Goldman Sachs

Quantitative Strategies Technical Notes

April 1997

Stochastic Implied Trees: Arbitrage Pricing With Stochastic Term and Strike Structure of Volatility

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SUMMARY

In this paper we present an arbitrage pricing framework for valuing and hedging contingent equity index claims in the presence of a stochastic term and strike structure of volatility. Our approach to stochastic volatility is similar to the Heath-Jarrow-Morton (HJM) approach to stochastic interest rates. Starting from an initial set of index options prices and their associated local volatility surface, we show how to construct a family of continuous time stochastic processes which define the *arbitrage-free* evolution of this local volatility surface through time. The no-arbitrage conditions are similar to, but more involved than, the HJM conditions for arbitrage-free stochastic movements of the interest rate curve. They guarantee that even under a general stochastic volatility evolution the initial options prices, or their equivalent Black-Scholes implied volatilities, remain fair.

We introduce stochastic implied trees as discrete implementations of our family of continuous time models. The nodes of a stochastic implied tree remain fixed as time passes. During each discrete time step the index moves randomly from its initial node to some node at the next time level, while the local transition probabilities between the nodes also vary. The change in transition probabilities corresponds to a general (multifactor) stochastic variation of the local volatility surface. Starting from any node, the future movements of the index and the local volatilities must be restricted so that the transition probabilities to all future nodes are simultaneously martingales. This guarantees that initial options prices remain fair. On the tree, these martingale conditions are effected through appropriate choices of the drift parameters for the transition probabilities at every future node, in such a way that the subsequent evolution of the index and of the local volatility surface do not lead to riskless arbitrage opportunities among different option and forward contracts or their underlying index.

You can use stochastic implied trees to value complex index options, or other derivative securities with payoffs that depend on index volatility, even when the volatility surface is both skewed and stochastic. The resulting security prices are consistent with the current market prices of all standard index options and forwards, and with the absence of future arbitrage opportunities in the framework. The calculated options values are independent of investor preferences and the market price of index or volatility risk. Stochastic implied trees can also be used to calculate hedge ratios for any contingent index security in terms of its underlying index and all standard options defined on that index.

We thank Indrajit Bardhan, Peter Carr, Michael Kamal and Joseph Zou for helpful conversations. We are also grateful to Barbara Dunn for her careful review of the manuscript.

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The Black-Scholes theory of options pricing [Black 1973] assumes that stock prices are stochastic and vary lognormally, but that future stock volatilities, interest rates and dividend yields are known and deterministic. The theory is based on the exclusion of arbitrage: an option's payoff can be replicated by that of a time-varying portfolio of stock and riskless bonds, and must therefore at any time have the same value as the portfolio. The most compelling consequence of this *arbitrage-free* approach is that options values are *preference-free*: investors of all risk preferences can agree on the unique fair value of an option. This transcendent quality of the theory has led to its great practical success, spawning more than two decades of intensive research that extended it to other underlyers and relaxed its basic assumptions so as to better match the observed behavior of options markets and underlyers. The current generation of models, even though they treat underlyers more realistically and can be calibrated to prevailing options market prices, are still based on an arbitrage-free approach, admitting no arbitrage opportunities in their theoretical framework.

The history of interest rate options pricing illustrates this development. Original models were simple adaptations the Black-Scholes formula with bonds, rather than stocks, as the underlyers. Today, most interest rate options pricing models assume interest rates themselves are stochastic and mean-reverting, allow for several stochastic factors, and can be calibrated to observed initial bond prices (and their volatilities), while constraining future interest-rate evolution to be arbitrage-free. These models fall into two basic families. *Equilibrium* models[1] consider interest rate processes depending on one or more state variables and are derived from general equilibrium arguments. The market prices of risk are then derived from associated characteristics of the yield curve (such as level, slope, curvature, etc.) or bond prices. In general these models are not calibrated to all current bond prices, and may therefore contain initial arbitrage violations. *Arbitrage-free* models, in contrast, are calibrated to all initial bond prices and also admit no future arbitrage violations. They achieve this in two different ways. The first class[2] use stochastic interest rate processes that automatically generate arbitrage-free future scenarios, and equip the process with enough parameters to be forcibly calibrated to the initial traded bond prices. The second class[3], instead, start with exogenously specified stochastic process for bond prices or forward rates. They then derive constraints on the evolution of bond prices or forward rates so that no future arbitrages occur.

The history of stochastic volatility modeling is shorter but still similar to the history of stochastic interest rates. Existing stochastic volatility models fall into two basic families. *Complete-market* models[4] specify conditions under which the financial market is complete in the presence of the volatility risk. They posit (if necessary) hypothetical traded volatility instruments that can be used to hedge the volatility risk and complete the market. Contingent claim prices in these models depend critically on the price dynamics of the volatility instruments and may also implicitly depend

1. See, for example, Cox, Ingersoll and Ross [1985].
2. See, for example, Vasicek [1977], Black, Derman and Toy [1990].
3. See, for example, Ho and Lee [1986], Heath-Jarrow and Morton [1992].
4. See, for example, Merton [1973], Cox and Ross [1976], Johnson and Shanno [1987], Eisenberg and Jarrow [1994].

on the market price(s) of volatility risk. *Equilibrium* models[5] tend to assume (rather than derive) some parametric form for the stochastic evolution of the index and its volatility in equilibrium, and then derive implicit options valuation formulas which depend on the parameters of the process. The traded options prices are then inverted for the unknown parameters.

Complete-market models can be somewhat arbitrary and sometimes unnatural because of the specific assumptions they make about the hypothetical volatility instruments. The equilibrium volatility models have the drawback that the choice of the parametric form for the underlying stochastic processes remains largely arbitrary. In addition, it is usually difficult to invert complex and non-linear options prices to obtain the parameters. Finally, *ad hoc* specification of the market prices of risk can lead to violations of arbitrage[6].

In this paper we propose a new arbitrage-based approach to contingent claims valuation with stochastic volatility[7], similar to the Heath-Jarrow-Morton (HJM) methodology for stochastic interest rates[8]. We begin with a continuous time economy with multiple factors. We work with local (forward) volatilities, instead of implied volatilities (or option prices), imposing an exogenous stochastic structure on the local volatility surface. The primacy of the local volatility surface in our work is analogous to that of the forward rate curve in the HJM framework. Our model takes as given the initial local volatility surface and posits a general multi-factor continuous time stochastic process for its evolution across time. To ensure that this process is consistent with an arbitrage-free economy we characterize the conditions which guarantee absence of explicit arbitrage opportunities (at any future time) among the various option (and futures) contracts defined and traded on the underlying index. Under these conditions markets are complete and contingent claim valuation is preference-free. Unfortunately, in contrast to the HJM conditions, here the arbitrage-free conditions are complex and non-linear (integral) equations, which are difficult to use in their continuous form.

We then introduce *Stochastic Implied Trees* as a *discrete-time* framework where the volatility surface undergoes multi-factor (arbitrage-free) stochastic variations. Here we work with trinomial stochastic implied trees[9]. The location of the nodes in this kind of tree are fixed but the transition probabilities vary stochastically as time changes and index level moves. As time evolves, the index level moves randomly from node to node while local volatilities (and concurrently the transition probabilities) fluctuate stochastically across the tree. Starting from any initial node, the future

5. See, for example, Wiggins [1977], Hull and White [1977], Stein and Stein [1991].

6. See Cox, Ingersoll and Ross [1985], Heath, Jarrow and Morton [1992].

7. Presented in Risk Advanced Mathematics for Derivatives Conference, New York, December 1997.

8. For attempts in this direction see, for example, Dupire [1993] and Bruno Dupire in the Proceedings of Risk Derivatives Conference, Brussels, February 1997.

9. See Derman, Kani and Chriss [1996], Kani, Derman and Kamal [1996].

movements of the index and the local volatility surface must be restricted so that total transition probabilities to all future nodes are simultaneously martingales. On the tree, these martingale conditions can be satisfied by making an appropriate choice of the *drift parameter* for every future node. In the discrete time framework defined by the stochastic implied tree, this process step-by-step guarantees absence of arbitrage opportunities among different option (and forward) contracts and the underlying index.

We draw extensively on the analogy between interest rates and volatility throughout this paper. We begin by reviewing the concept of the local (forward) volatility surface and the *effective theory* of volatility which it defines. The local volatility surface is the options world analogue of the forward interest rate curve. Standard option prices calculated using today's local volatility surface match their market prices, just as the bond prices calculated from today's forward rate curve match their market prices. The dynamics of standard option prices, as defined by today's local volatility surface, albeit arbitrage-free, is based on the assumption of non-stochastic volatility, as portrayed by the static (non-random) nature of the local volatility surface. This *effective dynamics* of option prices is analogous to the deterministic, but arbitrage-free, bond price dynamics which result from a static forward rate curve. To allow *stochastic dynamics* we introduce exogenous stochastic structure on the effective theory. This is to say that we allow general (multi-factor) fluctuations of the local volatility surface as time and spot index level change. We impose dynamical conditions which explicitly guarantee absence of arbitrage among standard options, forwards and the underlying index. This process will augment an effective theory of volatility to a full *stochastic theory* of volatility in a manner which is the hallmark of the HJM approach to stochastic interest rates.

)CAL VOLATILITY SUR- .CE: THE EFFECTIVE THE- RY OF VOLATILITY

We can think of local volatility $\sigma_{K,T}$ as the market's consensus estimate of instantaneous volatility at the future market level K and future time T. Local volatilities corresponding to different future market levels and times together comprise the *local volatility surface.* The local volatility surface indicates the fair value of future index volatility at future market levels and times as implied by the spectrum of available standard option (and forward contract) prices.

The relationship between the local volatilities and option prices (or implied volatilities) in the options world is analogous to the relationship between the forward rates and bond prices (or yield-to-maturities) in the fixed income world. We can calculate the forward interest rates f_T corresponding to the future times T from the spectrum of zero-coupon bond prices B_T with different maturities T, using a well-known formula

$$f_T = -\frac{1}{B_T}\frac{dB_T}{dT} \qquad \text{(EQ 1)}$$

Similarly, we can calculate the local volatility $\sigma_{K,T}$ corresponding to the future market level K and time T from the spectrum of standard option prices $C_{K,T}$, with different strikes K and maturities T, using the formula

$$\sigma^2{}_{K,T} = 2\frac{\left\{\frac{\partial C_{K,T}}{\partial T} + (r-\delta)K\frac{\partial C_{K,T}}{\partial K} + \delta C_{K,T}\right\}}{K^2\frac{\partial^2 C_{K,T}}{\partial K^2}} \quad \textbf{(EQ 2)}$$

The riskfree discount rate r and the dividend yield δ in Equation 2 are both assumed to be constant. Also, the quantities which we will discuss throughout this paper are usually evaluated at a specific times t or spot prices S, and contain other explicit or implicit (deterministic or stochastic) parameters which we may omit for brevity. For example, the quantities in Equations 1 and 2 are evaluated at the present time and spot price, hence $f_T = f_T(t_0)$, $\sigma_{K,T} = \sigma_{K,T}(t_0, S_0)$ etc.

Equations 1 often serves as a general definition for forward rates, regardless of the specific nature of the interest rate process. It can be shown[10] that under very general assumptions, forward rates are risk-adjusted expectations of future short rates

$$f_T = E^{(T)}[r(T)] \quad \textbf{(EQ 3)}$$

The expectation $E^{(T)}[\ldots]$ is performed at the present time and with respect to a measure known as the *T-maturity forward risk-adjusted measure.* The precise description of this measure is not necessary for our purposes here. The only thing to remember is that Equation 1 gives us a direct way for extracting these expectations of future short rates from the traded bond prices.

Similarly, it can be shown that local volatilities are risk-adjusted expectations of future instantaneous volatilities. More precisely, local variance $\sigma^2{}_{K,T}$ is a risk-adjusted expectation of future instantaneous variance $\sigma^2(T)$ at time T as

$$\sigma^2_{K,T} = E^{(K,T)}[\sigma^2(T)] \quad \textbf{(EQ 4)}$$

Here the expectation $E^{(K,T)}[\ldots]$ is performed at the present time and market level, and with respect to a new measure which we call the *K-strike and T-maturity forward risk-adjusted measure*, as described in Appendix A. Again the precise details about the measure are unimportant at this point, only that these expectations can be directly extracted from the market prices of standard options, as given by Equation 2.

A static (non-random) local volatility surface defines an *effective theory* of volatility in the same way as a static forward rate curve defines an effective theory for interest rates. In an effective theory, specific expectations (or *integrals*) of some or all of the underlying stochastic variables are extracted from the current prices of the traded assets, and are subsequently assumed to remain unchanged

10. See, for example, Jamshidian [1993].

as time evolves. The effective dynamics which results is based on some of the sources of uncertainty being "effectively" integrated out of the full stochastic theory. Let us briefly review the interest rate case first.

The Effective Interest Rate Theory

In the effective interest rate setting, the forward rate curve is evaluated from the available bond prices at time t_0, and is assumed to remain unchanged thereafter as time t evolves, thus for all $t \geq t_0$:

$$f_T(t) = f_T \qquad \textbf{(EQ 5)}$$

As Figure 1 illustrates, this procedure integrates all sources of interest rate stochasticity out of the original theory, and therefore, the effective dynamics of the rates in the effective theory is completely deterministic. As physical time t elapses, the spot rate (or short rate) $r(t)$ rolls along the static forward rate curve, coinciding with the forward rate at time t:

$$r(t) = f_t \qquad \textbf{(EQ 6)}$$

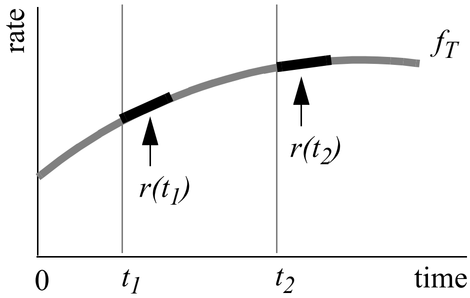
The dynamics of zero-coupon bond prices is also deterministic and is described by a simple *backward equation*:

$$\left(\frac{d}{dt} - f_t\right) B_T(t) = 0 \qquad \textbf{(EQ 7)}$$

This equation, with the aid of Equation 6, shows that the asset price dynamics in the effective theory is *local* and arbitrage-free. Equation 7 is also the dual of the *forward equation* satisfied by the zero-coupon bond prices:

$$\left(\frac{d}{dT} + f_T\right) B_T(t) = 0 \qquad \textbf{(EQ 8)}$$

FIGURE 1. In an effective theory defined by a static forward rate curve, short rate follows the instantaneous forward rates.



The forward equation is merely a restatement of Equation 1, and holds by the definition of the forward rates regardless of specific assumptions concerning the behavior of interest rates.

The backward equation describes propagation *forward* in physical time, for a fixed maturity. More precisely, it relates the prices of a T-maturity bond at different time points t, with earlier times in terms of the later ones. This is best understood by introducing the *forward propagator (*or *forward Green's function) $p_{t,t'}$*, which relates bond prices at times t and t', with $t \le t'$, for any T-maturity bond, through a simple relationship:

$$B_T(t) = p_{t,t'} B_T(t') \qquad \textbf{(EQ 9)}$$

The forward propagator $p_{t,t'}$ describes bond price evolution forward in physical time, as illustrated by Figure 2(a). It satisfies the backward and forward differential equations with boundary conditions $p_{t,t} = 1$:

$$\left(\frac{d}{dt} - f_t\right) p_{t,t'} = 0 \quad ; \quad \left(\frac{d}{dt'} + f_{t'}\right) p_{t,t'} = 0 \qquad \textbf{(EQ 10)}$$

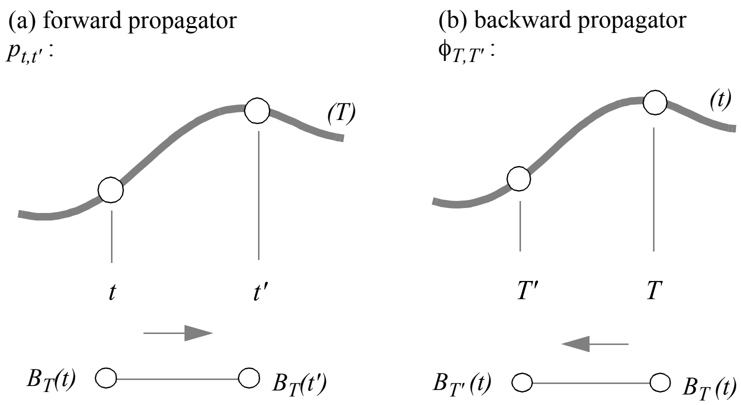
and for any $t \le \tilde{t} \le t'$, the composition relation:

$$p(t, t') = p(t, \tilde{t}) p(\tilde{t}, t') \qquad \textbf{(EQ 11)}$$

Similarly, the forward equation describes propagation *backward* in maturity time, for a fixed physical time. More precisely, it relates the prices of bonds with different maturities T, but at a fixed time t, with longer maturity bonds in terms of the shorter maturity ones. The *backward propagator*[11] $\phi_{T,T'}$ relates

11. The forward and backward propagators for a static yield curve are both simply equal to the discount function i.e $p_{u,v} = \phi_{u,v} = \exp\left(-\int_u^v f_\tau d\tau\right)$.

