Differentiation of Integrals

$$\frac{d}{dx} f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx}$$

The chain rule for partial differentiation states that if we have a bivariate function $g(u(x), v(x))$, where u and v are both functions of x, then

$$\frac{dg}{dx} = \frac{\partial g}{\partial u} u'(x) + \frac{\partial g}{\partial v} v'(x).$$

Now consider the case where the integrand is $f(x,t)$ and t is the dummy variable. Let F denote the anti-derivative of f, we can write it as

$$\int_{u(x)}^{v(x)} f(x,t)dt = F(x, v(x)) - F(x, u(x)).$$

Let's define

$$I(v(x), u(x), x) = F(x, v(x)) - F(x, u(x)),$$

we can see that

$$\frac{\partial I}{\partial v} = f(x, v(x)), \qquad \text{and} \qquad \frac{\partial I}{\partial u} = -f(x, u(x)).$$

Leibnitz's Rule

Now differentiating I with respect to x, we obtain

$$\begin{aligned}
\frac{dI}{dx} &= \frac{\partial I}{\partial v}v'(x) + \frac{\partial I}{\partial u}u'(x) + \frac{\partial I}{\partial x}\cancelto{1}{\frac{dx}{dx}} \\
&= f(x, v(x))v'(x) - f(x, u(x))u'(x) + \frac{\partial}{\partial x}\int_{u(x)}^{v(x)} f(x,t)\, dt \\
&= f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\, dt.
\end{aligned}$$

Under the special case where the integrand isn't a function of x, then we recover the corollary integration relationship:

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\, dt\right] = f(v(x))v'(x) - f(u(x))u'(x).$$

Modelling

Assumptions

Specify model (dynamic)

Calibration

calibrated model

price exotics

Market Data

Model-free

distribution?

What is the "model-free" framework?

In a **model-free** formulation, we let $f(s)$ denote the risk-neutral probability density function of the stock price at time T, we can price a vanilla European call option maturing at time T as follows:

PDF

$$\frac{C_0}{B_0} = \mathbb{E}^*\left[\frac{C_T}{B_T}\right]$$

$$C(K) = e^{-rT}\mathbb{E}^*[(S_T - K)^+] = e^{-rT}\int_K^\infty (s-K)f(s)\,ds.$$

In earlier modelling approach, we will attempt to specify a model for the stock price process. A typical example is the Black-Scholes model, which will lead to:

$$C(K) = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^\infty \left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K\right)e^{-\frac{x^2}{2}}\,dx.$$

We could also have used the Bachelier model, the displaced-diffusion model, or the SABR model.

$\Rightarrow$ Once a model is chosen, the risk-neutral density is also determined, by calibrating the model to market option data.

What is the "model-free" framework?

Suppose we have sufficient liquid option quotes in the market, can we skip over the step of using a model to specify the stock price process, but instead **extract the risk-neutral density function** directly?

$f(\ ) = ?$

Calls / put's

Market Price	Model-Free Formula
$C(K_1)$	$e^{-rT}\int_{K_1}^{\infty}(s-K_1)f(s)\,ds$
$C(K_2)$	$e^{-rT}\int_{K_2}^{\infty}(s-K_2)f(s)\,ds$
$C(K_3)$	$e^{-rT}\int_{K_3}^{\infty}(s-K_3)f(s)\,ds$
$C(K_4)$	$e^{-rT}\int_{K_4}^{\infty}(s-K_4)f(s)\,ds$
$\vdots$	$\vdots$

Implied Risk-Neutral Density

- Black-Scholes model for European call and put options allow us to determine their prices by taking expectation of the option payoff on maturity, discount back to today.

- The model **assumes a lognormal process for the stock price** following
$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$
under $\mathbb{Q}^*$, where the volatility σ is a model parameter that we need to determine.

- Since the vanilla option market is very liquid, we do not need to rely on any mathematical models to calculate the prices of options.

