Mathematical Definition of Brownian Motion

Example Let W_t denote a Brownian process. Conditional on $W_1 > 0$, what is the probability that $W_2 < 0$?

0.25

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$$

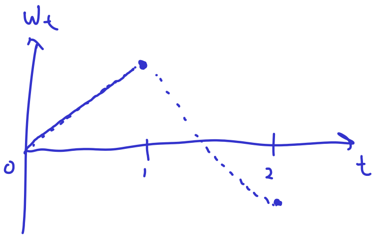
Example Let W_t denote a Brownian process. Determine the probability of $\mathbb{P}(W_1 \times W_2 > 0)$.

$$= \mathbb{P}(W_2 < 0 \cap W_1 < 0) + \mathbb{P}(W_2 > 0 \cap W_1 > 0)$$

$$= \mathbb{P}(W_2 < 0 \mid W_1 < 0)\,\mathbb{P}(W_1 < 0) + \mathbb{P}(W_2 > 0 \mid W_1 > 0)\,\mathbb{P}(W_1 > 0)$$

$$= 0.75 \cdot 0.5 + 0.75 \cdot 0.5 = 0.75$$

$$\mathbb{P}\left(W_2 < 0 \,\middle|\, W_1 > 0 \right)$$



$$= \mathbb{P}\left(\text{step down} \cap \begin{array}{c}\text{2nd step bigger} \\ \text{then 1st step}\end{array} \,\middle|\, W_1 > 0 \right)$$

$$= \mathbb{P}\left(\{W_2 < W_1\} \cap \{|W_2 - W_1| > |W_1 - W_0|\} \,\middle|\, W_1 > 0 \right)$$

$$= \mathbb{P}\left(\{W_2 < W_1\} \,\middle|\, W_1 > 0 \right) \times \mathbb{P}\left(\{|W_2 - W_1| > |W_1 - W_0|\} \,\middle|\, W_1 > 0 \right)$$

$= \quad 0.5 \times 0.5 = 0.25$

$$\mathbb{P}(w_2 > 0 \mid w_1 > 0)$$

0.5

$$= \mathbb{P}(\text{steps up} \mid w_1 > 0)$$

0.5 × 0.5

$$+ \mathbb{P}\left(\text{steps down} \cap \begin{array}{c}\text{2nd step smaller} \\ \text{than first step}\end{array} \;\middle|\; w_1 > 0\right)$$

$$= 0.75$$

Brownian Motion is a Markov Process

We say that a stochastic process exhibits **Markov property** if the conditional distribution of its future state depends only on its present state.

$\Rightarrow$ All relevant information is already subsumed in the present state.

Mathematically, a stochastic process X_t is **Markovian** if

$$\mathbb{E}_t[X_T] = \mathbb{E}[X_T | X_t, X_{t-1}, X_{t-2}, \cdots, X_0] = \mathbb{E}[X_T | X_t]$$

Brownian motion is a Markov process. Suppose $0 < s < t$, note that

$W_t - W_0 \sim N(0, t-s)$

$$\begin{aligned} \mathbb{E}_s[W_t] &= \mathbb{E}_s[W_t - W_s + W_s] \\ &= \mathbb{E}_s[\cancel{W_t - W_s}^{\,0}] + \mathbb{E}_s[W_s] \\ &= W_s \end{aligned}$$

const.

This means that to "predict" W_t given all the information up until time s, we only need to consider the value of the process at time s, i.e. W_s.

This should not be surprising since it is an independent increment process. Note that all independent increment processes exhibit Markov property.

$Y \sim N(\mu, \sigma^2)$

$M_Y(\theta) = \mathbb{E}[e^{\theta Y}]$

$= e^{\mu\theta + \frac{1}{2}\theta^2\sigma^2}$

Geometric Brownian Motion

Let W_t denote a Brownian motion. We say that

$$X_t = e^{W_t}$$

is a **geometric Brownian motion** (GBM)—a continuous-time stochastic process in which its logarithm follows a Brownian motion. We use this process to model asset prices in the **Black-Scholes model**.