FIGURE 2. Forward propagator describes the evolution of bond prices forward in physical time. Backward propagator describes evolution of bond prices backward in maturity time.



zero-coupon bond prices of maturities T and T', with $T' \le T$, at any fixed time t, using the relation

$$B_T(t) = \phi_{T,T}B_{T'}(t) \tag{EQ 12}$$

The backward propagator $\phi_{T,T'}$ describes bond price evolution backward in maturity time, as depicted by Figure 2(b). It also satisfies the forward and backward equations with boundary conditions $\phi_{T,T} = 1$:

$$\left(\frac{d}{dT}+f_T\right)\phi_{T,T} = 0 \quad ; \quad \left(\frac{d}{dT}-f_T\right)\phi_{T,T} = 0 \tag{EQ 13}$$

and, for any $T \le \tilde{T} \le T$, the composition relation

$$\phi_{T,T} = \phi_{T,\tilde{T}}\phi_{\tilde{T},T} \tag{EQ 14}$$

The Effective Volatility Theory

In the effective volatility setting, the local volatility surface is calculated using the spectrum of available option prices (and futures) at time t_0, and is assumed to remain unchanged thereafter as time t and index price S change:

$$\sigma_{K,T}(t,S) = \sigma_{K,T} \tag{EQ 15}$$

This procedure amounts to averaging out all sources of stochastic volatility, leaving the index price uncertainty as the only source of uncertainty left within the theory. The resulting effective dynamics only depends on the index price and time and, as a function of these variables, is deterministic. As the physical time t elapses and index price S_t moves, the instantaneous volatility $\sigma(t)$ follows along the local volatility surface, as depicted in Figure 3, coinciding with the local volatility at time t and level S_t:

$$\sigma(t) = \sigma_{t,\,S_t} \tag{EQ 16}$$

This is consistent with an equilibrium (effective) index price process described by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_{t,\,S_t} dZ_t \tag{EQ 17}$$

where μ_t is the index's expected return and dZ_t is the standard Wiener measure at time t. In this process the instantaneous volatility is a known (deterministic) function of time t and index price S_t. Implied Tree models are the discrete frameworks for implementing the (effective) dynamics represented by Equation 17. The dynamics of standard option prices in the effective theory is described by the *backward equation*:

$$\left(\frac{\partial}{\partial t} + (r-\delta)S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2_{S,\,t}S^2\frac{\partial^2}{\partial S^2} - r\right)C_{K,\,T}(t, S) = 0 \tag{EQ 18}$$

Since the only remaining source of uncertainty is the index price, the standard options are completely hedgeable (using index as the hedge) within the effective theory. Equations 16 and 18 then show that the option price dynamics in this theory is arbitrage-free. Equation 18 is also the dual of the *forward equation* satisfied by the standard option prices:

FIGURE 3. In an effective theory represented by a static local volatility surface, instantaneous volatility $\sigma(t)$ at time t follows the local volatility at time t and index price S_t.

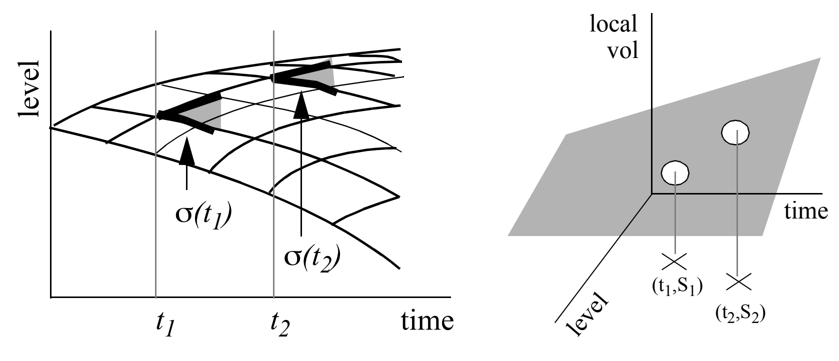
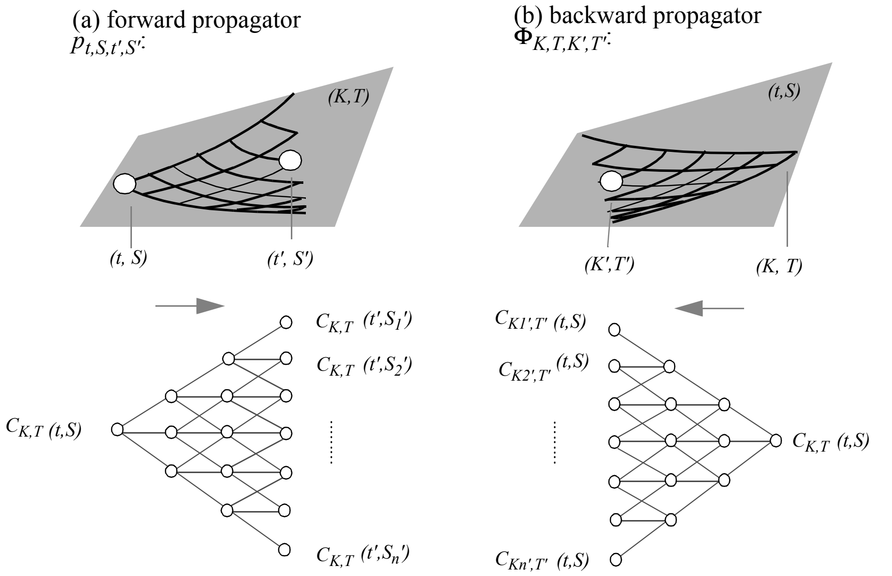


FIGURE 4. Forward propagator describes the evolution of standard prices in physical time and index price. Backward propagator describes the evolution of option prices in maturity time and strike price.



$$\left(\frac{\partial}{\partial T} + (r-\delta)K\frac{\partial}{\partial K} - \frac{1}{2}\sigma^2{}_{K,T}K^2\frac{\partial^2}{\partial K^2} + \delta\right)C_{K,T}(t,S) = 0 \qquad \textbf{(EQ 19)}$$

This forward equation is the same as Equation 2 and holds by the definition of local volatility, regardless of any specific assumptions about the behavior of volatility.

The *forward propagator* $p_{t,S,t',S'}$ describes the relationship between the option prices at the two points *(t, S)* and *(t', S')*, with $t \le t'$, for any *K*-strike and *T*-maturity standard option, through the relation

$$C_{K,T}(t,S) = \int_0^{\infty} p_{t,S,t',S'} C_{K,T}(t',S')dS' \qquad \textbf{(EQ 20)}$$

The forward propagator $p_{t,S,t',S'}$ describes option price evolution forward in time and index price, as illustrated by Figure 4(a). We can define the forward transition probability density function *p(t,S,t',S')* in terms of the forward propagator as $p(t,S,t',S') = e^{r(t'-t)} p_{t,S,t',S'}$. It describes the total probability that the index price will reach level *S'* at time *t'*, given that the index price at time *t* is *S.* The mathematical properties of $p_{t,S,t',S'}$ and *p(t,S,t',S')* are discussed in Appendix B.

The *backward propagator* $\Phi_{K,T,K',T'}$ describes the relationship between prices of two standard options corresponding to strike-maturity pairs (K,T) and (K',T'), with $T' \leq T$, at a fixed time t and index price S, as

$$C_{K,T}(t,S) = \int_0^{\infty} \Phi_{K,T,K',T'} C_{K',T'}(t,S) dK' \qquad \textbf{(EQ 21)}$$

As Figure 4(b) illustrates, We can also define the effective theory backward transition probability density function $\Phi(K,T,K',T')$ in terms of the backward propagator as $\Phi(K,T,K',T') = e^{\delta(T-T')} \Phi_{K,T,K',T'}$. Appendix B discusses some of the mathematical properties of $\Phi_{K,T,K',T'}$ and $\Phi(K,T,K',T')$.

We can use Equation 17, either by performing simulations or by using implied tree methods, to price and hedge complex options, with the knowledge that the standard options initially used to derive the local volatility surface will have model prices which match their market values. In spite of this calibration, if the volatility has a substantial stochastic behavior, the prices and hedge ratios of most options with path-dependent or volatility-dependent payoffs will not be accurately represented by the effective theory results. The reason is simply that effective theory results are based on the assumption that local volatilities are *static* or, equivalently, that the instantaneous volatility is substantially a function of the market level (and time). This is a good assumption in situations where the volatility exhibits strong correlation to the market level and, hence, can be viewed predominantly as a function of it. For most equity index option markets, for example, this more or less holds, specially for shorter dated options. On the contrary, in the currency options markets or in longer dated equity (and most other) options markets, the volatility is predominantly stochastic and the effective theory of static local volatilities is not valid. We must therefore move towards a full stochastic framework by allowing general multi-factor stochastic variations of the volatility surface.

TOWARDS A STOCHASTIC THEORY OF VOLATILITY

To allow for stochastic dynamics we must introduce exogenous stochastic structure on the effective theory. In general, there are few restrictions on the choice of this structure. One important restriction, which is the cornerstone of the arbitrage framework, is the absence of any explicit future arbitrage opportunities in the final stochastic theory. Another restriction is how close the number or the behavior of the stochastic factors are to what is empirically observed. For now, we will consider very general (but sufficiently regular) stochastic structures and discuss the conditions which must be imposed upon them to guarantee the absence of arbitrage. Let us briefly examine the stochastic interest rate theory first.

The Stochastic Interest Rate Theory

Figure 5 illustrates the dynamics of the forward rates in the stochastic framework. Here, the forward rate curve is allowed to fluctuate stochastically with several independent stochastic factors represented by Brownian motions W^i, $i = 1, \ldots, n$, with factor volatilities $\vartheta^i{}_T(t)$ generally depending on both maturity T and time t, according to the stochastic differential equation:

$$df_T(t) = \alpha_T(t)dt + \sum_{i=1}^{n} \vartheta^i_T(t)dW^i_t \qquad \textbf{(EQ 22)}$$

In the family of processes described by Equation 22, the volatility coefficients $\vartheta^i{}_T(t)$ reflect the sensitivities of specific maturity forward rates to the random shocks introduced by the Brownian motions W^i. These coefficients are left unrestricted, except for mild measurability and integrability conditions, and can depend on the past histories of the Brownian motions W^i. The drift coefficients $\alpha_T(t)$ must also satisfy mild measurability and integrability conditions, but must be further constrained by the no-arbitrage requirement.

The *spot rate* at time t, $r(t)$, is the instantaneous forward rate at time t, i.e, $r(t) = f_t(t)$. The stochastic integral equation satisfied by the spot rate is found by integrating Equation 22 and evaluating the result at $T = t$. It is given by

$$r(t) = f_t(0) + \int_0^t \alpha_t(u)du + \sum_{i=1}^{n}\int_0^t \vartheta^i{}_t(u)dW^i{}_u \qquad \textbf{(EQ 23)}$$

It has been argued by Heath, Jarrow and Morton, that there will be no explicit arbitrage opportunities in the theory defined by Equation 23 if (and only if) the drift coefficients are of the form:

FIGURE 5. In a stochastic interest rate theory spot rate *r(t)* follows the instantaneous forward rate $f_t(t)$.

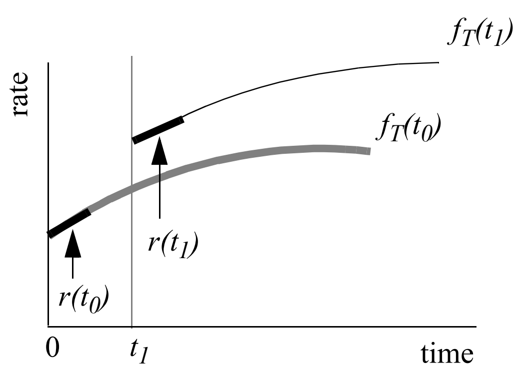
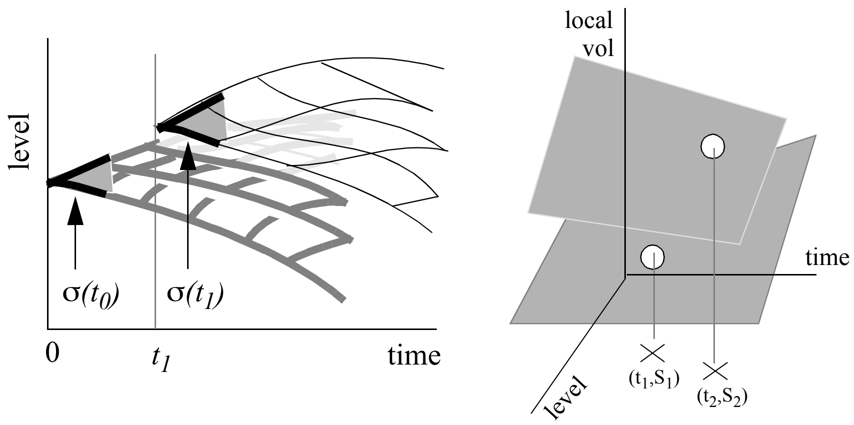


FIGURE 6. In a stochastic volatility theory instantaneous volatility $\sigma(t)$ follows the local volatility $\sigma_{S_t,t}(t,S_t)$, at time t and index price S_t.



$$\alpha_T(t) = \sum_{i=1}^{n} \vartheta_T^i(t)\left(\int_t^T \vartheta_u^i(t)du + \lambda^i(t)\right) \qquad \textbf{(EQ 24)}$$

Here $\lambda^i(t)$, $i = 1, ..., n$, denote the *market prices of risk,* which can not explicitly depend on maturity T but are otherwise arbitrary. Under these conditions, they have shown that markets are complete and contingent claims prices are independent of the market prices of risk.

The Stochastic Volatility Theory

Our goal is to introduce a similar stochastic structure on the local volatility surface. To do so, we allow the surface to undergo stochastic fluctuations with several independent stochastic factors, $W^0, W^1, ..., W^n$, based on the following stochastic differential equation:

$$d\sigma^2_{K,T}(t,S) = \alpha_{K,T}(t,S)dt + \sum_{i=0}^{n} \theta^i_{K,T}(t,S)dW^i_t \qquad \textbf{(EQ 25)}$$

We include $W^0 = Z$, the index price's source of uncertainty, among the factors so that the stochastic variations of the local volatility surface may depend on the prevailing market level. The family of processes of Equation 25 defines a multifactor dynamics for the local volatility surface, as illustrated by Figure 6. These processes can be integrated, starting from a fixed (non-random) initial local volatility surface $\sigma_{K,T}(0,S_0)$ at time $t = 0$, as

$$\sigma^2_{K,T}(t, S_t) = \sigma^2_{K,T}(0, S_0) + \int_0^t \alpha_{K,T}(u, S_u)du + \sum_{i=0}^{n} \int_0^t \theta^i_{K,T}(u, S_u)dW^i_u \qquad \text{(EQ 26)}$$

The factor volatility $\theta^i_{K,T}(t, S)$ reflects the sensitivity of local volatilities $\sigma_{K,T}(t,S)$, across the whole surface, to the shock introduced by the Brownian motion W^i. Except for mild measurability and integrability conditions[12], the family of factor volatilities are unrestricted, generally depending on time and index price, and on the factors or their past histories. However, for the sake of brevity we have omitted explicit references to all variables other than time t and index price S from the expressions for factor volatilities, and we will do the same for other quantities such as drift coefficients and local volatilities.

The *spot volatility* (or instantaneous volatility) at time t, $\sigma(t)$, is the instantaneous local volatility at time t and level S_t, i.e

$$\sigma(t) = \sigma_{S_t, t}(t, S_t) \qquad \text{(EQ 27)}$$

It describes the variability of index price return process, as given by the differential equation

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(t)dW^0_t \qquad \text{(EQ 28)}$$

or its integral form

$$S_t = S_0 + \int_0^t \mu_u S_u du + \int_0^t \sigma(u)S_u dW^0_u \qquad \text{(EQ 29)}$$

where μ_t is the index's expected return. Setting $T = t$ and $K = S_t$ in Equation 26 we find the stochastic integral equation satisfied by the spot volatility as

$$\sigma^2(t) = \sigma^2_{t,S_t}(0, S_0) + \int_0^t \alpha_{t,S_t}(u, S_u)du + \sum_{i=0}^{n} \int_0^t \theta^i_{t,S_t}(u, S_u)dW^i_u \qquad \text{(EQ 30)}$$

The drift coefficients $\alpha_{K,T}(t, S)$ must also satisfy mild measurability and integrability conditions, but they must be further restricted by the requirement that the stochastic theory described by Equations 28 and 30 disallows explicit arbitrage opportunities among the standard options, forwards and their underlying index. This is similar to the HJM arbitrage conditions on the spot rate process.

