QF620 Additional Examples
Session 2: Binomial Tree and the Risk-Neutral Measure

1 Examples

1. Consider a one-step binomial tree model with $S_0 = 50$, $u = 2$, $r = 1\%$. The actual probability is given by $p = 2/3$, $q = 1/3$.

   (a) Write down the risk-neutral probabilities.

   (b) Evaluate the European call and put option prices struck at $K = 50$ at time $t = 1$.

2. Consider a three-step binomial process with $S_0 = \$4$, $u = 2$, $d = 0.5$, $p = \frac{2}{3}$, $q = \frac{1}{3}$. Use the law of iterated expectation to demonstrate the following relationship:

$$\mathbb{E}_1[\mathbb{E}_2[S_3]] = \mathbb{E}_1[S_3].$$

3. Consider a three-step binomial tree with parameters: $S_0 = \$4$, $u = 2$, $d = \frac{1}{2}$, $r = 0.25$. Determine the value of

   (a) An European put option struck at $K = \$10$ at time $t = 3$.

   (b) An American put option struck at $K = \$10$ at time $t = 3$.

   (c) An European call option struck at $K = \$10$ at time $t = 3$.

   (d) An American call option struck at $K = \$10$ at time $t = 3$.

4. Consider a binomial tree model where $S_0 = 10$, $u = 2$, $d = 1/2$, $r = 25\%$. The real world probabilities of up and down ticks are given by: $p = 3/4$, $q = 1/4$.

   (a) Explain why should the inequality $0 < d < 1 + r < u$ hold.

   (b) Determine the expectation $\mathbb{E}[S_3]$ under the real world probability measure, and $\mathbb{E}^*[S_3]$ under the risk-neutral probability measure.

   (c) Early exercise premium of an American option is defined as

$$\text{Early Exercise Premium} = V_A - V_E,$$

   where V_A is the American option premium while V_E is the European option premium, both struck at the same level and expire at the same time. Determine the early exercise premium of an American put option struck at $K = 10$ expiring on the 3^{rd} time step.

5. A stock is worth 5 today and the parameters of our binomial tree are $u = 2, d = 1/2, r = 5\%$. The actual probability is $p = 0.75, q = 0.25$.

   (a) What is $\mathbb{E}[S_3]$ under the real-world probability measure?

   (b) Write down the Radon-Nikodym derivative process and use this to show that the discounted stock price is a martingale under the risk-neutral measure.

6. Let S_i denote the stock price at time step i. If the stock price process can be described by the following equation

$$\log\left(\frac{S_{i+1}}{S_i}\right) = \left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z_i,$$

where $i = 0$ to $N-1$, and $Z_i \sim \text{iid}(0,1)$. Derive the expression for S_N.

7. Consider a binomial tree model where $S_0 = 16$, $u = 2$, $d = 1/2$, $r = 25\%$. The real world probabilities of up and down ticks are given by $p = 2/3$, $q = 1/3$. Write down the Radon-Nikodym derivative process $\frac{d\mathbb{Q}}{d\mathbb{P}}$, and show that

$$\mathbb{E}\left[S_3 \times \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^*\left[S_3\right].$$

8. **Discussion*** How are u and d in the binomial tree model related to σ, the volatility of the stock? Here, we connect the up and down jump parameters, u and d to the distribution of the underlying asset postulated by our model. To this end, we solve for the u and d parameters that will allow us to match the first and second moments of the underlying asset, respectively. Consider a small time step of Δt, our goal is to ensure that over this time period our binomial tree is consistent with

$$\begin{aligned}\mathbb{E}[S_{\Delta t}] &= p \times u \times S + (1-p) \times d \times S = Se^{r\Delta t}\\ \mathbb{E}[S^2_{\Delta t}] &= p \times u^2 \times S^2 + (1-p) \times d^2 \times S^2 = S^2 e^{(2r+\sigma^2)\Delta t}.\end{aligned}$$

