Single-Name Credit Modelling

QF622 Credit Risk Models

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Recap and Review

Definition of credit risk

- Definition of default varies with the context.
- Credit risk is the risk fo losses owing the fact that borrowers or counterparties may be unwilling or unable to fulfill their contractual obligations.
- Default risk vs. recovery risk vs. spread risk.
- Lending risk vs. issuer risk vs. counterparty credit risk.
- Credit contagion – defaults are correlated and the correlation could shoot up.

Traded credit products

- Securities including fixed coupon bonds and floating rate notes.
- Securities could serve as underlying of repo agreements – a form of collateralised lending.
- Securitisation: ABS and MBS securities – a form of credit risk transfer.
- Credit derivatives: asset swaps, TRS, CDS, n-to-default basket, CDO.

Historical vs. market

- The higher the credit rating, the lower the default probability.
- Default rates are not homogenous in time.
- Recovery rates are negatively correlated with default rates.
  They are random with a spread-out distribution.
- The market charges a risk premium over historical default rates.
- The credit triangle $s = \lambda(1 - R)$.

The basics on Risk-Neutral Pricing

- Dynamic hedging: a net portfolio of a derivative and hedging positions which is instantaneously insensitive to market movements.
- The portfolio is risk-free and should grow at risk-free rate.
- The cost of the derivative is essentially the cost of hedging it, with hedging instruments funded at risk-free rate.
- Unrealistic assumptions include continuous hedging, zero transaction costs and zero counterparty credit risk.

Pricing with a conditional expectation

Discounted expectation under the risk-neutral measure:

$$V(0) = \mathbb{E}\left[\frac{V(T)}{\beta(T)}\right]$$

The value of risk-free money market account with \$1 at time 0:

$$\beta(T) = \exp\left(\int_0^T r(s)ds\right)$$

Zero-coupon bond vs. discount factor:

$$Z(0,T) = \mathbb{E}\left[\frac{1}{\beta(T)}\right] = \mathbb{E}\left[\exp\left(-\int_0^T r(s)ds\right)\right]$$

Some (not so) random remarks on risk-neutral pricing

- Risk-neutral pricing is feasible for a reasonably liquid and complete market.
- With premise of dynamic hedging, a workable pricing model should reprice all hedging instruments.
- A pricing model must be calibrated to a set of market instruments before being used. In some sense, it is an interpolator / extrapolator.
- Market is incomplete and some risk cannot be hedged with liquid market instruments, which requires the use of empirical data and risk mitigants such as reserves.

Structural Models

Corporate default and capital structure (I)

Pioneering work by Merton based on the idea to model corporate default through its simplified capital structure with

- liability represented by a zero-coupon bond with expiry T and face value F $D(t)$
- equity paying no dividend with total value of E

asset value

equity value

$$\underbrace{A(t)}_{\text{asset value}} = \underbrace{D(t)}_{\text{debt paying } F \text{ at } T} + \underbrace{E(t)}_{\text{equity value}}$$

debt paying F at T

Price of the debt and default risk

A risky bond could be priced knowing the risk-free yield-to-maturity and the yield spread. When the risk-free rate is continuously compounded, we simply have

risk-free rate

$$D(t) = Fe^{-(r + s)(T-t)}$$

credit spread (diff between risky yield & risk free yield)

Equivalently, we have

$$s = -\frac{1}{T-t} \ln\left(\frac{D(t)}{F}\right) - r$$

or we need to determine the value of the debt to have knowledge on the credit spread.