12. The factor volatility functions $\theta^i_{K,T}(t, S)$ are assumed to be positive, adapted and jointly measurable with respect to the Borel σ-algebra restricted to $0 \le t \le T \le T^*$, for some fixed maximum time T*. They must also satisfy $\int_0^t (\theta^i_{K,T})^2(u, S_u)du < \infty$, $i = 0, \ldots, n$, to assure regularity of spot volatility process, and certain additional integrability conditions to assure regularity of the standard option price processes.

Let us briefly examine (a variation of) the HJM argument below.

The HJM Conditions and the Stochastic Theory of Interest Rates

The bond price dynamics corresponding to the forward rate process of Equation 85 is, by applying Ito's lemma, described by the stochastic integral equation

$$dB_T(t) = r(t)B_T(t)dt + \int_t^T \frac{\delta B_T(t)}{\delta f_u(t)} df_u(t)du + \frac{1}{2}\int_t^T\int_t^T \frac{\delta^2 B_T(t)}{\delta f_u(t)\delta f_{u'}(t)} df_u(t)df_{u'}(t)dudu' \quad \text{(EQ 31)}$$

The symbol $\frac{\delta}{\delta f_u}$ here denotes the variational (or functional) derivative with respect to the function f evaluated at u. The first term in this equation describes precisely the effective theory bond price dynamics restricted to the fixed forward rate curve $f_T(t)$ at time t. The next two terms describe the bond price dynamics resulting from the stochastic variations of the effective theory (defined by $f_T(t)$) during the next infinitesimal time interval dt.

It follows from the definition of the forward rates (Equation 1) that the price of a T-maturity zero-coupon bond with unit face, at time t, is given by

$$B_T(t) = \exp\left(-\int_t^T f_u(t)du\right) \quad \text{(EQ 32)}$$

From this expression it is simple to see that for any u ($t \le u \le T$):

$$\frac{\delta B_T(t)}{\delta f_u(t)} = -B_T(t) \quad \text{(EQ 33)}$$

Another way of seeing this is by noticing how the forward and backward propagators, $p_{t,t'}$ and $\phi_{T,T'}$, corresponding to an otherwise fixed (non-random) forward rate curve, respond to sudden changes of a specific forward rate f_u along the curve. It is simple to see that $p_{t,t'}$ satisfies the following relation, as depicted in Figure 7(a):

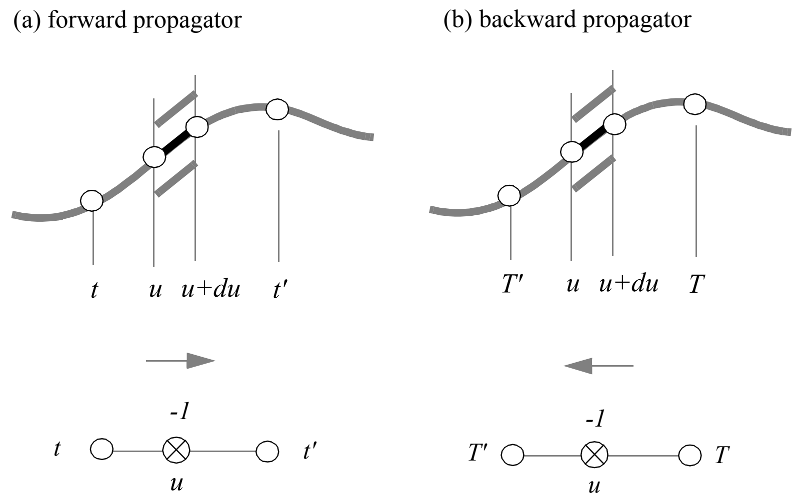
$$\frac{\delta p(t, t')}{\delta f_u} = -p(t, u)p(u, t') = -p(t, t') \quad \text{(EQ 34)}$$

and, as shown in Figure 7(b), that $\phi_{T,T'}$ satisfies the relation:

$$\frac{\delta \phi_{T, T}}{\delta f_u} = -\phi_{T, u}\phi_{u, T} = -\phi_{T, T} \quad \text{(EQ 35)}$$

These relations combined, respectively, with Equations 9 and 12, again lead to Equation 33.

FIGURE 7. Sensitivity of the forward and backward propagators $p_{t,t'}$ and $\phi_{T,T'}$ to the sudden changes of the forward rate f_u.



Similarly, we can show that for $t \le u \le u' \le T$ the second order variational derivatives are given by:

$$\frac{\delta^2 B_T(t)}{\delta f_u(t) \delta f_{u'}(t)} = B_T(t) \qquad \text{(EQ 36)}$$

The special f_u-independent form of variational relations 33-36 can be directly attributed to the special form of the functional relationship between the zero-coupon bond prices and the forward rates as described by Equation 32. This feature underlies the special simplicity of no-arbitrage conditions in the HJM framework.

Using Equations 22, 33 and 36 inside Equation 31 we find

$$\frac{dB_T(t)}{B_T(t)} = r(t)dt - \sum_{i=0}^{n} \left(\int_t^T \vartheta_u^i(t) du \right) dW_t^i - \left(\int_t^T \left[\alpha_u(t) - \sum_{i=1}^{n} \vartheta_u^i(t) \int_t^u \vartheta_v^i(t) dv \right] du \right) dt \qquad \text{(EQ 37)}$$

If the drift coefficients $\alpha_T(t)$ satisfy the no-arbitrage conditions of Equation 24 for some set of market prices of risk $\lambda^i(t)$, then Equation 37 shows that in terms of the equivalent measure $d\overline{W}^i = dW^i + \lambda^i dt$, defined by the Brownian motions $\overline{W}^i{}_t = W^i{}_t + \int_0^t \lambda^i(u) du$, $i = 1, \ldots, n$, the dynamics of zero-coupon bond prices is:

$$\frac{dB_T(t)}{B_T(t)} = r(t)dt - \sum_{i=1}^{n}\left(\int_t^T \vartheta_u^i(t)du\right)d\overline{W}_t^i \qquad \textbf{(EQ 38)}$$

Therefore, $\{dW^i\ ;\ i = 1,...,n\}$ defines an equivalent martingale measure under which the rescaled bond prices $B_T(t)\exp\left(-\int_0^t r(u)du\right)$ for all maturities T are jointly martingale. Under this measure the interest rate contingent claims prices are independent of the market prices of risk and, hence, remain preference-free.

THE NO-ARBITRAGE CONDITION AND THE STOCHASTIC THEORY OF VOLATILITY

The standard option prices $C_{K,T}(t,S)$ are functionals of the local volatilities at time t and market level S, just as bond prices $B_T(t)$ are functionals of the forward rates at time t. As a result, the dynamical variations of the local volatility surface induce correpsonding dynamical variations of the standard option prices. During a time interval dt, the index price moves and the local volatilities also change. We can think of the local volatility changes as comprised of two components. A predictable component, due to movements of time and index price restricted to the static local volatility surface $\sigma_{K,T}(t,S)$ at time t and level S, and a non-predictable (stochastic) component due to dynamic fluctuations away from this surface. It is somewhat simpler, but entirely equivalent, to work with the transition probabilities, instead of option prices. The transition probability, $P_{K,T}(t,S)$, describes the total probability that the index price will reach level K at time T, given that the index price at time t is S, when both the index price and volatility are stochastic. It is related to the option prices $C_{K,T}(t,S)$ through a general and well-known[13] formula:

$$P_{K,T}(t,S) = e^{r(T-t)}\frac{\partial^2}{\partial K^2}C_{K,T}(t,S) \qquad \textbf{(EQ 39)}$$

The dynamical evolution of transition probabilities $P_{K,T}(t,S)$ based on the local volatility process of Equation 26 is given by the stochastic integral equation:

(EQ 40)

$$dP_{K,T} = \left[\left(\frac{\partial P_{K,T}}{\partial t} + \mu(t)S\frac{\partial P_{K,T}}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 P_{K,T}}{\partial S^2}\right)dt + \sigma(t)S\frac{\partial P_{K,T}}{\partial S}dW^0(t)\right]_{(t,S)} +$$

$$\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T'}}d\sigma^2_{K',T'}dK'dT' +$$

$$\frac{1}{2}\int_t^T\int_t^T\int_0^\infty\int_0^\infty \frac{\delta^2 P_{K,T}}{\delta\sigma^2_{K',T'}\delta\sigma^2_{K'',T''}}d\sigma^2_{K',T'}d\sigma^2_{K'',T''}dK'dK''dT'dT''$$

13. See Breeden and Litzenberger [1978].

All the probability and local volatility expressions in this equation are evaluated at *(t,S)*. The first term describes the effective dynamics of the transition probabilities $P_{K,T}(t,S)$ restricted to the fixed local volatility surface $\sigma_{K,T}(t,S)$, prevailing at time t and level S. The bracket symbol, $[\ldots]_{(t,S)}$, therefore, expresses the fact that in this term the future volatility is a deterministic function of the future time T and market level K, given by $\sigma_{K,T}(t,S)$ viewed as function of these two variables. The next two terms describe the dynamical variations of the transition probabilities resulting from the stochastic fluctuations of the local volatility surface during the next instant of time dt.

Contrary to Equation 32, in general there are no explicit expressions describing the functional relationship between option prices and local volatilities. Therefore, we can not directly compute the variational derivatives in Equation 40. Instead, we can look at the variations of the forward and backward transition probabilities with respect to the specific local volatilities. As shown in Appendix C and illustrated in Figure 8, the forward transition probability $p(t,S,t',S')$, associated with the non-random local volatility surface $\sigma_{K,T}(t,S)$ prevailing at time t and spot price S, has the following variational derivative with respect to the local volatility $\sigma_{v,u}(t,S)$ on the surface, corresponding to future maturity u and market level v:

$$\frac{\delta p(t, S, t', S')}{\delta\sigma^2_{v,u}} = \frac{1}{2}p(t, S, u, v)v^2\frac{\partial^2}{\partial v^2}p(u, v, t', S') \qquad \textbf{(EQ 41)}$$

This relation holds for any u in the range $t \le u \le t'$, otherwise the variational derivative is equal to zero. Similarly, the backward transition probability $\Phi(K,T,K',T')$ satisfies, for $T' \le u \le T$, the relation

$$\frac{\delta\Phi(K, T, K', T')}{\delta\sigma^2_{v,u}} = \frac{1}{2}\Phi(K, T, v, u)v^2\frac{\partial^2}{\partial v^2}\Phi(v, u, K', T') \qquad \textbf{(EQ 42)}$$

and zero otherwise. Using Equations 21 and 39, the standard option prices $C_{K,T}(t,S)$ and transition probabilities $P_{K,T}(t,S)$ satisfy similar relationships for $t \le u \le T$:

$$\frac{\delta C_{K,T}(t, S)}{\delta\sigma^2_{v,u}} = \frac{1}{2}\Phi(K, T, v, u)v^2\frac{\partial^2}{\partial v^2}C_{v,u}(t, S) \qquad \textbf{(EQ 43)}$$

and

$$\frac{\delta P_{K,T}(t, S)}{\delta\sigma^2_{v,u}} = \frac{1}{2}p(t, S, u, v)v^2\frac{\partial^2}{\partial v^2}p(u, v, T, K) \qquad \textbf{(EQ 44)}$$

in which the effective transition probabilities $p(\ldots)$ and $\Phi(\ldots)$ correspond to the static local volatility surface $\sigma_{K,T}(t,S)$ prevailing at time t and market level S. In arriving at Equations 43 and 44 we have also used the following identities:

$$P_{K,T}(t,S) = p(t,S,T,K) \quad \textbf{(EQ 45)}$$

$$p(t,S,T,K) = e^{r(T-t)}\frac{\partial^2}{\partial K^2}C_{K,T}(t,S) \quad \textbf{(EQ 46)}$$

$$\Phi(K,T,S,t) = e^{\delta(T-t)}\frac{\partial^2}{\partial S^2}C_{K,T}(t,S) \quad \textbf{(EQ 47)}$$

FIGURE 8. Sensitivity of the forward and backward transition probabilities *p(t,S,t',S')* and Φ*(K,T,K',T')* to the sudden changes of the local volatility $\sigma_{v,u}$.

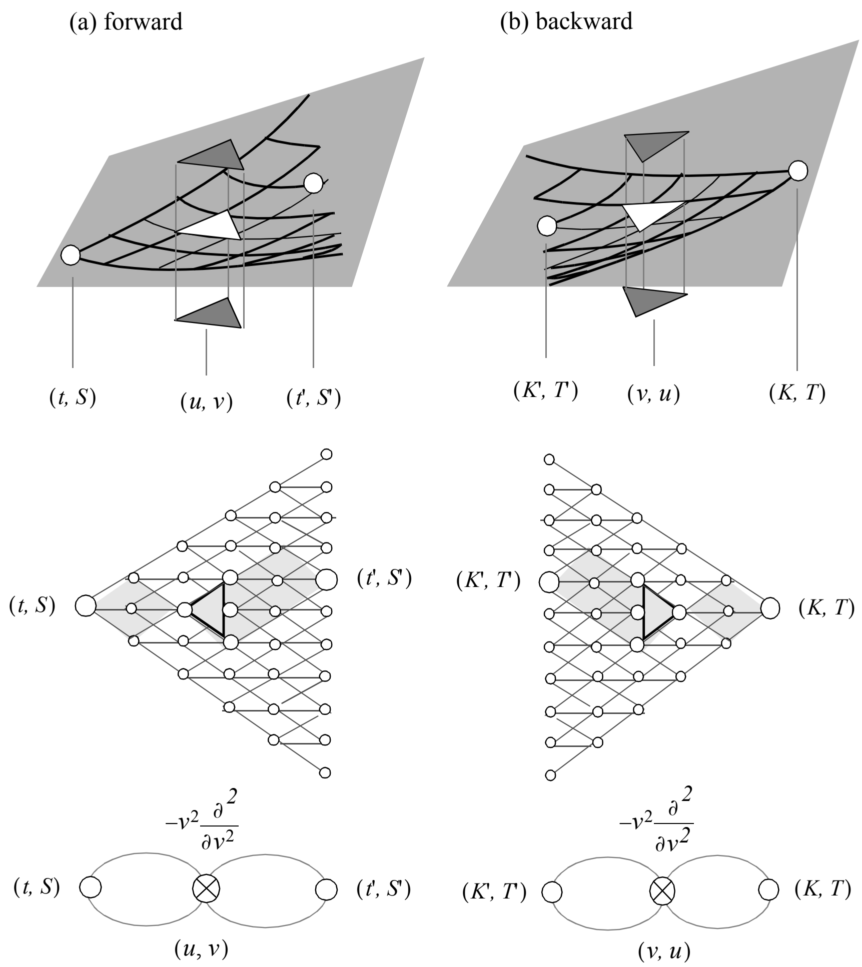
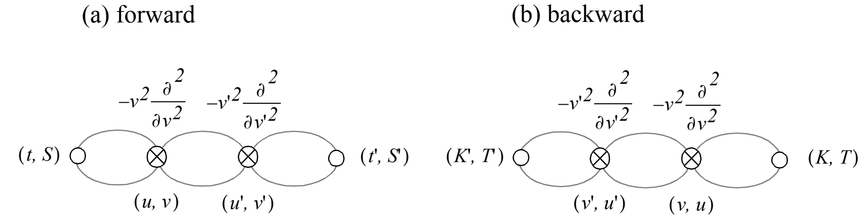


FIGURE 9. Second order variational derivatives of the forward and backward transition probabilities *p(t,S,t',S')* and $\Phi(K,T,K',T')$ with respect to the local volatilities.



As discussed in Appendix B, these identities are all consequences of the fact that the effective theory associated with $\sigma_{K,T}(t,S)$ embodies all the information necessary for pricing standard options of all strikes and maturities correctly.

