

QF620 Stochastic Modelling

Mid-term Revision Pack

1 Practice Questions

1. Let W_t denote a Brownian motion. Derive the stochastic differential equation for dX_t and group the drift and diffusion coefficients together for the following stochastic processes:

- (a) $X_t = W_t^2$
- (b) $X_t = t + e^{W_t}$
- (c) $X_t = W_t^3 - 3tW_t$
- (d) $X_t = e^{t+W_t}$
- (e) $X_t = e^{\frac{t}{2}} \sin(W_t)$
- (f) $X_t = e^{W_t - \frac{t}{2}}$

2. Consider 2 stochastic processes Y_t and Z_t , following the dynamics

$$\begin{cases} dY_t = b(t)Y_t dW_t \\ dZ_t = A(t)dt + B(t)dW_t. \end{cases}$$

Define a new stochastic process X_t as $X_t = Y_t Z_t$, write down the stochastic differential equation for dX_t .

3. Let W_t and \tilde{W}_t denote two independent Brownian motions, derive the SDE for the stochastic variable $Y_t = \frac{W_t}{\tilde{W}_t}$.

4. Consider an interest rate model following the stochastic differential equation:

$$dr_t = \theta dt + \sigma dW_t,$$

where θ and σ are both constants. Determine

- (a)
- (b)

$$\mathbb{E} \left[\int_0^T r_t dt \right]$$

$$V \left[\int_0^T r_t dt \right]$$

5. Let W_t denote a standard Brownian motion. Derive the stochastic differential equations for dX_t using Itô's formula for the following processes:

- (a) $X_t = t + W_t^2$

- (b) $X_t = 5^{W_t}$
 (c) $X_t = e^{\int_0^t u W_u du}$. *Hint: The Riemann integral is a function of time, $f(t) = \int_0^t u W_u du$.*
 (d) $X_t = \frac{W_t}{Z_t}$, where Z_t is another Brownian motion, and $dW_t dZ_t = \rho dt$.

6. Solve the following stochastic differential equations:

- (a) $dX_t = 5X_t dt + 0.1X_t dW_t$.
 (b) $dX_t = \mu X_t dt + \sigma dW_t$. *Hint: Apply Itô's formula to $f(t, x) = e^{-\mu t} x$.*
 (c) $dX_t = \frac{1}{4}(X_t - 1)dW_t$. *Hint: Displaced-diffusion process.*

7. Let W_t denote a standard Brownian motion.

- (a) Evaluate $\mathbb{E}[W_t^2]$ and $\mathbb{E}[W_t^4]$.
 (b) Evaluate the expectation $\mathbb{E}[|W_T|]$, where $|\cdot|$ denote the absolute value.
 (c) Determine the mean and variance of the following integrals:

i.

$$\int_t^T u dW_u$$

ii.

$$\int_t^T W_u du$$

8. Let W_t denote a standard Brownian motion. Derive the stochastic differential equations for dX_t using Itô's formula for the following processes:

- (a) $X_t = W_t^3 - 3tW_t$
 (b) $X_t = t^2 W_t - 2 \int_0^t u W_u du$
 (c) $X_t = W_t \tilde{W}_t$ where W_t and \tilde{W}_t are independent Brownian motions
 (d) $X_t = W_t Z_t$ where $dW_t dZ_t = \rho dt$

9. Consider the following 2 stochastic differential equations

$$\begin{cases} dX_t = \sigma_X X_t dW_t^X \\ dY_t = \sigma_Y Y_t dW_t^Y \end{cases}$$

where W_t^X and W_t^Y are two standard-Brownian motions.

- (a) Solve the two SDEs using Itô's formula, and write down the expression for $X_T Y_T$.
 (b) If W_t^X and W_t^Y are independent, i.e. $\text{Cov}(W_T^X, W_T^Y) = 0$, evaluate $\mathbb{E}[X_T Y_T]$.
 (c) If W_t^X and W_t^Y are correlated, i.e. $\text{Cov}(W_T^X, W_T^Y) = \rho T$, evaluate $\mathbb{E}[X_T Y_T]$.

