QF620 Stochastic Modelling in Finance
Solution to Assignment 4/4

1. (a) The solution to the Black-Scholes sde is given by

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*}$$
$$\Rightarrow \quad \sqrt{S_T} = \sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*}$$

Under martingale valuation framework, we obtain

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^*\left[\sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*}\right] \\ &= e^{-rT}\sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma^2 T}{8}} \\ &= \sqrt{S_0} e^{-\frac{rT}{2}} e^{-\frac{\sigma^2 T}{8}} \quad \triangleleft \end{aligned}$$

(b) The payoff is twice differentiable:

$$h(S_T) = \sqrt{S_T}, \qquad h'(S_T) = \frac{1}{2S_T^{1/2}}, \qquad h''(S_T) = -\frac{1}{4S_T^{3/2}}$$

Using Breeden-Litzenberger formula, we have

$$\begin{aligned} V_0 &= e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK \\ &= \sqrt{S_0}e^{-\frac{rT}{2}} - \int_0^F \frac{1}{4K^{3/2}}P(K)dK - \int_F^\infty \frac{1}{4K^{3/2}}C(K)dK \quad \triangleleft \end{aligned}$$