Displaced-Diffusion Model – Shifted Lognormal

In 1983, Mark Rubinstein introduced the **displaced-diffusion model**. Consider the following forward price process:

Black:
$$dF_t = \sigma F_t dW_t$$

We say that F_T follows a **lognormal** distribution. Based on this definition, we call the following a **shifted lognormal** (or displaced-diffusion) process:

$$d(F_t + \alpha) = \sigma(F_t + \alpha)dW_t, \qquad \alpha \in \mathbb{R}.$$

Since α is a constant, the process can be written as

$$d(F_t + \alpha) = dF_t = \sigma(F_t + \alpha)dW_t$$

$\because d\alpha = 0$

Let $X_t = F_t + \alpha$, we can readily see that:

$$dX_t = \sigma X_t dW_t, \qquad X_T = F_T + \alpha.$$

$$dF_t = \sigma\left[\beta F_t + (1-\beta) F_0\right] dW_t$$

$$= \underbrace{\sigma \beta F_t \, dW_t}_{\text{geometric}} + \underbrace{\sigma(1-\beta) F_0 \, dW_t}_{\text{arithmetic}}$$

$dF_t = \sigma F_t \, dW_t$, Itô's to $X_t = \log(F_t) = f(F_t)$

$$dF_t = \sigma\left[\beta F_t + (1-\beta)F_0\right] dW_t$$

$$X_t = \log\left[\beta F_t + (1-\beta) F_0\right] = f(F_t)$$

$$f(x) = \log\left[\beta x + (1-\beta) F_0\right]$$

$$f'(x) = \frac{1}{\beta x + (1-\beta) F_0} \cdot \beta$$

$$f''(x) = \frac{1}{\left[\beta x + (1-\beta) F_0\right]^2} \cdot -\beta^2$$

Itô's:

$$dX_t = f'(F_t)\, dF_t + \frac{1}{2} \cdot f''(F_t)\, (dF_t)^2$$

$$= \frac{\beta}{\cancel{\beta F_t + (1-\beta) F_0}} \cdot \sigma \cancel{\left[\beta F_t + (1-\beta) F_0\right]}\, dW_t$$

$$- \frac{1}{2} \cdot \frac{\beta^2}{\cancel{\left[\beta F_t + (1-\beta) F_0\right]^2}} \sigma^2 \cancel{\left[\beta F_t + (1-\beta) F_0\right]^2}\, dt$$

$$dX_t = \beta \sigma\, dW_t - \frac{1}{2} \beta^2 \sigma^2\, dt$$

$$\int_0^T dX_t = \int_0^T \beta \sigma\, dW_t - \int_0^T \frac{1}{2} \beta^2 \sigma^2\, dt$$

$$X_T - X_0 = \beta \sigma \cdot W_T - \frac{1}{2}\beta^2 \sigma^2 T$$

$$\log\left[\beta F_t + (1-\beta) F_0\right] - \log\left[\overset{F_0}{\cancel{\beta F_0}} + (1-\cancel{\beta}) F_0\right] = \text{''} - \text{''}$$

$$\log \frac{\beta F_t + (1-\beta) F_0}{F_0} = \text{''} - \text{''}$$

$$\frac{\beta F_t + (1-\beta) F_0}{F_0} = e^{\beta \sigma W_T - \frac{1}{2}\beta^2 \sigma^2 T}$$

Displaced-Diffusion Model – Option Pricing

The following stochastic differential equation is the most commonly used form for displaced-diffusion process

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t, \qquad \beta \in [0,1].$$

Note that the SDE now comprises a **geometric** and an **arithmetic** Brownian motion.

To solve this, we apply Itô formula to the function

$$X_t = f(F_t), \qquad \text{where } f(x) = \log[\beta x + (1-\beta)F_0]$$

to obtain

$$F_T = \frac{F_0}{\beta}\exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta}F_0.$$

Displaced-Diffusion Model – Option Pricing

Question Given that

$$\text{Black} : F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

$$\text{Displaced-Diffusion} : F_T = \frac{F_0}{\beta} \exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta} F_0.$$

suppose we have implemented the option pricing function

`BlackCall(`F`, `K`, `σ`, `T`)`

can we price a European call option price under displaced-diffusion model using the same `BlackCall` function?

