QF623: Portfolio management I
Portfolio construction II

Alternative portfolio construction methods

- Regularization of optimized portfolios
  - Introducing portfolio constraints
  - Shrinking the covariance matrix

- Risk-based weighting schemes
  - Robust, independent of expected returns
  - Minimum variance, equal risk contribution
  - Volatility targeting

- **Black-Litterman (BL) approach**

- **Fundamental law of active management**

Black-Litterman model

- Most important input in mean-variance optimization is the vector of expected returns.
- Best and Grauer (1991) demonstrate that a small increase in the expected return of a single asset can dramatically increase its weight, resulting in unintuitive concentrated portfolios.
- Need to search for a reasonable (stable) starting (neutral) point for expected returns, i.e. the equilibrium returns.
- Put in other words, in the absence of views, what will a rational investor hold?

Black-Litterman model

- Combines different well-known concepts
  - CAPM (Sharpe (1964))
  - Reverse optimization (Sharpe (1974))
  - Mixed estimation (Theil (1971,1978))
  - Mean-variance optimization (Markowitz (1952))

- Through CAPM and reverse optimization, the BL model provides an intuitive prior, the equilibrium market portfolio, as a starting point for estimating asset returns.

- Provides a clear way to specify investor views (relative or absolute, partial or complete) on returns and to blend these views with prior information, i.e. market-implied asset returns.

Black-Litterman model

- Flexibility in the specification of investor views
  - Relative or absolute
  - Partial or complete
  - Span arbitrary and overlapping sets of assets

- Enables investors to combine their unique views on different assets in a manner that results in intuitive, diversified portfolios.

- Lee (2000) show that the BL model "largely mitigates" the problem of estimation error-maximization by spreading the errors across the expected returns.

Black-Litterman model

- Start with normally distributed expected returns:

$$r \sim N(\mu, \Sigma) \quad (1)$$

- Goal of the BL model is to model these expected returns, which are assumed to be normally distributed with mean μ and variance Σ.
- Define μ, the unknown mean return, as a random variable distributed as:

$$\mu \sim N(\Pi, \Sigma_\Pi) \quad (2)$$

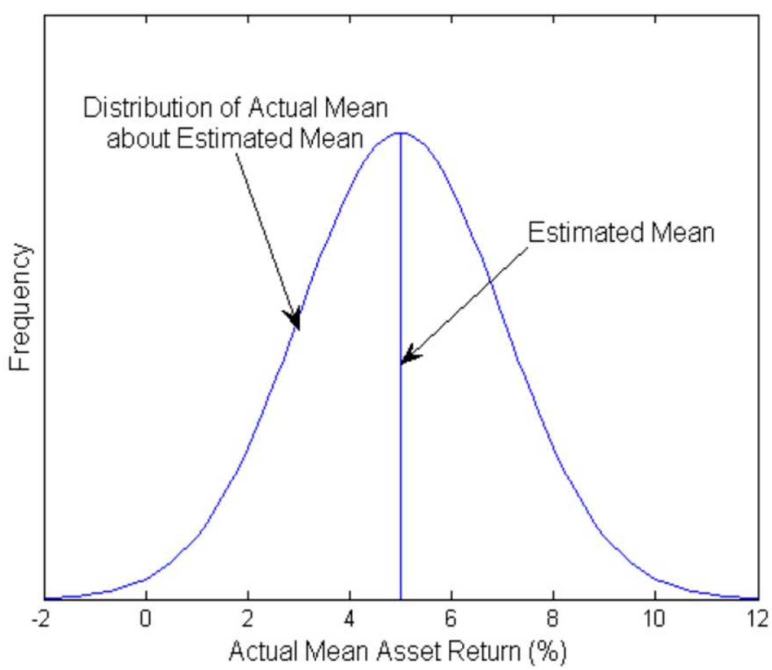
where Π is the estimate of the mean and Σ_Π is the variance of the unknown mean μ.

Black-Litterman model

- This is equivalent to saying that the prior returns are normally distributed around Π with some noise term ε:

$$\mu = \Pi + \varepsilon \quad (3)$$

Figure 1 - Distribution of Actual Mean about Estimated Mean



Source: Walters, J (2014) – The Black-Litterman model in detail

Black-Litterman model

- ε is normally distributed with mean 0 and variance Σ_Π and is assumed to be uncorrelated with μ

- Define the variance of the returns about the estimate Π as Σ_r. The independence assumption between ε and μ implies:
$$\Sigma_r = \Sigma + \Sigma_\Pi \quad (4)$$

- In the absence of estimation error, i.e. $\varepsilon \equiv 0$, then $\Sigma_r = \Sigma$.

- Canonical reference model for BL expected return is:
$$r \sim N(\Pi, \Sigma_r) \quad (5)$$

Black-Litterman model
Computing equilibrium returns

- Model starts with a neutral equilibrium portfolio for the prior estimate of returns, i.e. returns before any investor views are incorporated

- Candidate for a neutral portfolio is the well-known CAPM market portfolio. Under the CAPM framework, the prior distribution for the BL model is the estimated mean excess (over the risk-free rate) return from the market portfolio.

$$E(r) = r_f + \alpha + \beta r_m$$

where r_f is the risk-free rate, r_m is the market portfolio return, α is the residual or asset specific return and β is the asset's sensitivity to the market portfolio.

Black-Litterman model
Computing equilibrium returns

- Under CAPM, the asset specific risk is uncorrelated with other assets, and this risk can be diversified away => An investor is rewarded for taking on systematic risk measured by β.

- Because all investors should hold the same risky portfolio in the CAPM world, at equilibrium the market capitalization of the various assets will determine their weights in the market portfolio.

Black-Litterman model
Reverse optimization

- Consider the following quadratic utility function:

$$U = w^T \Pi - \frac{\delta}{2} w^T \Sigma w$$

where w is the vector of asset weights, Π is the vector of equilibrium asset excess return, δ is the risk aversion parameter and Σ is the covariance matrix of asset excess returns.

- First order condition with no constraints yields the implied equilibrium excess returns (equation 1):

$$\frac{dU}{dw} = \Pi - \delta \Sigma w = 0$$

$$\Pi = \delta \Sigma w \quad (6)$$

Black-Litterman model
Reverse optimization

- Calibrating δ: Multiply both sides of (5) by w^T:

$$w^T\Pi = w^T\delta\Sigma w$$
$$r - r_f = \delta\sigma^2$$
$$\delta = \frac{r - r_f}{\sigma^2} = \frac{SR}{\sigma}$$

  where $r = w^T\Pi + r_f$ is the total return on the market portfolio, r_f is the risk-free rate, σ^2 is the variance of the market portfolio and SR is the Sharpe ratio of the market portfolio.

- Black and Litterman (1992) assume a Sharpe ratio close to 0.5 in their example to calibrate the market risk aversion coefficient.

