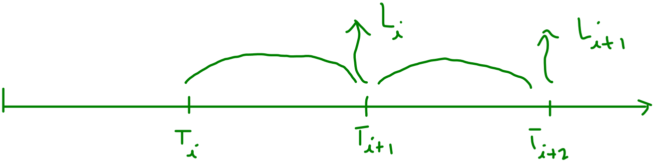
For hedge funds and other institutional clients, they use CMS products to speculate on the movement of the yield curve.

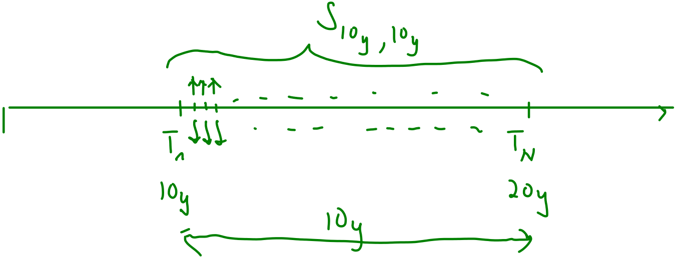
- Receive long-maturity CMS rate if they think yield will steepen
  $\Rightarrow$ Spread trade: Receive 10y pay 2y CMS
- Pay long-maturity CMS rate if they think yield will flatten
  $\Rightarrow$ Spread trade: Pay 10y receive 2y CMS
- CMS spread options

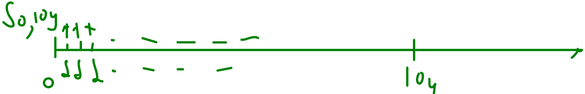
Libor



$$L_i(t) = \frac{1}{\Delta_i} \cdot \frac{D_i(t) - D_{i+1}(t)}{D_i(t)}$$

Swaps





CMS

Risk-Neutral Density of the Forward Swap Rate

The value of a **CMS payoff** is a function of the distribution of the swap rate.

The standard practice in the market is to use the **static-replication** method to obtain a **model-independent convexity correction**.

model-free

Let us begin with an IRR-settled payer swaption:

$$V^{pay}(K) = D(t,T)\int_K^\infty \text{IRR}(s)\cdot(s-K)\;f(s)\;ds$$

Differentiating the formula twice yields

$$\frac{\partial V^{pay}(K)}{\partial K} = -D(t,T)\int_K^\infty \text{IRR}(s)\;f(s)\;ds$$

$$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(t,T)\;\text{IRR}(K)\;f(K)$$

This can be rewritten as

$$f(K) = \frac{\partial^2 V^{pay}(K)}{\partial K^2} \times \frac{1}{D(t,T)\;\text{IRR}(K)}$$

Static Replication Approach

We will also obtain the same result by differentiating the IRR-settled receiver swaption formula twice.

Suppose we want to value a contract paying $g(S_{n,N}(T))$ at time T, we let

$$h(K) = \frac{g(K)}{\text{IRR}(K)},$$

and write (let $F = S_{n,N}(0)$ denote the forward swap rate)

$$\begin{aligned} V_0 &= D(0,T) \int_0^\infty g(K)\ f(K)\ dK \\ &= \int_0^F h(K)\ \frac{\partial^2 V^{rec}(K)}{\partial K^2}\ dK + \int_F^\infty h(K)\ \frac{\partial^2 V^{pay}(K)}{\partial K^2}\ dK \end{aligned}$$

Integration-by-parts twice, we will get

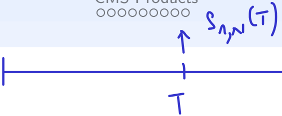
$$\begin{aligned} V_0 &= D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] \\ &\quad + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

Static Replication of CMS Payoffs

Using **quotient rule**, the first and second order derivatives of $h(K)$ are given by:

$$\begin{aligned} h(K) &= \frac{g(K)}{\mathrm{IRR}(K)} \\ h'(K) &= \frac{\mathrm{IRR}(K)g'(K) - g(K)\mathrm{IRR}'(K)}{\mathrm{IRR}(K)^2} \\ h''(K) &= \frac{\mathrm{IRR}(K)g''(K) - \mathrm{IRR}''(K)g(K) - 2\cdot\mathrm{IRR}'(K)g'(K)}{\mathrm{IRR}(K)^2} \\ &\quad + \frac{2\cdot\mathrm{IRR}'(K)^2 g(K)}{\mathrm{IRR}(K)^3}. \end{aligned}$$

CMS Rate



Example Show that a CMS rate payment for the swap rate $S_{n,N}(T)$ at time T can be valued as (where $F = S_{n,N}(0)$)