Hint: Combine $W_T^X + W_T^Y$ into a single normally distributed random variable, then apply the moment generating function method to evaluate the expectation.

10. Let W_t denote a Brownian motion. Evaluate the stochastic integral

$$I_t = \int_0^t W_u dW_u,$$

and proceed to find the mean and variance of the stochastic integral I_t .

11. Determine whether the following processes X_t are martingales

(a) $X_t = W_t + 4t$

(b) $X_t = W_t^2$

(c) $X_t = W_t Z_t$ where W_t and Z_t are independent Brownian motion

12. Consider a stock price following the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a Brownian motion under the real world probability measure, and a risk-free bond following the differential equation

$$dB_t = r B_t dt.$$

Determine the following

(a) $\mathbb{E}[S_T]$, where the expectation is taken under the real world measure.

(b) $\mathbb{E}^*[S_T]$, where the expectation is taken under the risk-neutral measure associated to the risk-free bond numeraire.

13. Let W_t denote a standard Brownian motion. The stochastic variable X_t follows the process

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Derive the stochastic differential equation for dY_t if:

(a)

$$Y_t = 10^{W_t}$$

(c)

$$Y_t = \frac{X_t}{W_t}$$

(b)

$$Y_t = t^2 + W_t^2$$

(d)

$$Y_t = \frac{t}{W_t}$$

14. Use Itô's formula to show that

$$\int_0^t W_u^2 dW_u = \frac{W_t^3}{3} - \int_0^t W_u du.$$

2 Suggested Solutions

1. (a) $dX_t = dt + 2W_t dW_t$
 (b) $dX_t = \left(1 + \frac{1}{2}e^{W_t}\right) dt + e^{W_t} dW_t$
 (c) $dX_t = 3(W_t^2 - t) dW_t$
 (d) $dX_t = \frac{3}{2}X_t dt + X_t dW_t$
 (e) $dX_t = \cos(W_t) e^{\frac{t}{2}} dW_t$
 (f) $dX_t = e^{W_t - \frac{t}{2}} dW_t$

2. Applying product rule for stochastic calculus, we have

$$\begin{aligned} dX_t &= d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t + b(t) Y_t B(t) dt \\ &= Y_t (A(t) dt + B(t) dW_t) + Z_t b(t) Y_t dW_t + b(t) Y_t B(t) dt \end{aligned}$$

Collecting dt and dW_t terms together, we obtain:

$$dX_t = (A(t) Y_t + b(t) Y_t B(t)) dt + (Y_t B(t) + X_t b(t)) dW_t$$

3. Let $Y_t = \frac{W_t}{\tilde{W}_t} = f(W_t, \tilde{W}_t)$, we have By Itô's lemma

$$dY_t = \frac{1}{\tilde{W}_t} dW_t - \frac{W_t}{\tilde{W}_t^2} d\tilde{W}_t + \frac{W_t}{\tilde{W}_t^3} dt \triangleleft$$

4. First we integrate the arithmetic stochastic differential equation from 0 to t :

$$\int_0^t dr_u = \int_0^t \theta du + \int_0^t \sigma dW_u \quad \Rightarrow \quad r_t = r_0 + \theta t + \sigma W_t$$

Next, we integrate r_t from 0 to T to obtain

$$\begin{aligned} \int_0^T r_t dt &= r_0 T + \int_0^T \theta t dt + \int_0^T \sigma W_t dt \\ &= r_0 T + \frac{\theta T^2}{2} + \sigma \int_0^T W_t dt \end{aligned}$$

- (a) We have

$$\mathbb{E} \left[\int_0^T r_t dt \right] = r_0 T + \frac{\theta T^2}{2} \triangleleft$$

since we have established that

$$\int_0^T W_t dt \sim N \left(0, \frac{T^3}{3} \right)$$

(b) We have

$$\begin{aligned}
V \left[\int_0^T r_t dt \right] &= V \left[\sigma \int_0^T W_t dt \right] \\
&= \sigma^2 V \left[\int_0^T W_t dt \right] \\
&= \sigma^2 \mathbb{E} \left[\left(\int_0^T (T-t) dW_t \right)^2 \right] \\
&= \sigma^2 \int_0^T (T-t)^2 dt \\
&= \frac{\sigma^2 T^3}{3} \triangleleft
\end{aligned}$$

