Machine Learning and Financial Applications

Lecture 3
Modern Portfolio Management

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Learning outcomes

- Get to know modern portfolio theory
- Able to implement common portfolios
- Know the pros and cons of common portfolio strategies

w1 + w2 + w3 = 1 (budget constraint)

optional (wi >= 0 (no-short constraint))

objective

What is Portfolio Management?

Portfolio Risk and Return

Q: Do we have access to asset risk and return parameters?

n assets

Portfolio Return:

$$R_p = \sum_{i=1}^{n} w_i R_i$$

w^T R

- R_p: Expected return of the portfolio
- w_i: Weight of asset i in the portfolio
- R_i: Expected return of asset i

w^T sigma (w = [w1, w2, w3])

Portfolio Variance (Risk):

rho_ij < 1

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

- σ_p^2: Variance of portfolio returns
- σ_{ij}: Covariance between returns of asset i and asset j

Portfolio Standard Deviation (Volatility):

$$\sigma_p = \sqrt{\sigma_p^2}$$

- σ_p: Standard deviation (volatility) of the portfolio

Mean-Variance Optimization

max Rp s.t. sigma_p^2 <= sigma_0^2

$$\left(\text{Minimize } \sigma_p^2 \quad \text{subject to} \quad R_p \geq \overset{\text{constant}}{R_t}, \quad \sum_{i=1}^{n} w_i = 1 \right) \rightarrow w^*$$

- R_t: Target return
- Optimization problem balances risk (σ_p^2) and return (R_p).

Q:

- What is an alternative formulation?
- How does the risk aversion parameter come into the picture?

Risk-Adjusted Return

Efficient Frontier:

- Set of portfolios offering the maximum expected return for a given level of risk (or minimum risk for a given return).

Sharpe Ratio:

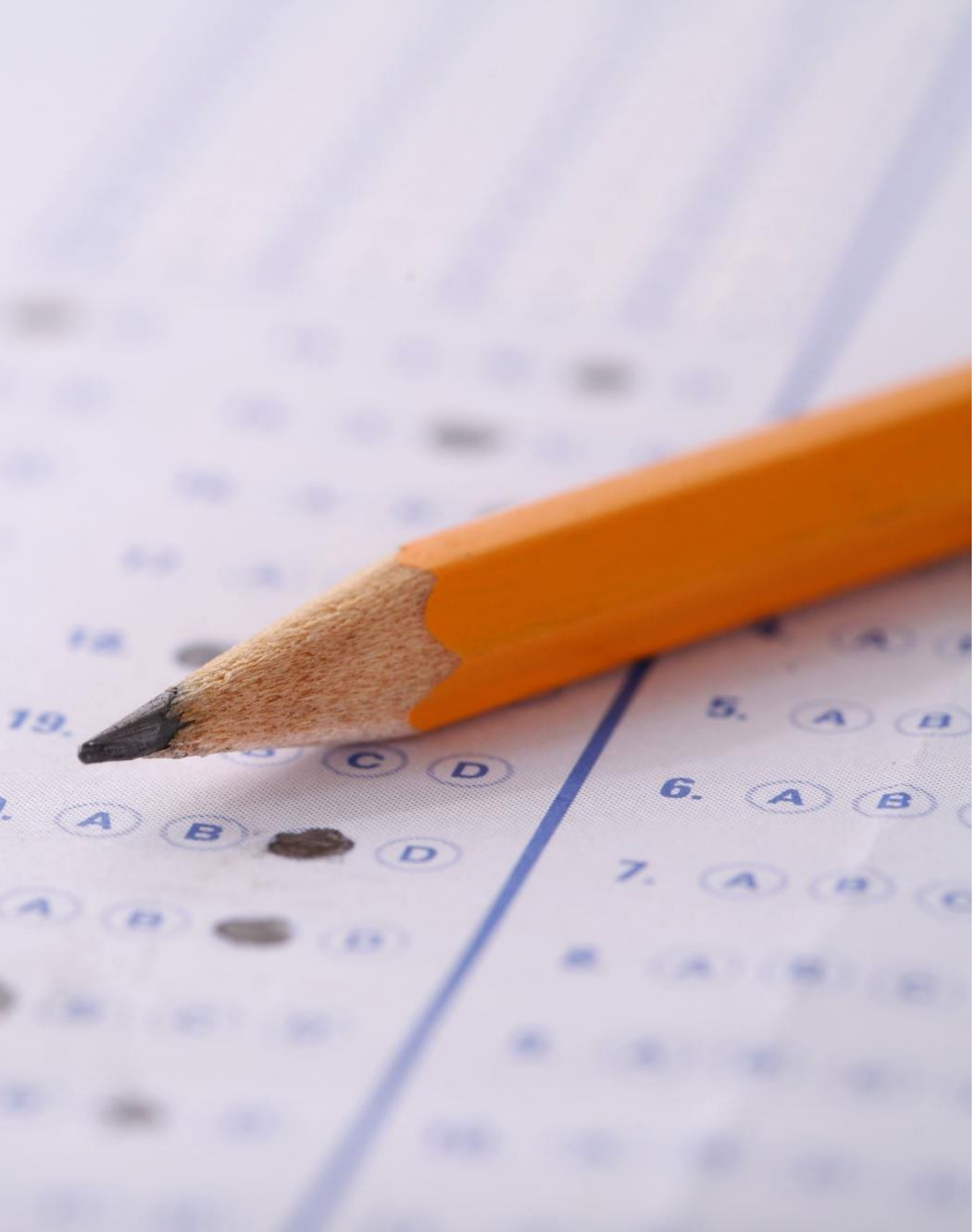
Rf (benchmark)

