QF623: Portfolio management I
Review of portfolio theory

The efficient frontier

- Let $x = (x_1, \dots . x_n)$ be the vector of weights in a portfolio
- Assume the long-only portfolio is fully invested: $\sum_{i=1}^{n} x_i = 1$
- Denote $R = (R_1, \dots . R_n)$ as the vector of asset returns where R_i is the return of asset *i*
- Taking expectations, we have $\mu = E(R)$as the vector of expected returns and $\Sigma = E[(R-\mu)(R-\mu)^T]$ as the covariance matrix of asset returns

The efficient frontier

- The portfolio expected return is:

$$\mu(x) = E[R(x)] = x^T E[R] = x^T \mu$$

- The portfolio variance is:

$$\begin{aligned}\sigma^2(x) &= E\left[\big(R(x) - \mu(x)\big)\big(R(x) - \mu(x)\big)^T\right] \\ &= E[(\, x^T R - x^T \mu)(\, x^T R - x^T \mu)^T] \\ &= \; x^T E[(R - \mu)(R - \mu)^T] x \\ &= \; x^T \Sigma x\end{aligned}$$

The efficient frontier

- The Markowitz problem (MPT or MVA) can be formulated as:
  - Maximizing the portfolio expected return under a volatility constraint (σ-problem)

$$\max\ \mu(x) \qquad \mathrm{u.c.} \qquad \sigma(x) \leq \sigma^*$$

  - Minimizing portfolio volatility under a return constraint (μ-problem)

$$\min \sigma(x) \qquad \mathrm{u.c.} \qquad \mu(x) \geq \mu^*$$

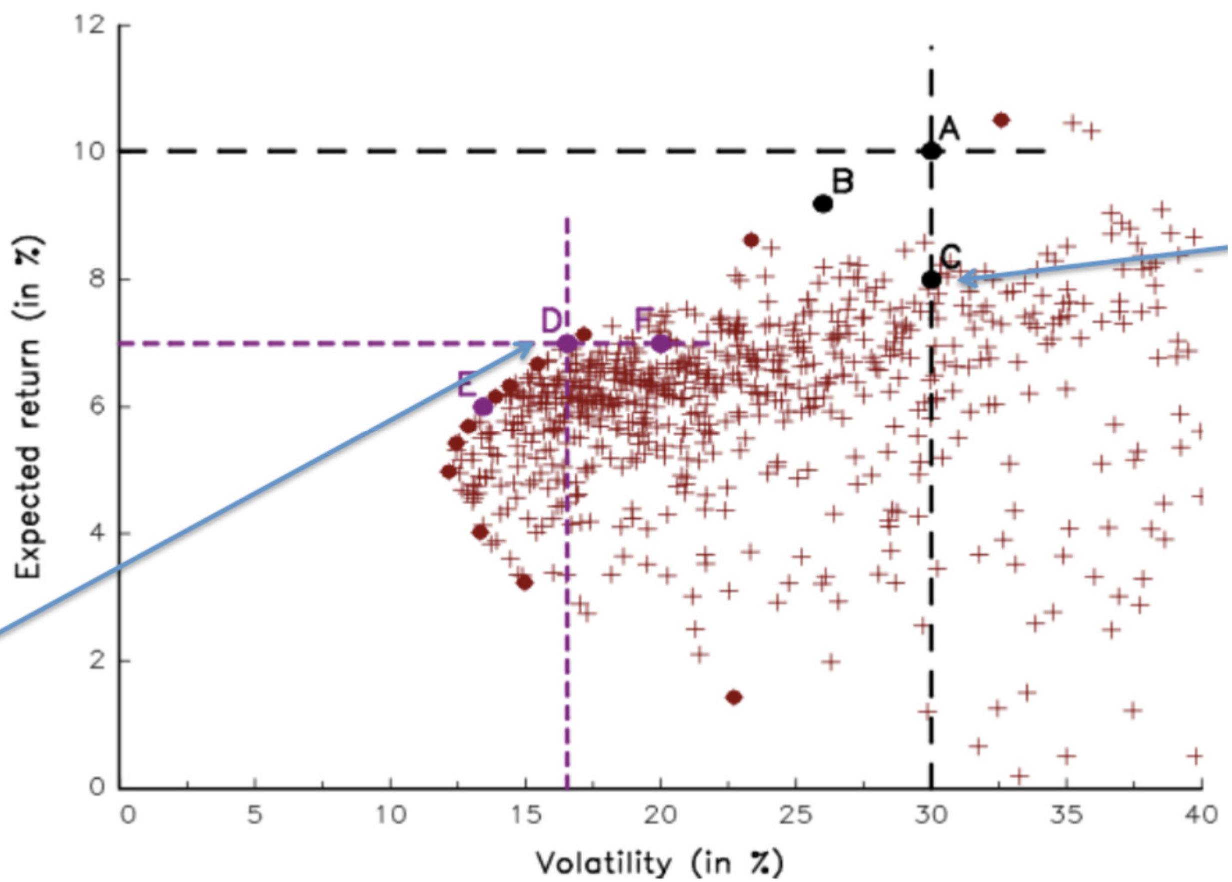
The efficient frontier - Example

- Consider four risky assets. Their expected returns are 5%, 6%, 8% and 6% respectively, while their volatilities are 15%, 20%, 25% and 30%. The corresponding correlation matrix is given by the following matrix:

$$\rho = \begin{pmatrix} 1 & & & \\ 0.1 & 1 & & \\ 0.4 & 0.7 & 1 & \\ 0.5 & 0.4 & 0.8 & 1 \end{pmatrix}$$

- We simulate 1000 portfolios, and solve 2 problems.
  - σ-problem: Maximize portfolio return conditional on 30% volatility
  - μ-problem: Minimize portfolio volatility conditional on 7% return

The efficient frontier – Example



For the μ-problem, portfolio D dominates

For the σ-problem, portfolio A dominates portfolio C, although both are feasible solutions

Generating the efficient frontier

- By considering all portfolios belonging to the simplex set defined by $\{x \in [0,1]^n \; \mathbf{1}^T x = 1\}$ we can compute the expected return and volatility bounds of the portfolio: $\mu^- \leq \mu(x) \leq \mu^+$ and $\sigma^- \leq \sigma(x) \leq \sigma^+$

- The σ-problem has a solution if $\sigma^* \geq \sigma^-$. The μ-problem has a solution if $\mu^* \geq \mu^+$.

Generating the efficient frontier

Optimized Markowitz portfolios

- The earlier investor problem essentially maximizes the Sharpe Ratio $(\mu/_{\sigma})$, which is a non-linear optimization problem.

- Markowitz proposed to transform the above problem to a quadratic optimization problem which is easier to solve

- σ-problem becomes:

$$x^*(\phi) = argmax\ x^T\mu - \frac{\phi}{2}x^T\Sigma x \quad u.c. \quad \mathbf{1}^T x = 1 \ \ for\ some\ \phi \geq 0$$

- μ-problem becomes:

$$x^*(\gamma) = argmin \frac{1}{2}x^T\Sigma x - \gamma x^T\mu \qquad u.c. \quad \mathbf{1}^T x = 1; \gamma = \phi^{-1}$$

Generating the efficient frontier

Optimized Markowitz portfolios

- σ-problem is equal to the μ-problem. We will look at the σ-problem where investor wants to maximize expected return for some risk aversion parameter, ϕ.

- If $\phi = 0$, the quadratic program completely emphasizes on return maximization and we have $\mu(x^*(\phi)) = \mu^+$.

