QF620 Additional Examples
Session 4: Stochastic Integrals and Itô Formula

1 Examples

1. If $X_t = \exp(W_t)$, determine the stochastic differential equation satisfied by dX_t.

2. Consider the function
$$g : \mathbb{R}^2 \to \mathbb{R},\ g(x,t) = \exp\left(-\frac{t}{2} + x\right)$$
and the stochastic process defined as $X_t = \exp\left(-\frac{t}{2} + W_t\right)$. Show that $dX_t = X_t dW_t$.

3. The following question provides a comparison between ordinary calculus and stochastic calculus:

   (a) Evaluate the integral
   $$\int_0^t d(u^2)$$

   (b) Let W_t denote a standard Brownian motion. Evaluate the integral
   $$\int_0^t d(W_u^2)$$

4. Let W_t denote a Brownian motion. Consider the stochastic process X_t defined as
$$X_t = \exp\left(\mu t + \sigma W_t\right).$$

   (a) Derive the stochastic differential equation for dX_t by applying Itô's formula to the function
   $$f : \mathbb{R}^2 \to \mathbb{R},\ f(t,x) = e^{\mu t + \sigma x}.$$

   (b) Derive the stochastic differential equation for dX_t using chain rule by first letting $Y_t = \mu t + \sigma W_t$, and then apply Itô's formula to the function
   $$g : \mathbb{R} \to \mathbb{R},\ g(x) = e^x.$$

5. (a) Consider the function
   $$f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{x^2}{2}.$$
   Let W_t denote a standard Brownian motion. Show that
   $$\int_0^T W_u dW_u = \frac{W_T^2 - T}{2}$$
   by applying Itô's formula to the process $X_t = f(W_t) = \frac{W_t^2}{2}$.

(b) Based on the previous question, we can generalize by considering the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^n.$$

Define $X_t = f(W_t) = W_t^n$, derive the stochastic differential equation for dX_t, and show that

$$\int_0^T W_u^{n-1} dW_u = \frac{W_T^n}{n} - \frac{(n-1)}{2} \int_0^T W_u^{n-2} du.$$

6. Consider the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$$

Let W_t denote a standard Brownian motion. If

$$dX_t = \sigma X_t dW_t,$$

write down the stochastic differential equation satisfied by $Y_t = f(X_t)$.

7. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,t) = x^2 + t.$$

Let W_t denote a standard Brownian motion. If

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

write down the stochastic differential equation satisfied by $Y_t = f(X_t)$.

8. Consider the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x).$$

Let W_t denote a standard Brownian motion. If

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

write down the stochastic differential equation satisfied by $Y_t = f(X_t)$.

9. Consider the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \sqrt{x}.$$

Let W_t denote a standard Brownian motion. If

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

write down the stochastic differential equation satisfied by $Y_t = f(X_t)$.

10. If $dX_t = \mu X_t dt + \sigma X_t dW_t$, let $Y_t = \sin(X_t)$, write down the stochastic differential equation for dY_t.

11. If $dX_t = \mu X_t dt + \sigma X_t dW_t$, let $Y_t = \frac{1}{X_t}$, write down the stochastic differential equation for dY_t.

12. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,t) = e^{x+t}.$$

Let W_t denote a standard Brownian motion. If

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

write down the stochastic differential equation satisfied by $Y_t = f(X_t)$.

13. Determine the mean and variance of the stochastic integral

$$I = \int_0^T W_u \, dW_u.$$

14. Show that

$$\int_0^T W_u \, dW_u = \frac{W_T^2}{2} - \frac{T}{2}$$

by applying Itô's formula to the function $f(W_t) = W_t^2$.