In Cox-Rubinstein-Ross binomial tree, the relationship between the up and down jump is taken to be $u = \frac{1}{d}$. The relationship of the first moment allows us to solve for

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

The relationship of the second moment allows us to derive the following

$$\begin{aligned} pu^2 + (1-p)d^2 &= e^{(2r+\sigma^2)\Delta t}\\ p(u^2 - d^2) + d^2 &= e^{(2r+\sigma^2)\Delta t}\\ ue^{r\Delta t} + de^{r\Delta t} - 1 - d^2 + d^2 &= e^{(2r+\sigma^2)\Delta t}\\ u^2 + 1 - ue^{-r\Delta t} - ue^{(r+\sigma^2)\Delta t} &= 0\\ u^2 - u(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}) + 1 &= 0\end{aligned}$$

Solving the quadratic equation, we find

$$u = \frac{e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}}{2} + \frac{1}{2}\sqrt{(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t})^2 - 4}.$$

As an approximation, we write down the exponential functions in the solution as Taylor expansion around 0 since the time step is typically selected to be sufficiently small in the actual implementation of the model. Expanding up to the 1^{st} order, we have

$$\begin{aligned} u &\approx \frac{1}{2}(1 - r\Delta t + 1 + r\Delta t + \sigma^2\Delta t) + \frac{1}{2}\sqrt{(1 - r\Delta t + 1 + r\Delta t + \sigma^2\Delta t)^2 - 4}\\ &= 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + \dots\\ &= e^{\sigma\sqrt{\Delta t}},\end{aligned}$$

and by the construction of our model,

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}.$$

2 Suggested Solutions

1. (a) The risk-neutral probabilities are given by

$$p^* = \frac{(1+r)-d}{u-d} = 0.34$$
$$q^* = \frac{u-(1+r)}{u-d} = 0.66$$

   (b) The European call price is

$$C = \frac{1}{1+r}[p^* \times (100-50)^+ + q^* \times (25-50)^+] = 16.83$$

   The European put price is

$$P = \frac{1}{1+r}[p^* \times (50-100)^+ + q^* \times (50-25)^+] = 16.34$$

2. First we write down the two-step binomial tree model for the stock price:

$$\begin{tikzcd}
 & & & 32 \\
 & & 16 \arrow[ur] \arrow[dr] & \\
 & 8 \arrow[ur] \arrow[dr] & & 8 \\
4 \arrow[ur] \arrow[dr] & & 4 \arrow[ur] \arrow[dr] & \\
 & 2 \arrow[ur] \arrow[dr] & & 2 \\
 & & 1 \arrow[ur] \arrow[dr] & \\
 & & & 0.5
\end{tikzcd}$$

For the right hand side of the relationship, we have:

$$\mathbb{E}_1[S_3] = \begin{cases} \mathbb{E}_1[S_3|S_1 = 8] = \frac{2}{3} \cdot \frac{2}{3} \cdot 32 + 2 \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot 8 + \frac{1}{3} \cdot \frac{1}{3} \cdot 2 = 18 \\ \\ \mathbb{E}_1[S_3|S_1 = 2] = \frac{2}{3} \cdot \frac{2}{3} \cdot 8 + 2 \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot \frac{1}{3} \cdot 0.5 = 4.5 \end{cases}$$

The left hand side is given by an inner and an outer expectations:

$$\overbrace{\mathbb{E}_1\Big[\underbrace{\mathbb{E}_2[S_3]}_{\text{inner}}\Big]}^{\text{outer}}$$

First, we evaluate the inner expectation:

$$\mathbb{E}_2[S_3] = \begin{cases} \mathbb{E}_2[S_3|S_2 = 16] = \frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8 = 24 \\ \mathbb{E}_2[S_3|S_2 = 4] = \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2 = 6 \\ \mathbb{E}_2[S_3|S_2 = 1] = \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 0.5 = 1.5 \end{cases}$$

Next, we proceed to evaluate the outer expectation:

$$\mathbb{E}_1\Big[\mathbb{E}_2[S_3]\Big] \begin{cases} \mathbb{E}_1\Big[\mathbb{E}_2[S_3]\Big| S_1 = 8\Big] = \frac{2}{3} \cdot 24 + \frac{1}{3} \cdot 6 = 18 \\ \mathbb{E}_1\Big[\mathbb{E}_2[S_3]\Big| S_1 = 2\Big] = \frac{2}{3} \cdot 6 + \frac{1}{3} \cdot 1.5 = 4.5 \end{cases}$$

We have verified that both sides of the iterated expectation relationship are identical.