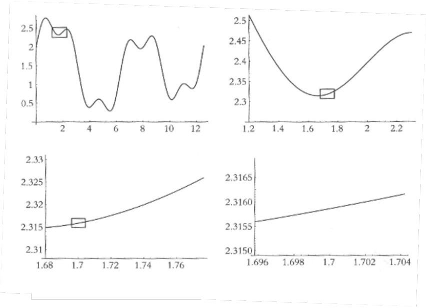
$W_t \sim N(0, t)$

Example Let W_t denote a Brownian motion, and let $X_t = e^{W_t}$. Evaluate the following expectations:

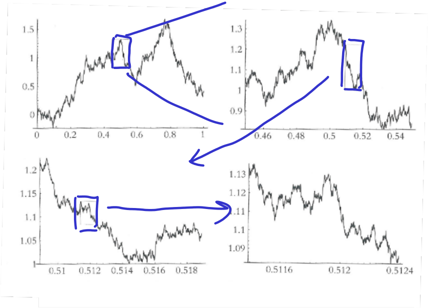
1. $\mathbb{E}[X_T]$ $= \mathbb{E}[e^{W_T}] = e^{0\cdot 1 + \frac{1}{2}1^2 \cdot T}$ $= e^{\frac{T}{2}}$
2. $\mathbb{E}[X_T^2]$

$= \mathbb{E}[(e^{W_T})^2] = \mathbb{E}[e^{2W_T}] = e^{0\cdot 2 + \frac{1}{2}2^2\cdot T}$ $= e^{2T}$

Brownian Motion Properties—Differentiability



Smooth functions are differentiable.



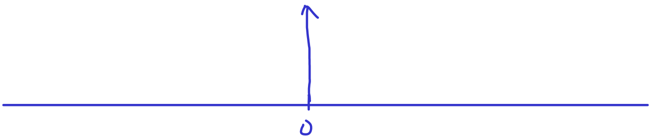
Brownian motions are nowhere differentiable, and self-similar.

Brownian Motion Properties—Fractal

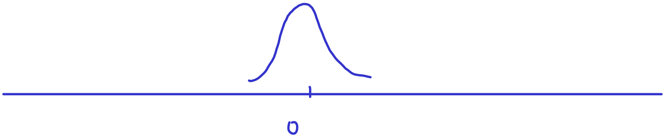
Here's a list of Brownian motion properties that might seem odd at first glance:

- Although W_t is **continuous everywhere**, it is **differentiable nowhere**.
- Brownian motion will eventually hit any and every real value no matter how large or how negative. No matter how far above the axis, it will (with probability one) be back down to zero at some later time.
- Once Brownian motion hits a value, it immediately hits it again **infinitely often**, and will continue to return after arbitrarily large times.
- It doesn't matter what scale you examine Brownian motion on - it looks just the same. Brownian motion is **fractal**.

