An Overview of Common Portfolio Selection Strategies

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In the realm of financial decision-making, obtaining a nuanced comprehension of the various portfolio selection methodologies is pivotal for penetrating the complexities inherent in the decision-making dynamics. Such dynamics are often modeled based on the first two statistical moments of the return distribution: mean (expected return) and variance (risk). These strategies exhibit distinctions in their application within the objective function—commonly referred to as the utility function—and the constraints delineated during the formulation of the optimization problem. This chapter aims to cover prevalent portfolio selection strategies, commencing with an exposition on the Mean-Variance Optimization (MVO) framework and its associated variants. These include the Maximum Return Portfolio, which endeavors to maximize expected returns, and the Global Minimum Variance Portfolio, which seeks to minimize risk without regard for expected returns, and the Tangency Portfolio (also called the Maximum Sharpe Ratio Portfolio), where the solution lies at the tangent point between the Capital Market Line (CML) and the efficient frontier.

Subsequently, we will transition the Index Tracking Portfolio strategy, which focuses on minimizing the tracking error relative to a specified benchmark, allowing for the replication of benchmark performance while managing deviation risk.

Moreover, we will extend to the Mean-CVaR Portfolio strategy, which imposes a constraint on the Conditional Value at Risk (CVaR). CVaR is increasingly recognized as a more comprehensive measure of downside risk, as it not only considers the likelihood of extreme losses but also the magnitude of those losses beyond a certain threshold. This strategy is particularly beneficial for investors seeking to mitigate the impact of tail-end risks on their portfolios.

Through the lens of these strategies, this chapter aims to provide an in-depth understanding of how different portfolio selection methodologies are theoretically underpinned and practically applied within the domain of financial optimization, offering a granular perspective on the strategic nuances that guide portfolio construction and optimization.

1 Mean-Variance Optimization

Mean-Variance Optimization (MVO), first proposed by Markowitz in his seminal 1952 work [Mar52], fundamentally revolutionized portfolio management by introducing a quantitative framework to evaluate the risk-return tradeoff inherent in portfolio selection. At its core, MVO seeks to achieve an optimal balance between expected portfolio return and risk, defined as the variance of portfolio returns. This optimization can be pursued in two primary forms: maximizing portfolio return for a given level of risk or minimizing risk for a specified minimum return.

Let us start with the minimum variance approach and try to derive its closed-form solution. Besides minimizing the portfolio variance, two constraints are considered: equating the portfolio return to a target return, and making the portfolio weights sum to one. The optimization problem can be expressed as:

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \mathbf{w}^T \Sigma \mathbf{w} \\
\text{s.t.} \quad & \mathbf{w}^T \boldsymbol{\mu} = \mu_0 \\
& \mathbf{w}^T \mathbf{1} = 1
\end{aligned} \tag{1}
$$

We can obtain the following Lagrangian:

$$
\mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^T \Sigma \mathbf{w} + \lambda_1(\mu_0 - \mathbf{w}^T \boldsymbol{\mu}) + \lambda_2(1 - \mathbf{w}^T \mathbf{1})
$$

Continuing from the formulation of the Lagrangian, to obtain the first-order conditions (FOCs) necessary for finding the optimal solution to the minimum variance portfolio problem, we differentiate the Lagrangian function $\mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2)$ with respect to the portfolio weights vector $\mathbf{w}$ and each of the Lagrange multipliers (λ_1 and λ_2). The differentiation yields:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = 2\Sigma\mathbf{w} - \lambda_1\boldsymbol{\mu} - \lambda_2\mathbf{1} = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \mu_0 - \mathbf{w}^T\boldsymbol{\mu} = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 1 - \mathbf{w}^T\mathbf{1} = 0. \tag{4}$$

These conditions are essential for determining the critical points of the Lagrangian, which correspond to the portfolio weights that minimize variance while satisfying the constraints on portfolio return and total weight.

To proceed with the derivation, we will substitute the expression for $\mathbf{w}$ into the second and third FOCs to solve for the Lagrange multipliers λ_1 and λ_2. Recall that the second FOC equates the portfolio's expected return to the target return μ_0, and the third FOC ensures that the sum of the portfolio weights equals one. Substituting the expression for $\mathbf{w}$ derived from the differentiation of the Lagrangian with respect to $\mathbf{w}$, we have:

$$\mathbf{w}^* = \frac{1}{2}\lambda_1\Sigma^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2\Sigma^{-1}\mathbf{1},$$

into the second and third conditions, we obtain:

$$\mu_0 = \left(\frac{1}{2}\lambda_1\Sigma^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2\Sigma^{-1}\mathbf{1}\right)^T \boldsymbol{\mu}, \tag{5}$$

$$1 = \left(\frac{1}{2}\lambda_1\Sigma^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2\Sigma^{-1}\mathbf{1}\right)^T \mathbf{1}. \tag{6}$$

By rearranging these equations, we aim to solve for λ_1 and λ_2. Let's denote $A = \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}$, $B = \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1}$ (by symmetry of Σ^{-1}), and $C = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, where $\mathbf{1}$ is the vector of ones. These scalars represent the inverse-variance weighted sums of expected returns and unity, respectively, facilitating a simplified representation of the system.

Substituting these definitions into the system yields:

$$\mu_0 = \frac{1}{2}\lambda_1 A + \frac{1}{2}\lambda_2 B, \tag{7}$$

$$1 = \frac{1}{2}\lambda_1 B + \frac{1}{2}\lambda_2 C. \tag{8}$$

To solve for λ_1 and λ_2, we find:

$$\lambda_1^* = \frac{-2B + 2C\mu_0}{AC - B^2}, \tag{9}$$

$$\lambda_2^* = \frac{2A - 2B\mu_0}{AC - B^2}. \tag{10}$$

The denominators $AC - B^2$ ensure that the system of equations is solvable, provided that this determinant is non-zero, which is a requisite condition for the existence of a unique solution in such a linear system.

Now plugging λ_1^* and λ_2^* into the expression of $\mathbf{w}^*$, we have:

$$\mathbf{w}^* = \frac{1}{2}\left(\frac{-2B + 2C\mu_0}{AC - B^2}\right)\Sigma^{-1}\boldsymbol{\mu} + \frac{1}{2}\left(\frac{2A - 2B\mu_0}{AC - B^2}\right)\Sigma^{-1}\mathbf{1},$$

Simplifying the expression by distributing $\Sigma^{-1}\boldsymbol{\mu}$ and $\Sigma^{-1}\mathbf{1}$ and combining terms gives us the closed-form solution for the optimal weights:

$$\mathbf{w}^* = \left(\frac{C\mu_0 - B}{AC - B^2}\right)\Sigma^{-1}\boldsymbol{\mu} + \left(\frac{A - B\mu_0}{AC - B^2}\right)\Sigma^{-1}\mathbf{1},$$

This solution precisely characterizes the distribution of portfolio weights across the risky assets and the risk-free asset that minimizes the portfolio variance for a given target return μ_0, subject to the constraints that the expected portfolio return meets the target and the sum of the weights equals one.

An alternative representation of MVO using risk aversion parameter λ is formulated as follows:

where $\mathbf{w}$ represents the portfolio weights. The constraint ensures that the portfolio is fully invested $(\mathbf{w}^T\mathbf{1} = 1)$.

Let us discuss a few special portfolios based on this formulation, including the Maximum Return Portfolio (MRP), the Global Minimum Variance Portfolio (GMVP), and the Maximum Sharpe Ratio Portfolio (MSRP). Note that in all these portfolios, we can optionally add a no-short-selling constraint, that is, $\mathbf{w} \geq 0$.

1.1 Implementing the Mean-Variance Portfolio

We now attempt to implement the mean-variance portfolio. We start by selecting a representative set of stocks from various sectors, including technology, healthcare, financial services, industrials, and energy. This selection aims to capture a wide range of risk-return profiles to exploit the benefits of diversification, as advocated by Modern Portfolio Theory (MPT). The stocks chosen—Microsoft (MSFT), Johnson & Johnson (JNJ), JPMorgan Chase (JPM), The Walt Disney Company (DIS), Boeing (BA), and Chevron (CVX)—span essential sectors of the economy, providing a comprehensive view of the market dynamics over the selected time range from January 1, 2013, to January 1, 2024.

