Three-State Model

We have shown that the risk-neutral price for a vanilla call option in a two-state world is an **arbitrage-free price**.

- Note that both the **risk-neutral approach** and the hedging argument give a boundary on the set of arbitrage-free prices.
- In the two-state world, the upper and lower bound agree, and we have a **unique arbitrage-free price**.

An extension to our binomial tree model is to allow for more states, the most simple of which is a **trinomial tree model**. Let us add another market state in which the stock price takes on the same value of 100 on the next time step:

$$\begin{tikzcd} & \begin{matrix}110\\10\end{matrix} \\ \begin{matrix}100\\C_0\end{matrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{matrix}100\\0\end{matrix} \\ & \begin{matrix}90\\0\end{matrix} \end{tikzcd}$$

Three-State Model

Suppose we buy Δ of stocks today to hedge the sale of one call option, then our portfolio will be worth $110\Delta - 10$, 100Δ or 90Δ in the next time step.

Since we only have one free variable, and we have 2 constraints (which two?), we do not have sufficient degree of freedom to enforce our portfolio to be worth the same in all three states.

If we try to match only the $S_1 = 100$ and $S_1 = 90$ states, then Δ is 0 and we have not hedged at all.

If we try to match the $S_1 = 110$ and $S_1 = 90$ states, then we are back to the 2-state world, and $\Delta = \frac{1}{2}$.

⇒ The portfolio will be worth 45, 50 and 45 respectively in the upper, middle and lower states.

⇒ It is no longer risk-free. We know that it must be worth at least 45 tomorrow, so it must be worth more than 45 today, since interest rate is 0. This implies

$$50 - C_0 > 45 \quad \Rightarrow \quad C_0 < 5.$$

⇒ This is just an upper bound, and is <u>not a unique price</u>.

tomorrow: $45 < \text{Hedged Portfolio} < 50$

today: $45 < 100 \times \frac{1}{2} - C < 50$

$45 < 50 - C$

$C < 5$

$50 - C < 50$

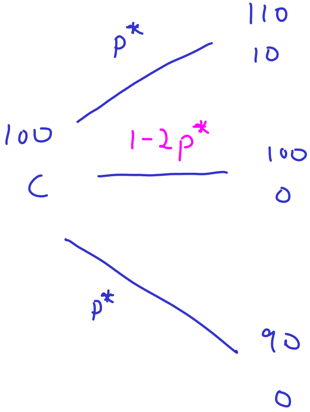
$0 < C$

Three-State Model

Let us now try the risk-neutral probability approach:

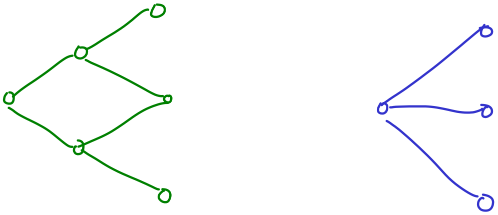
- In order for the expected value of the stock tomorrow to equal the price today, we must have the probability of state $S_1 = 110$ equal that of state $S_1 = 90$.
- Unlike the binomial case in the previous section, this can be achieved by any probability p^* as long as $0 < p^* < \frac{1}{2}$. The corresponding probability for $S_1 = 100$ is then given by $1 - 2p^*$.
- This range of p^* yields an option price of $0 < C_0 < 5$. So we can only conclude that the set of arbitrage-free prices for the option is between 0 and 5.

The three-state market is an example of an **incomplete market**. The characteristic of an incomplete market is that portfolios **cannot be set up to replicate** precisely the desired payoff, and the price of an option can only be shown to <u>lie within an interval</u>, and is <u>not a unique value</u>.



$$0 < 1-2p^* < 1$$

$$\mathbb{E}[S_1] = S_0 = 100$$



Session 2: Binomial Tree and the Risk-Neutral Measure

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Changing Probability Measure

A **probability space** is used to model experiments with different possible outcomes.

It is made up of the **sample space** Ω, which is a set of all possible outcomes, and a **probability measure** $\mathbb{P}$, which assigns a probability to each element of Ω.

In short, $(\Omega, \mathbb{P})$ denotes a probability space. A random variable can take either value of Ω, and the likelihood of it taking a specific value $\omega \in \Omega$ is determined by the probability measure $\mathbb{P}$.

In other words, Ω lists the set of possible outcomes, and $\mathbb{P}$ determines the distribution.

