Quadratic Variation and Box Calculus

In this supplementary note, we want to use the Law of Large Numbers to argue that $(\Delta W_t)^2$ and Δt are of the same order, leading to the Box calculus rule of $dW_t \cdot dW_t = dt$.

Suppose W_t is a Brownian motion, and we have the time partition $\{t_0, t_1, t_2, \ldots, t_n\}$, where $t_0 = 0$ and $t_n = T$. Consider any two timestamps t_i and t_{i-1}, note that

$$\mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})^2\right] = t_i - t_{i-1}$$

Let us define $\Delta W_t = W_{t_i} - W_{t_{i-1}}$, and $\Delta t = t_i - t_{i-1}$. We have

$$\mathbb{E}[(\Delta W_t)^2] = \Delta t$$

Also note that a quick rearrangement yields

$$\mathbb{E}\left[\frac{(\Delta W_t)^2}{\Delta t}\right] = 1.$$

The Law of Large Numbers asserts that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n} = 1.$$

Now we write,

$$(\Delta W_t)^2 = \Delta t \cdot \frac{(\Delta W_t)^2}{\Delta t} = \frac{T}{n} \cdot \frac{(\Delta W_t)^2}{\Delta t} = T \cdot \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n}$$

Summing up and sending the limit of $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta W_t)^2 = T \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n} = T.$$

Compare this with the expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta t = T,$$

and using the differential notation, we arrive at the rule of

$$(dW_t)^2 = dt.$$

Note that this does not mean that every single $(\Delta W_t)^2$ is exactly equal to Δt. But when summed over a time interval, the Law of Large Numbers asserts that it converges to the mean.