After downloading the stock price data, we focus on extracting and processing the adjusted close prices. We then linearly interpolate missing values and compute daily log returns, which will later serve as inputs for calculating expected returns, variances, and covariances needed to solve the mean-variance optimization problem. See the following code list for stock data download and processing.

```

import yfinance as yf
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

# Set the precision to 5 decimal places
pd.set_option('display.precision', 5)

# Set the precision to 5 decimal places for numpy
np.set_printoptions(precision=5)

# Define the stock symbols for different sectors
stock_symbols = ['MSFT', 'JNJ', 'JPM', 'DIS', 'BA', 'CVX']

# Define the date range
start_date = '2013-01-01'
end_date = '2024-01-01'

# Download data for each stock symbol
# We will store the adjusted close prices in a DataFrame
prices = pd.DataFrame()

for symbol in stock_symbols:
    ticker = yf.Ticker(symbol)
    hist_data = ticker.history(start=start_date, end=end_date)
    # Extracting adjusted close prices and renaming the column to the stock symbol
    prices[symbol] = hist_data['Close'].rename(symbol)

# Handling missing values by interpolation
prices.interpolate(method='linear', inplace=True)

# Assuming 'prices' is the DataFrame obtained from the previous step that contains adjusted
    close prices
```

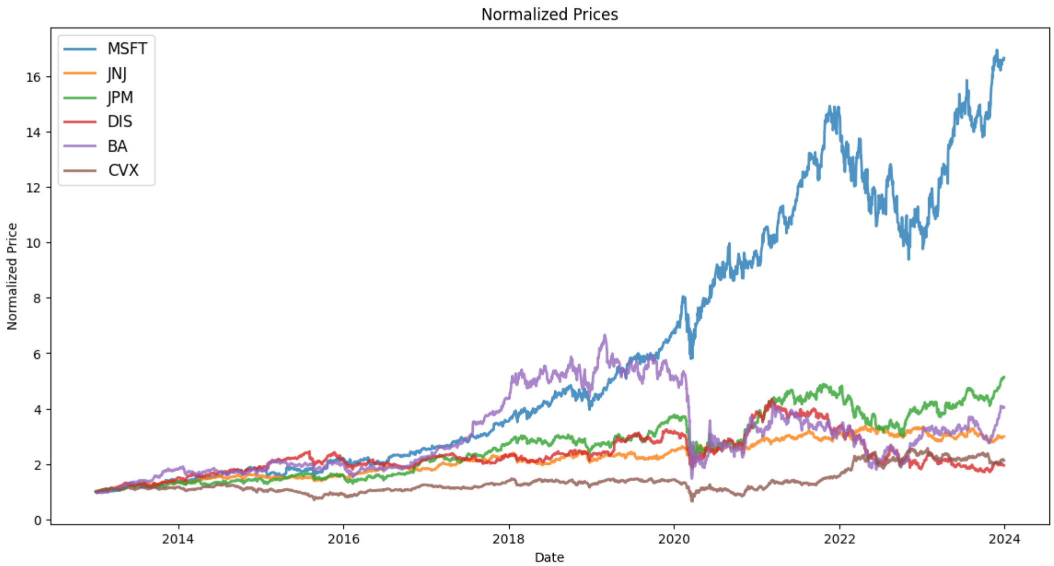

Figure 1: Normalized price curve of each stock.

```python

# Compute the data matrix with log-returns
log_returns = np.log(prices).diff().iloc[1:]

# Number of stocks and number of days
N = log_returns.shape[1]  # Number of columns (stocks)
T = log_returns.shape[0]  # Number of rows (days)

# Normalize the prices
normalized_prices = prices / prices.iloc[0]

# Plot the normalized prices
plt.figure(figsize=(14, 7))
for c in normalized_prices.columns.values:
    plt.plot(normalized_prices.index, normalized_prices[c], lw=2, alpha=0.8, label=c)

plt.legend(loc='upper left', fontsize=12)
plt.title('Normalized Prices')
plt.xlabel('Date')
plt.ylabel('Normalized Price')
plt.show()
```

Listing 1: Stock data downloading and processing.

Running these commands generates Figure 1, where we plot the normalized price curves of each stock. The figure suggests that Microsoft has been showing the highest return compared to other stocks, despite a high degree of volatility between 2022 and 2023.

Next, we calculate the sample mean and the covariance matrix of the daily log returns, and then adapt these metrics to an annual scale to align with long-term investment strategies. The following code snippet illustrates the computation process, translating daily statistics into their annual counterparts.

```python
# Calculate the sample mean vector of the log returns
mu_daily = log_returns.mean()

# Calculate the sample covariance matrix of the log returns
Sigma_daily = log_returns.cov()

# Convert daily values to annual
mu = mu_daily * 252
Sigma = Sigma_daily * 252

# Display the results
print("Annual expected return vector (mu):\n", mu)
```

```
print("\nAnnual covariance matrix (Sigma):\n", Sigma)
```

Listing 2: Calculating annual expected return and covariance matrix.

Executing the provided commands generates Table \ref{tab:expected_return}, showcasing the annual expected return vector, and Table \ref{tab:cov_matrix}, displaying the annual covariance matrix. These sample-based estimates are empirical approximations of expected returns and risk and play an important role in various portfolio strategies.

Table 1: Annual Expected Return Vector (μ)

Stock	Expected Return
MSFT	0.25618
JNJ	0.09984
JPM	0.14912
DIS	0.06082
BA	0.12707
CVX	0.06859

Table 2: Annual Covariance Matrix (Σ)

	MSFT	JNJ	JPM	DIS	BA	CVX
MSFT	0.07216	0.01861	0.03175	0.03211	0.03919	0.02738
JNJ	0.01861	0.03100	0.01916	0.01538	0.02041	0.01852
JPM	0.03175	0.01916	0.06906	0.03839	0.05680	0.04511
DIS	0.03211	0.01538	0.03839	0.07003	0.05032	0.03268
BA	0.03919	0.02041	0.05680	0.05032	0.14485	0.05456
CVX	0.02738	0.01852	0.04511	0.03268	0.05456	0.08165

Using the mean-variance optimization, we aim to construct a portfolio that not only aligns with a specified risk-return profile but also adheres to the principles of MPT. As part of our strategic approach, we introduce a target portfolio return of 20% annually as a crucial constraint. This target serves as a benchmark for expected performance, guiding the optimization process. Additionally, we enforce a full investment constraint to ensure the portfolio weights sum to unity, reflecting a fully invested portfolio without leverage. The core of our optimization lies in the objective function designed to minimize the portfolio's variance, thereby optimizing the trade-off between risk and return.

```
from scipy.optimize import minimize

# Define the target portfolio return
mu_0 = 0.2  # 20% target return

# Define the objective function for portfolio variance minimization
def objective_function_mvo(w, Sigma):
    return w.T @ Sigma @ w  # Portfolio variance

# Constraints include achieving the target return and ensuring full investment
constraints = [
    {'type': 'eq', 'fun': lambda w: w @ mu - mu_0},  # Target portfolio return constraint
    {'type': 'eq', 'fun': lambda w: np.sum(w) - 1}   # Full investment constraint
]

# Initial guess with equal weighting across assets
w0 = np.ones(len(mu)) / len(mu)

# Bounds to prevent short selling by constraining weights between 0 and 1
bounds = tuple((0, 1) for _ in range(len(mu)))

# Perform the optimization using Sequential Least Squares Programming (SLSQP)
result = minimize(fun=objective_function_mvo, x0=w0, args=(Sigma,), method='SLSQP',
    constraints=constraints, bounds=bounds)

# Extract the optimal portfolio weights
```

```
w_star = result.x

# Display the calculated optimal weights
print("Optimal weights (w*):\n", w_star)

===== OUTPUT =====
Optimal weights (w*):
 [6.09207e-01 2.90990e-01 9.98038e-02 7.15573e-18 2.12504e-17 0.00000e+00]
```

Listing 3: Executing Mean-Variance Optimization.

The analysis suggests allocating a substantial portion of the portfolio to the first asset, likely due to its superior return and acceptable risk profile. To confirm that the portfolio meets the target return constraint, we can utilize the following code snippet.

```
# Calculate the portfolio return
portfolio_return = np.dot(w_star, mu)
print("Expected Portfolio Return:", portfolio_return)
===== OUTPUT =====
Expected Portfolio Return: 0.20000000004529658
```

Listing 4: Printing the optimized target return

This output confirms that the target return constraint has been met successfully.

2 Maximum Return Portfolio

The Maximum Return Portfolio (MRP) targets achieving the highest possible expected return from the set of available investments. This portfolio does not primarily concern itself with the risk (variance) associated with the returns but focuses on maximizing the expected return. The formulation for the MRP is straightforward:

$$\max_{\mathbf{w}} \mathbf{w}^T \boldsymbol{\mu}$$

subject to:

$$\mathbf{w}^T \mathbf{1} = 1,$$

where the objective is to maximize the expected portfolio return $\mathbf{w}^T \boldsymbol{\mu}$ subject to the constraint that the sum of portfolio weights equals 1. Given its indifference to risk, the optimal strategy under this formulation typically allocates the entirety of the investment to the asset with the maximum expected return, assigning zero weights to all others. Mathematically, if we denote the asset with the highest expected return by i^*, then the optimal weight vector $\mathbf{w}_{MRP}$ can be expressed as:

$$w_{MRP,j} = \begin{cases} 1 & \text{if } j = i^*, \\ 0 & \text{otherwise}, \end{cases}$$

where $i^* = \arg\max_j \{\mu_j\}$ and μ_j is the expected return of asset j. This means that for the asset i^* with the highest expected return, $w_{i^*} = 1$, and for all other assets $j \neq i^*$, $w_j = 0$. This portfolio typically resides at one end of the efficient frontier, representing the highest expected return but also potentially carrying the highest risk.

