# QF620 Stochastic Modelling in Finance
# Solution to Assignment (2 of 4)

1. (a) The squared stock price process can be derived via Itô's formula with $X_t = S_t^2 = f(S_t)$,

$$dX_t = dS_t^2 = (2r + \sigma^2)X_t dt + 2\sigma X_t dW_t.$$

(b) Applying Itô's formula to $f(S_t) = \log(S_t)$, the stock price process is given by

$$dS_t = rS_t dt + \sigma S_t dW_t \quad \Rightarrow \quad S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Since $X_t = S_t^2$ also follows a geometric Brownian motion, we apply Itô's formula to the log function $g(X_t) = \log(X_t)$ to obtain the solution

$$S_t^2 = S_0^2 \exp\left[(2r - \sigma^2)T + 2\sigma W_t\right].$$

(c) First we have

$$\begin{aligned}\mathbb{E}[S_t] &= S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t} \cdot \mathbb{E}[e^{\sigma W_t}] \\ &= S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t} \cdot e^{\frac{\sigma^2 t}{2}} = S_0 e^{rt}\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{E}[S_t^2] &= S_0^2 e^{(2r - \sigma^2)t} \cdot \mathbb{E}[e^{2\sigma W_t}] \\ &= S_0^2 e^{(2r - \sigma^2)t} \cdot e^{\frac{4\sigma^2 t}{2}} = S_0^2 e^{(2r + \sigma)t}\end{aligned}$$

2. Using the Black-Scholes model, the stock price follows a lognormal process

$$\begin{aligned}dS_t &= rS_t dt + \sigma S_t dW_t^* \\ S_T &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T} \\ &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}X}, \qquad X \sim N(0,1).\end{aligned}$$

We can price the cash-or-nothing digital call option as

$$\begin{aligned}V_{\text{Cash Digital}}(0) = e^{-rT}\mathbb{E}^*\left[\mathbb{1}_{S_T > K}\right] &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{S_T > K} e^{-\frac{x^2}{2}} dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).\end{aligned}$$

3. The same approach can be applied to the asset-or-nothing digital call option. Again based on Black-Scholes model

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t^* \\
S_T &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T} \\
&= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}X}, \qquad X \sim N(0,1).
\end{aligned}
$$

The asset-or-nothing digital call option can be priced as

$$
\begin{aligned}
V_{\text{Asset Digital}}(0) &= e^{-rT}\mathbb{E}^*\left[S_T \mathbb{1}_{S_T>K}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_T \mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}} dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx \\
&= S_0 \Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$