Modelling of Multi-Name Credit Products

QF622 Credit Risk Models

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Recap and Review

Bond and FRN

Bond

1. Pricing with yield to maturity, with risk-free rate and credit spread separated.
2. Equally sensitive to interest rate and credit spread.
3. Inverse relationship between yield and price, with positive convexity.

FRN

1. Par floater spread as an indicator of credit worthiness.
2. More of a pure credit instrument, with a much higher sensitivity to par floater spread than to interest rate.

Asset swap

1. Long a fixed coupon bond and long a payer IRS.
2. Priced with asset swap spread to par at trade inception.
3. Asset swap spread is an indicator of credit worthiness.
4. A synthetic position in an FRN of the same issuer in the absence of default.
5. At the time of default, the IRS remains in place.

Credit default swap

1. Protection buyer (seller) shorts (longs) credit risk.
2. Premium leg is strip of zero recovery risky zero coupon bonds, if accrued interests at default are not considered.
3. Protection leg is a fixed payment at default, assuming a fixed recovery at par.
4. Pricing with prevailing CDS spread and risky annuity.
5. Standardisation of CDS contracts – roll dates, running spreads and upfront payments.
6. Calibration instruments for hazard rate curves.

The Gaussian Latent Variable Model

The single-name case – specification

The latent variable

- A random variable A_i for credit name i, which is drawn from a standard normal distribution.
- Similar to the Merton model, we assume default occurs before time T if A_i is less than a time dependent threshold $C_i(T)$.

$$Pr(\tau_i \leq T) = Pr(A_i \leq C_i(T)) = \Phi(C_i(T))$$

- The model is specified by the term structure of $C_i(T)$, which is calibrated to issuer i's survival curve by

$$\Phi(C_i(T)) = 1 - Q_i(T) \quad \text{or} \quad C_i(T) = \Phi^{-1}(1 - Q_i(T)).$$

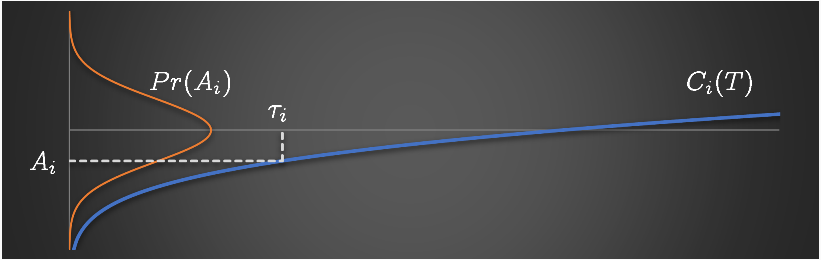
The single-name case – calibration

	$\lambda_i = 1\%$		$\lambda_i = 5\%$		$\lambda_i = 8\%$	
T	$Q_i(T)$	$C_i(T)$	$Q_i(T)$	$C_i(T)$	$Q_i(T)$	$C_i(T)$
0.1	99.90%	−3.0904	99.50%	−2.5767	99.20%	−2.4104
1	99.00%	−2.3282	95.12%	−1.6569	92.31%	−1.4264
5	95.12%	−1.6569	77.88%	−0.7681	67.03%	−0.4408
10	90.48%	−1.3096	60.65%	−0.2703	44.93%	0.1274
100	36.79%	0.3375	0.67%	2.4709	0.03%	3.4012

- When $Q_i(T) \to 1$, $C_i(T)$ is negative and high in magnitude.
- When $Q_i(T) \to 0$, $C_i(T)$ is positive and high in magnitude.
- The poorer the credit is, the higher $C_i(T)$ is, for a given T.
- The longer T is, the higher $C_i(T)$ is, for a given credit i.

lower hazard rate $\lambda_i \Rightarrow$ higher survival prob $Q_i(T)$

The single-name case – from latent variable to default time



- A_i has no dynamics through time. It has no financial meaning and cannot be observed. It is therefore latent.
- Knowledge of A_i is sufficient to pin down the default time of credit i.

$$Q_i(\tau_i) = 1 - \Phi(C_i(\tau)) = 1 - \Phi(A_i) \quad \text{or} \quad \tau_i = Q_i^{-1}(1 - \Phi(A_i))$$

The single-name case – simulating default time

$1 - \Phi(A_i)$ is uniformly distributed, we have for $u \in (0, 1)$,

$$Pr(1 - \Phi(A_i) \leq u) = Pr(\Phi(A_i) \geq 1 - u) = Pr(A_i \geq \Phi^{-1}(1 - u))$$
$$= 1 - \Phi(\Phi^{-1}(1 - u)) = u$$

We could simulate default time with a Monte Carlo simulation. For each path p of the total P paths: draw a uniform random number u^p and solve for $\tau_i^p = Q_i^{-1}(u)$. With all P paths, the expected default time maybe calculated as the average across all τ_i^p.

Simulated expected default time with different flat hazard rates.

λ_i	1%	5%	8%
$\mathbb{E}(\tau_i)$	99.15	19.82	12.55

Do the results make sense?

⇒ lower hazard rate ⇒ lower time to default

Extension to multi-name cases - the market factor

- To extend the model to a multi-name setting, we introduce an A_i for each of the $i = 1, \cdots, N_C$ credits.
- Defaults dependency is achieved by correlating A_i and A_j.
- This could be done through a single factor model: (instead of looking at all the pairwise variables)

as $\beta_i \to 1$, A_i becomes solely determined by the market factor

a common **market factor**, Gaussian distributed independently from Z_i

$$A_i = \beta_i Z + \sqrt{1 - \beta_i^2} Z_i$$

Gaussian distributed, as in the single-name case

the **idiosyncratic factor** specific to credit i, Gaussian distributed independently from Z

