# QF620 Additional Examples
# Session 3: Brownian Motion and Martingale

## 1 Examples

1. Let $S_n = \sum_{i=1}^{n} X_i$ denote an $n$-step random walk, where $X_i$ is independent and identically distributed with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$. Evaluate
   (a) $\mathbb{E}[S_n]$.
   (b) $V[S_n]$.
   (c) $\mathbb{E}_m[S_n]$ (conditional on $S_m$, where $m < n$)

2. If $W_t$ and $W_s$ are Brownian processes, and $t > s$, determine
$$\mathbb{E}[(W_t - W_s)^4].$$

3. If $W_t$ is a Brownian motion, show that $e^{\theta W_t - \frac{\theta^2 t}{2}}$ is a martingale.

4. Let $X_i$ denote a sequence of random variables taking the values of either $+1$ or $-1$ with equal probability $\frac{1}{2}$, and let $S_n = \sum_{i}^{n} X_i$ where $n \in \mathbb{N}$. If $m \in \mathbb{N},\ m < n$, show that $\mathbb{E}[S_n - S_m] = 0$ and $\mathrm{Cov}(S_n - S_m, S_m) = 0$.

5. Let $W_t$ denote a Brownian motion, write down the probability density function of $W_t$. Let $0 < s < t$, write down the probability density function of $W_t - W_s$.

6. Let $W_t$ denote a Brownian motion. Evaluate the expectation $\mathbb{E}[W_t]$, $\mathbb{E}[W_t^2]$, and $\mathbb{E}[W_t^4]$.

7. Let $W_t$ denote a Brownian motion. Evaluate the expectation $\mathbb{E}[W_t]$ ***in full by making use of its probability density function.***

8. Let $W_t$ denote a Brownian motion. Find $\mathbb{E}[W_t^{125}]$.

9. Let $W_t$ denote a Brownian motion. If $W_1 > 0$, what is the probability $\mathbb{P}(W_2 > 0 | W_1 > 0)$?

10. Let $W_t$ denote a Brownian motion. What is the probability $\mathbb{P}(W_1 \times W_2 > 0)$?

Chapters 1-3 Review

11. Let $X$ be a standard normally distributed random variable, i.e. $X \sim N(0, 1)$, show that its mean, mode and median is given by
$$\text{Mean} = 0, \quad \text{Mode} = 0, \quad \text{Median} = 0.$$

12. Let $X$ be a normally distributed random variable, i.e. $X \sim N(\mu, \sigma^2)$, show that its mean, mode and median is given by
$$\text{Mean} = \mu, \quad \text{Mode} = \mu, \quad \text{Median} = \mu.$$

13. Show that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu.$$

14. Show that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma^2.$$

15. Consider a normally distributed random variable $X \sim N(\mu, \sigma^2)$. Define $Y = e^X$, show that

$$\mathbb{E}[Y] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = e^{\mu+\frac{1}{2}\sigma^2}.$$

16. Consider a normally distributed random variable $X \sim N(\mu, \sigma^2)$. Evaluate $\mathbb{E}[e^{\theta X}]$, where $\theta \in \mathbb{R}$ is a constant, using the following method:
    (a) completing the square.
    (b) moment generating function.

17. If $X \sim N(0,1)$. Let $Y_t = \sqrt{t}X$, show that $V[Y_t] = t$.

18. Let $f(t,x) = tx^2$. Work out the Taylor expansion up to the $2^{nd}$ order.

## 2 Suggested Solutions

1. First, given the probability distribution of $X_i$, we note that

$$\begin{aligned}\mathbb{E}[X_i] &= p \times (1) + (1-p) \times (-1)\\ &= 2p - 1\end{aligned}$$

and

$$\begin{aligned}V[X_i] &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2\\ &= \left[p \times (1)^2 + (1-p) \times (-1)^2\right] - (2p-1)^2\\ &= 4p(1-p)\end{aligned}$$

(a) The unconditional expectation of the random walk is given by

$$\begin{aligned}\mathbb{E}[S_n] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n]\\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]\\ &= \underbrace{(2p-1) + (2p-1) + \cdots + (2p-1)}_{\text{n terms}} = n(2p-1)\end{aligned}$$

