Machine Learning and Financial Applications

Lecture 9
Robust Portfolio Optimization

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Robust Portfolio Optimization

Concept Overview:

- Robust Portfolio Optimization seeks to construct portfolios that are less sensitive to uncertainties in the input parameters, such as asset returns ($\boldsymbol{\mu}$), covariance matrices (Σ), and risk aversion levels (λ).
- The objective is to optimize the portfolio performance under worst-case scenarios, considering different types of uncertainties in these parameters.

Mathematical Formulation:

- The goal is to maximize the worst-case performance of the portfolio when there is uncertainty in the model parameters:

$$\max_{\mathbf{w}} \min_{\boldsymbol{\mu} \in \mathcal{U}_{\mu}, \Sigma \in \mathcal{U}_{\Sigma}, \lambda \in \mathcal{U}_{\lambda}} \left(\mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2} \lambda \mathbf{w}^T \Sigma \mathbf{w} \right)$$

- Here, $\mathcal{U}_{\mu}$, $\mathcal{U}_{\Sigma}$, and $\mathcal{U}_{\lambda}$ represent the uncertainty sets for the expected returns, covariance matrix, and risk aversion coefficient, respectively.

- **Key Benefits**:
  - **Enhanced Stability**: Provides portfolios that are less sensitive to errors in parameter estimation, ensuring stable performance across different scenarios.
  - **Improved Risk Management**: Offers protection against worst-case outcomes by incorporating parameter uncertainties directly into the optimization process.
  - **Flexibility**: Adaptable to different types of uncertainty sets and levels of investor risk aversion.

Uncertainty in Expected Return

- Assuming uncertainty only in $\boldsymbol{\mu}$, the problem simplifies to:

$$\max_{\mathbf{w}} \left(\min_{\boldsymbol{\mu} \in \mathcal{U}_\mu} \mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2} \lambda \mathbf{w}^T \Sigma \mathbf{w} \right)$$

- This can be solved using robust optimization techniques that convert the problem into a convex optimization problem under ellipsoidal or polyhedral sets.

- **Ellipsoidal Uncertainty:** $\mathcal{U}_\mu = \{\boldsymbol{\mu} : (\boldsymbol{\mu} - \boldsymbol{\mu}_0)^T \Psi^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0) \leq \rho^2\}$
  - This set captures uncertainty within a confidence region centered around $\boldsymbol{\mu}_0$, the estimated returns.
- **Polyhedral Uncertainty:** $\mathcal{U}_\mu = \{\boldsymbol{\mu} : \|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|_\infty \leq \rho\}$
  - This set allows for bounded deviations from $\boldsymbol{\mu}_0$ in all directions, representing worst-case scenarios with maximum deviation ρ.

Uncertainty in Covariance Matrix

When only Σ is uncertain:

$$\max_{\mathbf{w}} \left(\mathbf{w}^T \boldsymbol{\mu} - \frac{1}{2} \lambda \max_{\Sigma \in \mathcal{U}_\Sigma} \mathbf{w}^T \Sigma \mathbf{w} \right)$$

This problem is addressed by robust quadratic programming techniques, which handle uncertainties in the covariance matrix to maintain portfolio robustness.

Spectral Norm Uncertainty: $\mathcal{U}_\Sigma = \{\Sigma : \|\Sigma - \Sigma_0\|_2 \leq \eta\}$

- This set constrains the covariance matrix to lie within an ellipsoidal region around the estimated covariance Σ_0, using spectral norm η as the measure of uncertainty.

Component-wise Uncertainty: $\mathcal{U}_\Sigma = \{\Sigma : |\Sigma_{ij} - \Sigma_{0,ij}| \leq \eta_{ij}, \forall i, j\}$

- Each element of the covariance matrix can vary within a specified range $[\Sigma_{0,ij} - \eta_{ij}, \Sigma_{0,ij} + \eta_{ij}]$, allowing for more granular control over the uncertainty.

Case Study

- **Returns and Risk**:
  - Mean returns: $\mu_i = 1.15 + \frac{i \cdot 0.05}{150}$
  - Standard deviations: $\sigma_i = \frac{0.05}{450} \cdot (2i \cdot n \cdot (n+1))^{0.5}$
  - Covariance matrix Σ: Diagonal with variances, assuming no correlation.
- **Revised MVO Model**:

$$\begin{aligned} \underset{\mathbf{w}}{\text{maximize}} \quad & \sum_{i=1}^{n} \mu_i w_i - \lambda \sum_{i=1}^{n} \sigma_i^2 w_i^2 \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i = 1, \\ & w_i \geq 0 \quad \forall i. \end{aligned}$$

- **Practical Note**: Parameters typically estimated from historical data or financial models.