The correlation structure

Given $A_i = -A_j$

$\rho_{ij}(T) = P(A_i \le C_i(T) \cap A_j \le C_j(T))$

$= P(A_i \le C_i(T) \cap -A_i \le C_j(T))$

$= P(A_i \le C_i(T) \cap A_i \ge -C_j(T))$

$= P(-C_j(T) \le A_i \le C_i(T))$

$= \Phi(C_i(T)) - \Phi(C_j(T))$

Correlation between latent variables

The correlation between A_i and A_j is given by

$$\rho_{i,j} = \frac{\mathbb{E}(A_i A_j) - \mathbb{E}(A_i)\mathbb{E}(A_j)}{\sqrt{\mathbb{V}(A_i)}\sqrt{\mathbb{V}(A_j)}} = \beta_i \beta_j .$$

Joint default as a bivariate Guassian distribution

For credit i, its default is modelled by a Gaussian variate A_i: default happens before T is $A_i \le C_i(T)$. Likewise for credit j. Therefore, the joint probability of default before T is given by

$$p_{i,j}(T) = \Phi_2(C_i(T), C_j(T), \rho_{i,j}) .$$

The correlation structure – limiting cases

Minimum dependence

In this case, $\beta_i = -\beta_j = 1$ and $\rho_{i,j} = -1$. Clearly, $A_i = -A_j$ and

$$p_{i,j}(T) = \max\left(1 - Q_i(T) - Q_j(T), 0\right).$$

Independence

In this case, $\beta_i = \beta_j = 0$ and $\rho_{i,j} = 0$. Clearly, A_i and A_j are independent and

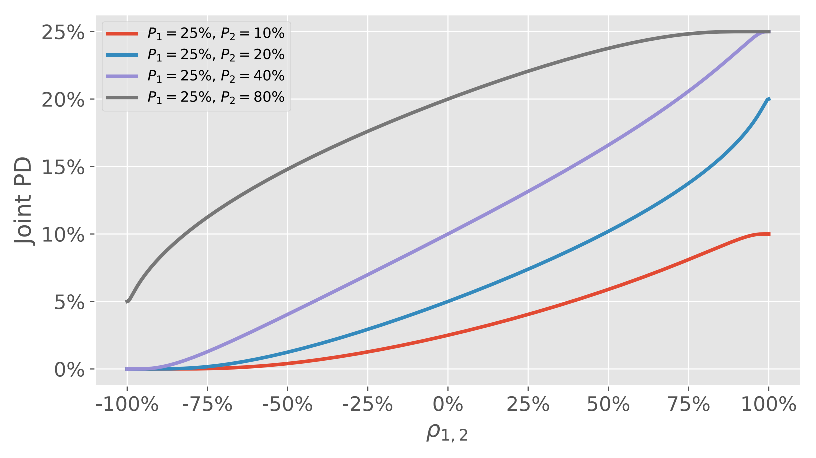
$$p_{i,j}(T) = \left(1 - Q_i(T)\right)\left(1 - Q_j(T)\right).$$

Maximum dependence

In this case, $\beta_i = \beta_j = 1$ and $\rho_{i,j} = 1$. Clearly, $A_i = A_j$ and

$$p_{i,j}(T) = \min\left(1 - Q_i(T), 1 - Q_j(T)\right).$$

The correlation structure – real examples



⇒ higher corr ⇒ higher default prob

- As $\rho_{i,j} \to -1$, $p_{i,j}(T) \to \max(P_1 + P_2 - 1, 0)$.
- As $\rho_{i,j} \to 1$, $p_{i,j}(T) \to \min(P_1, P_2)$.
- When $\rho_{i,j} = 0$, $p_{i,j}(T) = P_1 P_2$.

The conditional hazard rates (I)

Credit i defaults before T if $A_i = \beta_i Z + \sqrt{1-\beta_i^2} Z_i \le C_i(T)$, which could be written as

$$Z_i \le \frac{C_i(T) - \beta_i Z}{\sqrt{1-\beta_i^2}}.$$

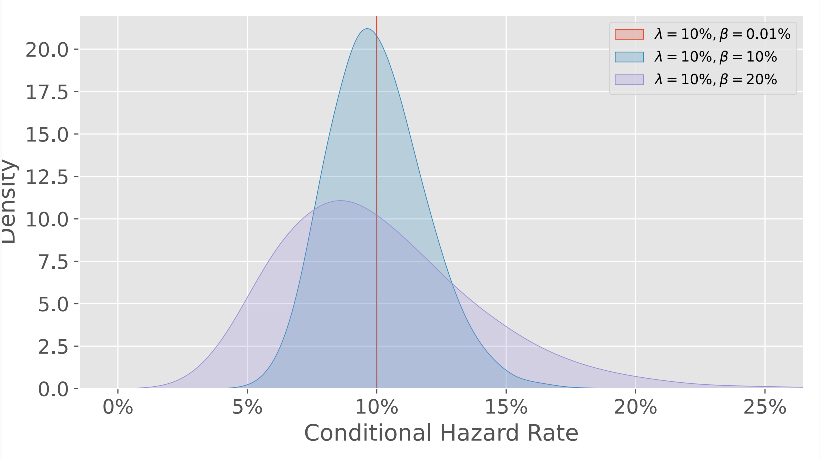
It follows that the default probability of credit i for time horizon T conditional on the realisation of market factor Z is

$$p_i(T|Z) = 1 - Q_i(T|Z) = \Phi\left(\frac{C_i(T) - \beta_i Z}{\sqrt{1-\beta_i^2}}\right).$$

With a flat conditional hazard rate or $Q_i(T|Z) = \exp\left(-\lambda_i(T|Z)T\right)$, we have

$$\lambda_i(T|Z) = -\frac{1}{T} \ln \Phi\left(\frac{\beta_i Z - C_i(T)}{\sqrt{1-\beta_i^2}}\right).$$

The conditional hazard rates (II)



- The higher β_i, the more spread out the conditional hazard rate.
- As $\beta_i \to 0$, the conditional hazard rate becomes deterministic and converges to the unconditional hazard rate.