Corporate default and capital structure (II)

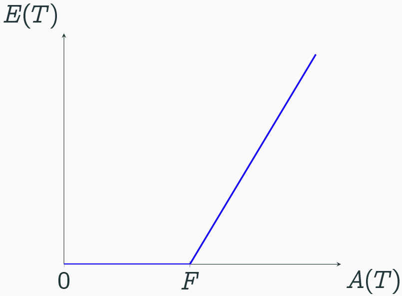
Assumption

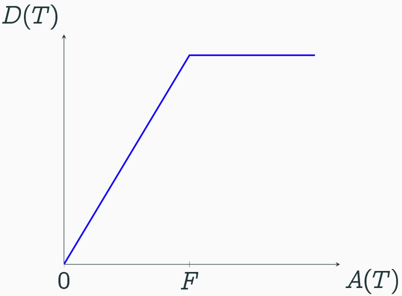
Default could only happen at T, which is the maturity of the debt.

1. If $A(T) \geq F$
   - the asset value of the firm is sufficient to repay the debt
   - bond holder is paid F, leaving $A(T) - F$ to equity holders

2. If $A(T) < F$
   - the asset value of the corporate is insufficient to repay the debt
   - bond holder has a claim on $A(T)$, leaving nothing to equity holders

Corporate default and capital structure (III)





- $E(T) = \max(A(T) - F, 0)$
- a long European call on A struck at F with expiry T

- $D(T) = \min(A(T), F)$ or $F - \max(F - A(T), 0)$
- a short European put on A struck at F with expiry T and a cash position in F

Back to Black-Scholes (I)

To determine the value of $D(t)$, we just need to price a European put with the usual Black-Scholes framework under the risk-neutral measure, where the asset A grows at the risk-free rate r.

asset value volatility

$$\frac{dA(t)}{A(t)} = r dt + \boxed{\sigma_A} dW_t$$

Back to Black-Scholes (II)

Given $D(T) = F - \max(F - A(T), 0)$, we have

$$
\begin{aligned}
D(t) &= \underbrace{Fe^{-r(T-t)}}_{\text{long cash}} \underbrace{- \left(Fe^{-r(T-t)}\Phi(-d_2) - A(t)\Phi(-d_1)\right)}_{\text{short put}} \\
&= Fe^{-r(T-t)}\Phi(d_2) + A(t)\Phi(-d_1)
\end{aligned}
$$

with

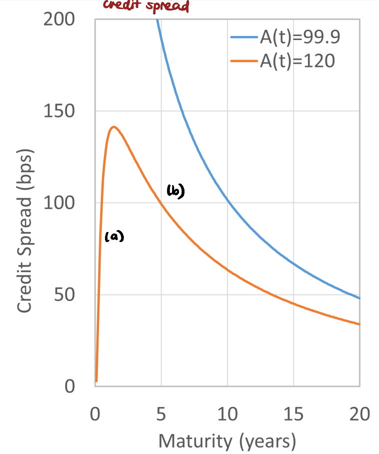
$$
\begin{aligned}
d_1 &= \frac{\ln(A(t)/F) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}} \\
d_2 &= d_1 - \sigma_A\sqrt{T-t}
\end{aligned}
$$

With $E(T)$ being a long European call payoff on A, we also have

$$
E(t) = A(t)\Phi(d_1) - Fe^{-r(T-t)}\Phi(d_2)
$$

Analysis on implied credit spread

by pricing debt using BS, we can obtain implied credit spread



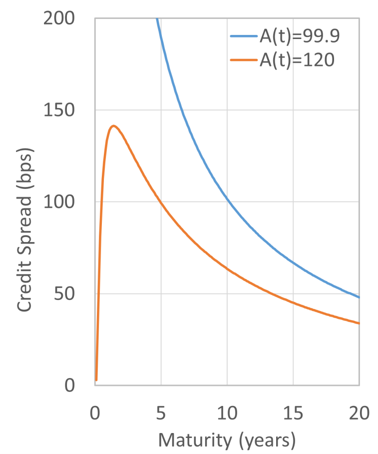
with $F = 100$, $r = 5\%$, and $\sigma_A = 20\%$