$$S_p = \frac{R_p - R_f}{\sigma_p}$$

excess return = Rp - Rf

- S_p: Sharpe ratio of the portfolio
- R_f: Risk-free rate
- Measures risk-adjusted return; higher values indicate better performance.

- Risk and return constitute two objectives
- Often there is a tradeoff in these two
- To analyze both metrics together, we can use:
  - Efficient frontier
  - Sharpe ratio
  - Etc…



In-class quiz

- Q1-3

Common Portfolio Strategies

- Maximum return portfolio
- Minimum variance portfolio
- Mean-variance portfolio
- Maximum Sharpe ratio portfolio
- Etc…

Group Discussion

How to properly evaluate and assess different portfolio strategies?

Maximum Return Portfolio (MRP)

Example:

mu1 = 10%, mu2 = 20%, mu3 = 15%

w = [0 1 0]

Objective: Maximize the expected portfolio return

Rp

$$\text{maximize}_{\mathbf{w}} \ \mathbf{w}^T \boldsymbol{\mu}$$

- $\mathbf{w}$: Weight vector of assets in the portfolio
- $\boldsymbol{\mu}$: Expected return vector of the assets

Constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

- Ensures the sum of the portfolio weights equals 1 (full investment).

Optimal Strategy:

- Allocate entire investment to the asset with the highest expected return.

Optimal Weight Vector ($\mathbf{w}_{MRP}$):

$$w_{MRP,j} = \begin{cases} 1 & \text{if } j = i^* \\ 0 & \text{otherwise} \end{cases}$$

- $i^* = \arg\max_j \{\mu_j\}$
- μ_j: Expected return of asset j
- Asset i^* has the highest expected return, so $w_{i^*} = 1$ and $w_j = 0$ for all $j \neq i^*$.

Implications:

- The MRP is risk-indifferent; it maximizes return without considering variance.
- It typically resides on one end of the efficient frontier, representing the highest potential return but also the highest risk.

Global Minimum Variance Portfolio (GMVP)

Q: Do we need to predict asset returns?

Objective: Minimize Portfolio Variance

$$\min_{\mathbf{w}} \ \mathbf{w}^T \Sigma \mathbf{w}$$

Constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

- Ensures the portfolio is fully invested.

Lagrangian Formulation:

$$\mathcal{L}(\mathbf{w}, \gamma) = \mathbf{w}^T \Sigma \mathbf{w} + \gamma(1 - \mathbf{w}^T \mathbf{1})$$

- γ: Lagrange multiplier for the constraint.