Conditional loss distribution (I)

- For the pricing of certain correlation products, it is crucial to model the portfolio loss distribution.
- With a one factor model, credits are conditionally independent.
- The portfolio loss distribution is obtained by integrating the conditional portfolio loss distribution over the market factor:

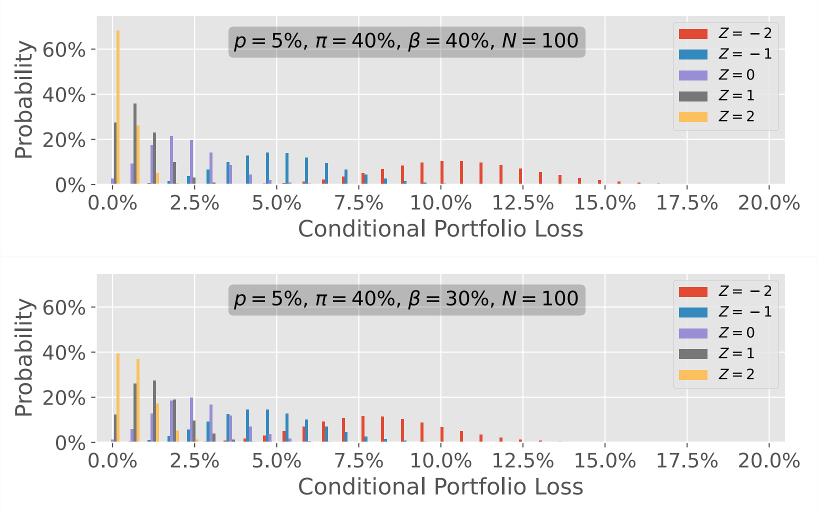
$$f(L(T)) = \int_{-\infty}^{+\infty} f(L(T)|Z)\phi(Z)dZ.$$

- For a homogenous pool of N credits with the same survival curve and recovery rate, we have

$$f(L(T)|Z) = Pr\left(L(T) = \frac{n(1-\pi)}{N} \middle| Z \right) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n},$$

where $p = p(T|Z) = \Phi\left(\frac{C(T) - \beta Z}{\sqrt{1-\beta^2}} \right)$.

Conditional loss distribution (II)



Loss distribution of a large homogenous pool (I)

- For a homogenous pool of credits, the conditional loss distribution is a binomial distribution with conditional default probability p.
- By the law of large numbers, the conditional loss distribution converges to a unit point mass of probability at the conditional expected portfolio loss of

$$(1-\pi)p.$$

- By integrating the conditional portfolio loss over the market factor, we could derive the unconditional loss distribution function.

Loss distribution of a large homogenous pool (II)

Denoting the cumulative loss distribution function as

$$F(K) = Pr(L(T) \leq K),$$

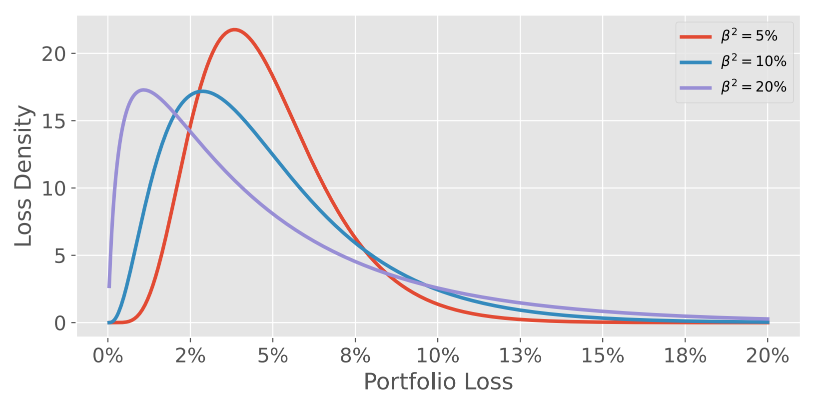
we have

$$\begin{aligned} F(K) &= Pr\left((1-\pi)\Phi\left(\frac{C(T)-\beta Z}{\sqrt{1-\beta^2}}\right) \leq K\right) \\ &= Pr\left(Z \geq \frac{1}{\beta}\left(C(T) - \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{K}{1-\pi}\right)\right)\right) \\ &= 1 - \Phi\left(\frac{1}{\beta}\left(C(T) - \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{K}{1-\pi}\right)\right)\right) = 1 - \Phi(A(K)). \end{aligned}$$

The probability density function is given by

$$f(K) = \frac{\partial F(K)}{\partial K} = \frac{\phi(A(K))}{1-\pi}\frac{\sqrt{1-\beta^2}}{\beta}\left(\phi\left(\Phi^{-1}\left(\frac{K}{1-\pi}\right)\right)\right)^{-1}.$$

Loss distribution of a large homogenous pool (II)



- Loss distribution at 5Y generated with credit spread of 100bps and 40% recovery rate.
- Higher correlation $\Rightarrow$ more skewed to the left and more likely to attain a higher loss.

Simulating multi-name defaults

Monte Carlo simulation

1. Calculate $C_i(T) = \Phi^{-1}(1 - Q_i(T))$ for all $i = 1, \cdots, N$ credits.
2. Generate $p = 1, \cdots, P$ independent Gaussian random numbers Z^p, and PN independent Gaussian random numbers Z_i^p.
3. Calculate PN values of $A_i^p = \beta_i Z^p + \sqrt{1 - \beta_i^2} Z_i^p$.
4. Calculate PN values of $u_i^p = 1 - \Phi\left(A_i^p\right)$.
5. For each credit and path, compute $\tau_i^p = Q_i^{-1}\left(u_i^p\right)$.

Having the knowledge of path-wise τ_i^p, loss distribution may be built easily.

