$$\mathbb{E}^{\mathbb{P}^{HF}}[S_T] \gg S_0$$

$$\mathbb{E}^{\mathbb{P}^{\text{hopeless}}}[S_T] \ggg S_0$$

$\mathbb{Q}$

risk-neutral

if can hedge

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$$

$$\mathbb{E}^{\mathbb{P}^{\text{bull}}}[S_T] > S_0$$

$$\mathbb{E}^{\mathbb{P}^{\text{bear}}}[S_T] < S_0$$

American Options under Binomial Tree

An **American option** can be exercised on any time before expiry. We can use binomial tree to capture the early exercise premium:

$\Rightarrow$ We should only exercise if we make more money by exercising the option.

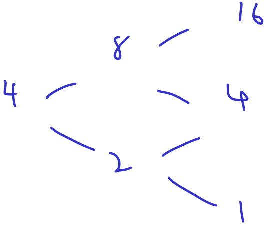
For European options, risk-neutral expectation is given by

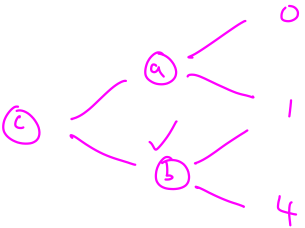
$$V_n^E = \frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d]$$

For American options, this should become

$$V_n^A = \max\left\{\frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d],\ (K - S_n)^+\right\}$$

Example Determine the price of an American put option for the same example in the previous page.





$$\text{a} = \max\left\{ \frac{1}{1.25} \times \left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 \right),\ (5-8)^+ \right\}$$

$$= \max\left\{ \frac{4}{5} \times \frac{1}{2},\ 0 \right\} = \frac{2}{5}$$

$$\text{b} = \max\left\{ \frac{4}{5} \times \left(\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4 \right),\ (5-2)^+ \right\}$$

$$= \max\left\{ \frac{2}{5} \times 5,\ 3 \right\} = 3$$

c $= \max\left\{ \frac{4}{5} \times \left(\frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot 3 \right), (5-4)^+ \right\}$

$= \max\left\{ \frac{2}{5} \times \left(\frac{2}{5} + 3 \right), 1 \right\}$

$= \frac{34}{25}$

V_A

$V_E = \frac{24}{25}$

$V_A - V_E = \frac{10}{25} = \underline{E}\,\underline{E}\,P$

early exercise premium

Central Limit Theorem for Geometric Process

Central Limit Theorem can be used to determine the distribution of the product of independent random variables X_i. Consider the product

$$W = X_1 \times X_2 \times \cdots X_n,$$

as long as $X_i \in (0, \infty)$, we can write

$$\ln W = \ln X_1 + \ln X_2 + \cdots + \ln X_n.$$

This is just the sum of n independent random variables $\ln X_i$.

As long as each of these random variables has a finite mean and variance, the distribution of $\ln W$ will tend towards a normal distribution under the limit $n \to \infty$.

In other words, W will follow a **lognormal distribution** by Central Limit Theorem.

Lognormal (Geometric) Binomial Tree

Suppose we now model the **log-price process**, and also take interest rate into consideration. We want

1. $\mathbb{E}[\ln(S_T)] = \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$
2. $V[\ln(S_T)] = \sigma^2 T$

Following the same approach as the normal binomial tree, we divide the time interval into n uniform steps, and have the following relationship:

$$\ln(S_i) = \ln(S_{i-1}) + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}X_i,$$

where X_i takes on values ± 1 with equal probability $\frac{1}{2}$. Adding them up, we have

$$\ln(S_T) = \ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i.$$

→ CLT

Lognormal (Geometric) Binomial Tree

Once again, we can apply Central Limit Theorem to show that

$$\ln(S_T) = \ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z, \qquad Z \sim N(0,1)$$

$$\Rightarrow S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right].$$

Black-Scholes stock price model

$\ln S_T - \ln S_0 = \ln \frac{S_T}{S_0}$

Example Show that the distribution of the log stock price satisfies the requirement of $\mathbb{E}[\ln(S_T)] = \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T$ and $V[\ln(S_T)] = \sigma^2 T$.

