Black-Scholes Assumptions

The Black-Scholes market model contains two differential equations

$\mathbb{P}$ W_t

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = rB_t dt \end{cases}$$

The context is that the market contains

1. A **risky asset** S_t, typically a stock price process.
2. A **risk-free asset** B_t, typically a risk-free bond.

Assumptions made include:

1. Underlying is lognormal with constant mean and variance.
2. The risk-free rate r is a constant.
3. No dividend is paid during the life of the option.
4. Short selling is permitted.
5. No risk-free arbitrage opportunities.
6. Trading is possible in continuous time.
7. No transaction costs, no taxes and no trading limits.

$\mathbb{P}$: $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$

$$dB_t = r B_t \, dt$$

$$\int_0^T \frac{dB_t}{B_t} = \int_0^T r \, dt$$

$$\left[\log B_t \right]_0^T = rT$$

$$\log \frac{B_T}{B_0} = rT$$

$$B_T = B_0 e^{rT}$$

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

$$\mathbb{E}^P\left[S_T \right] = S_0 e^{\mu T}$$

we hope $\mu > r$

$$\mathbb{E}^P\left[\frac{S_T}{B_T} \right] > \frac{S_0}{B_0}$$

$\mathbb{Q}^*$:

$$\frac{S_0}{B_0} = \mathbb{E}^{\mathbb{Q}^*}\left[\frac{S_T}{B_T}\right]$$

$B_T = B_0 e^{rT}$

$$P_0 = e^{-rT}\,\mathbb{E}^*\left[P_T\right] \Leftarrow \frac{P_0}{B_0} = \mathbb{E}^{\mathbb{Q}^*}\left[\frac{P_T}{B_T}\right]$$

$$C_0 = e^{-rT}\,\mathbb{E}^*\left[C_T\right] \qquad \frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}^*}\left[\frac{C_T}{B_T}\right]$$

Choice of Numeraire

Based on previous discussions on the stock price process, we have established that it is growing at the risk-free rate under the risk-neutral measure:

$$\mathbb{E}_t^*[S_{t+\Delta t}] = S_t e^{r\Delta t}.$$

The expectation notation $\mathbb{E}^*$ is used to indicate that the expectation is evaluated under the risk-neutral measure $\mathbb{Q}^*$. This relationship can be rearranged into

$$\frac{S_t}{e^{rt}} = \mathbb{E}_t^*\left[\frac{S_{t+\Delta t}}{e^{r(t+\Delta t)}}\right].$$

- In words, this means that the best estimate of the price ratio on the subsequent time step is just the price ratio on the current time step.
- The security in the denominator of the price ratio expression is called the **numeraire** security.
- The only requirement for a particular security to qualify as a numeraire security is that it has to be **strictly positive** at all times.
- The risk-free money market account paying an interest of r is a popular choice of numeraire.

Equivalent Martingale Measure

Key concepts:

- In a complete market, any derivative security is attainable. Since we can hedge a derivative product perfectly, the derivative security **loses its randomness** and **behaves like a risk-less bond**.
- So real world probabilities do not come into the picture in a risk-neutral valuation framework at all.
- If we hedge according to our risk-neutral valuation framework, then all risk is eliminated, and the hedged portfolio grows at a risk-free rate.
- Consequently, the hedged portfolio divided by the risk-free bond is a **martingale**.
- Two probabilities measures are equivalent if they agree on what is possible and what is impossible.

Equivalent Martingale Measure

- In other words, if one portfolio is an arbitrage in one measure, then it is an arbitrage in all other equivalent measures.

- If the option price we determined under the risk-neutral measure is **arbitrage-free**, then it is arbitrage-free in the real world.

- If we can express security price processes discounted by a numeraire security as a martingale, then there can be no arbitrage opportunities.

- Under the risk-neutral probabilities associated to this numeraire security, the discounted option price is also a martingale, and we can therefore determine its present value.

- The risk-free money market account $B_t = B_0 e^{rt}$ is a common choice for numeraire (used by Harrison and Kreps (1979)), but **the choice is arbitrary**.