First-Order Condition:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = 2\Sigma \mathbf{w} - \gamma \mathbf{1} = 0$$

Solving for $\mathbf{w}$:

$$\mathbf{w} = \frac{\gamma}{2} \Sigma^{-1} \mathbf{1}$$

Determine γ Using Full Investment Constraint:

$$\frac{\gamma}{2} \mathbf{1}^T \Sigma^{-1} \mathbf{1} = 1 \implies \gamma = \frac{2}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

GMVP Weight Vector:

$$\mathbf{w}_{GMVP} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

Expected Return and Variance of GMVP

Expected Return of GMVP:

$$\mu_{GMVP} = \mathbf{w}_{GMVP}^T \boldsymbol{\mu}$$

- $\boldsymbol{\mu}$: Vector of expected returns for the assets.

Substitute GMVP Weights:

$$\mu_{GMVP} = \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right)^T \boldsymbol{\mu} = \frac{\mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

Variance of GMVP:

$$\sigma_{GMVP}^2 = \mathbf{w}_{GMVP}^T \Sigma \mathbf{w}_{GMVP}$$

Substitute GMVP Weights into Variance Expression:

$$\sigma_{GMVP}^2 = \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right)^T \Sigma \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right)$$

Simplify Variance Formula:

$$\sigma_{GMVP}^2 = \frac{1}{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})^2} \left(\mathbf{1}^T\Sigma^{-1}\Sigma\Sigma^{-1}\mathbf{1} \right)$$

Final Variance Expression:

$$\sigma_{GMVP}^2 = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

Mean-Variance Optimization (MVO)

Objective: Maximize the trade-off between expected return and risk for a portfolio of assets.

$$\max_{\mathbf{w}} \left(\mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2} \lambda \mathbf{w}^T \Sigma \mathbf{w} \right)$$

Subject to:

$$\mathbf{w}^T \mathbf{1} = 1$$

Parameters:

- $\mathbf{w}$: Portfolio weights for assets.
- $\boldsymbol{\mu}$: Vector of expected returns.
- Σ: Covariance matrix of returns.
- λ: Risk-aversion coefficient.

Interpretation: Balances maximizing expected returns and minimizing risk, controlled by λ.

Solving the MVO Using Lagrangian Multipliers

Lagrangian Function:

$$\mathcal{L}(\mathbf{w}, \gamma) = \mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2} \lambda \mathbf{w}^T \Sigma \mathbf{w} + \gamma (1 - \mathbf{w}^T \mathbf{1})$$

First-Order Conditions (FOCs):