2.1 Implementing the Maximum Return Portfolio

The formulation of the Maximum Return Portfolio (MRP) is straightforward. It involves identifying the asset with the highest expected return and allocating all investment to that asset. The following code snippet demonstrates this implementation:

```
# Find the index of the asset with the highest expected return
i_star = np.argmax(mu)

# Allocate 100% to the asset with the highest expected return
w_mrp = np.zeros(len(mu))
w_mrp[i_star] = 1
```

```

print("Optimal weights for Maximum Return Portfolio (w_MRP):\n", w_mrp)
print("Corresponding stock ticker:\n", stock_symbols[i_star])
===== OUTPUT =====
Optimal weights for Maximum Return Portfolio (w_MRP):
 [1. 0. 0. 0. 0. 0.]
Corresponding stock ticker:
 MSFT
```

Listing 5: Implementing the Maximum Return Portfolio

This output confirms that Microsoft (MSFT) has the highest expected return in our portfolio analysis, so we allocate all our investments to this asset.

3 Global Minimum Variance Portfolio

The Global Minimum Variance Portfolio (GMVP) is the portfolio with the lowest possible risk (variance) among all portfolios of assets. It is especially appealing for risk-averse investors. The GMVP can be formulated as:

$$\min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}$$

subject to:

$$\mathbf{w}^T \mathbf{1} = 1,$$

where the objective is to minimize the portfolio variance $\mathbf{w}^T \Sigma \mathbf{w}$ while ensuring the portfolio is fully invested. The GMVP is located at the tip of the efficient frontier, representing the lowest-risk portfolio irrespective of return.

To derive the closed-form solution $\mathbf{w}_{GMVP}$ for the GMVP, we leverage the method of Lagrange multipliers, which facilitates the optimization of a function subject to equality constraints. The corresponding Lagrangian for this optimization problem is given by:

$$\mathcal{L}(\mathbf{w}, \gamma) = \mathbf{w}^T \Sigma \mathbf{w} + \gamma(1 - \mathbf{w}^T \mathbf{1}),$$

where γ represents the Lagrange multiplier associated with the investment constraint.

Differentiating $\mathcal{L}$ with respect to $\mathbf{w}$ and setting the derivative equal to zero yields the first-order condition:

$$2\Sigma \mathbf{w} - \gamma \mathbf{1} = 0.$$

Solving this equation for $\mathbf{w}$ involves isolating the portfolio weights vector:

$$\mathbf{w} = \frac{\gamma}{2} \Sigma^{-1} \mathbf{1}.$$

To find the exact value of γ, we apply the full investment constraint $\mathbf{w}^T \mathbf{1} = 1$, leading to:

$$\frac{\gamma}{2} \mathbf{1}^T \Sigma^{-1} \mathbf{1} = 1.$$

Solving for γ gives:

$$\gamma = \frac{2}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

Substituting back into the expression for $\mathbf{w}$, we obtain the closed-form solution for the GMVP:

$$\mathbf{w}_{GMVP} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

This result shows the inverse relationship between the portfolio's optimal weight distribution and the assets' covariance matrix in GMVP. This portfolio does not chase returns but focuses solely on the minimization of risk, adhering to the most conservative investment rationale.

The expected return on the GMVP, μ_{GMVP}, can be found by calculating the weighted sum of the individual expected returns of the assets in the portfolio using the GMVP weights:

$$\mu_{GMVP} = \mathbf{w}_{GMVP}^T \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$ is the vector of expected returns for the assets. Substituting the closed-form solution for $\mathbf{w}_{GMVP}$ we have:

$$\mu_{GMVP} = \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right)^T \boldsymbol{\mu} = \frac{\mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}.$$

The variance of the GMVP, σ_{GMVP}^2, is then:

$$\begin{aligned}
\sigma_{GMVP}^2 &= \mathbf{w}_{GMVP}^T \Sigma \mathbf{w}_{GMVP} \\
&= \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right)^T \Sigma \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \right) \\
&= \frac{1}{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})^2} (\mathbf{1}^T\Sigma^{-1})\Sigma(\Sigma^{-1}\mathbf{1}) \\
&= \frac{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}{(\mathbf{1}^T\Sigma^{-1}\mathbf{1})^2} \\
&= \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}
\end{aligned}$$

where we have used the fact that the transpose of a scalar is the scalar itself and that matrix multiplication is associative. Hence, the GMVP is characterized by its expected return μ_{GMVP} and its variance σ_{GMVP}^2. These coordinates determine the position of the GMVP on the efficient frontier in mean-variance space.

3.1 Implementing the Global Minimum Variance Portfolio

The Global Minimum Variance Portfolio (GMVP) is designed to minimize portfolio variance, representing the least risky portfolio, assuming that no expected returns are specified. We define the objective function as the portfolio variance, which is a quadratic form of the weights and the covariance matrix. Below is the implementation to optimize the GMVP under the constraint that the weights sum to one, optionally enforcing no short selling by bounding the weights between 0 and 1.

```python
# Objective function for GMVP (portfolio variance)
def objective_function_gmvp(w, Sigma):
    return w.T @ Sigma @ w

# Constraint for the optimization (sum of weights equals 1)
constraints = [
    {'type': 'eq', 'fun': lambda w: np.sum(w) - 1}  # Sum of weights = 1
]

# Initial guess (equal weighting)
w0 = np.ones(len(Sigma)) / len(Sigma)

# Optional: Bounds for the weights to ensure no short-selling
bounds = tuple((0, 1) for _ in range(len(Sigma)))

# Perform the optimization using Sequential Least Squares Programming (SLSQP)
result = minimize(fun=objective_function_gmvp, x0=w0, args=(Sigma,), method='SLSQP',
    constraints=constraints, bounds=bounds)

# Extract the optimal weights
w_gmvp = result.x

print("Optimal weights for GMVP (w_GMVP):\n", w_gmvp)
===== OUTPUT =====
Optimal weights for GMVP (w_GMVP):
```

```
[9.29626e-02 6.69021e-01 3.69898e-02 1.32877e-01 5.20417e-18 6.81492e-02]
```

Listing 6: Implementing the Global Minimum Variance Portfolio

This result shows the optimal allocation for minimizing variance in a portfolio context, where the major allocation is to the second asset, reflecting its lower risk contribution relative to others.

4 Maximum Sharpe Ratio Portfolio

The Maximum Sharpe Ratio Portfolio (MSRP), or the Tangency Portfolio, was first proposed by [Sha66], aiming to maximize the Sharpe ratio of the portfolio. This is a more intuitive choice as investors are often more interested in the Sharpe ratio than setting the risk-aversion parameter. The portfolio that maximizes the Sharpe Ratio is the point where the Capital Market Line (CML) is tangent to the efficient frontier. Here, CML is the line that connects the risk-free asset and the tangency point on the efficient frontier. It represents the set of portfolios that optimally combine the risk-free asset and the market portfolio (tangency portfolio). Investors can choose their desired risk level along this line, balancing between the risk-free asset and the market portfolio. It represents the best risk-adjusted return when considering the ability to lend and borrow at the risk-free rate.

The Sharpe Ratio is defined as:

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p},$$

where R_p denotes the portfolio return, σ_p is the portfolio risk, and R_f is the risk-free interest rate. The MSRP is thus formulated as follows.

$$\max_{\mathbf{w}} \frac{R_p(\mathbf{w}) - R_f}{\sigma_p(\mathbf{w})} = \max_{\mathbf{w}} \frac{\mathbf{w}^T\boldsymbol{\mu} - R_f}{\sqrt{\mathbf{w}^T\Sigma\mathbf{w}}},$$

subject to:

$$\mathbf{w}^T\mathbf{1} = 1.$$

This portfolio has the highest Sharpe Ratio and is considered optimal in the sense of providing the highest expected return per unit of risk, assuming lending and borrowing at the risk-free rate is allowed. This corresponds to the point with the highest slope on the efficient frontier; that is, the point after which the marginal increment in excess return starts to decrease for the same unit increase in risk.

Geometrically, this optimization corresponds to the tangency portfolio on the efficient frontier in the Mean-Variance space. The efficient frontier represents the set of portfolios that provide the highest expected return for a given level of risk or the lowest risk for a given level of expected return. The point on this frontier that maximizes the Sharpe Ratio is where a line originating from the risk-free rate (R_f) tangentially intersects the frontier. This point signifies the optimal risk-return trade-off and is hence deemed the most efficient portfolio. See Figure 2 for an illustration on the three portfolios (MRP, GMVP, and MSRP) on the efficient frontier, which intersects with CML at the MSRP.

However, the objective function of maximizing the Sharpe Ratio is non-linear due to the square root in the denominator. This can potentially make the optimization problem non-convex, complicating the search for a global maximum. A workaround is to start from the unconstrained MVO problem using the expected excess returns[1]:

$$\max_{\mathbf{w}} \mathbf{w}^T(\boldsymbol{\mu} - R_f\mathbf{1}) - \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w}$$

The Lagrangian $\mathcal{L}$ for this problem, without considering any constraint, is:

$$\mathcal{L}(\mathbf{w}) = \mathbf{w}^T(\boldsymbol{\mu} - R_f\mathbf{1}) - \frac{\lambda}{2}\mathbf{w}^T\Sigma\mathbf{w}.$$

[1] Under some conditions, the optimal mean-variance portfolio fully invested will equal the maximum Sharpe ratio portfolio.