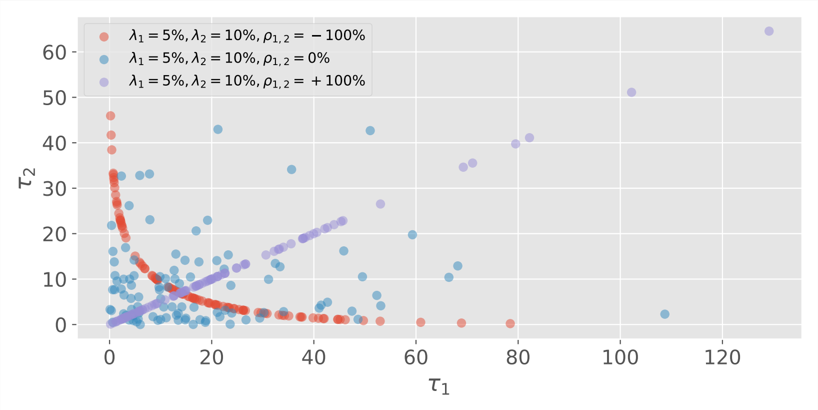
Default clustering (I)

- Defaults are rare events. But they usually happen together as a cluster.
- We could explore how well the model captures such clustering of defaults.
- Assuming flat hazard rate λ_1 and λ_2 for credits 1 and 2, respectively, we have

$$\tau_1 = -\frac{\ln \Phi(-A_1)}{\lambda_1} \quad \text{and} \quad \tau_2 = -\frac{\ln \Phi(-A_2)}{\lambda_2}$$

where A_1 and A_2 are two Gaussian random variates with correlation ρ.

Default clustering (II)



With the maximum level of dependence with $\rho = 100\%$, we have $A_1 = A_2$, and therefore

$$\frac{\tau_1}{\tau_2} = \frac{\lambda_2}{\lambda_1}.$$

Calibrating the correlations (I)

The problem

The latent variables are not observable, let alone their correlations.

Revisting Merton's model

- Equity price could be seen as a call option on the issuer i's asset value. Hence,
$$\frac{dE_i}{dV_i} = \Delta_i.$$
- If we assume Δ_i is constant over a small amount of time over which we are estimating the correlation, we have
$$\rho\left(\frac{dE_i}{E_i}, \frac{dE_j}{E_j}\right) \approx \rho\left(\frac{dV_i}{V_i}, \frac{dV_j}{V_j}\right).$$
- A latent variable is Gaussian distributed with zero mean and may be interpreted as an asset return, i.e., similar to $\frac{dV_i}{V_i}$.

Calibrating the correlations (II)

Revisting Merton's model (cont'd)

- Correlations among Gaussian latent variables of issuers may be estimated from correlations among their equity returns.
- Say we have the correlation matrix $\mathbf{C}$ of equity returns for N issuers, we need to

$$\min \sum_{i=1}^{N} \sum_{j=1}^{i-1} (C_{i,j} - \beta_i \beta_j)$$
$$s.t. \quad \beta_i^2 < 1$$

Calibration to market instruments

When prices of correlation products are observable on the market, the correlation may be directly calibrated.

Modelling Default Times Using Copulas

What's a copula?

- For a random vector $(X_1, \dots, X_N)$ whose marginal CDFs $F_i(x) = \Pr(X_i \leq x)$ are continuous. We know that

$$(U_1, \dots, U_N) = (F_1(X_1), \dots, F_d(X_N))$$

  has uniformly distributed marginals on the interval $[0, 1]$.
- A joint CDF of $(U_1, \dots, U_N)$

$$C(u_1, \dots, u_N) = \Pr(U_1 \leq u_1, \dots, U_d \leq u_N])$$

  is a copula of $(X_1, \dots, X_d)$.
- While C determines the dependence structure among $(X_1, \dots, X_N)$, the marginal distribution of X_i is specified by F_i.

Sklar's Theorem

- Any multi-variate distribution can be written as a unique copula if the marginal distributions are continuous.
- Loosely speaking, for any multi-variate distribution, there is an equivalent copula function.
- The advantage of working with copula is that it separates the choice of the marginal distribution from the choice of the dependence structure.

A more intuitive interpretation

Marginal distributions of the default times τ_i and τ_j of issuers i and j are known from their survival curves calibrated to the CDS market.

1. We know how to correlate two Gaussian variates, whose marginal CDFs are Φ. Each could be mapped to and from the interval $[0, 1]$, using Φ and Φ^{-1}.
2. We know that τ_i and τ_j could also be mapped to the interval $[0, 1]$, using the their respective CDF and inverse CDF.
3. Therefore, the correlation structure embedded in the bi-variate Gaussian distribution could be imposed on the default time marginals through the interval $[0, 1]$.
4. The use of Gaussian variates means that this is a Gaussian copula of default times, while other copulas are possible.

The latent variable model as a Guassian copula

The multi-variate distribution of default time is

$$\begin{aligned} &\Pr(\tau_1 \le t_1, \cdots, \tau_N \le t_N) \\ =&\Pr\left(A_1 \le C_1(t_1), \cdots, A_N \le C_N(t_N)\right) \\ =&\Phi_N\left(\Phi^{-1}\left(1 - Q_1(t_1)\right), \cdots, \Phi^{-1}\left(1 - Q_N(t_N)\right), \rho\right) \\ =&C\left(u_1(t_1), \cdots, u_N(t_N)\right) \end{aligned}$$

where $\rho_{i,j} = \beta_i\beta_j$ for any $i \ne j$.

- The copula function is a multi-variate Gaussian cumulative distribution function.
- The marginals are Gaussian distributions.
- In this case, the correlation matrix is constrained to a one-factor setup, which could easily be generalised.

Limitation of a one-factor correlation structure (I)

One-factor correlation structure

Under the one-factor model, all credits are driven, although to different extents parametrised by β_i, by a single common factor Z.

- For N credits, the correlation structure is parametrised by N parameters.
- For a full correlation matrix, there are $\sum_1^{N-1} i = \frac{N(N-1)}{2}$ degrees of freedom subject to the positive-definite condition.
- What if we need to model cross-sector credits?