Changing the probability measure affects the **likelihood** of each price-path being realized. Given two probability measures, we define the **Radon-Nikodym derivative** of the probability $\mathbb{P}$ with respect to the probability measure $\mathbb{Q}$ as

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega) = \frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)},$$

which relates the two probability measure.

Radon-Nikodym Derivative

From the Radon-Nikodym derivative, we can derive $\mathbb{Q}$ from $\mathbb{P}$, or vice versa.

Consider the case where $\mathbb{P}(\omega_i) = 0$, but $\forall i : \mathbb{Q}(\omega_i) \neq 0$. In this case, not all of the ratio $\frac{\mathbb{P}(\omega_i)}{\mathbb{Q}(\omega_i)}$ is defined, hence the Radon-Nikodym derivative does not exist.

If we take them away from our analysis, then we will be losing information: these paths may be $\mathbb{P}$-impossible, but they are $\mathbb{Q}$-possible.

Throwing them away will cause us to lose information about $\mathbb{Q}$. In short, $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is undefined if $\mathbb{Q}$ allows something which $\mathbb{P}$ doesn't, and vice versa. This leads us to the important concept of **equivalence**.

Equivalence of Probability Measure

Two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if they operate on the same sample space Ω and they agree on what is possible and impossible. If ω is any event in the sample space, then

$$\mathbb{P}(\omega) > 0 \quad \Leftrightarrow \quad \mathbb{Q}(\omega) > 0.$$

Radon-Nikodym Derivative

Example A stock is worth \$100 today. Assume interest rate is 0. Consider a two-state model where the stock can increase/decrease its value by \$10.

$\mathbb{P}$
- A hedge fund manager has a bullish view on this speculative stock, thinking that the stock price will increase with 80% probability per period.

$\mathbb{Q}$
- A market maker takes a risk-neutral view on the same stock, under which the expected stock price two periods later is the same as today.

Note that the fund manager works under an empirical probability measure $\mathbb{P}$, and the market maker works under the risk-neutral measure $\mathbb{Q}$.
The stock process and the Radon-Nikodym process $\frac{d\mathbb{Q}}{d\mathbb{P}}$ are:

$$\begin{tikzcd}
 & & 120 \\
 & 110 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 100 \\
 & 90 \arrow[ur] \arrow[dr] & \\
 & & 80
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \frac{25}{64} \\
 & \frac{5}{8} \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & \frac{25}{16} \\
 & \frac{5}{2} \arrow[ur] \arrow[dr] & \\
 & & \frac{25}{4}
\end{tikzcd}$$

$$\frac{dQ}{dp} \uparrow$$

$$1$$

$$\frac{\frac{1}{2}}{\frac{4}{5}} = \frac{5}{8}$$

$$\frac{\frac{1}{2}}{\frac{1}{5}} = \frac{5}{2}$$

$$\frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{4}{5} \cdot \frac{4}{5}} = \frac{25}{64}$$

$$\frac{2 \times \frac{1}{2} \times \frac{1}{2}}{2 \times \frac{4}{5} \times \frac{1}{5}} = \frac{25}{16}$$

$$\frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{5} \cdot \frac{1}{5}} = \frac{25}{4}$$

Radon-Nikodym Derivative

Let S_i denote the stock price at time period $i = 0, 1, 2$. The hedge fund manager has the following expectation:

$$\mathbb{E}^P[S_2] = 0.8^2 \times 120 + 2 \times 0.8 \times 0.2 \times 100 + 0.2^2 \times 80 = 112$$

The market maker, under risk-neutral measures, expects

$$\mathbb{E}^Q[S_2] = 0.5^2 \times 120 + 2 \times 0.5 \times 0.5 \times 100 + 0.5^2 \times 80 = 100$$

The market maker can **enforce** this expectation by borrowing 100 and buy the stock today. At $t = 2$, the market maker has a debt of exactly 100 to repay.

The hedge fund manager's expectation of 112 is **non-enforceable**—there is risk associated.