Lognormal (Geometric) Binomial Tree

$X \sim N(\mu, \sigma^2)$

$M_X(\theta) = \mathbb{E}[e^{\theta X}]$

$= e^{\mu\theta + \frac{1}{2}\theta^2\sigma^2}$

Example If

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right], \quad Z \sim N(0,1),$$

$\theta : \to \sigma\sqrt{T}$

$\mu : \to 0$

$\sigma^2 : \to 1$

use the MGF for normal distribution to show that $\mathbb{E}[S_T] = S_0 e^{rT}$, thus satisfying the no-arbitrage condition under the risk-neutral measure.

$$\bar{\mathbb{E}}[S_T] = \bar{\mathbb{E}}\left[S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}\right]$$

$$= S_0 e^{(r - \frac{\sigma^2}{2})T} \, \bar{\mathbb{E}}\left[e^{\sigma\sqrt{T}Z}\right]$$

$$= S_0 e^{(r - \frac{\sigma^2}{2})T} \, M_X(\sigma\sqrt{T}) = S_0 e^{(r - \cancel{\frac{\sigma^2}{2}})T} \cdot e^{0 + \cancel{\frac{1}{2}\sigma^2} T \cdot 1}$$

$S_0 e^{rT}$

$=$

Changing Measure – Example

Toss a coin for 3 times, and define 2 random variables:

1. $X =$ total number of heads
2. $Y =$ total number of tails

The random variables can be specified without knowing the probability measure (what are they?).

If we specify the probability measure of a fair coin $p = \frac{1}{2}$, we can determine the distribution of X and Y — they have the same distribution.

If we specify the probability measure of a biased coin $p = \frac{2}{3}$, then we will get a different distribution.

Let X be a random variable defined on a finite probability space $(\Omega, \mathbb{P})$, the expectation of X is defined as

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega).$$

$p^* \cdot S_0 \cdot u + (1-p^*) \cdot S_0 \cdot d = \mathbb{E}^*[S_1] = S_0(1+r)$

Conditional Expectation

In our binomial pricing model, we have chosen the risk-neutral probability measure

$$p^* = \frac{(1+r)-d}{u-d}, \qquad q^* = \frac{u-(1+r)}{u-d}. \quad = 1 - p^*$$

At every time-step t_n, we have

$$S_n = \frac{1}{1+r}[p^* S_{n+1}^u + q^* S_{n+1}^d].$$

Using the expectation notation, we can write

$$\mathbb{E}_n^*[S_{n+1}] = p^* S_{n+1}^u + q^* S_{n+1}^d$$
$$\Rightarrow \quad \therefore \; S_n = \frac{1}{1+r}\mathbb{E}_n^*[S_{n+1}].$$

The notation $\mathbb{E}_n^*[S_{n+1}]$ denotes the **conditional expectation** of S_{n+1} based on the information at time t_n.

