Properties of Stochastic Integrals

Consider I_T defined as the stochastic integral:

$$I_T = \int_0^T f(u, W_u)\, dW_u.$$

Below are the key properties of stochastic integrals I_T:

1. $\mathbb{E}[I_T] = 0$ mean
2. $\mathbb{E}[I_T^2] = \mathbb{E}\left[\left(\int_0^T f(u, W_u)\, dW_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u, W_u)^2\, du\right]$ variance
3. If f is a deterministic function, then $I_T \sim N\left(0, \int_0^T f(u)^2\, du\right)$.
4. $\mathbb{E}\left[\int_0^T f(u)\, dW_u \times \int_0^T g(s)\, dW_s\right] = \mathbb{E}\left[\int_0^T f(u)g(u)\, du\right]$
5. **Itô's Isometry theorem** states that $\mathbb{E}\left[\left(\int_0^T X_t\, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2\, dt\right]$.

$$\int t \, dt \qquad \int t \, dW_t$$

$$\int W_t \, dt \qquad \int W_t \, dW_t$$

$$\int S_t \, dt \qquad \int S_t \, dW_t$$

Properties of Stochastic Integrals: Proof of (1)

Recall that our first principle definition of the stochastic integrals is given by

$$\int_0^T f(t, W_t)\, dW_t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})$$

Note that Brownian motion's increments are independent, so $(W_{t_i} - W_{t_{i-1}})$ is independent from $W_{t_{i-1}}$. Hence, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T f(t, W_t)\, dW_t\right] &= \mathbb{E}\left[\lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=i}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}})\right] \times \mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}})\right] \times 0 = 0
\end{aligned}$$

Properties of Stochastic Integrals: Proof of (2)

Let $\Delta W_{t_{i-1}} = W_{t_i} - W_{t_{i-1}}$ and $\Delta W_{t_{j-1}} = W_{t_j} - W_{t_{j-1}}$, note that

$$\mathbb{E}\Big[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}}\Big] = \begin{cases} \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) & i = j \\ 0 & i \neq j \end{cases}$$

We have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T f(t, W_t)\, dW_t\right)^2\right] &= \mathbb{E}\left[\int_0^T f(s, W_s)\, dW_s \times \int_0^T f(u, W_u)\, dW_u\right] \\
&= \lim_{n\to\infty} \sum_{i=1, j=1}^{n} \mathbb{E}\Big[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}}\Big] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) \\
&= \mathbb{E}\left[\int_0^T f(t, W_t)^2\, dt\right]
\end{aligned}$$

$$\mathbb{E}\left[\Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}}\right]$$

$$= Cov\left(\Delta W_{t_{i-1}}, \Delta W_{t_{j-1}}\right) = \begin{cases} i = j : & V\left[\Delta W_{t_{i-1}}\right] = t_i - t_{i-1} \\ i \neq j : & 0 \end{cases}$$

Stochastic Integrals

$$\int_0^T W_t\, dW_t \neq \left[\frac{W_t^2}{2}\right]_0^T = \frac{W_T^2}{2}$$

Example Consider the stochastic integral

$$I_T = \int_0^T W_t\, dW_t,$$

determine its mean $\mathbb{E}[I_T]$ and variance $V[I_T]$.

$$\mathbb{E}\left[\int_0^T W_t\, dW_t\right] = 0$$

$$V\left[\int_0^T W_t\, dW_t\right] = \mathbb{E}\left[\left(\int_0^T W_t\, dW_t\right)^2\right]$$

Itô's Isometry

$$= \mathbb{E}\left[\int_0^T W_t^2\, dt\right] = \int_0^T \mathbb{E}[W_t^2]\, dt = \int_0^T t\, dt$$

$$= \frac{t^2}{2}\Big|_0^T = \frac{T^2}{2}$$

Box Calculus

Strong Law of Large Numbers

Kolmogorov's strong law states that the average of a sequence of independent random variables having a common distribution will with probability 1 converge to the mean of that distribution:

$$n \to \infty \quad \Rightarrow \quad \mathbb{P}\left(\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \mu \right) = 1.$$

For example, if $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the Strong Law states that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = 0.$$

$\mathbb{E}[(\Delta W_t)^2] = \Delta t$

We can use the Law of Large Numbers to argue that $(\Delta W_t)^2$ and Δt are of the same order, leading to the Box calculus rule of $dW_t \cdot dW_t = dt$.