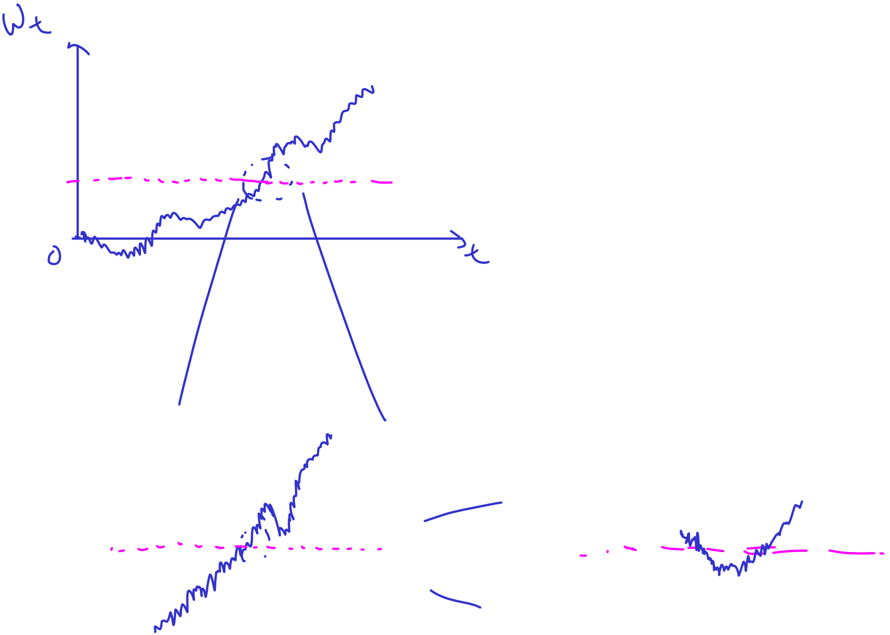
$t = 0$



$t > 0$



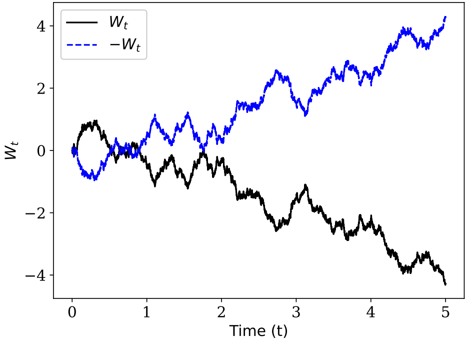
$t \to \infty$


W_t
0
t

Brownian Transformation—Reflection

The properties of a Brownian motion are invariant under a number of transformations — it remain a Brownian motion after these transformations.

Brownian motion properties are **invariant** under reflection. In other words, $W_t :\to -W_t$, **reflection along the x-axis** (multiplied by -1) will not affect the properties of W_t as a Brownian motion since the distribution is symmetric around 0.



Brownian Martingales

If M_t is a stochastic process, we say that it is a **martingale** if

$$\forall T \geq t : \quad M_t = \mathbb{E}_t[M_T].$$

In words, the expected value of this process, conditional on information up to t, is equal to its value taken at time t.

Example Let W_t be a Brownian motion, show that W_t is a martingale.

Solution Consider $0 \leq s \leq t$, we can show that

$$\begin{aligned} \mathbb{E}_s[W_t] &= \mathbb{E}_s[W_t - W_s + W_s] \\ &= \mathbb{E}_s[W_t - W_s] + \mathbb{E}_s[W_s] \\ &= W_s. \quad \triangleleft \end{aligned}$$

$W_t - W_s \sim N(0, t-s)$

Brownian Martingales

$\mathbb{V}[W_t - W_s] = \mathbb{E}[(W_t - W_s)^2] - \cancel{\mathbb{E}[W_t - W_s]}^{0}{}^2$

Example Let W_t be a Brownian motion, show that $W_t^2 - t$ is a martingale.

Solution Consider $0 \le s \le t$, we can show that

$$\begin{aligned}\mathbb{E}_s[W_t^2 - t] &= \mathbb{E}_s[(W_t - W_s + W_s)^2] - t \\ &= W_s^2 - s. \quad \triangleleft\end{aligned}$$

$$\mathbb{E}_s\left[(W_t - W_s + W_s)^2 \right] = \mathbb{E}_s\left[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 \right]$$

$$= (t - s) + 0 + W_s^2$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Brownian Martingales

Example If W_t is a Brownian motion, show that W_t^3 isn't a martingale.

Solution To show this, we check the conditional expectation

$$\begin{aligned}\mathbb{E}_s[W_t^3] &= \mathbb{E}_s[(W_t - W_s + W_s)^3] \\ &= \mathbb{E}_s[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3] \\ &= 3(t-s)W_s + W_s^3.\end{aligned}$$

From here we observe that W_t^3 doesn't satisfy the definition of a standard Brownian motion. ◁