when $t \uparrow$, credit spread $\downarrow$

- With $A(t) = 99.9$, the firm is in a "default" state, with with $s \to +\infty$ as $T \to t$. With increasing T, $A(T)$ may grow to a higher value, or the debt is more likely to be repaid. Therefore, credit spread falls with increasing maturity.
- (healthy balance sheet) $A(t) > F$
  With $A(t) = 120$, the firm has a positive equity value, with $s \to 0$ as $T \to t$. With increasing T, $A(T)$ may fall below F and s increases. driven by diffusion $\sigma\sqrt{t}$ (a)
  However, as A grows at risk-free rate, the default risk of debt eventually declines as maturity increases. driven by drift rt (effect of drift > effect of diffusion) (b)

driv

Analysis on implied credit spread



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- With $A(t) = 99.9$, the firm is in a “default” state, with with $s \to +\infty$ as $T \to t$. With increasing T, $A(T)$ may grow to a higher value, or the debt is more likely to be repaid. Therefore, credit spread falls with increasing maturity.
- With $A(t) = 120$, the firm has a positive equity value, with $s \to 0$ as $T \to t$. With increasing T, $A(T)$ may fall below F and s increases. However, as A grows at risk-free rate, the default risk of debt eventually declines as maturity increases.

Default probability and recovery rate

The price of the zero-coupon debt is also a probability weighted discounted payoff at maturity.

$$D(t) = Fe^{-r(T-t)}\left(\overbrace{\Phi(d_2)}^{\substack{\text{survival prob.}\\ Pr\,(A(T)\geq F)}} + \overbrace{\pi}^{\text{recovery rate}}\ \underbrace{\Phi(-d_2)}_{\substack{Pr\,(A(T)<F)\\ \text{default prob.}}}\right)$$

(similar to digital all or nth option)

Substituting it into the Black-Scholes price of $D(t)$, we have

$$\text{recovery rate}: \ \pi = \frac{A(t)\Phi\left(-d_1\right)}{Fe^{-r(T-t)}\Phi\left(-d_2\right)}.$$

Estimating asset volatlity from equity price

What do we know about σ_A? Nearly nothing! However, given

$$dA(t) = rA(t)dt + \sigma_A A(t)dW_t,$$

we know know that $E(t)$ as a function of $A(t)$ follows

$$dE(t) = (\cdots)\,dt + \frac{\partial E(t)}{\partial A(t)}\sigma_A A(t) dW_t.$$

And assuming $E(t)$ follows a lognormal distribution, we have

$$dE(t) = \mu E(t)dt + \sigma_E E(t) dW_t.$$

This leads to

Black-Scholes Delta or $\Phi\,(d_1)$

$$\sigma_E = \boxed{\frac{\partial E(t)}{\partial A(t)}}\frac{A(t)}{E(t)}\sigma_A$$

A concrete example

The value of a company's equity is \$3m and its volatility is 80%. It needs to pay its debt of \$10m in one year. The risk-free rate is 5%. Price the company's debt. Solve for its default probability and recovery rate.

$$\begin{cases} E(t) = A(t)\Phi\left(d_1\right) - Fe^{-r(T-t)}\Phi\left(d_2\right) \\ \sigma_E = \Phi\left(d_1\right)\dfrac{A(t)}{E(t)}\sigma_A \\ d_1 = \dfrac{\ln\left(A(t)/F\right) + \left(r + \frac{1}{2}{\sigma_A}^2\right)(T-t)}{\sigma_A\sqrt{T-t}} \\ d_2 = d_1 - \sigma_A\sqrt{T-t} \end{cases}$$

With $E(t) = 3$, $\sigma_E = 0.8$, $r = 0.05$, $F = 10$, $T - t = 1$ and solving this system of non-linear equations:

$A(t) = 12.40$ and $\sigma_A = 21.23\%$

$D(t) = A(t) - E(t) = 9.40$

$Pr\left(A(T) < F\right) = \Phi\left(-d_2\right) = 12.70\%$

$R = \frac{A(t)\Phi(-d_1)}{Fe^{-r(T-t)}\Phi(-d_2)} = 90.32\%$

Remarks on Merton's model

- Simple, intuitive, and relatable to corporate capital structure.
- The capital structure is too simple to be realistic.
- A single possible default time at T while its distribution is continuous in reality.
- Corporate asset values are not transparent.
- Credit spread always approaches zero as $T \to t$ but even the healthiest companies have positive spreads at very short maturities.
- Various enhancements are possible.
- Typically not applied within the risk-neutral framework and not used for pricing credit derivatives.