5. (a) Let $f(t, x) = t + x^2$, we have $\frac{\partial f}{\partial t} = 1$, $\frac{\partial f}{\partial x} = 2x$, and $\frac{\partial^2 f}{\partial x^2} = 2$. By Itô's formula, we obtain

$$dX_t = dt + 2W_t dW_t + \frac{1}{2} \cdot 2dt = 2dt + 2W_t dW_t. \triangleleft$$

- (b) Let $f(W_t) = 5^{W_t}$, we have $f'(W_t) = 5^{W_t} \log(5)$, and $f''(W_t) = 5^{W_t} (\log(5))^2$. By Itô's formula, we obtain

$$dX_t = X_t \log(5) dW_t + \frac{1}{2} X_t (\log(5))^2 dt. \triangleleft$$

- (c) Let $f(t) = e^{\int_0^t u W_u du}$, we have $f'(t) = e^{\int_0^t u W_u du} \cdot t W_t$. By Itô's formula, we obtain

$$dX_t = t W_t X_t dt. \triangleleft$$

- (d) Let $f(w, z) = \frac{w}{z}$, we have

$$\frac{\partial f}{\partial w} = \frac{1}{z}, \quad \frac{\partial^2 f}{\partial w^2} = 0, \quad \frac{\partial f}{\partial z} = -\frac{w}{z^2}, \quad \frac{\partial^2 f}{\partial z^2} = \frac{2w}{z^3}, \quad \frac{\partial^2 f}{\partial w \partial z} = -\frac{1}{z^2}.$$

By Itô's formula, we obtain

$$\begin{aligned}
dX_t &= \frac{1}{Z_t} dW_t - \frac{W_t}{Z_t^2} dZ_t + \frac{1}{2} \frac{2W_t}{Z_t^3} dt - \frac{1}{Z_t^2} \rho dt \\
&= \left(\frac{W_t}{Z_t^3} - \frac{1}{Z_t^2} \rho \right) dt + \frac{1}{Z_t} dW_t - \frac{W_t}{Z_t^2} dZ_t. \triangleleft
\end{aligned}$$

6. (a) Consider the sde

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Applying Itô's formula to $\log(X_t)$ and integrating both sides, the solution is given by

$$X_T = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

Substituting $\mu = 5$ and $\sigma = 0.1$, we obtain

$$X_T = X_0 e^{4.995T + 0.1W_T} \quad \triangleleft$$

(b)

$$\begin{aligned} d(e^{-\mu t} X_t) &= -\mu e^{-\mu t} X_t dt + e^{-\mu t} dX_t \\ &= \sigma e^{-\mu t} dW_t \\ X_T &= X_0 e^{\mu T} + \sigma \int_0^T e^{-\mu(s-T)} dW_s. \quad \triangleleft \end{aligned}$$

(c) This is a displaced-diffusion stochastic differential equation. Let $Y_t = f(X_t) = \log(X_t - 1)$, we have

$$f'(X_t) = \frac{1}{X_t - 1}, \quad f''(X_t) = -\frac{1}{(X_t - 1)^2}.$$

By Itô's Lemma,

$$dY_t = \frac{1}{4} dW_t - \frac{1}{32} dt.$$

Integrating both sides and simplifying, we have

$$X_T = 1 + (X_0 - 1) e^{-\frac{T}{32} + \frac{W_T}{4}}. \quad \triangleleft$$

7. (a) Given that $W_t \sim N(0, t)$, let $X \sim N(0, 1)$, we have

$$\begin{aligned}\mathbb{E}[W_t^2] &= \mathbb{E}[tX^2] = t \\ \mathbb{E}[W_t^4] &= \mathbb{E}[t^2X^4] = 3t^2. \quad \triangleleft\end{aligned}$$