- If $\phi = \infty$, the optimization problem becomes:

$$x^*(\infty) = argmin \frac{1}{2} x^T \Sigma x \qquad u.c. \quad \mathbf{1}^T x = 1$$

and we have $\sigma(x^*(\infty)) = \sigma^-$, the minimum variance portfolio

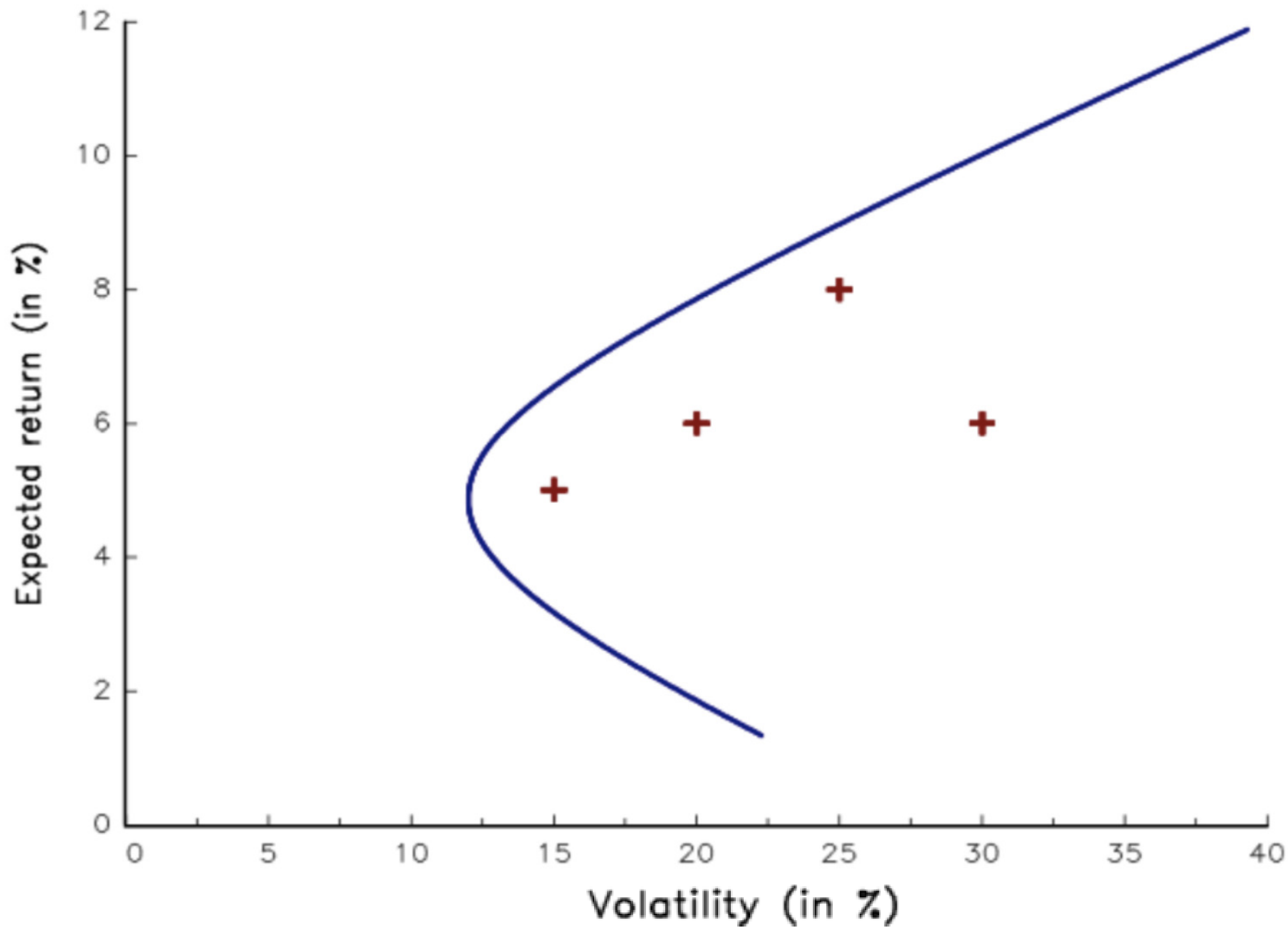
Solving the unconstrained ϕ-problem

Table: Solving the ϕ-problem

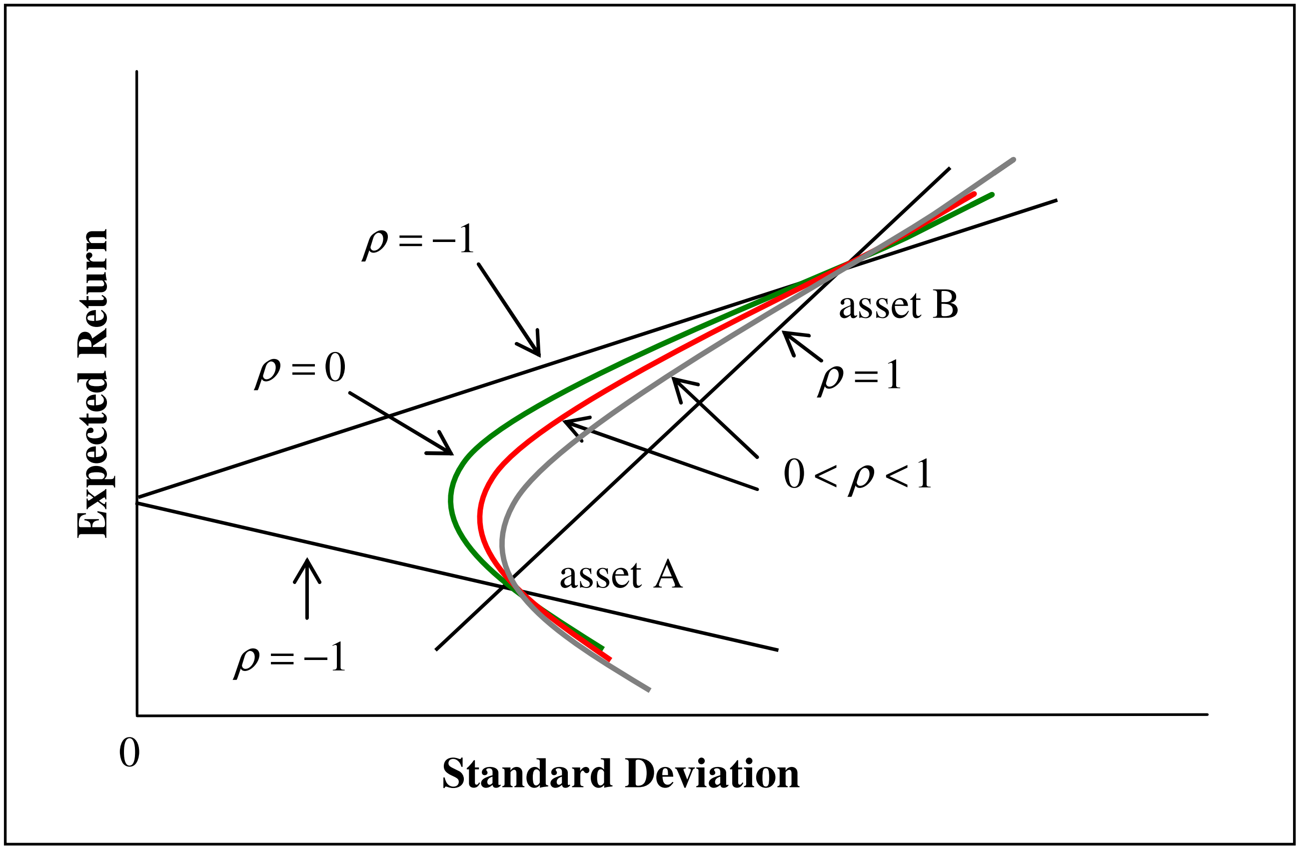
ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(x^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(x^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

