$$

The instantaneous forward rate $f(t,T)$ is defined as the limit of $\delta \to 0$, i.e.

$$f(t,T) = \lim_{\delta\to 0} F(t;T,T+\delta) = -\frac{\partial \log D(t,T)}{\partial T},$$

where we have made use of the first principles definition of differentiation.

Integrating this instantaneous forward rate from t to T yields

$$\int_t^T f(t,u)\,du = -\log D(t,T) \quad \Rightarrow \quad D(t,T) = e^{-\int_t^T f(t,u)du}.$$

Ho-Lee Model: A Recap

Ho-Lee interest rate model is given by

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a Brownian motion under the measure $\mathbb{Q}^*$. To fit the initial term structure, we require that

$$\theta(T) = -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.$$

From earlier discussions, we have also established that the integrated rate can be written as:

$$\begin{aligned}\int_0^T r_u\,du &= \int_0^T r_0\,du + \int_0^T\int_0^u \theta(s)\,ds\,du + \int_0^T\int_0^u \sigma\,dW_s^*\,du \\ &= r_0T + \int_0^T \theta(s)(T-s)ds + \int_0^T \sigma(T-s)dW_s^*.\end{aligned}$$

With mean and variance:

$$\mathbb{E}\left[\int_0^T r_u\,du\right] = r_0T + \int_0^T \theta(s)(T-s)\,ds, \qquad V\left[\int_0^T r_u\,du\right] = \frac{1}{3}\sigma^2T^3.$$

Ho-Lee Model: A Recap

The discount factor can now be reconstructed:

$$D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u\, du}\right] = \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right].$$

This yields an affine short rate model:

$$\begin{aligned} D(0,T) &= e^{A(0,T) - r_0 B(0,T)}, \\ A(0,T) &= -\int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3, \\ B(0,T) &= T, \end{aligned}$$

or more generally

$$\begin{aligned} D(t,T) &= e^{A(t,T) - r_t B(t,T)}, \\ A(t,T) &= -\int_t^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 (T-t)^3, \\ B(t,T) &= T - t. \end{aligned}$$

Ho-Lee Model: Changing to T-forward Measure

Since

$$\int_0^T \theta(s)(T-s)\,ds = -\log D(0,T) - r_0 T + \frac{1}{6}\sigma^2 T^3,$$

we can also express the integrated rate as

$$\int_0^T r_u\,du = -\log D(0,T) + \frac{1}{6}\sigma^2 T^3 + \int_0^T \sigma(T-s)\,dW_s^*.$$

The Radon-Nikodym derivative relating $\mathbb{Q}^*$ and $\mathbb{Q}^T$ is given by

$$\begin{aligned}
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^*} = \frac{D(T,T)/D(0,T)}{B(T)/B(0)} = \frac{e^{-\int_0^T r_u\,du}}{D(0,T)} &= \exp\left[-\log D(0,T) - \int_0^T r_u\,du\right] \\
&= \exp\left[-\frac{1}{6}\sigma^2 T^3 - \int_0^T \sigma(T-s)\,dW_s^*\right] \\
&= \exp\left[-\frac{1}{2}\int_0^T \sigma^2(T-s)^2\,ds - \int_0^T \sigma(T-s)\,dW_s^*\right].
\end{aligned}$$

Ho-Lee Model: Changing to T-forward Measure

By Girsanov theorem, $\mathbb{Q}^T$ and $\mathbb{Q}^*$ are equivalent, and there exist a standard Brownian motion under $\mathbb{Q}^T$ given by:

$$dW_t^T = dW_t^* + \sigma(T-t)dt,$$

where W_t^T is a Brownian motion under the forward measure $\mathbb{Q}^T$. So we can also write the Ho-Lee interest rate process under the forward measure as

$$dr_t = \left[\theta(t) - \sigma^2(T-t)\right] dt + \sigma dW_t^T.$$

Given that

$$\begin{aligned} \frac{\partial}{\partial T} \log D(t,T) &= -r_t - \int_t^T \theta(s)\, ds + \frac{1}{2}\sigma^2(T-t)^2 \\ -f(t,T) &= -r_t - \int_t^T \theta(s)\, ds + \frac{1}{2}\sigma^2(T-t)^2 \\ f(t,T) &= r_t + \int_t^T \theta(s)\, ds - \frac{1}{2}\sigma^2(T-t)^2 \end{aligned}$$