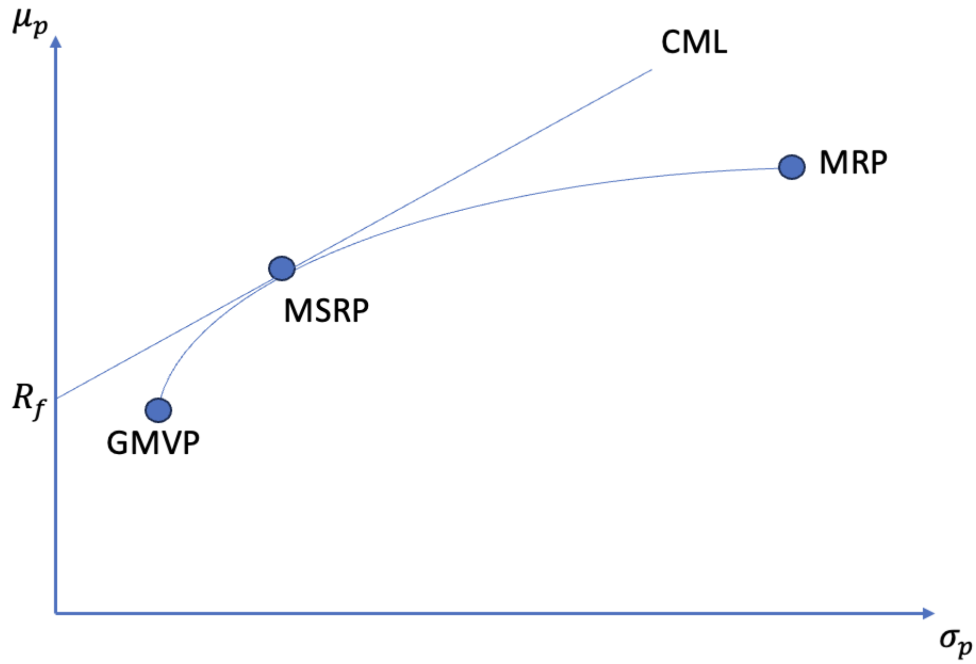

Figure 2: Illustration of the three special portfolios on the efficient frontier along with the CML.

We take the gradient of $\mathcal{L}$ with respect to $\mathbf{w}$ and set it to zero to find the First-Order Condition (FOC):

$$\nabla_{\mathbf{w}}\mathcal{L} = (\boldsymbol{\mu} - R_f\mathbf{1}) - \lambda\Sigma\mathbf{w} = 0.$$

We can solve for $\mathbf{w}$ to get the following closed-form solution:

$$\mathbf{w}^*_{unconstrained} = \frac{1}{\lambda}\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}).$$

Now we consider the budget constraint that requires $\mathbf{1}^T\mathbf{w} = 1$. To achieve this, we simply need to divide each element in $\mathbf{w}^*_{unconstrained}$ by the sum of all the elements, i.e., $\mathbf{1}^T\mathbf{w}^*_{unconstrained}$. Therefore, the scaled solution that maps to the original MSRP formulation is:

$$\begin{aligned}\mathbf{w}_{MSRP} &= \frac{\mathbf{w}^*_{unconstrained}}{\mathbf{1}^T\mathbf{w}^*_{unconstrained}} \\ &= \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}\end{aligned} \tag{11}$$

where the risk-aversion parameter is canceled out. We note a similarity between $\mathbf{w}_{MSRP}$ and $\mathbf{w}_{GMVP}$, where both have Σ^{-1} in the numerator and $\mathbf{1}^T\Sigma^{-1}$ in the denominator.

4.1 An Alternative Derivation

We can also derive the closed-form solution using an alternative approach based on Lagrangian multiplier and first-order conditions. This process starts with a portfolio scenario comprising N risky assets, characterized by a return vector $\mathbf{r} = [r_1, r_2, \ldots, r_N]^T$, in addition to a singular risk-free asset—exemplified here by Treasury bills—with a constant return denoted as R_f. It is pertinent to note that, in general, vectors are denoted by bold lowercase letters, and matrices by bold uppercase letters, with the exception being the risk-free rate R_f, a deviation made to maintain consistency with established conventions in financial literature.

The portfolio weights allocated to the N risky assets are represented by $\mathbf{w} \in R^N$, while w_f denotes the weight attributed to the risk-free asset. These weights are subject to a budget constraint $\mathbf{w}^T\mathbf{1} + w_f = 1$, which ensures that the total allocation across all assets equates to one, thereby allocating the total wealth to these assets. We start with this general case and later make $w_f = 0$ to recover the Tangency portfolio with all risky assets.

The portfolio return is thus a weighted sum of risky assets plus the risk-free asset:

$$\begin{aligned} r_p &= \mathbf{w}^T\mathbf{r} + w_f R_f \\ &= \mathbf{w}^T\mathbf{r} + (1 - \mathbf{w}^T\mathbf{1})R_f \\ &= \mathbf{w}^T(\mathbf{r} - R_f\mathbf{1}) + R_f \end{aligned}$$

Thus the excess return $\tilde{r}_p$ of the portfolio is:

$$\tilde{r}_p = r_p - R_f = \mathbf{w}^T(\mathbf{r} - R_f\mathbf{1}) = \mathbf{w}^T\tilde{\mathbf{r}}$$

where $\tilde{\mathbf{r}} = \mathbf{r} - R_f\mathbf{1}$ denotes the vector of excess return random variables. This expression makes sense as the total excess return of the portfolio is a weighted sum of the individual excess returns for the risky assets. The corresponding expected excess return $\tilde{\mu}_p$ and variance σ_p^2 of the portfolio are:

$$\begin{aligned} \tilde{\mu}_p = \mu_p - R_f = \mathbf{w}^T(\boldsymbol{\mu} - R_f\mathbf{1}) = \mathbf{w}^T\tilde{\boldsymbol{\mu}} \\ \sigma_p^2 = \mathbf{w}^T\Sigma\mathbf{w} \end{aligned}$$

where $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - R_f\mathbf{1}$ denotes the vector of expected asset returns.

We can now form a similar minimum variance portfolio as formulation 1:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T\Sigma\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T\tilde{\boldsymbol{\mu}} = \tilde{\mu}_0 \end{aligned} \tag{12}$$

where we left out the budget constraint $\mathbf{w}^T\mathbf{1} = 1$ since not all weights need to be allocated to the risky assets in this general analysis.

The Lagrangian function, $\mathcal{L}(\mathbf{w}, \lambda)$ is thus given by:

$$\mathcal{L}(\mathbf{w}, \lambda) = \mathbf{w}^T\Sigma\mathbf{w} + \lambda(\tilde{\mu}_0 - \mathbf{w}^T\tilde{\boldsymbol{\mu}}).$$

Taking the derivative with respect to $\mathbf{w}$, we obtain:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{w}} = 2\Sigma\mathbf{w} - \lambda\tilde{\boldsymbol{\mu}} = 0.$$

Solving for $\mathbf{w}$ yields:

$$\mathbf{w} = \frac{1}{2}\lambda\Sigma^{-1}\tilde{\boldsymbol{\mu}}.$$

Taking the derivative with respect to λ gives us the constraint itself:

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \tilde{\mu}_0 - \mathbf{w}^T\tilde{\boldsymbol{\mu}} = 0.$$

Substituting the expression for $\mathbf{w}$ from the first derivative into this equation, we obtain an equation that allows us to solve for the Lagrange multiplier λ:

$$\begin{aligned} \tilde{\mu}_0 = \left(\frac{1}{2}\lambda\Sigma^{-1}\tilde{\boldsymbol{\mu}}\right)^T\tilde{\boldsymbol{\mu}} = \frac{1}{2}\lambda\tilde{\boldsymbol{\mu}}^T\Sigma^{-1}\tilde{\boldsymbol{\mu}} \\ \lambda = \frac{2\tilde{\mu}_0}{\tilde{\boldsymbol{\mu}}^T\Sigma^{-1}\tilde{\boldsymbol{\mu}}}. \end{aligned}$$

Once we have λ, we can substitute back into the expression for $\mathbf{w}$ to obtain the optimal weights:

$$\mathbf{w}^* = \tilde{\mu}_0 \frac{\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}^T \Sigma^{-1}\tilde{\boldsymbol{\mu}}}.$$

Now, suppose our portfolio consists of all risky assets, and the previous budget constraint $\mathbf{1}^T\mathbf{w} = 1$ applies. Plugging in $\mathbf{w}^*$ to this constraint gives:

$$\tilde{\mu}_0 = \frac{\tilde{\boldsymbol{\mu}}^T \Sigma^{-1}\tilde{\boldsymbol{\mu}}}{\mathbf{1}^T \Sigma^{-1}\tilde{\boldsymbol{\mu}}}$$

which gives the target excess return that corresponds to the case of all risky assets. Plugging it back into $\mathbf{w}^*$ gives:

$$\mathbf{w}_{MSRP} = \frac{\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{\mathbf{1}^T \Sigma^{-1}\tilde{\boldsymbol{\mu}}} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}{\mathbf{1}^T \Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}$$

which corresponds to the same closed-form solution for MSRP as in 11.