Taking the variational derivatives of both sides of Equations 41 and 42 with respect to the local volatility $\sigma_{v',u'}$ we find the second order variational derivatives as

$$\frac{\delta p(t, S, t', S')}{\delta\sigma^2_{v,u}\delta\sigma^2_{v',u'}} = \frac{1}{4}p(t, S, u, v)v^2\frac{\partial^2}{\partial v^2}p(u, v, u', v')v'^2\frac{\partial^2}{\partial v'^2}p(u', v', t', S') \qquad \text{(EQ 48)}$$

for any $t \le u \le u' \le t'$, and

$$\frac{\delta\Phi(K, T, K', T')}{\delta\sigma^2_{v,u}\delta\sigma^2_{v',u'}} = \frac{1}{4}\Phi(K, T, v, u)v^2\frac{\partial^2}{\partial v^2}\Phi(v, u, v', u')v'^2\frac{\partial^2}{\partial v'^2}\Phi(v', u', K', T') \qquad \text{(EQ 49)}$$

for $T' \le u' \le u \le T$. Figure 9 gives a graphical depiction of these identities. The standard option prices $C_{K,T}(t,S)$ and transition probabilities $P_{K,T}(t,S)$ satisfy similar relationships for $t \le u \le u' \le T$:

$$\frac{\delta C_{K,T}(t, S)}{\delta\sigma^2_{v,u}\delta\sigma^2_{v',u'}} = \frac{1}{4}p(t, S, u, v)v^2\frac{\partial^2}{\partial v^2}p(u, v, u', v')v'^2\frac{\partial^2}{\partial v'^2}C_{v',u'}(t, S) \qquad \text{(EQ 50)}$$

$$\frac{\delta P_{K,T}(t, S)}{\delta\sigma^2_{v,u}\delta\sigma^2_{v',u'}} = \frac{1}{4}p(t, S, u, v)v^2\frac{\partial^2}{\partial v^2}p(u, v, u', v')v'^2\frac{\partial^2}{\partial v'^2}p(u', v', T, K) \qquad \text{(EQ 51)}$$

Using these relations, Appendix D proves that Equation 40 leads to

$$dP_{K,T} = \sigma(t)S\frac{\partial P_{K,T}}{\partial S}d\overline{W}^0 + \sum_{i=0}^{n}\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T'}}\sigma^2_{K',T'}\theta^i_{K',T'}dK'dT'd\overline{W}^i \qquad \textbf{(EQ 52)}$$

if and only if, for any *S, K* and $t \le T$, the drift functions $\alpha_{K,T}(t,S)$ satisfy the following *no-arbitrage conditions*

$$\alpha_{K,T}(t,S) = -\sum_{i=0}^{n}\theta^i_{K,T}(t,S)\left\{\frac{1}{p(t,S,T,K)}\int_t^T\int_0^\infty \theta^i_{K',T'}(t,S)p(t,S,T',K')K'^2\frac{\partial^2}{\partial K'^2}p(T',K',T,K)dK'dT' - \Pi^i\right\} \qquad \textbf{(EQ 53)}$$

where $\Pi^0 = 0$ and Π^i for $i = 1, ..., n$ are arbitrary but independent of *K* and *T*, and where the equivalent measure $\{\overline{W}^i\}$ is defined by

$$d\overline{W}^0 = dW^0 + \frac{(\mu(t) - r + \delta)}{\sigma(t)}dt \ ; \ d\overline{W}^i = dW^i + \Pi^i dt \qquad \textbf{(EQ 54)}$$

The quantities Π^i denote the *market prices of risk* associated with the volatility risk factors W^i, $i = 1, ..., n$, while μ - $(r$-$\delta)$ is the market price of risk associated with the index price risk factor W^0. Equation 52 shows that under the no-arbitrage conditions the measure $\{d\overline{W}^i ; i = 1, ..., n\}$ is an *equivalent martingale measure*, with respect to which the rescaled index price and rescaled option prices for all strikes and maturities are simultaneously martingales.

These no-arbitrage conditions in the present case are significantly more involved than the HJM no-arbitrage conditions described in the previous section. The basic reason is that local volatilities span a (two-dimensional) surface on which (forward and backward) propagation depends, in a rather complicated and non-linear manner, on the structure of local volatilities across the whole surface. This is evident by the apparent complexity of Equations 44 and 51 as compared to the simplicity of the corresponding Equations 33 and 36 in the interest rate framework. It is, therefore, rather difficult to use the no-arbitrage conditions for stochastic volatility in their continuous form directly.

In the next section we introduce *Stochastic Implied Trees* as a discrete-time framework for describing arbitrage-free stochastic variations of the local volatility surface.

StOCHASTIC IMPLIED TREES

Figure 11 gives a schematic illustration of the dynamics in a stochastic volatility theory. As the physical time moves forward, the index price changes and, simultaneously, all local volatilities on the volatility surface undergo multi-factor stochastic variations.

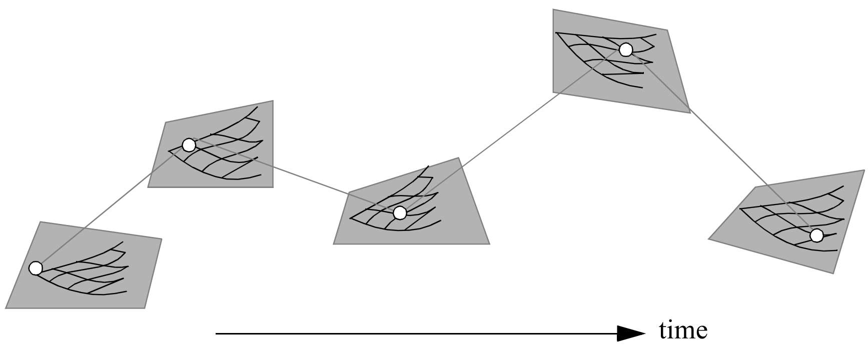
To provide a more quantitative description of this stochastic dynamics we choose to work within a discrete-time framework described by a *Stochastic*

Implied Tree. These trees are extensions of the standard (non-stochastic) implied trees, which are used to describe effective volatility models (see Derman, Kani and Chriss [1996]). Figure 12 shows an example of a 1-year, 5-period standard implied trinomial tree which is calibrated to a market where at-the-money implied volatility is 25% and there is an implied volatility skew of 0.5% point per 10 strike points. In an implied trinomial tree the location of the nodes, or the *state space*, is more or less arbitrarily. Once the state space is fixed, however, the transition probabilities at different nodes are determined from the requirement that standard options and forwards with strike prices coinciding with those nodes and maturing at different periods of the tree all have prices using the tree which match their market prices. Since local volatility at any node depends on the nodal levels and the transition probabilities to the nearby nodes, the local volatilities at different nodes are also determined in this way.

Stochastic implied trinomial trees are extensions of the implied trinomial tree in which the transition probabilities are, in addition, allowed to vary stochastically, with several stochastic factors, as time elapses and index level moves. The index level is allowed to move randomly from node to node, while the local volatilities, and simultaneously the transition probabilities corresponding to the future nodes, all vary stochastically across the tree. This behavior is shown in Figure 13.

Starting from any initial node, the possible future movements of the local volatility surface must be restricted to guarantee absence of any arbitrage opportunities in the discrete theory represented by the stochastic implied tree. As discussed earlier, this is equivalent to the requirement that the total transition probabilties to all future nodes be simultaneously martingales on the tree. This

FIGURE 11. Schematic illustration of the dynamics of the index price and local volatility surface in a stochastic volatility theory.



is also the same as the requirement that all rescaled standard option prices be simultaneously martingales on the tree. As Figure 14 shows, during the time interval Δt, the spot price will move randomly (by amount ΔS) to one of the nearby nodes and, at the same time, the local volatility surface will assume one of its N possible configurations, $w^1, ..., w^N$. As a result, the total transition probability $P_{K,T}(t,S)$ to any given future node (K,T) also moves to one of its several possible values $P^{(i)}_{K,T}(t+\Delta t, S+\Delta S)$, $i = 1, ..., M$, during this time interval. To guarantee no-arbitrage, $P_{K,T}$ must be a martingale (fair game), that is it must equal the expectation , under some (equivalent) measure, of its future values

FIGURE 12. Example of an Implied Trinomial Tree describing an effective volatility theory.

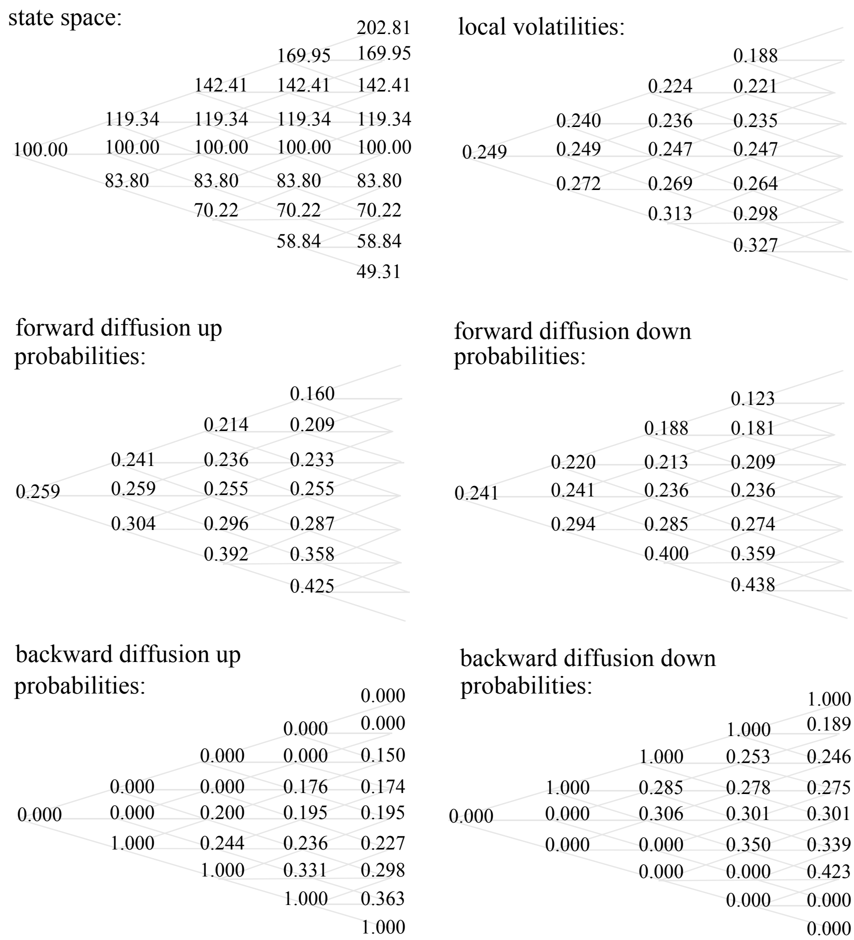


FIGURE 13. **In a Stochastic Implied Tree, as the index moves from node A to node B in a single time step, the local volatilities and transition probabilities, for every node on the future subtree beginning at node B, vary stochastically with multiple stochastic factors.**

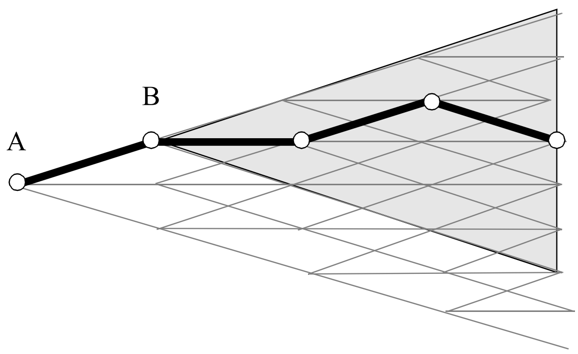
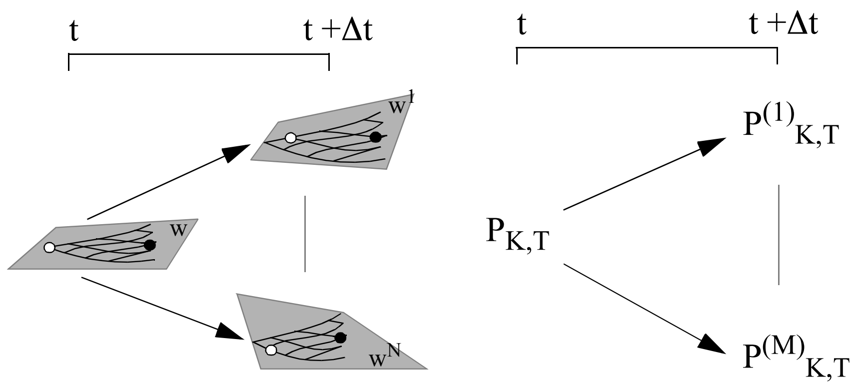


FIGURE 14. **During a time step Δt, the total transition probability $P_{K,T}$ will move to one of M values $P^{(i)}_{K,T}$, $i = 1, ...,M$, as index price moves randomly to one of the nearby nodes and the local volatility surface assumes one of N possible configurations.**



Our Notation in Discrete Time

$P^{(i)}_{K,T}$ for all the future nodes (K,T) on the tree.

To make positivity manifest, it is more convenient to redefine the drift and volatility functions in Equation 25 as $\alpha_{K,T} \to \alpha_{K,T}\sigma^2_{K,T}$ and $\theta^l_{K,T} \to \theta^l_{K,T}\sigma^2_{K,T}$, $l = 0,$

..., *n*, and begin by discretizing the following continuous-time differential equation:

$$\frac{d\sigma^2_{K,T}(t,S)}{\sigma^2_{K,T}(t,S)} = \alpha_{K,T}(t,S)dt + \sum_{l=0}^{n} \theta^l_{K,T}(t,S)dW^l_t \qquad \textbf{(EQ 55)}$$

We let the integer pair *(i,j)* label the node (t_i,S_j) describing the current location (i.e *(t,S)*) of the index at the i^{th} step of the simulation. We also let the pair *(n,m)* label the future node (t_n,S_m) corresponding the future time and level (i.e *(T,K)*). Then the discrete form of Equation 55 can be written as

$$\Delta\sigma^2_{m,n}(i,j) = \sigma^2_{m,n}(i,j)\left[\alpha_{m,n}(i,j)\Delta t_i + \sum_{l=0}^{n} \theta^l_{m,n}(i,j)\Delta W^l_i\right] \qquad \textbf{(EQ 56)}$$

The vector $(\Delta W^0_i, \Delta W^1_i, ..., \Delta W^n_i)$ is random and is drawn, at time *i*, from the sample space of the increments of *n* independent Brownian motions W^l.

The volatility parameters $\theta^l_{m,n}(i,j)$ are pre-specified but the drift parameters $\alpha_{m,n}(i,j)$ must be determined from the no-arbitrage requirements that the total probabilities $P_{m,n}(i,j)$ of arriving at the future node *(n,m)* from the (fixed) initial node *(i,j)* must be jointly martingales for all future nodes *(n,m)*. As we shall argue below, these martingale conditions are precisely enough to completely determine all the drift parameters step by step during the simulation process.

A Stochastic Implied Tree simulation begins with the construction of a trinomial implied tree calibrated to today's prices of standard options and forwards. The simulation begins at the node *(0,0)* of this tree. During the first simulation step the drift parameters $\alpha_{m,n}(0,0)$, for all future nodes *(m,n)*, are determined from the martingale conditions on the total probabilities $P_{m,n}(0,0)$. Figure 15 illustrates that the drift parameter $\alpha_{0,0}(0,0)$ is determined from the martingale condition for $P_{1,2}(0,0)$ This also guarantees that the transition probabilities $P_{1,1}(0,0)$ and $P_{1,0}(0,0)$ are martingales. The reason is that these probabilities are constrained by two extra conditions which must hold irrespective of the specific behavior of the local volatilities:

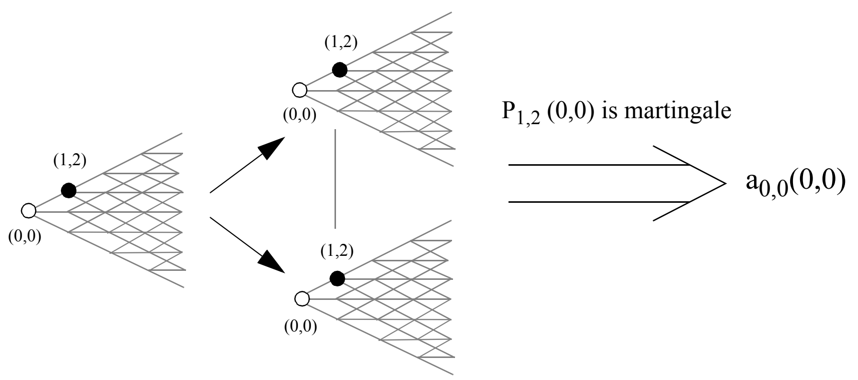
$$P_{1,0}(0,0) + P_{1,1}(0,0) + P_{1,2}(0,0) = 1 \qquad \textbf{(EQ 57)}$$

$$P_{1,0}(0,0)S_{1,0} + P_{1,1}(0,0)S_{1,1} + P_{1,2}(0,0)S_{1,2} = S_{0,0}e^{(r-\delta)(t_1-t_0)}$$

The first condition is the normalization condition, requiring that the sum of the three total transition probabilities at time t_1 must be unity. The second is the forward condition, requiring that the t_1-maturity forward price at time t_0 must match its risk-neutral value.

