

# QF620 Stochastic Modelling in Finance

## Solution to Assignment (1 of 4)

1. We can directly substitute for the MGF for normal distribution

$$M_X(\theta) = \mathbb{E} \left[ e^{\theta X} \right] = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}.$$

(a)

$$\mathbb{E} [e^X] = \mathbb{E} [e^{1 \cdot X}] = M_X(1) = e^{\mu + \frac{1}{2}\sigma^2}. \quad \triangleleft$$

(b)

$$\mathbb{E} [e^{2X}] = M_X(2) = e^{2\mu + 2\sigma^2}. \quad \triangleleft$$

2. Since  $W_t$  follows a normal distribution, we can once again use the MGF relationship. Note that in this case, given the mean and variance of  $W_t$ , the MGF is given by

$$M_{W_t}(\theta) = \mathbb{E} \left[ e^{\theta W_t} \right] = e^{\frac{1}{2}\theta^2 t}.$$

Hence

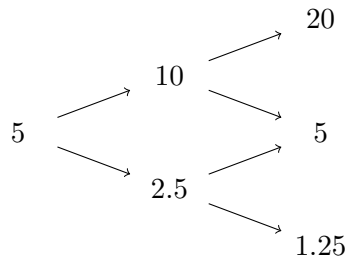
(a)

$$\mathbb{E} [e^{W_t}] = M_{W_t}(1) = e^{\frac{1}{2}t}. \quad \triangleleft$$

(b)

$$\mathbb{E} [e^{\sigma W_t}] = M_{W_t}(\sigma) = e^{\frac{1}{2}\sigma^2 t}. \quad \triangleleft$$

3. The binomial tree is given by:



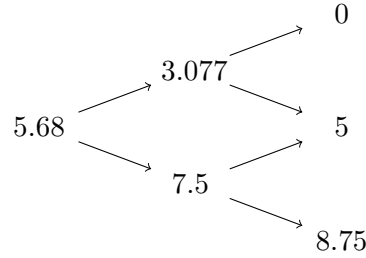
The risk-neutral probabilities can be calculated as follows:

$$p^* = \frac{(1+r) - d}{u - d} = \frac{1.04 - 0.5}{2 - 0.5} = 0.36, \quad q^* = 1 - p^* = 0.64.$$

(a) The European put option can be valued directly as

$$\begin{aligned} P_0^E &= \frac{1}{(1+r)^2} \mathbb{E}^* [P_2^E] \\ &= \frac{1}{1.04^2} \times [0.36^2 \cdot 0 + 2 \cdot 0.36 \cdot 0.64 \cdot 5 + 0.64^2 \cdot 8.75] \\ &\approx 5.44 \quad \triangleleft \end{aligned}$$

(b) The American put option is 5.68. The binomial tree of the American put is given by:



(c) The European call option can be valued directly as

$$\begin{aligned} C_0^E &= \frac{1}{(1+r)^2} \mathbb{E}^* [C_2^E] \\ &= \frac{1}{1.04^2} \times [0.36^2 \cdot 10 + 2 \cdot 0.36 \cdot 0.64 \cdot 0 + 0.64^2 \cdot 0] \\ &\approx 1.198 \quad \triangleleft \end{aligned}$$

(d) The American call option price is 1.198, the same value as the European call.

4. (a)

$$\begin{aligned} \mathbb{P}(W_2 < 0 | W_1 > 0) &= \mathbb{P}(|W_2 - W_1| > |W_1 - W_0|) \times \mathbb{P}(W_2 < W_1) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \quad \triangleleft \end{aligned}$$

(b)

$$\begin{aligned} \mathbb{P}(W_1 \times W_2 < 0) &= \mathbb{P}(W_1 < 0, W_2 > 0) + \mathbb{P}(W_1 > 0, W_2 < 0) \\ &= \mathbb{P}(W_2 > 0 | W_1 < 0) \mathbb{P}(W_1 < 0) + \mathbb{P}(W_2 < 0 | W_1 > 0) \mathbb{P}(W_1 > 0) \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4} \quad \triangleleft \end{aligned}$$

(c)

$$\begin{aligned} \mathbb{P}(W_1 < 0 \cap W_2 < 0) &= \mathbb{P}(W_2 < 0 | W_1 < 0) \mathbb{P}(W_1 < 0) \\ &= \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} \quad \triangleleft \end{aligned}$$

5. First we note that if  $X \sim N(0, 1)$ , then we have the following (using Moment Generating Function)

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = 1, \quad \mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^4] = 3.$$

Next, note that

$$\begin{aligned} (W_t - W_s)^2 &\sim N(0, (t-s))^2 = (t-s)N(0, 1)^2 = (t-s)X^2 \\ (W_t - W_s)^4 &\sim N(0, (t-s))^4 = (t-s)^2 N(0, 1)^4 = (t-s)^2 X^4 \end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[(W_t - W_s)^2] &= \mathbb{E}[(t - s)X^2] = (t - s) \\ \mathbb{E}[(W_t - W_s)^4] &= \mathbb{E}[(t - s)^2 X^4] = 3(t - s)^2\end{aligned}$$

So we have

$$\begin{aligned}V[(W_t - W_s)^2] &= \mathbb{E}[(W_t - W_s)^4] - \mathbb{E}[(W_t - W_s)^2]^2 \\ &= 3(t - s)^2 - (t - s)^2 = 2(t - s)^2 \quad \triangleleft\end{aligned}$$

6. We know that  $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$ . Let  $X \sim N(0, 1)$ , we have

$$\begin{aligned}\mathbb{E}[|W_{t+\Delta t} - W_t|] &= \sqrt{\Delta t} \mathbb{E}[|X|] = \sqrt{\frac{\Delta t}{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} dx \\ &= 2\sqrt{\frac{\Delta t}{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} dx \\ &= 2\sqrt{\frac{2\Delta t}{2\pi}} \int_0^{\infty} e^{-u} du = \sqrt{\frac{2\Delta t}{\pi}} \quad \triangleleft\end{aligned}$$