3 Given $S_0 = \$12$, $u = \frac{1}{d} = 2$ and $r = 25\%$, we construct a binomial tree with the following stock prices:

$$S_1^u = \$24,\ S_1^d = \$6,$$
$$S_2^{uu} = \$48,\ S_2^{ud} = S_2^{du} = \$12,\ S_2^{dd} = \$3,$$
$$S_3^{uuu} = \$96,\ S_3^{uud} = S_3^{udu} = S_3^{duu} = \$24,\ S_3^{udd} = S_3^{dud} = S_3^{ddu} = \$6,\ S_3^{ddd} = \$1.50.$$

Moreover, by $\mathbb{E}^*[S_1] = S_1^u p^* + S_1^d(1-p^*) = S_0(1+r)$, we have:

$$p^* = \frac{S_0(1+r) - S_1^d}{S_1^u - S_1^d} = \frac{S_0(1+r) - S_0 d}{S_0 u - S_0 d} = \frac{1+r-d}{u-d} = \frac{0.75}{1.5} = \frac{1}{2}.$$

(a) For an American put option struck at $K = \$10$, the payoff is $(K - S_n)^+$. Specifically,

$$V_3^{uuu} = \$0,\ V_3^{uud} = V_3^{udu} = V_3^{duu} = \$0,\ V_3^{udd} = V_3^{dud} = V_3^{ddu} = \$4,\ V_3^{ddd} = \$8.50.$$

Using these values, we compute the following:

$$V_2^{uu} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{uuu}p^*}_{=\$0} + \underbrace{V_3^{uud}(1-p^*)}_{=\$0}\right], \underbrace{(K - S_2^{uu})^+}_{=\$0}\right\} = \$0,$$

$$V_2^{ud} = V_2^{du} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{udu}p^*}_{=\$0} + \underbrace{V_3^{udd}(1-p^*)}_{=\$2}\right], \underbrace{(K - S_2^{ud})^+}_{=\$0}\right\} = \$1.60,$$

$$V_2^{dd} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{ddu}p^*}_{=\$2} + \underbrace{V_3^{ddd}(1-p^*)}_{=\$4.25}\right], \underbrace{(K - S_2^{dd})^+}_{=\$7}\right\} = \$7,$$

$$V_1^{u} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_2^{uu}p^*}_{=\$0} + \underbrace{V_2^{ud}(1-p^*)}_{=\$0.80}\right], \underbrace{(K - S_1^{u})^+}_{=\$0}\right\} = \$0.64,$$

$$V_1^{d} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_2^{du}p^*}_{=\$0.80} + \underbrace{V_2^{dd}(1-p^*)}_{=\$3.50}\right], \underbrace{(K - S_1^{d})^+}_{=\$4}\right\} = \$4,$$

$$\therefore V_0^{A} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_1^{u}p^*}_{=\$0.32} + \underbrace{V_1^{d}(1-p^*)}_{=\$2}\right], \underbrace{(K - S_0)^+}_{=\$0}\right\} = \$1.856.$$

Similarly, for a corresponding European put option, its values $V_n^E = \frac{1}{1+r}\mathbb{E}^*[V_{n+1}^E]$ are computed as:

$$V_2^{uu,E} = \$0,\ V_2^{ud,E} = V_2^{du,E} = \$1.60,\ V_2^{dd,E} = \$5,\ V_1^{u,E} = \$0.64,\ V_1^{d,E} = \$2.64,\ V_0^E = \$1.312.$$

Hence, the required early exercise premium is $V_0^A - V_0^E = \$0.544 \approx \$0.54$ (2 d.p.).

(b) For an American cash-or-nothing digital call option with strike price \$20 and a fixed payoff of \$1, we have the following corresponding values:

$$V_3^{uuu} = \$1,\ V_3^{uud} = V_3^{udu} = V_3^{duu} = \$1,\ V_3^{udd} = V_3^{dud} = V_3^{ddu} = \$0,\ V_3^{ddd} = \$0.$$

Using these values, we compute the following:

$$V_2^{uu} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{uuu}p^*}_{=\$0.50} + \underbrace{V_3^{uud}(1-p^*)}_{=\$0.50}\right], \$1\right\} = \$1,$$

$$V_2^{ud} = V_2^{du} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{udu}p^*}_{=\$0.50} + \underbrace{V_3^{udd}(1-p^*)}_{=\$0}\right], \$0\right\} = \$0.40,$$

$$V_2^{dd} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_3^{ddu}p^*}_{=\$0} + \underbrace{V_3^{ddd}(1-p^*)}_{=\$0}\right], \$0\right\} = \$0,$$

$$V_1^{u} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_2^{uu}p^*}_{=\$0.50} + \underbrace{V_2^{ud}(1-p^*)}_{=\$0.20}\right], \$1\right\} = \$1,$$

$$V_1^{d} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_2^{du}p^*}_{=\$0.20} + \underbrace{V_2^{dd}(1-p^*)}_{=\$0}\right], \$0\right\} = \$0.16,$$

$$\therefore V_0^{A} = \max\left\{\frac{1}{1+r}\left[\underbrace{V_1^{u}p^*}_{=\$0.50} + \underbrace{V_1^{d}(1-p^*)}_{=\$0.08}\right], \$0\right\} = \$0.464.$$

Similarly, for a corresponding European put option, its values $V_n^E = \frac{1}{1+r}\mathbb{E}^*[V_{n+1}^E]$ are computed as:

$$V_2^{uu,E} = \$0.80,\ V_2^{ud,E} = V_2^{du,E} = \$0.40,\ V_2^{dd,E} = \$0,\ V_1^{u,E} = \$0.48,\ V_1^{d,E} = \$0.16,\ V_0^{E} = \$0.256.$$

Hence, the required early exercise premium is $V_0^A - V_0^E = \$0.208 \approx \$0.21$ (2 d.p.). Unfortunately, most candidates were not prepared and hence unable to tackle this question with full workings made, given the absence of additional examples for that part.

8 Given the payoff $\alpha + \beta\log\left(\frac{S_T}{2}\right) - \log(K)$ under the risk-neutral measure $\mathbb{Q}^*$ modelled using the Black-Scholes model $dS_t = rS_t\,dt + \sigma S_t\,dW_t^*$ with solution $S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*}$, we have, **assuming the valuation formula for the European option**, the value:

$$V_0^E = e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\alpha + \beta\log\left(\frac{S_T}{2}\right) - \log(K)\right] = e^{-rT}\left[\alpha + \beta\log\left(\frac{S_0}{2}\right) + \beta\left(r - \frac{\sigma^2}{2}\right)T - \log(K)\right].$$

For the **corresponding American option** (which some candidates interpreted), we set $\alpha+\beta\log\left(\frac{S_T}{2}\right) - \log(K) > 0$ and then perform the necessary integration steps to obtain the corresponding valuation formula for V_0^A. So long as candidates performed all steps correctly and obtained the correct formula for the American option, it should be fine.

10 Under the displaced-diffusion model, the final answer is $e^{-rT}\Phi(d_2)$, where d_2 is the constant obtained from the displaced-diffusion sub-part of Part 1 of the group project using the valuation formula for the cash-or-nothing digital call option. Unfortunately, most candidates were not prepared and hence unable to tackle this question with full workings made, given the absence of additional examples for that part.