4.2 Implementing the Maximum Sharpe Ratio Portfolio

We implement the Maximum Sharpe Ratio Portfolio (MSRP), which is designed to maximize the Sharpe Ratio, indicating the best possible excess return per unit of risk. We assume a daily risk-free rate of 0.5% for this computation. The Sharpe Ratio is defined as the difference between the portfolio's return and the risk-free rate, divided by the portfolio's standard deviation. The following code snippet outlines the optimization process to maximize the Sharpe Ratio by minimizing its negative value.

```
Rf = 0.005  # Placeholder for the risk-free rate

# Objective function for MSRP (negative Sharpe Ratio)
def objective_function_msrp(w, mu, Sigma, Rf):
    Rp = np.dot(w, mu)  # Portfolio return
    sigma_p = np.sqrt(np.dot(w.T, np.dot(Sigma, w)))  # Portfolio standard deviation
    sharpe_ratio = (Rp - Rf) / sigma_p
    return -sharpe_ratio  # Minimize negative Sharpe Ratio

# Constraint for the optimization (sum of weights equals 1)
constraints = [
    {'type': 'eq', 'fun': lambda w: np.sum(w) - 1}  # Sum of weights = 1
]

# Initial guess (equal weighting)
w0 = np.ones(len(mu)) / len(mu)

# Optional: Bounds for the weights to prevent short-selling
bounds = tuple((0, 1) for _ in range(len(mu)))

# Perform the optimization
result = minimize(fun=objective_function_msrp, x0=w0, args=(mu, Sigma, Rf), method='SLSQP',
    constraints=constraints, bounds=bounds)

# Extract the optimal weights
w_msrp = result.x

print("Optimal weights for MSRP (w_MSRP):\n", w_msrp)
===== OUTPUT =====
Optimal weights for MSRP (w_MSRP): [6.86661e-01 2.19200e-01 9.41398e-02 4.08744e-17 0.00000e+00 1.99493e-17]
```

Listing 7: Implementing the Maximum Sharpe Ratio Portfolio

This output illustrates the optimal asset allocation to maximize the Sharpe Ratio. The majority of the portfolio is concentrated in the first three assets, reflecting their superior risk-adjusted returns compared to the others.

5 Solving the MVO Formulation

We now turn to the general MVO formulation, which is stated here again for ease of reference.

where:

- $\mathbf{w}$ is the vector of portfolio weights for the assets.
- Σ is the covariance matrix of asset returns.
- $\boldsymbol{\mu}$ is the vector of expected returns for the assets.
- λ is the risk-aversion coefficient; higher values of λ place more emphasis on risk relative to the expected return.

The objective function consists of two terms: the portfolio variance $\mathbf{w}^T\Sigma\mathbf{w}$ which we want to minimize, and the expected return $\mathbf{w}^T\boldsymbol{\mu}$ which we want to maximize. By adjusting λ, the investor can control the expected return level relative to the level of risk they are willing to assume. Constraint $\mathbf{w}^T\mathbf{1} = 1$ ensures that the sum of the weights is equal to 1, which places a budget constraint that requires all capital to be invested.

We can solve for the optimal portfolio weights using the same technique that combines Lagrangian with FOCs. Specifically, the Lagrangian for this optimization problem is given by:

$$\mathcal{L}(\mathbf{w}, \gamma) = \mathbf{w}^T\boldsymbol{\mu} - \frac{1}{2}\lambda\mathbf{w}^T\Sigma\mathbf{w} + \gamma(1 - \mathbf{w}^T\mathbf{1}),$$

where γ is the Lagrange multiplier associated with the budget constraint.

The FOCs for this optimization are obtained by taking the partial derivatives of $\mathcal{L}$ with respect to $\mathbf{w}$ and γ, and setting them to zero. For $\mathbf{w}$, we have:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{w}} = \boldsymbol{\mu} - \lambda\Sigma\mathbf{w} - \gamma\mathbf{1} = 0.$$

This can be rearranged to solve for $\mathbf{w}$:

$$\lambda\Sigma\mathbf{w} = \boldsymbol{\mu} - \gamma\mathbf{1},$$

$$\mathbf{w} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \frac{\gamma}{\lambda}\Sigma^{-1}\mathbf{1}.$$

To find γ, we use the budget constraint:

$$\mathbf{w}^T\mathbf{1} = 1,$$

Substituting the expression for $\mathbf{w}$ gives:

$$\left(\frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \frac{\gamma}{\lambda}\Sigma^{-1}\mathbf{1}\right)^T\mathbf{1} = 1,$$

$$\frac{1}{\lambda}\boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1} - \frac{\gamma}{\lambda}\mathbf{1}^T\Sigma^{-1}\mathbf{1} = 1.$$

Solving for γ yields:

$$\gamma = \frac{1}{\lambda}\left(\boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1} - \lambda\right)\Big/\left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right).$$

Substituting γ back into the equation for $\mathbf{w}$, we get the optimal weights:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \left(\frac{1}{\lambda}\left(\boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1} - \lambda\right)\Big/\left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)\right)\Sigma^{-1}\mathbf{1}.$$

This formula provides the closed-form solution for the optimal portfolio weights in a mean-variance optimization framework, given a particular level of risk aversion (λ), expected returns ($\boldsymbol{\mu}$), and the covariance matrix of the returns (Σ). The solution balances the trade-off between maximizing expected returns and minimizing risk, adjusted by the investor's risk tolerance.

We can also express w_{MVO} as a function of $\mathbf{w}_{GMVP}$, $\mathbf{w}_{MSRP}$, μ_{GMVP}, and σ^2_{GMVP}, where:

- $\mathbf{w}_{GMVP} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$ is the optimal weight vector of GMVP.
- $\mathbf{w}_{MSRP} = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}}$ is the optimal weight vector of MSRP, assuming $R_f = 0$ for simplification.
- $\mu_{GMVP} = \frac{\mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$ is the expected return of the GMVP.
- $\sigma^2_{GMVP} = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$ is the variance of the GMVP.

The original expression for $\mathbf{w}_{MVO}$ is:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \left(\frac{1}{\lambda}\left(\boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1} - \lambda\right) \Big/ \left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)\right)\Sigma^{-1}\mathbf{1}.$$

We can introduce $a = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$ (which is also $1/\sigma^2_{GMVP}$) and $b = \boldsymbol{\mu}^T\Sigma^{-1}\mathbf{1}$ (which is also $\mu_{GMVP}/\sigma^2_{GMVP}$). Using a and b, we rewrite the expression for $\mathbf{w}_{MVO}$ as:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \left(\frac{b-\lambda}{\lambda a}\right)\Sigma^{-1}\mathbf{1}.$$

Since $\mathbf{w}_{GMVP} = \Sigma^{-1}\mathbf{1}/a$, we can substitute $\Sigma^{-1}\mathbf{1}$ with $a\mathbf{w}_{GMVP}$:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu} - \left(\frac{b-\lambda}{\lambda}\right)\mathbf{w}_{GMVP}.$$

For the MSRP, given that $\mathbf{w}_{MSRP} = \Sigma^{-1}\boldsymbol{\mu}/b$, we can solve for $\Sigma^{-1}\boldsymbol{\mu}$ in terms of $\mathbf{w}_{MSRP}$ and b:

$$\Sigma^{-1}\boldsymbol{\mu} = b\mathbf{w}_{MSRP}.$$

Now, substituting $\Sigma^{-1}\boldsymbol{\mu}$ into the expression for $\mathbf{w}_{MVO}$, we get:

$$\mathbf{w}_{MVO} = \frac{b}{\lambda}\mathbf{w}_{MSRP} - \left(\frac{b-\lambda}{\lambda}\right)\mathbf{w}_{GMVP}.$$

Simplifying this expression:

$$\mathbf{w}_{MVO} = \left(\frac{b}{\lambda}\right)\mathbf{w}_{MSRP} + \left(1 - \frac{b}{\lambda}\right)\mathbf{w}_{GMVP}.$$

This shows that the weights for the MVO can be viewed as a linear combination of the weights of the GMVP and the MSRP. The coefficients of this combination depend on the investor's risk aversion coefficient λ, the expected return on the GMVP μ_{GMVP}, and the variance of the GMVP σ^2_{GMVP}, with R_f assumed to be zero.