(b) The unconditional variance of the random walk is given by

$$\begin{aligned}V[S_n] &= V[X_1] + V[X_2] + \cdots + V[X_n]\\ &= \underbrace{4p(1-p) + 4p(1-p) + \cdots + 4p(1-p)}_{\text{n terms}} = 4np(1-p)\end{aligned}$$

(c) The conditional expectation of the random walk is given by

$$\begin{aligned}\mathbb{E}_m[S_n] &= \mathbb{E}_m\left[S_m + \sum_{i=m+1}^{n} X_i\right]\\ &= S_m + \mathbb{E}_m[X_{m+1}] + \mathbb{E}_m[X_{m+2}] + \cdots + \mathbb{E}_m[X_n]\\ &= S_m + (n-m)(2p-1)\end{aligned}$$

2. First we note that if $X \sim N(0,1)$, then we have the following

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = 1, \quad \mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^4] = 3.$$

Next, note that

$$(W_t - W_s)^4 \sim N(0, (t-s))^4 = (t-s)^2 N(0,1)^4 = (t-s)^2 X^4.$$

Hence,

$$\mathbb{E}[(W_t - W_s)^4] = \mathbb{E}[(t-s)^2 X^4] = 3(t-s)^2.$$

3. We can show that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\theta W_t - \frac{1}{2}\theta^2 t\right)\bigg| s\right] &= \mathbb{E}\left[\exp\left(\theta(W_t - W_s)\right)\exp\left(\theta W_s - \frac{1}{2}\theta^2 t\right)\bigg| s\right]\\ &= \exp\left(\theta W_s - \frac{1}{2}\theta^2 t\right)\mathbb{E}\left[\exp(\theta(W_t - W_s))|s\right]\\ &= \exp\left(\theta W_s - \frac{1}{2}\theta^2 t\right)\exp\left[\frac{1}{2}\theta^2(t-s)\right]\\ &= \exp\left(\theta W_s - \frac{1}{2}\theta^2 s\right).\end{aligned}$$

4. Since $S_n$ is made up of a sequence of identical and independently distributed Bernoulli trial with mean 0 and variance 1, we have

$$\forall n \in \mathbb{N}: \ \mathbb{E}[S_n] = 0.$$

And so

$$\mathbb{E}[S_n - S_m] = \mathbb{E}[S_n] - \mathbb{E}[S_m] = 0.$$

Next, we note that

$$\begin{aligned}
\mathrm{Cov}(S_n - S_m, S_m) &= \mathbb{E}[(S_n - S_m)S_m] - \mathbb{E}[S_n - S_m]\mathbb{E}[S_m] \\
&= \mathbb{E}[(S_n - S_m)S_m] - 0 \cdot 0 \\
&= \mathbb{E}[S_n S_m] - \mathbb{E}[S_m^2] \\
&= \mathbb{E}[\mathbb{E}[S_n S_m|m]]] - m \\
&= \mathbb{E}[S_m \mathbb{E}[S_n|m]] - m \\
&= \mathbb{E}[S_m^2] - m \\
&= m - m = 0. \quad \lhd
\end{aligned}$$

5. We know that $W_t \sim N(0, t)$, i.e. it is normally distributed with 0 mean and a variance of $t$, which measures the time elapsed. A normal probability density function $N(\mu, \sigma^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

So the probability density function for $W_t$ is

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

By the definition of Brownian motion, $W_t - W_s \sim N(0, t - s)$. So the probability density function of $W_t - W_s$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}. \quad \lhd$$

6. Since $W_t \sim N(0, t)$, we can already conclude that

$$\mathbb{E}[W_t] = 0 \quad \because \text{ mean is } 0$$

and

$$\begin{aligned}
V[W_t] = t &= \mathbb{E}[W_t^2] - \mathbb{E}[W_t]^2 \quad \because \text{ variance is } t \\
&= \mathbb{E}[W_t^2] - 0 \\
\Rightarrow \quad t &= \mathbb{E}[W_t^2]. \quad \lhd
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathbb{E}[W_t^4] &= \mathbb{E}[t^2 X^4] \quad \text{where } X \sim N(0,1) \\
&= 3t^2. \quad \lhd
\end{aligned}$$