Radon-Nikodym derivative gives us a mean to relate this two expectations. We have

$$\mathbb{E}^Q[S_2] = \mathbb{E}^P\left[S_2 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \quad \text{and} \quad \mathbb{E}^P[S_2] = \mathbb{E}^Q\left[S_2 \cdot \frac{d\mathbb{P}}{d\mathbb{Q}}\right]$$

$$\mathbb{E}^{\varphi}\left[S_2 \right] \neq \mathbb{E}^{P}\left[S_2 \right]$$

$$\mathbb{E}^{\varphi}\left[S_2 \right] = \mathbb{E}^{\cancel{P}}\left[S_2 \cdot \frac{d\varphi}{d\cancel{P}} \right]$$

$$\mathbb{E}^{P}\left[S_2 \right] = \mathbb{E}^{\cancel{\varphi}}\left[S_2 \cdot \frac{dP}{d\cancel{\varphi}} \right]$$

Radon-Nikodym Derivative

$$\frac{\frac{1}{2}\cdot\frac{1}{2}}{0.8\cdot 0.8} \qquad \frac{\frac{1}{2}\cdot\frac{1}{2}}{0.8\cdot 0.2} \qquad \frac{\frac{1}{2}\cdot\frac{1}{2}}{0.2\cdot 0.2}$$

To see this relationship, note that

$$\mathbb{E}^P\left[S_2 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 0.8^2 \cdot 120 \cdot \frac{25}{64} + 2 \cdot 0.8 \cdot 0.2 \cdot 100 \cdot \frac{25}{16} + 0.2^2 \cdot 80 \cdot \frac{25}{4}$$
$$= 100 = \mathbb{E}^Q[S_2].$$

Notice how the Radon-Nikodym derivative reduces the weight on the higher stock values, and increases the weight on the lower stock values, bringing down the expectation.

Similarly, we have

$$\mathbb{E}^Q\left[S_2 \cdot \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = 0.5^2 \cdot 120 \cdot \frac{64}{25} + 2 \cdot 0.5 \cdot 0.5 \cdot 100 \cdot \frac{16}{25} + 0.5^2 \cdot 80 \cdot \frac{4}{25}$$
$$= 112 = \mathbb{E}^P[S_2].$$

Radon-Nikodym derivative allows us to relate the expectation of one probability measure to the other.

Normal (Arithmetic) Binomial Tree

Suppose that interest rate is 0. Consider a binomial tree model where in each time step the stock moves up or down with the same probability. Denote today's stock price as S_0, on the maturity date T, we want:

1. $\mathbb{E}[S_T] = S_0$ $\mathbb{P}$
2. $V[S_T] = \sigma^2 T$ $\sigma \rightarrow$ annualized

To this end, we:

- Divide the time interval from 0 to T into n uniform steps
- For each time step we require a mean of 0 and a variance of $\frac{\sigma^2 T}{n}$.
- Hence, for each time step, the stock moves up or down by $\sigma\sqrt{\frac{T}{n}}$ with probability $\frac{1}{2}$.

Let X_i denote a sequence of independent random variables taking values $+1$ or -1 with equal probability $\frac{1}{2}$, after n steps, the stock will be distributed as

$$S_T = S_0 + \sum_{i=1}^{n} \sigma\sqrt{\frac{T}{n}} X_i.$$

Normal (Arithmetic) Binomial Tree

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean μ and variance s^2, and let

$$Y_n = X_1 + X_2 + \cdots + X_n.$$

Then the distribution of

$$Z = \frac{Y_n - n\mu}{\sqrt{ns^2}}$$

converges to a standard normal random variable with $N(0,1)$ as $n \to \infty$.

Suppose we send the limit of $n \to \infty$, CLT states that:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \quad \sim \quad N(0,1).$$

Hence the distribution of the stock price converges to

$$S_T = S_0 + \sigma\sqrt{T} Z, \quad Z \sim N(0,1).$$

Normal (Arithmetic) Binomial Tree

$V[aX+b] = a^2 V[X]$

Example Show that the stock price distribution

$$S_T = S_0 + \sigma\sqrt{T}Z, \quad Z \sim N(0,1)$$

Bachelier's model

satisfies the requirements of $\mathbb{E}[S_T] = S_0$ and $V[S_T] = \sigma^2 T$.

$$\mathbb{E}[S_T] = \mathbb{E}[S_0 + \sigma\sqrt{T}Z]$$

$$= S_0 + \sigma\sqrt{T}\,\cancel{\mathbb{E}[Z]}^{\,0} = S_0$$

$$V[S_T] = V[S_0 + \sigma\sqrt{T}Z] = \sigma^2 T\, V[Z] = \sigma^2 T$$

Non-zero Interest Rates

$\underline{r}$: $1 \quad 1+r$

$\underline{S}$: $S_0 \quad S_1^u \quad S_1^d$

Suppose the market also consists of a discretely compounded risk-free interest rate r, such that investing \$1 at time $t = 0$ will yield a return of $\$(1 + r)$ one period later.