Now we re-express b in terms of μ_{GMVP} and σ^2_{GMVP}. We know that $b = \mu_{GMVP} \cdot a$ because $\mu_{GMVP} = \frac{b}{a}$. Therefore, we can express b as $b = \mu_{GMVP} \cdot \frac{1}{\sigma^2_{GMVP}}$. Substitute b in the expression for $\mathbf{w}_{MVO}$ gives:

$$\mathbf{w}_{MVO} = \frac{\mu_{GMVP}}{\lambda\sigma^2_{GMVP}}\mathbf{w}_{MSRP} + \left(1 - \frac{\mu_{GMVP}}{\lambda\sigma^2_{GMVP}}\right)\mathbf{w}_{GMVP}$$

Finally, we factor out $\mathbf{w}_{GMVP}$ to get:

$$\mathbf{w}_{MVO} = \mathbf{w}_{GMVP} + \left(\frac{\mu_{GMVP}}{\lambda\sigma^2_{GMVP}}\right)(\mathbf{w}_{MSRP} - \mathbf{w}_{GMVP})$$

This expression shows that the MVO portfolio weights $\mathbf{w}_{MVO}$ are a combination of the GMVP weights $\mathbf{w}_{GMVP}$ and an adjustment in the direction of $\mathbf{w}_{MSRP} - \mathbf{w}_{GMVP}$, scaled by the ratio $\frac{\mu_{GMVP}}{\lambda\sigma^2_{GMVP}}$. This ratio represents the trade-off between risk (as captured by the variance of the GMVP, σ^2_{GMVP}) and return (as captured by μ_{GMVP}), modulated by the investor's risk aversion parameter λ.

6 Index Tracking Portfolio

Index-tracking portfolios represent a cornerstone of passive investment strategies, whereby the primary aim is to replicate the performance of a predetermined market index, such as the S&P 500 or the Dow Jones Industrial Average (often referred to as the Dow 30). This approach is fundamentally distinct from active investment strategies, where the goal is to outperform the market through asset selection and timing. The essence of index tracking lies in its focus on achieving the same returns as the index, thereby offering investors exposure to a broad market segment or the market as a whole, depending on the index being tracked.

Recall the Efficient Market Hypothesis (EMH), which posits that at any given time, asset prices fully reflect all available information. According to the EMH, it's challenging, if not impossible, for investors to consistently achieve higher returns than the market average without taking on additional risk. This hypothesis, therefore, provides a theoretical justification for index tracking, suggesting that matching the market performance is a prudent strategy, especially when considering the difficulty and costs associated with attempting to outperform the market.

Executing an index-tracking strategy requires first selecting a target index to be tracked, then selecting the assets that are constituents of the target index and determining their respective weights. The most straightforward approach is to replicate the index exactly, purchasing all the assets in the same proportions as they are found in the index. However, full replication can be impractical or costly for indices with a large number of constituents or for those that include illiquid assets.

To address the challenges of full replication, optimization techniques are employed to select a sparse subset of assets and determine their weights so that the tracking error, which measures the deviation of the portfolio's returns from the index returns, is minimized. This optimization problem can be complex, especially when incorporating transaction costs, tax considerations, and regulatory constraints.

In addition, the actual weights of assets in the portfolio will drift from their target weights due to differing rates of return over time. Periodic rebalancing is thus required to adjust the portfolio's composition back to its target weights, thereby ensuring continued alignment with the index's performance. The frequency and methodology of rebalancing can significantly affect transaction costs and tax efficiency.

6.1 Mathematical Formulation

The mathematical formulation of an index-tracking portfolio focuses on minimizing the tracking error through a constrained optimization problem. The tracking error is typically quantified as the standard deviation of the difference between the returns of the portfolio and the index over a specified time period. This section delves into the details of formulating this optimization problem.

Denote N as the number of assets in the universe from which the tracking portfolio is constructed, r_i^t as the return of asset i at time t for $i = 1, \ldots, N$, w_i as the weight of asset i in the tracking portfolio, r_M^t as the return of the index M at time t, and T as the total number of periods considered for tracking. The objective is to minimize the tracking error (TE), which is often defined as the standard deviation of the difference between the portfolio's returns and the index's returns over the considered time period. TE is a measure of how closely the portfolio follows the index and is defined as:

$$TE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N} w_i r_i^t - r_M^t\right)^2}$$

The goal of the optimization problem is to find the set of weights $\mathbf{w}^T = \{w_1, w_2, ..., w_N\}$ (held constant over the T periods) that minimizes the tracking error, subject to constraints that reflect investment limitations and strategies.

The optimization problem can be formally stated as:

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & TE \\
\text{s.t.} \quad & \sum_{i=1}^{N} w_i = 1 \\
& \mathbf{w}^T \boldsymbol{\mu} = \mu_0
\end{aligned}
$$

where $\mathbf{w} = [w_1, w_2, \ldots, w_N]^T$ is the vector of weights to be optimized. Here, we consider the full investment constraint ($\sum_{i=1}^{N} w_i = 1$) that the entire portfolio is fully allocated, and no cash is held, and the target return constraint ($\mathbf{w}^T \boldsymbol{\mu} = \mu_0$).

Treating both the portfolio return r_p and market index r_M as random variables, we can define the equivalent TE as the variance of their difference $\mathrm{Var}(r_p - r_M)$, given the portfolio's target return level. Following [Edi13], we can derive the closed-form optimal solution of the portfolio weights based on KKT conditions. We first introduce β_i to denote the beta of asset i relative to the market index M:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

We can express the portfolio beta as $\beta_p = \sum_{i=1}^{N} \beta_i w_i = \beta^T \mathbf{w}$. Thus the previous TE can be re-expressed as follows:

where we have used the definition of $r_p = \sum_{i=1}^{N} w_i r_i$ in the second equality.

Considering the independence of σ_M^2 and noting that $\sigma_p^2 = \mathbf{w}^T \Sigma \mathbf{w}$, we can reformulate the objective of the index tracking portfolio as follows.

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T \Sigma \mathbf{w} - \sigma_M^2 \beta^T \mathbf{w} \\
\text{s.t.} \quad & \mathbf{w}^T \boldsymbol{\mu} = \mu_0 \\
& \mathbf{w}^T \mathbf{1} = 1
\end{aligned}
$$

where we scaled down the original objective by half to ease the derivation. We can write the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}\mathbf{w}^T \Sigma \mathbf{w} - \sigma_M^2 \beta^T \mathbf{w} - \lambda(\mathbf{w}^T \boldsymbol{\mu} - \mu_0) - \gamma(\mathbf{w}^T \mathbf{1} - 1)$$

where $\mathbf{g} = \begin{bmatrix} g_1 & g_2 \end{bmatrix}^T$ are the Lagrange multipliers.

To obtain the FOCs, we can take the derivative of $\mathcal{L}$ with respect to $\mathbf{w}$ and setting it equal to zero to obtain the stationarity condition:

$$\Sigma \mathbf{w} - \sigma_M^2 \beta - g_1 \boldsymbol{\mu} - g_2 \mathbf{1} = 0$$

The constraints in matrix form are:

$$\mathbf{B}^T \mathbf{w} = \mathbf{m}, \quad \mathbf{B} = \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}$$

We then combine the stationarity condition and the constraints into the augmented system:

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \Sigma & -\mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \sigma_M^2 \beta \\ \mathbf{m} \end{bmatrix}$$

Using block matrix inversion, we solve for $\mathbf{x}$ which contains $\mathbf{w}^*$ and $\mathbf{g}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

We express $\mathbf{A}^{-1}$ using the block matrix inversion formula where:

$$\mathbf{Q} = (\Sigma + \mathbf{B}(\mathbf{B}^T \Sigma^{-1} \mathbf{B})^{-1} \mathbf{B}^T)^{-1}$$

and hence:

$$\mathbf{w}^* = \mathbf{Q}(\sigma_M^2 \beta - \mathbf{B}(\mathbf{B}^T \mathbf{Q} \mathbf{B})^{-1} \mathbf{m})$$

This expression takes into account the impact of market variance and constraints on the optimal portfolio weights. The derived $\mathbf{w}^*$ provides a set of portfolio weights that minimize the tracking error of the portfolio while adhering to the constraints of the desired return (μ_0) and the sum of weights equal to one. This is achieved by balancing the market variance-weighted vector $\sigma_M^2 \beta$ and the constraints encoded by $\mathbf{B}$ and $\mathbf{m}$.

6.2 Implementing the Index Tracking Portfolio

The goal of an Index Tracking Portfolio is to mimic the performance of a benchmark index, in this case, the Dow Jones Industrial Average (DJIA). The process begins by acquiring the historical price data of the DJIA and the stocks considered for the tracking portfolio. We then compute and visualize their cumulative returns to analyze the performance similarity.

```python
# Import required libraries
import yfinance as yf
import matplotlib.pyplot as plt

# Define the DJIA symbol
djia_symbol = '^DJI'

# Download historical data for DJIA
djia_ticker = yf.Ticker(djia_symbol)
djia_hist_data = djia_ticker.history(start='start_date', end='end_date')

# Extract closing prices and handle missing values
djia_prices = djia_hist_data['Close'].interpolate(method='linear')

# Calculate daily and cumulative returns for the DJIA
djia_daily_returns = djia_prices.pct_change().dropna()
djia_cumulative_returns = (1 + djia_daily_returns).cumprod()

# Assuming 'prices' contains the closing prices of the stocks
stock_daily_returns = prices.pct_change().dropna()
stock_cumulative_returns = (1 + stock_daily_returns).cumprod()

# Add DJIA cumulative returns to the stocks' cumulative returns DataFrame
stock_cumulative_returns['DJIA'] = djia_cumulative_returns

# Plot cumulative returns
plt.figure(figsize=(14, 7))
for column in stock_cumulative_returns.columns:
    plt.plot(stock_cumulative_returns.index, stock_cumulative_returns[column], lw=2, label=
     column)
plt.title('Cumulative Returns of Stocks and DJIA Index')
plt.xlabel('Date')
plt.ylabel('Cumulative Returns')
plt.legend(loc='upper left', fontsize=12)
plt.show()
```

Listing 8: Data preparation for index tracking portfolio

This visualization of cumulative returns, shown in Figure 3, shows the relative performance of the stocks against the DJIA. The subsequent step involves developing portfolio weights that minimize the average tracking error, which is the average squared deviation between the portfolio's and the DJIA's returns across all time points.