7. We need to evaluate

$$\mathbb{E}[W_t] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} w e^{\frac{-w^2}{2t}} dw.$$

Let

$$u = \frac{w^2}{2t} \quad \Rightarrow \quad du = \frac{w}{t} dw.$$

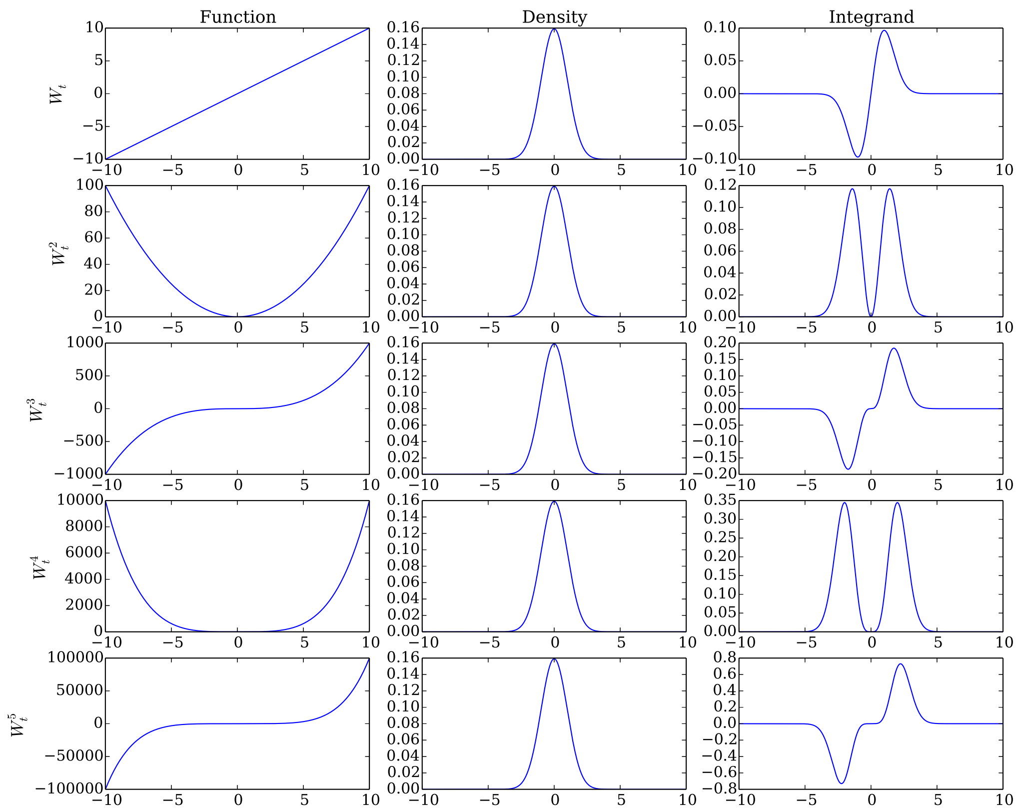
We have

$$\frac{1}{\sqrt{2\pi t}} \int w e^{-\frac{w^2}{2t}} dw = \frac{\sqrt{t}}{\sqrt{2\pi}} \int e^{-\frac{w^2}{2t}} \frac{w}{t} dw = \frac{\sqrt{t}}{\sqrt{2\pi}} \int e^{-u} du = \frac{\sqrt{t}}{\sqrt{2\pi}} [-e^{-u} + C] = \frac{\sqrt{t}}{\sqrt{2\pi}} \left[ -e^{-\frac{w^2}{2t}} + C \right].$$

So back to our definite integral

$$\begin{aligned} \mathbb{E}[W_t] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} w e^{-\frac{w^2}{2t}} dw \\ &= \frac{\sqrt{t}}{\sqrt{2\pi}} \left[ -e^{-\frac{w^2}{2t}} \right]_{-\infty}^{\infty} \\ &= 0. \quad \triangleleft \end{aligned}$$

8. You should have observed a pattern in the previous question – when taking the expectation of $W_t$, all odd-powered expectations evaluate to 0. This is due to the fact that $W_t$ has symmetrical probability density function:



Since taking expectation involves integrating the random variable weighted by the density function (which is symmetrical across the $y$-axis), the integrand, given by

$$x^n \times e^{-\frac{x^2}{2t}}, \quad n = 1, 2, 3, \cdots$$

when $n$ is odd will always remain 0, as we would end up integrating equal area above and below the $x$-axis. So $\mathbb{E}[W_t^{125}] = 0$. $\triangleleft$

9. Given that $W_1 > 0$, two cases will yield the required event: $\{W_2 > 0|W_1 > 0\}$. The first is that $W_2$ is an upward step, which occurs with probability $\frac{1}{2}$. The second case is when $W_2$ steps down, but the step size is not as large as $W_1$, so that $W_2$ is still above the $x$-axis. The event

$$|W_2 - W_1| < |W_1 - W_0|$$

occurs with probability $\frac{1}{2}$. So

$$\begin{aligned}\mathbb{P}(W_2 > 0|W_1 > 0) &= \mathbb{P}(W_2 > W_1) + \mathbb{P}(\{W_2 < W_1\} \cap \{|W_2 - W_1| < |W_1 - W_0|\}) \\ &= \frac{1}{2} + \mathbb{P}(\{|W_2 - W_1| < |W_1 - W_0|\}|\{W_2 < W_1\})\mathbb{P}\Big(\{W_2 < W_1\}\Big) \\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}. \quad \triangleleft\end{aligned}$$

10. In order to have the event $\{W_1 \times W_2 > 0\}$, $W_1$ and $W_2$ would both need to be simultaneously positive or negative. This probability can be calculated as follow:

$$\begin{aligned}\mathbb{P}(W_1 \times W_2 > 0) &= \mathbb{P}\Big(\{W_2 > 0\} \cap \{W_1 > 0\}\Big) + \mathbb{P}\Big(\{W_2 < 0\} \cap \{W_1 < 0\}\Big) \\ &= \mathbb{P}\Big(\{W_2 > 0\}|\{W_1 > 0\}\Big)\mathbb{P}\Big(W_1 > 0\Big) + \mathbb{P}\Big(\{W_2 < 0\}|\{W_1 < 0\}\Big)\mathbb{P}\Big(W_1 < 0\Big) \\ &= \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} = \frac{3}{4}. \quad \triangleleft\end{aligned}$$

※ Without thinking it through, it might be tempting to guess that $\mathbb{P}(W_1 \times W_2 > 0) = \mathbb{P}(W_1 \times W_2 < 0) = \frac{1}{2}$, given the independent increment property. Why is this "intuition" wrong?

11. If $X \sim N(0, 1)$, then the mean is given by $\mathbb{E}[X] = 0$. The mode is determined as follow:

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ \phi'(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times (-x) = 0 \qquad \Rightarrow \qquad x = 0.\end{aligned}$$

The median is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du = 0.5.$$

Since the probability density function of the standard normal distribution is symmetric across the $y$-axis, we can infer that $\Phi(0) = 0.5$. $\triangleleft$

12. For $X \sim N(\mu, \sigma^2)$, mean is given by $\mathbb{E}[X] = \mu$. The mode is given by

$$f'(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \times \left[-\frac{(x-\mu)}{\sigma^2}\right] = 0 \qquad \Rightarrow \qquad x = \mu.$$

Again due to the symmetric property of normal distribution, we know that

$$F(m) = \int_{-\infty}^{m} f(x)dx = 0.5 \qquad \Rightarrow \qquad m = F^{-1}(0.5) = \mu. \quad \lhd$$

13. This can be worked out using either moment generating function (MGF) or just basic integration. Under the MGF approach, we see that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mathbb{E}[X], \quad X \sim N(\mu, \sigma^2).$$

The MGF for $X$, a normally distributed random variable, is given by

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Taking the first derivative, we obtain

$$\frac{dM_X(t)}{dt} = (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

We can therefore conclude that

$$\mathbb{E}[X] = \frac{dM_X(0)}{dt} = \mu.$$

Alternatively, using basic integration approach, we can write

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{\text{first integral}} + \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mu e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{\text{second integral}}$$