Ho-Lee Model: Instantaneous Forward Process

From this we observe that

$$df(t,T) = \sigma dW_t^T,$$

i.e. the instantaneous forward rate is a martingale under $\mathbb{Q}^T$. So when fitting the initial term structure, the two expressions are the same:

$$\begin{aligned}\theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T \\ &= \frac{\partial f(0,T)}{\partial T} + \sigma^2 T.\end{aligned}$$

where

$$D(0,T) = \exp\left[\int_0^T f(0,u)\,du\right].$$

Finally, we write

$$D(t,T) = \frac{D(0,T)}{D(0,t)}\exp\left[-r_t(T-t) + (T-t)f(0,t) - \frac{\sigma^2 t}{2}(T-t)^2\right].$$

Ho-Lee Model: Zero-Coupon Bond Options

Using Itô's formula, we can show that the zero-coupon bond price process under $\mathbb{Q}^*$ is given by

$$dD(t,T) = r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*.$$

This is a lognormal process, of which the solution is given by

$$D(t,T) = D(0,T)\exp\left[\int_0^t r_u\,du - \frac{\sigma^2}{2}\int_0^t (T-u)^2\,du - \sigma\int_0^t (T-u)\,dW_u^*\right].$$

We have previously established that for Ho-Lee model, we have

$$\begin{aligned}\int_0^t r_u\,du &= -\log D(0,t) + \frac{1}{6}\sigma^2 t^3 + \sigma\int_0^t (t-u)\,dW_u^* \\ &= -\log D(0,t) + \frac{\sigma^2}{2}\int_0^t (t-u)^2\,du + \sigma\int_0^t (t-u)\,dW_u^*.\end{aligned}$$

Ho-Lee Model: Zero-Coupon Bond Options

Substituting this into the solution for $D(t,T)$, we obtain

$$D(t,T) = \frac{D(0,T)}{D(0,t)} \exp\left\{-\frac{\sigma^2}{2}\int_0^t \left[(T-u)^2-(t-u)^2\right] du - \sigma\int_0^t \left[(T-u)-(t-u)\right] dW_u^*\right\}.$$

In the previous discussion, we have also shown that changing measure from $\mathbb{Q}^*$ to $\mathbb{Q}^T$, Girsanov theorem tells us that

$$\begin{aligned} dW_t^T &= dW_t^* + \sigma(T-t)dt \\ dW_u^t &= dW_u^* + \sigma(t-u)du \end{aligned}$$

is a Brownian motion. So we have

$$\begin{aligned} &\frac{D(0,T)}{D(0,t)} \exp\left\{-\frac{\sigma^2}{2}\int_0^t \left[(T-u)^2-(t-u)^2\right] du - \sigma\int_0^t \left[(T-u)-(t-u)\right]\Big(dW_u^t - \sigma(t-u)\,du\Big)\right\} \\ &= \frac{D(0,T)}{D(0,t)} \exp\left\{-\frac{\sigma^2}{2}\int_0^t \left[(T-u)-(t-u)\right]^2 du - \sigma\int_0^t \left[(T-u)-(t-u)\right] dW_u^t\right\} \\ &= \frac{D(0,T)}{D(0,t)} \exp\left\{-\frac{\sigma^2}{2}(T-t)^2 t - \sigma(T-t)W_t^t\right\}. \end{aligned}$$