Efficient frontier of Markowitz

By making ϕ increments sufficiently granular



Diversification benefits under different correlation structures



Analytical solution

- Suppose we require the portfolio weights to sum to 100%, i.e. $\mathbf{1}^T x = 1$

- The Lagrange function of the Markowitz optimization is:

$$\mathcal{L}(x; \lambda_0) = x^T \mu - \frac{\phi}{2} x^T \Sigma x + \lambda_0 (\mathbf{1}^T x - 1)$$

where λ_0 = Lagrange multiplier associated with the constraint

- We have the following first-order conditions:

$$\partial_x \mathcal{L}(x; \lambda_0) = \mu - \phi \Sigma x + \lambda_0 \mathbf{1} = 0$$
$$\partial_{\lambda_0} \mathcal{L}(x; \lambda_0) = \mathbf{1}^T x - 1 = 0$$

- We obtain $\mathrm{x} = \phi^{-1} \Sigma^{-1} (\mu + \lambda_0 \mathbf{1})$ from rearranging the first-order condition w.r.t. x

Analytical solution

- By substituition, we get $\mathbf{1}^T\phi^{-1}\Sigma^{-1}\mu + \lambda_0(\mathbf{1}^T\phi^{-1}\Sigma^{-1}\mathbf{1}) = 1$. It follows that

$$\lambda_0 = \frac{1 - \mathbf{1}^T\phi^{-1}\Sigma^{-1}\mu}{\mathbf{1}^T\phi^{-1}\Sigma^{-1}\mathbf{1}}$$

- The analytical solution is then

$$x^*(\phi) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} + \frac{1}{\phi}.\frac{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mu - (\mathbf{1}^T\Sigma^{-1}\mu)\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

- If $\phi \to \infty$, we have the minimum variance portfolio

$$x_{mv} = x^*(\infty) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

Two-fund theorem

- The entire efficient frontier can be generated from only two portfolios (funds) which lie on the efficient frontier.

- Allocation between the two portfolios is driven entirely by the investor's risk preferences

- Proof: Assume two portfolios P_1 and P_2 lie on the efficient frontier. Let ϕ_1 and ϕ_2 be the risk aversion parameters associated with P_1 and P_2 respectively. Consider another portfolio P_3 which is a weighted average of P_1 and P_2. Let α be the proportion of P_3 invested in P_1. We can then write:

Two-fund theorem

$$
\begin{aligned}
P_3 &= \alpha P_1 + (1-\alpha)P_2 \\
&= \alpha x^*(\phi_1) + (1-\alpha)x^*(\phi_2) \\
&= \alpha\left\{\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} + \frac{1}{\phi_1}.\frac{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mu - (\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right\} \\
&\quad + (1-\alpha)\left\{\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} + \frac{1}{\phi_2}.\frac{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mu - (\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right\} \\
&= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} + \frac{1}{\phi_3}.\frac{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mu - (\mathbf{1}^T\Sigma^{-1}\mathbf{1})\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \\
&= x^*(\phi_3)
\end{aligned}
$$

Where $\phi_3 = {}^{\phi_1\phi_2}/_{\alpha\phi_2+(1-\alpha)\phi_1} > 0$. Hence P_3 is also a solution to the Markowitz problem.

- The investor's choice of α reflects ϕ, his/ her risk preferences.

Adding practical constraints

- Constraints compresses the efficient frontier naturally.
  - Equality vs. inequality
  - Linear vs. non-linear

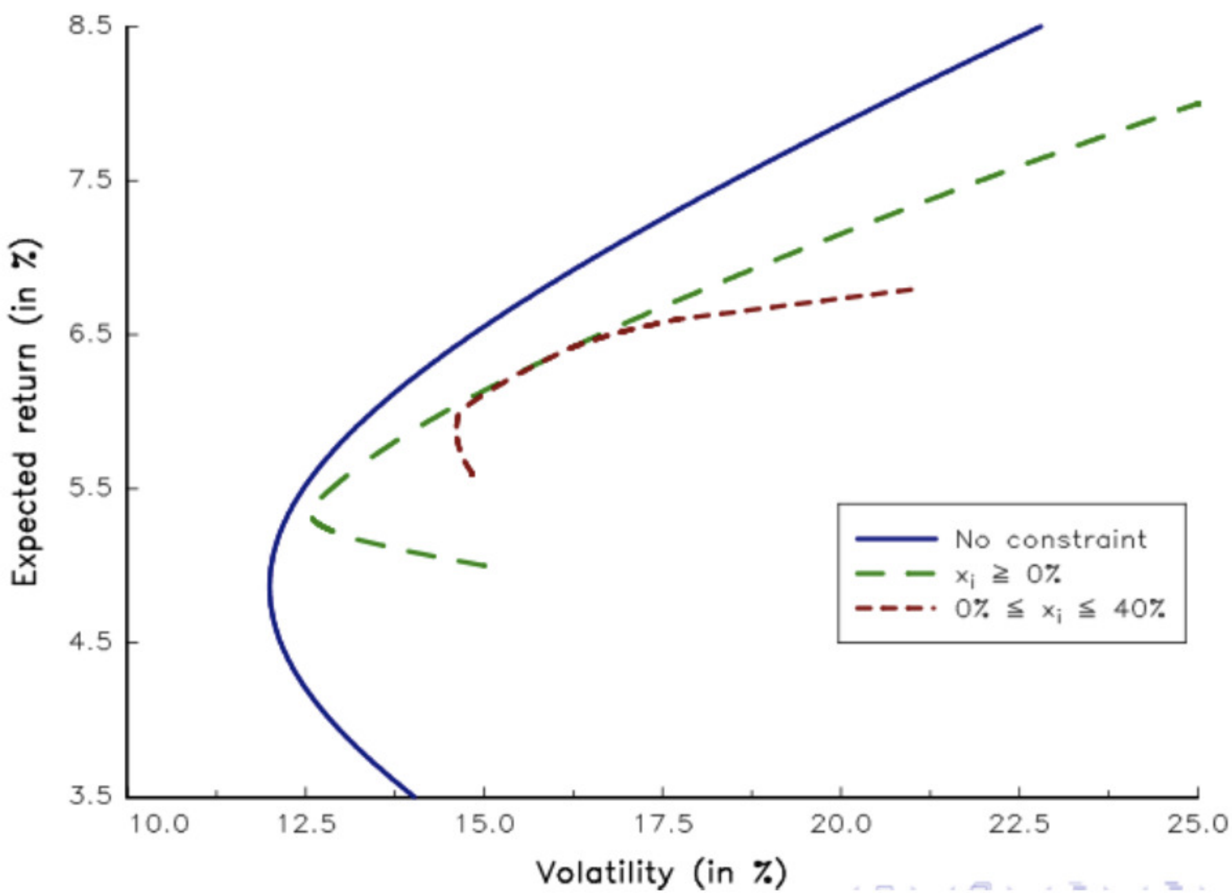
- No short sale restrictions $(\boldsymbol{x} \geq 0)$ – Often used by practitioners
  - Exchanges may prevent short sales in certain assets
  - Mandate restrictions (e.g. pension funds, retirement accounts)
  - Short selling often requires substantial credit qualifications

- With inequality constraints, the Lagrange multiplier method no longer works because it imposes an equality in the constraint

- The efficient frontier has to be computed by ‘brute force’ for each portfolio

The efficient frontier with constraints

Figure: The efficient frontier with some weight constraints



Solving the σ-problem with weight constraints

Table: Solving the σ-problem with weight constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	−10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	−37.01	−64.41	0.00	0.00	0.00	13.87
$\mu(x^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

The tangency portfolio

- Tobin (1958) showed that a single optimized portfolio dominates all others if there exists a risk-free asset.