3. (a) To value an European put option struck at $K = \$10$ on the last (3^{rd}) time step, first draw out the 3-step binomial tree, work out the risk-neutral probabilities p^* and q^*, and proceed as follow:

$$\begin{aligned}
V_P^E(0) &= \frac{1}{(1+r)^3}\mathbb{E}^*[V_P^E(3)] \\
&= \left(\frac{4}{5}\right)^3 \left[\frac{1}{2^3} \times 1 \times 0 + \frac{1}{2^3} \times 3 \times 2 + \frac{1}{2^3} \times 3 \times 8 + \frac{1}{2^3} \times 1 \times 9.5\right] \\
&= 2.528.
\end{aligned}$$

(b) To value an American put option struck at $K = \$10$ on the last (3^{rd}) time step, we need to apply the backward induction method as follow:

$$\begin{tikzcd}
 & & & 0 \\
 & & (1) \arrow[ur] \arrow[dr] & \\
 & (4) \arrow[ur] \arrow[dr] & & 2 \\
V_P^A \arrow[ur] \arrow[dr] & & (2) \arrow[ur] \arrow[dr] & \\
 & (5) \arrow[ur] \arrow[dr] & & 8 \\
 & & (3) \arrow[ur] \arrow[dr] & \\
 & & & 9.5
\end{tikzcd}$$

where the backward induction steps are given by

$$\begin{aligned}
(1) &: \max\left\{(10-16)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times 0 + \frac{1}{2} \times 2\right)\right\} = \frac{4}{5} \\
(2) &: \max\left\{(10-4)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times 2 + \frac{1}{2} \times 8\right)\right\} = 6 \\
(3) &: \max\left\{(10-1)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times 8 + \frac{1}{2} \times 9.5\right)\right\} = 9 \\
(4) &: \max\left\{(10-8)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times \frac{4}{5} + \frac{1}{2} \times 6\right)\right\} = 2.72 \\
(5) &: \max\left\{(10-2)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times 6 + \frac{1}{2} \times 9\right)\right\} = 8 \\
V_P^A &: \max\left\{(10-4)^+,\ \frac{4}{5} \times \left(\frac{1}{2} \times 2.72 + \frac{1}{2} \times 8\right)\right\} = 6
\end{aligned}$$

(c) To value an European call option struck at $K = \$10$ on the last (3^{rd}) time step, we can proceed as follow:

$$\begin{aligned}
V_C^E(0) &= \frac{1}{(1+r)^3}\mathbb{E}^*[V_C^E(3)] \\
&= \left(\frac{4}{5}\right)^3 \left[\frac{1}{2^3} \times 1 \times 22 + \frac{1}{2^3} \times 3 \times 0 + \frac{1}{2^3} \times 3 \times 0 + \frac{1}{2^3} \times 1 \times 0\right] \\
&= 1.408
\end{aligned}$$