The second integral evaluates to $\mu$. The first integral evaluates to 0. To see this, let $u = \frac{x-\mu}{\sqrt{2}\sigma}$, we have $du = \frac{dx}{\sqrt{2}\sigma}$, and hence the first integral becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} u e^{-u^2} du,$$

which is 0. $\lhd$

14. Similar to the previous question, this can be worked out using either MGF or just basic integration technique. To use the MGF approach, first expand the integral, and we obtain

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 \\
&= \mathbb{E}[X^2] - \mu^2
\end{aligned}$$

Using the MGF approach, taking the $2^{nd}$ derivative of the normal random variable's moment generating function with respect to $t$, we obtain

$$\begin{aligned}
\frac{dM_X(t)}{dt} &= (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2} \\
\frac{d^2 M_X(t)}{dt^2} &= (\sigma^2 + (\mu + \sigma^2 t)^2)e^{\mu t + \frac{1}{2}\sigma^2 t^2}
\end{aligned}$$

Hence

$$\mathbb{E}[X^2] = \frac{d^2 M_X(0)}{dt^2} = \sigma^2 + \mu^2$$

And so we've shown that the integral evaluates to $\sigma^2$. Alternatively, we can just use integration by parts as follow:

$$\int u \, dv = uv - \int v \, du,$$

where

$$\begin{aligned} u = x - \mu \quad &\Rightarrow \quad du = dx, \\ v = -\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad &\Rightarrow \quad dv = (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u \, dv \\ &= -\frac{1}{\sqrt{2\pi}\sigma} (x-\mu)\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= 0 + \sigma^2 \times 1 = \sigma^2. \quad \triangleleft \end{aligned}$$

15. The MGF of a normally distributed random variable $X \sim N(\mu, \sigma^2)$ is given by

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Hence we have

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{tX}]\big|_{t=1} = e^{\mu + \frac{1}{2}\sigma^2}$$

Alternatively, we can solve the question by completing the square (see the next question). $\triangleleft$

16. (a) By completing the square, we have

$$\begin{aligned} \mathbb{E}[e^{\theta X}] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\theta x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\theta x - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2 \theta x}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu + \sigma^2\theta)x + \mu^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2\theta)x + (\mu+\sigma^2\theta)^2 - (\mu+\sigma^2\theta)^2 + \mu^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2 - (\mu+\sigma^2\theta)^2 + \mu^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2 - \mu^2 - 2\mu\sigma^2\theta - \sigma^4\theta^2 + \mu^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2}{2\sigma^2}} e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} dx \\ &= e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} \quad \triangleleft \end{aligned}$$

(b) By the MGF approach, we have

$$\mathbb{E}[e^{\theta X}] = \mathbb{E}[e^{tX}]\big|_{t=\theta} = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} \quad \triangleleft$$

17. Since $X \sim N(0,1)$, we know that $V[X] = 1$, as the variance of $X$ is already given to be equal to 1. Therefore,

$$V[Y_t] = V[\sqrt{t}X] = t \cdot V[X] = t.$$

18. First work out the partial derivatives:

$$\frac{\partial f}{\partial t} = x^2, \quad \frac{\partial f}{\partial x} = 2tx, \quad \frac{\partial^2 f}{\partial t^2} = 0, \quad \frac{\partial^2 f}{\partial x^2} = 2t, \quad \frac{\partial^2 f}{\partial t \partial x} = 2x.$$

Expanding around $(t_0, x_0)$, we obtain

$$\begin{aligned} f(t,x)|_{(t_0,x_0)} = f(t_0,x_0) &+ \frac{\partial f}{\partial t}(t-t_0) + \frac{\partial f}{\partial x}(x-x_0) \\ &+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial t^2}(t-t_0)^2 + 2\frac{\partial^2 f}{\partial t \partial x}(t-t_0)(x-x_0) + \frac{\partial^2 f}{\partial x^2}(x-x_0)^2\right] \\ &+ \cdots \end{aligned}$$

Writing

$$\begin{aligned} \Delta t &= t - t_0, \\ \Delta x &= x - x_0, \\ \Delta f &= f(t,x) - f(t_0,x_0), \end{aligned}$$

we have

$$\Delta f \approx x_0^2 \Delta t + 2t_0 x_0 \Delta x + 2x_0 \Delta t \Delta x + t_0 \Delta x^2. \quad \triangleleft$$