15. Determine how is the following stochastic integral distributed:

$$I = \int_0^T u^2 \, dW_u.$$

16. Apply Itô formula to the function $X_t = f(t, W_t) = tW_t$, and show that

$$\int_0^T W_t \, dt = TW_T - \int_0^T t \, dW_t = \int_0^T (T-t) \, dW_t.$$

Use this to show that

$$V\left[\int_0^T W_t \, dt\right] = \frac{T^3}{3}.$$

2 Suggested Solutions

1. Using Itô's formula, we have

$$dX_t = \exp(W_t)dW_t + \frac{1}{2}\exp(W_t)dt = \frac{1}{2}X_t dt + X_t dW_t.$$

2. We have the derivatives

$$\begin{aligned}\frac{\partial g}{\partial x}(W_t, t) &= \frac{\partial^2 g}{\partial x^2}(W_t, t) = \exp\left(-\frac{t}{2} + W_t\right), \\ \frac{\partial g}{\partial t}(W_t, t) &= -\frac{1}{2}\exp\left(-\frac{t}{2} + W_t\right).\end{aligned}$$

Using Itô's formula, we obtain

$$dX_t = -\frac{1}{2}X_t dt + X_t dW_t + \frac{1}{2}X_t dt = X_t dW_t.$$

3. (a) Under ordinary calculus, we write

$$d(u^2) = 2u\ du,$$

hence the integral is

$$\int_0^t 2u\ du = t^2.$$

(b) Under stochastic calculus, we can show that

$$\begin{aligned}\Delta W_t^2 &= 2W_t \Delta W_t + \frac{1}{2!} \times 2 \times (\Delta W_t)^2 + \cdots \\ \therefore \quad d(W_t^2) &= 2W_t dW_t + dt \quad \text{as } \Delta t \to 0^+\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^t d(W_u^2) &= 2\int_0^t W_u\ dW_u + \int_0^t du \\ W_t^2 - \cancelto{0}{W_0^2} &= 2\int_0^t W_u\ dW_u + t.\end{aligned}$$

4. (a) Given the definition of f, it should be straightforward to apply Itô's formula to work out that

$$dX_t = \left(\mu + \frac{1}{2}\sigma^2\right) X_t dt + \sigma X_t dW_t.$$

(b) Let $Y_t = \mu t + \sigma W_t$, we can write down the stochastic differential equation for dY_t:

$$dY_t = \mu dt + \sigma dW_t.$$

Next, we apply Itô formula to the function

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \exp(x).$$

to obtain

$$\begin{aligned}
dX_t &= g'(Y_t)dY_t + \frac{1}{2}g''(Y_t)(dY_t)^2 \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) X_t dt + \sigma X_t dW_t.
\end{aligned}$$

5. (a) The function $f(x)$ has the following derivatives:

$$f'(W_t) = W_t, \quad f''(W_t) = 1.$$

By Itô's formula,

$$\begin{aligned}
dX_t &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 \\
&= W_t dW_t + \frac{1}{2}dt.
\end{aligned}$$

Integrating both sides from 0 to T, we obtain

$$\int_0^T W_u dW_u = \frac{W_T^2 - T}{2}.$$

(b) Using Itô's formula, we can derive the stochastic differential equation

$$dX_t = nW_t^{n-1}dW_t + \frac{1}{2}n(n-1)W_t^{n-2}dt.$$

Integrating both sides from 0 to T gives

$$\begin{aligned}
\int_0^T dX_u &= \int_0^T nW_u^{n-1}dW_u + \int_0^T \frac{1}{2}n(n-1)W_u^{n-2}du \\
X_T &= \int_0^T nW_u^{n-1}dW_u + \frac{n(n-1)}{2}\int_0^T W_u^{n-2}du \\
\int_0^T W_u^{n-1}dW_u &= \frac{W_T^n}{n} - \frac{(n-1)}{2}\int_0^T W_u^{n-2}du.
\end{aligned}$$