Application of EMM — Black-Scholes

Under the **Black-Scholes economy**, let B_t denote the value of the money-market account with $B_0 = 1$, and let S_t denote the stock price process. The following differential equations described their dynamics:

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Here W_t is a $\mathbb{P}$-Brownian motion under the real-world measure, and μ is its (unknown) drift coefficient.

Question Which is the most difficult parameter to estimate among r, μ, and σ?

Source: Google Finance

$\underline{\mathbb{P}}$:
$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$
$$dB_t = r B_t \, dt$$

$$f(b,x) = \frac{x}{b}$$
$$f_b = -\frac{x}{b^2}$$
$$f_x = \frac{1}{b}, \quad f_{xx} = 0$$

Let $\quad X_t = \dfrac{S_t}{B_t} = f(B_t, S_t)$

$\underline{\text{Itô's}}$:
$$dX_t = f_b(B_t, S_t)\, dB_t + f_x(B_t, S_t)\, dS_t + \frac{1}{2} f_{xx}(B_t, S_t)(dS_t)^2$$
$$= -\frac{S_t}{B_t^{\cancel{2}}} \cdot r \cancel{B_t}\, dt + \frac{1}{B_t}\left(\mu S_t \, dt + \sigma S_t \, dW_t\right)$$

$$dX_t = (\mu - r) X_t \, dt + \sigma X_t \, dW_t$$

$$dX_t = \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right) \qquad \mathbb{P}$$

Girsanov : $\mathbb{P} \to \mathbb{Q}^*$

$$= \sigma X_t \, dW_t^* \qquad \mathbb{Q}^*$$

$$\boxed{dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt}$$

$\mathbb{P}$: $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$

$\mathbb{Q}^*$: $= \mu S_t \, dt + \sigma S_t \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right)$

$$= r S_t \, dt + \sigma S_t \, dW_t^*$$

Application of EMM — Black-Scholes

The value of B_t is strictly positive and can be used as a numeraire. Define the relative price process $X_t = \frac{S_t}{B_t} = f(S_t, B_t)$, we can apply Itô's formula to obtain

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

To identify the equivalent martingale measure we apply **Girsanov's theorem** with $\kappa = \frac{\mu - r}{\sigma}$ to obtain:

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt,$$

where W_t^* is a standard Brownian motion under probability measure $\mathbb{Q}^*$. Here the * notation is used to indicate we have chosen the **risk-free account** B_t **as our numeraire**, which is the most common choice. Substituting, we obtain

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^*. \end{aligned}$$

Application of EMM — Black-Scholes

This is the only measure which turns the relative price process into martingale. We can now determine what is the stock price process under this unique **martingale measure** $\mathbb{Q}^*$:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\
&= r S_t dt + \sigma S_t dW_t^*.
\end{aligned}
$$

Under the equivalent martingale measure, the drift of the stock μ is irrelevant and is replaced by the risk-free interest rate r. The solution to this stochastic differential equation is

$$
S_T = S_0 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T^* \right].
$$

Application of EMM — Black-Scholes

$W_T^* \sim N(0,T) = \sqrt{T}\, N(0,1)$

A European call option with strike K and maturing at time T where $V_T = (S_T - K)^+$ can be evaluated by **martingale pricing theorem** as follow

$$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right] = \mathbb{E}^*\left[\frac{(S_T-K)^+}{B_T}\right] \quad B_0 e^{rT}$$

$$\rightarrow V_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} - K\right]^+ e^{-\frac{x^2}{2}}\,dx$$

$$\rightarrow V_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_1 - \sigma\sqrt{T}), \quad d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

We have already learned how to derive the Black-Scholes **option pricing formula** by evaluating the expectation.

Session 8: Static Replication of European Payoffs

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Behavior of Model Parameters – ρ

Implication on Distribution

- The correlation parameter ρ is proportional to the skewness of stock returns.
- Intuitively, a negative correlation results in high volatility when the stock price drops, and this spreads the left tail of the probability density. The right tail is associated with low volatility and is not spread out.
- A negative correlation creates a fat left tail and a thin right tail in the stock return distribution.