- (b) First we note that $W_T \sim N(0, T)$. Let $X \sim N(0, 1)$, we have

$$\begin{aligned}\mathbb{E}[|W_T|] &= \mathbb{E}[\sqrt{T}|X|] \\ &= \frac{\sqrt{T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} dx \\ &= \frac{\sqrt{T}}{\sqrt{2\pi}} \left[\int_{-\infty}^0 -xe^{-\frac{x^2}{2}} dx + \int_0^{\infty} xe^{-\frac{x^2}{2}} dx \right] \\ &= \frac{\sqrt{T}}{\sqrt{2\pi}} \times 2 \times \underbrace{\int_0^{\infty} xe^{-\frac{x^2}{2}} dx}_1 \\ &= \sqrt{\frac{2T}{\pi}}. \quad \triangleleft\end{aligned}$$

- (c) i.

$$\begin{aligned}\mathbb{E} \left[\int_t^T u dW_u \right] &= 0 \quad \because \text{Stochastic integrals have zero mean.} \\ V \left[\int_t^T u dW_u \right] &= \int_t^T u^2 du \quad \because \text{It\^o's Isometry} \\ &= \frac{T^3 - t^3}{3} \quad \triangleleft\end{aligned}$$

- ii. Integration of W_t across dt is a Riemann integral, and can be expressed as infinite sum. We show that it can be rearranged into a weighted sum of normal random variable with 0 mean, and hence the expectation is 0:

$$\mathbb{E} \left[\int_t^T W_u du \right] = 0$$

For variance, we have

$$\begin{aligned}V \left[\int_t^T W_u du \right] &= \mathbb{E} \left[\int_t^T W_u du \times \int_t^T W_s ds \right] \\ &= \mathbb{E} \left[\int_t^T \int_t^T W_u W_s du ds \right] \\ &= \int_t^T \int_t^T \mathbb{E}[W_u W_s] du ds \\ &= \int_t^T \int_t^T \text{Cov}(W_u, W_s) du ds \\ &= \int_t^T \int_t^T \min\{u, s\} du ds \\ &= \int_t^T \left(\int_t^s u du + \int_s^T s du \right) ds \\ &= \frac{T^3}{3} + \frac{2t^3}{3} - t^2T \quad \triangleleft\end{aligned}$$

8. We apply Itô's formula to derive the stochastic differential equations.

(a) Let $f(t, x) = x^3 - 3tx$, first evaluate the partial derivatives

$$\frac{\partial f}{\partial t} = -3x, \quad \frac{\partial f}{\partial x} = 3x^2 - 3t, \quad \frac{\partial^2 f}{\partial x^2} = 6x.$$

The s.d.e. is given by

$$\begin{aligned} dX_t &= -3W_t dt + (3W_t^2 - 3t)dW_t + \frac{1}{2} \cdot 6 \cdot W_t dt \\ &= 3(W_t^2 - t)dW_t. \quad \triangleleft \end{aligned}$$

(b) Let $f(t, W_t) = t^2 W_t - 2 \int_0^t u W_u du$, first evaluate the partial derivatives

$$\frac{\partial f}{\partial t}(t, W_t) = 2tW_t - 2tW_t = 0, \quad \frac{\partial f}{\partial x}(t, W_t) = t^2, \quad \frac{\partial^2 f}{\partial x^2}(t, W_t) = 0.$$