Ho-Lee Model: Zero-Coupon Bond Options

The payoff of a zero-coupon bond option with option expiry t and bond maturity T is given by $(D(t,T)-K)^+$. To value this option, we write

$$
\begin{aligned}
V_0 &= \mathbb{E}^*\left[\frac{B_0\,(D(t,T)-K)^+}{B_t}\right] \\
&= D(0,t)\mathbb{E}^T\left[(D(t,T)-K)^+\right] \\
&= D(0,T)\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}+\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}\right) \\
&\qquad - D(0,t)K\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}-\frac{1}{2}\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt{t}}\right) \\
&= D(0,T)\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}+\frac{1}{2}\sigma_{HL}^2}{\sigma_{HL}}\right) - D(0,t)K\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}-\frac{1}{2}\sigma_{HL}^2}{\sigma_{HL}}\right),
\end{aligned}
$$

where $\sigma_{HL}=\sigma(T-t)\sqrt{t}$ is the Ho-Lee integrated volatility.

Vasicek Model: Recap

Consider the Vasicek model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

We have

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)}\, dW_s^*,$$

with mean and variance

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$
$$V[r_t] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

we can write the integrated short rate as:

$$\int_0^t r_u\, du = \int_0^t r_0 e^{-\kappa u}\, du + \int_0^t \theta\left(1 - e^{-\kappa u}\right)\, du + \frac{\sigma}{\kappa}\int_0^t \left(1 - e^{\kappa(u-t)}\right)\, dW_u^*$$

Vasicek Model: Integrated Short Rate

Now we proceed to work out the mean and variance of the integrated short rate. Taking expectation on both sides and, we can write:

$$\begin{aligned}\mathbb{E}^*\left[\int_0^t r_u\, du\right] &= \int_0^t r_0 e^{-\kappa u}\, du + \int_0^t \theta\left(1-e^{-\kappa u}\right)\, du \\ &= \frac{r_0}{\kappa}\left(1-e^{-\kappa t}\right) + \theta t - \frac{\theta}{\kappa}\left(1-e^{-\kappa t}\right)\end{aligned}$$

Let

$$\begin{aligned} B(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa}, \\ \therefore \quad \mathbb{E}^*\left[\int_0^t r_u\, du\right] &= r_0 B(0,t) + \theta t - \theta B(0,t).\end{aligned}$$

Next, we evaluate its variance

$$V\left[\int_0^t r_u\, du\right] = V\left[\frac{\sigma}{\kappa}\int_0^t \left(1-e^{\kappa(u-t)}\right)\, dW_u^*\right].$$

Vasicek Model: Integrated Short Rate

Using Itô's Isometry, we obtain

$$
\begin{aligned}
&= \frac{\sigma^2}{\kappa^2}\int_0^t \left(1-e^{\kappa(u-t)}\right)^2\ du \\
&= \frac{\sigma^2}{\kappa^2}\int_0^t \left(1-2e^{\kappa(u-t)}+e^{2\kappa(u-t)}\right)\ du \\
&= \frac{\sigma^2}{\kappa^2}\left[t-\frac{2}{\kappa}\left(1-e^{-\kappa t}\right)+\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)\right] \\
&= \frac{\sigma^2}{2\kappa^3}\left[2\kappa t-4+4e^{-\kappa t}+1-e^{-2\kappa t}\right] \\
&= \frac{\sigma^2}{2\kappa^3}\left[2\kappa t-2+2e^{-\kappa t}-\left(1-2e^{-\kappa t}+e^{-2\kappa t}\right)\right] \\
&= -\frac{\sigma^2}{2\kappa^3}\left[-2\kappa t+2(1-e^{-\kappa t})+(1-e^{-\kappa t})^2\right] \\
&= \frac{\sigma^2}{2\kappa^2}2t-\frac{\sigma^2}{2\kappa^2}2B(0,t)-\frac{\sigma^2}{2\kappa}B(0,t)^2
\end{aligned}
$$

Vasicek Model: Discount Factor Reconstruction

Reconstructing the discount factor, we write

$$D(0,t) = \mathbb{E}^* \left[e^{-\int_0^t r_u \, du} \right] = e^{A(0,t) - r_0 B(0,t)},$$

where

$$A(0,t) = \exp\left\{ \left(\theta - \frac{\sigma^2}{2\kappa^2} \right) \Big[B(0,t) - t \Big] - \frac{\sigma^2}{4\kappa} B(0,t)^2 \right\}$$

$$B(0,t) = \frac{1 - e^{-\kappa t}}{\kappa}.$$

or more generally:

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right] = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = \exp\left\{ \left(\theta - \frac{\sigma^2}{2\kappa^2} \right) \Big[B(t,T) - T + t \Big] - \frac{\sigma^2}{4\kappa} B(t,T)^2 \right\}$$

$$B(t,T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}.$$

Vasicek Model: Changing to T-forward Measure

The Radon-Nikodym derivative of measure $\mathbb{Q}^T$ with respect to measure $\mathbb{Q}^*$ is given by

$$\begin{aligned}
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^*} &= \frac{\exp\left[-\int_0^T r_u\, du\right]}{D(0,T)} \\
&= \exp\left[-\log D(0,T) - \int_0^T r_u\, du\right] \\
&= \exp\left[-\frac{\sigma^2}{2\kappa^2}\int_0^T \left(1-e^{\kappa(u-T)}\right)^2 du - \frac{\sigma}{\kappa}\int_0^T \left(1-e^{\kappa(u-T)}\right) dW_u^*\right].
\end{aligned}$$

Therefore W_t^T is a martingale under $\mathbb{Q}^T$ where

$$\begin{aligned}
dW_t^T &= dW_t^* + \frac{\sigma}{\kappa}(1-e^{\kappa(t-T)})dt \\
&= dW_t^* + \sigma B(t,T)dt.
\end{aligned}$$

Under $\mathbb{Q}^T$, the short rate model becomes

$$dr_t = \left[\kappa\theta - \sigma^2 B(t,T) - \kappa r_t\right]dt + \sigma dW_t^T.$$

Vasicek Model: Instantaneous Forward Process

The instantaneous forward rate is given by

$$f(t,T) = \left[\kappa\theta - \frac{\sigma^2}{2}B(t,T)\right]B(t,T) + r_t e^{-\kappa(T-t)},$$

following the stochastic differential equation

$$df(t,T) = \sigma e^{-\kappa(T-t)}dW_t^T.$$

The discount factor follows the SDE

$$\begin{aligned} dD(t,T) &= r_t D(t,T)dt - \frac{\sigma}{\kappa}\left(1 - e^{\kappa(t-T)}\right)D(t,T)dW_t^* \\ &= r_t D(t,T)dt - \sigma B(t,T)D(t,T)dW_t^*. \end{aligned}$$

The zero-coupon bond follows a lognormal process, and is solved by

$$D(t,T) = D(0,T)\exp\left[\int_0^t r_u du - \frac{\sigma^2}{2\kappa^2}\int_0^t \left(1 - e^{\kappa(u-T)}\right)^2 du - \frac{\sigma}{\kappa}\int_0^t \left(1 - e^{\kappa(u-T)}\right)dW_u^*\right].$$

Vasicek Model: Zero-Coupon Bond Options

We have previously established that

$$\int_0^t r_u du = -\log D(0,t) + \frac{\sigma^2}{2\kappa^2}\int_0^t \left(1-e^{\kappa(u-t)}\right)^2 du + \frac{\sigma}{\kappa}\int_0^t \left(1-e^{\kappa(u-t)}\right) dW_u^*.$$

Substituting into the solution for $D(t,T)$, we obtain

$$\begin{aligned}\frac{D(0,T)}{D(0,t)} \times \exp\Bigg[& -\frac{\sigma^2}{2\kappa^2}\int_0^t \left[\left(1-e^{\kappa(u-T)}\right)^2 - \left(1-e^{\kappa(u-t)}\right)^2\right] du \\ & -\frac{\sigma}{\kappa}\int_0^t \left[\left(1-e^{\kappa(u-T)}\right) - \left(1-e^{\kappa(u-t)}\right)\right] dW_u^* \Bigg].\end{aligned}$$

Using Girsanov Theorem to change the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^T$,

$$dW_t^T = dW_t^* + \sigma\frac{1-e^{\kappa(T-t)}}{\kappa}dt = dW_t^* + \sigma B(t,T)dt$$

is a $\mathbb{Q}^T$-Brownian motion.

