Stochastic Integrals with Constant Integrand

We have seen that we can evaluate stochastic integrals with constant integrand, for instance

$$\int_0^T dW_t = W_T - W_0 = W_T,$$

or if $\alpha \in \mathbb{R}$ is a constant, then

$$\int_0^T \alpha \, dW_t = \alpha W_T.$$

The same applies to other stochastic variables as well, for example

$$\int_0^T dX_t = X_T - X_0.$$

This is because of the definition of stochastic integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1}) \times (W_{t_i} - W_{t_{i-1}}).$$

where we have partitioned time into n intervals, with $t_0 = 0$ and $t_n = T$. When the function f is a constant α, then it can be written out of the summation, and we have

$$\begin{aligned}
\alpha \lim_{n\to\infty} \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}}) &= \alpha \lim_{n\to\infty} \Big[\big(W_{t_1} - W_{t_0}\big) + \big(W_{t_2} - W_{t_1}\big) + \cdots + \big(W_{t_n} - W_{t_{n-1}}\big) \Big] \\
&= \alpha \lim_{n\to\infty} \Big[W_{t_n} - W_{t_0} \Big] \\
&= \alpha \Big[W_T - W_0 \Big] \\
&= \alpha W_T.
\end{aligned}$$

Note that this “telescoping series” property no longer holds when f is not merely a constant, but a function of time or Brownian motion (or both).