Limitation of a one-factor correlation structure (II)

Example

We have four credits, two in sector A and two in sector B. For simplicity, let's assume that $\beta_1 = \beta_2 = \beta_A$ and $\beta_3 = \beta_4 = \beta_B$.

$$\rho = \left(\begin{array}{cc|cc} 1 & \beta_A^2 & \beta_A\beta_B & \beta_A\beta_B \\ \beta_A^2 & 1 & \beta_A\beta_B & \beta_A\beta_B \\ \hline \beta_A\beta_B & \beta_A\beta_B & 1 & \beta_B^2 \\ \beta_A\beta_B & \beta_A\beta_B & \beta_B^2 & 1 \end{array}\right)$$

We further look at the case of $\beta_A \geq \beta_B > 0$, which implies that $\beta_A^2 \geq \beta_A\beta_B \geq \beta_B^2$.

Inter-sector correlation $\beta_A\beta_B$ is even higher than the intra-sector correlation β_B^2!

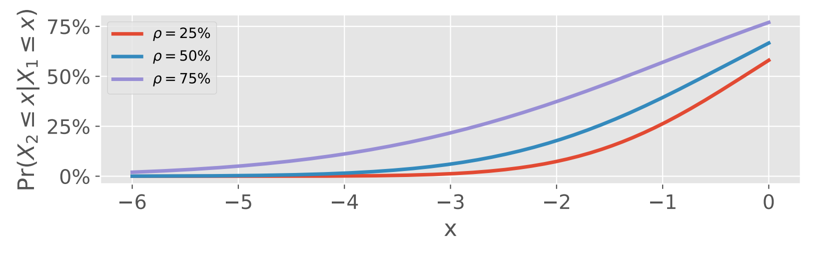
Simulating correlated default times using a Gaussian copula

1. Generate P sets of correlated Gaussian random variables Z_i^p, where $i = 1, \cdots, N$ and $p = 1, \cdots, P$.
2. Map each correlated random Gaussian variable Z_i^p to a uniform random variable using the marginal $u_i^p = \Phi(Z_i^p)$.
3. Map each of the uniform random variables to a default time by solving $\tau_i^p = Q_i^{-1}(u_i^p)$.

Tail dependence

Bi-variate Gaussian distribution

The conditional probability $\Pr(X_2 \leq x | X_1 \leq x) \to 0$ as $x \to -\infty$, i.e., asymptotically independent. This is lower tail independence.



- We are often concerned about (joint) distribution in the tail.
- Lower and upper tail dependence may be measured.
- Gaussian copula is asymptotically tail independent.
- Other popular choices (e.g., student-t copula) may provide tail dependence.

GFC and the Gaussian copula model

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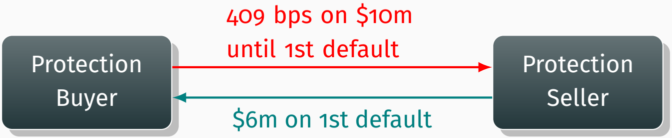
Default Baskets

nth-to-default baskets (I)

- A default basket contract is almost the same as a CDS.
- The only difference lies in the credit event that terminates the contract, defined as the nth default in a basket of N credits.
- The total number of credits are typically fewer than ten.
- The premium leg functions in exactly the same way as a CDS and is terminated when the nth default occurs.
- The protection leg pays $1 - \pi_{i,n}$, where i is the index of the credit the nth to default since the inception of the contract.

nth-to-default baskets (II)



Reference Basket
$10m credit A, 90 bps
$10m credit B, 120 bps
$10m credit C, 120 bps
$10m credit D, 120 bps
$10m credit E, 150 bps

n	spread
1	409 bps
2	147 bps
3	60 bps
4	22 bps
5	6 bps

Understanding default dependence

The two-credit case

For credits A and B with default times τ_A and τ_B, respectively:

$$P_i(T) = \mathbb{E}(\mathbf{1}_{\tau_i \leq T}) = 1 - Q_i(T), \quad \text{for } i \in \{A, B\}$$

all of which are extracted from single-name CDS markets. Let $P_{A,B}(T) = \mathbb{E}(\mathbf{1}_{\tau_A \leq T}\mathbf{1}_{\tau_B \leq T})$ be the probability of joint defaults and

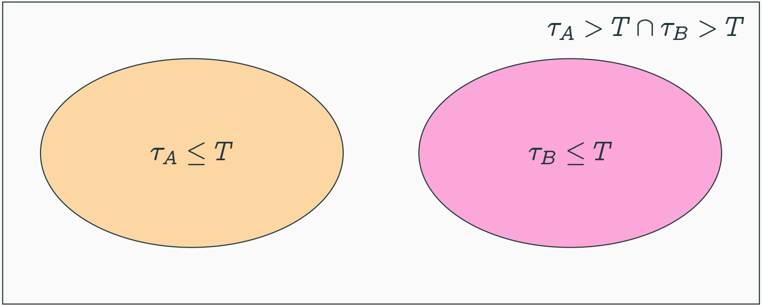
$$Q_{A,B}(T) = 1 - P_A(T) - P_B(T) + P_{A,B}(T).$$