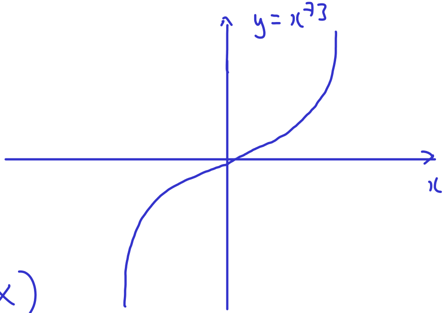
$$\mathbb{E}_s\left[(W_t - W_s)^3\right] + 3\,\mathbb{E}_s\left[(W_t - W_s)^2\right]W_s + 3\,\mathbb{E}_s\left[(W_t - W_s)\right]W_s^2 + W_s^3$$

$$0 \quad + \quad 3(t-s)\,W_s \quad + \quad 0 \quad + \quad W_s^3$$

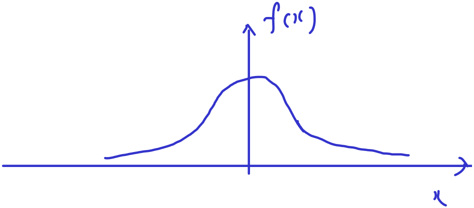
$$\mathbb{E}\left[W_t^{73}\right] = 0$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} x^{73} \cdot f(x)\, dx$$



$(\times)$



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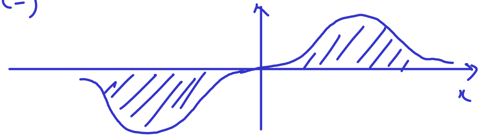

Brownian Martingales

Recall that if $X \sim N(\mu, \sigma^2)$, then by the Moment Generating Function of normal distribution, we have shown that

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

Similarly, for standard normal random variable $Z \sim N(0,1)$, the same method will give us

$$\mathbb{E}\left[e^{\theta Z}\right] = e^{\frac{1}{2}\theta^2}.$$

Example Given that $W_t \sim N(0,t)$, evaluate the following expectation:

$$\mathbb{E}\left[e^{\sigma W_t}\right] = e^{\frac{1}{2}\sigma^2 \cdot t}$$

where $\sigma \in \mathbb{R}$ is a real number.

Brownian Martingales

M_t

$=$

Example If W_t is a Brownian motion, show that $\exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right)$ is an **exponential martingale**.

Solution

M_t

$$\mathbb{E}_s\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right] = \mathbb{E}_s\left[e^{\sigma(W_t - W_s)} \cdot e^{\sigma W_s - \frac{1}{2}\sigma^2 t}\right]$$

$$(W_t - W_s + W_s)$$

$$= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \cdot \mathbb{E}_s\left[e^{\sigma(W_t - W_s)}\right]$$

$$= e^{\sigma W_s - \cancel{\frac{1}{2}\sigma^2 t}} \cdot e^{\frac{1}{2}\sigma^2(\cancel{t} - s)}$$

$$= e^{\sigma W_s - \frac{1}{2}\sigma^2 s}. \quad \triangleleft$$

M_s

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Session 4: Stochastic Integrals and Itô's Formula

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Building Stochastic Models

Our objective is to formulate a model for the stock price process. Let W_t denote a Brownian motion. We know that just W_t itself is not going to be a particularly good model for a stock, since we will also need to be able to

1. control its **drift** over time, and
2. control its **volatility**.

To that end, with t denoting time, we let

$$S_t = S_0 + \mu t + \sigma W_t, \quad \mu \in \mathbb{R},\ \sigma \in \mathbb{R}.$$

This now looks to be a more reasonable model for a stock price process.

We shall look at the distribution of the stock price S_t under our model. The stock price process is normally distributed as $S_t \sim N(S_0 + \mu t, \sigma^2 t)$. We can verify that:

$$\mathbb{E}[S_t] = S_0 + \mu t.$$

$$V[S_t] = V[\sigma W_t] = \sigma^2 t.$$

Building Stochastic Models

Example Consider the stock model

$$S_t = S_0 + \mu t + \sigma W_t$$

described in the previous page. Show that in this model, there is a non-zero probability for S_t to take on negative values.