Conditional Expectation Properties

Linearity: for all constants a and b, we have

$$\mathbb{E}_n[aX + bY] = a\mathbb{E}_n[X] + b\mathbb{E}_n[Y].$$

Extracting known variable: if X only depends on the first n tosses, then

$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y].$$

Iterated expectation: if $0 \leq n \leq m \leq N$, then

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X].$$

Independence: if X only depends on the $(n+1)^{th}$ toss, then

$$\mathbb{E}_n[X] = \mathbb{E}[X].$$

$\mathbb{E}[S_T] = S_0 e^{\mu T}$ $\mathbb{E}^*[S_T] = S_0 e^{rT}$

Martingales

We have derived the conditional expectation earlier as follow

$\mathbb{E}^*_n[S_{n+1}] = S_n(1+r)$

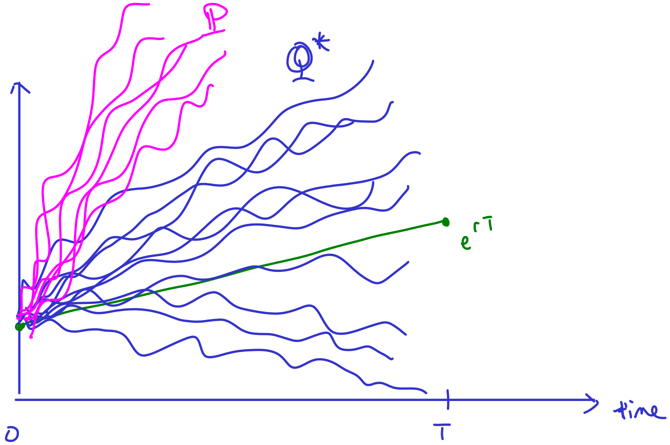
$$S_n = \frac{1}{1+r}\mathbb{E}^*_n[S_{n+1}].$$

Dividing both sides by $(1+r)^n$, we obtain

$$M_n = \frac{S_n}{(1+r)^n} = \mathbb{E}^*_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right].$$

Under the risk-neutral measure, the best estimate based on the information at time t_n of the value of the discounted stock price at time t_{n+1} is the discounted stock price at time t_n:

$$M_n = \mathbb{E}^*_n[M_{n+1}], \quad n = 0, 1, \ldots, N-1$$
$$\therefore\ M_0 = \mathbb{E}^*[M_N]$$



$$\mathbb{E}^{P}\left[S_T\right] > \mathbb{E}^{Q}\left[S_T\right] > S_0$$

$$\mathbb{E}^{P}\left[\frac{S_T}{(1+r)^T}\right] > \mathbb{E}^{Q}\left[\frac{S_T}{(1+r)^T}\right] = \frac{S_0}{(1+r)^0}$$

Martingales

$$\mathbb{E}^*_n\left[\frac{S_{n+1}}{S_n}\right] = \frac{p^*\cdot u\cdot \cancel{S_n} + (1-p^*)\cdot d\cdot \cancel{S_n}}{\cancel{S_n}}$$

Consider the general binomial model with $0 < d < 1 + r < u$, with the risk-neutral probabilities

$$p^* = \frac{(1+r)-d}{u-d}, \qquad q^* = \frac{u-(1+r)}{u-d}.$$

Under the risk-neutral measure, the discounted stock price is a **martingale**

$$\begin{aligned}
\mathbb{E}^*_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \mathbb{E}^*_n\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{1}{1+r}\mathbb{E}^*_n\left[\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{p^*\times u + q^*\times d}{1+r} \\
&= \frac{S_n}{(1+r)^n}.
\end{aligned}$$

Risk-Neutral Pricing Formula

Consider an N-period binomial tree pricing model with $0 < d < 1 + r < u$, and with the risk-neutral probability measure. Let V_N be a derivative security with payoff at time t_N, depending on the outcomes of coin tosses.

$$\frac{S_n}{(1+r)^n} = \mathbb{E}^*_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]$$

For $0 \leq n \leq N$, the discounted price of the derivative security is a martingale:

$$\frac{V_n}{(1+r)^n} = \mathbb{E}^*_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right].$$

Furthermore, the price of the derivative security at time t_n is given by the risk-neutral pricing formula

$$V_n = \mathbb{E}^*_n\left[\frac{V_N}{(1+r)^{N-n}}\right].$$

Arbitrage & Complete Markets

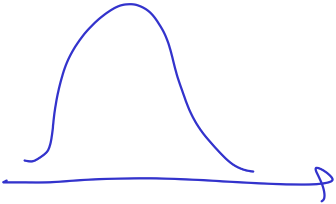
- A **complete market** is one in which a derivative product can be **replicated** from more basic instruments, such as cash and the underlying asset.
- This usually involves **dynamically rebalancing** a portfolio of the simpler instruments, according to some formula or algorithm, to replicate the more complicated product, the derivative.
- In an **incomplete market** (e.g. with trading frictions), you cannot replicate the option with simpler instruments.
- In a complete market you can replicate derivatives with the simpler instruments. But you can also turn this on its head so that you can **hedge** the derivative with the underlying instruments to make a risk-free instrument.
- In the binomial model you can replicate an option from stock and cash, or you can hedge the option with the stock to make cash.