- Instead, the traded price of these options are published real-time by exchanges globally, and the process can now be reversed—given that an option traded at a particular price, what is the implied volatility that we should substitute into our Black-Scholes formula to give us this price, assuming that the underlying stock price is indeed following a lognormal process?

Implied Risk-Neutral Density

- One option price allows us to determine one implied volatility for a particular **strike** and **maturity**.
- The market is constantly providing live information about option prices across a wide range of strikes for a given maturity.
- Given this information, we can now bring our analysis to the next level—instead of asking for just one single implied volatility to match one option price, we want to determine, for a given maturity, the **implied risk-neutral distribution**, that allows us to match the market volatility smile or skew.
- To this end, we need to apply **Leibniz's rule**:

$$I(x) = \int_{u(x)}^{v(x)} g(x,t)dt$$

$$\frac{dI(x)}{dx} = g(x,v(x))\frac{dv}{dx} - g(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial g(x,t)}{\partial x}dt$$

Implied Risk-Neutral Density

This allows us to **extract the risk-neutral probability density function** from market-traded vanilla option prices.

Let $f(s)$ denote the risk-neutral probability density, we can apply Leibniz's rule to obtain:

$$
\begin{aligned}
\overset{I(K)}{C(K)} &= e^{-rT}\mathbb{E}[(S_T-K)^+] = e^{-rT}\int_{K = u(K)}^{\overset{v(K)}{\overset{\shortparallel}{\infty}}} \overset{g(K,\,s)}{(s-K)f(s)}\,ds \\
\frac{\partial C(K)}{\partial K} &= e^{-rT}\left[\lim_{x\to\infty}(x-K)f(x)\frac{dx}{dK} - (K-K)f(K)\frac{dK}{dK} - \int_K^\infty f(s)\,ds\right] \\
&= -e^{-rT}\int_K^\infty f(s)\,ds \\
\frac{\partial^2 C(K)}{\partial K^2} &= -e^{-rT}\left[\lim_{x\to\infty} f(x)\frac{dx}{dK} - f(K)\frac{dK}{dK} + \int_K^\infty \frac{\partial f(s)}{\partial K}\,ds\right] \\
&= e^{-rT}f(K).
\end{aligned}
$$

Implied Risk-Neutral Density

We can also carry out the same procedure to the put options:

$$P(K) = e^{-rT}\mathbb{E}[(K - S_T)^+] = e^{-rT}\int_0^K (K-s)f(s)\,ds$$

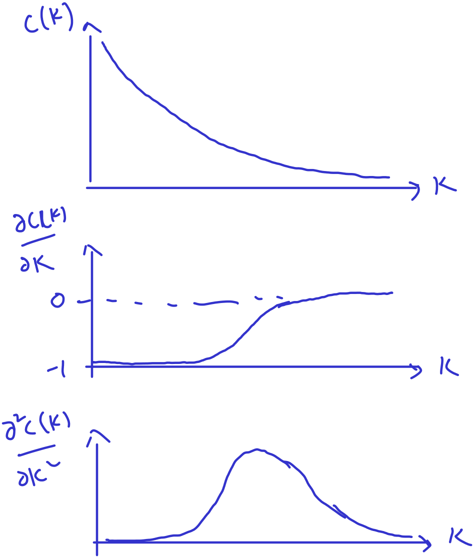
These give us

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-rT}f(K) \qquad \text{and} \qquad \frac{\partial^2 P(K)}{\partial K^2} = e^{-rT}f(K).$$

This is the **Breeden-Litzenberger formula**, which showed in 1978 that the terminal distribution of the stock price implicit in the option prices, also known as the **implied distribution**, can be obtained by differentiating the call & put option prices twice with respect to the strike price.