Solution Since $S_t \sim N(S_0 + \mu t, \sigma^2 t)$, its probability density function is given by

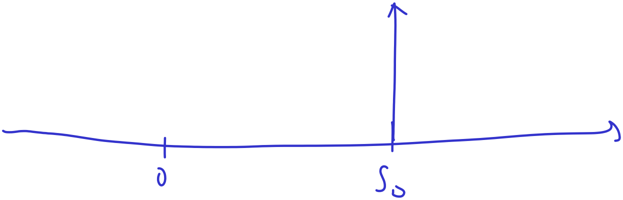
$$f(s) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\left[-\frac{(s - S_0 - \mu t)^2}{2\sigma^2 t}\right].$$

The probability of the event $S_t < 0$ can then be evaluated as

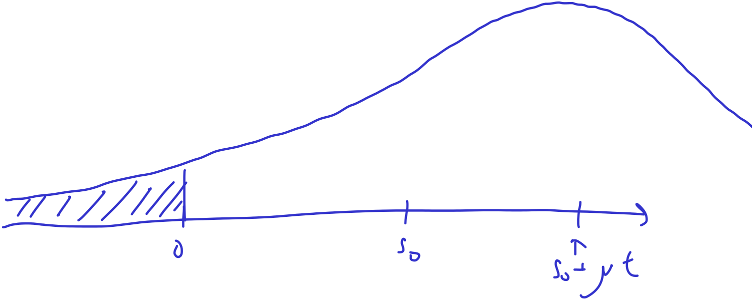
$$\mathbb{P}(S_t < 0) = \frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{0} e^{-\frac{(s - S_0 - \mu t)^2}{2\sigma^2 t}} ds.$$

If $\mu > 0$, then as t increases this probability will decrease, but it remains a non-zero positive value, since the density for $S_t < 0$ is non-zero.

$t = 0$



$t \uparrow$



Building Stochastic Models

A stochastic process is a continuous process that can be written either in the **integral form**

$$S_t = S_0 + \int_0^t \sigma \, dW_u + \int_0^t \mu \, du,$$

or equivalently in the **differential form**

$$dS_t = \mu \, dt + \sigma \, dW_t.$$

In most of the models we encounter in quantitative finance, σ and μ are functions of S_t and t only, so we write

$$dS_t = \mu(t, S_t) \, dt + \sigma(t, S_t) \, dW_t.$$

Suppose μ and σ are both constants, such that $\mu(t, S_t) = \mu$ and $\sigma(t, S_t) = \sigma$, then we can solve the SDE for this simple case and obtain

$$S_t = S_0 + \mu t + \sigma W_t.$$

From Brownian Motions to Stochastic Processes

Brownian motion is the natural candidate to be used to model the evolution of the stock price process S_t. We can write the future stock price as its present price plus a **deterministic** and a **stochastic components**:

$$\underbrace{S_{t+\Delta t}}_{\text{future price}} = \underbrace{S_t}_{\text{present price}} + \underbrace{\mu(t, S_t)\Delta t}_{\text{deterministic}} + \underbrace{\sigma(t, S_t)(W_{t+\Delta t} - W_t)}_{\text{stochastic}}$$

Question Having defined the drift coefficient as $\mu(t, S_t)$, how can the drift term be considered as deterministic, given that S_t is a stochastic process?

$$S_{t+\Delta t} - S_t = \mu(t, S_t)\,\Delta t + \sigma(t, S_t)(W_{t+\Delta t} - W_t)$$

$$\Delta S_t = \mu(t, S_t)\,\Delta t + \sigma(t, S_t)\,\Delta W_t$$

From Brownian Motions to Stochastic Processes

By using Brownian motion increment to form the stochastic component of our model, we are effectively using independent normally distributed increment to drive our stock price process.