6. The derivatives are given by

$$f'(x) = 2x, \quad f''(x) = 2.$$

By Itô's formula, we have

$$\begin{aligned}
dY_t = dX_t^2 &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\
&= 2X_t(\sigma X_t dW_t) + \frac{1}{2} \times 2 \times \sigma^2 X_t^2 dt \\
&= \sigma^2 X_t^2 dt + 2\sigma X_t^2 dW_t \\
&= \sigma^2 Y_t dt + 2\sigma Y_t dW_t.
\end{aligned}$$

7. The partial derivatives of $f(x,t) = x^2 + t$ are given by

$$\frac{\partial f}{\partial t} = 1, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2.$$

By Itô's formula, we have

$$
\begin{aligned}
dY_t &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2 \\
&= dt + 2X_t dX_t + \frac{1}{2}\cdot 2 \cdot (dX_t)^2 \\
&= dt + 2X_t(\mu X_t dt + \sigma X_t dW_t) + \sigma^2 X_t^2 dt \\
&= (1 + \sigma^2 X_t^2 + 2\mu X_t^2)dt + 2\sigma X_t^2 dW_t
\end{aligned}
$$

8. The derivatives of $f(x) = \log(x)$ are given by

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}.$$

By Itô's formula, we have

$$
\begin{aligned}
dY_t = d\log X_t &= \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t.
\end{aligned}
$$

9. The derivatives of $f(x) = \sqrt{x}$ are given by

$$f'(x) = \frac{1}{2\sqrt{X_t}}, \quad f''(x) = -\frac{1}{4\sqrt{x^3}}.$$

By Itô's formula, we have

$$
\begin{aligned}
d\sqrt{X_t} = dY_t &= \frac{1}{2\sqrt{X_t}}dX_t - \frac{1}{2} \times \frac{1}{4\sqrt{X_t^3}} \times (dX_t)^2 \\
&= \left(\frac{\mu\sqrt{X_t}}{2} - \frac{1}{8\sqrt{X_t^3}}\sigma^2 X_t^2\right)dt + \frac{1}{2}\sqrt{X_t}\sigma dW_t \\
&= \frac{1}{2}\left(\mu - \frac{1}{4}\sigma^2\right)Y_t dt + \frac{1}{2}\sigma Y_t dW_t.
\end{aligned}
$$

10. The derivatives of $f(x) = \sin(x)$ are given by

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x).$$

By Itô's formula, we have

$$
\begin{aligned}
dY_t &= \cos(X_t)dX_t - \frac{1}{2}\sin(X_t)(dX_t)^2 \\
&= \cos(X_t)(\mu X_t dt + \sigma X_t dW_t) - \frac{1}{2}\sin(X_t)\sigma^2 X_t^2 dt \\
&= \left(\mu\cos(X_t)X_t - \frac{\sigma^2}{2}\sin(X_t)X_t^2\right)dt + \sigma\cos(X_t)X_t dW_t.
\end{aligned}
$$

11. The derivatives of $f(x) = \frac{1}{x}$ are given by

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

By Itô's formula, we have

$$\begin{aligned} d\frac{1}{X_t} = dY_t &= -\frac{1}{X_t^2}dX_t + \frac{1}{2}\frac{2}{X_t^3}(dX_t)^2 \\ &= -\frac{1}{X_t^2}(\mu X_t dt + \sigma X_t dW_t) + \frac{1}{2}\frac{2}{X_t^3}(\sigma^2 X_t^2 dt) \\ &= (\sigma^2 - \mu)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t \\ &= (\sigma^2 - \mu)Y_t dt - \sigma Y_t dW_t. \end{aligned}$$

12. The partial derivatives of $f(x,t) = e^{x+t}$ are given by

$$\frac{\partial f}{\partial t} = e^{x+t}, \quad \frac{\partial f}{\partial x} = e^{x+t}, \quad \frac{\partial^2 f}{\partial x^2} = e^{x+t}.$$