Black-Litterman model
Reverse optimization

- Black and Litterman make the simplifying assumption that the structure of the covariance matrix of the estimate is proportional to the covariance of the returns, i.e. $\Sigma_\Pi = \tau\Sigma$

- We can rewrite (2) as:

$$\mu \sim N(\Pi, \tau\Sigma)$$

and rewrite (5) as:

$$r \sim N(\Pi, (1+\tau)\Sigma)$$

Black-Litterman model
Reverse optimization

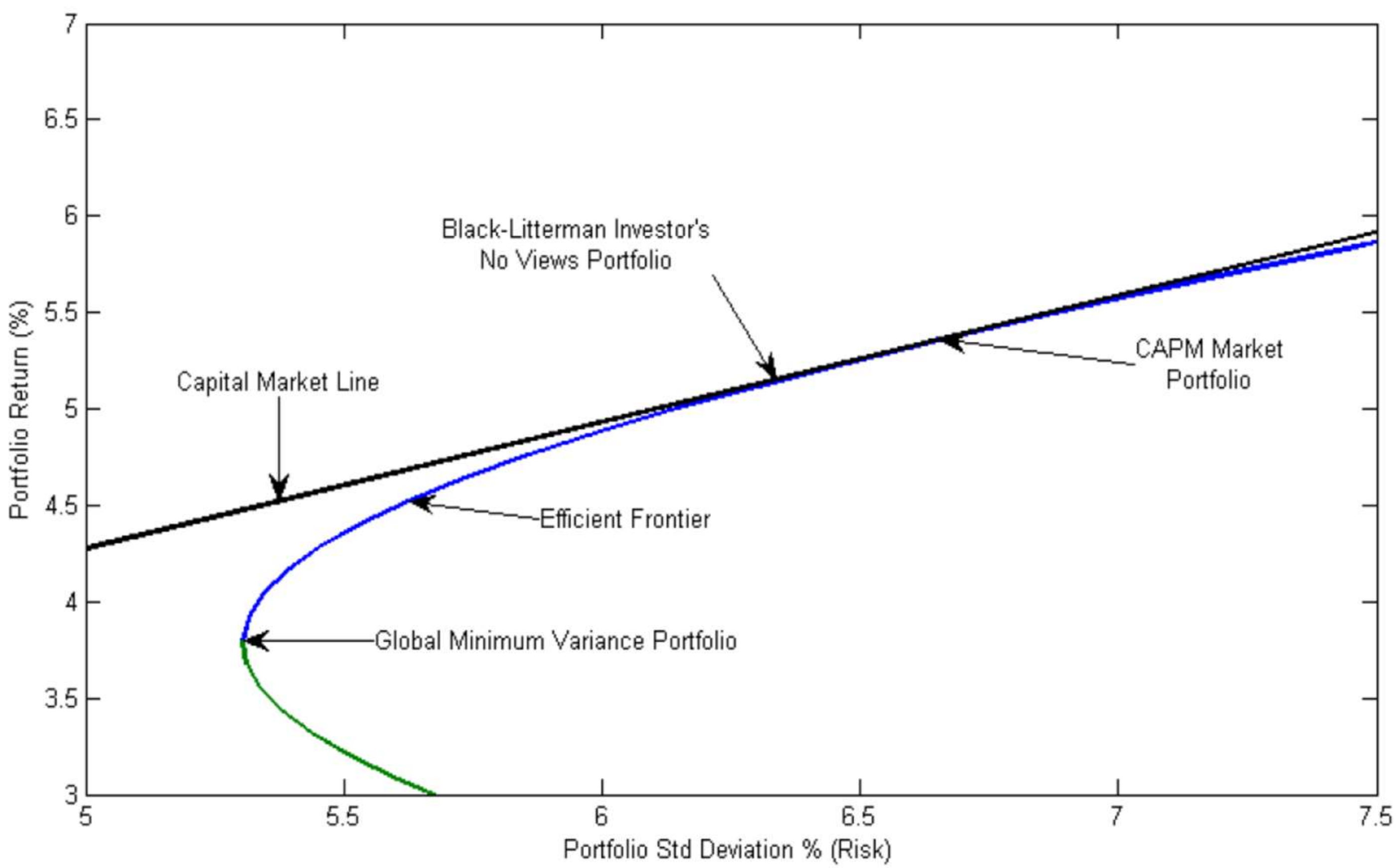
- From (6), we can write:

$$\Pi = \delta\Sigma w$$
$$w = (\delta\Sigma)^{-1}\Pi$$
$$\hat{w} = (\delta(1+\tau)\Sigma)^{-1}\Pi = (1/(1+\tau))(\delta\Sigma)^{-1}\Pi$$
$$\hat{w} = (1/(1+\tau))w$$

- Because of uncertainty in the estimates, an investor may hold $1/(1+\tau)$ in the neutral portfolio and $\tau/(1+\tau)$ in the risk-free asset.

Neutral portfolio on the efficient frontier

Figure 3 - Investor's Portfolio in the Absence of Views



Source: Walters, J (2014) – The Black-Litterman model in detail

Black-Litterman empirical example
Idzorek (2005) – Starting neutral point

Table 1 Expected Excess Return Vectors

Asset Class	Historical μ_{Hist}	CAPM GSMI μ_{GSMI}	CAPM Portfolio μ_P	Implied Equilibrium Return Vector Π
US Bonds	3.15%	0.02%	0.08%	0.08%
Int'l Bonds	1.75%	0.18%	0.67%	0.67%
US Large Growth	-6.39%	5.57%	6.41%	6.41%
US Large Value	-2.86%	3.39%	4.08%	4.08%
US Small Growth	-6.75%	6.59%	7.43%	7.43%
US Small Value	-0.54%	3.16%	3.70%	3.70%
Int'l Dev. Equity	-6.75%	3.92%	4.80%	4.80%
Int'l Emerg. Equity	-5.26%	5.60%	6.60%	6.60%
Weighted Average	-1.97%	2.41%	3.00%	3.00%
Standard Deviation	3.73%	2.28%	2.53%	2.53%
High	3.15%	6.59%	7.43%	7.43%
Low	-6.75%	0.02%	0.08%	0.08%

** All four estimates are based on 60 months of excess returns over the risk-free rate. The two CAPM estimates are based on a risk premium of 3. Dividing the risk premium by the variance of the market (or benchmark) excess returns (σ^2) results in a risk-aversion coefficient (λ) of approximately 3.07.*

Neutral starting point for the BL model from a return vector perspective

Source: Idzorek, T.M. (2005) – A step-by-step guide to the Black-Litterman model
Note: GSMI is the UBS global securities markets index

Black-Litterman empirical example
Idzorek (2005) – Starting neutral point

Table 2 Recommended Portfolio Weights

Asset Class	Weight Based on Historical w_{Hist}	Weight Based on CAPM GSMI w_{GSMI}	Weight Based on Implied Equilibrium Return Vector Π	Market Capitalization Weight w_{mkt}
US Bonds	1144.32%	21.33%	19.34%	19.34%
Int'l Bonds	-104.59%	5.19%	26.13%	26.13%
US Large Growth	54.99%	10.80%	12.09%	12.09%
US Large Value	-5.29%	10.82%	12.09%	12.09%
US Small Growth	-60.52%	3.73%	1.34%	1.34%
US Small Value	81.47%	-0.49%	1.34%	1.34%
Int'l Dev. Equity	-104.36%	17.10%	24.18%	24.18%
Int'l Emerg. Equity	14.59%	2.14%	3.49%	3.49%
High	1144.32%	21.33%	26.13%	26.13%
Low	-104.59%	-0.49%	1.34%	1.34%

Neutral starting point for the BL model from the portfolio perspective

Source: Idzorek, T.M. (2005) – A step-by-step guide to the Black-Litterman model
Note: GSMI is the UBS global securities markets index

Black-Litterman model
Posterior return incorporating views

Let N be the number of assets and k be the number of investor views. Using the implied equilibrium excess returns Π as the starting point and Bayes Theorem, the posterior combined return vector[1] can be written as:

$$E[R] = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q] \quad (7)$$

where τ is a scalar, Σ is the covariance matrix of excess returns ($N \times N$ matrix), P is a matrix that identifies the assets involved in the views ($K \times N$ matrix), Ω is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view ($K \times K$ matrix), Π is the implied equilibrium excess return vector ($N \times 1$ column vector) and Q is the view vector ($K \times 1$ column vector).