(d) To value an American call option struck at $K = \$10$ on the last (3^{rd}) time step, we need to apply the backward induction method as follow:

$$\begin{tikzcd}
 & & & 22 \\
 & & (1) \arrow[ur] \arrow[dr] & \\
 & (4) \arrow[ur] \arrow[dr] & & 0 \\
V_P^A \arrow[ur] \arrow[dr] & & (2) \arrow[ur] \arrow[dr] & \\
 & (5) \arrow[ur] \arrow[dr] & & 0 \\
 & & (3) \arrow[ur] \arrow[dr] & \\
 & & & 0
\end{tikzcd}$$

where the backward induction steps are given by

$$\begin{aligned}
&(1) : \max\left\{(16-10)^+,\ \frac{4}{5}\times\left(\frac{1}{2}\times 22+\frac{1}{2}\times 0\right)\right\} = 8.8\\
&(2) : 0\\
&(3) : 0\\
&(4) : \max\left\{(8-10)^+,\ \frac{4}{5}\times\left(\frac{1}{2}\times 8.8+\frac{1}{2}\times 0\right)\right\} = 3.52\\
&(5) : 0\\
&V_C^A : \max\left\{(4-10)^+,\ \frac{4}{5}\times\left(\frac{1}{2}\times 3.52+\frac{1}{2}\times 0\right)\right\} = 1.408
\end{aligned}$$

As one would expect from derivative pricing and valuation, there is no incentive in early exercising an American call option on non-dividend-paying stocks. Here we obtain an equal price for the European and American call options, highlighting the fact that under the binomial tree model the early exercise premium of an American call is 0.

4. (a) No arbitrage. $\triangleleft$

   (b) Under the real world probability measure,

$$\begin{aligned}
\mathbb{E}[S_3] &= \left(\frac{3}{4}\right)^3\times 80 + 3\times\left(\frac{3}{4}\right)^2\times\left(\frac{1}{4}\right)\times 20 + 3\times\left(\frac{3}{4}\right)\times\left(\frac{1}{4}\right)^2\times 5 + \left(\frac{1}{4}\right)^3\times 1.25\\
&= 42.91
\end{aligned}$$

   Under the risk-neutral probability measure,

$$\begin{aligned}
\mathbb{E}^*[S_3] &= \left(\frac{1}{2}\right)^3\times 80 + 3\times\left(\frac{1}{2}\right)^3\times 20 + 3\times\left(\frac{1}{2}\right)^3\times 5 + \left(\frac{1}{2}\right)^3\times 1.25\\
&= 19.53 \quad \triangleleft
\end{aligned}$$

   (c) Using backward induction, we can obtain

$$\begin{aligned}
V_A &= 2.32\\
V_E &= 1.52
\end{aligned}$$

   and so $EEP = 0.8$. $\triangleleft$

5. (a) Under the actual probability measure, the expectation of the stock price at time step 3 is

$$\begin{aligned}\mathbb{E}[S_3] &= \left(\frac{3}{4}\right)^3 \times 40 + 3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \times 10 + 3\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \times 2.5 + \left(\frac{1}{4}\right)^3 \times 0.625 \\ &= 21.455.\end{aligned}$$

(b) The Radon-Nikodym derivative for a binomial process is given by

$$\begin{tikzcd}
 & & & 0.11685 \\
 & & 0.239 \arrow[ur] \arrow[dr] & \\
 & 0.4889 \arrow[ur] \arrow[dr] & & 0.6055 \\
1 \arrow[ur] \arrow[dr] & & 1.2385 \arrow[ur] \arrow[dr] & \\
 & 2.5333 \arrow[ur] \arrow[dr] & & 3.1376 \\
 & & 6.4178 \arrow[ur] \arrow[dr] & \\
 & & & 16.258
\end{tikzcd}$$

The expectation of the stock price at time step 3 under the risk-neutral probability measure is given by

$$\begin{aligned}\mathbb{E}^*[X] &= \left[\left(\frac{11}{30}\right)^3 \times 40 + 3\left(\frac{11}{30}\right)^2\left(\frac{19}{30}\right) \times 10 + 3\left(\frac{11}{30}\right)\left(\frac{19}{30}\right)^2 \times 2.5 + \left(\frac{19}{30}\right)^3 \times 0.625\right] \\ &= 5.7881.\end{aligned}$$