Subsequently, **Carr and Madan** showed in 1998 that any European payoff can be replicated using a portfolio of cash, forward contracts, and European call & put options.

$$f(K) = e^{rT} \cdot \frac{\partial^2 C(K)}{\partial K^2} = e^{rT} \frac{\partial^2 P(K)}{\partial K^2}$$



$$\frac{\partial^2 c(K)}{\partial K^2} \simeq \frac{c(K+\Delta K) - 2c(K) + c(K-\Delta K)}{(\Delta K)^2}$$

Static Replication of European Payoff

To replicate any twice differentiable European payoff $h(S_T)$, we write

$\frac{V_0}{B_0} = \mathbb{E}\left[\frac{V_T}{B_T}\right]$

$$V_0 = e^{-rT}\mathbb{E}[h(S_T)] = e^{-rT}\int_0^\infty h(s)f(s)\,ds.$$

Let $F = S_0 e^{rT}$, we have

$$V_0 = e^{-rT}\mathbb{E}[h(S_T)] = \underbrace{\int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}\,dK}_{(1)} + \underbrace{\int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}\,dK}_{(2)}$$

Note that

1. We have changed the dummy variable of the integral from s to K, as a reminder that the second-order derivatives of the call and put options are with respect to the strike.
2. We are using liquid OTM and ATM options, i.e. low-strike puts and high-strike calls, to extract the risk-neutral density.

$$\frac{d}{dx}\left[u(x)\cdot v(x)\right] = \frac{du}{dx}\cdot v(x) + u(x)\cdot\frac{dv}{dx}$$

$$(u\cdot v)' = u'v + uv'$$

$$\int (u\cdot v)' = \int u'v + \int uv'$$

$$u\cdot v = \int u'v + \int uv'$$

$$\int u\cdot v' = u\cdot v - \int u'v$$

Static Replication of European Payoff

Let us consider the call integral (2). Using integration-by-parts twice, we obtain

$$
\begin{aligned}
&\int_F^\infty \overset{u}{h(K)} \overset{v'}{\frac{\partial^2 C(K)}{\partial K^2}}\, dK \\
&= \left[\overset{u}{h(K)} \overset{v}{\frac{\partial C(K)}{\partial K}}\right]_F^\infty - \int_F^\infty \overset{u}{\overset{u'}{h'(K)}} \overset{v'}{\overset{v}{\frac{\partial C(K)}{\partial K}}}\, dK \\
&= \left[\cancelto{0}{h(\infty)\frac{\partial C(\infty)}{\partial K}} - h(F)\frac{\partial C(F)}{\partial K}\right] - \left[\overset{u}{h'(K)}\overset{v}{C(K)}\right]_F^\infty + \int_F^\infty \overset{u'}{h''(K)}\overset{v}{C(K)}\, dK \\
&= -h(F)\frac{\partial C(F)}{\partial K} - \left[\cancelto{0}{h'(\infty)C(\infty)} - h'(F)C(F)\right] + \int_F^\infty h''(K)C(K)\, dK \\
&= -h(F)\frac{\partial C(F)}{\partial K} + h'(F)C(F) + \int_F^\infty h''(K)C(K)\, dK.
\end{aligned}
$$

Static Replication of European Payoff

Applying the same steps to the put integral (1), we can obtain

$$\int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}\,dK = h(F)\frac{\partial P(F)}{\partial K} - h'(F)P(F) + \int_0^F h''(K)P(K)\,dK.$$

Combining both integrals (1) and (2), we have:

$$V_0 = h(F)\left[-\frac{\partial C(F)}{\partial K} + \frac{\partial P(F)}{\partial K}\right] + h'(F)\Big[C(F) - P(F)\Big]$$
$$+ \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\,dK.$$

e^{-rT}

This expression can be simplified further using **put-call parity**:

$$C(K) - P(K) = S_0 - Ke^{-rT}.$$

Static Replication of European Payoff

Note that differentiating both sides of the put-call parity with respect to K yields:

$$\frac{\partial C(K)}{\partial K} - \frac{\partial P(K)}{\partial K} = -e^{-rT}.$$

Also, when $K = F = S_0 e^{rT}$, the call and put options are worth the same, so that:

K $\qquad = S_0 - S_0 \cancel{e^{rT}}\, \cancel{e^{-rT}} = 0$

$$C(F) - P(F) = S_0 - Fe^{-rT} = 0.$$

Substituting both results, we arrive at the final static replication formula:

$$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)\, dK + \int_F^\infty h''(K)C(K)\, dK$$

Reminder Note that K in the integrals is a dummy variable — we use it to remind ourselves that the integrals are weighted across $P(K)$ and $C(K)$, i.e. put and call options across a wide range of strikes.