Reduced Form Models

Desired features

- Default risk captured with a default event at an unknown future time.
- Recovery may be uncertain.
- Credit spread may change even though default does not occur.
- Prices of market instruments may be repriced.
- Fast to calibrate and fast to price.

Building block – zero recovery risky zero coupon bond (I)

A bond paying no coupon, maturing at T with a face value of \$1 and a zero recovery rate in case of default.

1. If no default before T, the bond pays \$1 at T. payment @ maturity T (do not require dist. of τ)
2. Otherwise if default happens before T, the bond pays nothing.

The PV of the bond is given by

$$\hat{Z}(0,T) = \mathbb{E}\left(\exp\left(-\int_0^{T} r(s)ds\right)\mathbf{1}_{\tau>T}\right)$$

where

$$\mathbf{1}_{\tau>T} = \begin{cases} 1 & \text{if } \tau > T \\ 0 & \text{if } \tau \leq T \end{cases}$$

Building block – zero recovery risky zero coupon bond (II)

On default time τ:

- A credit can only default once.
- A credit will default sooner or later, if we extend the time horizon to infinity.

In this case, we do not need to know τ to price the zero recovery risky zero coupon bond, as there is no cash flow before maturity. We only need to know the probability of default happening before T, or $Pr\left(\tau \leq T\right)$.

Building block – fixed payment at default

A contract paying \$1 at default if default happens before expiry T.

payment @ default τ
(requires dist of τ)

1. If no default before T, the contract pays nothing.
2. Otherwise if default happens at $\tau < T$, the contract pays \$1.

The PV of the contract is given by

$$D(0,T) = \mathbb{E}\left(\exp\left(-\int_0^{\tau} r(s)ds\right) 1_{\tau \leq T}\right).$$

Payments like this occur as recovery payments and in protection leg of a CDS contract. And in this case, we do need to know the full distribution of τ, i.e., $Pr\,(t < \tau \leq t + dt)$, rather than just the probability of default happening before T, or $Pr\,(\tau \leq T)$.

Building block – random payment at default

A contract paying a random quantity $\pi(\tau)$ at default if default happens before expiry T.

1. If no default before T, the contract pays nothing.
2. Otherwise if default happens at $\tau < T$, the contract pays a random quantity $\pi(\tau)$.

The PV of the contract is given by

$$\hat{D}(0,T) = \mathbb{E}\left(\exp\left(-\int_0^{\tau} r(s)ds\right) \pi(\tau) \mathbf{1}_{\tau \leq T}\right).$$

Payments like this are akin to a random recovery at default. Again, we do need to know the full distribution of τ. Furthermore, the distribution of $\pi(\tau)$ may be linked to other risk factors.

Recap on Poisson distribution (I)

Poisson distribution

- A discrete probability distribution describes the probability of a given number of events occurring over a specified period if these events occur with a known average rate and independently of the time since the last event.
- If X is the random variable “number of occurrences in a given interval” for which the average number of occurrence is λ, the probability of k occurrences in that interval is given by

$$f(k, \lambda) = Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

- The first two moments of the distribution are given by

$$E(X) = Var(X) = \lambda.$$

Recap on Poisson distribution (II)

Example

The number of corporate default follows a Poisson distribution with an average number of default being 1 every 6 months.

1. What is the probability of exactly 3 default events in 1.5 years?
2. What is the probability of at least 2 default events in 1 year?
3. What is the probability of exactly 1 default in the 1st half of the year and exactly 1 default in the 2nd half of the year?