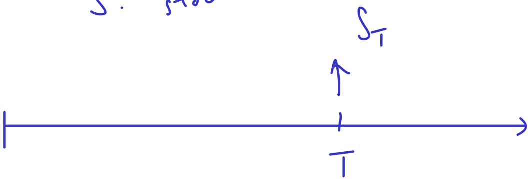
$$\boxed{D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK}$$

using static replication, where

$$h''(K) = \frac{-\text{IRR}''(K) \cdot K - 2 \cdot \text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 \cdot K}{\text{IRR}(K)^3}.$$

$g(S) = S \quad , \quad g'(S) = 1 \quad , \quad g''(S) = 0$

S: stock



$h(S) = S$, $h'(S) = 1$, $h''(S) = 0$

$V_0 = e^{-rT} \cdot F + 0 + 0$

$$-\frac{\partial c(k)}{\partial k} = \frac{c(k) - c(k+\Delta k)}{\Delta k}$$

CMS Caplet

ATM

Example Show that an at-the-money (ATM) CMS caplet struck at the forward swap rate $L = S_{n,N}(0) = F$ maturing at T can be valued as

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^{\infty} h''(K)V^{pay}(K)dK$$

using static replication, where

$$h'(K) = \frac{\text{IRR}(K) - \text{IRR}'(K) \cdot (K - L)}{\text{IRR}(K)^2}$$

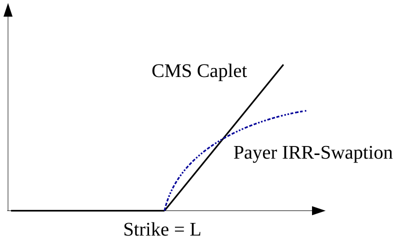
$$h''(K) = \frac{-\text{IRR}''(K)(K-L) - 2 \cdot \text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 \cdot (K-L)}{\text{IRR}(K)^3}$$

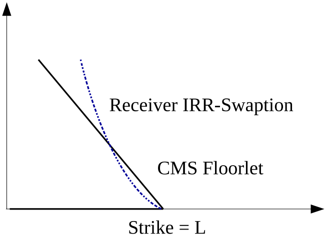
$$g(S) = S - L, \quad g'(S) = 1, \quad g''(S) = 0$$

CMS Replication – Intuition

Note that the CMS caplet payoff and the IRR-settled payer swaption payoff are both functions of the same swap rate $S_{n,N}(T)$'s distribution.

Beyond the strike rate of L, CMS caplet payoff is **linear** and payer swaption payoff is **concave** of the swap rate $S_{n,N}(T)$:





Since swaptions are vanilla derivatives and more liquid, we can **replicate the CMS caplet** payoff using a basket of IRR-settled payer swaptions with increasing strikes starting with the CMS caplet strike L.

CMS caplet $= (S - L)^+$

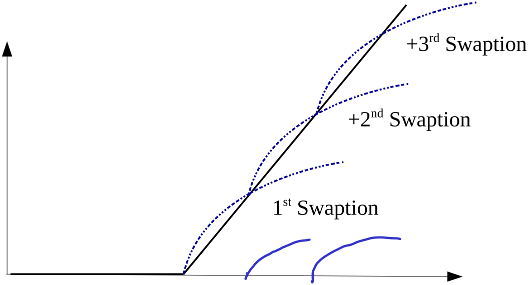
IRR payer swaption $= \sum_i \frac{\frac{1}{m}}{\left(1+\frac{S}{m}\right)^i} \cdot (S - L)^+$

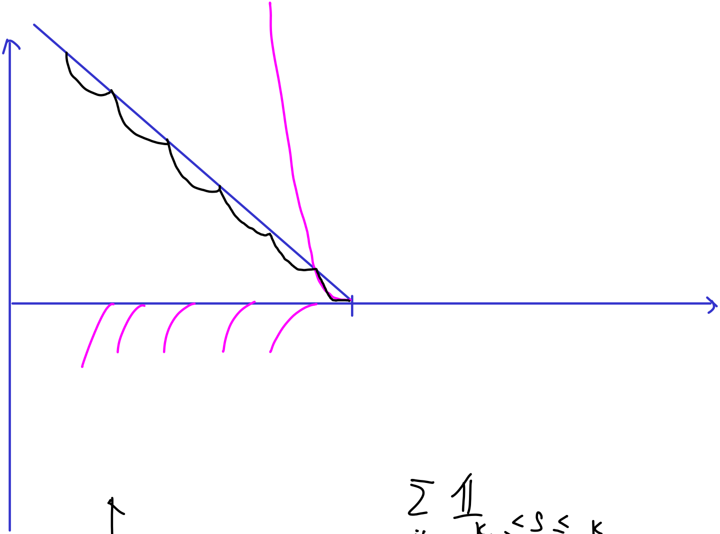
CMS Replication – Intuition

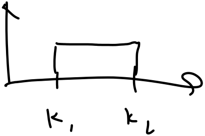
Using a series of IRR-settled payer swaptions, we can **statically replicate** the CMS caplet as follows:

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

$$\approx V^{pay}(L)h'(L) + \sum_{i=1}^{\infty} h''(L + i\cdot\Delta K)\, V^{pay}(L + \ i\cdot\Delta K)\,\Delta K$$







$$\sum_{i} \mathbb{1}_{k_1 \le S \le k_2}$$

Single-currency Change of Numeraire Theorem

Products requiring **convexity correction** are characterised by the fact that the underlying assets are paid at the wrong time or in the wrong denomination.

In this case, the forwards need to be adjusted to reflect the fact that they are paid incorrectly. This adjustment is known in the market as convexity correction.

inside $\mathbb{Q}$

We are already familiar with the **single-currency change of numeraire theorem**, which states that in an arbitrage-free economy, we have

$\mathbb{E}^{\mathbb{Q}}[X_T] = \mathbb{E}^{P}\left[X_T \cdot \frac{d\mathbb{Q}}{dP}\right]$

$$\mathbb{E}^N[H_T] = \mathbb{E}^M\left[H_T \frac{N_T/N_0}{M_T/M_0}\right], = \mathbb{E}^M\left[H_T \cdot \frac{d\mathbb{Q}^N}{d\mathbb{Q}^M}\right]$$

where

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{N_T/N_0}{M_T/M_0}.$$

Many of the products we are interested in will be multi-currency products. We can extend the single-currency theorem to include multi-currency economies.

$$dB_t = r_t B_t \, dt$$

$$B_T = B_0 \, e^{\int_0^T r_u \, du}$$

$$D(0,T) = \mathbb{E}^*\left[e^{-\int_0^T r_u \, du} \right]$$

$$\mathbb{Q}^*: \quad \frac{C_0}{B_0} = \mathbb{E}^*\left[\frac{C_T}{B_T}\right] \Rightarrow C_0 = B_0\, \mathbb{E}^*\left[\frac{C_T}{B_T}\right]$$

unique

$$\mathbb{Q}^T: \quad \frac{C_0}{D(0,T)} = \mathbb{E}^T\left[\frac{C_T}{\cancel{D(T,T)}^{\,1}}\right] \Rightarrow C_0 = D(0,T)\, \mathbb{E}^T\left[C_T\right]$$

$$D(0,T)\, \mathbb{E}^T\left[C_T\right] = B_0\, \mathbb{E}^*\left[\frac{C_T}{B_T}\right]$$

$$\mathbb{E}^T\left[C_T\right] = \mathbb{E}^*\left[C_T \cdot \frac{D(T,T) / D(0,T)}{B_T / B_0}\right]$$

$$\mathbb{E}^T\left[C_T\right] = \mathbb{E}^*\left[C_T \cdot \frac{d\mathbb{Q}^T}{d\mathbb{Q}^*}\right]$$

$$\mathbb{Q}^* : \quad \frac{V_0^{pay}}{B_0} = \mathbb{E}^*\left[\frac{V_T^{pay}}{B_T}\right] \Rightarrow V_0^{pay} = B_0\, \mathbb{E}^*\left[\frac{P_{n+1,N}(T)\,(S_T-K)^+}{B_T}\right]$$

unique

$$\mathbb{Q}^{n+1,N} : \quad \frac{V_0^{pay}}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_T^{pay}}{P_{n+1,N}(T)}\right] \Rightarrow V_0^{pay} = P_{n+1,N}(0)\, \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\,(S_T-K)^+}{\cancel{P_{n+1,N}(T)}}\right]$$

$$P_{n+1,N}(0)\, \mathbb{E}^{n+1,N}\left[(S_T-K)^+\right] = B_0\, \mathbb{E}^*\left[\frac{P_{n+1,N}(T)\,(S_T-K)^+}{B_T}\right]$$

$$\mathbb{E}^{n+1,N}\left[(S_T-K)^+\right] = \mathbb{E}^*\left[(S_T-K)^+ \cdot \frac{P_{n+1,N}(T) / P_{n+1,N}(0)}{B_T / B_0}\right]$$

$$\mathbb{E}^{n+1,N}\left[(S_T-K)^+\right] = \mathbb{E}^*\left[(S_T-K)^+ \cdot \frac{d\mathbb{Q}^{n+1,N}}{d\mathbb{Q}^*}\right]$$