Now we take the limit of $\Delta t \to 0^+$, and obtain the **stochastic differential equation (SDE)**

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

To solve this SDE, we wish we could write it in the following format:

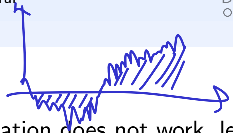
$$\frac{dS_t}{dt} = \mu(t, S_t) + \sigma(t, S_t)\frac{dW_t}{dt}.$$

However, this is not feasible given that one of Brownian motion's properties is that it is **nowhere differentiable**.

$$\int_0^T dS_t = \int_0^T \mu(t, S_t)\, dt + \int_0^T \sigma(t, S_t)\, dW_t$$

$$S_T - S_0 = \qquad '' \qquad + \qquad ''$$

Stochastic Integrals



Since the differentiation formulation does not work, let us try the integration formulation by expressing the stock price process as follow:

$$S_T = S_0 + \underbrace{\int_0^T \mu(u, S_u)\, du}_{\text{Riemann integral}} + \underbrace{\int_0^T \sigma(u, S_u)\, dW_u}_{\text{stochastic integral}} .$$

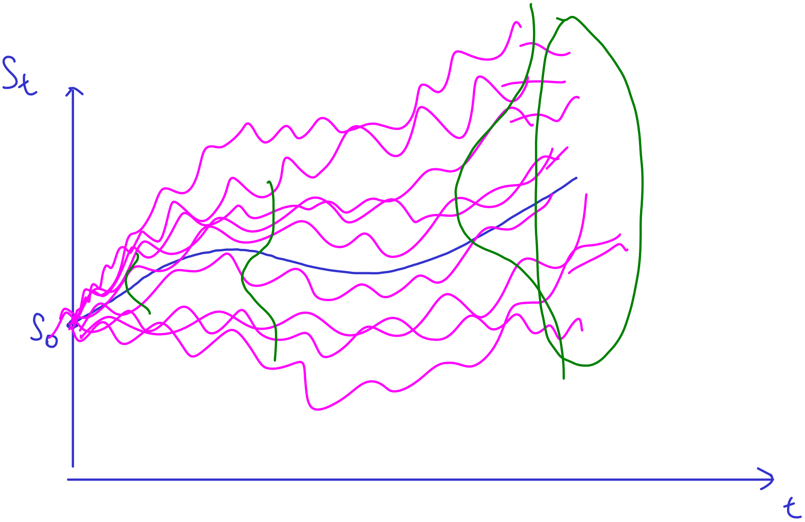
↑ R.V.

Note that on the right hand side, the first integral is a classic **Riemann integral**, and we know how to manage it. Recall the definition of a Riemann integral

Riemann Integration

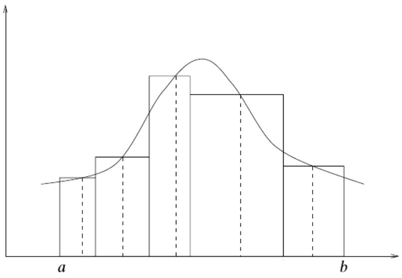
Let f be a regular function and P_n be a partition of the interval $[0, T]$, given by $\{t_0 = 0, t_1, t_2, \ldots, t_n = T\}$, then f is Riemann integrable if the following limit converges

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \times (t_i - t_{i-1}), \quad x_i \in [t_{i-1}, t_i].$$


S_t
S_0
t

Definition of Riemann Integrals

We can define and visualise a Riemann integral as the area under the curve. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which we would like to integral over the interval $[a, b]$.



Partitioning the interval into

$$\{t_0 = a,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5 = b\}$$

We can approximate the area as

$$S = \sum_{i=1}^{5} f(x_i)(t_i - t_{i-1}),$$

where

$$x_i \in [t_{i-1}, t_i].$$

The integral is therefore defined as $n \to \infty$.

Definition of Stochastic Integrals

What about the second integral? Let us define it using the same approach as the Riemann integral before. With the same notations, we define the **stochastic integral** as

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(t_{i-1}) \times (W_{t_i} - W_{t_{i-1}}).$$

This limit exists in an appropriate sense. With this definition for stochastic integral, we note that

- the result of a stochastic integral is a random variable, as opposed to what we get from Riemann integral.
- in Riemann integral of a function $f(x_i)$, x_i can be any point in the interval $[t_{i-1}, t_i]$, whereas in stochastic integral, x_i must be taken at the left side of each interval (t_{i-1}). This is due to the **previsibility** condition.
- **Previsibility**: the value of $f(x_i)$ is only known at the beginning of the interval. Taking $x_i \neq t_{i-1}$ lead to different results.