QF620 Stochastic Modelling in Finance

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AY2024-25 Assignment 1 Extra from 2023

1. Let W_t denote a Brownian motion. Derive the stochastic differential equation for dX_t and group the drift and diffusion coefficients together for the following stochastic processes:

   (a) $X_t = W_t^2$

   (b) $X_t = t + e^{W_t}$

   (c) $X_t = W_t^3 - 3tW_t$

   (d) $X_t = e^{t+W_t}$

   (e) $X_t = e^{\frac{t}{2}} \sin(W_t)$

   (f) $X_t = e^{W_t - \frac{t}{2}}$

   Solution: Let $X_t = f(t, W_t)$ and $0 \leq s < t$. Recall Itô's formula:

$$dX_t = f_t\,dt + f_{W_t}\,dW_t + \frac{f_{W_tW_t}}{2}(dW_t)^2 = \underbrace{\left(f_t + \frac{f_{W_tW_t}}{2}\right)}_{\text{drift coefficient}} dt + \underbrace{f_{W_t}}_{\text{diffusion coefficient}} dW_t.$$

   (a) $f_t = 0,\ f_{W_t} = 2W_t,\ f_{W_tW_t} = 2 \Rightarrow dX_t = dt + 2W_t\,dW_t$

   (b) $f_t = 1,\ f_{W_t} = f_{W_tW_t} = e^{W_t} \Rightarrow dX_t = \left(1 + \frac{e^{W_t}}{2}\right) dt + e^{W_t}\,dW_t$

   (c) $f_t = -3W_t,\ f_{W_t} = 3W_t^2 - 3t,\ f_{W_tW_t} = 6W_t \Rightarrow dX_t = (-3W_t + 3W_t)\ dt + 3(W_t^2 - t)\,dW_t$

   (d) $f_t = f_{W_t} = f_{W_tW_t} = X_t \Rightarrow dX_t = X_t\left(\frac{3}{2}\,dt + dW_t\right)$

   (e) $f_t = \frac{X_t}{2},\ f_{W_t} = e^{\frac{t}{2}}\cos(W_t),\ f_{W_tW_t} = -X_t \Rightarrow dX_t = e^{\frac{t}{2}}\cos(W_t)\,dW_t$

   (f) $f_t = -\frac{X_t}{2},\ f_{W_t} = X_t,\ f_{W_tW_t} = X_t \Rightarrow dX_t = X_t\,dW_t$

2. Consider 2 stochastic processes Y_t and Z_t, following the dynamics

$$\begin{cases} dY_t = b(t) Y_t \, dW_t \\ dZ_t = A(t)\, dt + B(t)\, dW_t. \end{cases}$$

Defining a new stochastic process X_t as $X_t = Y_t Z_t$, write down the stochastic differential equation for dX_t.

Solution: Let $X_t = Y_t Z_t$. Then, we have:

$$\begin{aligned} dX_t &= Z_t \, dY_t + Y_t \, dZ_t + dY_t \, dZ_t \\ &= b(t) X_t \, dW_t + Y_t [A(t)\, dt + B(t)\, dW_t] + b(t) Y_t \, dW_t [A(t)\, dt + B(t)\, dW_t] \\ &= A(t) Y_t \, dt + [b(t) X_t + B(t) Y_t]\, dW_t + A(t) b(t) Y_t \underbrace{dW_t \, dt}_{=0} + b(t) B(t) Y_t \underbrace{(dW_t)^2}_{=dt} \\ &= [A(t) + b(t) B(t)] Y_t \, dt + [b(t) X_t + B(t) Y_t]\, dW_t. \end{aligned}$$

3. Let W_t and $\tilde{W}_t$ denote two independent Brownian motions. Derive the SDE for the stochastic variable $Y_t = \frac{W_t}{\tilde{W}_t}$.