Implication on Pricing

- This increases the prices of out-of-the-money puts and decreases the prices of out-of-the-money calls relative to the Black-Scholes model price.
- Intuitively, out-of-the-money put options benefit substantially from a fat left tail.
- A positive correlation will have completely opposite effects—it creates a fat right tail and a thin left tail.

SABR :

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} \, dW_t^F \\ d\alpha_t = \nu \alpha_t \, dW_t^{\alpha} \end{cases}$$

$$dW_t^F \, dW_t^{\alpha} = \rho \, dt$$

$$\alpha \, , \quad \beta \, , \quad \rho \, , \quad \nu$$

$\uparrow$

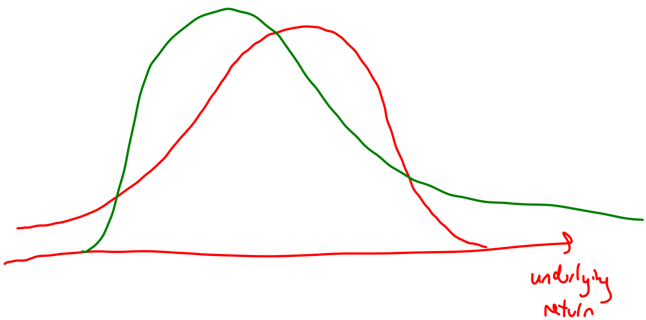
usually fixed

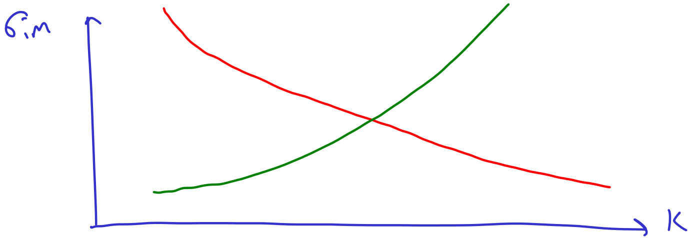
if $\rho > 0$: $F_t \uparrow, \alpha_t \uparrow$; $F_t \downarrow, \alpha_t \downarrow$

$\Rightarrow \rho < 0$: $F_t \uparrow, \alpha_t \downarrow$; $F_t \downarrow, \alpha_t \uparrow$

neg ρ

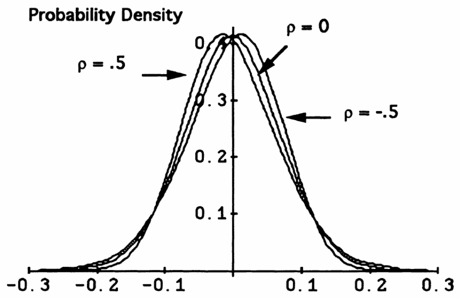
pos ρ





σ, m
K

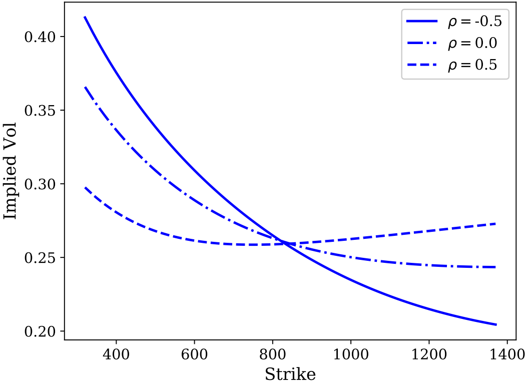
Behavior of Model Parameters – ρ



$\Rightarrow$ **Positive correlation** between stock and volatility is associated with **positive skew** in return distribution.

$\Rightarrow$ **Negative correlation** between stock and volatility is associated with **negative skew** in return distribution.

Behavior of Model Parameters – ρ



Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.