Displaced-Diffusion Model

From the graph, it appears that the implied volatility smile we observed in the market is between normal and lognormal.

We have seen earlier that the **displaced-diffusion** (shifted lognormal) model comprises features of the normal and lognormal models. Under a displaced-diffusion model, we have:

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t^*$$

Recall that the solution is given by

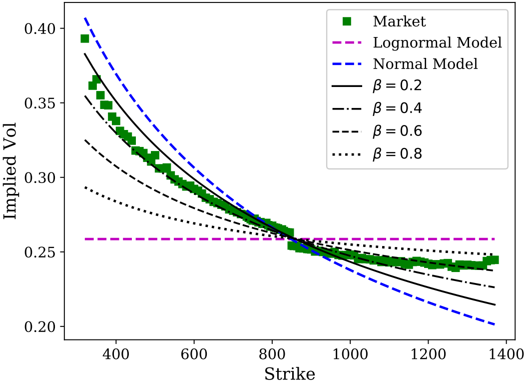
$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T^*} - \frac{1-\beta}{\beta} F_0$$

The option price under the displaced-diffusion model is

$$\text{Displaced-Diffusion} = \text{Black}\left(\frac{F_0}{\beta},\ K + \frac{1-\beta}{\beta}F_0,\ \sigma\beta,\ T\right)$$

Fitting Market Implied Volatilities

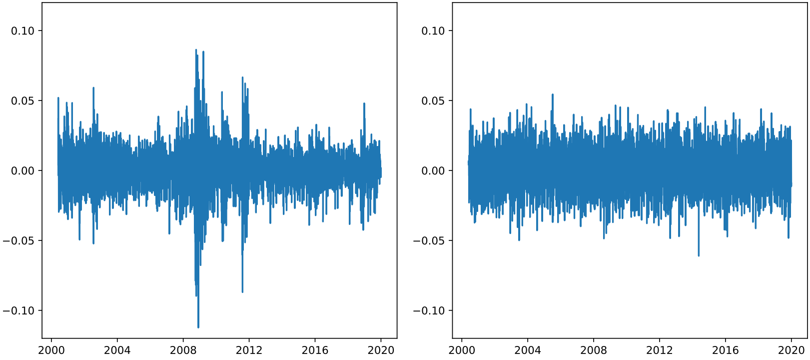
Observe that we are able to obtain a closer fit to the market using the displaced diffusion model by choosing the right β parameter.



However, the fit is still not sufficiently accurate. How can we improve this?

Fitting Market Implied Volatilities

Which one is the “real” returns of a financial asset?



Stochastic Volatility

Direct observation of the daily log-return of any underlying should convince us that volatility is stochastic instead of deterministic.

In other words, instead of treating it as a constant, it should also be described by a stochastic differential equation. As a simple extension, we let it follow a driftless lognormal process

$$d\sigma_t = \nu \sigma_t dW_t^{\sigma},$$

where ν is the **volatility of volatility**. We can solve this SDE to obtain the volatility process

$$\begin{aligned} \sigma_T &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu W_T^{\sigma}\right] \\ &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu\sqrt{T} N(0,1)\right]. \end{aligned}$$

In other words, instead of letting volatility be a constant, it is now evolving according to its own SDE, hence σ is also a stochastic process.

Heston Model

The **Heston Model** a stochastic volatility model formulated by Steven Heston in 1993, and is given by the stochastic differential equations:

$\sigma S_t \, dW_t$

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^V \end{cases}$$

where $dW_t^S dW_t^V = \rho \, dt$.

Heston models the variance as a stochastic process, following a mean-reverting square-root diffusion process.

The value of vanilla European options are determined by a 1-d integral which has to be evaluated numerically.

Heston model is popular among the equity desks.

SABR Model

The **SABR Model** (stochastic alpha-beta-rho) is pioneered by Patrick Hagan in 2002, and is characterised by the SDEs

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t = \nu \alpha_t dW_t^{\alpha} \end{cases}$$

where $dW_t^F dW_t^{\alpha} = \rho\, dt$.