Solution: Note that $W_t, \tilde{W}_t \sim N(0, t)$ for some $t > 0$. Using Itô's formula, we have:

$$d\left(\frac{W_t}{\tilde{W}_t}\right) = \frac{dW_t}{\tilde{W}_t} - \frac{W_t}{\tilde{W}_t^2}\, d\tilde{W}_t - \frac{1}{2}\left(\frac{W_t}{\tilde{W}_t^2}\right)' (d\tilde{W}_t)^2 = \frac{dW_t}{\tilde{W}_t} - \frac{W_t}{\tilde{W}_t^2}\, d\tilde{W}_t + \frac{W_t}{\tilde{W}_t^3}\, dt.$$

4. Consider an interest rate model following the stochastic differential equation: $dr_t = \theta\, dt + \sigma\, dW_t$, where θ and σ are both constants. Determine (a) $\mathbb{E}\left[\int_0^T r_t\, dt\right]$, (b) $V\left[\int_0^T r_t\, dt\right]$.

Solution: Firstly, $\displaystyle\int_0^T dr_t = \theta \int_0^T dt + \sigma \int_0^T dW_t \Leftrightarrow r_T - r_0 = \theta(T-0) + \sigma(W_T - W_0)$. This then implies $r_T = r_0 + \theta T + \sigma W_T$.

(a) $\displaystyle\mathbb{E}\left[\int_0^T r_t\, dt\right] = \mathbb{E}\left[\int_0^T (r_0 + \theta t)\, dt\right] = \mathbb{E}\left[r_0 T + \tfrac{\theta}{2}T^2\right] = r_0 T + \tfrac{\theta}{2}T^2$

(b) We have:

$$
\begin{aligned}
V\left[\int_0^T r_t\, dt\right] &= \mathbb{E}\left[\left(\int_0^T r_t\, dt\right)^2\right] - \left(\mathbb{E}\left[\int_0^T r_t\, dt\right]\right)^2 = \mathbb{E}\left[\int_0^T r_t^2\, dt\right] - \left(r_0 T + \frac{\theta}{2}T^2\right)^2 \\
&= \mathbb{E}\left[\left(\int_0^T (r_0 + \theta t + \sigma W_t)\, dt\right)^2\right] - T^2\left(r_0 + \frac{\theta}{2}T\right)^2 \\
&= \mathbb{E}\left[\left(r_0 T + \frac{\theta}{2}T^2 + \sigma \int_0^T W_t\, dt\right)^2\right] - T^2\left(r_0 + \frac{\theta}{2}T\right)^2 \\
&= \mathbb{E}\left[T^2\left(r_0 + \frac{\theta}{2}T\right)^2\right] + 2T\left(r_0 + \frac{\theta}{2}T\right)\sigma \mathbb{E}\left[\int_0^T W_t\, dt\right] \\
&+ \sigma^2 \mathbb{E}\left[\left(\int_0^T W_t\, dt\right)^2\right] - T^2\left(r_0 + \frac{\theta}{2}T\right) \\
&= 2T\left(r_0 + \frac{\theta}{2}T\right)\sigma \int_0^T \underbrace{\mathbb{E}[W_t]}_{=0}\, dt + \sigma^2 \mathbb{E}\left[\int_0^T W_t^2\, dt\right] \text{ (by Itô's isometry theorem)} \\
&= \sigma^2 \mathbb{E}\left[\int_0^T \int_0^T W_t W_u\, dt\, du\right] = \sigma^2 \int_0^T \int_0^T \mathbb{E}[W_t W_u]\, dt\, du \\
&= \sigma^2 \int_0^T \int_0^T \mathrm{cov}[W_t, W_u]\, dt\, du = \sigma^2 \int_0^T \int_0^T \min\{W_t, W_u\}\, dt\, du \\
&= \sigma^2 \int_0^T \left(\int_0^u t\, dt + \int_u^T u\, dt\right) du = \sigma^2 \int_0^T \left(\frac{1}{2}u^2 + u(T-u)\right) du \\
&= \sigma^2 \left(\frac{1}{6}T^3 + \frac{1}{2}T^3 - \frac{1}{3}T^3\right) = \frac{\sigma^2 T^3}{3}.
\end{aligned}
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