Note: the purpose of sample questions is to give an idea of the types of exam questions. Some of them are more difficult than the actual exam questions, some of them are easier. The total number of questions in the exam will also be different. And please remember to bring a caculator to the exam. During the exam, the relevant formulas and slides to the questions will also be provided in a cheat sheet.

Topics NOT covered by the exam

- Details of floating point representation
- Multi-dimensional binomial tree
- Multi-dimensional root searching
- FX market conventions
- Derivation of Fokker Planck equation and local volatility
- Box-Muller, acceptance rejection, and polor rejection method
- Stratified sampling, importance sampling and Quasi Monte Carlo
- Douglas scheme

Sample Question 1

a) Design an algorithm that uses false position method to compute $x = \sqrt{c}$, $c > 1$, using the initial points $x_0 = \frac{1+c}{4}$, $x_1 = c$.

b) Write a python function `mySqrt(c)` that implements the algorithm in **a)**. Explain the corresponding algorithmic steps in the comment. Note that minor punctuation and syntactic mistakes are tolerated in the answer.

c) Calculate the first 3 iterations for solving $x = \sqrt{7}$, and list also the first three step of bisection method.

a) The problem is in fact root search for $f(x) = x^2 - c = 0$.

```
REQUIRE c > 1
x0 <- (1+c)/4
x1 <- c
WHILE x1-x0>tol
  x2 <- (x0(x1*x1-c)-x1(x0*x0-c))/(x1*x1 - x0*x0}
  IF x2*x2 - c == 0
     return x2
  ELSIF (x0*x0-c)(x2*x2-c) < 0
     x1 <- x2
  ELSE
     x0 <- x2
  ENDIF
ENDWHILE
```

b)

```
def mySqrt(c, tol):
  assert c > 1
  x0 = (1+c)/4
  x1 = c
  while x1 - x0 > tol:
    x2 = (x0*(x1*x1 -c) - x1*(x0 * x0 -c)) / (x1*x1 - x0*x0)
    if math.abs(x2*x2 - c) < tol:
```

```
      return x2
    else:
      if (x0*x0 - c)(x2*x2-c) < 0:
         x1 = x2
      else:
         x0 = x2
```

c) c = 7, x0 = 2, x1 = 7

mySqrt:

step 1: $x_2 = (2 \times (49 - 7) - 7 \times (4 - 7))/(49 - 4) \approx 2.33$, *next bracket (2.33, 7)*

step 2: $x_2 = (2.33 \times (49 - 7) - 7 \times (5.44 - 7))/(49 - 5.44) \approx 2.50$, *next brackete (2.50, 7)*

step 3: $x_2 = (2.50 \times (49 - 7) - 7 \times (6.25 - 7))/(49 - 6.25) \approx 2.58$

bisection:

step1: $x_2 = (7 + 2)/2 = 4.5$, *next bracket (2, 4.5)*

step2: $x_2 = (4.5 + 2)/2 = 3.25$, *next bracket (2, 3.25)*

step3: $x_2 = (3.25 + 2)/2 = 2.625$

Sample Question 2

We want to model the process of stock price using the following SDE:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW_t$$

and would like to use $X(t) = \ln(S(t))$ as the state variable.

a) What is the SDE for $X(t)$?

$dX(t) = (\mu(t) - \frac{1}{2}\sigma(t)^2)dt + \sigma(t)dW_t$

b) Given a standard normal random variable $z \sim N(0, 1)$ and time step Δt, write down the equation that evolves X from time t to $t + \Delta t$ using Euler scheme, i.e., the function f that $X_{t+\Delta t} = f(X_t, \Delta(t), z)$.

$X_{t+\Delta t} = X_t + (\mu(t) - \frac{1}{2}\sigma(t)^2)\Delta t + \sigma(t)\sqrt{\Delta t}z$

Sample Question 3

Consider the following one step binomial tree model of the stock price process:

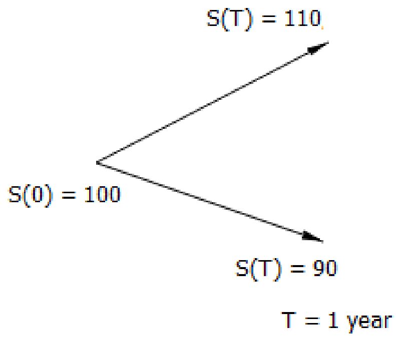
Recall: Call option payoff (S_1 - K)^+

a) Assume interest rate is 0%, compute the price of a call option struck at 100 at maturity 1 year using the method of replicating portfolio. Calculate the amount of stock shares you need to hold to replicate the option payoff. V_0

Holding Δ *share of the stock and* β *cash in the portfolio, to replicate the payoff of call we have*

$$\Delta S_u + \beta = 10$$
$$\Delta S_d + \beta = 0$$

beta = V_0 - Delta S_0



So $\Delta = 0.5, \beta = -45$, *the option worths* $0.5 \times 100 + (-45) = 5$.

b) Assume interest rate is 5%, compute the price of a call option struck at 100 at maturity 1 year using the method of replicating portfolio. Calculate the amount of stock shares you need to hold to replicate the option payoff.

Holding Δ *share of the stock and* β *cash in the portfolio, to replicate the payoff of call we have*

$$\Delta S_u + \beta e^{rT} = 10$$
$$\Delta S_d + \beta e^{rT} = 0$$

So $\Delta = 0.5, \beta = -45e^{-0.05} \approx -42.805$, *the option worths* $0.5 \times 100 + (-42.805) \approx 7.195$.

Sample Question 4

Convert the following `fixed<9, 4>` binary numbers to real numbers (use two's complement for negative numbers):

101101100

$-16 + 2^{6-4} + 2^{5-4} + 2^{3-4} + 2^{2-4} = -16 + 4 + 2 + 0.5 + 0.25 = 9.25$

011011010

$2^{7-4} + 2^{6-4} + 2^{4-4} + 2^{3-4} + 2^{1-4} = 8 + 4 + 1 + 0.5 + 0.125 = 13.625$

001110010

$2^{6-4} + 2^{5-4} + 2^{4-4} + 2^{1-4} = 4 + 2 + 1 + 0.125 = 7.125$

Sample Question 5

Consider the process X_t

$$dX_t = \sigma dW_t$$

where W_t is a standard Brownian motion and σ is the constant volatility. Denote V_t the price of a derivative on X_t. Derive the partial differential equation satisfied by $V(x, t)$.

Assume the money market account process is $\frac{dB_t}{B_t} = rdt$, *where* r *is the risk free interest rate.*

$$d(\tfrac{V_t}{B_t}) = V_t \times (-\tfrac{r}{B_t})dt + \tfrac{1}{B_t}(\tfrac{\partial V_t}{\partial X_t} dX_t + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial X_t^2} dX_t^2) = \tfrac{1}{B_T}(\tfrac{1}{2}\sigma^2 \tfrac{\partial^2 V_t}{\partial X_t^2} - rV_t)dt + \tfrac{1}{B_t}\tfrac{\partial V_t}{\partial X_t}\sigma dW_t$$

V_t is the price of a tradeable instrument, $\frac{V_t}{B_t}$'s drift under risk neutural measure should be martingale, so the PDE to satisfy is:

$\frac{1}{2}\sigma^2 \frac{\partial^2 V_t}{\partial X_t^2} - rV_t = 0$