6.3 Implementing the Index Tracking Portfolio

To construct an Index Tracking Portfolio, we aim to minimize the tracking error, which measures the deviation between the portfolio returns and the returns of the benchmark index, in this case, the Dow Jones Industrial Average (DJIA). Additionally, we set a target return of 20%, while ensuring full investment and no short selling.

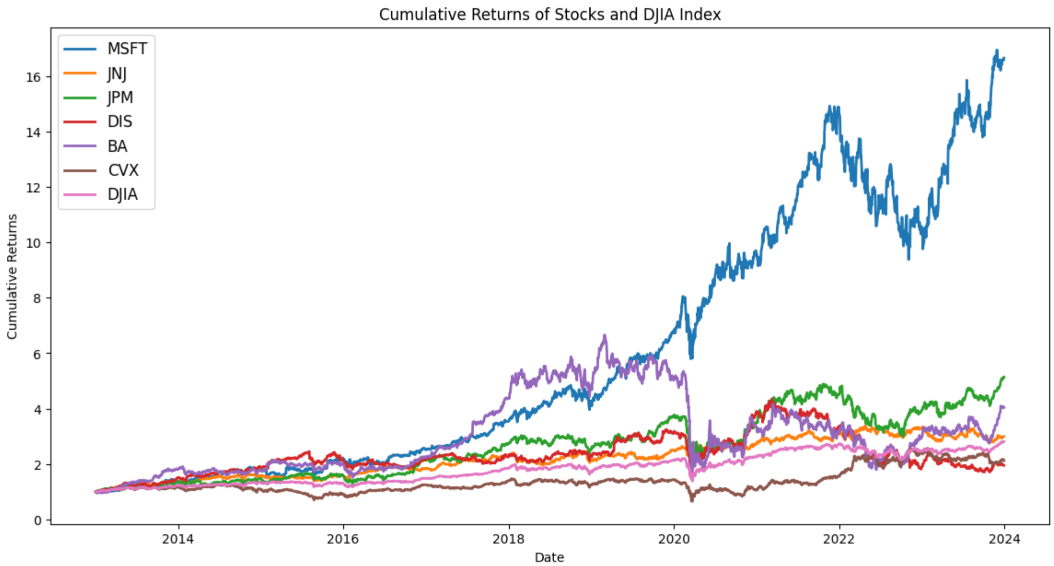

Figure 3: Cumulative returns of all stocks and the DJIA index.

```python
# Define your target return
target_return = 0.2

def tracking_error_optimization(weights):
    # Calculate portfolio returns as the dot product of weights and stock returns
    portfolio_returns = np.dot(stock_daily_returns, weights)
    # Calculate the tracking error as the mean squared difference between
    # portfolio returns and index returns
    tracking_error = np.mean((portfolio_returns - djia_daily_returns.values) ** 2)
    return tracking_error

# Constraints for the optimization
# Sum of weights equals 1 and portfolio return equals target_return
constraints = [
    {'type': 'eq', 'fun': lambda weights: np.sum(weights) - 1},
    {'type': 'eq', 'fun': lambda weights: np.dot(weights, mu) - target_return}
]

# Bounds to ensure weights are between 0 and 1 (no short selling)
bounds = tuple((0, 1) for asset in stock_symbols)

# Number of assets
n_assets = len(stock_symbols)

# Initial guess (equal weights)
initial_weights = np.ones(n_assets) / n_assets

# Perform the optimization
result = minimize(tracking_error_optimization, initial_weights, method='SLSQP', bounds=
    bounds, constraints=constraints)

# Extract the optimal weights
optimal_weights = result.x

# Display the optimal weights
print("Optimal weights for Index Tracking Portfolio:\n", optimal_weights)
===== Output =====
Optimal weights for Index Tracking Portfolio:
 [0.54284 0.07634 0.22333 0.      0.15748 0.     ]
```

Listing 9: Implementing the Index Tracking Portfolio

The output reveals the optimal allocation of assets in the Index Tracking Portfolio, with the majority of the allocation going to the first asset, while the fourth and sixth assets receive zero allocation,

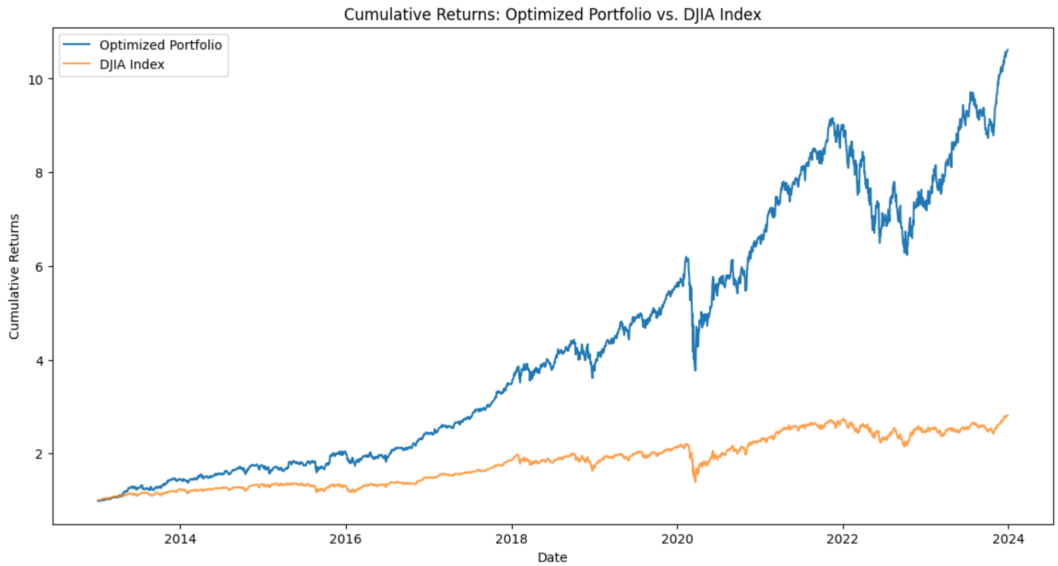

Figure 4: Cumulative returns comparison between the optimized index-tracking portfolio and the DJIA index.

aligning with our constraints and objectives.

To evaluate the effectiveness of the Index Tracking Portfolio, we calculate the tracking error as follows.

```
# Compute portfolio returns using the optimal weights
portfolio_returns = np.dot(stock_daily_returns, optimal_weights)

# Calculate the tracking error as the mean squared difference between
# portfolio returns and DJIA index returns
tracking_error = np.mean((portfolio_returns - djia_daily_returns.values) ** 2)
print("Tracking Error:", tracking_error)
===== Output =====
Tracking Error: 4.531914054915266e-05
```

Listing 10: Calculating and Displaying Tracking Error

To further understand the performance, we plot the cumulative returns of both the portfolio and the DJIA index. This visual comparison, as shown in Figure 4, helps to visualize how well the portfolio tracks the benchmark over time.

We can also look at the daily and annual tracking error in percentage terms, as shown in the following code snippet after calculating the daily excess return time series.

```
# Calculate excess returns of the portfolio over the index
excess_returns = portfolio_daily_returns_series - djia_daily_returns

# Calculate the standard deviation of the excess returns (daily tracking error)
tracking_error_daily = excess_returns.std()

# Annualize the tracking error
tracking_error_annualized = tracking_error_daily * np.sqrt(252)

# Convert to percentage
tracking_error_annualized_pct = tracking_error_annualized * 100

print("Daily Tracking Error: {:.2f}%".format(tracking_error_daily * 100))
print("Annualized Tracking Error: {:.2f}%".format(tracking_error_annualized_pct))
===== Output =====
Daily Tracking Error: 0.67%
Annualized Tracking Error: 10.66%
```

Listing 11: Calculating the tracking error in percentage term

7 Mean-CVaR Portfolio

The MVO framework we have been working with so far uses variance to measure the portfolio risk. However, this may not necessarily align with the practical interests of most investors, who prefer upside risk more than downside risk. Several downside risk measures have been proposed, including VaR (value-at-risk) and CVaR (conditional value-at-risk). Unlike variance, which treats deviations above and below the mean equally, VaR and CVaR provide a more nuanced view by specifically targeting the left tail of the distribution of returns, where losses reside. In this section, we will focus on these two risk measures and introduce the formulation of the mean CVaR portfolio.