1. $f(3,3) = \dfrac{3^3 e^{-3}}{3!}$
2. $1 - f(0,2) - f(1,2) = 1 - \dfrac{2^0 e^{-2}}{0!} - \dfrac{2^1 e^{-2}}{1!}$
3. $f(1,1) \times f(1,1) = \left(\dfrac{1^1 e^{-1}}{1!}\right)^2$

Recap on Exponential distribution (I)

Exponential distribution

- A continuous probability distribution of the time between events in a Poisson process, i.e., a process in which events occur continuously and independently at a constant average rate.
- With rate parameter λ, its probability density function is

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

- Its cumulative distribution function is

$$F(x, \lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Recap on Exponential distribution (II)

Example

The time one spends in a bank is exponentially distributed with mean 10 min , i.e., $\lambda = 0.1$. Find the probability of the following:

1. Someone spending more than 10 min in the bank:
   $Pr(x > 10) = 1 - F(10, 0.1) = e^{-10\times 0.1} = \frac{1}{e}$.
2. Someone spending more than 20 min in the bank given that she is still in the bank after 10 min:
   $Pr(x > 20|x > 10) = \frac{Pr(x > 20)}{Pr(x > 10)} = \frac{e^{-20\times 0.1}}{e^{-10\times 0.1}} = \frac{1}{e} = Pr(x > 10)$

Memoryless property

$$Pr(X > t + x|X > t) = \frac{Pr(X > t + x \cap X > t)}{Pr(X > t)} = \frac{Pr(X > t + x)}{Pr(X > t)}$$
$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = Pr(X > x)$$

Modelling default as as a Poisson process

Hazard rate model

The distribution of τ may be modelled as the first arrival time of a Poisson process parameterised by a hazard rate $\lambda(t)$ (aka credit intensity), which may be calibrated to prices of market instruments.

The hazard rate is a function of time, i.e., this is an inhomogeneous Poisson process. And the resulting distribution for the first arrival time is no loner an exponential distribution.

Distribution of the default time (I)

The time to first arrival in a Poisson process is a random variable whose distribution is characterised as follows.

Cumulative distribution function (CDF) is

$$F(t) = \begin{cases} 1 - \exp\left(-\int_0^t \lambda(s)ds\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Unconditional default probability up to time t is

$$Pr(\tau \leq t) = F(t) = 1 - \exp\left(-\int_0^t \lambda(s)ds\right)$$

Unconditional survival probability up to time t is

$$Q(t) = Pr(\tau > t) = 1 - F(t) = \exp\left(-\int_0^t \lambda(s)ds\right)$$

Distribution of the default time (II)

Probability density function (PDF) is

$$f(t) = \begin{cases} \lambda(t) \exp\left(-\int_0^t \lambda(s) ds\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

With the above, we also have

$$\lim_{dt \to 0} \frac{Pr\left(\tau \leq t + dt \middle| \tau > t\right)}{dt} = \frac{f(t)}{1 - F(t)} = \lambda(t)$$

In other words, $\lambda(t)dt$ is the default probability from t to $t + dt$ conditional on survival up to time t.

Degenerated case of constant hazard rate

Assumption

Hazard rate is constant, i.e., $\lambda(t) = \lambda$.

- Default probability: $Pr(\tau \leq t) = 1 - \exp(-\lambda t)$
- Survival probability: $Q(t) = Pr(\tau > t) = \exp(-\lambda t)$
- Probability density function: $Pr\left(t < \tau \leq t + dt\right) = \lambda \exp(-\lambda t)\, dt$
- Expected default time: $\mathbb{E}(\tau) = \int_0^{+\infty} \lambda s \exp(-\lambda s)\, ds = \frac{1}{\lambda}$
- Variance: $\mathbb{E}\left(\tau^2\right) - \left(\mathbb{E}(\tau)\right)^2 = \int_0^{+\infty} \lambda s^2 \exp(-\lambda s)\, ds - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

A concrete example (I)

Example

An issuer with a 5% constant hazard rate.

