SMU
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Session 9: Static Replication of Payoffs with Discontinuities

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QF620 Stochastic Modelling in Finance

Replicating Payoffs with Discontinuities

So far we have looked at static replication of European payoffs which are twice differentiable.

But what if the European payoff contains **discontinuities**?

$\Rightarrow$ We can still use static replication—we just need to **start from integrating the payoff weighted by the risk-neutral density**, instead of applying the Carr-Madan formula directly.

To this end, we start with

$$\begin{aligned}V_0 = e^{-rT}\mathbb{E}[h(S_T)] &= e^{-rT}\int_0^\infty h(K)f(K)\,dK \\ &= \int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}\,dK + \int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}\,dK,\end{aligned}$$

and perform integration-by-parts twice to obtain the replication formula.

Replicating Cash-or-Nothing Options

Example Use static replication to value a cash-or-nothing digital call option with payoff:

$$h(S_T) = \mathbb{1}_{S_T \geq \bar{K}} = \begin{cases} 1, & \text{if } S_T \geq \bar{K} \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{K} \geq F = S_0 e^{rT}$.

Solution In this case, we start with

$$\int_{\bar{K}}^{\infty} h(K) \frac{\partial^2 C(K)}{\partial K^2}\, dK.$$

We note that for $K \geq \bar{K}$, $h(K) = 1$, $h'(K) = 0$, and $h''(K) = 0$. Using integration by parts, we obtain

$$\left[h(K) \frac{\partial C(K)}{\partial K} \right]_{\bar{K}}^{\infty} - \left[h'(K) C(K) \right]_{\bar{K}}^{\infty} + \int_{\bar{K}}^{\infty} h''(K) C(K)\, dK = -\frac{\partial C(\bar{K})}{\partial K}$$

Replicating Cash-or-Nothing Options: Implementation

Example Call spread for $\bar{K} = 50$:

$$-\frac{\partial C(\bar{K})}{\partial K} \approx \frac{C(\bar{K} - \Delta K) - C(\bar{K})}{\Delta K}$$