QF620 Stochastic Modelling in Finance
Assignment 2/4
Due Date: 23-Oct-2024

1. Assume that a stock price S_t follows the Black-Scholes lognormal process:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

   (a) Use Itô's formula to derive the stochastic differential equation for the process S_t^2.

   (b) Solve the stochastic differential equations of S_t and S_t^2.

   (c) Use your solutions to verify the following expectations:

$$\mathbb{E}[S_t] = Se^{rt}$$
$$\mathbb{E}[S_t^2] = S^2 e^{(2r+\sigma^2)t},$$

2. Digital (or binary) options are characterized by the binary state of their payoffs. A cash-or-nothing digital call option pays \$1 if $S_T > K$ on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Cash Digital}}(T) = \mathbb{1}_{S_T > K}.$$

   Using the Black-Scholes model, the stock price follows a lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

   Derive a valuation formula for this call option:

$$V_{\text{Cash Digital}}(0) = e^{-rT}\mathbb{E}\left[\mathbb{1}_{S_T > K}\right].$$

3. An asset-or-nothing digital call option pays S_T if $S_T > K$ on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Asset Digital}}(T) = S_T \mathbb{1}_{S_T > K}.$$

   Again using the Black-Scholes model, derive a valuation formula for the asset-or-nothing digital call option:

$$V_{\text{Asset Digital}}(0) = e^{-rT}\mathbb{E}\left[S_T \mathbb{1}_{S_T > K}\right].$$