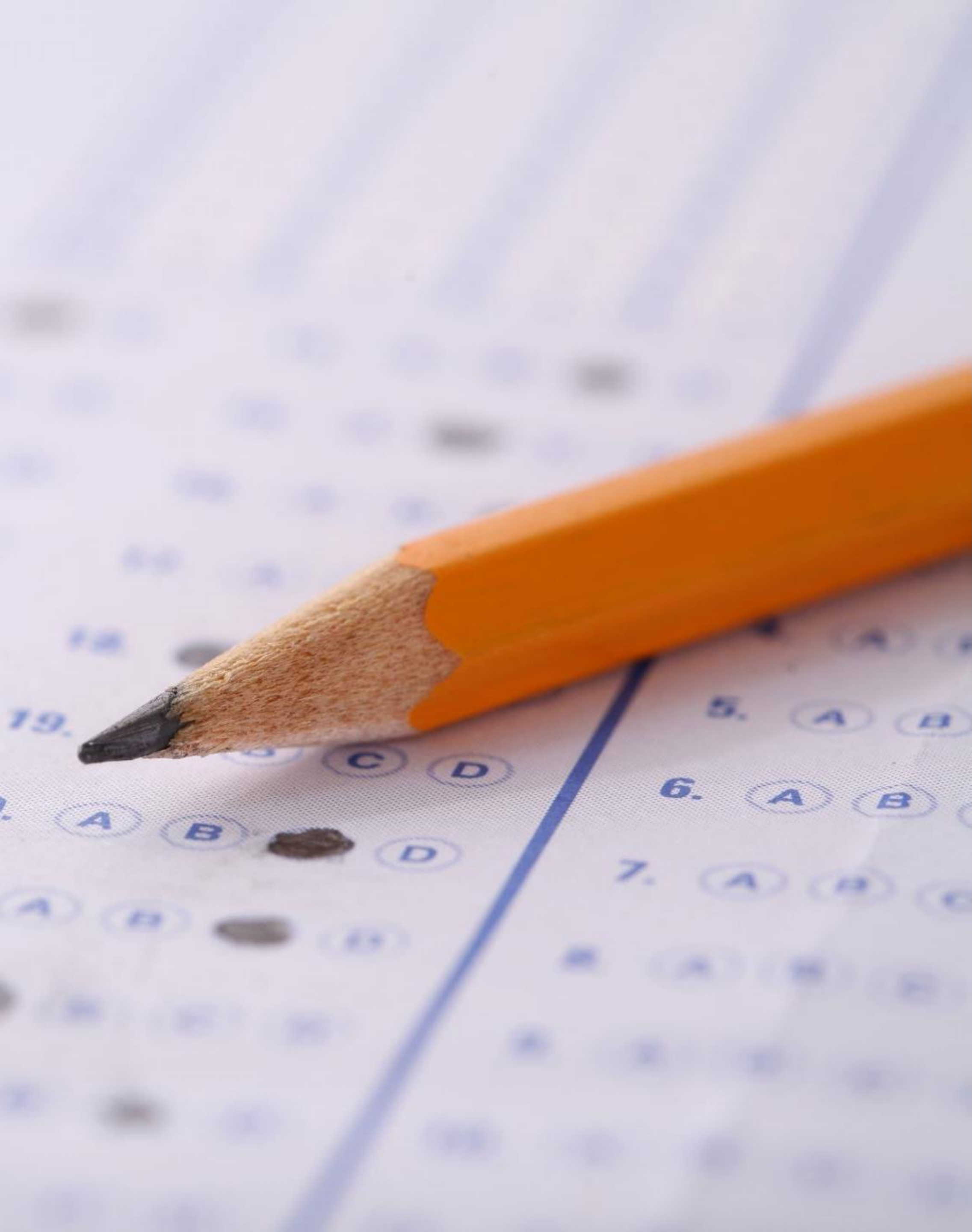
$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \boldsymbol{\mu} - \lambda \Sigma \mathbf{w} - \gamma \mathbf{1} = 0$$

Optimal Weights:

$$\mathbf{w} = \frac{1}{\lambda} \Sigma^{-1} \boldsymbol{\mu} - \frac{\gamma}{\lambda} \Sigma^{-1} \mathbf{1}$$

Determine γ using Budget Constraint:

$$\gamma = \frac{1}{\lambda} \left(\boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1} - \lambda \right) \Big/ \left(\mathbf{1}^T \Sigma^{-1} \mathbf{1} \right)$$



In-class quiz

- Q4-7

Maximum Sharpe Ratio Portfolio (MSRP)

Objective: Maximize Sharpe Ratio

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p}$$

- R_p: Portfolio return
- R_f: Risk-free rate
- σ_p: Portfolio standard deviation (risk)

Optimization Problem:

$$\max_{\mathbf{w}} \frac{\mathbf{w}^T \boldsymbol{\mu} - R_f}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}}$$

- $\boldsymbol{\mu}$: Expected returns vector
- Σ: Covariance matrix of asset returns
- Subject to: $\mathbf{w}^T \mathbf{1} = 1$

Challenge:

- Non-linear objective function due to square root in the denominator, making the problem potentially non-convex.

Workaround - Using Excess Returns:

$$\max_{\mathbf{w}} \left(\mathbf{w}^T (\boldsymbol{\mu} - R_f \mathbf{1}) - \frac{\boldsymbol{\lambda}}{2} \mathbf{w}^T \Sigma \mathbf{w} \right)$$

- Simplifies the optimization problem using excess returns over the risk-free rate.

Maximum Sharpe Ratio Portfolio (MSRP) (optimal)

Lagrangian Formulation:

$$\mathcal{L}(\mathbf{w}) = \mathbf{w}^T(\boldsymbol{\mu} - R_f\mathbf{1}) - \frac{\lambda}{2}\mathbf{w}^T\Sigma\mathbf{w}$$

- λ: Risk aversion parameter

First-Order Condition (FOC):

$$\nabla_{\mathbf{w}}\mathcal{L} = (\boldsymbol{\mu} - R_f\mathbf{1}) - \lambda\Sigma\mathbf{w} = 0$$

- Solve for $\mathbf{w}$ to find the unconstrained solution:

$$\mathbf{w}^*_{unconstrained} = \frac{1}{\lambda}\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})$$

Normalization for Full Investment:

$$\mathbf{w}_{MSRP} = \frac{\mathbf{w}^*_{unconstrained}}{\mathbf{1}^T\mathbf{w}^*_{unconstrained}} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}$$

An Alternative Derivation for MSRP (optional)

Portfolio Composition:

- N risky assets with return vector $\mathbf{r} = [r_1, r_2, \ldots, r_N]^T$.
- One risk-free asset (e.g., Treasury bills) with constant return R_f.