Box Calculus Rules

We note that $(\Delta W_t)^2$ is in the order of Δt, and hence ΔW_t is in the order of $\sqrt{\Delta t}$. Under the limit of $\Delta t \to 0$, we use differential notation. This yields

$$
\begin{aligned}
\Delta t &\to dt \\
(\Delta t)^2 &\to (dt)^2 = 0 \\
\Delta W_t &\to dW_t \\
(\Delta W_t)^2 &\to dW_t^2 = dt \\
(\Delta t \cdot \Delta W_t) &\to dt \cdot dW_t = 0
\end{aligned}
$$

$\sqrt{dt} = dt^{0.5}$

$dt \cdot dt^{0.5} = dt^{1.5}$

These rules are based on the formalism of **Box Calculus**:

$\cdot$	dt	dW_t
dt	0	0
dW_t	0	dt

$(dX_t)^2 = (\mu\, dt + \sigma\, dW_t)^2$

$= \cancelto{0}{\mu^2 (dt)^2} + \cancelto{0}{2\mu\sigma\, dt\, dW_t} + \sigma^2 \cancelto{dt}{(dW_t)^2}$

So for example, we can write:

$$dX_t = \mu\, dt + \sigma\, dW_t \quad \Rightarrow \quad (dX_t)^2 = \sigma^2 dt.$$

The History...

Brown, a botanist, discovered the motion of pollen particles in water in 1827.

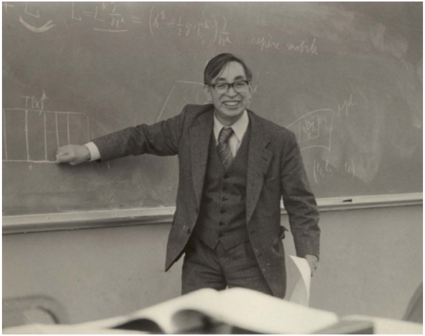
At the beginning of the 20^{th} *century, Brownian motion was studied by Einstein, Perrin and other physicists.*

In 1923, against this scientific background, Wiener defined probability measures in path spaces, and used the concept of Lebesgue integrals to lay the mathematical foundations of stochastic analysis.

In 1942, Dr. Itô began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.

Kiyosi Itô (1915–2008)

> *...I finally devised stochastic differential equations, after painstaking solitary endeavours.*
>
> — Kiyosi Itô (1915–2008)



Why Itô's Formula?

From a theoretical viewpoint:

- Now that we have defined the stochastic integral, we want to be able to manipulate it without coming back to the definition.
- General rules similar to ordinary calculus chain rules, product rules etc. will be very handy.

From a practical viewpoint:

- We have now defined the stock price process, and knowing that an option price is a function of that random process,
- We want to study and understand the infinitesimal evolution of the option price process as well.

The purpose of computing is insight, not numbers.

— Richard Hamming (1915–1998)

There is no doubt that the field of quantitative finance has been thoroughly transformed by the basic insights provided by Ito's calculus, both on a conceptual and on a computational level.

— Hans Föllmer (b. 1941)

Itô's lemma

$\boxed{\text{Theorem / Proposition}}$

↓

Lemma

↓

manuscript

↓

footnote

Formulation of Itô's Formula

Under ordinary calculus, if we define $f(t)$ as a function of time t, **Taylor expansion** yields

$$f(t+\Delta t) = f(t) + f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots$$

If we let $x = f(t)$, a simple rearrangement yields

$$\begin{aligned} f(t+\Delta t) - f(t) &= f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots \\ \Delta x &= f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots \end{aligned}$$

Sending the limit of $\Delta t \to 0^+$, we obtain

$$\frac{dx}{dt} = f'(t).$$

Formulation of Itô's Formula

Now consider the case for stochastic calculus, involving a function of Brownian motion $f(W_t)$. We have

$$f(W_t + \Delta W_t) = f(W_t) + f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots$$

If we let $X_t = f(W_t)$, a simple rearrangement now yields

$$\begin{aligned} f(W_t + \Delta W_t) - f(W_t) &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots \\ \Delta X_t &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots \end{aligned}$$