Both FtD and StD triggering probabilities are a function of the joint probability of default:

$$P_{FtD}(T) = 1 - Q_{A,B}(T) = P_A(T) + P_B(T) - P_{A,B}(T)$$

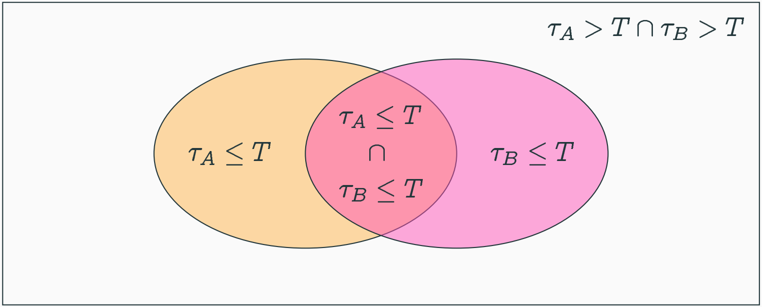
$$P_{StD}(T) = P_{A,B}(T)$$

Understanding default dependence – minimum dependence



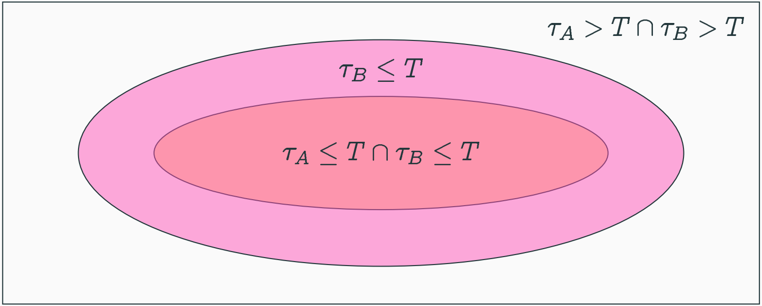
- $P_{A,B}(T) = \max\left(P_A(T) + P_B(T) - 1, 0\right)$.
- Defaults are A and B are typically mutually exclusively.
- With $P_{A,B}(T) = 0$, $P_{FtD}(T) = P_A(T) + P_B(T)$ and $P_{StD}(T) = 0$.
- Otherwise, $P_{FtD}(T) = 1$ and $P_{StD}(T) = P_A(T) + P_B(T) - 1$.

Understanding default dependence – independence



- Defaults are independent, i.e., $P_{A,B}(T) = P_A(T)P_B(T)$.
- $P_{FtD}(T) = P_A(T) + P_B(T) - P_A(T)P_B(T)$.
- $P_{StD}(T) = P_A(T)P_B(T)$.

Understanding default dependence – maximum dependence



- When the better quality credit defaults, the lower quality credit also defaults, i.e., $P_{A,B}(T) = \min\left(P_A(T), P_B(T)\right)$.
- $P_{FtD}(T) = \max\left(P_A(T), P_B(T)\right)$.
- $P_{StD}(T) = \min\left(P_A(T), P_B(T)\right)$.

Understanding default dependence – observations

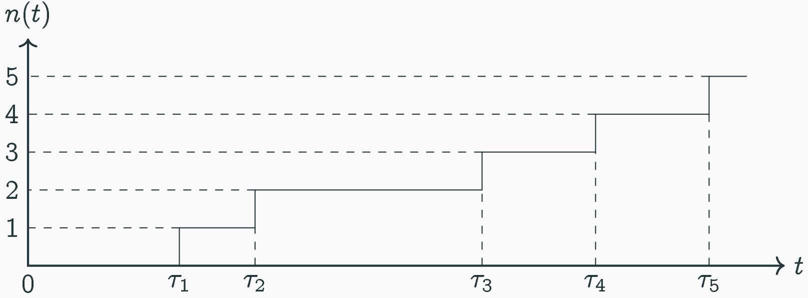
	FtD	StD
Minimum Dependence	$P_A(T) + P_B(T)$	0
Independence	$P_A(T) + P_B(T) - P_A(T)P_B(T)$	$P_A(T)P_B(T)$
Maximum Dependence	$\max\left(P_A(T), P_B(T)\right)$	$\min\left(P_A(T), P_B(T)\right)$

- $P_{FtD}(T)$ decreases with increasing default dependency.
- $P_{StD}(T)$ increases with increasing default dependency.

The basket survival curve

For a basket of homogenous credits (with the same recovery rate π), the pricing of nth-to-default basket boils down to finding the basket survival curve.

$$\Pr\left(n(t) \geq n\right) = 1 - Q\left(n(t) < n\right)$$



Basket spreads and dependency (I)

The two-credit case

For credits A and B, we have their single-name CDSes:

kth credit's risky annuity
$\hat{A}_k(T) = \sum \Delta_i Q_k(t_i) Z(t_i)$

$$(1-\pi)\int_0^T Z(s)dQ_k(s) + S_k(T)\,\hat{A}_k(T) = 0, \quad \text{for } k \in \{A, B\}.$$

Having built the basket survival curves $Q_n(t) = Q\left(n(t) < n\right)$ for $n \in \{1, 2\}$, the nth-to-default baskets:

$$(1-\pi)\int_0^T Z(s)dQ_n(s) + S_n(T)\,\hat{A}_n(T) = 0, \quad \text{for } n \in \{1, 2\}.$$

nth-to-default basket's risky annuity
$\hat{A}_n(T) = \sum \Delta_i Q_n(t_i) Z(t_i)$

Basket spreads and dependency (II)

Minimum dependence

Assuming $P_A(t) + P_B(t) < 1$ or $Q_A(t) + Q_B(t) > 1$ with mutually exclusive defaults:

$$Q_1(t) = Q_A(t) + Q_B(t) - 1 \quad \text{and} \quad Q_2(t) = 1$$
$$dQ_1(t) = dQ_A(t) + dQ_B(t) \quad \text{and} \quad dQ_2(t) = 0$$

Therefore,

risk-free annuity $A(T) = \sum \Delta_i Z(t_i)$

$$S_1(T)\left(\hat{A}_A(T) + \hat{A}_B(T) - \boxed{A(T)}\right) = S_A(T)\hat{A}_A(T) + S_B(T)\hat{A}_B(T)$$

It follows that $S_1(T) > S_A(T) + S_B(T)$. And it's straightforward to see that $S_2(T) = 0$.