7.1 Value-at-Risk (VaR)

Value-at-risk (VaR) quantifies the potential maximum loss a portfolio might incur over a specified period at a given confidence level. Consider a confidence level β set at 95%. The corresponding significance level is then $1-\beta = 5\%$. In this context, VaR_β for $\beta = 95\%$, delineates the loss threshold within the highest 5% of all possible losses, underscoring the maximal loss anticipated to occur with 95% probability. For example, if $VaR_\beta = \$1,000$ for $\beta = 95\%$ over one month, this implies a 95% confidence that the portfolio's losses will not exceed \$1,000 within the next month.

The computation of VaR is based on the quantile of the portfolio's return distribution, formulated as:

$$VaR_\beta(r_p) = \inf \{c \in R : P(r_p \leq c) \geq 1-\beta\}$$

where c is chosen as the minimum portfolio return (which may be negative in the left tail of the return distribution) that satisfies the threshold condition.

7.2 Conditional Value-at-Risk (CVaR)

The Conditional Value-at-Risk (CVaR), also referred to as the Expected Shortfall (ES), addresses the limitations inherent in the Value-at-Risk (VaR) metric by estimating the expected loss on the condition that losses surpass the VaR threshold. Differing from VaR, which only identifies a threshold for potential losses, CVaR quantifies the mean of losses that exceed this threshold. This results in a nuanced and coherent depiction of tail risk[2]:

$$\text{CVaR}_\beta(r_p) = E[r_p | r_p \leq VaR_\beta(r_p)]$$

Thus, CVaR provides an enhanced understanding of potential extreme losses, making it a pivotal measure for investors with a risk-averse orientation.

Given the intrinsic complexity in directly minimizing CVaR due to its non-convex nature, a seminal advancement by [RU00] reformulated this into an equivalent, convex optimization problem:

$$\text{CVaR}_\beta(\mathbf{w}, \alpha) = \min_{\alpha \in R} \alpha + \frac{1}{1-\beta} \int_{\mathbf{r} \in R^N} \max(-\mathbf{w}^T\mathbf{r} - \alpha, 0) f(\mathbf{r}) d\mathbf{r}$$

In this formulation, $f(\mathbf{r})$ represents the continuous probability density function of i.i.d asset returns encapsulated within the vector $\mathbf{r}$. The notation $r_p = \mathbf{w}^T\mathbf{r}$ signifies the portfolio returns, which are negated to denote losses. For a designated VaR cutoff threshold α along the loss/return distribution, the expression $\max(-\mathbf{w}^T\mathbf{r} - \alpha, 0)$ captures the excess loss beyond α. Integrating this expression over various loss/return scenarios facilitates the evaluation of the aggregate loss exceeding α. Consequently, $\text{CVaR}_\beta(\mathbf{w}, \alpha)$ emerges as convex with respect to both $\mathbf{w}$ and α. This transformation allows the original non-convex metric $\text{CVaR}_\beta(\mathbf{w})$, when defined solely over $\mathbf{w}$, to be optimized as a convex function $\text{CVaR}_\beta(\mathbf{w}, \alpha)$, encompassing both $\mathbf{w}$ and α.

By approximating $\text{CVaR}_\beta(\mathbf{w}, \alpha)$ with a sequence of T daily realizations of $\mathbf{r}$ based on its pdf $f(\mathbf{r})$, we can then obtain the following mean-CVaR formulation:

[2]Formally, CVaR adheres to the four coherence axioms defined by [ADEH99]: sub-additivity, translation invariance, monotonicity, and positive homogeneity. In contrast, VaR lacks sub-additivity, undermining its capacity to benefit from diversification. Diversification posits that the risk associated with a consolidated portfolio ought to be equal to or less than the aggregate risks of its constituent portfolios. A function f exhibits sub-additivity if it satisfies $f(x+y) \leq f(x) + f(y)$.

$$
\begin{aligned}
\min_{\mathbf{w},\alpha} \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} (-\mathbf{w}^T \mathbf{r}_t - \alpha)^+ \\
\text{s.t.} \quad & \mathbf{w}^T \boldsymbol{\mu} \geq \mu_0 \\
& \mathbf{w}^T \mathbf{1} = 1
\end{aligned}
$$

where $(-\mathbf{w}^T \mathbf{r}_t - \alpha)^+ = \max(-\mathbf{w}^T \mathbf{r}_t - \alpha, 0)$. By introducing the auxiliary variables $\{u_t\}_{t=1}^{T}$, we can further rewrite the original formulation as the following linear expression:

$$
\begin{aligned}
\min_{\mathbf{w},\alpha} \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} u_t \\
\text{s.t.} \quad & u_t \geq 0, \forall t = 1, \ldots, T \\
& u_t \geq -\mathbf{w}^T \mathbf{r}_t - \alpha, \forall t = 1, \ldots, T \\
& \mathbf{w}^T \boldsymbol{\mu} \geq \mu_0 \\
& \mathbf{w}^T \mathbf{1} = 1
\end{aligned}
$$

7.3 Implementing the Mean-CVaR Portfolio

The Mean-CVaR (Conditional Value at Risk) Portfolio optimization aims to minimize the expected shortfall beyond a given confidence level, incorporating both the optimization of portfolio weights and the determination of an optimal cutoff value, α. In the following code listing, we implement a Mean-CVaR model with a daily return target of 0.001% and a confidence level of 95%.

```python
import numpy as np
from scipy.optimize import minimize

# Define target return and confidence level
mu_0 = 0.001  # Target daily return, 0.001%
beta = 0.95   # Confidence level for CVaR

# Placeholder for daily expected returns
expected_returns = mu_daily

# Define the objective function for Mean-CVaR optimization
def mean_cvar_objective(x):
    w = x[:-1]  # Extract weights
    alpha = x[-1]  # Extract alpha (CVaR cutoff)
    portfolio_losses = -log_returns @ w - alpha
    u = np.maximum(portfolio_losses, 0)  # Positive parts of losses beyond alpha
    mean_cvar = alpha + np.mean(u) / (1 - beta)  # Mean-CVaR calculation
    return mean_cvar

# Define constraints: weights sum to 1 and ensure portfolio return >= mu_0
constraints = [
    {'type': 'eq', 'fun': lambda x: np.sum(x[:-1]) - 1},  # Sum of weights equals 1
    {'type': 'ineq', 'fun': lambda x: np.dot(x[:-1], expected_returns) - mu_0}  # Target return constraint
]

# Bounds for weights (no short selling) and alpha (no bounds)
bounds = tuple((0, 1) for _ in expected_returns) + ((None, None),)

# Initial guess for weights and alpha
initial_guess = np.ones(len(expected_returns) + 1) / (len(expected_returns) + 1)

# Solve the optimization problem using SLSQP method
result = minimize(mean_cvar_objective, initial_guess, method='SLSQP', bounds=bounds, constraints=constraints)

# Extract the optimal weights and display the result
optimal_weights = result.x[:-1]
```

```
print("Optimal weights for Mean-CVaR Portfolio:\n", optimal_weights)
===== Output =====
Optimal weights for Mean-CVaR Portfolio:
[9.62097e-01 2.43284e-03 3.54701e-02 3.34070e-18 1.35525e-19 2.05321e-18]
```

Listing 12: Implementing the Mean-CVaR Portfolio

The results suggest that a significant portion of the portfolio is allocated to the first asset, highlighting its role in minimizing the expected shortfall at the set confidence level.

8 Summary

In this chapter, we have explored a variety of portfolio selection strategies that cater to different investor preferences and risk tolerances. Each strategy is underpinned by a unique set of mathematical principles and objectives, ranging from risk minimization to the maximization of returns relative to a benchmark.

The Mean-Variance Optimization (MVO) strategy, pioneered by Harry Markowitz, remains a foundational concept in modern portfolio theory, advocating for an optimal balance between risk and return. This strategy's versatility is evident in its various derivatives, including the Maximum Return Portfolio, which seeks the highest returns irrespective of risk, and the Global Minimum Variance Portfolio, which focuses solely on risk minimization.

The Maximum Sharpe Ratio Portfolio, or the Tangency Portfolio, extends these ideas by considering the ratio of excess return to volatility, thus enhancing the economic rationale of the risk-return trade-off by incorporating the risk-free rate into its calculation.

The Index Tracking Portfolio strategy diverges from these approaches by prioritizing the replication of a benchmark's returns, thus catering to investors who prefer a passive management strategy that aims to mirror market performance.

Moving into advanced risk assessment strategies, the Mean-CVaR Portfolio introduces a focus on the Conditional Value at Risk, providing a sophisticated measure of tail risk that is particularly valuable for risk-averse investors concerned with potential significant losses.

The practical implementations of these strategies, demonstrated through Python simulations, offer a bridge between theoretical finance and real-world portfolio management. These examples underscore the importance of computational tools and techniques in executing complex optimization problems and achieving precise investment targets.

This exploration of portfolio selection strategies showcases a rich array of methodologies available to investors, highlighting the dynamic interplay between mathematical theory and investment practice. Moreover, the diverse portfolio optimization problems discussed serve as practical test beds for the application of advanced modeling techniques, including reinforcement learning. In the upcoming chapters, we will introduce common RL algorithms and explore their integration into various portfolio choice problems. This will demonstrate how RL can be leveraged to optimize decision-making processes in financial environments.

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