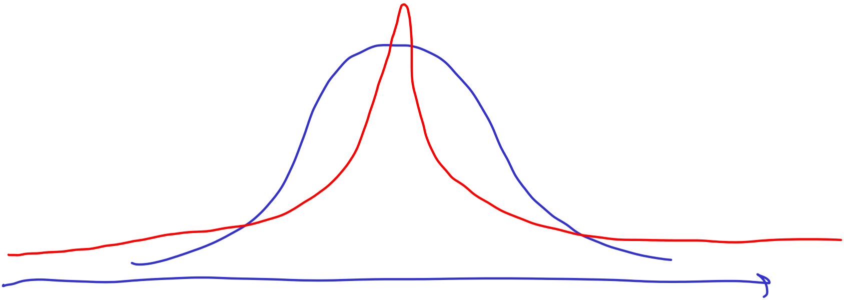
Behavior of Model Parameters – ν

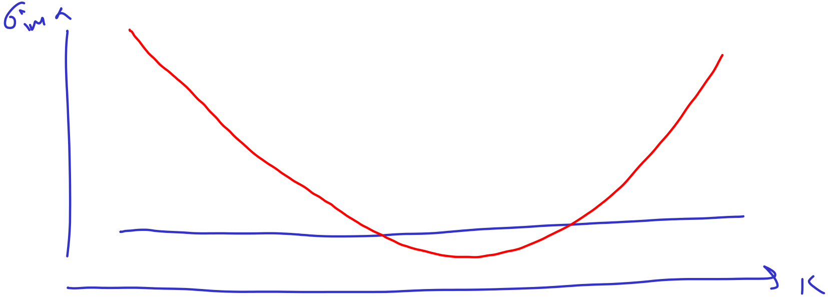
Implication on Distribution

- When the volatility of volatility parameter is 0, we are back to a normal log-return distribution (if $\beta = 0$).
- Otherwise, it increases the kurtosis of stock returns, creating two fat tails in both ends of the distribution.
- This has the effect of raising out-of-the-money puts and out-of-the-money call prices.

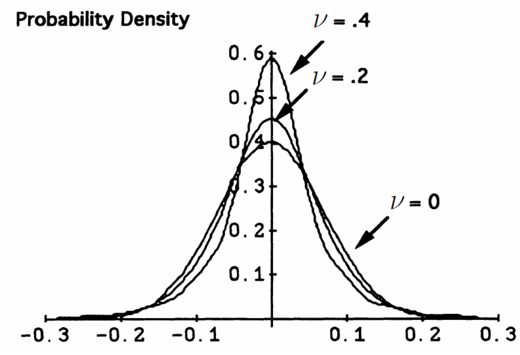
Implication on Pricing

- If volatility is uncorrelated with stock return, then increasing the volatility of volatility only increases the kurtosis of spot return.
- In this case, random volatility is associated with increases in the prices of far-from-the-money options relative to near-the-money options.
- In contrast, the correlation of volatility with the spot return produces skewness.




σ_{im}
K

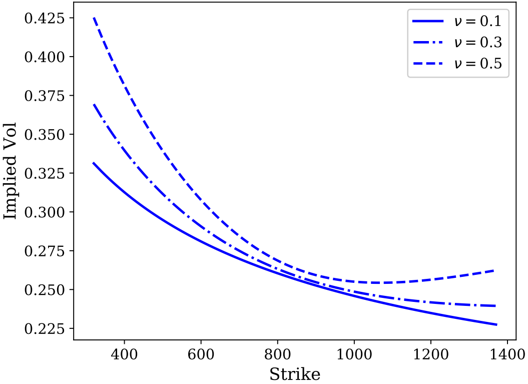
Behavior of Model Parameters – ν



⇒ **Increasing volatility-of-volatility** has the effect of **increasing the kurtosis** of return.

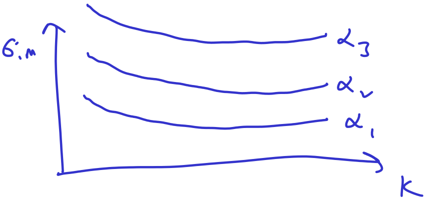
⇒ When the volatility-of-volatility parameter is 0, volatility will be deterministic.

Behavior of Model Parameters – ν



Larger volatility-of-volatility ν increases the price of out-of-the-money call and put options.