Vasicek Model: Zero-Coupon Bond Options

Therefore, we obtain

$$
\begin{aligned}
&\frac{D(0,T)}{D(0,t)}\exp\left[-\frac{\sigma^2}{2\kappa^2}\int_0^t\left[\left(1-e^{\kappa(u-T)}\right)-\left(1-e^{\kappa(u-t)}\right)\right]^2du\right.\\
&\qquad\qquad\left.-\frac{\sigma}{\kappa}\int_0^t\left[\left(1-e^{\kappa(u-T)}\right)-\left(1-e^{\kappa(u-t)}\right)\right]dW_u^t\right]\\
&=\frac{D(0,T)}{D(0,t)}\exp\left[-\frac{\sigma^2}{2\kappa^2}\int_0^t\left(e^{\kappa(u-t)}-e^{\kappa(u-T)}\right)^2du-\frac{\sigma}{\kappa}\int_0^t\left(e^{\kappa(u-t)}-e^{\kappa(u-T)}\right)dW_u^t\right].
\end{aligned}
$$

The price of a call option on a zero-coupon bond with strike K and maturity T can be written as

$$
D(0,T)\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}+\frac{1}{2}\sigma_V^2}{\sigma_V}\right)-D(0,t)K\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}-\frac{1}{2}\sigma_V^2}{\sigma_V}\right),
$$

where the Vasicek integrated volatility is

$$
\sigma_V=\sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}\frac{1-e^{-\kappa(T-t)}}{\kappa}=\sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}B(t,T).
$$

Hull-White Model: Discount Factor Reconstruction

Consider the stochastic differential equation

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t^*,$$

we have

$$r_t = r_0 e^{-\kappa t} + \kappa \int_0^t \theta(s) e^{-\kappa t}\, ds + \sigma \int_0^t e^{-\kappa t}\, dW_s^*,$$

and more generally

$$r_T = r_t e^{-\kappa(T-t)} + \kappa \int_t^T \theta(s) e^{-\kappa(T-s)}\, ds + \sigma \int_t^T e^{-\kappa(T-s)}\, dW_s^*.$$

Note that the short rate is normally distributed. Apart from $\theta(t)$ being a function of time, all other components of the Hull-White model are identical to that of the Vasicek model.

$\Rightarrow$ We can proceed to integrate r_t to obtain the integrated short rate.

Hull-White Model: Instantaneous Forward Rate

The discount factor maturing at time T can be reconstructed as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = \underbrace{\exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\left[B(t,t) - T + t\right] - \frac{\sigma^2}{4\kappa}B(t,T)^2\right\}}_{\text{Vasicek's } A(t,T)} \cdot \exp\left\{-\kappa \int_t^T \theta(s)B(s,T)\, ds\right\}$$

and

$$B(t,T) = \underbrace{\frac{1 - e^{-\kappa(T-t)}}{\kappa}}_{\text{Vasicek's } B(t,T)}.$$

The instantaneous forward rate is given by

$$f(t,T) = \kappa \int_t^T \theta(s) e^{-\kappa(T-s)}\, ds - \frac{\sigma^2}{2} B(t,T)^2 + r_t e^{-\kappa(T-t)}.$$

To fit the Hull-White model to the initial term structure, we do:

$$\theta(t) = f(0,t) + \frac{1}{\kappa}\frac{\partial f(0,t)}{\partial t} + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-2\kappa t}\right).$$

Hull-White Model: Zero-Coupon Bond Options

Discount factor can then be reconstructed as

$$D(t,T) = \frac{D(0,T)}{D(0,t)} \exp\left\{-r_t B(t,T) + B(t,T) f(0,t) - \frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa t}\right) B(t,T)^2\right\}.$$

