QF620 Additional Examples
Session 10: Greeks and Dynamic Hedging
(supplementary, no exam)

1 Examples

1. Suppose the stock price S_t follows the Black-Scholes process:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Show that the value of a European contract paying $\log \frac{S_T}{K}$ on maturity T can be written as:

$$V_0 = e^{-rT}\left[\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right].$$

Use this formula to derive the delta (Δ) and gamma (Γ) of this contract.

2. Consider the Black-Scholes process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion. The call option formula is given by

$$C(S_0, K, r, \sigma, T) = S_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Find $\frac{\partial C}{\partial K}$.

3. In the real world probability measure $\mathbb{P}$, the stock price process follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. In the market there is also a risk-free bond following the deterministic process $dB_t = rB_t dt$. We form a portfolio $V_t = \phi_t S_t + \psi_t B_t$ following the trading strategy (ϕ_t, ψ_t). If $\forall t : \phi_t = 1$, find an appropriate choice of ψ_t so that V_t is a self-financing portfolio.

4. **Discussions** Suppose we have access to the following tradables in the market

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = r B_t dt \end{cases}$$

We form a portfolio $V_t = \phi_t S_t + \psi_t B_t$ following the trading strategy (ϕ_t, ψ_t). If the strategy is self-financing, then we have

$$\begin{aligned} dV_t &= \phi_t dS_t + \psi_t dB_t \\ &= \phi_t(\mu S_t dt + \sigma S_t dW_t) + \psi_t(r B_t dt) \\ &= (\phi_t \mu S_t + \psi_t r B_t)dt + \phi_t \sigma S_t dW_t. \end{aligned}$$

The self-financing portfolio V_t can also be thought of as a function of time and stock, such that $V_t = f(t, S_t)$, where $f(t, x)$ is a bivariate function. Applying Itô's Formula, we obtain

$$\begin{aligned} dV_t &= \frac{\partial f}{\partial t}(t, S_t)dt + \frac{\partial f}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2 \\ &= \left(\frac{\partial f}{\partial t}(t, S_t) + \mu S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t)\right) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t)dW_t. \end{aligned}$$

Comparing the dW_t term, we get $\phi_t = \frac{\partial f}{\partial x}(t, S_t)$. Next we compare the dt term and obtain

$$\begin{aligned} \frac{\partial f}{\partial x}(t, S_t)\mu S_t + \psi_t r B_t &= \frac{\partial f}{\partial t}(t, S_t) + \mu S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \\ \psi_t &= \frac{\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t)}{r B_t}. \quad \triangleleft \end{aligned}$$

5. Consider the Black-Scholes process

$$dS_t = r S_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion. The call option formula is given by

$$C(S_0, K, r, \sigma, T) = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Find $\frac{\partial C}{\partial K}$.

6. Volga (or vomma) measures the vega's sensitivity to volatility. Let C denote the value of a call option, volga is defined as

$$\frac{\partial^2 C}{\partial \sigma^2}.$$

Under Black-Scholes model, the value of a European call option is expressed as

$$C(S_0, K, r, \sigma, T) = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Derive the expression for volga under Black-Scholes model.

2 Suggested Solutions

1. Taking the partial derivatives of V_0 with respect to S_0, we obtain

$$\Delta = \frac{e^{-rT}}{S_0}, \qquad \Gamma = -\frac{e^{-rT}}{S^2}. \quad \triangleleft$$

2. Noting that

$$Ke^{-rT} = S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \quad \text{and} \quad \frac{\partial d_1}{\partial K} = \frac{1}{S_0/K} \cdot \left(-\frac{S_0}{K^2}\right) \cdot \frac{1}{\sigma\sqrt{T}} = -\frac{1}{K\sigma\sqrt{T}} = \frac{\partial d_2}{\partial K},$$

we can proceed with the partial derivative with respect to K as follow:

$$\begin{aligned}
\frac{\partial C}{\partial K} &= S_0 \frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial K} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial K} - e^{-rT}\Phi(d_2) \\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - e^{-rT}\Phi(d_2) \\
&= \underbrace{S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma\sqrt{T} d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right)}_{0} - e^{-rT}\Phi(d_2) \\
&= -e^{-rT}\Phi(d_2).
\end{aligned}$$

