Name: ______________________

SINGAPORE MANAGEMENT UNIVERSITY

MASTERS

Master of Science in Quantitative Finance

QF607

Numerical Methods

INSTRUCTIONS TO STUDENTS

1. The time allowed for this examination paper is **2.5** hours.
2. This examination paper contains **6** questions. You are **REQUIRED** to answer **ALL** questions. No marks will be deducted for wrong answers.
3. This is an open-book examination. You are allowed to use a financial calculator.
4. There are **seven (7)** printed pages including this instruction sheet.
5. You are required to return the full set of question paper at the end of the examination. Please write your name on the top right hand corner of this instruction sheet.

Question 1 [8 marks]

Convert the following numbers to fixed point representation `fixed<9, 3>`. Use two's complement for negative numbers. Calculate the round-off error for each case, and point out potential problem if there is any.

a) 18.56

b) -7.5

c) 22.87

d) 36.5

Question 2 [15 marks]

The price of a dividend free stock S follows a geometric Brownian motion with volatility 15%. The risk free continuous compounding interest rate is 5%. The stock price today is 100 dollars.

a) [10 marks] Set up a two step JRRN (Jarrow, Rudd risk neutral) binomial tree to price option with 2 year maturity using the log of stock price as state variable. You should list and compute the parameters of the tree, and draw or list the nodes of the tree with their associated stock price.

b) [5 marks] Price a 2 year European put option (strike = 105 dollars) on the tree you have set up in **a)**. List the option price at each node of the tree.

Question 3 [15 marks]

Root search the following equation with starting two points at $x_1 = 0, x_2 = 4$:

$$x^2 = x + 1$$

a) [5 marks] Apply three steps of bisection method to solve the above equation.

b) [5 marks] Apply two steps of secant method to solve the above equation.

c) [5 marks] Apply two steps of false position method to solve the above equation.

Question 4 [12 marks]

We would like to numerically integrate the function $f(x) = \frac{1}{2}e^{-\frac{1}{2}x^2}$ over $[0, 2]$:

$$\int_0^2 f(x)dx, \quad \text{where } f(x) = \frac{1}{2}e^{-\frac{1}{2}x^2}$$

a) [4 marks] Describe in pseudo code the algorithm to evaluate the integral using Monte-Carlo simulation.

b) [4 marks] Define $g(x)$ as the second order Taylor expansion of $f(x)$ at $x = 0$, write down the expression of $g(x)$ and compute $\int_0^2 g(x)dx$ analytically. Note that $\int_0^2 g(x)dx$ can be used to derive $E[g(x)]$ for uniformly distributed random variable x in $[0, 2]$.

c) [4 marks] Write a python function `mcIntegrate(f, a, b, g, g_integral)` to calculate Monte-Carlo integration of a function $f(x)$ over the interval $[a, b]$, where $g(x)$ is the function who's integral over $[a, b]$ is known as `g_integral`, and can be used as control variate to reduce the Monte-Carlo noise.

Question 5 [20 marks]

Under the Black-Scholes model

$$dS_t = (r - q)dt + \sigma dW_t$$

/S_t

we have derived the PDE that the price of any derivative instrument, denoted as $V(S_t, t)$, has to satisfy:

$$\frac{\partial V}{\partial t} + (r - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

a) [8 marks] Let $X_t = \ln \frac{S_t}{B_t}$, $\hat{V}_t = \frac{V_t}{B_t}$, where B_t is the deterministic process of the money account: $dB_t = rB_t dt$. Using X_t as state variable, derive the PDE that $\hat{V}_t$ has to satisfy.

c) [8 marks] Derive the linear system we need to solve under implicit Euler scheme at each time step t_i.

b) [4 marks] In terms of numerical stability and implementation, what are the advantages of the PDE derived in a) compared to the original Black-Scholes PDE?

Question 6 [30 marks]

Consider the Ornstein-Uhlenbeck process X_t following the SDE:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

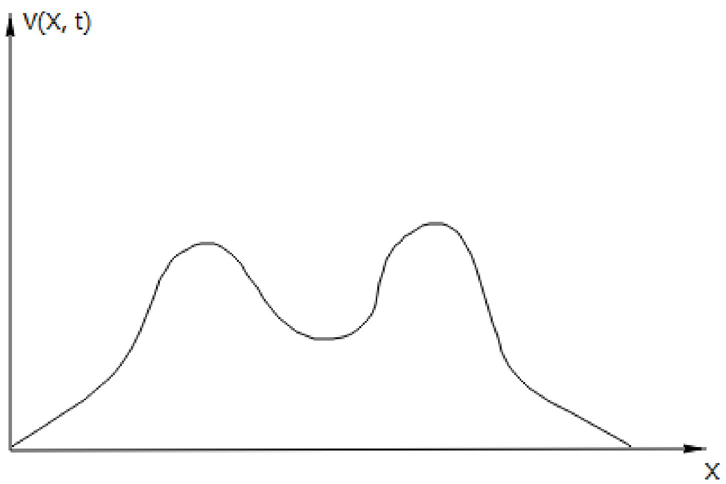
where θ, μ and σ are all positive constants. The risk-free interest rate r is constant and the money market account process is $\frac{dB_t}{B_t} = rdt$.

a) [10 marks] We want to simulate X_t using Monte Carlo so we need to discretize the SDE. Given a standard normal random variable $z \sim N(0,1)$ and time step Δt, write down the equation that evolves X from the i-th step to $i+1$ using Euler scheme, i.e., the function f that $X_{i+1} = f(X_i, \Delta(t), z)$.

b) [10 marks] Derive the PDE that the price of a derivative on X_t, denoted as V_t, should satisfy, using the fact that $\frac{V_t}{B_t}$ is a martingale under risk neutral measure.

c) [5 marks] Discretize the partial differential equation using central difference in X dimension and one-sided difference with explicit Euler scheme in temporal dimension, write down the equation that computes $V(x, t_i)$ from the time slice t_{i+1}.

d) [5 marks] Following the result of c), assume the interest rate r and the parameter θ are 0, and suppose that the derivative's price at t_{i+1} is of the form shown in the below chart, i.e., $V(x, t_{i+1})$ is known, sketch the rough shape of $V(x, t_i)$ on top of the plot and explain the rationale.



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