[1] *See for example Walters, J. (2014) – The Black-Litterman model in detail for the proof* of the formula for the combined return vector

Black-Litterman model
Posterior return incorporating views

- (7) can be re-written in a more intuitive form as:
$$E[R] = \Pi + (\tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1})(Q - P\Pi) \quad (8)$$

- If there is 100% certainty of the views, then $\Omega \rightarrow 0$ and (8) becomes:
$$E[R] = \Pi + (\tau\Sigma P^T(P\tau\Sigma P^T)^{-1})(Q - P\Pi)$$
If P is invertible, i.e. a view on every asset has been offered, then the above equation becomes:
$$E[R] = P^{-1}Q$$

- If the investor is completely unsure of his views (i.e. equivalently to having no views), then $\Omega \rightarrow \infty$ and (8) becomes:
$$E[R] = \Pi$$

Black-Litterman empirical example
Idzorek (2005) – Investor views

- Example showing absolute and relative views on single and pairs of assets, as well as a view involving multiple groups of assets.
- Absolute - View 1: International developed equity will have an absolute excess return of 5.25% (confidence of 25%)
- Relative – View 2: International bonds will outperform US bonds by 25 bps (confidence of 50%)
- Multiple assets – View 3: US large growth and US small growth will outperform US large value and US small value by 2% (confidence of 65%)

Black-Litterman empirical example
Idzorek (2005) – Investor view 1

- View 1: The implied equilibrium excess return is 4.8% which is lower than the absolute expectation of 5.25%.

- Absolute views lead to a long bias in the portfolio.

Table 1 Expected Excess Return Vectors

Asset Class	Historical μ_{Hist}	CAPM GSMI μ_{GSMI}	CAPM Portfolio μ_P	Implied Equilibrium Return Vector Π
US Bonds	3.15%	0.02%	0.08%	0.08%
Int'l Bonds	1.75%	0.18%	0.67%	0.67%
US Large Growth	-6.39%	5.57%	6.41%	6.41%
US Large Value	-2.86%	3.39%	4.08%	4.08%
US Small Growth	-6.75%	6.59%	7.43%	7.43%
US Small Value	-0.54%	3.16%	3.70%	3.70%
Int'l Dev. Equity	-6.75%	3.92%	4.80%	4.80%
Int'l Emerg. Equity	-5.26%	5.60%	6.60%	6.60%
Weighted Average	-1.97%	2.41%	3.00%	3.00%
Standard Deviation	3.73%	2.28%	2.53%	2.53%
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Low	-6.75%	0.02%	0.08%	0.08%

** All four estimates are based on 60 months of excess returns over the risk-free rate. The two CAPM estimates are based on a risk premium of 3. Dividing the risk premium by the variance of the market (or benchmark) excess returns (σ^2) results in a risk-aversion coefficient (λ) of approximately 3.07.*

Black-Litterman empirical example
Idzorek (2005) – Investor view 2

- Relative views align more closely to the way investment managers think.

- View 2 states that the return of international bonds is 25 bps greater than that of the US. Implied equilibrium spread between international bonds and US bonds is +59 bps, lower than what is expressed in the view.

- Relative to the neutral portfolio, we expect to underweight international bonds and overweight US bonds.

Table 1 Expected Excess Return Vectors

Asset Class	Historical μ_{Hist}	CAPM GSMI μ_{GSMI}	CAPM Portfolio μ_P	Implied Equilibrium Return Vector Π
US Bonds	3.15%	0.02%	0.08%	0.08%
Int'l Bonds	1.75%	0.18%	0.67%	0.67%
US Large Growth	-6.39%	5.57%	6.41%	6.41%
US Large Value	-2.86%	3.39%	4.08%	4.08%
US Small Growth	-6.75%	6.59%	7.43%	7.43%
US Small Value	-0.54%	3.16%	3.70%	3.70%
Int'l Dev. Equity	-6.75%	3.92%	4.80%	4.80%
Int'l Emerg. Equity	-5.26%	5.60%	6.60%	6.60%
Weighted Average	-1.97%	2.41%	3.00%	3.00%
Standard Deviation	3.73%	2.28%	2.53%	2.53%
High	3.15%	6.59%	7.43%	7.43%
Low	-6.75%	0.02%	0.08%	0.08%

** All four estimates are based on 60 months of excess returns over the risk-free rate. The two CAPM estimates are based on a risk premium of 3. Dividing the risk premium by the variance of the market (or benchmark) excess returns (σ^2) results in a risk-aversion coefficient (λ) of approximately 3.07.*

Black-Litterman empirical example
Idzorek (2005) – Investor view 3

- View requires one to think in terms of 2 distinct sub-portfolios.
- Market-cap weighted within each sub-portfolio
- Weighted average equilibrium spread between Growth and Value is 2.47%, versus the view that it is expected to be 2%.
- Tilt away from Growth and towards Value

Table 3a View 3 – Nominally "Outperforming" Assets

Asset Class	Market Capitalization (Billions)	Relative Weight	Implied Equilibrium Return Vector Π	Weighted Excess Return
US Large Growth	\$5,174	90.00%	6.41%	5.77%
US Small Growth	\$575	10.00%	7.43%	0.74%
	\$5,749	100.00%	Total	6.52%

Table 3b View 3 – Nominally "Underperforming" Assets

Asset Class	Market Capitalization (Billions)	Relative Weight	Implied Equilibrium Return Vector Π	Weighted Excess Return
US Large Value	\$5,174	90.00%	4.08%	3.67%
US Small Value	\$575	10.00%	3.70%	0.37%
	\$5,749	100.00%	Total	4.04%

Black-Litterman empirical example

Idzorek (2005) - Inputting views into the model

- Constructing the Q vector of K views $(K = 3)$
- ε , the error term corresponding to each view, features in BL's expected return vector through Ω

General Case:

$$Q + \varepsilon = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

Example:

$$Q + \varepsilon = \begin{bmatrix} 5.25 \\ 0.25 \\ 2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

Absolute view 1
Relative view 2
View 3

Black-Litterman empirical example

Idzorek (2005) - Inputting views into the model

- Constructing Ω, the diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view.

- In our case, P is a 3 views by 8 assets matrix.