The s.d.e. is given by

$$dX_t = t^2 dW_t. \quad \triangleleft$$

(c) Let $f(W_t, \tilde{W}_t) = W_t \tilde{W}_t$, first evaluate the partial derivatives, the s.d.e. is given by

$$dX_t = \tilde{W}_t dW_t + W_t d\tilde{W}_t. \quad \triangleleft$$

(d) Let $f(W_t, Z_t) = W_t Z_t$, first evaluate the partial derivatives, the s.d.e. is given by

$$dX_t = Z_t dW_t + W_t dZ_t + \rho dt. \quad \triangleleft$$

9. (a) The solutions to the pair of stochastic differential equations are given by

$$\begin{aligned} X_T &= X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} \\ Y_T &= Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y} \end{aligned}$$

hence

$$X_T Y_T = X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}.$$

- (b) Suppose W_t^X and W_t^Y are independent Brownian motions. First we observe that

$$\mathbb{E}[\sigma_X W_T^X + \sigma_Y W_T^Y] = 0$$

and

$$V[\sigma_X W_T^X + \sigma_Y W_T^Y] = \sigma_X^2 T + \sigma_Y^2 T.$$

And so

$$\sigma_X W_T^X + \sigma_Y W_T^Y \sim N\left(0, \sigma_X^2 T + \sigma_Y^2 T\right).$$

Now let $Z = \sigma_X W_T^X + \sigma_Y W_T^Y \sim N\left(0, \sigma_X^2 T + \sigma_Y^2 T\right)$, we have

$$\begin{aligned} \mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] &= \mathbb{E}\left[e^Z\right] \\ &= \exp\left(\frac{\sigma_X^2 T + \sigma_Y^2 T}{2}\right). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[X_T Y_T] &= \mathbb{E}\left[X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} \mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} e^{\frac{\sigma_X^2 T + \sigma_Y^2 T}{2}} \\ &= X_0 Y_0. \end{aligned}$$

- (c) Now W_t^X and W_t^Y are correlated with a correlation coefficient of ρ . Again we observe that

$$\mathbb{E}[\sigma_X W_T^X + \sigma_Y W_T^Y] = 0$$

and

$$V[\sigma_X W_T^X + \sigma_Y W_T^Y] = \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T.$$

And so

$$\sigma_X W_T^X + \sigma_Y W_T^Y \sim N\left(0, \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T\right).$$

Now let $Z = \sigma_X W_T^X + \sigma_Y W_T^Y \sim N\left(0, \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T\right)$, we have

$$\begin{aligned}\mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] &= \mathbb{E}\left[e^Z\right] \\ &= \exp\left(\frac{\sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}[X_T Y_T] &= \mathbb{E}\left[X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} \mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} e^{\frac{\sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T}{2}} \\ &= X_0 Y_0 e^{\sigma_X \sigma_Y \rho T}.\end{aligned}$$

10. First use Itô's lemma to show that

$$I_t = \int_0^t W_u dW_u = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

Then proceed to evaluate

$$\mathbb{E}[I_t] = 0$$

$$V[I_t] = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}\left[\int_0^t W_u^2 du\right] = \frac{1}{2}t^2 \quad \text{by Itô's isometry}$$

11. (a) Apply Itô's lemma to the function $X_t = f(t, W_t) = W_t + 4t$, we have

$$dX_t = 4dt + dW_t \Rightarrow \mathbb{E}[dX_t] = 4dt \neq 0.$$

So this process is not a martingale.

(b) Apply Itô's lemma to the function $X_t = f(W_t) = W_t^2$, we have

$$dX_t = dt + 2W_t dW_t \Rightarrow \mathbb{E}[dX_t] = dt \neq 0.$$

So this process is not a martingale.

(c) Apply Itô's lemma to the function $X_t = f(W_t, Z_t)$, we have

$$\begin{aligned} dX_t &= W_t dZ_t + Z_t dW_t + dW_t dZ_t \\ &= W_t dZ_t + Z_t dW_t \\ \Rightarrow \mathbb{E}[dX_t] &= 0. \end{aligned}$$

Since W_t and Z_t are independent Brownian motions. So this process is a martingale.

