QF620 Stochastic Modelling in Finance
Solution to Assignment (1 of 4)

1. We can directly substitute for the MGF for normal distribution

$$M_X(\theta) = \mathbb{E}\left[e^{\theta X}\right] = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}.$$

   (a)

$$\mathbb{E}\left[e^{X}\right] = \mathbb{E}\left[e^{1\cdot X}\right] = M_X(1) = e^{\mu+\frac{1}{2}\sigma^2}. \quad \triangleleft$$

   (b)

$$\mathbb{E}\left[e^{2X}\right] = M_X(2) = e^{2\mu+2\sigma^2}. \quad \triangleleft$$

2. Since W_t follows a normal distribution, we can once again use the MGF relationship. Note that in this case, given the mean and variance of W_t, the MGF is given by

$$M_{W_t}(\theta) = \mathbb{E}\left[e^{\theta W_t}\right] = e^{\frac{1}{2}\theta^2 t}.$$

   Hence

   (a)

$$\mathbb{E}\left[e^{W_t}\right] = M_{W_t}(1) = e^{\frac{1}{2}t}. \quad \triangleleft$$

   (b)

$$\mathbb{E}\left[e^{\sigma W_t}\right] = M_{W_t}(\sigma) = e^{\frac{1}{2}\sigma^2 t}. \quad \triangleleft$$

3. The binomial tree is given by:

$$\begin{tikzcd}
 & & 20 \\
 & 10 \arrow[ur] \arrow[dr] & \\
5 \arrow[ur] \arrow[dr] & & 5 \\
 & 2.5 \arrow[ur] \arrow[dr] & \\
 & & 1.25
\end{tikzcd}$$

   The risk-neutral probabilities can be calculated as follows:

$$p^* = \frac{(1+r)-d}{u-d} = \frac{1.04-0.5}{2-0.5} = 0.36, \qquad q^* = 1-p^* = 0.64.$$

(a) The European put option can be valued directly as

$$
\begin{aligned}
P_0^E &= \frac{1}{(1+r)^2}\mathbb{E}^*\left[P_2^E\right] \\
&= \frac{1}{1.04^2} \times \left[0.36^2 \cdot 0 + 2 \cdot 0.36 \cdot 0.64 \cdot 5 + 0.64^2 \cdot 8.75\right] \\
&\approx 5.44 \quad \triangleleft
\end{aligned}
$$

(b) The American put option is 5.68. The binomial tree of the American put is given by:

$$
\begin{tikzcd}
 & & 0 \\
 & 3.077 \arrow[ur] \arrow[dr] & \\
5.68 \arrow[ur] \arrow[dr] & & 5 \\
 & 7.5 \arrow[ur] \arrow[dr] & \\
 & & 8.75
\end{tikzcd}
$$

(c) The European call option can be valued directly as

$$
\begin{aligned}
C_0^E &= \frac{1}{(1+r)^2}\mathbb{E}^*\left[C_2^E\right] \\
&= \frac{1}{1.04^2} \times \left[0.36^2 \cdot 10 + 2 \cdot 0.36 \cdot 0.64 \cdot 0 + 0.64^2 \cdot 0\right] \\
&\approx 1.198 \quad \triangleleft
\end{aligned}
$$

(d) The American call option price is 1.198, the same value as the European call.

4. (a)

$$
\begin{aligned}
\mathbb{P}(W_2 < 0 | W_1 > 0) &= \mathbb{P}(|W_2 - W_1| > |W_1 - W_0|) \times \mathbb{P}(W_2 < W_1) \\
&= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \quad \triangleleft
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbb{P}(W_1 \times W_2 < 0) &= \mathbb{P}(W_1 < 0, W_2 > 0) + \mathbb{P}(W_1 > 0, W_2 < 0) \\
&= \mathbb{P}(W_2 > 0 | W_1 < 0)\mathbb{P}(W_1 < 0) + \mathbb{P}(W_2 < 0 | W_1 > 0)\mathbb{P}(W_1 > 0) \\
&= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4} \quad \triangleleft
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathbb{P}(W_1 < 0 \cap W_2 < 0) &= \mathbb{P}(W_2 < 0 | W_1 < 0)\mathbb{P}(W_1 < 0) \\
&= \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} \quad \triangleleft
\end{aligned}
$$

5. First we note that if $X \sim N(0,1)$, then we have the following (using Moment Generating Function)

$$
\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = 1, \quad \mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^4] = 3.
$$

Next, note that

$$
\begin{aligned}
(W_t - W_s)^2 &\sim N(0, (t-s))^2 = (t-s)N(0,1)^2 = (t-s)X^2 \\
(W_t - W_s)^4 &\sim N(0, (t-s))^4 = (t-s)^2 N(0,1)^4 = (t-s)^2 X^4
\end{aligned}
$$

Hence,

$$\mathbb{E}[(W_t - W_s)^2] = \mathbb{E}[(t-s)X^2] = (t-s)$$
$$\mathbb{E}[(W_t - W_s)^4] = \mathbb{E}[(t-s)^2 X^4] = 3(t-s)^2$$

So we have

$$\begin{aligned} V[(W_t - W_s)^2] &= \mathbb{E}[(W_t - W_s)^4] - \mathbb{E}[(W_t - W_s)^2]^2 \\ &= 3(t-s)^2 - (t-s)^2 = 2(t-s)^2 \quad \triangleleft \end{aligned}$$

6. We know that $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$. Let $X \sim N(0,1)$, we have

$$\begin{aligned} \mathbb{E}[|W_{t+\Delta t} - W_t|] = \sqrt{\Delta} t \mathbb{E}[|X|] &= \sqrt{\frac{\Delta t}{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} \, dx \\ &= 2\sqrt{\frac{\Delta t}{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} \, dx \\ &= 2\sqrt{\frac{2\Delta t}{2\pi}} \int_0^{\infty} e^{-u} du = \sqrt{\frac{2\Delta t}{\pi}} \quad \triangleleft \end{aligned}$$