Default time distribution

- Default time is exponentially distributed with parameter λ
- Expected default time: $\mathbb{E}\left(\tau\right) = \dfrac{1}{\lambda} = 20$ years
- Variance of default time: $\mathbb{E}\left(\tau^2\right) - \left(\mathbb{E}\left(\tau\right)\right)^2 = \dfrac{1}{\lambda^2} = 400$ years

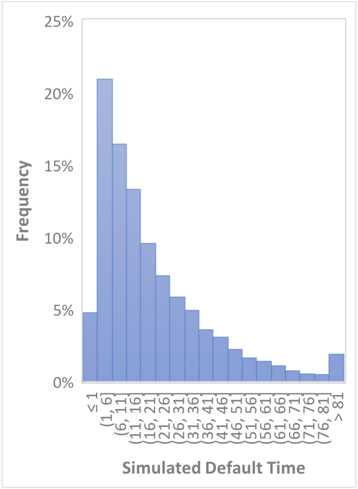
Credit spread

- Assuming a recovery rate of $R = 40\%$
- Credit spread: $s = \lambda\left(1 - R\right) = 300$ bps

A concrete example (II)

- Monte Carlo simulation of τ with inverse CDF
  1. draw $u \sim \mathcal{U}_{[0,1]}$ so that $\exp(-\lambda\tau) = u$
  2. find $\tau = -\frac{1}{\lambda}\ln(u)$
  3. repeat until convergence

- RHS histogram generated using `rand()` in Excel.
  - average: 20.13 years
  - variance: 415.81 years
  - non-negligible short-term default probability



Unpredictable default time

- Contrasting with the Merton's model, the default time in the reduced form model is not predicable.
- Even when we are an instant of time before default, when it happens, it is still a surprise.
- It helps to model the short end of the credit curve: a non-zero spread at the short maturities for a healthy corporate may be explained by the probability of a sudden default.

Modelling default as a Cox process

So far, we have assumed $\lambda(t)$ to be time dependent but deterministic.

- It allows us to model a random default time.
- It does not capture the fact that $\lambda(t)$ changes over time as new information in the market may change the expectations about the default risk.

The hazard rate $\lambda(t)$ may be assumed to be stochastic to generalise the model.

- Two sources of randomness: Poisson jumps and evolution of $\lambda(t)$. The model is doubly stochastic.
- With $\lambda(t)$ being stochastic, the Poison process is known as a Cox process.

A generic modelling framework

In the modelling framework we build so far, the stochastic processes for the short rate and the hazard rate are left to be specified.

Stochastic processes for $r(t)$ and $\lambda(t)$

$$dr(t) = \theta\left(r(t), t\right) dt + \sigma^r\left(r(t), t\right) dW_t^r$$

$$d\lambda(t) = \mu\left(\lambda(t), t\right) dt + \sigma^\lambda\left(\lambda(t), t\right) dW_t^\lambda$$

$$dW_t^r dW_t^\lambda = \rho dt$$

- Drift terms $\theta\left(r(t), t\right)$ and $\mu\left(\lambda(t), t\right)$ may be used to fit the initial term structures of interest rate and credit curves.
- Diffusion terms $\sigma^r\left(r(t), t\right)$ and $\sigma^\lambda\left(\lambda(t), t\right)$ may be calibrated to the prices of vanilla options (assuming there are liquid quotes).
- Correlation ρ is usually estimated historically – incomplete market.

Choice of models

- Could interest rates and hazard rates be negative?
  - Gaussian family of models: the state variable is normally distributed and may attain negative values.
  - Cox, Ingersoll, and Ross model: $\mu\left(\lambda(t), t\right) = \kappa\left(\mu - \lambda(t)\right)$ and $\sigma^{\lambda}\left(\lambda(t), t\right) = \sigma^{\lambda}\sqrt{\lambda(t)}$, where the state variable is always non-negative.
  - Lognormal models?