- Consider a portfolio, y, which is a combination of a risk-free asset and a risky portfolio, x. Denote r_f as the return of the risk-free asset. The return of portfolio y is then

$$R(\boldsymbol{y}) = (1-\alpha)r_f + \alpha R(\boldsymbol{x}) \qquad y = \begin{bmatrix} 1-\alpha \\ \alpha \boldsymbol{x} \end{bmatrix}$$

where $\alpha \geq 0$ is the proportion of wealth invested in the risky portfolio and y is a $(n+1)$ vector of asset weights

The tangency portfolio

- Taking expectations, we obtain

$$\mu(y) = (1-\alpha)r + \alpha\mu(x) = r + \alpha(\mu(x) - r)$$
$$\sigma^2(y) = \alpha^2\sigma^2(x)$$

- By substitution, we get

$$\mu(y) = r + \frac{(\mu(x) - r)}{\sigma(x)}\sigma(y) \qquad (1)$$

- Define the Sharpe ratio of portfolio x as

$$SR(x|r) = \frac{\mu(x) - r}{\sigma(x)}$$

- We can re-write (1) as

$$\frac{\mu(y) - r}{\sigma(y)} = \frac{\mu(x) - r}{\sigma(x)} \Leftrightarrow SR(y|r) = SR(x|r)$$

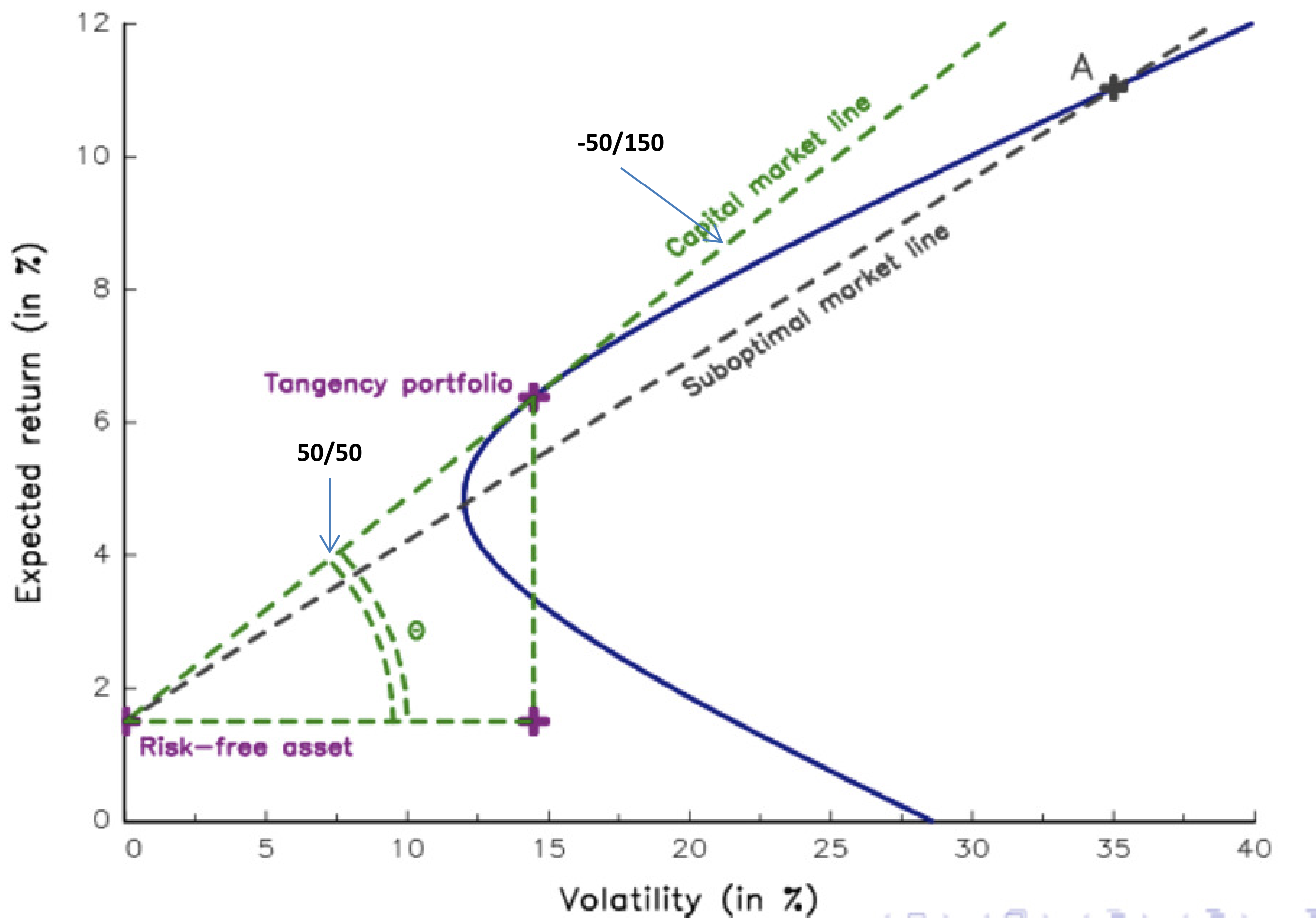
The capital market line

- The tangency portfolio is the risky portfolio that maximizes the Sharpe ratio in the presence of a risk-free asset.

- Any portfolio belonging to the capital market line has the same maximum Sharpe ratio, i.e. any linear combination of risk-free asset and tangency portfolio will generate the same maximum Sharpe ratio.

The capital markets line

Figure: The capital market line



The separation theorem

- Consider a portfolio *x* of risky assets and a risk-free asset, *r*. Denote

$$\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix} \qquad \tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \qquad \tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

- The Markowitz problem in the presence of a risk-free asset is

$$\tilde{x}^*(\phi) = argmax \tilde{x}^T \tilde{\mu} - \frac{\phi}{2} \tilde{x}^T \tilde{\Sigma} \tilde{x} \qquad u.c.\, \mathbf{1}^T \tilde{x} = 1$$

- We can rewrite the objective function as

$$f(x) = x^T \mu + x_r r - \frac{\phi}{2} x^T \Sigma x$$

and the constraint becomes $\mathbf{1}^T x + x_r = 1$

The separation theorem

- The corresponding Lagrange function is

$$\mathcal{L}(x; x_r; \lambda_0) = x^T\mu + x_r r - \frac{\phi}{2} x^T \Sigma x + \lambda_0(\mathbf{1}^T x + x_r - 1)$$

- We have the following first order conditions

$$\partial_x \mathcal{L}(x; x_r; \lambda_0) = \mu - \phi\Sigma x + \lambda_0 \mathbf{1} = 0$$
$$\partial_{x_r} \mathcal{L}(x; x_r; \lambda_0) = r - \lambda_0 = 0$$
$$\partial_{\lambda_0} \mathcal{L}(x; x_r; \lambda_0) = \mathbf{1}^T x + x_r - 1 = 0$$

- The solution is

$$x^* = \phi^{-1}\Sigma^{-1}(\mu - r\mathbf{1})$$
$$\lambda_0{}^* = r$$
$$x_r{}^* = 1 - \phi^{-1}\mathbf{1}^T\Sigma^{-1}(\mu - r\mathbf{1})$$

