QF620 Stochastic Modelling in Finance

AY2024-25 Session 5 Examples

1 SDEs and Martingale

Example Use Itô's formula to derive the stochastic differential equations of the following processes, and determine which of them are martingales:

1. $X_t = W_t^2$
2. $X_t = 2 + t + e^{W_t}$
3. $X_t = W_t^2 + \tilde{W}_t^2$, where W_t and $\tilde{W}_t$ are independent Brownian motions
4. $X_t = W_t^2 - t$
5. $X_t = W_t^3$
6. $X_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$

Solution: Let $X_t = f(t, W_t)$ and $0 \leq s < t$. Recall Itô's formula:

$$dX_t = f_t\,dt + f_{W_t}\,dW_t + \frac{f_{W_tW_t}}{2}(dW_t)^2 = \left(f_t + \frac{f_{W_tW_t}}{2}\right)dt + f_{W_t}\,dW_t.$$

1. $f_t = 0,\ f_{W_t} = 2W_t,\ f_{W_tW_t} = 2 \Rightarrow dX_t = dt + 2W_t\,dW_t$

$$\mathbb{E}_s[X_t] = \mathbb{E}_s[W_t^2] = \mathbb{E}_s[(W_t - W_s)^2 + 2W_tW_s - W_s^2] = \underbrace{\mathbb{E}_s[(W_t - W_s)^2]}_{=t-s} + 2\underbrace{\mathbb{E}_s[W_tW_s]}_{=W_s^2} - \underbrace{\mathbb{E}_s[W_s^2]}_{=W_s^2}$$

$$= t - s + W_s^2 \neq W_s^2 \Rightarrow X_t \text{ is not a martingale}$$

Verification: $\mathbb{E}_s[dX_t] = \mathbb{E}_s[dt + 2W_t\,dW_t] = \underbrace{\mathbb{E}_s[dt]}_{=dt} + 2\underbrace{\mathbb{E}_s[W_t\,dW_t]}_{=0} = dt \neq 0$

2. $f_t = 1,\ f_{W_t} = f_{W_tW_t} = e^{W_t} \Rightarrow dX_t = \left(1 + \frac{e^{W_t}}{2}\right)dt + e^{W_t}\,dW_t$

$$\mathbb{E}_s[X_t] = \mathbb{E}_s[2 + t + e^{W_t}] = 2 + t + \mathbb{E}_s[e^{(W_t - W_s) + W_s}] = 2 + t + e^{W_s}\underbrace{\mathbb{E}_s[e^{(W_t - W_s)}]}_{=e^{\frac{1}{2}(t-s)}}$$

$$= 2 + t + e^{W_s + \frac{1}{2}(t-s)} \neq 2 + s + e^{W_s} \Rightarrow X_t \text{ is not a martingale}$$

Verification: $\mathbb{E}_s[dX_t] = \mathbb{E}_s\left[\left(1 + \frac{e^{W_t}}{2}\right)dt + e^{W_t}\,dW_t\right] = \left[\left(1 + \frac{1}{2}e^{W_s + \frac{1}{2}(t-s)}\right)dt + e^{W_s + \frac{1}{2}(t-s)}dW_s\right] \neq 0$

3. Similar to Q1, we have $dX_t = 2\,dt + 2W_t\,dW_t + 2\tilde{W}_t\,d\tilde{W}_t$, $\mathbb{E}[X_t] = 2(t-s) + W_s^2 + \tilde{W}_s^2 \neq W_s^2 + \tilde{W}_s^2$, so X_t is not a martingale. To verify this, we have $\mathbb{E}_s[dX_t] = 2\,dt \neq 0$.

4. Similar to Q1, we have $dX_t = 2W_t\,dW_t$, $\mathbb{E}[X_t] = t - s + W_s^2 - t = W_s^2 - s$, so X_t is a martingale. To verify this, we have $\mathbb{E}_s[dX_t] = 2\mathbb{E}_s[W_t\,dW_t] = 0$.

5. $f_t = 0$, $f_{W_t} = 3W_t^2$, $f_{W_tW_t} = 6W_t \Rightarrow dX_t = 3W_t\,dt + 3W_t^2\,dW_t$

$$\begin{aligned}
\mathbb{E}_s[X_t] &= \mathbb{E}_s[W_t^3] = \mathbb{E}_s[(W_t - W_s + W_s)^3] \\
&= \underbrace{\mathbb{E}_s[(W_t - W_s)^3]}_{=0} + 3\underbrace{\mathbb{E}_s[(W_t - W_s)^2 W_s]}_{=3(t-s)W_s} - 3\underbrace{\mathbb{E}_s[(W_t - W_s)W_s^2]}_{=0} + W_s^3 \\
&= 3(t-s)W_s + W_s^3 \neq W_s^3 \Rightarrow X_t \text{ is not a martingale}
\end{aligned}$$

Verification: $\mathbb{E}_s[dX_t] = \mathbb{E}_s[3W_t\,dt + 3W_t^2\,dW_t] = 3W_s\,dt \neq 0$

6. $f_t = -\frac{\theta^2}{2}X_t$, $f_{W_t} = \theta X_t$, $f_{W_tW_t} = \theta^2 X_t \Rightarrow dX_t = \theta X_t\,dW_t$
Refer to page 25 of QF620 Session 3 slides for the proof that proves the martingale property. To verify this, we have $\mathbb{E}_s[dX_t] = \theta\mathbb{E}_s[X_t\,dW_t] = \theta X_s \underbrace{\mathbb{E}_s[dW_t]}_{=0} = 0$.