Portfolio Weights:

- Weights for risky assets: $\mathbf{w} \in \mathbb{R}^N$.
- Weight for risk-free asset: w_f.
- Budget constraint: $\mathbf{w}^T\mathbf{1} + w_f = 1$.

Portfolio Return:

$$r_p = \mathbf{w}^T\mathbf{r} + w_f R_f = \mathbf{w}^T(\mathbf{r} - R_f\mathbf{1}) + R_f$$

Excess Return:

$$\tilde{r}_p = r_p - R_f = \mathbf{w}^T(\mathbf{r} - R_f\mathbf{1}) = \mathbf{w}^T\tilde{\mathbf{r}}$$

- $\tilde{\mathbf{r}} = \mathbf{r} - R_f\mathbf{1}$: Vector of excess return random variables.

Expected Excess Return and Variance:

$$\tilde{\mu}_p = \mathbf{w}^T\tilde{\boldsymbol{\mu}}, \quad \sigma_p^2 = \mathbf{w}^T\Sigma\mathbf{w}$$

- $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - R_f\mathbf{1}$: Vector of expected excess returns.

Optimization Problem:

$$\min_{\mathbf{w}} \ \mathbf{w}^T\Sigma\mathbf{w}, \quad \text{subject to } \mathbf{w}^T\tilde{\boldsymbol{\mu}} = \tilde{\mu}_0$$

Lagrangian Function:

$$\mathcal{L}(\mathbf{w}, \lambda) = \mathbf{w}^T\Sigma\mathbf{w} + \lambda(\tilde{\mu}_0 - \mathbf{w}^T\tilde{\boldsymbol{\mu}})$$

First-Order Condition (FOC):

- Derivative w.r.t. $\mathbf{w}$:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{w}} = 2\Sigma\mathbf{w} - \lambda\tilde{\boldsymbol{\mu}} = 0$$

Solve for $\mathbf{w}$:

$$\mathbf{w} = \frac{1}{2}\lambda\Sigma^{-1}\tilde{\boldsymbol{\mu}}$$

Determine λ Using Constraint:

- Derivative w.r.t. λ:

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \tilde{\mu}_0 - \mathbf{w}^T\tilde{\boldsymbol{\mu}} = 0$$

- Substitution yields:

$$\lambda = \frac{2\tilde{\mu}_0}{\tilde{\boldsymbol{\mu}}^T\Sigma^{-1}\tilde{\boldsymbol{\mu}}}$$

Optimal Weights:

$$\mathbf{w}^* = \tilde{\mu}_0\frac{\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}^T\Sigma^{-1}\tilde{\boldsymbol{\mu}}}$$

Applying Full Investment Constraint:

- For $\mathbf{1}^T\mathbf{w} = 1$:

$$\mathbf{w}_{MSRP} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}$$

- Confirms the closed-form solution for MSRP.

Connecting MVO Weights with Other Portfolios

Final Expression for Optimal Weights:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \left(\frac{1}{\lambda}\left(\boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1} - \lambda\right) \Big/ \left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)\right)\Sigma^{-1}\mathbf{1}$$