In a similar way, the three drift parameters $\alpha_{1,2}(0,0)$, $\alpha_{1,1}(0,0)$ and $\alpha_{1,0}(0,0)$ are determined from the martingale conditions of the three total transition probabilities $P_{2,4}(0,0)$, $P_{2,3}(0,0)$ and $P_{2,2}(0,0)$. The remaining transition probabilities $P_{2,1}(0,0)$ and $P_{2,0}(0,0)$ will then also be martingales due to the normalization and forward conditions at time t_2. In this way all drift parameters $\alpha_{m,n}(0,0)$ will be determined during the first simulation step. Finally, to complete this step we draw a random vector $(\Delta W_0^0, \Delta W_0^1, ..., \Delta W_0^n)$ from the sample space of the increments of W^i at time t_0, and use this vector to simultaneously arrive at a (random) new location for the index price and new values for all future local volatilities. Equation 56 is used directly with $i = j = 0$ to calculate the new local volatility values from this choice of the random vector. As for the index price, we use the random number ΔW_0^0 to determine which of the three possible future nodes (i.e *(1,2)*, *(1,1)* or *(1,0)*) does the index price moves to during time interval Δt. Figure 16 gives one simple possible method for doing this starting from an arbitrary initial node *(i,j)*. First ΔW_i^0 is renormalized to represent a uniformly-distributed random number between *0* and *1*. Let $P_u(i,j)$, $P_m(i,j)$ and $P_d(i,j)$ denote the one period transition probabilities, prevailing at time t_i and index price S_j, from the node *(i,j)* to the up, middle and down nodes at time t_{i+1}. We then compare our random number with these three probabilities. If it is smaller than $P_d(i,j)$, we move the index price to the down node. On the other hand, if the random number is greater than the sum $P_u(i,j) + P_m(i,j)$, we allow the index price to move to the up node. In every other case we move the index price to the middle node at the next time period.

FIGURE 15. The drift parameter $\alpha_{0,0}(0,0)$ in a Stochastic Implied Tree is determined from the martingale condition on the total transition probability $P_{1,2}(0,0)$.



We can continue this procedure, step-by-step, for any point *(i,j)* along a simulated path through the stochastic implied tree. First, all the drift parameters $\alpha_{m,n}(i,j)$ are determined from the martingale conditions on $P_{m,n}(i,j)$. Appendix E gives the necessary details for doing this calculation. Subsequently, these drift parameters are used to generate arbitrage-free (random) movements of the future local volatility surface as the index price moves randomly forward across the tree. We can generate many such sample paths through the tree. Along each path, the movements of the index price and the local volatility surface are random realizations of an arbitrage-free dynamics, which step-by-step guarantees absence of arbitrage opportunities among different standard option (and forward) contracts and their underlying index within the discrete time framework of the stochastic implied tree.

A SIMPLE EXAMPLE

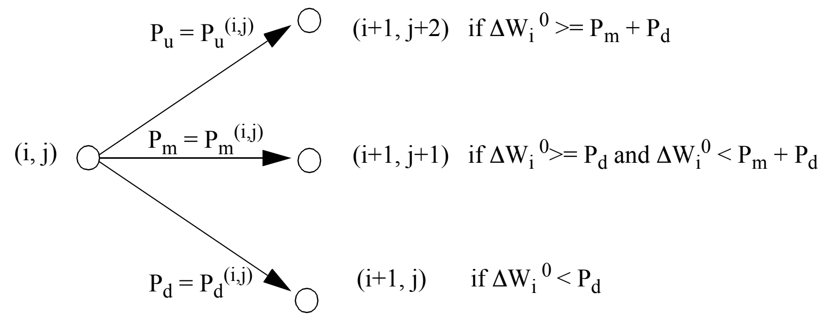
Consider a one-factor stochastic volatility model with a lognormal volatility of volatility structure, as described by the following pair of stochastic differential equations:

$$\frac{dS}{S} = \mu dt + \sigma dW^0$$

$$\frac{d\sigma^2_{K,T}}{\sigma^2_{K,T}} = \alpha_{K,T}dt + \theta dW^1$$

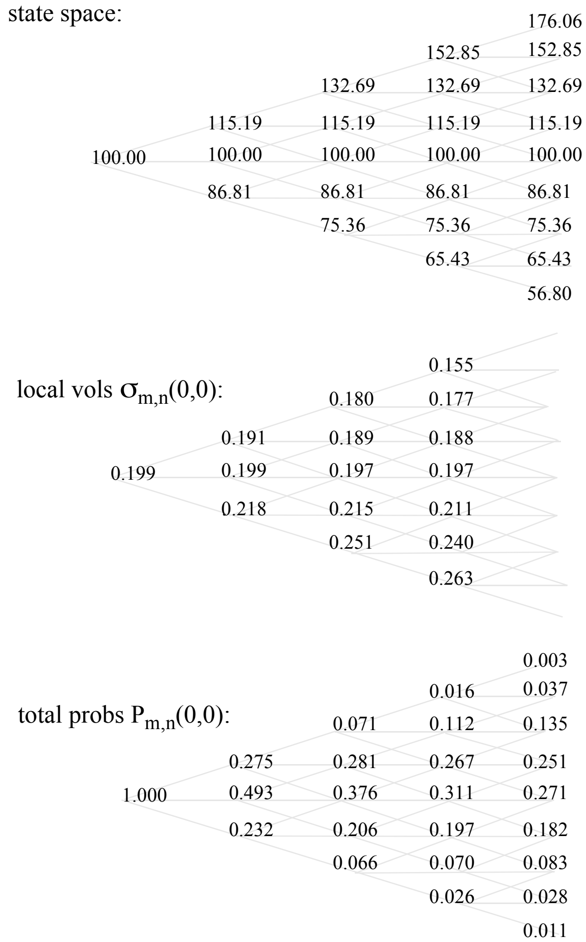
where $\sigma(t) = \sigma_{t,S_t}(t, S_t)$. For the purpose of this example we take the volatility coefficient θ to be constant, so that the factor W^1 has the interpretation of a simultaneous constant (proportional) shift in all local volatilities. All the other quantities can depend on *t*, *S*, factors W^0 and W^1 or their past values. More specifically, we consider a 1-year, 5-period example with the initial term and strike structure of volatility given by an at-the-money implied volatility of 20% and a constant skewness of 0.5% per 10 strike points. For instance, initially a 80 strike option of any maturity has implied volatility of 21%. Let the riskfree discount rate be equal to 10%, dividend yield 5% and the volatility (of volatility) param-

FIGURE 16. Determining which node the index price will go to during one simulation step using the renormalized random number ΔW_i^0.



eter $\theta = 30\%$. We choose the state space of the stochastic implied trinomial tree to be the same as a standard (CRR- type) trinomial tree with constant volatility of 20%. Figure 17 shows this state space. It also shows the local volatilities and total transition probabilities, corresponding to various nodes of this tree, at the initial time $t = 0$. As we expect, local volatilities increase as the index level decreases roughly twice as fast as implied volatilities. Also the probability distribution is skewed (around the forward price) towards the lower index levels. The first step toward the construction of the stochastic implied tree is to determine the drift coefficients $\alpha_{m,n}(0,0)$ at time $t_0 = 0$. Appendix E gives the formulas for directly calculating these coefficients, which are shown in Figure 18.

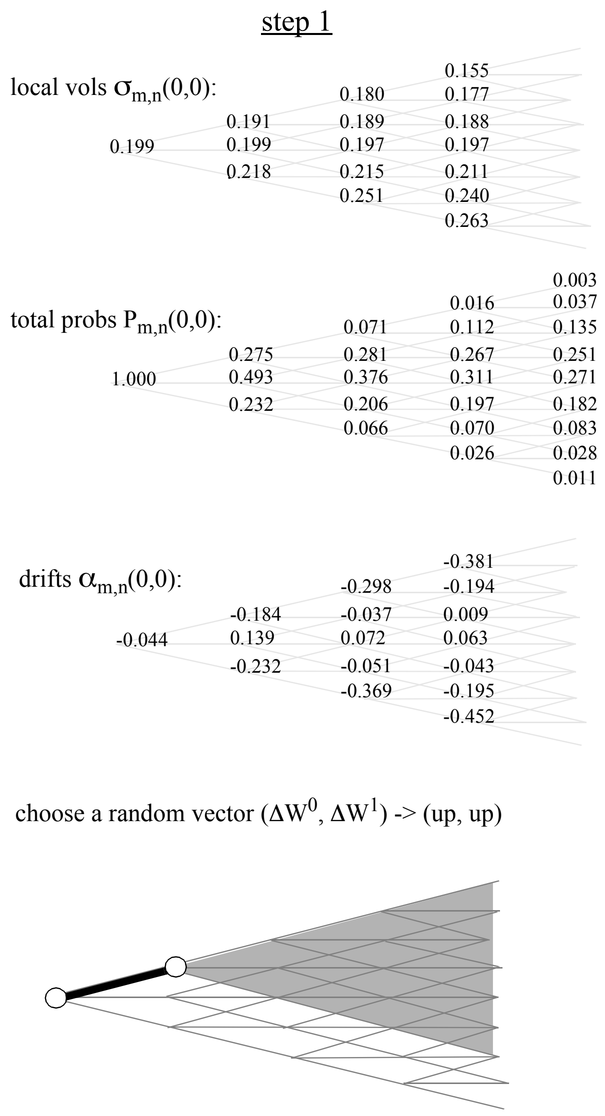
FIGURE 17. The state space of a Stochastic Implied Trinomial Tree, the local volatility surface and the total transition probability distribution on the tree at the initial time *t = 0*.



We can justify the numbers by examining what can happen to the total transition probabilities during the next time interval Δt. All local volatilities will simultaneously move, with probability of 1/2, to their up values, $\sigma^{(u)}{}_{m,n}(0,0)$, or their down values, $\sigma^{(d)}{}_{m,n}(0,0)$, as given by

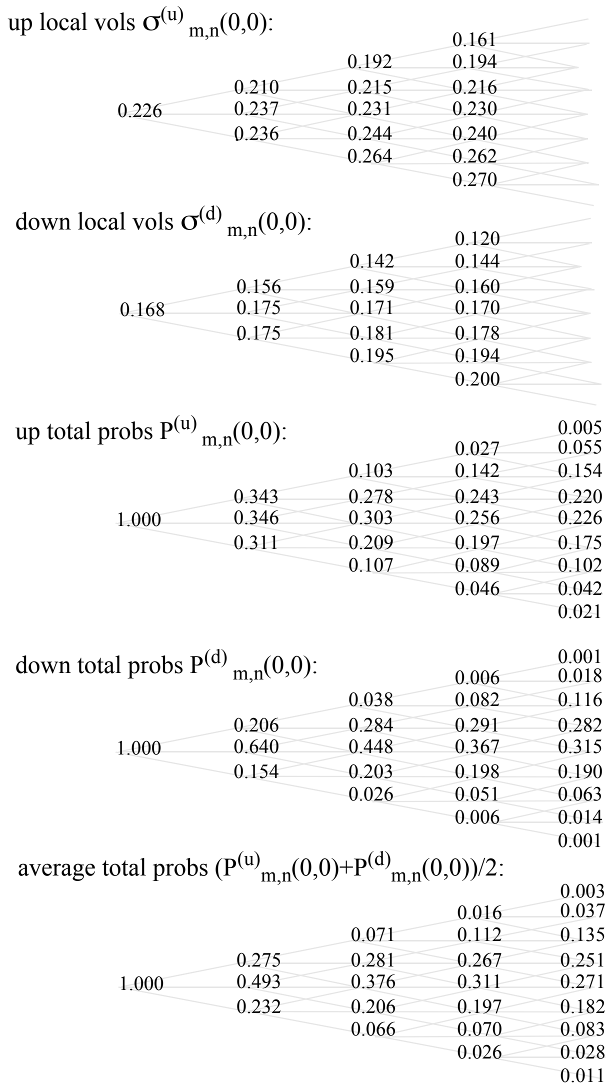
$$\sigma^{(u,d)}{}_{m,n}(0,0) = \sigma_{m,n}(0,0)\exp\left\{\left(\alpha_{m,n}(0,0)-\frac{1}{2}\theta^2\right)\Delta t \pm \theta\sqrt{\Delta t}\right\}$$

FIGURE 18. The first step of the Stochastic Implied Tree construction consists of determining all the drift coefficients $\alpha_{m,n}(0,0)$, at time $t_0 = 0$, from the martingale conditions for the total probabilities $P_{m,n}(0,0)$.



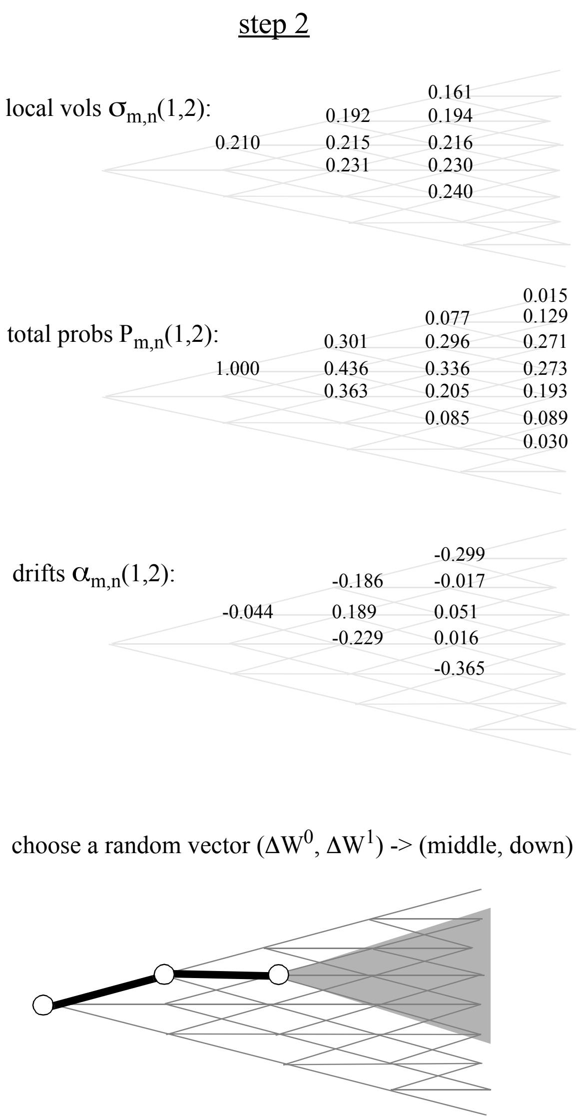
with $\alpha_{m,n}(0,0)$ given in Figure 18. As a result, all transition probabilities also change across the tree, simultaneously moving to their up values, $P^{(u)}{}_{m,n}(0,0)$, or to their down values, $P^{(d)}{}_{m,n}(0,0)$, each with probability of 1/2. Figure 19 shows that with the present choice of drift coefficients the initial total probabilities are precisely equal to the average value of their up and down values. To complete the step 1 we draw a pair of independent random numbers between *0* and *1*, say (0.853, 0.612). Since 0.853 is greater than the sum of prevailing down and middle probabilities, 0.493+0.232 = 0.725, as discussed in Figure 16 we move the index to the node *(1,2)*. Also, since 0.612 is greater than 1/2 we

FIGURE 19. Up- and down- values of local volatilities and total transition probabilities corresponding the first simulation step.



move all local volatilities to their up values, before we begin the next simulation step. The step 2 of the simulation is precisely the same as step 1, except confined to the subtree that begins at the node *(1,2)*. As shown in Figure 20, again the martingale conditions on the total probabilities $P_{m,n}(1,2)$ are used to solve for the drift coefficients $\alpha_{m,n}(1,2)$ at time $t_1 = 0.25$, and then these coefficients, together with a pair of random numbers, are used to determine jointly the new values for the index price and the future local volatilities. Steps 3 and 4 are also

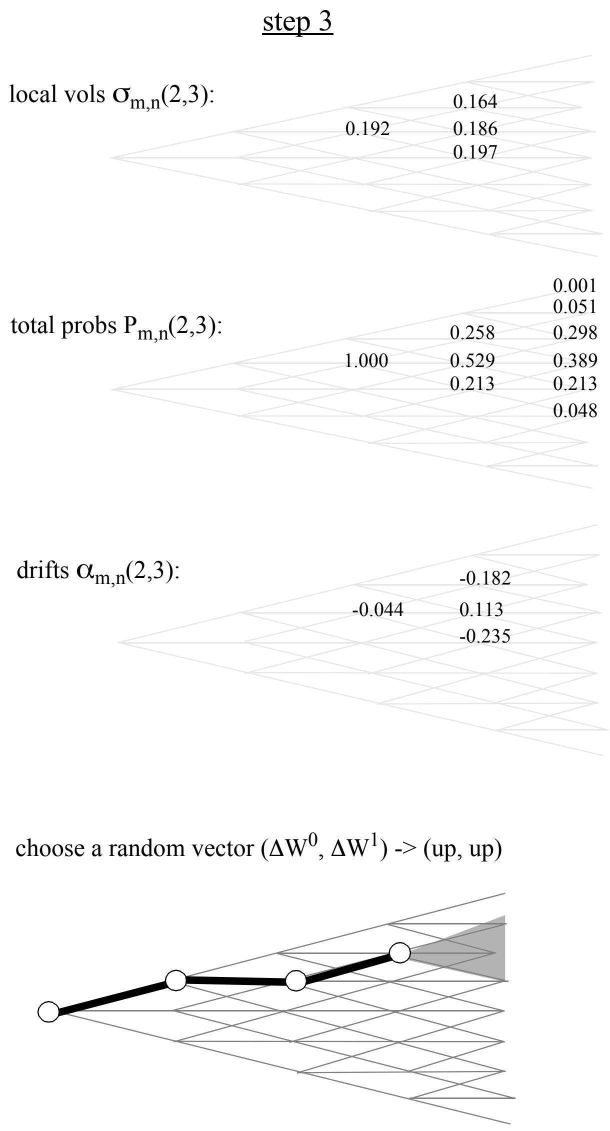
FIGURE 20. During the step 2 of the simulation, the drift coefficients $\alpha_{m,n}(1,2)$, at time $t_1 = 0.25$, are determined from the martingale conditions for the total probabilities $P_{m,n}(1,2)$.



quite similar and their results have been shown in Figures 21 and 22, repsectively.