The separation theorem

- Let $\alpha = \phi^{-1}\mathbf{1}^T\Sigma^{-1}(\mu - r\mathbf{1})$
- The solution becomes

$$x^* = \alpha\left(\frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\mu - r\mathbf{1})}\right) = \alpha x_m^* \ \ where\ x_m^* = \frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\mu - r\mathbf{1})}$$

$$\lambda_0{}^* = r$$

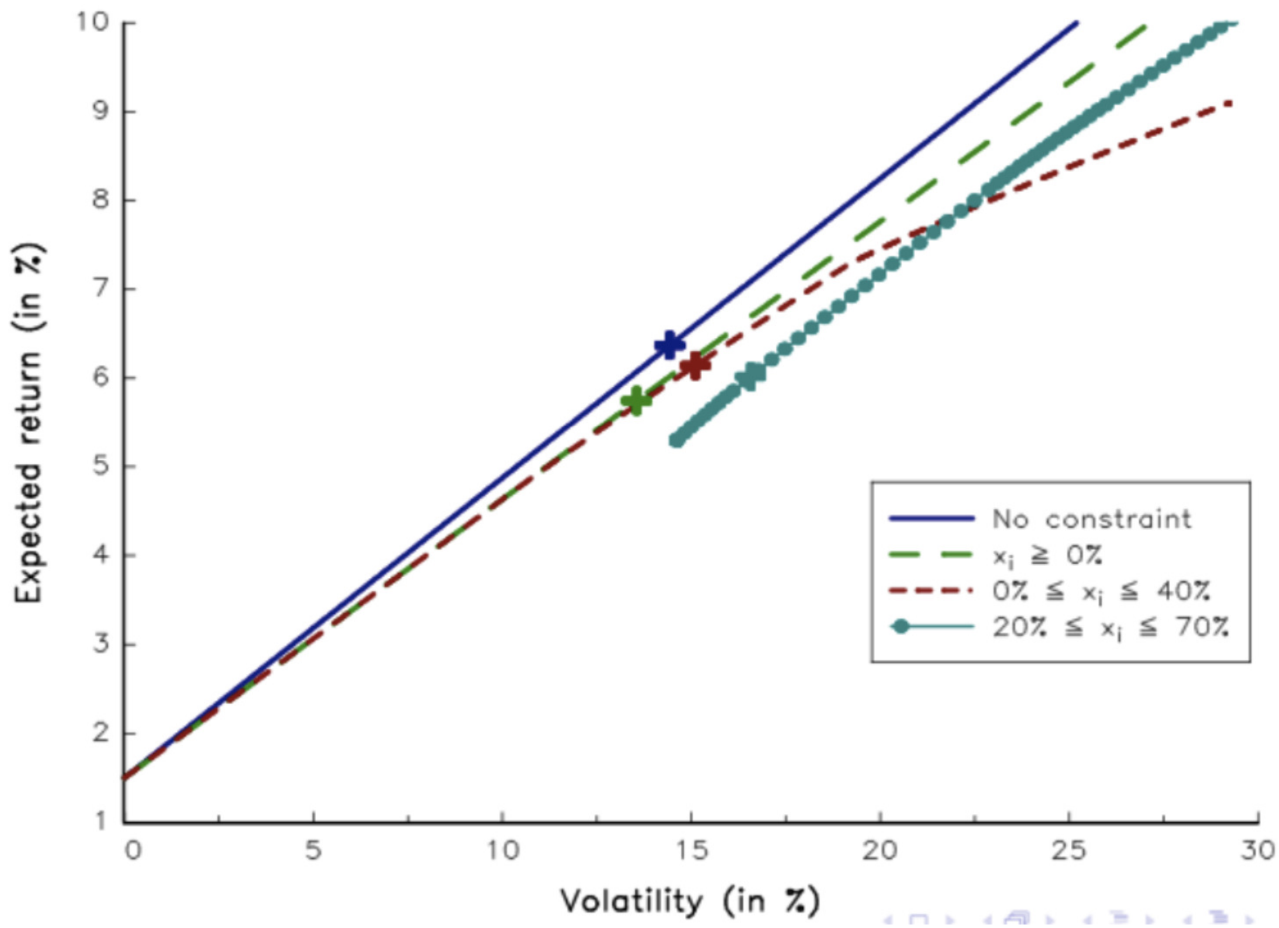
$$x_r{}^* = 1 - \alpha$$

- We obtain the separation theorem or one-fund theorem

$$\tilde{x}^* = \alpha\begin{pmatrix} x_m^* \\ 0 \end{pmatrix} + (1-\alpha)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Constrained efficient frontier with a risk-free asset

Figure: The efficient frontier with a risk-free asset



Market equilibrium and CAPM

- CAPM formula: For any asset i

$$r_i - r_f = \beta_i\left(r_m - r_f\right)$$

where r_m is the market (tangency portfolio) return and $\beta_i = \sigma_{i,m}/\sigma_M^2$ is the beta of asset i. This concept can be generalized to any portfolio $p = (x_1, \cdots, x_n)$ of risky assets.

- Implications:
  - This beta serves as an important measure of systematic, undiversifiable risk for any asset or portfolio
  - If $\beta_i = 0$, then the asset is uncorrelated with the market risk factor and leaves us with only idiosyncratic risk which can be diversified away by holding on to a large enough uncorrelated assets ($\rho\left(r_i, r_j\right) = 0 \; \forall \; i \neq j$ if β_i,$\beta_j = 0$)
  - One only get compensated for taking on risk that cannot be diversified away $\Longrightarrow$ risk premia ("quant") strategies
  - CAPM model can be extended to include other systematic risk factors (e.g. Fama-French 3 factor model)

Market equilibrium and CAPM

- CAPM proof: For an individual at her optimum portfolio, consider a small additional borrowing to finance the purchase of asset i. The portfolio mean is then $r_p = w_i(r_i - r_f) + r_m$ and the portfolio variance is $\sigma_p^2 = \sigma_m^2 + w_i^2\sigma_i^2 + 2w_i\sigma_{im}$ where w_i is the weight of asset i.

- Consider the derivative of the portfolio mean and variance w.r.t. w_i, the weight of asset i evaluated at the optimal point, i.e. $w_i = 0$

$$\left.\frac{\partial r_p}{\partial w_i}\right|_{w_i=0} = r_i - r_f \qquad \left.\frac{\partial \sigma_p^2}{\partial w_i}\right|_{w_i=0} = 2\sigma_{im}$$

Market equilibrium and CAPM

- At the optimum, the marginal change in portfolio expected return per unit change in portfolio variance must be equal for all risky assets, i.e. ratio of incremental return to risk must be equal across all assets.

- If this is not the case, then a risk arbitrage opportunity arises. Suppose Asset 1 has a higher expected return per unit of risk than Asset 2. Investors will sell Asset 2 [Asset 2 price ↓ (Asset 2 expected return ↑)] and buy Asset 1 [Asset 1 price ↑ (Asset 1 expected return ↓)]. This process will occur until the ratio is equalized across all assets.

$$\left.\frac{\partial r_p/\partial w_i}{\partial \sigma_p^2/\partial w_i}\right|_{w_i=0} = \frac{(r_i - r_f)}{2\sigma_{im}} = \frac{(r_j - r_f)}{2\sigma_{jm}} = \frac{(r_m - r_f)}{2\sigma_{mm}}$$

$$r_i - r_f = \beta_i(r_m - r_f) \quad where\ \beta_i = \frac{\sigma_{im}}{\sigma_{mm}}$$

Computation of beta

- Let $r_{i,t}$ and $r_{m,t}$ be the returns of asset i and the benchmark (market portfolio) at time t. Consider the following linear regression:

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \epsilon_{i,t}$$

- $\hat{\beta}_i$ is the ordinary least squares (OLS) estimate of asset i's beta. This approach can be extended to any asset or portfolio.