Basket spreads and dependency (III)

Maximum dependence

Assuming $Q_A(t) \geq Q_B(t)$ or $Q_B(t) \geq Q_A(t)$ for $t \in [0, T]$,

$$Q_1(t) = \min\left(Q_A(t), Q_B(t)\right)$$
$$Q_2(t) = \max\left(Q_A(t), Q_B(t)\right)$$

Therefore,

$$S_1(T) = \max\left(S_A(T), S_B(T)\right)$$
$$S_2(T) = \min\left(S_A(T), S_B(T)\right)$$

Basket spreads and dependency (IV)

Independence

With independent default, we have

$$\begin{aligned} Q_1(t) &= Q_A(t)Q_B(t) \\ Q_2(t) &= Q_A(t) + Q_B(t) - Q_A(t)Q_B(t) \end{aligned}$$

No exact relationship could be derived. But results are in between the two boundary cases.

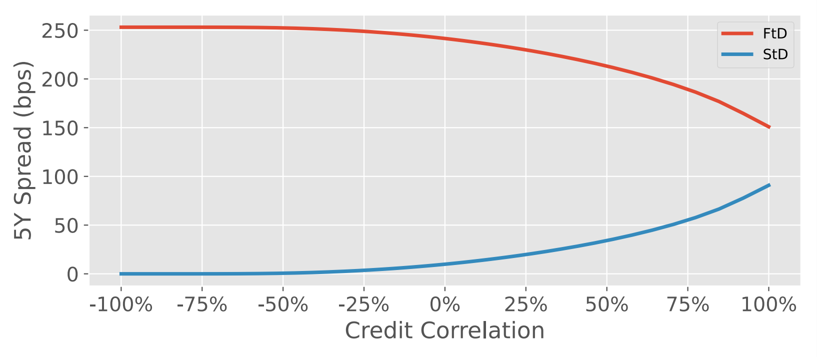
Observations

- FtD breakeven spread decreases with increasing correlation: long protection $\Rightarrow$ short correlation.
- StD breakeven spread increases with increasing correlation: long protection $\Rightarrow$ long correlation.

Basket spreads and dependency (V)

Example

- Generated with flat $S_A(T) = 150\ bps$ and $S_B(T) = 90\ bps$.
- With $\rho_{A,B} = -1$, $S_1(T) = 253\ bps$ and $S_2(T) = 0\ bp$.
- With $\rho_{A,B} = +1$, $S_1(T) = 150\ bps$ and $S_2(T) = 90\ bps$.
- With $\rho_{A,B} = 0$, $S_1(T) = 242\ bps$ and $S_2(T) = 10\ bps$.



Pricing Default Baskets

Pricing FtD homogenous baskets (I)

As long as we have the analytics to price single-name CDS, the task of pricing homogenous baskets boils down to finding the basket survival curve. In the case of FtD baskets, the survival curve is

$$Q(n(t)=0)=\int_{-\infty}^{+\infty}\phi(Z)Q(n(t)=0|Z)dZ$$

and

$$\begin{aligned}
Q(n(t)=0|Z) &= \Pr(\tau_1>t,\cdots,\tau_N>t|Z)\\
&= \prod_{i=1}^{N}\Pr(A_i>C_t(t)|Z)=\prod_{i=1}^{N}\Pr\left(Z_i>\frac{C_i(t)-\beta_i Z}{\sqrt{1-\beta_i^2}}\middle| Z\right)\\
&= \prod_{i=1}^{N}\Phi\left(\frac{\beta_i Z-C_i(t)}{\sqrt{1-\beta_i^2}}\right)
\end{aligned}$$

Pricing FtD homogenous baskets (II)

The survival probability of the FtD at time t is therefore

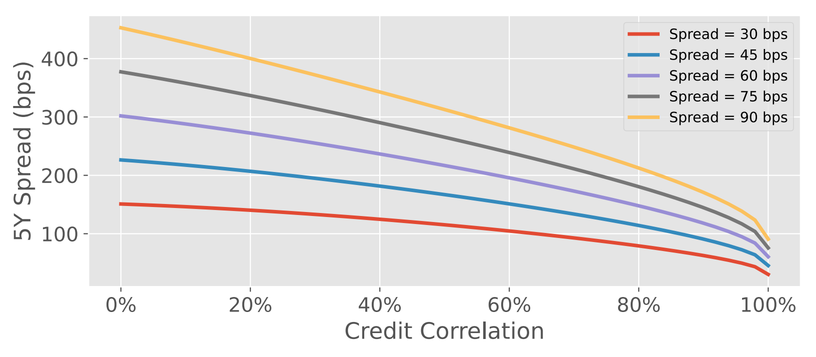
$$Q(n(t) = 0) = \int_{-\infty}^{+\infty} \phi(Z) \prod_{i=1}^{N} \Phi\left(\frac{\beta_i Z - C_i(t)}{\sqrt{1-\beta_i^2}}\right) dZ.$$

Numerical implementation

- This one-dimensional integral could be calculated numerically for each t_i on a time grid spanning $[0, T]$.
- The time discretisation needs to be chosen appropriately.
- Combined with an interpolation scheme, this forms a survival curve, which could project the survival probability at any t in the range of $[0, T]$.

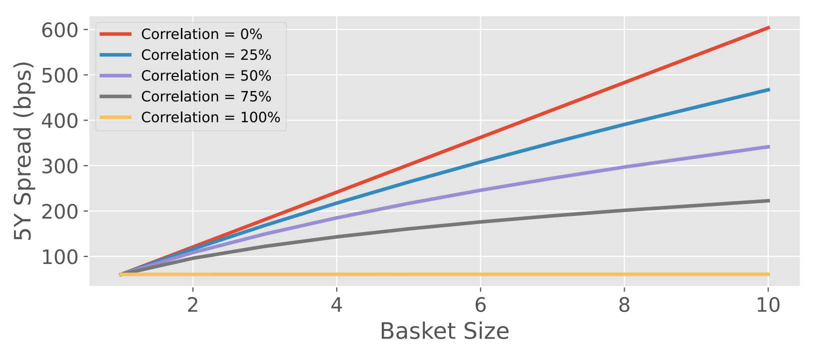
FtD homogenous baskets – correlation sensitivity

- FtD spread falls with increasing correlation. Long protection $\Rightarrow$ short correlation.
- As $\rho \to 0$, FtD spread tends to the sum of individual spreads.
- As $\rho \to 1$, FtD spread tends to the spread of the riskiest credit.