Static Replication of European Payoff

Example A financial contract pays aS_T^b on maturity date T, where a, $b \in \mathbb{R}^+$ are positive real numbers. Use the static replication method to replicate this payoff using vanilla European call and put options.

Solution With $h(S_T) = aS_T^b$, we have

$$h'(S_T) = abS_T^{b-1}, \quad h''(S_T) = ab(b-1)S_T^{b-2}.$$

Hence, the payoff, which is twice differentiable, can be static replicated with a portfolio of options as follow:

$$\begin{aligned} V_0 = e^{-rT}aF^b &+ \int_0^F ab(b-1)K^{b-2}P(K)\,dK \\ &+ \int_F^\infty ab(b-1)K^{b-2}C(K)\,dK. \end{aligned}$$

Static Replication of a Log Contract

Example Suppose we want to derive the valuation formula for a log contract paying $\log \frac{S_T}{S_0}$ at maturity T, where S_t is the value of a stock.

1. Derive the valuation formula under Black-Scholes model.
2. Formulate the static replication portfolio using the Carr-Madan approach.

1

$$S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma W_T^*}$$

$$\log \frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*$$

$$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right] \Rightarrow V_0 = e^{-rT}\,\mathbb{E}^*\left[\log \frac{S_T}{S_0}\right]$$

$$= e^{-rT}\left(r - \frac{\sigma^2}{2}\right)T$$

② $h(S_T) = \log \frac{S_T}{S_0} = \log S_T - \log S_0$

$$h'(S_T) = \frac{1}{S_T}, \quad h''(S_T) = -\frac{1}{S_T^2}$$

Carr-Madan: $V_0 = e^{-rT}\, \mathbb{E}^*\left[\log \frac{S_T}{S_0}\right]$

$$V_0 = e^{-rT} h(F) + \int_0^F h''(K) \cdot P(K)\,dK + \int_F^\infty h''(K)\, c(K)\,dK$$

$$= e^{-rT} \log \cdot \frac{\overset{= S_0 e^{rT}}{F}}{S_0} - \int_0^F \frac{1}{K^2} P(K)\,dK - \int_F^\infty \frac{1}{K^2} \cdot c(K)\,dK$$

$$= e^{-rT} \cdot rT - \int_0^F \frac{1}{K^2} P(K)\,dK - \int_F^\infty \frac{1}{K^2} c(K)\,dK$$

Variance Swaps

FYI

Variance swaps are contracts which allow us to gain explicit volatility (and variance) exposure. This frees us from the need to worry about delta or gamma hedging if we were to use vanilla options to gain volatility exposure.

The payoff of a variance swap is given by

$$\text{Var Swap} = \text{Notional} \times \left(\sigma_R^2 - \sigma_K^2\right),$$

where σ_R^2 is the **realized variance** of the stock and σ_K^2 is the **strike variance**.

The realized variance σ_R^2 is quantified as

$$\sigma_R^2 = \frac{252}{N}\sum_{i=1}^{N}\left(\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right)^2,$$

where i labels the value of the stock on each day and N is the total number of days in the contract. Variance swaps capture the realized variance of the underlying asset. It is an intuitive contract based on the definition of historical variance. The contract is often described in terms of the fair strike σ_K^2.

