$\int w\, dt = wt + C$

Itô's Formula: Applications

Example Apply Itô formula to the function $X_t = f(t, W_t) = tW_t$, and show that

$$\int_0^T W_t\, dt = TW_T - \int_0^T t\, dW_t = \int_0^T (T-t)\, dW_t.$$

Use this to show that

$$V\left[\int_0^T W_t\, dt\right] = \frac{T^3}{3}.$$

$$= \mathbb{E}\left[\left(\int_0^T (T-t)\, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T (T-t)^2\, dt\right]$$

$$= \int_0^T (T-t)^2\, dt$$

$$X_t = f(t, W_t) = t \cdot W_t$$

$$f(t,x) = t \cdot x$$

$$f_t = x, \quad f_x = t, \quad f_{xx} = 0$$

Itô's:

$$dX_t = f_t(t,W_t)\,dt + f_x(t,W_t)\,dW_t + \frac{1}{2} f_{xx}(t,W_t)\,(dW_t)^2$$

$$\int_0^T dX_t = \int_0^T W_t\,dt + \int_0^T t\,dW_t + \cancelto{0}{\frac{1}{2}\, 0} \cdot dt$$

$$\Big[X_t \Big]_0^T = \int_0^T W_t\,dt + \int_0^T t\,dW_t$$

$$T W_T - 0 = \int_0^T W_t\,dt + \int_0^T t\,dW_t$$

$$\int_0^T W_t \, dt = T W_T - \int_0^T t \, dW_t$$

$$\boxed{\int_0^T dW_t = W_T}$$

$$= \int_0^T T \, dW_t - \int_0^T t \, dW_t$$

$$= \int_0^T (T - t) \, dW_t$$