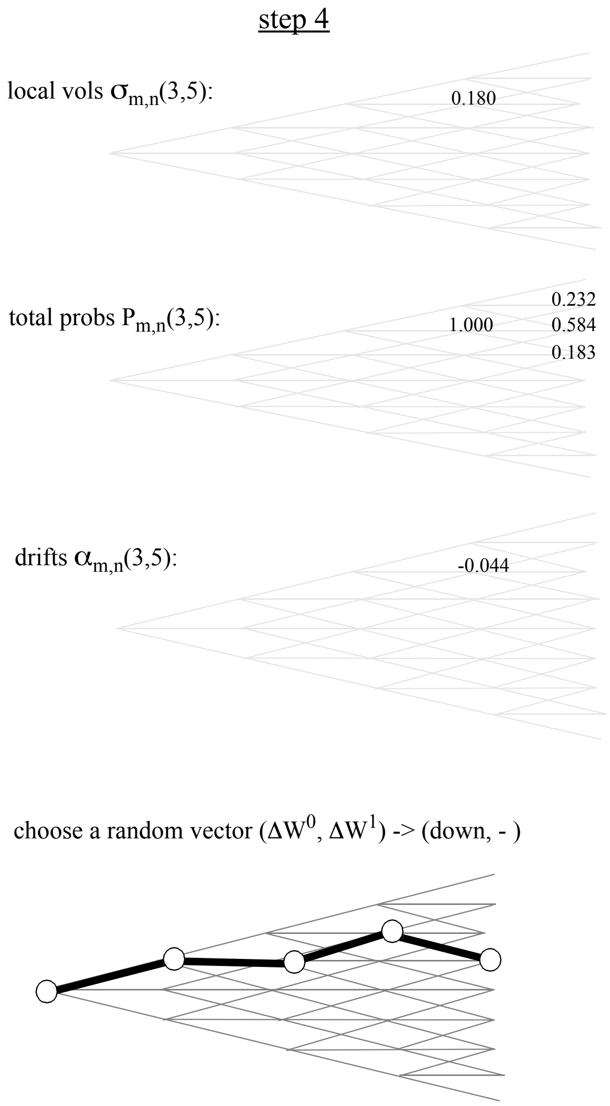
In this example, we chose a simple two-state (up and down) representation for the stochastic movements of the local volatility surface during the time step Δt.

FIGURE 21. During the step 3 of the simulation, the drift coefficients $\alpha_{m,n}(2,3)$, at time $t_2 = 0.50$, are determined from the martingale conditions for the total probabilities $P_{m,n}(2,3)$.



We could instead choose any equivalent representation of the same process with m states, for any integer $m > 1$. There are infinite number of equivalent representations for any choice of m. If the model is well-behaved, these discrete representations should all converge to the same continuous-time process as Δt goes to zero. However, a representation with large number of states may converge substantially faster than the two-state representation we chose here. Table 1

FIGURE 22. During the step 4 of the simulation, the drift coefficients $\alpha_{m,n}(3,5)$, at time $t_3 = 0.75$, are determined from the martingale conditions for the total probabilities $P_{m,n}(3,5)$.



shows the calibration results for a 50000 path simulation on the 5-period tree described above.

TABLE 1. Calibration results of a 50000-path simulation on a 1-year, 5-period Stochastic Implied Tree.

Strike Price	Option Type	Black-Scholes Price	Standard Implied Tree Price	Stochastic Implied Tree Price
130	call	1.142	1.118	1.176
120	call	2.629	2.764	2.775
110	call	5.332	5.529	5.556
100	call	9.628	9.395	9.432
90	put	2.452	2.566	2.556
80	put	0.840	0.936	0.928
70	put	0.202	0.244	0.230

The fourth and fifth columns give, respectively, the standard (non-stochastic) implied trinomial tree and the Stochastic Implied Tree results for a series of standard European-style call and put options used to calibrate the trees. The results are seen to agree well.

CAVIAT: Since the location of the nodes (i.e the state space) of the stochastic implied trinomial tree is fixed throughout, it may not be possible to fit very large local volatilities, which may occur at various nodes and at different times during the simulation, with transition probabilities which lie between *0* and *1*. In such cases, we must *overwrite* the unacceptable transition probabilities (or, equivalently, the local volatilities) at those nodes[14]. Even though, this overwrite procedure makes for an imperfect calibration to the initial smile (and, theoretically, a violation of arbitrage), it must be diligently adhered to, in order to keep the simulation process meaningful. We can define overwrite ratio as the number of overwrites per future node, per simulation path. In the previous example, the overwrite ratio for 5 periods and 50000 paths is found to be 2.7%, indicating

14. This also occurs in the standard implied trees. See, for instance, Derman, Kani and Chriss [1996].

that only a relatively small portion of the calculated local volatilities have been overwritten.

Pricing of Some Contracts with Payoffs Based on Realized Volatility

Consider a *realized variance forward contract*[15], defined as a forward contract on the realized variance of index returns, Σ^2, with strike price K and payoff $(\Sigma^2 - K)$ at the contract maturity. Table 2 shows the valuation results for a 1-year realized variance contract with zero strike price, using 20-period, 10000 path stochastic implied tree simulations with four different volatility of volatility parameters θ = 0%, 20%, 30%, 50%. To make the results more clear, we choose a flat initial volatility smile with a constant implied volatility of 20% for all standard European options. Also the discount rate and dividend yield are both chosen to be zero.

TABLE 2. **Prices of a zero-strike realized variance forward contract for different values of the volatility of volatility parameter.**

θ	0%	20%	30%	50%
price	399.81	400.37	401.10	400.69

It is clear from this table that the price of a realized variance forward contract is independent of the volatility of volatility parameter, and is what one would expect from a *static* 20% flat initial implied volatility surface. In fact, it can be shown that under very general conditions (see footnote 14) the price of this forward contract depends only on the initial volatility surface and not on the specific stochastic aspects of the volatility process. More precisely, it's price equals the discounted value of the expected (equilibrium) total index return variance during the life of the contract. As discussed earlier, this expectation is fully embodies in today's local volatility surface. Therefore, we are able to price this forward contract by using an effective theory ($\theta = 0$), as the second column in the table indicates. This is quite analogous to our ability to price index forwards contracts using the static initial forward curve without any specific knowledge of the stochastic behavior of the future index prices, or to price straight bonds using the initial yield curve with no specific knowledge of the behavior of future interest rates.

Now consider a *realized variance (call) option contract* with strike price K whose payoff at maturity is given by $Max(\Sigma^2 - K, 0)$. Table 3 shows the valua-

15. See also *Investing in Volatility*, Derman *etal.* [1996].

tion results for 1-year realized variance call options with different strike prices, under precisely the same conditions as before.

TABLE 3. Prices of realized variance call option contracts with different strike prices and volatility of volatility parameters.

	0%	20%	30%	50%
400	0.00	48.336	65.784	95.742
500	0.00	14.745	31.221	56.096
600	0.00	3.391	11.780	25.211
700	0.00	0.203	1.682	4.654

According to this table the price of a realized variance option contract increases with the volatility of volatility parameter. This result should be expected as most options' prices increase when their underlyer price becomes more volatile. Furthermore, like most options, the pricing and hedging of a realized variance option contract depends crucially on our choice of the stochastic volatility model.

HEDGING THE INDEX AND VOLATILITY RISKS IN STOCHASTIC VOLATILITY MODELS

Appendix D gives conditions for the existence and uniqueness of an equivalent martingale measure in multi-factor stochastic volatility models. Under these conditions the markets are complete and, given a contingent claim *C,* there exists an admissible self-financing trading strategy (Harrison and Pliska [1981]) involving the index *S*, the money market account *B,* and (any) *n* different standard options $C_i = C_{K_i, T_i}$, $i = 1,\ldots,n,$ which replicate this contingent claim:

$$N_B B + N_S S + \sum_{i=1}^{n} N_i C_i = C \quad a.e.$$

This replication strategy is dynamical, so that the hedge ratios N_B, N_S and N_i, $i = 1, \ldots, n,$ are in general functions of time and other dynamical variables.

To find the hedge ratios N_S and N_i we must separately move the index price *S* and introduce *n* independent shocks W^i (possibly corresponding to the *n* independent factors) to the initial local volatility surface, and subsequently reprice the contingent claim and the *n* hedge standard options. For the simple model in our examples, we find these hedge ratios from solving the following system of equations

$$\begin{bmatrix} 1 & \partial C_1/\partial S \\ 0 & \partial C_1/\partial W^1 \end{bmatrix} \begin{bmatrix} N_S \\ N_1 \end{bmatrix} = \begin{bmatrix} \partial C/\partial S \\ \partial C/\partial W^1 \end{bmatrix}$$

This system has a unique solution if the sensitivity matrix on the right hand side is non-singular. This is true if $\partial C_1/\partial W^1 > 0$, i.e when the sensitivity of the option to a parallel shift in the local volatility surface is positive. This condition holds for any standard option with non-zero strike price.

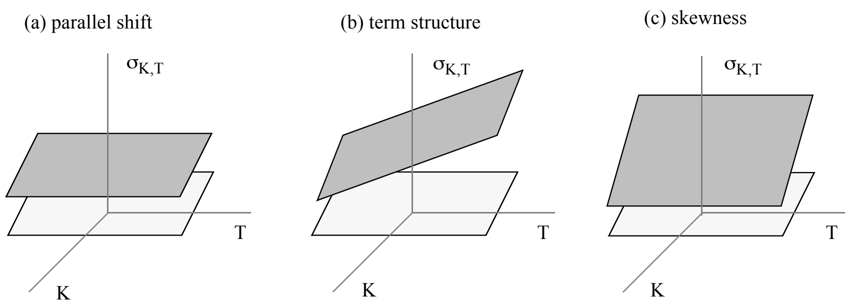
MORE REALISTIC STOCHASTIC VOLATILITY MODELS

In the previous example, all instantaneous changes of the volatility surface are caused by a single source of randomness, a *parallel shift factor*. However, empirical analyses[16] of the daily changes of the volatility surface for index options reveals other important sources of randomness. A more realistic stochastic volatility model may be, for example, given by the following pair of stochastic differential equations:

$$\frac{dS}{S} = \mu dt + \sigma dW^0$$

$$\frac{d\sigma^2_{K,T}(t,S)}{\sigma^2_{K,T}(t,S)} = \alpha_{K,T}(t,S)dt + \theta_1 dW^1 + \theta_2 e^{-\lambda(T-t)}dW^2 + \theta_3 e^{-\eta(K-S)}dW^3$$

where $\sigma(t) = \sigma_{t,S_t}(t, S_t)$ and $\theta_1, \theta_2, \theta_3, \lambda, \eta$ are all constants. The first factor has the same interpretation as before, whereas the second and third factors may be interpreted as the *term structure factor* and *skewness factor*, respectively. The shocks to the local volatility surface resulting from these factors are shown in the figure below:



16. To be presented in an upcoming Quantitative Strategies Research Note.

SUMMARY

In this paper we discussed an arbitrage pricing approach to contingent claims valuation with stochastic volatility similar to the Heath-Jarrow-Morton (HJM) methodology for stochastic interest rates. We began with a continuous time economy with multiple factors, and posited a general multi-factor continuous time stochastic process for the evolution of the local volatility surface. We characterized the conditions which guarantee absence of arbitrage opportunities among the various option and forward contracts defined on the underlying index. Under these conditions markets are complete and contingent claim valuation is preference-free. However, these no-arbitrage conditions are non-linear and difficult to use in their continuous form. We then introduced the Stochastic Implied Tree as a discrete-time framework for implementing our family of models. Starting from any initial node, we can guarantee absence of future arbitrages by choosing appropriate drift parameters for every future node. This procedure guarantees arbitrage-free future movements of the index and local volatility surface in the discrete-time world defined by the stochastic implied tree. We can use Stochastic Implied Trees to price contingent claims with payoffs which depend on the index and index volatility, when the volatility surface is skewed and stochastic. The resulting contingent claim prices are independent of the market prices of risk. They are also consistent with the current market prices of all standard options and forwards defined on the underlying index and with the absence of any future arbitrage opportunities.

APPENDIX A: Expectation Definitions of Local Volatility

This appendix provides expectation definitions for local volatility. It begins with Equation 2 as the definition of local volatility and derives the expectation relationships between local volatilities and future instantaneous volatilities, given the assumption that the equivalent martingale measure exists.

