QF620 Stochastic Modelling in Finance
Solution to Assignment 3/4

1. (a) This is an arithmetic Brownian process. We can directly integrate both sides to obtain

$$X_T = X_0 + \mu T + \sigma W_T.$$

The mean and variance are given by

$$\begin{aligned}
\mathbb{E}[X_T] &= X_0 + \mu T \\
V[X_T] &= V[X_0 + \mu T + \sigma W_T] = V[\sigma W_T] = \sigma^2 T
\end{aligned}$$

(b) This is a geometric Brownian process. To solve this stochastic differential equation, we apply Itô's formula to $f(S_t) = \log(S_t)$ to obtain the solution

$$X_T = X_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right].$$

The mean and variance are given by

$$\begin{aligned}
\mathbb{E}[X_T] &= \mathbb{E}\left[X_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\right]\right] \\
&= X_0 e^{rT} \\
V[X_T] &= \mathbb{E}[X_T^2] - \mathbb{E}[X_T]^2 \\
&= \mathbb{E}\left[X_0^2 \exp\left[\left(2\mu - \sigma^2\right) T + 2\sigma W_T\right]\right] - X_0^2 e^{2rT} \\
&= X_0^2 e^{(2\mu-\sigma^2)T} \mathbb{E}\left[e^{2\sigma W_T}\right] - X_0^2 e^{2\mu T} \\
&= X_0^2 e^{(2\mu-\sigma^2)T} e^{2\sigma^2 T} - X_0^2 e^{2\mu T} \\
&= X_0^2 e^{(2\mu+\sigma^2)T} - X_0^2 e^{2\mu T}
\end{aligned}$$

(c) This is a mean-reverting process. To solve this stochastic differential equation, we apply Itô's formula to $f(t, X_t) = e^{\kappa t} X_t$ to obtain the solution

$$X_t = X_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)} \, dW_u.$$

The mean and variance are given by

$$\begin{aligned}
\mathbb{E}[X_T] &= X_0 e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right) \\
V[X_T] &= V\left[X_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)} \, dW_u\right] \\
&= V\left[\sigma \int_0^t e^{\kappa(u-t)} \, dW_u\right] \\
&= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} \, du\right] \\
&= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).
\end{aligned}$$

2. Consider the function $Z_t = \frac{X_t}{Y_t} = f(X_t, Y_t)$. The partial derivatives of $f(x, y) = \frac{x}{y}$ are given by

$$f_x = \frac{1}{y}, \quad f_{xx} = 0, \quad f_y = -\frac{x}{y^2}, \quad f_{yy} = \frac{2x}{y^3}, \quad f_{xy} = -\frac{1}{y^2}.$$

Applying Itô's formula to f, we obtain

$$\begin{aligned}
dZ_t &= f_x(X_t, Y_t)dX_t + \frac{1}{2}f_{xx}(X_t, Y_t)(dX_t)^2 + f_y(X_t, Y_t)dY_t + \frac{1}{2}f_{yy}(X_t, Y_t)(dY_t)^2 \\
&\qquad + f_{xy}(X_t, Y_t)dX_t dY_t \\
&= \frac{1}{Y_t}(rX_t dt + \sigma X_t dW_t) + 0 - \frac{X_t}{Y_t^2}(rY_t dt + \sigma Y_t d\tilde{W}_t) + \frac{X_t}{Y_t^3}\sigma^2 Y_t^2 dt \\
&\qquad - \frac{1}{Y_t^2} X_t Y_t \sigma^2 dW_t d\tilde{W}_t \\
&= rZ_t dt + \sigma Z_t dW_t - rZ_t dt - \sigma Z_t d\tilde{W}_t + \sigma^2 Z_t dt - \sigma^2 Z_t dW_t d\tilde{W}_t \\
&= \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t + \sigma^2 Z_t dt - \sigma^2 Z_t dW_t d\tilde{W}_t.
\end{aligned}$$

(a) If W_t and $\tilde{W}_t$ are independent, then by Box calculus, $dW_t d\tilde{W}_t = 0$, hence

$$dZ_t = \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t + \sigma^2 Z_t dt.$$

(b) If W_t and $\tilde{W}_t$ have a correlation of 1, then by Box calculus, $dW_t d\tilde{W}_t = dt$, hence

$$\begin{aligned}
dZ_t &= \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t \\
&= 0. \quad \because W_t = \tilde{W}_t
\end{aligned}$$

(c) If W_t and $\tilde{W}_t$ have a correlation of ρ, then by Box calculus, $dW_t d\tilde{W}_t = \rho dt$, hence

$$dZ_t = \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t + \sigma^2 Z_t dt - \sigma^2 Z_t \rho dt.$$