Arbitrage & Complete Markets

- In a complete market, all security prices are attainable, and there is no arbitrage opportunities.
- The absence of arbitrage implies the existence of the risk-neutral measure probabilities.
- Under the **real-world measure**, riskier assets tend to have higher expected return than less risky assets.
- Under the **risk-neutral measure**, asset prices grow at the risk-free rate, and the relative value of any two assets is a martingale over time.
- So far we have used the risk-free interest account as the denominator in our ratio. However, this is not the only choice — the choice is in fact arbitrary.

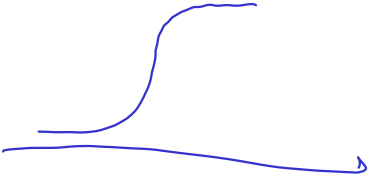
  $\Rightarrow$ Under the risk-neutral measure, all relative price processes should be a **martingale**.

PDF



distribution funct.   CDF

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$



Session 3: Brownian Motion and Martingale

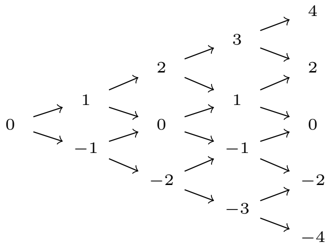
Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Random Walk on a Number Line

A <u>1-dimensional</u> **symmetric random walk** starts at 0 and goes up or down with an equal probability of 0.5.

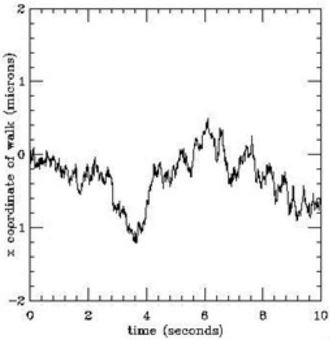
For instance, consider a 4-step 1-d symmetric random walk:

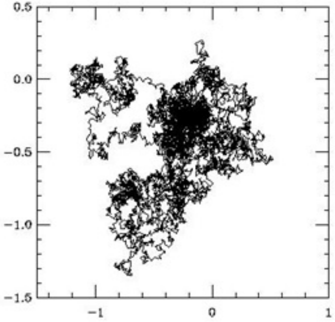


- Total number of paths $= 2^4$.
- Max. $= 4$, Min. $= -4$.
- $S_4 \sim \text{Bin}(4, 0.5)$.

These motions were such as to satisfy me, after frequently repeated observation, that they arose neither from currents in the fluid, nor from its gradual evaporation, but belonged to the particle itself.

— Robert Brown (1773–1858)





Brownian Motion

...the phenomenon of Brownian motion, the apparently erratic movement of tiny particles suspended in a liquid: Einstein showed that these movements satisfied a clear statistical law.

— C. P. Snow (1905–1980)

Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine: la loi de la probabilité.

— Louis Bachelier, Théorie de la spéculation, 1900

Rough translation:

If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

THE OPERATIONS OF THE STOCK EXCHANGE.

Stock Exchange Operations. — There are two kinds of forward-dated operations[1]:

- Forward contracts[2],
- Options[3].

These operations can be combined in infinite variety, especially since several types of options are dealt with frequently.

LES OPÉRATIONS DE BOURSE.

Opérations de bourse. — Il y a deux sortes d'opérations à terme :
Les opérations fermes ;
Les opérations à prime.
Ces opérations peuvent se combiner à l'infini, d'autant que l'on traite souvent plusieurs sortes de primes.

THÉORIE

DE

LA SPÉCULATION,

PAR M. L. BACHELIER.

Fig. 5.

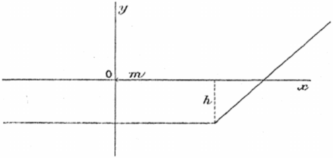


Fig. 7.

