QF620 Additional Examples
Session 6: Valuation Framework & Stochastic Volatility Models

1 Examples

1. In Bachelier's model, the stock price follows the arithmetic process

$$dS_t = \sigma S_0 dW_t.$$

   Derive the valuation formula for a call option struck at K. Assume interest rate is 0.

2. Using the same Bachelier's model in the previous question, derive the valuation formula for a digital cash-or-nothing call option struck at K. Assume interest rate is 0.

3. Under the risk-neutral measure $\mathbb{Q}^*$ associated to the risk-free bond numeraire, the stock price follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

   A financial contract pays $\log S_T$ on the expiry date T. Derive the valuation formula for this financial contract.

4. Under the risk-neutral measure $\mathbb{Q}^*$ associated to the risk-free bond numeraire, the stock price follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

   A financial contract pays $\log \frac{S_T}{K}$ on the expiry date T. Derive the valuation formula for this financial contract.

5. Under the risk-neutral measure $\mathbb{Q}^*$ associated to the risk-free bond numeraire, the stock price follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

   A financial contract pays $\max\left\{\log \frac{S_T}{K}, 0\right\}$ on the expiry date T. Derive the valuation formula for this financial contract.

2 Suggested Solutions

1. Solving the stochastic differential equation by directly integrating it, we obtain

$$
\begin{aligned}
\int_0^T dS_u &= \sigma S_0 \int_0^T dW_u \\
S_T - S_0 &= \sigma S_0 W_T \\
S_T &= S_0 + \sigma S_0 W_T.
\end{aligned}
$$

We want to evaluate

$$\mathbb{E}[(S_T - K)^+] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_0 + \sigma S_0 \sqrt{T} x - K\right)^+ e^{-\frac{x^2}{2}} dx$$

The inequality to satisfy is

$$
\begin{aligned}
S_T &> K \\
S_0 + \sigma S_0 \sqrt{T} x &> K \\
\Rightarrow x &> \frac{K - S_0}{\sigma S_0 \sqrt{T}} = x^*
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{E}[(S_T - K)^+] &= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 + \sigma S_0 \sqrt{T} x - K) e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}} dx + \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} \sigma S_0 \sqrt{T} x e^{-\frac{x^2}{2}} dx \\
&= (S_0 - K)\Phi(-x^*) + \frac{\sigma S_0 \sqrt{T}}{\sqrt{2\pi}} e^{\frac{-x^{*2}}{2}} \\
&= (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma S_0 \sqrt{T}}\right) + \sigma S_0 \sqrt{T} \phi\left(\frac{S_0 - K}{\sigma S_0 \sqrt{T}}\right)
\end{aligned}
$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du. \quad \triangleleft$$

2. This is straightforward. The inequality to satisfy is

$$
\begin{aligned}
S_T &> K \\
S_0 + S_0 \sigma \sqrt{T} x &> K \\
\Rightarrow \quad x &> \frac{K - S_0}{S_0 \sigma \sqrt{T}} = x^*.
\end{aligned}
$$

The option formula is

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{S_T > K} e^{-\frac{x^2}{2}} dx &= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= \Phi(-x^*) = \Phi\left(\frac{S_0 - K}{S_0 \sigma \sqrt{T}}\right). \quad \triangleleft
\end{aligned}
$$

3. The solution to the stochastic differential equation is given by

$$\begin{aligned}
S_T &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*} \\
\log S_T &= \log S_0 + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*.
\end{aligned}$$

Let V_t denote the value of this financial contract at time t. Under the $\mathbb{Q}^*$ measure, we have

$$\begin{aligned}
\frac{V_0}{B_0} &= \mathbb{E}^{\mathbb{Q}^*}\left[\frac{V_T}{B_T}\right] \\
V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\log S_T\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\log S_0 + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*\right] \\
&= e^{-rT}\left[\log S_0 + \left(r-\frac{\sigma^2}{2}\right)T\right]. \quad \triangleleft
\end{aligned}$$

4. We have

$$\begin{aligned}
\log S_T &= \log S_0 + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^* \\
\log S_T - \log K &= \log S_0 - \log K + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^* \\
\log \frac{S_T}{K} &= \log \frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*
\end{aligned}$$

Let V_t denote the value of this financial contract at time t. Under the $\mathbb{Q}^*$ measure, we have

$$\begin{aligned}
\frac{V_0}{B_0} &= \mathbb{E}^{\mathbb{Q}^*}\left[\frac{V_T}{B_T}\right] \\
V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\log \frac{S_T}{K}\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\log \frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*\right] \\
&= e^{-rT}\left[\log \frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T\right]. \quad \triangleleft
\end{aligned}$$

5. We have

$$\log \frac{S_T}{K} = \log \frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*.$$

To value $\max\left\{\log\left(\frac{S_T}{K}\right), 0\right\}$, we are interested in the inequality

$$\begin{aligned}
\log \frac{S_T}{K} &> 0 \\
\log S_0 - \log K + \left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x &> 0 \\
x &> \frac{\log \frac{K}{S_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*
\end{aligned}$$

Let V_t denote the value of this financial contract at time t. Under the $\mathbb{Q}^*$ measure, we have

$$
\begin{aligned}
\frac{V_0}{B_0} &= \mathbb{E}^{\mathbb{Q}^*}\left[\frac{V_T}{B_T}\right] \\
V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\max\left\{\log\frac{S_T}{K}, 0\right\}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left[\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right]e^{-\frac{x^2}{2}}\,dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\left[\int_{x^*}^{\infty}\left(\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right)e^{-\frac{x^2}{2}} + \int_{x^*}^{\infty}\sigma\sqrt{T}xe^{-\frac{x^2}{2}}\,dx\right] \\
&= e^{-rT}\phi(d_2)\sigma\sqrt{T} + e^{-rT}\left[\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right]\Phi(d_2).
\end{aligned}
$$

where $d_2 = \frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$. $\triangleleft$