The volatility is stochastic and follows a zero-drift lognormal dynamics. Hagan derived the formula for implied volatility σ_{SABR} as an analytical function of the model parameters.

To value vanilla European options, we just need to calculate σ_{SABR} and substitute this implied volatility into the Black formula to convert to price.

This is much quicker than the Heston model. SABR model is widely used across a range of asset classes.

SABR Model

$$
\begin{aligned}
&\sigma_{\mathsf{SABR}}(F_0, K, \alpha, \beta, \rho, \nu) \\
&= \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right) + \frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right) + \cdots\right\}} \\
&\quad \times \frac{z}{x(z)} \times \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right.
\end{aligned}
$$

where

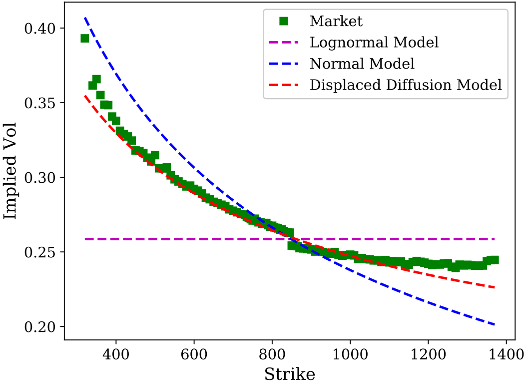
$$
z = \frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right),
$$

and

$$
x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].
$$

Code is provided in a separate Jupyter-Notebook

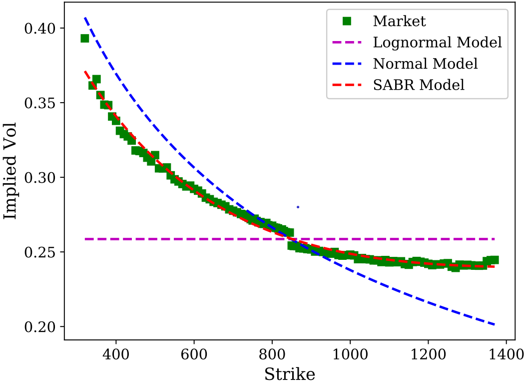
SABR Model



Displaced-diffusion model does not have sufficient **degree of freedom** to fit to market implied volatilities.

SABR Model

α, β, ρ, ν



SABR model is able to fit the implied volatility surface well—it is a popular model due to the ease of calculation.

Session 7: Equivalent Martingale Measure

Tee Chyng Wen

MSc in Quantitative Finance

Intuition behind Measure Change

Suppose we have a normally distributed stochastic process $X_t \sim N(-\kappa t, t)$, its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x+\kappa t)^2}{2t}\right].$$

Note that the process has a **drift coefficient** of $-\kappa$, which can be either positive or negative. For any bounded function $g(\cdot)$, we have the expectation

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Next, let us introduce another probability density function without drift:

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right].$$

Note that we can write the same expectation as:

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) \frac{f(x)}{\tilde{f}(x)} \tilde{f}(x) dx.$$

$$\exp\left[-\frac{(x+\kappa t)^2}{2t}+\frac{x^2}{2t}\right]=\exp\left[-\frac{x^2+2\kappa t x+\kappa^2t^2}{2t}+\frac{x^2}{2t}\right]$$

Intuition behind Measure Change

Since the probability density functions are non-zero, its ratio is well defined, and can be simplify into:

$$\frac{f(x)}{\tilde{f}(x)}=\frac{\cancel{\frac{1}{\sqrt{2\pi t}}}\exp\left[-\frac{(x+\kappa t)^2}{2t}\right]}{\cancel{\frac{1}{\sqrt{2\pi t}}}\exp\left[-\frac{x^2}{2t}\right]}=\exp\left(-\kappa x-\frac{1}{2}\kappa^2 t\right).$$

We can call this the **likelihood ratio**, or more commonly the **Radon-Nikodym derivative** in continuous-time model.