Now when we send the limit of $\Delta t \to 0^+$, we arrive at

$$dX_t = f'(W_t)dW_t + \underbrace{\frac{1}{2}f''(W_t)dt}_{\text{extra term}}.$$

$d S_t = \mu \, dt + \sigma \, dW_t$

Formulation of Itô's Formula

Building on this concept, suppose now we model the stock price as

$$\Delta S_t = \mu \, \Delta t + \sigma \, \Delta W_t.$$

Let $X_t = f(S_t)$, Taylor expansion yields

$$f(S_t + \Delta S_t) = f(S_t) + f'(S_t)\Delta S_t + \frac{1}{2!} f''(S_t)(\Delta S_t)^2 + \cdots$$

$$\Delta X_t = f'(S_t)\Delta S_t + \frac{1}{2!} f''(S_t)(\Delta S_t)^2 + \cdots$$

We have already defined the model for ΔS_t, hence

$$(\Delta S_t)^2 = \mu^2 \cdot (\Delta t)^2 + 2 \cdot \mu \cdot \sigma \cdot (\Delta t)(\Delta W_t) + \sigma^2 \cdot (\Delta W_t)^2$$

Using Box Calculus rules, under the limit of $\Delta t \to 0^+$, we have $(dS_t)^2 = \sigma^2 dt$. Substituting back to the Taylor expansion above, we obtain:

$$dX_t = f'(S_t) dS_t + \frac{1}{2} f''(S_t) \sigma^2 dt \quad = f'(S_t)\left(\mu \, dt + \sigma \, dW_t\right) + \frac{1}{2} f''(S_t)\sigma^2 dt$$

$$= \left(\mu f'(S_t) + \frac{\sigma^2}{2} f''(S_t)\right) dt + \sigma f'(S_t) \, dW_t.$$

Itô's Formula

This leads us to the famous **Itô's formula** (sometimes known as **Itô's lemma**):

Itô's Formula (Function of a Stochastic Process)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

and the function $f : \mathbb{R} \to \mathbb{R}$, f, f', f'' are continuous, then $Y_t = f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) \right) dt + \sigma_t f'(X_t) dW_t.$$

Example Suppose $dX_t = \mu \, dt + \sigma \, dW_t$, and $Y_t = X_t^2$. Derive the stochastic differential equation for dY_t.

$$Y_t = X_t^2 = f(X_t)$$

$f(x) = x^2$

$f'(x) = 2x \quad , \quad f''(x) = 2$

Itô's formula

$$\boxed{dY_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, (dX_t)^2}$$

$$= 2X_t \left(\mu\, dt + \sigma\, dW_t \right) + \frac{1}{2} \times 2 \times \sigma^2 \cdot dt$$

$$= \left(2\mu X_t + \sigma^2 \right) dt + 2\sigma X_t\, dW_t$$

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t$$

$$Y_t = f(X_t)$$

Itô's formula

$$\boxed{dY_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t) (dX_t)^2}$$

$$= f'(X_t)\left(\mu_t\, dt + \sigma_t\, dW_t\right) + \frac{1}{2} f''(X_t)\, \sigma_t^2\, dt$$

$$= \left(\mu_t f'(X_t) + \frac{1}{2} f''(X_t)\, \sigma_t^2\right) dt + \sigma_t f'(X_t)\, dW_t$$

Itô's Formula

More generally, Itô's Formula also allows us to write down the stochastic differential equation of a function of stochastic processes and time:

Itô's Formula (Function of a Stochastic Process & Time)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

and the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(t,x)$, g_t, g_x, g_{xx} are continuous, then $Y_t = g(t, X_t)$ is also a stochastic process and is given by

$$dY_t = \left[g_t(t,X_t) + \mu_t g_x(t,X_t) + \frac{1}{2}\sigma_t^2 g_{xx}(t,X_t) \right] dt + \sigma_t g_x(t,X_t) dW_t.$$

Example Suppose $dX_t = \mu \, dt + \sigma \, dW_t$, and $Y_t = e^{X_t + t}$. Derive the stochastic differential equation for dY_t.