$$S_{t_{i+1}} = S_{t_i} \exp\left[\left(r - \frac{\sigma_{t_i}^2}{2}\right)(t_{i+1} - t_i) + \sigma_{t_i} \cdot \left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

$$\log \frac{S_{t_{i+1}}}{S_{t_i}} = \left(r - \frac{\sigma_{t_i}^2}{2}\right)(t_{i+1} - t_i) + \sigma_{t_i} \cdot \left(W_{t_{i+1}} - W_{t_i}\right)$$

$$\left[\log \frac{S_{t_{i+1}}}{S_{t_i}}\right]^2 = \left[\qquad\qquad\qquad\right]^2$$

$$\simeq 0 + \cdot 0 + \sigma_{t_i}^2 \cdot (t_{i+1} - t_i)$$

let $X_t = \log(S_t) = f(S_t)$

Variance Swaps

$dX_t = f'(S_t)\, dS_t + \frac{1}{2} f''(S_t) (dS_t)^2$

To price a variance swap, we observe that the discrete sum over the log returns can be approximated by a continuous time integral

$$\sum_{i=1}^{N} \left[\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2 \approx \int_0^T \sigma_t^2 \, dt.$$

If we apply Itô's formula to a general stochastic differential equation

Not Black-Scholes.

$$dS_t = rS_t dt + \sigma_t S_t dW_t^*,$$

we obtain

dX_t

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma_t^2 dt \quad \Rightarrow \quad \sigma_t^2 dt = 2\left[\frac{dS_t}{S_t} - d\log S_t\right].$$

Integrating both sides and then take expectation, we obtain

$$\int_0^T \sigma_t^2 dt = 2\int_0^T \frac{dS_t}{S_t} - 2\log\left(\frac{S_T}{S_0}\right)$$

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\mathbb{E}^*\left[\int_0^T \frac{dS_t}{S_t}\right] - 2\mathbb{E}^*\left[\log\left(\frac{S_T}{S_0}\right)\right].$$

Variance Swaps

The first term on the RHS can be evaluated readily:

$$2\mathbb{E}^*\left[\int_0^T \frac{dS_t}{S_t}\right] = 2\mathbb{E}^*\left[\int_0^T \frac{r\cancel{S_t}\,dt + \sigma_t \cancel{S_t}\,dW_t^*}{\cancel{S_t}}\right] = 2rT.$$

The second term on the RHS is a static hedge of holding a log contract to expiry. It only depends on the initial stock price S_0 and the final stock price S_T. This is perfectly suited for the static replication approach, and is the same problem we have solved previously for the log contract:

$$2\mathbb{E}^*\left[\log\left(\frac{S_T}{S_0}\right)\right] = 2\log\left(\frac{F}{S_0}\right) - 2e^{rT}\int_0^F \frac{P(K)}{K^2}\,dK - 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}\,dK.$$

Since $F = S_0 e^{rT}$, this can be further simplified into

$$2\mathbb{E}^*\left[\log\left(\frac{S_T}{S_0}\right)\right] = 2rT - 2e^{rT}\int_0^F \frac{P(K)}{K^2}\,dK - 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}\,dK.$$

Variance Swaps

Note that

- ⇒ The log contract can be replicated using a portfolio of European put and call options.
- ⇒ The weighting of the options is $\frac{1}{K^2}$. The portfolio contains all possible strikes.
- ⇒ The portfolio has more weight for downside options than upside options—indicating skew sensitivity.
- ⇒ The portfolio is asking us to place a lot more weight on low strike puts, relative to high strike calls.

Finally, we obtain

$$\tilde{\sigma} T = \quad \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] = 2e^{rT}\int_0^F \frac{P(K)}{K^2}dK + 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}dK.$$

VIX Index

The generalized formula used in the VIX calculation[§] is:

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \qquad \textbf{(1)}$$

WHERE...

σ is	$VIX/100 \Rightarrow \text{VIX} = \sigma \times 100$
T	Time to expiration
F	Forward index level derived from index option prices
K_0	First strike below the forward index level, F
K_i	Strike price of i^{th} out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$.
ΔK_i	Interval between strike prices – half the difference between the strike on either side of K_i:

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$$

(*Note*: ΔK for the lowest strike is simply the difference between the lowest strike and the next higher strike. Likewise, ΔK for the highest strike is the difference between the highest strike and the next lower strike.)

R	Risk-free interest rate to expiration
$Q(K_i)$	The midpoint of the bid-ask spread for each option with strike K_i.