This is what we would expect to get — recall that the valuation formula for a cash-or-nothing digital option is given by

$$V_0 = e^{-rT}\Phi(d_2),$$

and the way to hedge this digital option is to form a call spread with strikes very close to each other, giving us the following relationship:

$$\lim_{\Delta K \to 0} \frac{C(K) - C(K + \Delta K)}{\Delta K} = -\lim_{\Delta K \to 0} \frac{C(K + \Delta K) - C(K)}{\Delta K} = -\frac{\partial C}{\partial K}. \quad \triangleleft$$

3. For V_t to be a self-financing portfolio, we need to have

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$

Given that $\phi_t = 1$, we have $d\phi_t = 0$. Applying Itô's Formula and stochastic calculus chain rule to V_t, we obtain

$$dV_t = dS_t + \psi_t dB_t + B_t d\psi_t + \cancelto{0}{d\psi_t dB_t}.$$

Comparing against the self-financing requirement, we note that V_t is self-financing if $d\psi_t = 0$. In other words, the self-financing requirement stipulates that we need $\psi_t = const.$ $\triangleleft$

Note that although this portfolio is self-financing, it is not a dynamic hedging strategy for a call option.

4. Noting that

$$Ke^{-rT} = S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \quad \text{and} \quad \frac{\partial d_1}{\partial K} = \frac{1}{S_0/K} \cdot \left(-\frac{S_0}{K^2}\right) \cdot \frac{1}{\sigma\sqrt{T}} = -\frac{1}{K\sigma\sqrt{T}} = \frac{\partial d_2}{\partial K},$$

we can proceed with the partial derivative with respect to K as follow:

$$\begin{aligned}
\frac{\partial C}{\partial K} &= S_0 \frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial K} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial K} - e^{-rT}\Phi(d_2) \\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T}d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - e^{-rT}\Phi(d_2) \\
&= \underbrace{S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T}d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma\sqrt{T}d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right)}_{0} - e^{-rT}\Phi(d_2) \\
&= -e^{-rT}\Phi(d_2).
\end{aligned}$$

This is what we would expect to get — recall that the valuation formula for a cash-or-nothing digital option is given by

$$V_0 = e^{-rT}\Phi(d_2),$$

and the way to hedge this digital option is to form a call spread with strikes very close to each other, giving us the following relationship:

$$\lim_{\Delta K \to 0} \frac{C(K) - C(K + \Delta K)}{\Delta K} = -\lim_{\Delta K \to 0} \frac{C(K + \Delta K) - C(K)}{\Delta K} = -\frac{\partial C}{\partial K}. \quad \triangleleft$$

5. First derive the expression for vega

$$
\begin{aligned}
C(S,K,r,\sigma,T) &= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)\\
\frac{\partial C}{\partial \sigma} &= S_0\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}\\
&= S_0\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\left(\sqrt{T}-\frac{d_1}{\sigma}\right) - Ke^{-rT}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\cdot\left(-\sqrt{T}-\frac{d_2}{\sigma}\right)\\
&= S_0\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\left(\sqrt{T}-\frac{d_1}{\sigma}\right) - S_0e^{\frac{\sigma^2T}{2}-\sigma\sqrt{T}d_1}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}+\sigma\sqrt{T}d_1-\frac{\sigma^2T}{2}}\cdot\left(-\sqrt{T}-\frac{d_2}{\sigma}\right)\\
&= \frac{S_0}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left[\sqrt{T}-\frac{d_1}{\sigma}+\sqrt{T}+\frac{d_2}{\sigma}\right]\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.
\end{aligned}
$$

Differentiating one more time with respect to σ, we obtain

$$
\begin{aligned}
\frac{\partial^2 C}{\partial \sigma^2} &= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot(-d_1)\cdot\frac{\partial d_1}{\partial \sigma}\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot(-d_1)\cdot\left(\sqrt{T}-\frac{d_1}{\sigma}\right)\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot(-d_1)\cdot\left(\sqrt{T}-\frac{d_2+\sigma\sqrt{T}}{\sigma}\right)\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{d_1 d_2}{\sigma} \quad \triangleleft
\end{aligned}
$$