- Another way is simply to apply the following relationship derived earlier:

$$\beta_i = \frac{\sigma_{im}}{\sigma_{mm}}$$

- Portfolio beta is linear in nature, i.e. $\beta_p = \sum_{i=1}^{N} \beta_i$

Importance of beta (to the market)

- Estimate of asset's sensitivity to the market return. The market risk factor is usually (time-varying) the main contributor to portfolio (long-only) variance.

- Market neutral and active funds (fund vs. benchmark) use beta extensively to monitor/ hedge their risk (sensitivity) to the market return.

- A "non-zero" beta suggests some element of market timing: >0 (<0) implies asset return tends to be positively (negatively) correlated to market return.

Markowitz problem in the active return space

- Consider a benchmark represented by portfolio q. The active return between the active portfolio x and the benchmark q is the return difference between the portfolio and the benchmark.

$$r_{x-q} = r_x - r_q = \sum_{i=1}^{N} x_i r_i - \sum_{i=1}^{N} q_i r_i = (x-q)^T r$$

- The active risk/ tracking error (volatility of active returns) is

$$\sigma_{x-q} = \sqrt{(x-q)^T \Sigma (x-q)}$$

Markowitz problem in the active return space

- Objective: Maximize excess return subject to tracking error constraint. There are 2 ways to formulate the problem:

- Approach 1: Solving for x

$$x^* = argmax(x-q)^T\mu \quad \text{s.t.}\, \mathbf{1}^T x = 1\; and\; \sigma_{x-q} \leq \sigma^*$$

Transforming it to the Markowitz problem (into a σ-problem) yields

$$x^*(\phi) = argmax\, f(x|q) u.c. \quad \mathbf{1}^T x = 1 \;\; for\ some\ \phi \geq 0 \quad where$$

$$f(x|q) = (x-q)^T\mu - {}^{\phi}/_{2}\,(x-q)^T\Sigma(x-q)$$

$$= x^T(\mu + \phi\Sigma q) - {}^{\phi}/_{2}\, x^T\Sigma x - \left({}^{\phi}/_{2}\, q^T\Sigma q + q^T\mu\right)$$

$$= x^T(\mu + \phi\Sigma q) - {}^{\phi}/_{2}\, x^T\Sigma x + constant$$

The Lagrange function of the optimization problem is:

$$\mathcal{L}(x, \lambda_0) = x^T(\mu + \phi\Sigma q) - {}^{\phi}/_{2}\, x^T\Sigma x + \lambda_0(\mathbf{1}^T x - 1)$$

Markowitz problem in the active return space

- Using the solution to the standard Markowitz problem derived earlier, we can simply substitute the μ with $\mu + \phi\Sigma q$ and note that $1^T q = 1$ to yield:

$$x^*(\phi) = q + \frac{1}{\phi}\Sigma^{-1}\mu - \frac{1}{\phi}\frac{1^T\Sigma^{-1}\mu\Sigma^{-1}1}{1^T\Sigma^{-1}1}$$

- Approach 2: Solving for Δx where $\Delta x = x - q$

$$\Delta x^* = argmax \Delta x^T \mu \quad \text{s.t. } \mathbf{1}^T \Delta x = 0 \ and \ \sigma_{\Delta x}^2 = T$$

The Lagrange function of the optimization problem is:

$$\mathcal{L}(\Delta x, \lambda_1, \lambda_2) = \Delta x^T \mu + \lambda_1 \mathbf{1}^T \Delta x + \frac{1}{2}\lambda_2(\Delta x^T \Sigma \Delta x - T)$$

Taking partial derivatives with respect to Δx yields

$$\Delta x^* = \frac{-1}{\lambda_2}\Sigma^{-1}[\mu + \lambda_1 \mathbf{1}]$$

Markowitz problem in the active return space

Following Merton (1972), we can define

$$a = \mu^T \Sigma^{-1} \mu \quad b = \mu^T \Sigma^{-1} \mathbf{1} \quad c = \mathbf{1}^T \Sigma^{-1} \mathbf{1} \quad d = a - {b^2}/{c}$$

Selecting values of λ so that the equality constraints are satisfied, we have

$$\Delta x^T \mathbf{1} = 0$$

$$\left(\frac{-1}{\lambda_2} \Sigma^{-1} [\mu - \lambda_1 \mathbf{1}] \right)^T \mathbf{1} = 0$$

$$\lambda_1 = {-\mu^T \Sigma^{-1} \mathbf{1}}/{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} = {-b}/{c}$$

$$\Delta x^T \Sigma \Delta x = T$$

$$\left(\frac{-1}{\lambda_2} \Sigma^{-1} \left[\mu - \frac{b}{c} \mathbf{1} \right] \right)^T \Sigma \left(\frac{-1}{\lambda_2} \Sigma^{-1} \left[\mu - \frac{b}{c} \mathbf{1} \right] \right) = T$$

$$\lambda_2^2 T = \left[\mu - \frac{b}{c} \mathbf{1} \right]^T \Sigma^{-1} \Sigma \Sigma^{-1} \left[\mu - \frac{b}{c} \mathbf{1} \right]$$

$$\lambda_2 = \sqrt{\frac{1}{T} \left(\mu^T \Sigma^{-1} - \frac{b}{c} \mathbf{1}^T \Sigma^{-1} \right) \left(\mu - \frac{b}{c} \mathbf{1} \right)} = \sqrt{\frac{1}{T} \left(a - \frac{b^2}{c} \right)} = \sqrt{d/T}$$

Markowitz problem in the active return space

Finally we get:

$$\Delta x^* = \sqrt{\frac{T}{d}}\Sigma^{-1}\left[\mu - \frac{b}{c}\mathbf{1}\right]$$

- Note that the solution is independent of the benchmark!

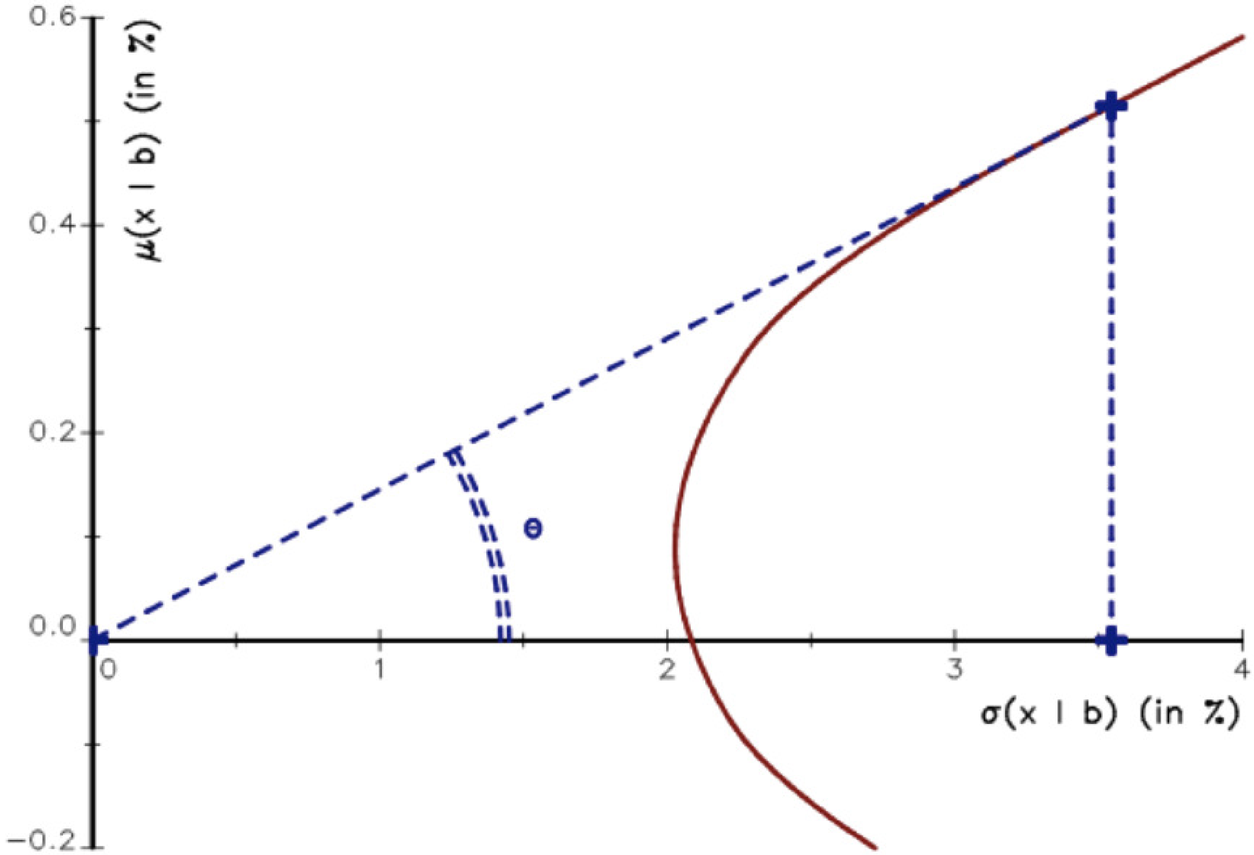
$$\mu_{\Delta x^*} = \sqrt{\frac{T}{d}}\left(a - \frac{b^2}{c}\right) = T\sqrt{d}$$