If a stock is worth S_0 today, and will be worth either S_1^u or S_1^d in the next period, then the following inequality must be satisfied for there to be no-arbitrage

$$S_1^d < S_0(1 + r) < S_1^u.$$

Going through the same replication argument as we have done earlier, under the risk-neutral formulation, we now require the risk-neutral probability to make the stock grow at the risk-free rate, i.e.

$$\mathbb{E}^*[S_1] = p^* S_1^u + (1 - p^*)S_1^d = S_0(1 + r).$$

solve for p^*

Rearranging, we have

$$p^* = \frac{S_0(1 + r) - S_1^d}{S_1^u - S_1^d}.$$

$$\frac{100 \times 1 - 90}{110 - 90} = \frac{1}{2}$$

Lognormal (Geometric) Binomial Tree

For the equity asset class, we tend to think in terms of a **geometric process** (i.e. multiplicative adjustment) rather than an **arithmetic process** (i.e. additive adjustment).

$$\begin{array}{ccc} & & S_1^u = u \times S_0 \\ & \nearrow & \\ S_0 & & \\ & \searrow & \\ & & S_1^d = d \times S_0 \end{array}$$

$u > 1$

$0 < d < 1$

where

$$u = \frac{S_1(H)}{S_0}, \qquad d = \frac{S_1(T)}{S_0}.$$

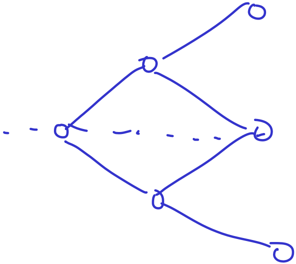
CRR

In this case, our risk-neutral probability of an up jump becomes

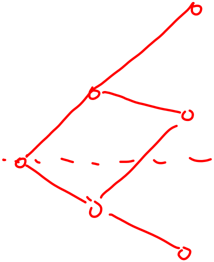
$$p^* = \frac{\cancel{S_0}(1+r) - \overset{d \times S_0}{S_1^d}}{\underset{u \times S_0}{S_1^u} - \underset{d \times S_0}{S_1^d}} = \frac{(1+r) - d}{u - d}.$$

Under the **Cox-Ross-Rubinstein** formulation, it is common to choose $d = \frac{1}{u}$.

$u = \frac{1}{d}$

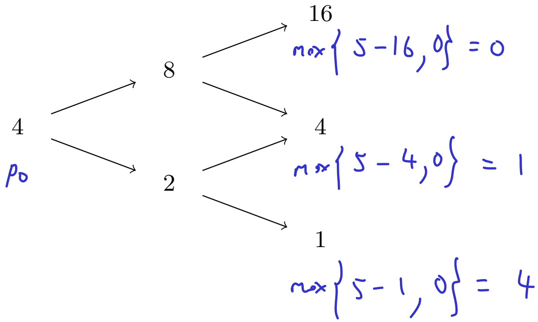


$u \neq \frac{1}{d}$



Lognormal (Geometric) Binomial Tree

Example Consider a stock currently worth \$4. Given the lognormal binomial tree parameter of $u = 2$, $d = 0.5$, $r = 25\%$, calculate the price of a European put option struck at \$5 today ($t = 0$), which expires 2 periods later ($t = 2$).



$$p^* = \frac{(1+r) - d}{u - d} = \frac{1.25 - 0.5}{2 - 0.5} = \frac{1}{2}$$

$$\text{put} = \frac{1}{(1+r)^2} \mathbb{E}^*\left[P_2 \right]$$

$$= \frac{1}{1.25^2}\left[\frac{1}{2} \times \frac{1}{2} \times 0 + 2 \times \frac{1}{2} \times \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} \times 4 \right]$$

$$= \frac{24}{25}$$

Call option $\quad \max\{S_T - K, 0\} \equiv (S_T - K)^+$

put option $\quad \max\{K - S_T, 0\} \equiv (K - S_T)^+$

American Options under Binomial Tree

An **American option** can be exercised on any time before expiry. We can use binomial tree to capture the early exercise premium:

$\Rightarrow$ We should only exercise if we make more money by exercising the option.

For European options, risk-neutral expectation is given by

$$V_n^E = \frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d]$$

For American options, this should become

$$V_n^A = \max\left\{\frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d],\ (K - S_n)^+\right\}$$

Example Determine the price of an American put option for the same example in the previous page.