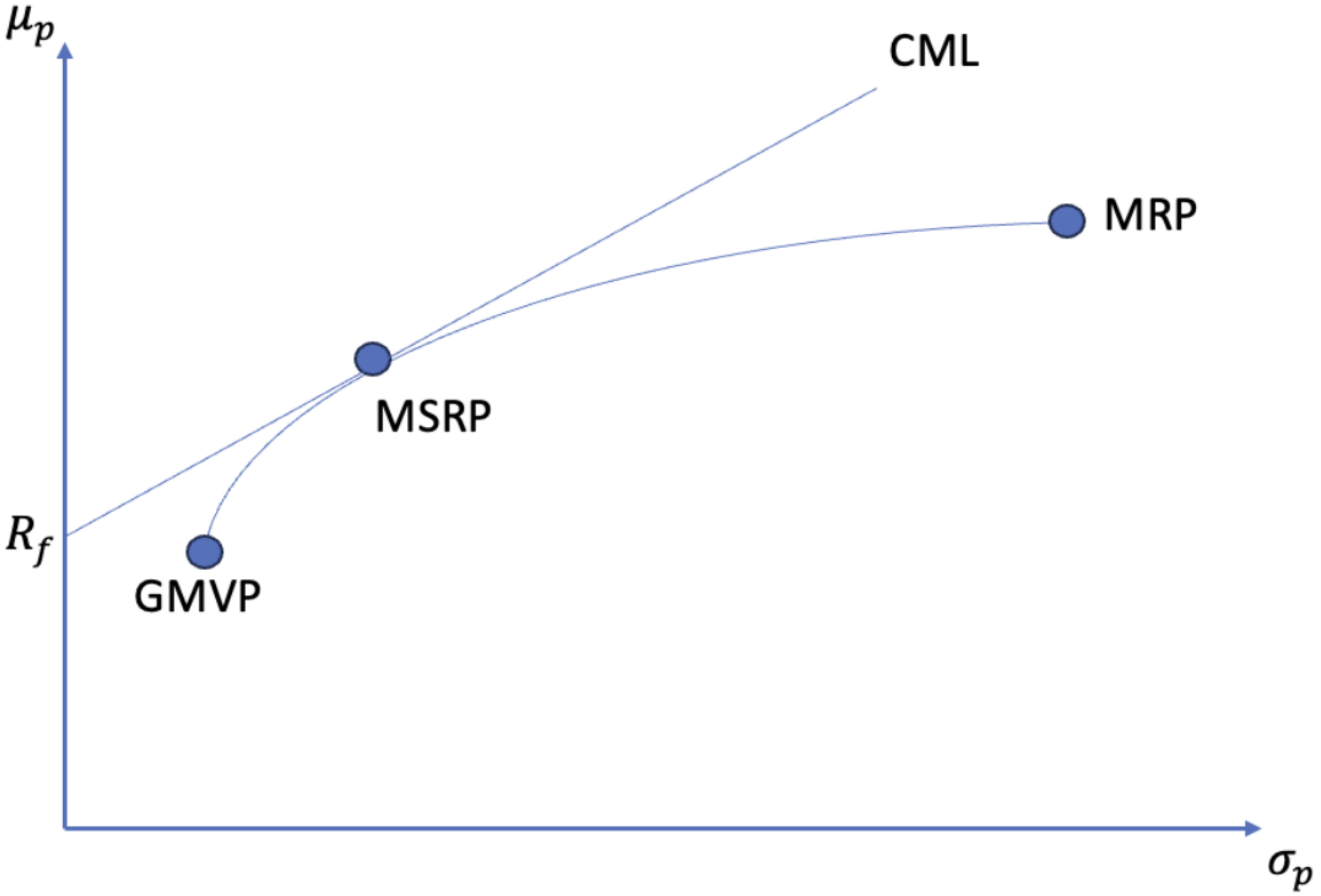
Weights as a Combination of GMVP and MSRP:

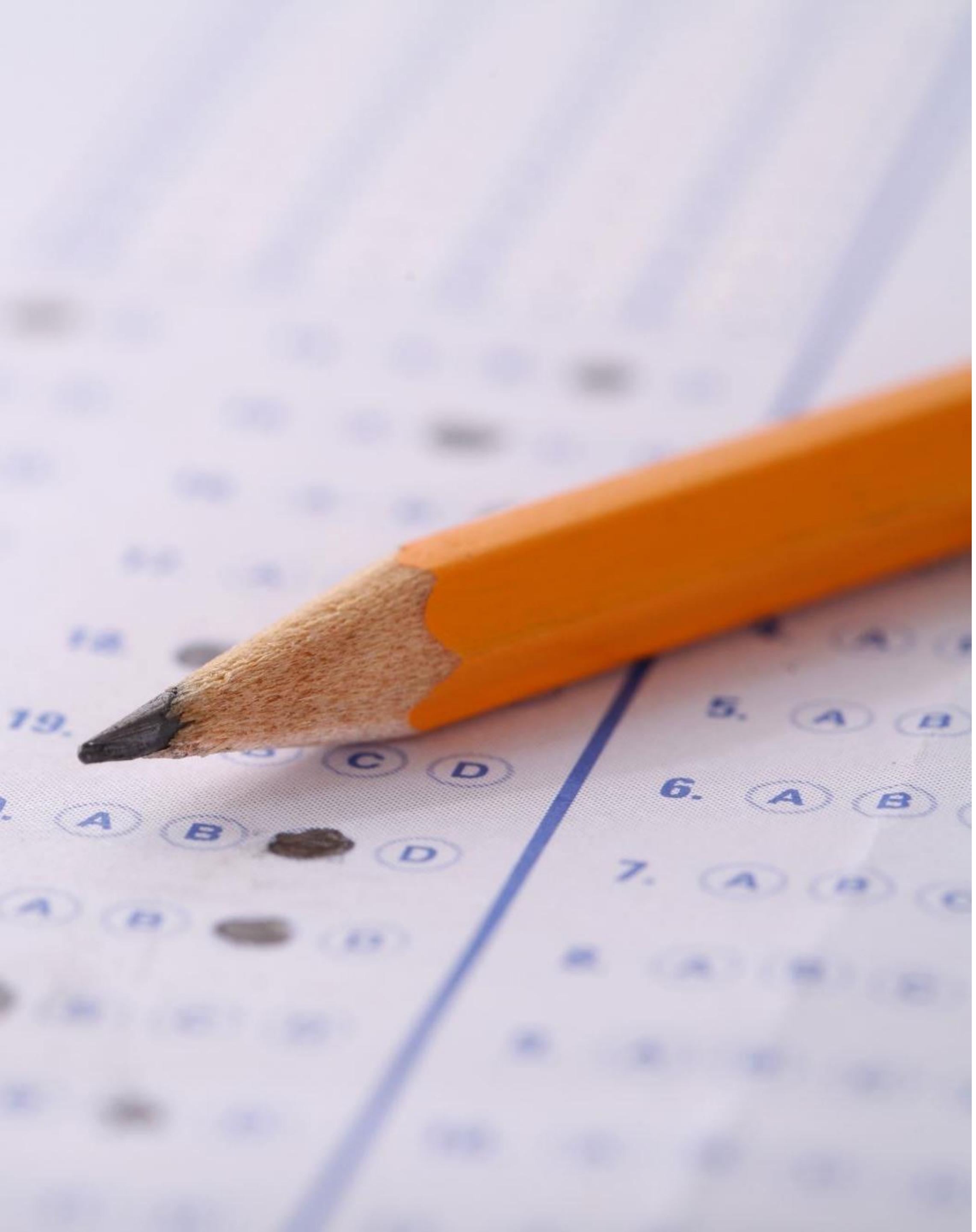
$$\mathbf{w}_{MVO} = \mathbf{w}_{GMVP} + \left(\frac{\mu_{GMVP}}{\lambda\sigma^2_{GMVP}}\right)\left(\mathbf{w}_{MSRP} - \mathbf{w}_{GMVP}\right)$$

Key Insights:

- $\mathbf{w}_{MVO}$ is a blend of the Global Minimum Variance Portfolio (GMVP) and Maximum Sharpe Ratio Portfolio (MSRP).
- The allocation adjustment between GMVP and MSRP depends on the risk-return trade-off parameter λ, GMVP's expected return (μ_{GMVP}), and its variance (σ^2_{GMVP}).

Comparing Common Portfolios





In-class quiz

- Q8-10

Coding session

Group Homework – Implementing Common Portfolio Strategies

- Obtain financial data (at least three assets) of your own choice
- Strategy Implementation:
  - Implement at least three strategies
  - Use rolling window approach to obtain out-of-sample results
  - Report final performance metrics (CAGR, annual volatility, Sharpe ratio, and max drawdown)

Due one day before class starts next week

Homework

- Watch/review video tutorials and class recording
- Post learning reflections and questions if any
- Complete group homework