$$Y_t = e^{X_t + t} = g(t, X_t)$$

$$g(t,x) = e^{x+t}$$

$$g_t = e^{x+t}$$

$$g_x = e^{x+t}, \quad g_{xx} = e^{x+t}$$

Itô's formula:

$$\boxed{dY_t = g_t(t, X_t)\,dt + g_x(t, X_t)\,dX_t + \frac{1}{2} g_{xx}(t, X_t)(dX_t)^2}$$

$$= e^{X_t+t} \cdot dt + e^{X_t+t}(\mu\,dt + \sigma\,dW_t) + \frac{1}{2} e^{X_t+t} \sigma^2 dt$$

$$= \left(1 + \mu + \frac{1}{2}\sigma^2\right) Y_t\,dt + \sigma Y_t\,dW_t$$

$$
\begin{aligned}
g(t+\Delta t,\, X_t+\Delta X_t) = {} & g(t, X_t) \\
& + g_t(t, X_t)\,\Delta t \quad + g_x(t, X_t)\,\Delta X_t \\
& + \frac{1}{2}\Big[g_{tt}(t, X_t)\,(\Delta t)^2 + 2 g_{tx}(t, X_t)\,\Delta t \cdot \Delta X_t \\
& \qquad + g_{xx}(t, X_t)\,(\Delta X_t)^2 \Big] \\
& + \frac{1}{3!}\Big[\;\cdots\; \Big]
\end{aligned}
$$

$\Delta t \to 0^+$

$$\underbrace{dY_t}_{g(t+\Delta t,\, X_t + \Delta X_t) - g(t, X_t)} = g_t(t, X_t)\,dt + g_x(t, X_t)\,dX_t + \frac{1}{2} g_{xx}(t, X_t)(dX_t)^2$$

$$= g_t(t, X_t)\,dt + g_x(t, X_t)(\mu_t\,dt + \sigma_t\,dW_t)$$

$$+ \frac{1}{2} g_{xx}(t, X_t)\,\sigma_t^2\,dt$$

$f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$

Itô's Formula: Applications

$\int x \, dx = \frac{x^2}{2} + C$

Example Show that

$$\int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

by applying Itô formula to $X_t = f(W_t) = W_t^2$.

$$dX_t = f'(W_t)\, dW_t + \frac{1}{2} f''(W_t) (dW_t)^2$$

$$= 2W_t \, dW_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$\int_0^T dX_t = \int_0^T 2W_t \, dW_t + \int_0^T dt$$

$$X_T - X_0 = 2\int_0^T W_t \, dW_t + T - 0$$

$$W_T^2 - \cancel{W_0^2}^{\,0} = 2\int_0^T W_t \, dW_t + T$$

$$\int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

Session 5: Stochastic Differential Equations

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

"Comments on the life and mathematical legacy of Wolfgang Doeblin":

> *One may invoke many reasons why the emergence of a specific branch of probability — the study of stochastic process — took a quite tortuous path throughout the* 20^{th} *century.*
>
> *On one hand, the pioneers were very often quite original mathematicians, such as Bachelier, Lévy, Itô, ..., whose novel ways of looking at things took a long time to be accepted.*

— Bernard Bru and Marc Yor (2002)

On the other hand, perhaps the fact that Brownian motion possesses so many properties, which we summarize as:

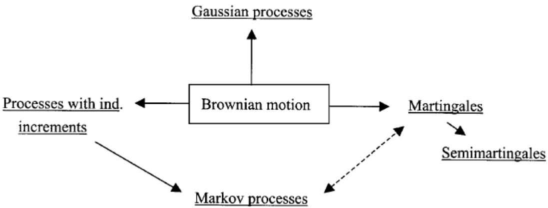
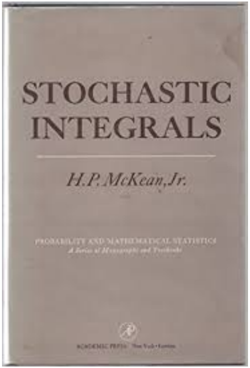


Fig. 1. Brownian motion and related processes

led many authors to develop studies of one or another special class of processes, thus giving a hard time to outsiders.

Itô's calculus took 25 years (1944-1969) to be accepted, the latter year being that of the publication of McKean's marvellous little book: Stochastic Integrals.

Bernard Bru and Marc Yor (2002)



SDEs and Martingale

Example Use Itô's formula to derive the stochastic differential equations of the following processes, and determine which of them are martingales:

1. $X_t = W_t^2$ (no)
2. $X_t = 2 + t + e^{W_t}$ (no)
3. $X_t = W_t^2 + \tilde{W}_t^2$, where W_t and $\tilde{W}_t$ are independent Brownian. (no)
4. $X_t = W_t^2 - t$ (yes)
5. $X_t = W_t^3$ (no)
6. $X_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$ (yes)

① check for $M_t = \mathbb{E}_t[M_T]$

② $dX_t = \underbrace{0}\,dt + \underbrace{\qquad}\,dW_t$

Bachelier Model (1900)

Louis Bachelier was the first mathematician to use random walk to analyse stock prices in 1900.