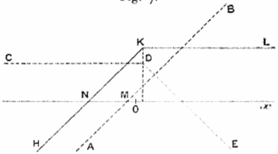
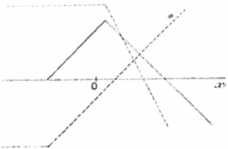


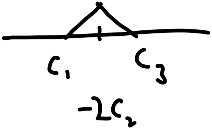
Fig. 10.



broken wing butterfly

Probabilité de l'opération en blanc.......... 0,30
» de bénéfice..................... 0,45
» de perte....................... 0,25

butterfly

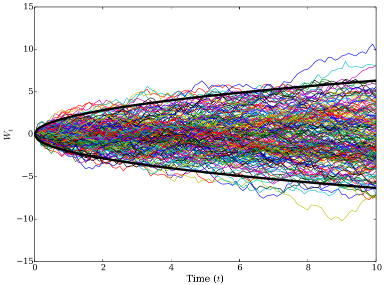


Brownian Motion

As early as 1900, **Louis Bachelier**, in his thesis "La Théorie de la Spéculation", proposed Brownian motion as a model of the fluctuations of stock prices.

Even today it is the building block from which we construct the basic reference model for a continuous time market.

We shall approach this topic by considering Brownian motion as an **infinitesimal random walk** in which smaller and smaller steps are taken at ever more frequent time intervals.



From Random Walk to Brownian Motion

Consider a positive integer n, define a **scaled random walk** $W_n(t)$ to have the following properties

1. $W_n(0) = 0$
2. time spacing is $\frac{t}{n}$
3. up and down jumps equal and of size $\sqrt{\frac{t}{n}}$
4. measure $\mathbb{P}$, given by up and down probabilities everywhere equal to $\frac{1}{2}$

In other words, $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the value of W_n on the i^{th} step is defined by

$$W_n\left(\frac{i}{n}\cdot t\right) = W_n\left(\frac{i-1}{n}\cdot t\right) + \sqrt{\frac{t}{n}}X_i, \quad 1 \leq i.$$

When n becomes large, W_n will not blow out due to the scaling of $\sqrt{\frac{t}{n}}$. It can be shown that

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}\right).$$

$$w_n\left(\frac{0}{n}\cdot t\right) = 0$$

$$w_n\left(\frac{1}{n}\cdot t\right) = 0 + \sqrt{\frac{t}{n}}\cdot X_1 = \sqrt{\frac{t}{n}}\cdot X_1$$

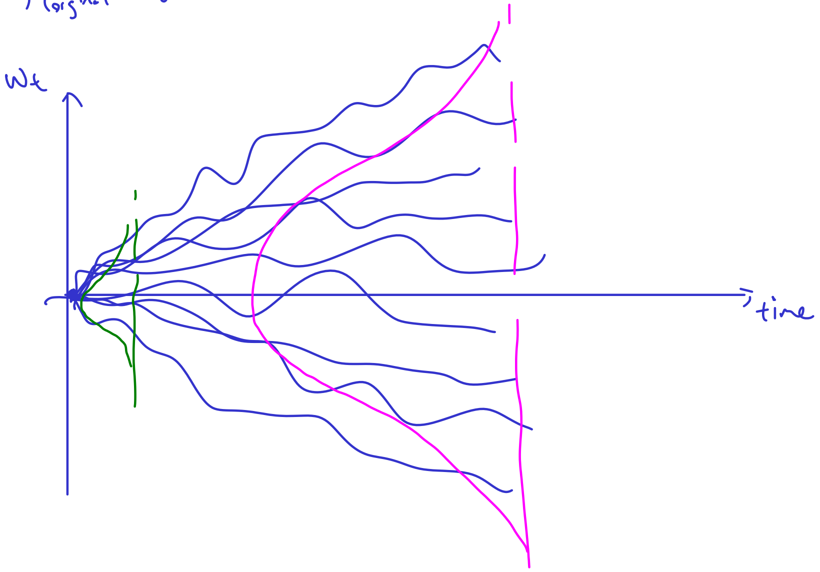
$$w_n\left(\frac{2}{n}\cdot t\right) = w_n\left(\frac{1}{n}t\right) + \sqrt{\frac{t}{n}}\cdot X_2 = \sqrt{\frac{t}{n}}\cdot\left(X_1 + X_2\right)$$

$$w_n\left(\frac{3}{n}\cdot t\right) = \ldots = \sqrt{\frac{t}{n}}\left(X_1 + X_2 + X_3\right)$$