FtD homogenous baskets – size of basket

- Generated with credits with spreads of 60 bps.
- FtD spread increases with increasing basket size.
- The lower the correlation, the steeper the slope.
- Linear at zero correlation and flat at perfect correlation.



Pricing NtD homogenous baskets (I)

In the case of ntD baskets, the survival curve $Q(n(t) \leq n)$ is more cumbersome to build. We need to calculate the probability of $n-1$ credits defaulting, which could happen in multiple ways.

Default distribution

- Let's define default distribution $f(k(t))$ as the probability of having k defaults at time t.
- The general idea is then to build the conditional default distribution $f(k(t)|Z)$ by recursion.
- Starting with an empty basket, names are successively added, with $f(k(t)|Z)$ updated, until all credits are introduced.

Pricing NtD homogenous baskets (II)

Building conditional default distribution

1. Initialise $f(k(t)|Z)$ as

$$f^{(0)}(k(t)|Z) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \leq k \leq N \end{cases}$$

2. Loop over credits $j = 1, \cdots, N$:
   - Update $f^{(j)}(0|Z)$, which requires the survival of credit j:

$$f^{(j)}(0|Z) = f^{(j-1)}(0|Z) \overbrace{q_j(t|Z)}^{\Phi\left(\frac{\beta_i Z - C_i(t)}{\sqrt{1-\beta_i^2}}\right)}.$$

   - Update for $k = 1$ up to $k = j$:

$$f^{(j)}(k(t)|Z) = \underbrace{f^{(j-1)}(k(t)|Z) q_j(t|Z)}_{\substack{\text{with } k(t) \text{ defaults,} \\ \text{credit } j \text{ must survive}}} + \underbrace{f^{(j-1)}(k(t)-1|Z) p_j(t|Z)}_{\substack{\text{with } k(t)-1 \text{ defaults,} \\ \text{credit } j \text{ must default}}}.$$

Pricing NtD homogenous baskets (III)

Building basket survival curve

1. Unconditional default distribution is

$$f(k(t)) = \int_{-\infty}^{+\infty} \phi(Z) f(k(t)|Z) dZ.$$

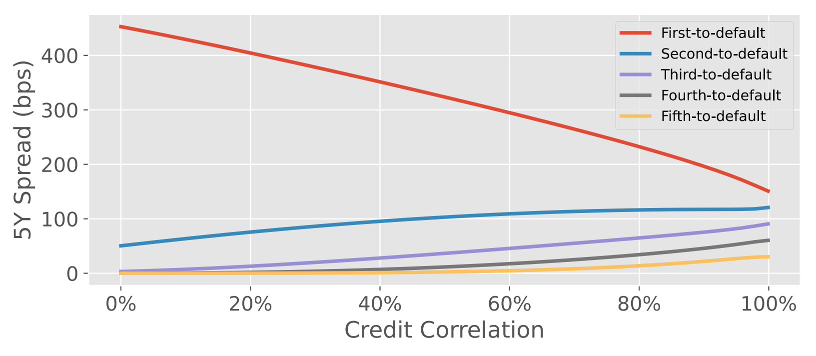
2. Basket survival probability at t is

$$Q(n(t) \leq n) = \sum_{k=0}^{n-1} f(k(t)).$$

3. Having $Q(n(t) \leq n)$ on a time grid spanning over $[0, T]$, a basket survival curve is built.

Pricing NtD homogenous baskets – in action

- Generated with credits with spreads of 30, 60, 90, 120, 150 bps.
- Long protection with an ntD ($n >= 2$) $\Rightarrow$ long correlation.
- As $\rho \to 1$, ntD spread tends to the spread of the nth riskiest credit.



Pricing inhomogeneous baskets

For an inhomogeneous baskets, where credits have different recovery rates:

- The pricing of the premium leg still relies solely on the basket survival curve.
- The pricing of the protection leg becomes more involved as we need to know the identity of the nth credit to default.
- For FtD contracts, analytical pricing is still possible. But when $n > 1$, it's simpler to use Monte Carlo simulations.

Monte Carlo simulations

Advantages

- A generic approach that could handle any basket payoff with homogenous or inhomogeneous credits.
- Very easy to implement and customise to various choices of model setup (number of factors and copula types).

Disadvantages

- Numerical convergence may be slow, results may be noisy, and Greeks may be unstable – variance reduction such as antithetic variables, importance sampling, etc.

Beyond the one-factor Gaussian copula model

Multi-factor Gaussian copula

The pricing framework could be extended to a multi-factor configuration. The essence is still the conditional independence.

For a payoff at time T of $\Theta(L(T))$, its price is given by

$$V = \int_{-\infty}^{+\infty} \left(\prod_{f=1}^{N_F} \phi(Z_f) \right) \Theta\left(L(T) | Z_1, \cdots, Z_{N_F}\right) dZ_1 \cdots dZ_{N_F}.$$

With a higher number of factors, it become prohibitive to perform the numerical integration while Monte Carlo simulations scale (roughly) linearly with the number of factors.

Such a configuration is meaningful when intra-sector correlation is very different from inter-sector correlation.

Student-t coplua

- Gaussian copula does not exhibit tail dependence although it's a feature of the financial market in reality.
- A popular choice is to use the student-t copula instead with a degree-of-freedom parameter controlling the tail dependency.
- Compared to the student-t copula, the Gaussian copula overprices FtD and underprices StD.
- Remember? Long protection with FtD is short correlation while long protection with StD is long correlation.
- Ultimately, market prices are determined by supply and demand not by a model.

Further Reading

1. Dominic O'Kane. *Modelling Single-name and Multi-name Credit Derivatives*. Wiley, 2008.

   Chapters 12-15.