In Bachelier model, the stock price process is a **symmetrical random walk**, correspond to a market under **equilibrium**. This follows a **normal distribution**:

$$S_T = S_0 + \sigma W_T, \quad W_T \sim N(0, T).$$

Given this definition, we can then proceed to derive valuation formulas for vanilla European options.

However, a shortcoming of this model is that the lack of a lower bound at 0.

In other words, while this is a reasonable model for interest rates, it leads to non-zero probability for negative stock prices.

Bachelier Model – Arithmetic Brownian Process

The Bachelier model for the stock price process is defined as

$$\int_0^T dS_t = \int_0^T \sigma \, dW_t. \quad \Rightarrow \quad S_T - S_0 = \sigma\left(W_T - \overset{0}{\cancel{W_0}}\right)$$

Integrating this stochastic equation, we can show that the terminal stock price is normally distributed as

$W_T \sim N(0, T)$

$\sqrt{T}\, N(0, 1)$

$$S_T \sim N(S_0, \sigma^2 T).$$

Let V_c denote the price of a European call option, we have:

$$\begin{aligned} V_c = e^{-rT}\mathbb{E}[(S_T - K)^+] &= e^{-rT}\mathbb{E}\left[(S_0 + \sigma W_T - K)^+\right] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_0 + \sigma\sqrt{T}x - K\right)^+ e^{-\frac{x^2}{2}} \, dx \end{aligned}$$

Note that $\left(S_0 + \sigma\sqrt{T}x - K\right)^+ = 0$ whenever $S_0 + \sigma\sqrt{T}x - K < 0$, and will only take on non-zero values when

$$S_0 + \sigma\sqrt{T}x - K > 0 \quad \Rightarrow \quad x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*.$$

Bachelier Model – Arithmetic Brownian Process

Hence, we now write

$$
\begin{aligned}
V_c &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \Big(S_0 + \sigma\sqrt{T}x - K\Big) e^{-\frac{x^2}{2}}\ dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}}\ dx + \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \sigma\sqrt{T} x e^{-\frac{x^2}{2}}\ dx \\
&= e^{-rT}(S_0 - K)\Big[\Phi(\infty) - \Phi(x^*)\Big] - \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \Big[e^{-\frac{x^2}{2}}\Big]_{x^*}^{\infty} \\
&= e^{-rT}\left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(x^*)^2}{2}}\right] \\
&= e^{-rT}\left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(-x^*)^2}{2}}\right] \\
&= e^{-rT}\left[(S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right].
\end{aligned}
$$

$\phi(-x^*)$

For **at-the-money (ATM)** options, we have $K = S_0$, and this formula reduces to $V_c = e^{-rT}\sigma\sqrt{\frac{T}{2\pi}}$.

$V_c^{ATM} \quad : \quad K = S_0$

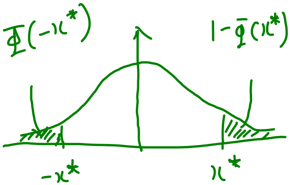
$$= e^{-rT}\left[\, 0 + \sigma\sqrt{T}\,\phi(0) \,\right]$$

$$= e^{-rT}\,\sigma\sqrt{T}\cdot\frac{1}{\sqrt{2\pi}}\,\cancelto{1}{e^{-\frac{0^2}{2}}}$$

$$\simeq \sigma\sqrt{T}\cdot 0.4$$

PDF $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

CDF $\Phi(x) = \int_{-\infty}^{x} \phi(u)\, du$



$$e^{-rT}(S_0 - K) \cdot \int_{x^*}^{\infty} \phi(x)\, dx$$

$$= e^{-rT}(S_0 - K) \cdot \left[\cancelto{1}{\Phi(\infty)} - \Phi(x^*) \right]$$

$$= e^{-rT}(S_0 - K)\, \Phi(-x^*)$$

$$\int x e^{-\frac{x^2}{2}} \, dx$$

let $\quad u = \frac{x^2}{2} \quad \Rightarrow \quad du = \frac{\cancel{2} x}{\cancel{2}} \, dx$

$$\int e^{-u} \, du = -\left[e^{-u} \right] = -\left[e^{-\frac{x^2}{2}} \right]$$