Under the equivalent martingale measure, the index price evolution is given by the stochastic differential equation

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma_t dZ_t \qquad \textbf{(EQ 58)}$$

where the riskless discount rate r and continuously compounded dividend yield δ are assumed to be constant, σ_t is the instantaneous volatility at time t and $Z = \overline{W}^0$ is a standard Brownian motion under this measure, as discussed in the text. Let $E = E_{t,S}$ denote the expectation corresponding to this measure, given the information available at time t (with the index level at $S = S_t$). Aside from the spot index level, the information at time t generally includes the past index levels, the values of the n (independent) stochastic factors W^i (governing the stochastic behavior of volatility) and their past histories. Under the equivalent martingale measure rescaled option prices are martingales. Therefore, the price,

$C_{K,T}(t,S)$, at time t and market level S of a standard European-style (call) option, with strike K and maturity T, with terminal value $C_{K,T}(T,S_T) = (S_T - K)^+$ is given by

$$C_{K,T}(t,S) = e^{-r(T-t)}E[(S_T - K)^+] \tag{EQ 59}$$

Differentiating this equation once with respect to K gives

$$\frac{\partial C_{K,T}}{\partial K} = -e^{-r(T-t)}E[\theta(S_T - K)] \tag{EQ 60}$$

where θ*(.)* is the *Heaviside* function. Differentiating twice with respect to K gives

$$\frac{\partial^2 C_{K,T}}{\partial K^2} = e^{-r(T-t)}E[\delta(S_T - K)] \tag{EQ 61}$$

where δ*(.)* is the *Dirac delta* function. Lastly, differentiating with respect to T gives

$$\frac{\partial C_{K,T}}{\partial T} = -rC_{K,T} + e^{-r(T-t)}\frac{\partial}{\partial T}E[(S_T - K)^+] \tag{EQ 62}$$

A formal application of Ito's lemma to the option's terminal payoff leads to the identity

$$d(S_T - K)^+ = \theta(S_T - K)dS_T + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S_T - K)dT \tag{EQ 63}$$

Taking Expectations of both sides of this equation and using Equation 58 leads to

$$dE[(S_T - K)^+] = (r - \delta)E[S_T\theta(S_T - K)]dT + \frac{1}{2}E[\sigma_T^2 S_T^2 \delta(S_T - K)]dT \tag{EQ 64}$$

The first term in this expression can be rewritten as

$$E[S_T\theta(S_T - K)] = E[(S_T - K)^+] + KE[\theta(S_T - K)] \tag{EQ 65}$$

Inserting this relation and multiplying both sides of Equation 64 by the discount factor $e^{-r(T-t)}$, and using Equations 59 and 60 we obtain

$$e^{-r(T-t)}\frac{\partial}{\partial T}E[(S_T - K)^+] = (r - \delta)\left\{C_{K,T} - K\frac{\partial C_{K,T}}{\partial K}\right\} + \frac{1}{2}e^{-r(T-t)}K^2 E[\sigma_T^2 \delta(S_T - K)] \tag{EQ 66}$$

Replacing this expression for the last term in Equation 62 gives

$$\frac{\partial C_{K,T}}{\partial T} = -(r-\delta)K\frac{\partial C_{K,T}}{\partial K} - \delta C_{K,T} + \frac{1}{2}e^{-r(T-t)}K^2 E[\sigma_T^2 \delta(S_T - K)] \qquad \text{(EQ 67)}$$

Finally, using conditional expectations we can write

$$E[\sigma_T^2 \delta(S_T - K)] = E[\sigma_T^2 | S_T = K] \cdot E[\delta(S_T - K)] \qquad \text{(EQ 68)}$$

which together with Equation 61, can be inserted back into Equation 67 to arrive at

$$E[\sigma_T^2 | S_T = K] = 2\frac{\left\{\frac{\partial C_{K,T}}{\partial T} + (r-\delta)K\frac{\partial C_{K,T}}{\partial K} + \delta C_{K,T}\right\}}{K^2 \frac{\partial^2 C_{K,T}}{\partial K^2}} \qquad \text{(EQ 69)}$$

The right hand side of this equation is precisely the definition of the local variance $\sigma^2_{K,T}(t,S)$, as defined in Equation 2. It follows that

$$\sigma^2_{K,T}(t,S) = E[\sigma_T^2 | S_T = K] = \frac{E[\sigma_T^2 \delta(S_T - K)]}{E[\delta(S_T - K)]} \qquad \text{(EQ 70)}$$

The local variance $\sigma^2_{K,T}(t,S)$ is, therefore, the conditional expectation of the instantaneous variance of index returns at the future time *T*, contingent on index level S_T being equal to *K*. If the instantaneous index volatility is only a function of the spot index level and time, i.e. if $\sigma_t = \sigma(S_t, t)$, then

$$\sigma^2_{K,T}(t,S) = E[\sigma_T^2 | S_T = K] = E[\sigma^2(S_T, T) | S_T = K] = \sigma^2(K,T) \qquad \text{(EQ 71)}$$

Since the right hand side is independent of *t* and *S*, in this case the local volatility surface remains *static* as time *t* evolves and index level *S* changes. This situation corresponds to an *effective theory* where all sources of volatility uncertainty, other than the future time *t* and the future index level *S*, are effectively averaged out of the theory, leaving an effective volatility which is only a function of *t* and *S*.

In the general stochastic setting the dynamics of local volatilities is described by the stochastic differential equation

$$\frac{d\sigma^2_{K,T}(t,S)}{\sigma^2_{K,T}(t,S)} = \alpha_{K,T}(t,S)dt + \sum_{i=0}^{n} \theta^i_{K,T}(t,S)d\overline{W}^i_t \qquad \text{(EQ 72)}$$

As discussed in the text, $\overline{W}^i$, $i = 0, ..., n$, are independent Brownian motions under the equivalent martingale measure and the volatility coefficients $\theta^i_{K,T}$ are some given functions of time t, index level S and factor values W^i_t, or the past histories of these variables. The drift coefficients $\alpha_{K,T}(t,S)$ have similar dependencies, but are constrained by the requirement of no-arbitrage. In an effective theory $d\sigma^2_{K,T}(t,S) = 0$ for all values of t and S, as seen by Equation 71, thus the drift and volatility coefficients are all identically equal to zero.

The denominator on the right hand side of Equation 70 is the total transition probability $P_{K,T}(t,S)$ (see Equation 39):

$$P_{K,T}(t,S) = E[\delta(S_T - K)] \tag{EQ 73}$$

Since $P_{K,T}$, for all values of K and T, are jointly martingales under the equivalent martingale measure, the stochastic differential equation governing their evolution has the form

$$\frac{dP_{K,T}(t,S)}{P_{K,T}(t,S)} = \sum_{i=0}^{n} \zeta^i_{K,T}(t,S)d\overline{W}^i_t \tag{EQ 74}$$

The numerator on the right hand side of Equation 70 is also a martingale under this measure. Therefore, by taking differentials of both sides of this equation and applying Ito's lemma we find that

$$\alpha_{K,T} + \sum_{i=0}^{n} \zeta^i_{K,T}\theta^i_{K,T} = 0 \tag{EQ 75}$$

Let us define a new measure, $d\hat{W}^i = d\overline{W}^i - \zeta^i_{K,T}dt$, specifically depending on K and T. From Equations 72 and 75 we observe that under this new measure local variance $\sigma^2_{K,T}$ is a martingale, i.e

$$\frac{d\sigma^2_{K,T}(t,S)}{\sigma^2_{K,T}(t,S)} = \sum_{i=0}^{n} \theta^i_{K,T}(t,S)d\hat{W}^i_t \tag{EQ 76}$$

We call this measure the *K-strike and T-maturity forward risk-adjusted measure* in analogy with T-maturity forward risk-neutral measure in interest rates (see Jamshidian [1993]). Letting $E^{(K,T)}[\ldots]$ denote expectations with respect to this new measure, we can write Equation 70 as

$$\sigma^2_{K,T} = E^{(K,T)}[\sigma_T^2] \tag{EQ 77}$$

Therefore, in the K-strike and T-maturity forward risk-adjusted measure the local variance $\sigma^2_{K,T}$ is the expectation of future spot variances σ_T^2 at time T. This is analogous to the similar situation in interest rate world where the forward rate f_T is the T-maturity forward risk-adjusted expectation of the future spot rates at time T.

APPENDIX B: Mathematics of Effective Theories

In this appendix we describe several mathematical relationships satisfied by the propagators, transition probabilities and the standard option prices in the effective volatility theories.

The forward propagator $p_{t, S, t', S'}$ in an effective volatility theory describes the (standard) option price evolution forward in time and index price. It satisfies the same backward equation as option prices, a dual forward equation and the boundary condition $p_{t, S, t, S'} = \delta(S - S')$ for all time t. Alternatively we can work with the forward transition probability density function, $p(t, S, t', S')$, which is defined in terms of the forward propagator as $p(t, S, t', S') = e^{r(t' - t)} p_{t, S, t', S'}$. The forward transition probability, with boundary condition $p(t, S, t, S') = \delta(S - S')$, satisfies the following backward equation

$$\left(\frac{\partial}{\partial t} + (r - \delta) S \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2{}_{S,\, t} S^2 \frac{\partial^2}{\partial S^2}\right) p(t, S, t', S') = 0 \qquad \textbf{(EQ 78)}$$

its dual forward equation

$$\frac{\partial}{\partial t'} p(t, S, t', S') + (r - \delta) \frac{\partial}{\partial S'} (S' p(t, S, t', S')) - \frac{1}{2} \frac{\partial^2}{\partial S'^2} ((\sigma^2{}_{S',\, t'} S'^2 p(t, S, t', S'))) = 0 \qquad \textbf{(EQ 79)}$$

and, for any $t \le \tilde{t} \le t'$, the Chapman-Kolmogorov relation

$$p(t, S, t', S') = \int_0^{\infty} p(t, S, \tilde{t}, \tilde{S}) p(\tilde{t}, \tilde{S}, t', S') d\tilde{S} \qquad \textbf{(EQ 80)}$$

The forward transition probability (propagator) relates prices of a standard option, with fixed strike K and maturity T, at different time and market levels according to

$$C_{K,T}(t, S) = e^{-r(T-t)}\int_0^\infty p(t, S, t', S')C_{K,T}(t', S')dS' \quad \textbf{(EQ 81)}$$

Differentiating this relation twice and evaluating it at $t' = T$ leads to

$$p(t, S, T, K) = e^{r(T-t)}\frac{\partial^2}{\partial K^2}C_{K,T}(t, S) \quad \textbf{(EQ 82)}$$

Similarly, the backward propagator $\Phi_{K,T,K',T'}$ describes option price evolution backward in maturity time and strike price. It satisfies the same forward equation as the option prices, its dual backward equation, and the boundary condition $\Phi_{K,T,K',T} = \delta(K-K')$ for all T. Alternatively we can work with the backward transition probability density function, $\Phi(K, T, K', T')$, which is defined in terms of the backward propagator as $\Phi(K, T, K', T') = e^{\delta(T-T')}\Phi_{K,T,K',T'}$. The backward transition probability density function, with the boundary condition $\Phi(K, T, K', T) = \delta(K-K')$, satisfies the following forward equation

$$\left(\frac{\partial}{\partial T} + (r-\delta)K\frac{\partial}{\partial K} - \frac{1}{2}\sigma^2_{K,T}K^2\frac{\partial^2}{\partial K^2}\right)\Phi(K, T, K', T') = 0 \quad \textbf{(EQ 83)}$$

its dual backward equation

$$\frac{\partial}{\partial T'}\Phi(K, T, K', T') + (r-\delta)\frac{\partial}{\partial K'}(K'\Phi(K, T, K', T')) + \frac{1}{2}\frac{\partial^2}{\partial K'^2}\left(\sigma^2_{K',T'}K'^2\Phi(K, T, K', T')\right) = 0 \quad \textbf{(EQ 84)}$$

and, for any $T' \le \tilde{T} \le T$, the chapman-Kolmogorov relation

$$\Phi(K, T, K', T') = \int_0^\infty \Phi(K, T, \tilde{K}, \tilde{T})\Phi(\tilde{K}, \tilde{T}, K', T')d\tilde{K} \quad \textbf{(EQ 85)}$$

The backward transition probability (propagator) relates prices of standard options, with different strikes and maturities, at a fixed time t, $t \le T' \le T$, and market level S according to

$$C_{K,T}(t, S) = e^{-\delta(T-T')}\int_0^\infty \Phi(K, T, K', T')C_{K',T'}(t, S) \quad \textbf{(EQ 86)}$$

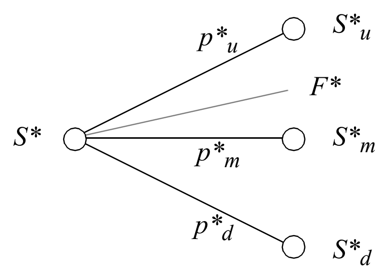
Differentiating this relation twice and evaluating it at $T' = t$ leads to

$$\Phi(K, T, S, t) = e^{\delta(T-t)} \frac{\partial^2}{\partial S^2} C_{K,T}(t, S) \qquad \textbf{(EQ 87)}$$

APPENDIX C: Local Volatility Variational Formulas in Effective Volatility Theories

In this appendix we derive variational formulas describing sensitivity of the transition probabilities (propagators) to a specific local volatility on the volatility surface. We work within the context of effective theories, (formally) changing the local volatility corresponding to a single future time and market level, while leaving all other local volatilities unchanged.

We begin by studying the relationship between transition probabilities and local volatilities in a discrete time setting. We then take the continuous-time limit by letting the spacing go to zero. Consider one period forward transition probabilities p^*_u, p^*_m and p^*_d, from the index level S^* at time t^* to the three nearby index levels S^*_u, S^*_m and S^*_d at time $t^* + \Delta t^*$, as shown in the figure below:



Let $F^* = S^* e^{(r-\delta)\Delta t^*}$ denote the one-step forward price and $\sigma^* = \sigma_{S^*, t^*}$ the local volatility, corresponding to the initial node (t^*, S^*). The three transition probabilities in the figure add up to one, and are further constrained by the forward and volatility conditions, i.e

$$p^*_u + p^*_m + p^*_d = 1 \qquad \textbf{(EQ 88)}$$

$$p^*_u S^*_u + p^*_m S^*_m + p^*_d S^*_d = F^* \qquad \textbf{(EQ 89)}$$

$$p^*_u (S^*_u - F^*)^2 + p^*_m (S^*_m - F^*)^2 + p^*_d (S^*_d - F^*)^2 = F^{*2} \sigma^{*2} \Delta t^* \qquad \textbf{(EQ 90)}$$

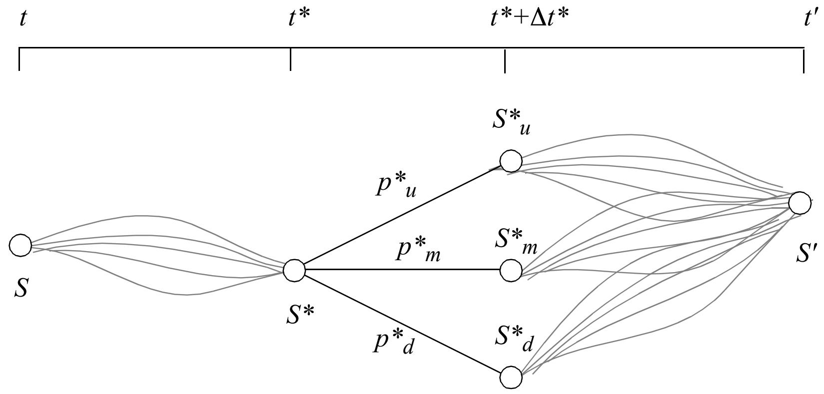
We can solve these expressions for transition probabilities in terms of the local volatility. The results are

$$p^*_u = \frac{(F^* - S^*_m)(F^* - S^*_d)}{(S^*_u - S^*_m)(S^*_u - S^*_d)} + \frac{F^{*2}}{(S^*_u - S^*_m)(S^*_u - S^*_d)} \sigma^{*2} \Delta t^* \qquad \textbf{(EQ 91)}$$

$$p^*_d = \frac{(F^* - S^*_m)(F^* - S^*_u)}{(S^*_m - S^*_d)(S^*_u - S^*_d)} + \frac{F^{*2}}{(S^*_m - S^*_d)(S^*_u - S^*_d)} \sigma^{*2} \Delta t^* \qquad \textbf{(EQ 92)}$$

and $p^*_m = 1 - p^*_u - p^*_d$.