General Case:

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix}$$

Example (Based on Satchell and Scowcroft (2000)):

$$\begin{array}{cl} P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & -.5 & .5 & -.5 & 0 & 0 \end{bmatrix} & \begin{array}{l} \text{Absolute view 1} \\ \text{Relative view 2} \\ \text{View 3} \end{array} \end{array}$$

Matrix P (Market capitalization method):

$$\begin{array}{cl} P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .9 & -.9 & .1 & -.1 & 0 & 0 \end{bmatrix} & \begin{array}{l} \text{Absolute view 1} \\ \text{Relative view 2} \\ \text{View 3} \end{array} \end{array}$$

Black-Litterman empirical example
Idzorek (2005) - Inputting views into the model

- Once P (matrix that identifies the assets involved in the views) is defined, we can calculate the variance of each individual view portfolio, $p_k \Sigma p_k^T$ where p_k is a $1 \times N$ row vector.

Table 4 Variance of the View Portfolios

View	Formula	Variance	
1	$p_1 \Sigma p_1'$	2.836%	Absolute view on developed equity
2	$p_2 \Sigma p_2'$	0.563%	Relative view between bonds
3	$p_3 \Sigma p_3'$	3.462%	Relative view between equity segments

Black-Litterman empirical example
Idzorek (2005) – Calibrating τ

- Magnitude of portfolio departure from their neutral market cap weights is controlled by the ratio of τ to the variance of the error term Ω

- Different authors propose different values of τ
  - Black and Litterman (1992) and Lee (2000) propose τ to be close to 0 as there is less uncertainty in the equilibrium returns relative to historical returns.

  - He and Litterman (1999) calibrate the confidence of the view so that Ω_{kk}/τ is equal to the variance of the k^{th} view portfolio, $p_k \Sigma p_k^T$. Under this calibration, τ becomes irrelevant (see equation 8).

Black-Litterman empirical example

Idzorek (2005) – Calibrating τ

- Assuming $\tau = 0.025$ and using the individual variances of the view portfolios from before, we have:

General Case:

$$\Omega = \begin{bmatrix} \left(p_1 \Sigma p_1^{'}\right) * \tau & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \left(p_k \Sigma p_k^{'}\right) * \tau \end{bmatrix}$$

Example:

$$\Omega = \begin{bmatrix} 0.000709 & 0 & 0 \\ 0 & 0.000141 & 0 \\ 0 & 0 & 0.000866 \end{bmatrix}$$

Black-Litterman empirical example

Idzorek (2005) – Final BL portfolio weights, $\hat{w}$

- A single view causes the posterior return of every asset in the portfolio to change due to the co-movements with other assets.

- One of the strongest features about the BL model is that weight changes only apply to assets with an expressed view.

Table 6 Return Vectors and Resulting Portfolio Weights

Asset Class	New Combined Return Vector $E[R]$	Implied Equilibrium Return Vector Π	Difference $E[R]-\Pi$	New Weight $\hat{w}$	Market Capitalization Weight w_{mkt}	Difference $\hat{w}-w_{mkt}$
US Bonds	0.07%	0.08%	-0.02%	29.88%	19.34%	10.54%
Int'l Bonds	0.50%	0.67%	-0.17%	15.59%	26.13%	-10.54%
US Large Growth	6.50%	6.41%	0.08%	9.35%	12.09%	-2.73%
US Large Value	4.32%	4.08%	0.24%	14.82%	12.09%	2.73%
US Small Growth	7.59%	7.43%	0.16%	1.04%	1.34%	-0.30%
US Small Value	3.94%	3.70%	0.23%	1.65%	1.34%	0.30%
Int'l Dev. Equity	4.93%	4.80%	0.13%	27.81%	24.18%	3.63%
Int'l Emerg. Equity	6.84%	6.60%	0.24%	3.49%	3.49%	0.00%
			Sum	103.63%	100.00%	3.63%

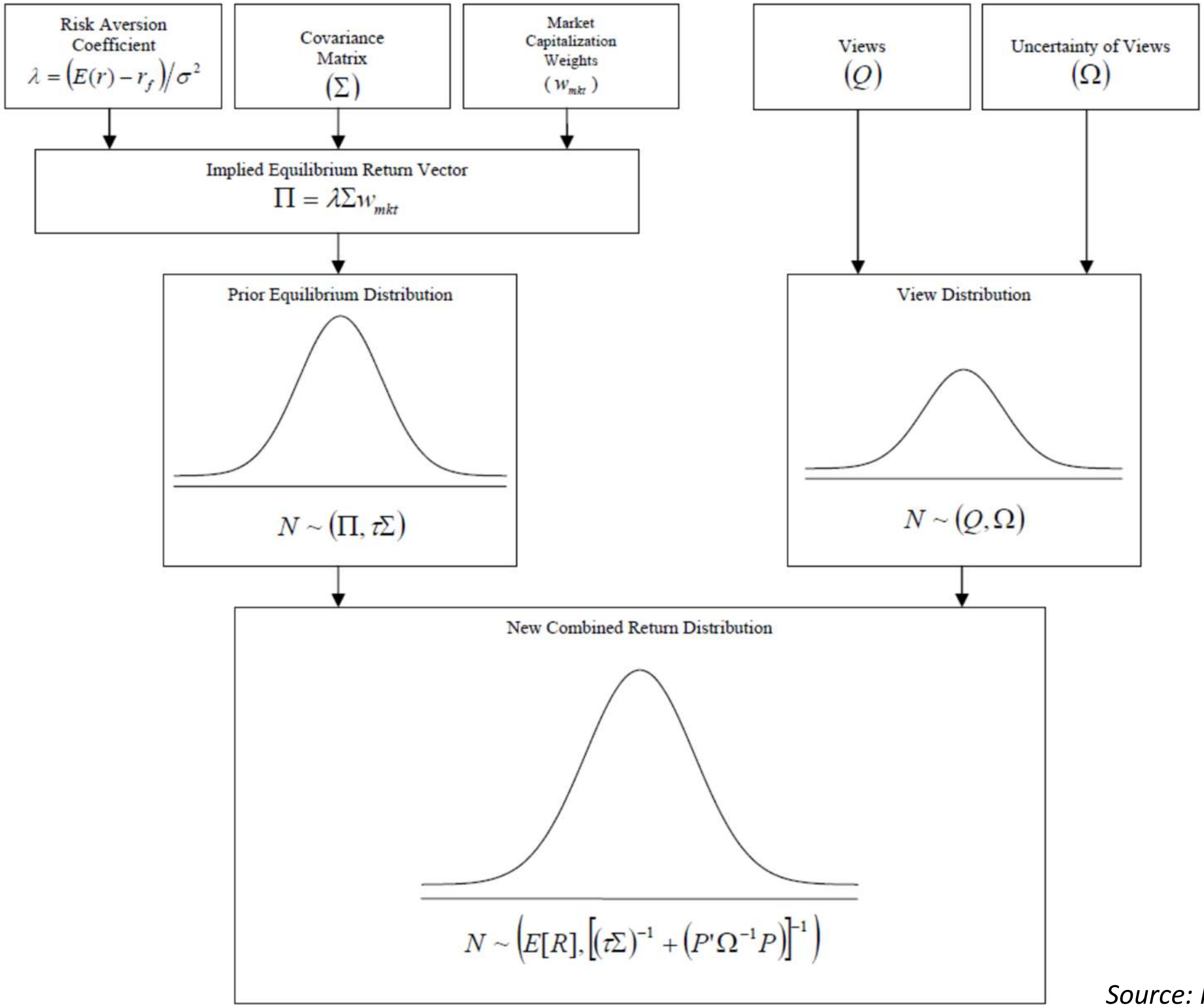
Black-Litterman empirical example
BL portfolio in the presence of constraints

- The intuitiveness of the BL model is less apparent with added portfolio constraints on unity, risk, beta, short selling, etc.