Radon-Nikodym derivative allows us to related expectations taken under the actual probability measure and the risk-neutral measure as

$$\mathbb{E}^*[X] = \mathbb{E}[XZ].$$

Verifying this relationship, we have

$$\begin{aligned}\mathbb{E}^*[X] = 5.7881 = \mathbb{E}[XZ] = 0.11685 &\times \left(\frac{3}{4}\right)^3 \times 40 \\ &+ 3 \times 0.6055 \times \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \times 10 \\ &+ 3 \times 3.1376 \times \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \times 2.5 \\ &+ 16.258 \times \left(\frac{1}{4}\right)^3 \times 0.625 = 5.7881 \quad \triangleleft\end{aligned}$$

Finally, to show that the discounted security price is a martingale under the risk-neutral measure, we have

Step 1: $\dfrac{\mathbb{E}[S_1]}{1.05} = \dfrac{0.4889 \times \frac{3}{4} \times 10 + 2.5333 \times \frac{1}{4} \times 2.5}{1.05} = 5$

Step 2: $\dfrac{\mathbb{E}[S_2]}{1.05^2} = \dfrac{0.23901 \times \left(\frac{3}{4}\right)^2 \times 20 + 2 \times 1.2385 \times \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) \times 5 + 6.4178 \times \left(\frac{1}{4}\right)^2 \times 1.25}{1.05^2} = 5$

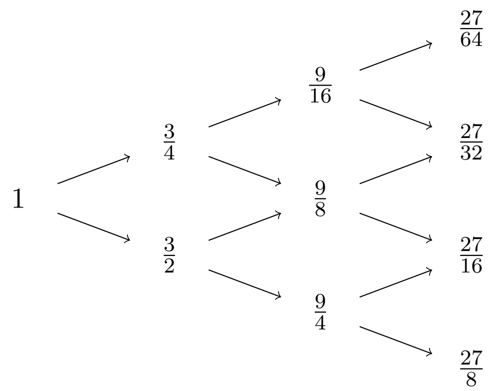
Step 3: $$\begin{aligned}\frac{\mathbb{E}[S_3]}{1.05^3} = \frac{1}{1.05^3}\Bigg(&0.11685 \times \left(\frac{3}{4}\right)^3 \times 40 + 3 \times 0.6055 \times \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \times 10 \\ &+ 3 \times 3.1376 \times \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \times 2.5 + 16.258 \times \left(\frac{1}{4}\right)^3 \times 0.625\Bigg) = 5.\end{aligned}$$

6. We sum both sides from 0 to $N-1$:

$$\begin{aligned}
\sum_{i=0}^{N-1} \log\left(\frac{S_{i+1}}{S_i}\right) &= \sum_{i=0}^{N-1}\left(r-\frac{\sigma^2}{2}\right)\Delta t + \sum_{i=0}^{N-1}\sigma\sqrt{\Delta t}Z_i \\
\log\left(\frac{S_1}{S_0}\right) + \log\left(\frac{S_2}{S_1}\right) + \cdots + \log\left(\frac{S_N}{S_{N-1}}\right) &= \left(r-\frac{\sigma^2}{2}\right)N\Delta t + \sigma\sqrt{\Delta t}X \\
\log S_N - \log S_0 &= \left(r-\frac{\sigma^2}{2}\right)N\Delta t + \sigma\sqrt{\Delta t}X \\
S_N &= S_0 \exp\left[\left(r-\frac{\sigma^2}{2}\right)N\Delta t + \sigma\sqrt{\Delta t}X\right]
\end{aligned}$$

where $X \sim \mathrm{iid}(0, N)$.

7. The Radon-Nikodym derivative process is given by



We have

$$\begin{aligned}
\mathbb{E}\left[S_3 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right] &= \left(\frac{2}{3}\right)^3 \cdot 128 \cdot \frac{27}{64} + 3\cdot\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\cdot 32\cdot\frac{27}{32} + 3\cdot\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2\cdot 8\cdot\frac{27}{16} + \left(\frac{1}{3}\right)^3\cdot 2\cdot\frac{27}{8} \\
&= 31.25 = \mathbb{E}^*\left[S_3\right] \quad \triangleleft
\end{aligned}$$