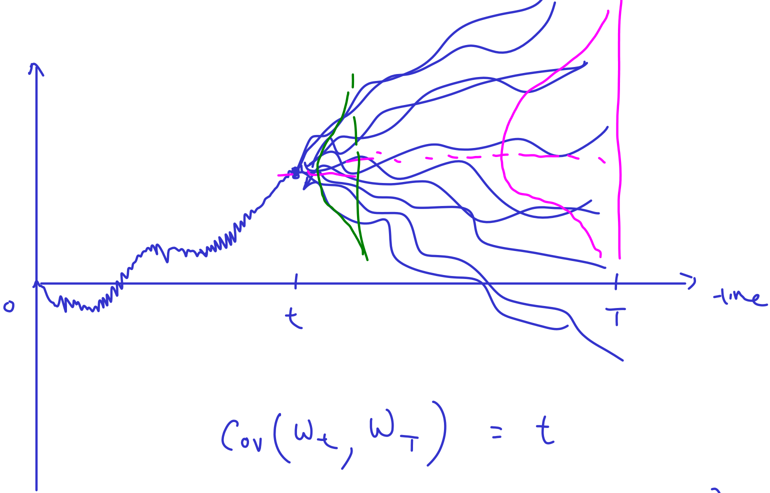
From Random Walk to Brownian Motion

- The distribution in the brackets tends to $N(0,1)$ by **central limit theorem**. Hence the distribution of $W_n(t)$ tends to $N(0,t)$.
- All the marginal distributions and conditional marginal distributions tend towards the same underlying normal distribution.
- Each random walk W_n has the property that its future movements away from a particular position are independent of where that position is, and indeed independent of its entire history of movements up to that time.
- Additionally, such a future displacement $W_n(s+t) - W_n(s)$ is binomially distributed with zero mean and variance t.
- Once again, the central limit theorem says that all conditional marginals tend towards a normal distribution of the same mean and variance.
- In other words, **Brownian motion is the limit of a scaled random walk** as $n \to \infty$.

Marginal dist



condition on t



$$\mathrm{Cov}(W_t, W_T) = t$$

$$= \mathrm{Cov}(W_t, W_T - W_t) + \mathrm{Cov}(W_t, W_t)$$

Mathematical Definition of Brownian Motion

The actual development of **Brownian motion** as a stochastic process did not appear until 1923 when mathematician **Norbert Weiner** established the modern mathematical framework of what is known today as the Brownian motion random process.

Brownian motion (or **Wiener process**) has the following properties:

1. $W_0 = 0$.
2. W_t is continuous for $t \geq 0$, there are no jumps.
3. It has stationary and independent increments.
4. For $0 \leq s \leq t$, $W_t - W_s$ follows normal distribution where $W_t - W_s \sim N(0, t-s)$.

These are necessary and sufficient conditions for a process to be identified as a Brownian motion.

Brownian motion is an important building block for modeling **continuous-time stochastic processes**. It has become an important framework for modeling financial markets.

$X_0 = \sqrt{0} \cdot Z = 0$, $X_t = \sqrt{t} Z$, $Z \sim N(0,1)$

Mathematical Definition of Brownian Motion

$X_t \sim \sqrt{t} \cdot N(0,1)$

Example If $Z \sim N(0,1)$, then $X_t = \sqrt{t}Z$ is continuous, and is marginally $\sim N(0,t)$ distributed as $N(0,t)$. Is X_t a Brownian motion?