The zero-coupon bond valuation formula is given by

$$\begin{aligned} V_0 &= \mathbb{E}^*\left[\frac{B_0\left(D(t,T)-K\right)^+}{B_t}\right] \\ &= D(0,t)\mathbb{E}^T\left[(D(t,T)-K)^+\right] \\ &= D(0,T)\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}+\frac{1}{2}\sigma_{IHW}^2}{\sigma_{IHW}}\right) - D(0,t)K\Phi\left(\frac{\log\frac{D(0,T)}{D(0,t)K}-\frac{1}{2}\sigma_{IHW}^2}{\sigma_{IHW}}\right), \end{aligned}$$

where σ_{IHW} is the integrated Hull-White volatility given by

$$\sigma_{HW} = \sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}\,\frac{1-e^{-\kappa(T-t)}}{\kappa}.$$

The Zero-Coupon Bond Dynamics

Let $D(t,T)$ denote the value at time t of a zero-coupon bond paying 1 at time T, and let r_t denote the spot interest rate at time t, then the zero-coupon bond dynamics is given by

$$dD(t,T) = r_t D(t,T)dt + \sigma(t,T)D(t,T)dW_t^*,$$

where W_t is a Brownian motion under the risk-neutral measure $\mathbb{Q}^*$. An application of Itô's lemma allows us to obtain the solution

$$D(t,T) = D(0,T)\exp\left[\int_0^t \left(r_u - \frac{1}{2}\sigma^2(u,T)\right)du + \int_0^t \sigma(u,T)dW_u^*\right].$$

Let $B_t = e^{\int_0^t r_u du}$ denote a money-market account paying interest r_t. We can write down the Radon-Nikodym derivative relating between the forward measure $\mathbb{Q}^T$ and the risk-neutral measure $\mathbb{Q}$ as follow

$$\begin{aligned}\frac{d\mathbb{Q}^T}{d\mathbb{Q}} &= \frac{\frac{D(t,T)}{D(0,T)}}{\frac{B_t}{B_0}} \\ &= \exp\left[-\int_0^t \frac{1}{2}\sigma^2(u,T)du + \int_0^t \sigma(u,T)dW_u^*\right].\end{aligned}$$

The Zero-Coupon Bond Dynamics

We can then use Girsanov's theorem to conclude that to change the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^T$, we use

$$dW_t^T = dW_t^* - \sigma(t,T)dt.$$

Note that choosing

$$\sigma(t,T) = -\sigma(T-t)$$

gives rise to the Ho-Lee model, while choosing

$$\sigma(t,T) = -\sigma\frac{1-e^{-\kappa(T-t)}}{\kappa}$$

yields the Hull-White model.

Two-Factor Models

All of the short rate models we have developed so far have just a single source of randomness—they are all driven by a 1-dimensional Brownian motion.

A consequence having only 1 source of randomness is that the zero rates $R(t,T)$ across all maturities T will be perfectly correlated.

$\Rightarrow$ The term structure will always move up and down in full synchronicity.

$\Rightarrow$ The term structure cannot invert.

$\Rightarrow$ The term structure cannot change its curvature.

In order to account for the possibility for the term structure to change its slope (or invert), we need to have a 2-factor model—using 2 Brownian motions to drive the randomness of our short rate model.

Two-Factor Hull-White Model

The two-factor Hull-White model is given by

$$\begin{cases} dr_t = \kappa(\theta(t) + x_t - r_t)dt + \sigma_r dW_t^* \\ dx_t = -\gamma x_t dt + \sigma_x dZ_t^* \\ dW_t^* dZ_t^* = \rho dt \end{cases}$$

In the long run, the short rate r_t will revert to the mean level of $\theta(t) + x_t$.

- $\Rightarrow$ Note that although $\theta(t) + x_t$ is now stochastic, this process is also mean reverting.
- $\Rightarrow$ $\theta(t)$ serves the same purpose as the one-factor model—it is used to match the initial term structure.
- $\Rightarrow$ x_t introduces a second source of randomness. By adjusting ρ, we can now control the possibility of the term structure to change its slope (or invert).