- In the unconstrained case, excess return is precisely proportional to the portfolio's tracking error (active risk) and manager skill (term *d* is the information ratio defined as the active return per unit of active risk).

- This linear efficient frontier is analogous to the capital market line (CML) which passes through the return-risk characteristics of the risk-free asset and tangency portfolio, where the tangency portfolio is determined by the Sharpe Ratio.

- Two key differences: In the active return space, the risk-free rate equals zero and the Sharpe Ratio is replaced with the Information Ratio.

Unconstrained linear active return frontier



Active return

Active risk/ tracking error

Efficient frontier - Going from active return to absolute return

- Let the return of an active return optimized portfolio *p** be

$$\mu_{p*} = (q + \Delta x^*)^T \mu = \mu_q + \mu_{\Delta x^*} = \mu_q + \sqrt{dT}$$

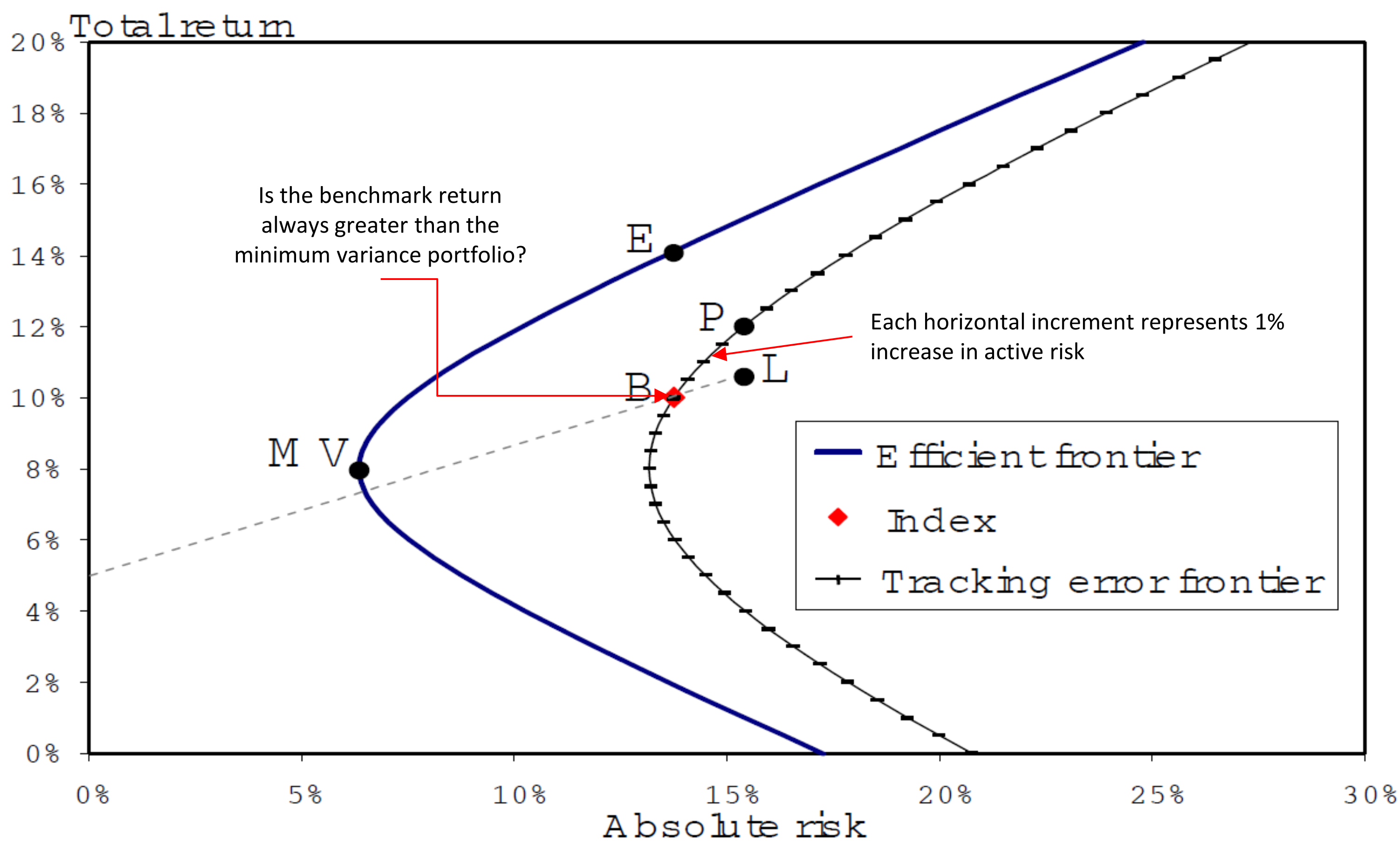
- The corresponding portfolio variance is then

$$\begin{aligned}
\sigma_{p*}^2 &= (q + \Delta x^*)^T \Sigma (q + \Delta x^*) \\
&= q^T \Sigma q + q^T \Sigma \Delta x^* + {\Delta x^*}^T \Sigma q + {\Delta x^*}^T \Sigma \Delta x^* \\
&= \sigma_q^2 + T + 2 q^T \Sigma \Delta x^* \\
&= \sigma_q^2 + T + 2 q^T \Sigma \sqrt{T/d}\, \Sigma^{-1}\left(\mu - b/c\, \mathbf{1}\right) \\
&= \sigma_q^2 + T + 2\sqrt{T/d}\left(q^T \mu - b/c\, q^T \mathbf{1}\right) \\
&= \sigma_q^2 + T + 2\sqrt{T/d}\left(\mu_q - \mu_{mv}\right)
\end{aligned}$$

by noting $q^T \mathbf{1} = 1$ and the minimum variance portfolio return is $\mu_{mv} = \mu^T x_{mv} = \mu^T \frac{\Sigma^{-1}\mathbf{1}}{1^T \Sigma^{-1} \mathbf{1}} = \frac{b}{c}$

- If we assume that $d > 0$ (some manager skill) and $\mu_q > \mu_{mv}$ (benchmark return greater than minimum variance portfolio return), then the above relationship suggests that increasing active returns necessarily increases portfolio variance.