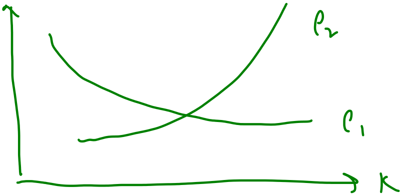
$\alpha \rightarrow$ level



$\alpha_1 < \alpha_2 < \alpha_3$

$\rho \rightarrow$ skew

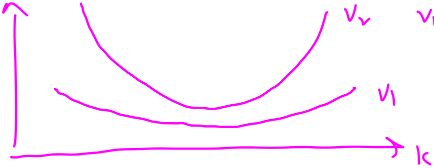
slope



$\rho_1 < 0 < \rho_2$

$\nu \rightarrow$ kurt

curvature



$\nu_1 < \nu_2$

Differentiation of Integrals

The chain rule for partial differentiation states that if we have a bivariate function $g(u(x), v(x))$, where u and v are both functions of x, then

$$\frac{dg}{dx} = \frac{\partial g}{\partial u} u'(x) + \frac{\partial g}{\partial v} v'(x).$$

Now consider the case where the integrand is $f(x,t)$ and t is the dummy variable. Let F denote the anti-derivative of f, we can write it as

$$\int_{u(x)}^{v(x)} f(x,t)dt = F(x, v(x)) - F(x, u(x)).$$

Let's define

$$I(v(x), u(x), x) = F(x, v(x)) - F(x, u(x)),$$

we can see that

$$\frac{\partial I}{\partial v} = f(x, v(x)), \qquad \text{and} \qquad \frac{\partial I}{\partial u} = -f(x, u(x)).$$

Leibnitz's Rule

Now differentiating I with respect to x, we obtain

$$\begin{aligned}\frac{dI}{dx} &= \frac{\partial I}{\partial v}v'(x) + \frac{\partial I}{\partial u}u'(x) + \frac{\partial I}{\partial x}\cancelto{1}{\frac{dx}{dx}} \\ &= f(x,v(x))v'(x) - f(x,u(x))u'(x) + \frac{\partial}{\partial x}\int_{u(x)}^{v(x)} f(x,t)\,dt \\ &= f(x,v(x))v'(x) - f(x,u(x))u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\,dt.\end{aligned}$$

Under the special case where the integrand isn't a function of x, then we recover the corollary integration relationship:

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))v'(x) - f(u(x))u'(x).$$

Modelling

Assumptions

↓

Specify model (dynamic)

↓

Calibration

↓

calibrated model

↓

price exotics

Market Data

Model-free

distribution ?

What is the "model-free" framework?

In a **model-free** formulation, we let $f(s)$ denote the risk-neutral probability density function of the stock price at time T, we can price a vanilla European call option maturing at time T as follows:

$$C(K) = e^{-rT}\mathbb{E}^*[(S_T - K)^+] = e^{-rT}\int_K^\infty (s-K)f(s)\,ds.$$

In earlier modelling approach, we will attempt to specify a model for the stock price process. A typical example is the Black-Scholes model, which will lead to:

$$C(K) = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^\infty \left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K\right)e^{-\frac{x^2}{2}}\,dx.$$

We could also have used the Bachelier model, the displaced-diffusion model, or the SABR model.

$\Rightarrow$ Once a model is chosen, the risk-neutral density is also determined, by calibrating the model to market option data.

What is the "model-free" framework?

Suppose we have sufficient liquid option quotes in the market, can we skip over the step of using a model to specify the stock price process, but instead **extract the risk-neutral density function** directly?

Market Price	Model-Free Formula
$C(K_1)$	$e^{-rT}\int_{K_1}^{\infty}(s-K_1)f(s)\,ds$
$C(K_2)$	$e^{-rT}\int_{K_2}^{\infty}(s-K_2)f(s)\,ds$
$C(K_3)$	$e^{-rT}\int_{K_3}^{\infty}(s-K_3)f(s)\,ds$
$C(K_4)$	$e^{-rT}\int_{K_4}^{\infty}(s-K_4)f(s)\,ds$
$\vdots$	$\vdots$