- He and Litterman (1999) and Litterman (2003) suggest inputting the derived posterior return vector into the constrained mean-variance optimizer.

- The idea is that one will get a constrained solution close enough to the ideal BL portfolio.

Deriving the posterior expected return
Overview



* *The variance of the New Combined Return Distribution is derived in Satchell and Scowcroft (2000).*

Source: Idzorek, T.M. (2005) – A step-by-step guide to the Black-Litterman model

Black-Litterman model
Incorporating investor confidence levels

- Herold (2003) notes that the major difficulty of the BL model is the requirement to specify a probability density function for each view.

- Idzorek (2005) present a method to determine implied confidence levels in the views and to allow for a 0%-100% user-specified confidence level for each view.

- This approach also removes the difficulty of specifying a value for τ.

Black-Litterman model
Incorporating investor confidence levels

- Recall that in the last example, the investor's confidence in a view depends only on the volatility of the view portfolio.

- In practice, additional factors can affect an investor's confidence in a view, such as the volatility regime (or more generally macro-economic regime) and historical accuracy of the model, screen or fundamental analyst.

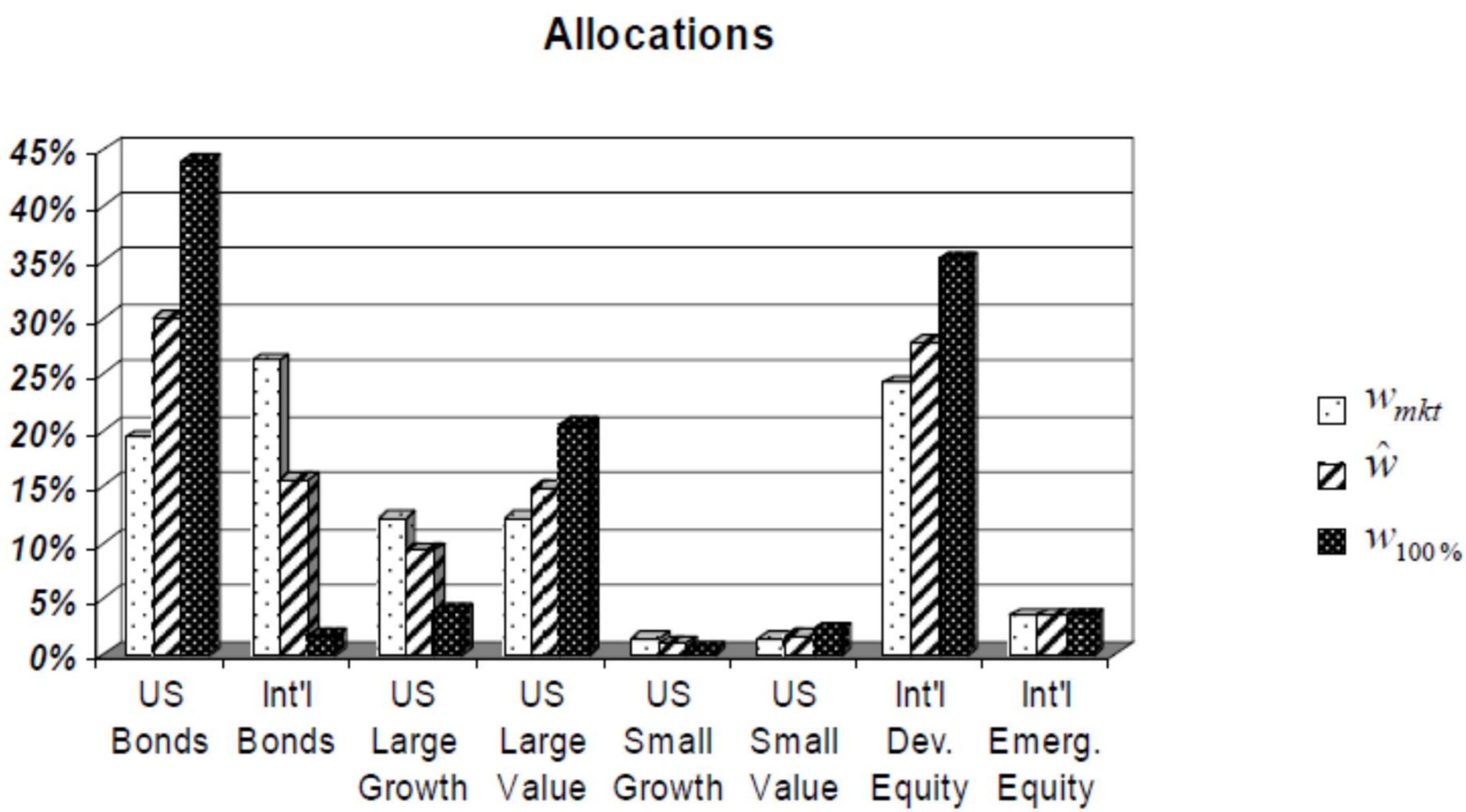
- When 100% confidence is specified for all of the views, the BL formula for the posterior return vector is:

$$E[R_{100\%}] = \Pi + (\tau\Sigma P^T(P\tau\Sigma P^T)^{-1})(Q - P\Pi)$$

which can be input into (6) to derive $w_{100\%}$

Black-Litterman empirical example

Idzorek (2005) – Comparing $w_{mkt}, \hat{w}, w_{100\%}$



Black-Litterman empirical example

Idzorek (2005) – Implied confidence levels

$$\text{Implied confidence level} = \frac{\widehat{w} - w_{mkt}}{w_{100\%} - w_{mkt}}$$

Table 7 Implied Confidence Level of Views

Asset Class	Market Capitalization Weights w_{mkt}	New Weight $\hat{w}$	Difference $\hat{w} - w_{mkt}$	New Weights (Based on 100% Confidence) $\hat{w}_{100\%}$	Difference $\hat{w}_{100\%} - w_{mkt}$	Implied Confidence Level $\frac{\hat{w} - w_{mkt}}{\hat{w}_{100\%} - w_{mkt}}$
US Bonds	19.34%	29.88%	10.54%	43.82%	24.48%	43.06%
Int'l Bonds	26.13%	15.59%	-10.54%	1.65%	-24.48%	43.06%
US Large Growth	12.09%	9.35%	-2.73%	3.81%	-8.28%	33.02%
US Large Value	12.09%	14.82%	2.73%	20.37%	8.28%	33.02%
US Small Growth	1.34%	1.04%	-0.30%	0.42%	-0.92%	33.02%
US Small Value	1.34%	1.65%	0.30%	2.26%	0.92%	33.02%
Int'l Dev. Equity	24.18%	27.81%	3.63%	35.21%	11.03%	32.94%
Int'l Emerg. Equity	3.49%	3.49%	--	3.49%	--	--