- How tractable is the model?
  - Could we derive analytical pricing formulae for our building blocks?
  - Could we derive analytical (approximate) pricing formulae for vanilla options for the calibration of volatility?
  - Numerical implementation?

Revisiting building blocks (I)

With a framework to model the default time τ, we could revisit the most basic credit instruments as our building blocks.

Zero recovery risky zero coupon bond

An instrument which pays \$1 at time T as long as the credit has not defaulted, or $\tau > T$. It pays nothing otherwise.

$$\begin{aligned}\hat{Z}(0,T) =& \mathbb{E}\left(\exp\left(-\int_0^T r(s)ds\right)\mathbf{1}_{\tau>T}\right)\\ =& \mathbb{E}\left(\exp\left(-\int_0^T \left(r(s)+\lambda(s)\right)ds\right)\right)\end{aligned}$$

Revisiting building blocks (II)

Fixed payment at default

An instrument which pays \$1 at the time of default τ if $\tau \leq T$.

$$D(0,T) = \mathbb{E}\left(\exp\left(-\int_0^\tau r(s)ds\right)\mathbf{1}_{\tau\leq T}\right)$$
$$= \mathbb{E}\left(\int_0^T \lambda(t)\exp\left(-\int_0^t (r(s)+\lambda(s))\,ds\right)dt\right)$$

Uncertain quantity at default

An instrument which pays an uncertain amount $\pi(\tau)$ at the time of default τ if $\tau \leq T$. If $\pi(\tau)$ is independent of interst rate and hazard rate:

$$\hat{D}(0,T) = \mathbb{E}\left(\exp\left(-\int_0^\tau r(s)ds\right)\pi(\tau)\,\mathbf{1}_{\tau\leq T}\right) = \mathbb{E}(\pi)\,D(0,T)$$

Remarks on reduced form models

- Captures risk of default as an event at an unknown time – the first jump of a Poisson process.
- Captures spread risk, i.e., the risk that credit spreads change even though default does not occur – the first jump of a Cox process.
- Captures the risk of an uncertain recovery payment at default – through one of our building blocks, the random quantity at default.
- Captures the co-movements in credit spreads and interst rates – through correlated stochastic processes.
- Most of the single-name credit derivatives we encounter today are tackled with the Hazard rate model with deterministic but time-dependent interest rates and credit intensity.

Recovery Models

Recovery modelling

- The modelling framework we have built so far allows us to plug in a recovery model.
- Recovering claims from a defaulted company is highly complex and we do not intend to model this process. Instead, we model the value of the default settlement.
- There are a variety of ways to model recovery rate. All of them are based on unrealistic (to different extents) assumptions.

Zero recovery

Assumption

All claims have a zero recovery at default.

- This is the simplest and most unrealistic specification.
- Pricing with the Cox process becomes less involved.
- It is an important benchmark which appears as building blocks.

Recovery of treasury

Assumption

At default, the recovery of the claim is a constant fraction, c, of the value of a risk-free asset which is otherwise equivalent to the claim.

- The equivalent risk-free asset does not need to be actively traded as long as it could be priced.
- Pricing becomes an interpolation exercise between the price of the risk-free asset and the price of the claim with zero recovery.
- A junk bond may recover more than its par value.

Example

A 10-year corporate bond paying 10% annual coupons is priced at par when the risk-free interest rate is zero. The price of the equivalent risk-free bond is $200. With any constant fraction greater than 50%, the recovery value is greater than par. What happens if the bond is of an even longer maturity?

Recovery of market value

Assumption

At default, the recovery of the claim is a constant fraction, $1 - q$, of its pre-default value.

- Similar to the recovery rule of portfolio of OTCDs.
- λq may be modelled as one quantity, without exact knowledge of either.
- The eventual recovery value depends on the path to default.

Example

An investment grade bond trading close to par with $q = 50\%$.