12. (a) W_t is a standard Brownian motion under the real world probability measure. The solution to the stock price sde is given by

$$S_T = S_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma W_T \right].$$

Taking expectation under the real world probability measure gives

$$\mathbb{E}[S_T] = S_0 \exp(\mu T)$$

- (b) If we want to take the expectation under the risk-neutral measure associated to the risk-free bond numeraire, we need to apply Girsanov's theorem to write down the stock price sde. Under the risk-neutral measure associated to the risk-free bond numeraire, the asset ratio $\frac{S_t}{B_t}$ is a martingale. Applying Itô's lemma to $X_t = f(S_t, B_t) = \frac{S_t}{B_t}$, we have

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

Since this process is expected to be a martingale under \mathbb{Q}^B , we can write

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t dW_t \\ &= \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^B \end{aligned}$$

where we've used Girsanov's theorem to define a new Brownian motion under the risk-neutral probability measure associated to the risk-free bond numeraire:

$$dW_t^B = dW_t + \frac{\mu - r}{\sigma} dt.$$

Substituting this to the stock price sde, we obtain

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^B \\ \Rightarrow S_T &= S_0 \exp \left[\left(r - \frac{\sigma^2}{2} \right) T + \sigma W_T \right]. \end{aligned}$$

Hence

$$\mathbb{E}^*[S_T] = S_0 \exp(rT).$$

13. (a) Let $Y_t = f(W_t) = 10^{W_t}$, we work out the following partial derivatives

$$f'(W_t) = 10^{W_t} \cdot \log(10), \quad f''(W_t) = 10^{W_t} \cdot (\log(10))^2.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 \\ &= \frac{(\log(10))^2}{2}10^{W_t}dt + 10^{W_t}\log(10)dW_t. \quad \triangleleft \end{aligned}$$

- (b) Let $Y_t = f(t, W_t) = t^2 + W_t^2$, we work out the following partial derivatives

$$f_t(t, W_t) = 2t, \quad f_x(t, W_t) = 2W_t, \quad f_{xx}(t, W_t) = 2.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, W_t)(dW_t)^2 \\ &= (1 + 2t)dt + 2W_t dW_t. \quad \triangleleft \end{aligned}$$

- (c) Let $Y_t = f(X_t, W_t) = \frac{X_t}{W_t}$, we work out the following partial derivatives

$$\begin{aligned} f_w(X_t, W_t) &= -\frac{X_t}{W_t^2}, \quad f_{ww}(X_t, W_t) = \frac{2X_t}{W_t^3}, \quad f_x(X_t, W_t) = \frac{1}{W_t}, \\ f_{xx}(X_t, W_t) &= 0, \quad f_{xw}(X_t, W_t) = -\frac{1}{W_t^2}. \end{aligned}$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_w dW_t + \frac{1}{2}f_{ww} (dW_t)^2 + f_x dX_t + \frac{1}{2}f_{xx} (dX_t)^2 + f_{xw} dX_t dW_t \\ &= \left(\frac{X_t}{W_t^3} - \frac{\sigma X_t}{W_t^2} + \frac{\mu X_t}{W_t} \right) dt + \left(\frac{\sigma X_t}{W_t} - \frac{X_t}{W_t^2} \right) dW_t. \quad \triangleleft \end{aligned}$$

- (d) Let $Y_t = f(t, W_t) = \frac{t}{W_t}$, we work out the following partial derivatives

$$f_t(t, W_t) = \frac{1}{W_t}, \quad f_x(t, W_t) = -\frac{t}{W_t^2}, \quad f_{xx}(t, W_t) = \frac{2t}{W_t^3}.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, W_t)(dW_t)^2 \\ &= \left(\frac{1}{W_t} + \frac{t}{W_t^3} \right) dt - \frac{t}{W_t^2} dW_t. \quad \triangleleft \end{aligned}$$

14. Let $X_t = f(W_t) = \frac{W_t^3}{3}$, we have

$$f'(W_t) = W_t^2, \quad f''(W_t) = 2W_t.$$

Applying Itô's lemma, we obtain the following stochastic differential equation

$$dX_t = W_t^2 dW_t + W_t dt.$$

Integrating both sides and rearranging, we obtain

$$\int_0^t W_u^2 dW_u = \frac{W_t^3}{3} - \int_0^t W_u du. \quad \triangleleft$$