Black-Litterman model
Incorporating investor confidence levels

- Idzorek (2005) propose that the diagonal elements of Ω be derived such that the user-specified confidence levels result in portfolio tilts which approximate:

$$Tilt_k \approx (w_{100\%} - w_{mkt}) \times C_k$$

where C_k is the confidence level associated with view k

- In the absence of other views, the approximate recommended weight vector resulting from the view is:

$$w_{100\%} \approx w_{mkt} + Tilt_k$$

Black-Litterman model
Idzorek (2005) method for incorporating C_k

- Step 1: For each view (k), calculate the posterior return vector using the BL formula under 100% certainty independently of other views.

$$E\left[R_{k,100\%}\right] = \Pi + \tau\Sigma p_k^T\left(p_k\tau\Sigma p_k^T\right)^{-1}(Q_k - p_k\Pi)$$

where $E\left[R_{k,100\%}\right]$ is the posterior return vector based on 100% confidence in the k^{th} view ($N \times 1$ column vector), p_k identifies the assets involved in the k^{th} view ($1 \times N$ row vector) and Q_k is the k^{th} view (1×1)

- Step 2: Calculate $w_{k,100\%}$, the weight vector based on 100% confidence in the k^{th} view, using the unconstrained mean-variance optimization formula

$$w_{k,100\%} = (\lambda\Sigma)^{-1}E\left[R_{k,100\%}\right]$$

Black-Litterman model
Incorporating investor confidence levels

- Step 3: Calculate the deviations from the neutral market cap weights caused by 100% confidence in the k^{th} view.

$$D_{k,100\%} = w_{k,100\%} - w_{mkt}$$

- Step 4: Multiply $D_{k,100\%}$ by the user-specified confidence (C_k) in the k^{th} view to estimate the desired tilt caused by the k^{th} view.

$$Tilt_k = D_{k,100\%} \times C_k$$

- Step 5: Estimate the target weight vector $\left(w_{k,\%}\right)$ based on the tilt.

$$w_{k,\%} = w_{mkt} + Tilt_k$$

Black-Litterman model
Incorporating investor confidence levels

- Step 6: Find the value of Ω_{kk} (the k^{th} diagonal element of Ω), representing the uncertainty in the k^{th} view, that minimizes the sum of squared differences between $w_{k,\%}$ and w_k.

$$min_{\Omega_{kk}} \sum \left(w_{k,\%} - w_k\right)^2$$

$$subject\ to\ \Omega_{kk} > 0$$

Where $w_k = (\lambda\Sigma)^{-1}\left[(\tau\Sigma)^{-1} + P_k^T \Omega_{kk}^{-1} P_k\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P_k^T \Omega_{kk}^{-1} Q_k\right]$

- Repeat steps 1 to 6 for the k views, and build the $k \times k$ diagonal Ω matrix incorporating the user-specified confidence levels. The final BL portfolio can be calculated as usual using (6) and (7).

Generalized fundamental law of active management

- Clarke et al (2002) define the generalized fundamental law of active management as:

$$IR \approx TC \times IC \times \sqrt{N}$$

where TC is the transfer coefficient defined as the cross-sectional correlation between active weights and forecast returns, IC is the information coefficient defined as the cross-sectional correlation between forecast returns and realized returns, commonly used as a proxy for manager skill and N = number of independent bets.

Generalized fundamental law of active management

- In terms of expected active return, the generalized fundamental law of active management becomes:

$$E(R_A) \approx TC \times IC \times \sqrt{N} \times \sigma_A$$

where σ_A = active risk of the portfolio.

- In correlation form, the law can be expressed as:

$$PC = TC \times IC$$

where PC is the performance coefficient, defined as the expected correlation between active weights and subsequent returns.

The correlation triangle

Figure 2: The correlation triangle

Source: Clarke et al (2002)

What drives portfolio performance?

- **Manager skill (IC)**
- Number of independent bets (N)
- Active risk (σ_A)
- **Transfer coefficient (TC)**
- Transfer coefficient and number of positions are somewhat related

What drives the TC?

- The transfer coefficient (TC) can be explicitly written as:

$$TC \equiv \frac{Cov(\boldsymbol{\alpha}, \boldsymbol{w^e})}{\sqrt{Var(\boldsymbol{\alpha})}\sqrt{Var(\boldsymbol{w^e})}} = \frac{E\big(\boldsymbol{\alpha} - E(\boldsymbol{\alpha})\big)E\big(\boldsymbol{w^e} - E(\boldsymbol{w^e})\big)}{\sqrt{Var(\boldsymbol{\alpha})}\sqrt{Var(\boldsymbol{w^e})}}$$

where $\boldsymbol{\alpha}$ is the vector of expected return of the i^{th} asset and $\boldsymbol{w^e} = \boldsymbol{w} - \boldsymbol{w^b}$ is the weight difference between the portfolio and the benchmark.

$$TC \equiv \frac{\sum_{i=1}^{N}(w_i^e - \bar{w}^e)(\alpha_i - \bar{\alpha})}{\sqrt{\sum_{i=1}^{N}\left(w_i^e - \bar{w}^e\right)^2}\sqrt{\sum_{i=1}^{N}(\alpha_i - \bar{\alpha})^2}}$$

where N is the number of assets.

Active portfolio construction problem with TC constraint

- Consider the following multi-factor model:

$$R_i = \alpha + \sum_{k=1}^{K} \beta_{i,k} F_k + \varepsilon_i, i = 1,2, \dots, n$$

where $F_k, k = 1,2, \dots, K$ are factor returns, $\beta_{i,k}$ are factor exposures of the i^{th} asset and ε_i is the idiosyncratic return, uncorrelated to R_i.

- The tracking error, using the multi-factor model is then:

$$TE = \sqrt{\sum_{k=1}^{K} \sum_{l=1}^{K} \beta_k \beta_l \sigma_{kl} + \sum_{i=1}^{n} \sigma_{\varepsilon_i}^2 \left(w_i^e\right)^2}$$

Where $\beta_k = \sum_{i=1}^{n} \beta_{ik} w_i^e$, $\sigma_{\varepsilon_i}^2 = Var(\varepsilon_i)$ and $\sigma_{kl} = cov(f_k, f_l)$

MV optimization with TC constraint

Yamamoto et al (2012)

$$Max \sum_{i=1}^{n} \alpha_i w_i^e$$

$$\text{subject to} = \sqrt{\sum_{k=1}^{K}\sum_{l=1}^{K} \beta_k \beta_l \sigma_{kl} + \sum_{i=1}^{n} \sigma_{\varepsilon_i}^2 \left(w_i^e\right)^2} \leq \sigma_{Ub}$$

$$\beta_k = \sum_{i=1}^{n} \beta_{ik} w_i^e \quad , k = 1,2, \dots , K$$

$$\sum_{i=1}^{n} w_i^e = 0; \quad -1\% \leq w_i^e \leq 1\%, i = 1,2, \dots , n; \quad w_i^e \geq -w_i^b, i = 1,2, \dots , n;$$