$V[aX] = a^2 V[X]$

Solution The increment $X_{s+t} - X_s$ is normally distributed with a mean of 0 and a variance of

$\mathbb{E}\left[(aX - a\mu)^2\right]$

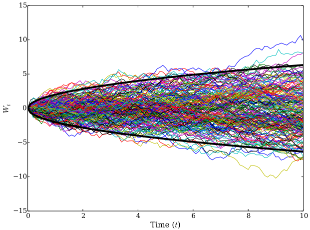
$$
\begin{aligned}
X_{s+t} - X_s &= \sqrt{s+t}Z - \sqrt{s}Z \\
&\sim \left(\sqrt{s+t} - \sqrt{s}\right) N(0,1) \qquad \because\ Z \text{ is the same RV} \\
&\sim N\left(0, \left(\sqrt{s+t} - \sqrt{s}\right)^2\right) \\
&\sim N\left(0, 2s + t - 2\sqrt{s(s+t)}\right).
\end{aligned}
$$

- For the process to be a Brownian motion, the increment needs to be of variance t.
- In addition, the increment is not independent of X_s.
- So X_t is not a Brownian motion. $\triangleleft$

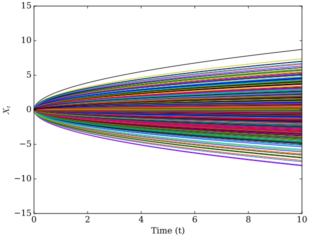
Mathematical Definition of Brownian Motion

- Consider $s < t$, it is important to realize that the condition that the distribution is normal of variance $t - s$ for every t and s and independent of the path up to time s, is much stronger than requiring W_t to be normally distributed with variance t for every t.

- For example, if we let $X_t = \sqrt{t}Z$, where Z is the same draw from a normal distribution for all t then we have that X_t is normally distributed with variance t.



vs



Mathematical Definition of Brownian Motion

Given this formal definition, we can proceed to derive the following statistical properties based on the properties of normal distribution:

$$\mathbb{E}[W_t] = 0, \quad W_t \sim N(0, t)$$

$$V[W_t] = \mathbb{E}[W_t^2] - \cancel{\mathbb{E}[W_t]}^{\,0\,2} = \mathbb{E}[W_t^2] = t$$

$$\mathbb{E}[W_t - W_s] = 0$$

$$V[W_t - W_s] = \mathbb{E}[(W_t - W_s)^2] = t - s$$

$$\mathrm{Cov}(W_s, W_t) = s, \quad s < t$$

For $t_1 < t_2 < t_3 < \cdots < t_n$, note that the Brownian motions $W_{t_1}, W_{t_2}, W_{t_3}, \ldots, W_{t_n}$ are **jointly normal** with mean 0 and covariance matrix

$$\begin{array}{cc} & \begin{matrix} t_1 & t_2 & t_3 & \cdots \end{matrix} \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \\ \vdots \end{matrix} & \begin{bmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_n \end{bmatrix} \end{array}.$$

$\mathrm{Cov}(W_{t_1}, W_{t_1}) \quad \mathrm{Cov}(W_{t_2}, W_{t_1})$

$\mathrm{Cov}(W_{t_1}, W_{t_2}) \quad \mathrm{Cov}(W_{t_2}, W_{t_2})$

$\vdots$

$$\mathbb{E}[W_t - W_s] = \mathbb{E}[W_t] - \mathbb{E}[W_s]$$

$$= 0 - 0 = 0$$

$$W_t - W_s \sim N(0, t-s)$$

$$\mathbb{E}[W_t - W_s] = \mathbb{E}[N(0, t-s)] = 0$$

$$
\begin{aligned}
\mathrm{Cov}(W_s, W_t) &= \mathrm{Cov}(W_s, \underbrace{W_t - W_s} + \underbrace{W_s}) \\
&= \mathrm{Cov}(W_s, W_t - W_s) + \mathrm{Cov}(W_s, W_s) \\
&= 0 + V[W_s] \\
&= s
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X, Y) \\
+ \mathrm{Cov}(X, Z)
\end{aligned}
$$