⇒ In discrete-time model, it is simply a ratio of two probabilities

⇒ In continuous-time model, it is a ratio of two probability density functions

To appreciate why it is often referred to as a "derivative", note that:

$d\mu = f(x)\,dx$

$$\int g\,d\mu=\int g\,\frac{d\mu}{d\nu}\,d\nu.$$

$d\nu = \tilde{f}(x)\,dx$

$\therefore \frac{d\mu}{d\nu}=\frac{f(x)}{\tilde{f}(x)}$

Intuition behind Measure Change

Let $\mathbb{P}$ denote the probability measure under the PDF $f(x)$, and let $\tilde{\mathbb{P}}$ denote the probability measure under the PDF $\tilde{f}(x)$.

Note that the Radom-Nikodym derivative is **strictly positive**:

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \frac{f(x)}{\tilde{f}(x)} > 0,$$

and that

$$\mathbb{E}^{\tilde{P}}\left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right] = 1. = \int_{-\infty}^{\infty} \frac{f(x)}{\cancel{\tilde{f}(x)}} \cdot \cancel{\tilde{f}(x)} \, dx = 1$$

The Radon-Nikodym deriative allows us to **change the probability measure** under which the expectation is evaluated:

$$\mathbb{E}^{P}[g(X_t)] = \mathbb{E}^{\tilde{P}}\left[g(X_t)\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right].$$

Note that the two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent (why?).

Girsanov Theorem

Using our definition of $\frac{d\mathbb{Q}}{d\mathbb{P}}$, we can show that if W_t is a standard Brownian motion under $\mathbb{P}$, then it becomes a Brownian motion with a drift coefficient $-\kappa$ under $\mathbb{Q}$, i.e. $W_t^* = W_t + \kappa t$. In addition, W_t^* follows the following $\mathbb{Q}$-Brownian motion properties:

1. $\mathbb{E}^Q[W_t^*] = 0$
2. $\mathbb{E}^Q[e^{\sigma W_t^*}] = e^{\frac{1}{2}\sigma^2 t}$
3. $\mathbb{E}^Q[e^{\sigma(W_{t+s}^* - W_s^*)}|s] = e^{\frac{1}{2}\sigma^2 t}$

Q P

W_t^* W_t

$\kappa_t : \to \kappa$

Girsanov Theorem

If W_t is a $\mathbb{P}$-Brownian motion and κ_t satisfies $\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \kappa_t^2 dt\right)\right] < \infty$, then there exists a measure $\mathbb{Q}$ such that

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$
2. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \kappa_t \, dW_t - \frac{1}{2}\int_0^T \kappa_t^2 \, dt\right)$ $\quad \exp\left(-\kappa W_t - \frac{1}{2}\kappa^2 t\right)$
3. $W_t^* = W_t + \int_0^t \kappa_u \, du$ is a $\mathbb{Q}$-Brownian motion.