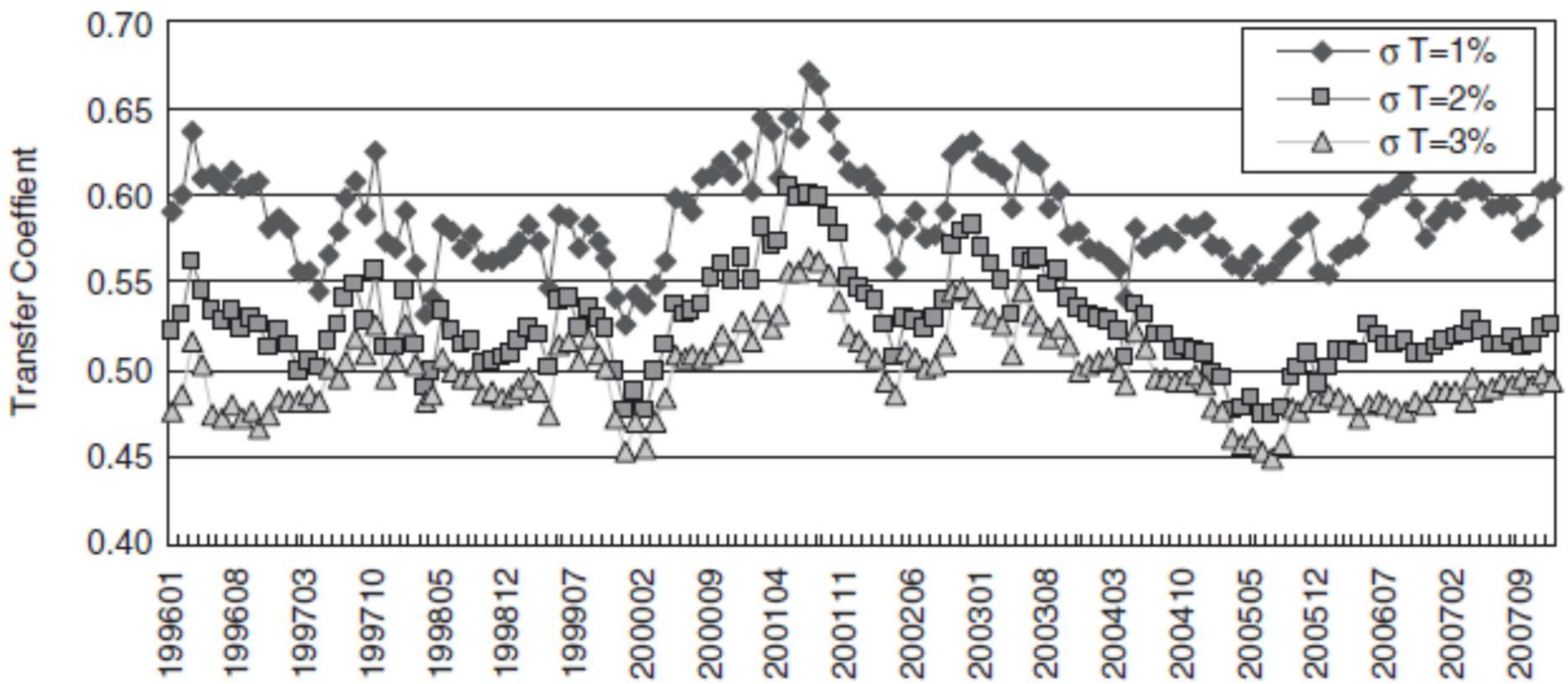
$$-1\% \leq \sum\nolimits_{i \in G_s} w_i^e \leq 1\% \text{ where } G_s \text{ is some sector grouping}$$

$$TC \equiv \frac{\sum_{i=1}^{N}(w_i^e - \bar{w}^e)(\alpha_i - \bar{\alpha})}{\sqrt{\sum_{i=1}^{N}\left(w_i^e - \bar{w}^e\right)^2}\sqrt{\sum_{i=1}^{N}(\alpha_i - \bar{\alpha})^2}} \geq \rho_{Lb}$$

Empirical study framework
Yamamoto et al (2012)

- Objective: To see if the addition of the TC constraint helps in portfolio construction.
- Data: Monthly data from Jan 1996-Dec 2007 of Tokyo stock exchange (TSE) consisting of around 1700 stocks.
- Use of Fama-French-Cahart 4 factor model as the risk model for tracking error calculation. Factor loadings estimated using 36 months of history.
- Alpha signals is a combination of price-to-earnings and price-to-book.
- Study carried out for TE of 1%, 2% and 3% annualized, and TC values of 0, 0.7, 0.8 and 0.9 (note that a TC value of 0 reverts back to the usual MV optimization)

TC profile of mean-variance optimized portfolios ($\rho = 0$)



Source: Yamamoto et al (2012) – Portfolio optimization under transfer coefficient constraint