- A sudden default implies a recovery value of around \$50.
- A prolonged default involving the bond price dropping to \$60 before the default implies a recovery value of around \$30.

Recovery of par value

Assumption

At default, the recovery of a bond is a constant fraction, R, of its face value.

- The face value of a bond is always known.
- The recovery value could be easily priced as a fixed payment at default, one of our building blocks.
- The eventual recovery value does not depend on the path to default.
- It is the most popular assumption for pricing simple credit derivatives, e.g., CDS.

Stochastic recovery

A motivating example

A pool of insurance against 125 issuers, all with recovery rate of 40% (of par). The insurance is sliced into 7 "tranches", with the first 3% of the credit loss hitting (and eventually wiping out) the first tranche, and so on and so forth according to the structure below.

Questions

- Is the "super senior" tranche of $[60\%, 100\%]$ risk-free?
- Should it really be risk-free?
- How should we model it so that it's not risk-free?

Deterministic Hazard Rates

A deterministic model (I)

In practice, for products that have very low or no dependency on the volatility, a deterministic model is adequate.

Risk-free zero-coupon bond

$$Z(0,T) = \exp\left(-\int_0^T r(s)ds\right)$$

Zero recovery risky zero coupon bond

$$\hat{Z}(0,T) = \exp\left(-\int_0^T (r(s) + \lambda(s))\, ds\right) = Z(0,T)Q(0,T)$$

Results under a deterministic model (II)

Survival probability

$$Q(0,T) = \exp\left(-\int_0^T \lambda(s)ds\right)$$

$$dQ(0,t) = -\lambda(t)\exp\left(-\int_0^t \lambda(s)ds\right)dt = -\lambda(t)Q(0,t)dt$$

Fixed payment at default

$$D(0,T) = \int_0^T \lambda(t)\exp\left(-\int_0^t (r(s)+\lambda(s))\,ds\right)dt = \int_0^T \hat{Z}(0,t)\lambda(t)dt$$
$$= \int_0^T Z(0,t)\lambda(t)Q(0,t)dt = -\int_0^T Z(0,t)dQ(0,t)$$

A concrete example (I)

The contract – a stylised CDS

- The payoff of the contract is linked to the default of an issuer.
- It pays $(1 - R)$ at the time of default if the default occurs before expiry T.
- It is funded by an insurance premium S which accrues and is paid continuously until expiry or default, whichever occurs first.

Dissecting the premium leg

- The premium S is paid continuously, i.e., Sdt is accrued between t and $t + dt$ if the issuer is not in default at $t + dt$.
- This resembles a zero recovery risky zero coupon bond with a face value of Sdt, whose price is $Z(0,t)Q(0,t)Sdt$.
- Integrating over the lifetime of the contract gives

  Premium Leg PV$(0,T) = S \int_0^T Z(0,t)Q(0,t)dt$.

A concrete example (II)

Dissecting the protection leg

- The payout of $(1-R)$ at default is a fixed payment at default.
- Its value is $(1-R)\int_0^T Z(0,t)\lambda(t)Q(0,t)dt$.
- Assuming $\lambda(t)=\lambda$ gives

  Protection Leg PV$(0,T) = \lambda(1-R)\int_0^T Z(0,t)Q(0,t)dt$.

Putting it all together

A fair contract should have a zero PV at inception:

$$S\int_0^T Z(0,t)Q(0,t)dt = \lambda(1-R)\int_0^T Z(0,t)Q(0,t)dt$$

$\Longrightarrow\ S=\lambda(1-R)$, which is the credit triangle.

Further Reading

1. Dominic O'Kane. *Modelling Single-name and Multi-name Credit Derivatives*. Wiley, 2008.

   Chapter 3 on Single-name credit modelling.

2. John C. Hull. *Options, Futures, and Other Derivatives*. Pretice-Hall.

   The chapter on Credit Risk in any edition.

3. Philipp J. Schöngucher. *Credit Derivatives Pricing Models*. Wiley, 2003.

   Chapters 3-7.