$W_t^* = W_t + \kappa t$

$W_t^* = W_t + \kappa t$

$W_t \to \mathbb{P}$

$W_t^* \to \mathbb{Q}$

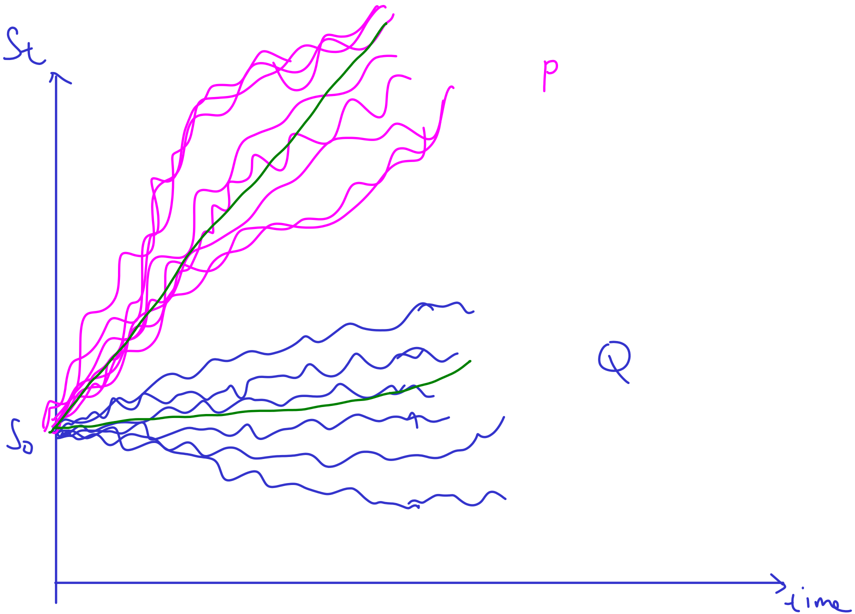
Girsanov Theorem — Example

Example Let W_t denote a $\mathbb{P}$-Brownian motion, and let W_t^* denote a $\mathbb{Q}$-Brownian motion. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and are related by the Radon-Nikodym derivative. Show that

1. $\mathbb{E}^P[W_t] = 0$
2. $\mathbb{E}^Q[W_t] = -\kappa t$
3. $\mathbb{E}^Q[W_t^*] = 0$
4. $\mathbb{E}^P[W_t^*] = \kappa t$

$\mathbb{E}^Q[W_t] = \mathbb{E}^Q[W_t^* - \kappa t] = 0 - \kappa t$

$\mathbb{E}^P[W_t^*] = \mathbb{E}^P[W_t + \kappa t] = 0 + \kappa t$



S_t
S_0
P
Q
time

Girsanov Theorem — Example

Example Consider a stochastic process X_t satisfying the following SDE

$$\mathbb{P}: \quad dX_t = \mu X_t dt + \sigma X_t dW_t, \quad = \nu X_t dt - \nu X_t dt + \mu X_t dt + \sigma X_t dW_t$$

where W_t is a $\mathbb{P}$-Brownian motion. Change the measure so that the drift coefficient of X_t is ν instead of μ.

nu mu

Solution Again, rewriting our SDE in the following format

$$dX_t = \nu X_t dt + \sigma X_t \left(dW_t + \frac{\mu - \nu}{\sigma} dt \right),$$

we let $\kappa = \frac{\mu - \nu}{\sigma}$, and apply Girsanov to get an equivalent measure $\mathbb{Q}$ under which

$$W_t^* = W_t + \frac{\mu - \nu}{\sigma} t \quad = W_t + \kappa t$$

is a $\mathbb{Q}$-Brownian motion. The process X_t satisfies the following SDE under this new measure

$$\mathbb{Q}: \quad dX_t = \nu X_t dt + \sigma X_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}$-Brownian motion. $\triangleleft$

$\mathbb{P}$

Before Black-Scholes:

Various people developed models of derivatives that are actuarial in that they define the value of an option as the empirical expected discounted value of its payoffs.

μ

This value does of course depend on the volatility of the stock. But they don't know what rate of return to use for growing the stock price into the future, and they don't know what rate to use for discounting the payoffs.

People who wanted to use this model had to forecast the return of the stock and figure out what discount rate to use as a consequence of its risk. It was personal.

— Emanuel Derman

Source: A Stylized History of Quantitative Finance

Black-Scholes (1971–3)

> *Hedge to eliminate stock risk from option. Require that hedged portfolio, which is riskless, earns the known riskless rate. Then we get the same formula for the option value as the actuarial one, but where all growth and discount rates are riskless rates.*
>
> *The value of the option does not depend on the expected return of the stock, since that has been hedged away. Instead it depends on the riskless rate, which is known, and on the future volatility of the stock.*

— Emanuel Derman

Source: A Stylized History of Quantitative Finance

Before Black-Scholes:

$$\text{Call} = S_0\Phi\left(\frac{\log\frac{S_0}{K}+\left(\mu+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-fT}\Phi\left(\frac{\log\frac{S_0}{K}+\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Black-Scholes:

$$\text{Call} = S_0\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

1997 Nobel Prize citation:

> *Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options.*
>
> *Their methodology has paved the way for economic valuations in many areas.*
>
> *It has also generated new types of financial instruments and facilitated more efficient risk management in society.*

— The Royal Swedish Academy of Sciences