By Itô's formula, we have

$$\begin{aligned} dY_t &= e^{X_t+t}dt + e^{X_t+t}dX_t + \frac{1}{2}e^{X_t+t}(dX_t)^2 \\ &= Y_t dt + Y_t(\mu X_t dt + \sigma X_t dW_t) + \frac{1}{2}Y_t\sigma^2 X_t^2 dt \\ &= \left(1 + \mu X_t + \frac{1}{2}\sigma^2 X_t^2\right) Y_t dt + \sigma X_t Y_t dW_t. \end{aligned}$$

13. Noting that I is a stochastic integral, we know that $\mathbb{E}[I] = 0$. Using Itô's Isometry, we have

$$\begin{aligned} V[I] &= \mathbb{E}[I^2] - \cancelto{0}{\mathbb{E}[I]^2} \\ &= \mathbb{E}\left[\left(\int_0^T W_u \, dW_u\right)^2\right] \\ &= \int_0^T \mathbb{E}[W_u^2] \, du = \int_0^T u \, du = \frac{T^2}{2}. \end{aligned}$$

14. The derivatives are given by

$$f'(w) = 2w, \quad f''(w) = 2.$$

Using Itô's formula, we have

$$\begin{aligned} dW_t^2 &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 \\ &= 2W_t dW_t + dt \\ \Rightarrow \quad W_t dW_t &= \frac{dW_t^2}{2} - \frac{dt}{2} \\ \int_0^T W_u \, dW_u &= \int_0^T \frac{dW_u^2}{2} - \int_0^T \frac{du}{2} = \frac{W_T^2 - \cancelto{0}{W_0^2}}{2} - \frac{T}{2}. \end{aligned}$$

15. Firstly, I being a stochastic integral, we know from its property that $\mathbb{E}[I] = 0$. To determine its variance, we make use of Itô's Isometry

$$\begin{aligned} V[I] &= \mathbb{E}[I^2] - \cancel{\mathbb{E}[I]^2}^{\,0} \\ &= \mathbb{E}\left[\left(\int_0^T u^2 \, dW_u\right)^2\right] \\ &= \int_0^T u^4 \, du = \frac{T^5}{5} \end{aligned}$$

So $I \sim N\left(0, \frac{T^5}{5}\right)$.

16. Consider the function $f(t, x) = t \cdot x$, we have the following partial derivatives:

$$f_t = x, \quad f_x = t, \quad f_{xx} = 0.$$

Applying Itô's formula, we obtain:

$$\begin{aligned} dX_t &= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2} f_{xx}(t, W_t)(dW_t)^2 \\ &= W_t dt + t dW_t. \end{aligned}$$

Integrating both sides from 0 to T:

$$\begin{aligned} \int_0^T dX_t &= \int_0^T W_t \, dt + \int_0^T t \, dW_t \\ [X_t]_0^T &= \int_0^T W_t \, dt + \int_0^T t \, dW_t \\ TW_T - 0 &= \int_0^T W_t \, dt + \int_0^T t \, dW_t \\ \int_0^T W_t \, dt &= TW_T - \int_0^T t \, dW_t \\ &= T \int_0^T dW_t - \int_0^T t \, dW_t \qquad \because W_T = \int_0^T dW_t \\ &= \int_0^T T \, dW_t - \int_0^T t \, dW_t \\ &= \int_0^T (T - t) \, dW_t. \quad \lhd \end{aligned}$$

Now, we can apply Itô's isometry to evaluate the variance

$$\begin{aligned} V\left[\int_0^T W_t \, dt\right] &= V\left[\int_0^T (T - t) \, dW_t\right] \\ &= \mathbb{E}\left[\left(\int_0^T (T - t) \, dW_t\right)^2\right] - \cancel{\mathbb{E}\left[\left(\int_0^T (T - t) \, dW_t\right)\right]^2}^{\,0} \\ &= \mathbb{E}\left[\int_0^T (T - t)^2 \, dt\right] \qquad \because \text{Itô's isometry} \\ &= \int_0^T (T - t)^2 \, dt \\ &= \frac{T^3}{3}. \quad \lhd \end{aligned}$$