Weight distribution under different TC values

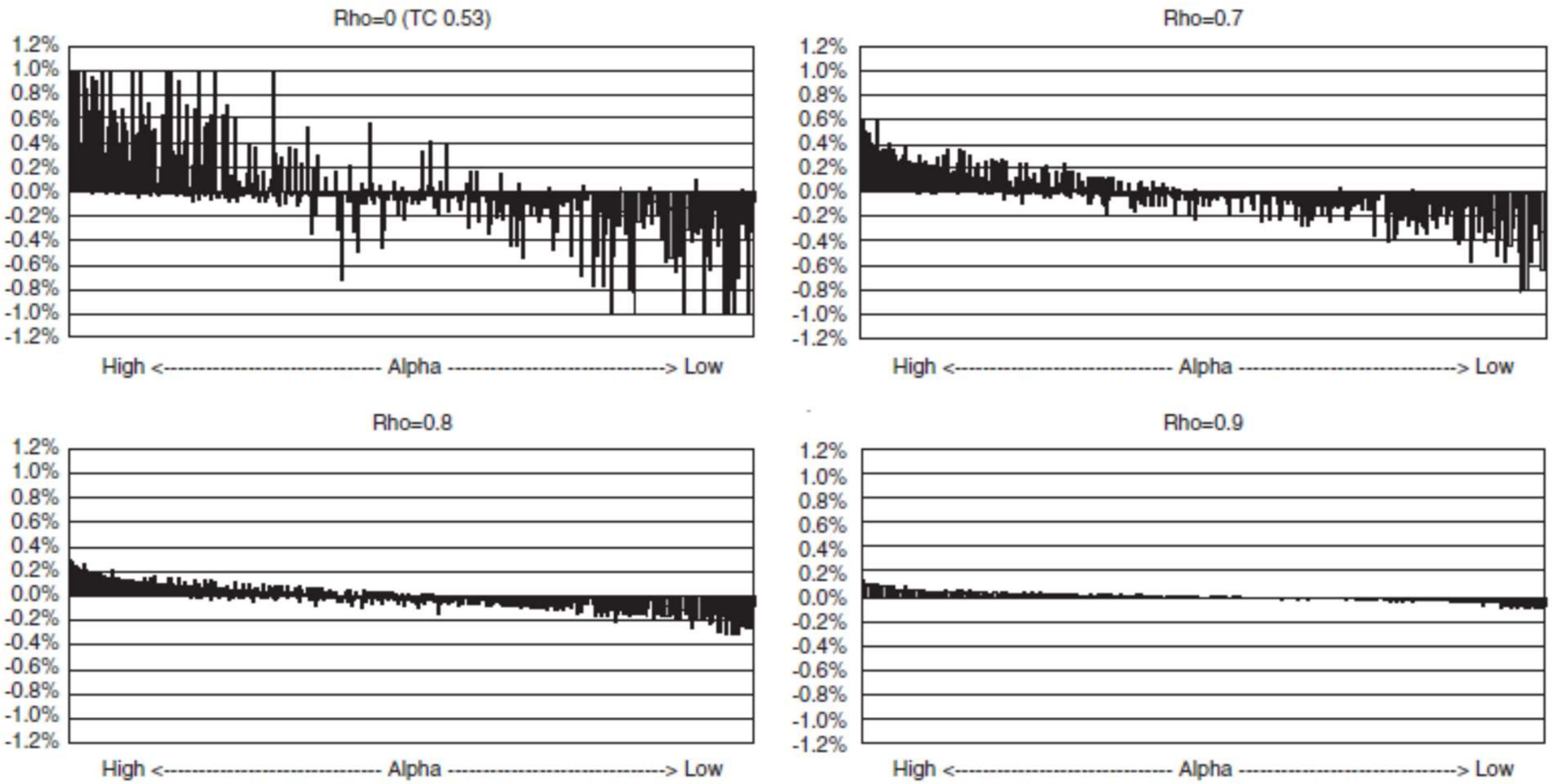


Figure 2: Portfolio weight ($\sigma_T = 2.0$ per cent, December, 2007).

Source: Yamamoto et al (2012) – Portfolio optimization under transfer coefficient constraint

Portfolio characteristics

Table 2: Average of portfolio characteristics

σ_T	ρ	*Objective value*	*Tracking error (%/ann.)*	*No. of assets*	*TC*
1 per cent	0.0	0.99	1.00	596.1	0.59
	0.7	0.92	1.00	649.1	0.70
	0.8	0.70	1.00	796.1	0.80
	0.9	0.33	1.00	1012.9	0.90
2 per cent	0.0	1.57	2.00	323.7	0.53
	0.7	1.25	2.00	542.4	0.70
	0.8	0.79	2.00	777.3	0.80
	0.9	0.34	1.97	1010.9	0.90
3 per cent	0.0	1.88	3.00	200.5	0.50
	0.7	1.34	3.00	524.6	0.70
	0.8	0.80	3.00	776.2	0.80
	0.9	0.35	2.87	1009.3	0.90

Source: Yamamoto et al (2012) – Portfolio optimization under transfer coefficient constraint

Ex-post performance

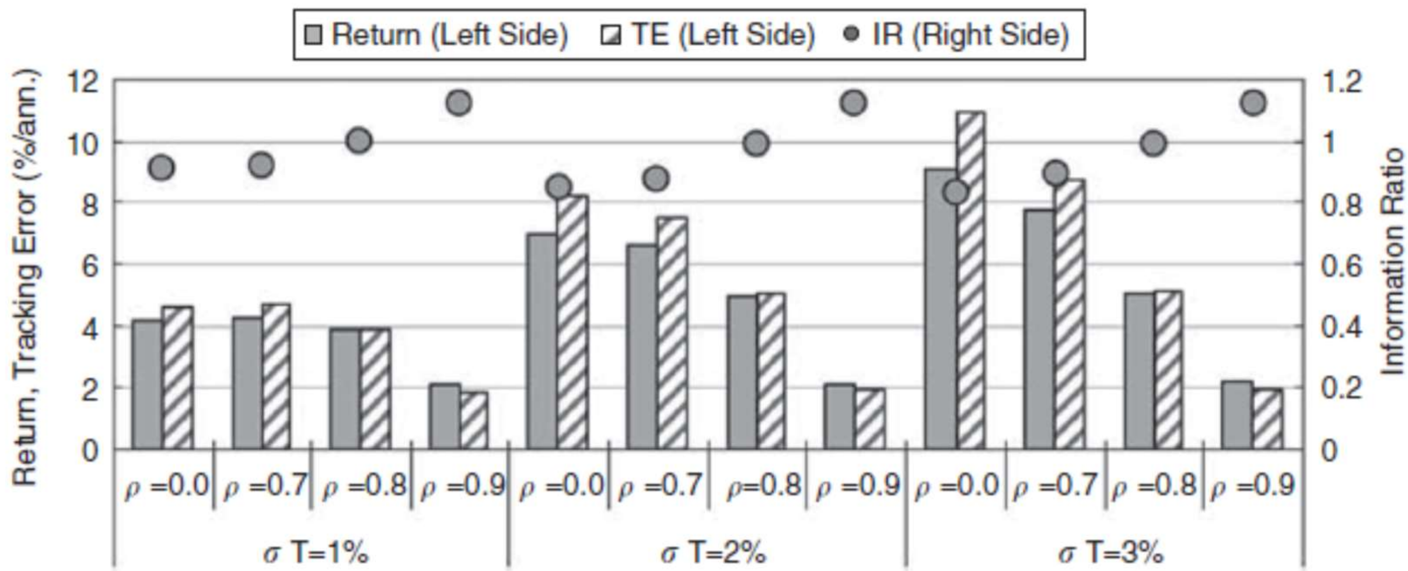


Figure 3: Ex-post performance.

Source: Yamamoto et al (2012) – Portfolio optimization under transfer coefficient constraint

Simulation framework

- For robustness, Yamamoto et al (2012) also conducted the same backtest but using randomly generated expected return vectors $\boldsymbol{\alpha} \in R^n$using $N(0,1)$ with $Cor(R_{t+1}, \alpha) = 0.03$.

- 100 backtests were run to smooth out the randomness.

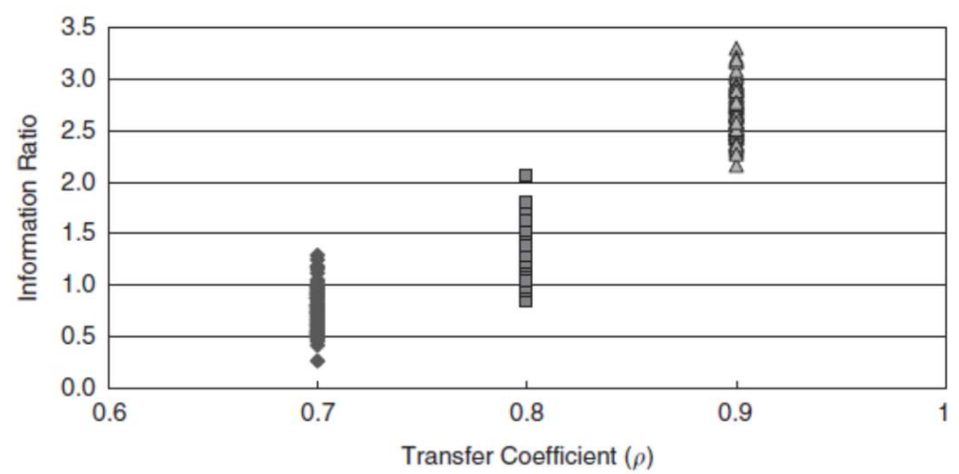


Figure 5: Information ratio ($\sigma_T = 2.0$ per cent).

Source: Yamamoto et al (2012) – Portfolio optimization under transfer coefficient constraint