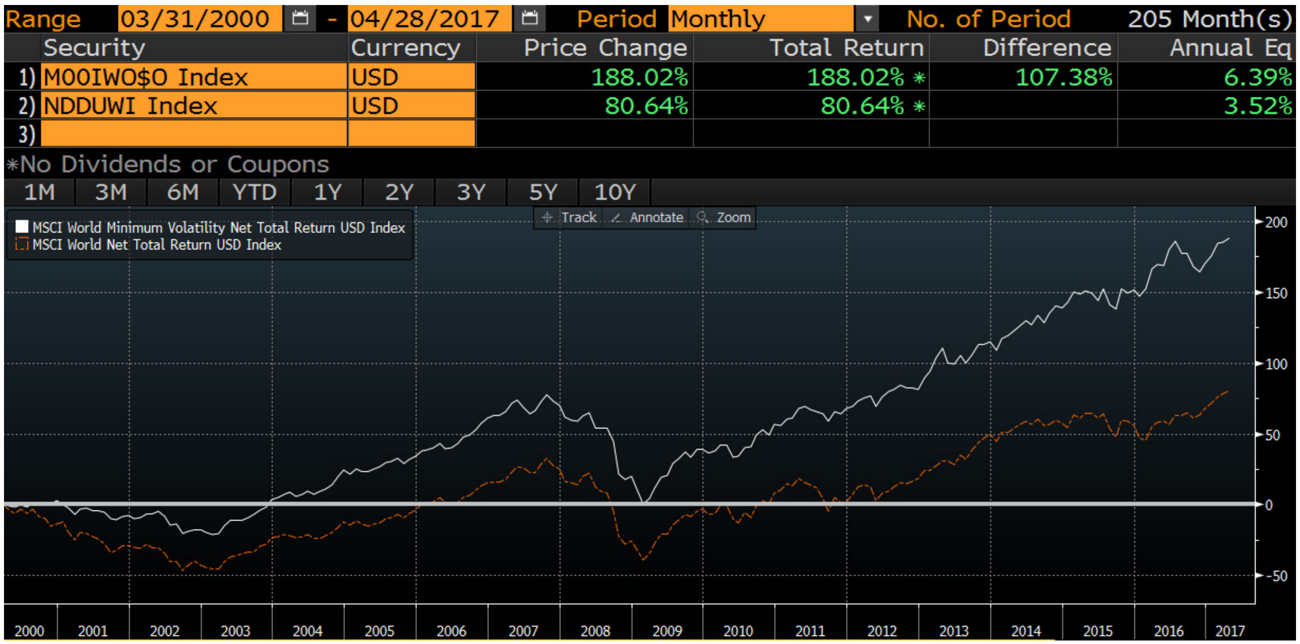
Inefficiency of active return optimized portfolios



Source: Jorion (2002) – Portfolio optimization with constraints on tracking error. Data used is the unhedged total returns in USD over the period 1980-2000 for the MSCI US, MSCI UK, MSCI Japan, MSCI Germany and Lehman Brothers US Aggregate Bond Index.

Inefficiency of active return optimized portfolios?

- $\mu_q > \mu_{mv}$? Empirical evidence suggests otherwise.



- $\mu_q < \mu_{mv}$ suggests that the actively managed portfolio will have a higher return and lower volatility than the benchmark, i.e. climb the frontier from left to right

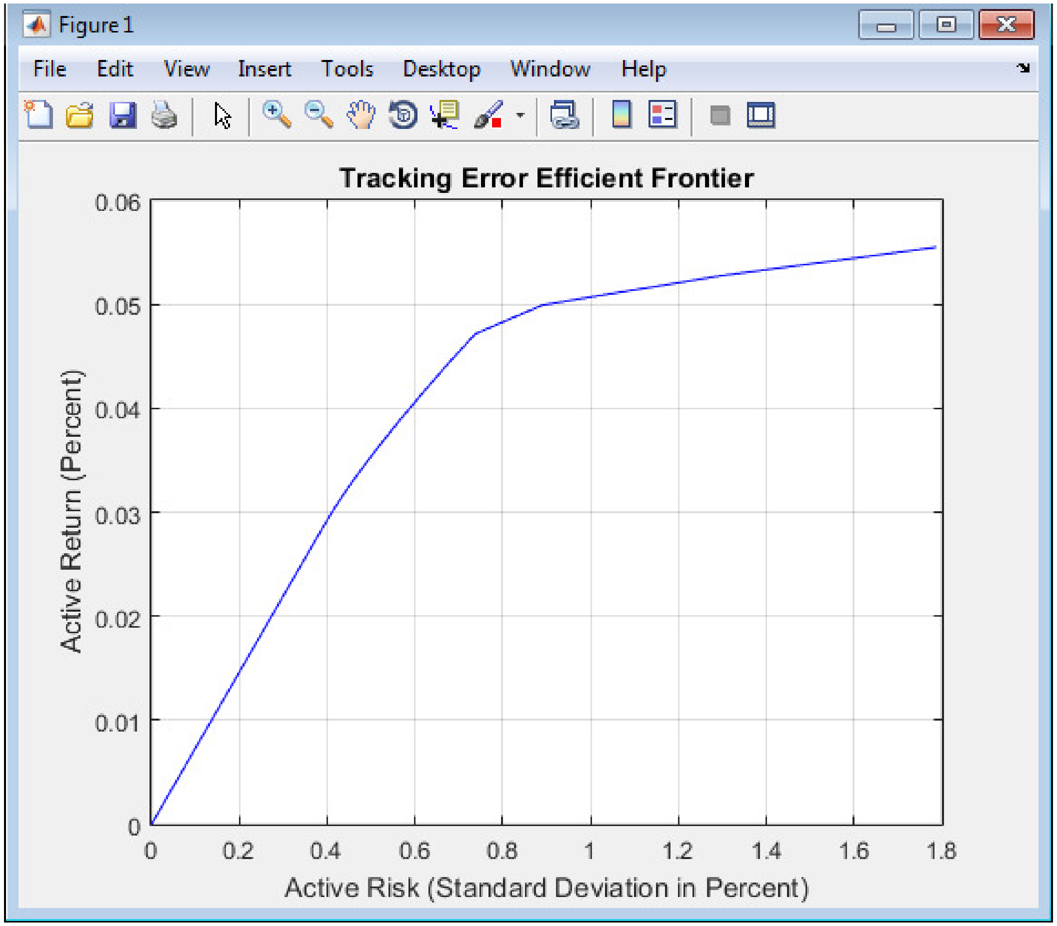
- May be true for multi-asset cases but less compelling in bottom-up security selection

Practical constraints on delta weights (Δx)

- No/ limited short selling (long-only, active extension)
- Maximum and minimum weight differences due to mandate restrictions
- Hedging along various risk dimensions (e.g. beta, country and sector neutrality, etc)
- Liquidity constraints
- Turnover constraints

Constrained active return frontier

Excess return



Tracking error

Markowitz framework – Pros and cons

- Pros
  - Investor preferences are described in a concise way – only the risk tolerance is required to determine an investor's optimal portfolio.
  - Only expected returns and covariance matrix of asset returns are required (blessing or curse?)
  - Powerful numerical algorithms exist for finding solutions to the quadratic programming problem.
  - There exists analytical solution to the Markowitz problem which makes it tractable.

- Cons
  - Mean-variance optimized portfolios often put excessive weights on assets with large expected returns (overly concentrated portfolios which is counterintuitive to notion of diversification), regardless of possible estimation errors in the input values (predicting expected returns is difficult if not impossible).
  - More noisy than inputs! Small changes in input = Large changes in weights => High turnover
  - Most investors think in terms of relative attractiveness between assets, less so in expected return terms.
  - Ignores preferences towards higher moments (skewness and kurtosis).