Now consider the forward transition probability $p(t, S, t', S')$, describing the total probability that starting with the level S at time t the index will move to the level S' at the future time t', in the effective theory context. We can isolate the sensitivity of this transition probability to a specific local volatility σ_{S^*, t^*}, corresponding to the future time $t \le t^* \le t'$ and future market level S^*, using the Chapman-Kolmogorov relation of Equation 80. In discrete-time this contribution is isolated in the following figure:



This figure describes the following decomposition of the total transition probability:

$$p(t, S, t', S') = p(t, S, t^*, S^*) p^*_u p(t^* + \Delta t^*, S^*_u, t', S') + \qquad \textbf{(EQ 93)}$$

$$p(t, S, t^*, S^*) p^*_m p(t^* + \Delta t^*, S^*_m, t', S') +$$

$$p(t, S, t^*, S^*)p^*_d p(t^* + \Delta t^*, S^*_d, t', S') +$$

$$\textit{terms with no sensitivity to } \sigma_{S^*, t^*}$$

Taking the variational derivative with respect to $\sigma^2_{S^*, t^*}$ gives

$$\frac{\delta p(t, S, t', S')}{\delta \sigma^2_{S^*, t^*}} = p(t, S, t^*, S^*)\left(\frac{\delta p^*_u}{\delta \sigma^2_{S^*, t^*}}\right)p(t^* + \Delta t^*, S^*_u, t', S') + \tag{EQ 94}$$

$$p(t, S, t^*, S^*)\left(\frac{\delta p^*_m}{\delta \sigma^2_{S^*, t^*}}\right)p(t^* + \Delta t^*, S^*_m, t', S') +$$

$$p(t, S, t^*, S^*)\left(\frac{\delta p^*_d}{\delta \sigma^2_{S^*, t^*}}\right)p(t^* + \Delta t^*, S^*_d, t', S')$$

From Equations 91 and 92, and ignoring *o(*Δt^**)* terms, we have

$$\frac{\delta p^*_u}{\delta \sigma^2_{S^*, t^*}} = \frac{S^{*2}}{(S^*_u - S^*_m)(S^*_u - S^*_d)} \sim \frac{1}{2}\left(\frac{S^*}{\Delta S^*}\right)^2 \tag{EQ 95}$$

$$\frac{\delta p^*_d}{\delta \sigma^2_{S^*, t^*}} = \frac{S^{*2}}{(S^*_m - S^*_d)(S^*_u - S^*_d)} \sim \frac{1}{2}\left(\frac{S^*}{\Delta S^*}\right)^2 \tag{EQ 96}$$

$$\frac{\delta p^*_m}{\delta \sigma^2_{S^*, t^*}} = -\left(\frac{\delta p^*_u}{\delta \sigma^2_{S^*, t^*}} + \frac{\delta p^*_d}{\delta \sigma^2_{S^*, t^*}}\right) \sim -\left(\frac{S^*}{\Delta S^*}\right)^2 \quad . \tag{EQ 97}$$

where we used the approximation $S^*_u - S^*_m \sim S^*_m - S^*_d = \Delta S^*$. Insert these relations back in Equation 94 leads to

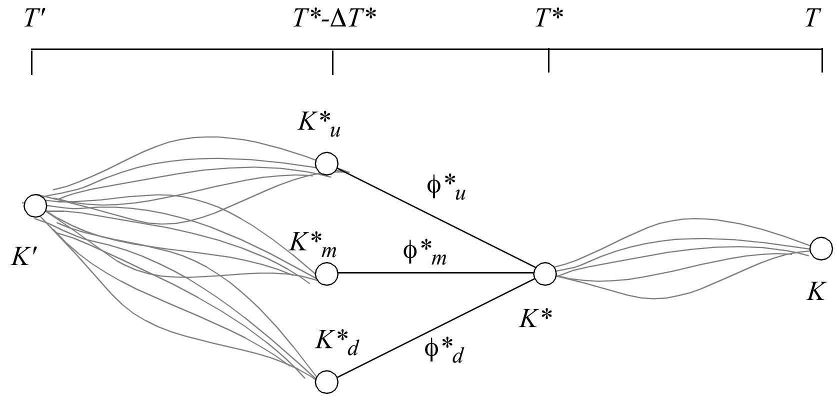
$$\frac{\delta p(t, S, t', S')}{\delta \sigma^2_{S^*, t^*}} = \frac{1}{2}p(t, S, t^*, S^*)S^{*2}\{p(t^* + \Delta t^*, S^* + \Delta S^*, t', S') - \tag{EQ 98}$$

$$2p(t^* + \Delta t^*, S^*, t', S') + p(t^* + \Delta t^*, S^* - \Delta S^*, t', S')\} / (\Delta S^*)^2$$

In the limit $\Delta S^* \to 0$ we find the desired result

$$\frac{\delta p(t, S, t', S')}{\delta \sigma^2_{S^*, t^*}} = \frac{1}{2}p(t, S, t^*, S^*)S^{*2}\frac{\partial^2}{\partial S^{*2}}p(t^*, S^*, t', S') \tag{EQ 99}$$

Similarly, the variational derivative of the backward transition probability $\Phi(K, T, K', T')$ to the local volatility σ_{K^*, T^*} with $T' \le T^* \le T$ is found using the figure:



and can be written as

$$\frac{\delta\Phi(K,T,K',T')}{\delta\sigma^2_{K^*,T^*}} = \frac{1}{2}\Phi(K,T,K^*,T^*)K^{*2}\frac{\partial^2}{\partial K^{*2}}\Phi(K^*,T^*,K',T') \qquad \textbf{(EQ 100)}$$

APPENDIX D: The No-Arbitrage Conditions and the Existence of the Equivalent Martingale Measure in Stochastic Volatility Theories

This appendix presents a proof of the no-arbitrage drift conditions of Equation 53. We also make the usual assumptions about the regularity, measurability and integrability of various quantities. A more rigorous treatment will need to address these issues.

Let us begin with Equation 40 in the text, describing the stochastic process followed by the total transition probability $P_{K,T}(t,S)$ in a stochastic volatility theory:

$$dP_{K,T} = \left[\left(\frac{\partial P_{K,T}}{\partial t} + \mu(t)S\frac{\partial P_{K,T}}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 P_{K,T}}{\partial S^2}\right)dt + \sigma(t)S\frac{\partial P_{K,T}}{\partial S}dW^0(t)\right]_{(t,S)} +$$

$$\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T'}}d\sigma^2_{K',T'}dK'dT' + \frac{1}{2}\int_t^T\int_t^T\int_0^\infty\int_0^\infty \frac{\delta^2 P_{K,T}}{\delta\sigma^2_{K',T'}\delta\sigma^2_{K'',T''}}d\sigma^2_{K',T'}d\sigma^2_{K'',T''}dK'dK''dT'dT''$$

The first term describes differential changes of the transition probability restricted to the effective theory defined by the (non-random) local volatility surface $\sigma_{K,T}(t,S)$ prevailing at time t and market level S. Restricted to this surface, $P_{K,T}(t,S)$ coincides with the effective theory total transition probability $p(t,S,K,T)$ and the instantaneous volatility $\sigma(t)$ coincides with the local volatility $\sigma_{t,S}$. Therefore, in view of Equation 78, the following backward equation holds:

$$\frac{\partial P_{K,T}}{\partial t} + (r-\delta)S\frac{\partial P_{K,T}}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 P_{K,T}}{\partial S^2} = 0$$

Using this expression, Equations 25 and 99, and some manipulations we arrive at

$$dP_{K,T} = \sigma(t)S\frac{\partial P_{K,T}}{\partial S}[dW^0(t) + (\mu(t) - r + \delta)dt] + \sum_{i=0}^{n}\left(\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T}}\theta^i_{K',T}dK'dT\right)dW^i_t$$

$$\left(\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T}}[\alpha_{K',T}(t,S) + \sum_{i=0}^{n}\theta^i_{K',T}(t,S)\left\{\frac{1}{p(t,S,T,K')}\times\right.\right.$$

$$\int_t^T\int_0^\infty \theta^i_{K'',T''}(t,S)p(t,S,T'',K'')K''^2\frac{\partial^2}{\partial K''^2}p(T'',K'',T,K')dK''dT''\}]dK'dT)dt$$

Note that the effective theory transition probabilities *p(..)* implicitly depend on *t* and *S*, whether or not they contain these variables explicitly.

Now assume that the drift parameters $\sigma_{K,T}(t,S)$ satisfy the following relations with $\Pi^0 = 0$ and (so far) arbitrary functions Π^i, $i = 1, .., n$:

$$\sigma_{K,T}(t,S) = -\sum_{i=0}^{n}\theta^i_{K,T}(t,S)\left\{\frac{1}{p(t,S,T,K)}\int_t^T\int_0^\infty \theta^i_{K',T}(t,S)p(t,S,T,K')K^{2'}\frac{\partial^2}{\partial K'^2}p(T,K',T,K)dK'dT - \Pi^i\right\}$$

Then we can define a new measure $\{d\overline{W}^i$, $i = 0, .., n\}$ by

$$d\overline{W}^0_t = dW^0_t + \frac{(\mu(t) - r + \delta)}{\sigma(t)}dt \quad ; \quad d\overline{W}^i_t = dW^i_t + \Pi^i_t dt \quad (i = 1, .., n)$$

in terms of which we have

$$dP_{K,T} = \sigma(t)S\frac{\partial P_{K,T}}{\partial S}d\overline{W}^0 + \sum_{i=0}^{n}\left(\int_t^T\int_0^\infty \frac{\delta P_{K,T}}{\delta\sigma^2_{K',T}}\theta^i_{K',T}dK'dT\right)d\overline{W}^i_t$$

The measure $\{d\overline{W}^i$, $i = 0, .., n\}$ is an equivalent martingale probability measure. Applying the arguments of Harrison and Kreps [Harrison 1979] we can show that this equivalent martingale measure is unique if (and only if) the market prices of risk $(\mu - r + \delta)/\sigma$ and Π^i,
$i = 1, .., n$, remain independent of strike price *K* and maturity *T*. Under these conditions the markets are complete and contingent claims valuation follows the standard methods of Harrison and Pliska [Harrison 1989] and remains independent of market prices of risk.

APPENDIX E: Computing Drift Parameters in Arbitrage-Free Stochastic Volatility Theories

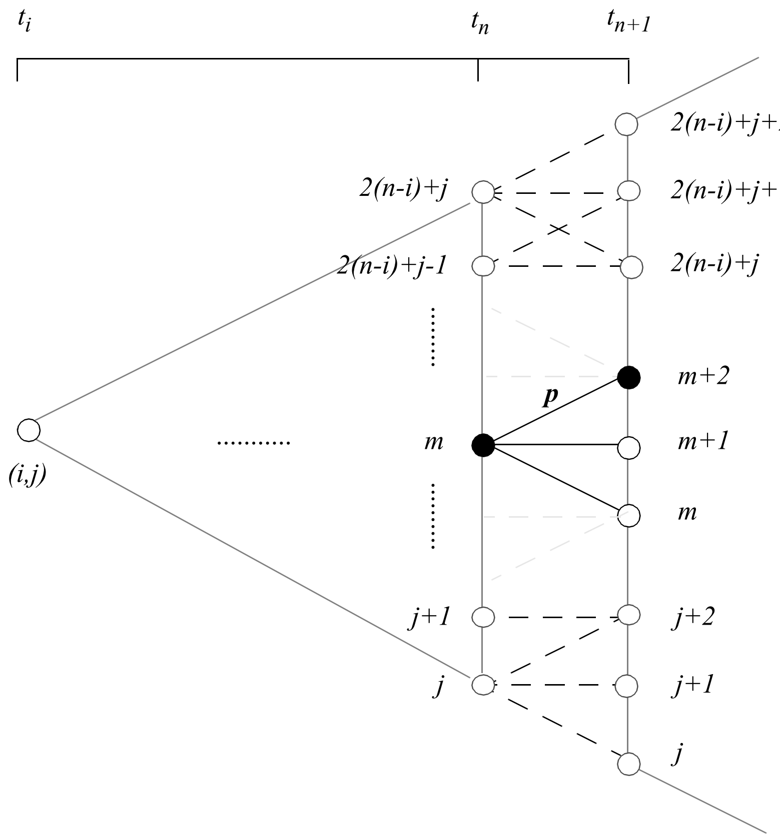
This appendix derives formulas for calculating drift parameters from the no-arbitrage conditions in stochastic volatility theories. We work in the discrete time context of the stochastic implied trinomial trees and show how to inductively calculate the arbitrage-free drift parameters for all future nodes from the martingale conditions on the total transition probabilities to the neighboring nodes at the next time step.

We begin our analysis at the *(i+1)*$^{\text{th}}$ step of the simulation, at time t_i, with index level at node *(i,j)* of the stochastic implied tree. Our objective is to calculate the arbitrage-free drift parameters $\alpha_{m,n}(i,j)$, to all future nodes *(n, m)* at future times t_n for $n \geq i$. We calculate the drift parameter $\alpha_{m,n}(i,j)$ iteratively, using the results of the previous iteration steps and the condition that the total transition probability $P_{m+2,n+1}(i,j)$, from the node *(i,j)* to the node *(n+1,m+2)*, is a martingale under all possible future movements of the local volatility surface. This situation is shown in the following figure:

The figure shows the subtree which starts at the initial node *(i,j)*. All the future movements of the index and local volatilities will be confined to the nodes of this subtree. Our iteration for calculating drift parameters $\alpha_{m,n}(i,j)$ for all subtree nodes begins with the calculation of the drift parameter at the initial node, $\alpha_{j,i}(i,j)$, and continues forward to subsequent time steps beginning with the highest node at each time step.

To make matter simple, for now let us assume that the only possible movements of the local volatility surface during the next instant Δt are up or down (proportionately), with some constant volatility θ, as in our example in the text, i.e

$$\sigma^{(u,d)}{}_{m,n}(i,j) = \sigma_{m,n}(i,j)\exp\left\{\left(\alpha_{m,n}(i,j) - \frac{1}{2}\theta^2\right)\Delta t \pm \theta\sqrt{\Delta t}\right\}$$



Suppose that we have calculated the drift parameters for every node before time t_n, and also for every nodes at time t_n which lies above the node *(n,m)*, shown in dark in the figure. We must now calculate $\alpha_{m,n}(i,j)$ from the previously known quantities and the martingale condition on the total probability $P_{m+2,n+1}(i,j)$, of arriving at the node *(n+1, m+2)* at the next time step. We can decompose the contributions to this probability into two components as follows:

$$P_{m+2,n+1}(i,j) = P_{m,n}(i,j)\boldsymbol{p} + \Lambda_{m,n}(i,j)$$

As in the figure, ***p*** denotes the one period up transition probability from the node *(n,m)* to the node *(n+1, m+2)*. The first term describes the contribution of the node *(n,m)* to the total transition probability, stemming from all the paths which go through this node before arriving at *(n+1,m+2)*. The second term describes the contribution of all the nodes lying above the node *(n,m)* to this transition probability.

Consider now the next instant Δt in time where all future local volatilities will simultaneously move either to their up state, $\sigma^{(u)}{}_{m,n}(i,j)$, or to their down state $\sigma^{(d)}{}_{m,n}(i,j)$. Since transition probabilities are direct functions of local volatilities, then all probabilities will also simultaneously move to their up or down states, i.e

$$P^{(u,d)}{}_{m+2,n+1}(i,j) = P^{(u,d)}{}_{m,n}(i,j)\boldsymbol{p}^{(u,d)} + \Lambda^{(u,d)}{}_{m,n}(i,j)$$

The quantities $P^{(u,d)}{}_{m,n}(i,j)$ and $\Lambda^{(u,d)}{}_{m,n}(i,j)$ depend on drift parameters and other quantities known from the previous iteration steps, but $\boldsymbol{p}^{(u,d)}$ remain unknown as they depend on the unknown drift parameter $\alpha_{m,n}(i,j)$. We have previously discussed the structure of this dependence in Equations 91 and 92. The one period transition probability $\boldsymbol{p}$ depends linearly on the local variance $\sigma^2{}_{m,n}(i,j)$, i.e $\boldsymbol{p} = A + B\sigma^2{}_{m,n}(i,j)$, with coefficients A and B depending only on the position of the nodes, which are fixed and known. Hence

$$\boldsymbol{p}^{(u,d)} = A + B\sigma^{(u,d)2}{}_{m,n}(i,j) =$$

$$= A + B\sigma^2{}_{m,n}(i,j)\exp\left\{2\left(\alpha_{m,n}(i,j) - \frac{1}{2}\theta^2\right)\Delta t \pm 2\theta\sqrt{\Delta t}\right\}$$

Using this and previous relations, we can now determine the unknown drift parameter $\alpha_{m,n}(i,j)$ from the martingale condition for the (known) total probability $P_{m+2,n+1}(i,j)$:

$$P_{m+2,n+1}(i,j) = \frac{1}{2}\{P^{(u)}{}_{m+2,n+1}(i,j) + P^{(d)}{}_{m+2,n+1}(i,j)\}$$

$$= \frac{1}{2}\{P^{(u)}{}_{m,n}(i,j)\boldsymbol{p}^{(u)} + P^{(d)}{}_{m,n}(i,j)\boldsymbol{p}^{(d)} + \Lambda^{(u)}{}_{m,n}(i,j) + \Lambda^{(d)}{}_{m,n}(i,j)\}$$

$$= \frac{1}{2}\{[P^{(u)}{}_{m,n}(i,j) + P^{(d)}{}_{m,n}(i,j)]A + \Lambda^{(u)}{}_{m,n}(i,j) + \Lambda^{(d)}{}_{m,n}(i,j)\} +$$

$$\frac{1}{2}B\sigma^2{}_{m,n}(i,j)e^{-\theta^2\Delta t}[P^{(u)}{}_{m,n}(i,j)e^{2\theta\sqrt{\Delta t}} + P^{(d)}{}_{m,n}(i,j)e^{-2\theta\sqrt{\Delta t}}]\exp\{2\alpha_{m,n}(i,j)\Delta t\}$$

Therefore the desired formula is

$$\alpha_{m,n}(i,j) = \left(\log\left[\frac{P_{m+2,n+1}(i,j) - \frac{1}{2}\left\{\left[P^{(u)}{}_{m,n}(i,j) + P^{(d)}{}_{m,n}(i,j)\right]A + \Lambda^{(u)}{}_{m,n}(i,j) + \Lambda^{(d)}{}_{m,n}(i,j)\right\}}{\frac{1}{2}B\sigma^2{}_{m,n}(i,j)e^{-\theta^2\Delta t}\left[P^{(u)}{}_{m,n}(i,j)e^{2\theta\sqrt{\Delta t}} + P^{(d)}{}_{m,n}(i,j)e^{-2\theta\sqrt{\Delta t}}\right]}\right]\right)/(2\Delta t)$$

This result can be readily extended to the cases where the local volatility surface can move to any number (more than two) of possible states during a time step, has multiple stochastic factors, or has factor volatilities which are more complicated functions of time, market level, factor values or their past histories.

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