Robust MVO Formulation

Problem Setup:

- In Robust MVO, the mean return vector μ and the covariance matrix Σ are **uncertain** and belong to **uncertainty sets** U_μ and U_Σ, respectively.
- Uncertain parameters:

$$\mu \in U_\mu, \quad \Sigma \in U_\Sigma$$

Robust Optimization Formulation:

- The goal is to optimize the portfolio weights $\mathbf{w}$ for the **worst-case** scenarios of expected returns and risks:

$$\begin{aligned} \underset{\mathbf{w}}{\text{maximize}} \quad & \min_{\mu \in U_\mu} \mu^T \mathbf{w} - \lambda \max_{\Sigma \in U_\Sigma} \mathbf{w}^T \Sigma \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{1} = 1, \\ & w_i \geq 0 \, \forall i \in \{1, \ldots, n\}. \end{aligned}$$

Key Considerations:

1. **Choosing Uncertainty Sets**:
   - U_μ: Sets for mean returns (e.g., ellipsoidal, polyhedral).
   - U_Σ: Sets for covariance matrices (e.g., spectral norm, component-wise).
2. **Solving the Optimization**:
   - Convert the robust MVO into a tractable convex optimization problem.
   - Use techniques like robust quadratic programming and duality theory.

Objective:

- Find a portfolio allocation that maximizes the worst-case return while minimizing the worst-case risk, ensuring stability and robustness against parameter uncertainties.

Robust MVO with Box Uncertainty for Expected Returns

Uncertainty Set for Mean Returns:

- Assume a fixed covariance matrix $\hat{\Sigma}$ and define the **box uncertainty set** for the mean return vector μ:

$$U_\mu = \{\mu \mid -\delta \leq \mu - \bar{\mu} \leq \delta\}$$

- **Parameters**:
  - $\bar{\mu}$: Center of the uncertainty set (pre-defined mean vector).
  - δ: Size of the uncertainty set (range of possible deviations).

Robust MVO Formulation:

- The goal is to optimize portfolio weights $\mathbf{w}$ considering the worst-case scenario for μ U_μ:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \min\limits_{\mu \in U_\mu} \mu^T \mathbf{w} - \lambda \mathbf{w}^T \hat{\Sigma} \mathbf{w} \\ \text{subject to} & \mathbf{w}^T \mathbf{1} = 1 \end{array}$$

Inner Minimization Problem:

- The worst-case expected return simplifies to:

$$\min_{\mu \in U_\mu} \mu^T \mathbf{w} = \bar{\mu}^T \mathbf{w} - \delta |\mathbf{w}|$$

- This expression reflects the worst-case return, occurring at the boundaries of the uncertainty set U_μ, based on the sign of each w_i.

Reformulating and Solving the Robust MVO Problem

(optional)

Equivalent Reformulation:

- Replace the inner minimization with its closed-form solution:

$$\begin{aligned} \underset{\mathbf{w}}{\text{maximize}} \quad & \bar{\mu}^T\mathbf{w} - \delta|\mathbf{w}| - \lambda\mathbf{w}^T\hat{\Sigma}\mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T\mathbf{1} = 1 \end{aligned}$$

Handling Absolute Value with Auxiliary Variables:

- Introduce auxiliary variables $\psi \in \mathbb{R}^N$ to replace $|\mathbf{w}|$:

$$\begin{aligned} \underset{\mathbf{w},\psi}{\text{maximize}} \quad & \bar{\mu}^T\mathbf{w} - \delta\psi - \lambda\mathbf{w}^T\hat{\Sigma}\mathbf{w} && (1) \\ \text{subject to} \quad & \mathbf{w}^T\mathbf{1} = 1, && (2) \\ & \psi_i \geq w_i, \quad \psi_i \geq -w_i, \quad i \in \{1, ..., N\} && (3) \end{aligned}$$

Optimization Strategy:

- Convert the robust optimization problem to a standard quadratic program using auxiliary variables, increasing the dimensionality of the decision variables but enabling efficient solving.
- For a long-only portfolio ($w_i \geq 0$), directly incorporate $|\mathbf{w}|$ for a simpler formulation.

Uncertainty in the Covariance Matrix

(optional)

Concept: Incorporate uncertainty in the covariance matrix Σ of asset returns to account for estimation errors and improve portfolio robustness.

Nominal Covariance Matrix: $\bar{\Sigma}$ (estimated from historical data).

Perturbation Matrix: Δ is a symmetric matrix that represents uncertainty in Σ.

Constraint on Δ:

$$|\Delta_{ij}| \leq \kappa(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}}, \quad \forall i, j$$

- **Interpretation**: The deviation Δ_{ij} is bounded by a scaled geometric mean of variances $(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{1/2}$, where κ controls the level of uncertainty (e.g., $\kappa = 0.02$ or 0.05).

Uncertainty Set for Σ:

$$U_{\Sigma} = \left\{ \Sigma \;\middle|\; \begin{array}{l} \bar{\Sigma}_{ij} - \kappa(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}} \leq \Sigma_{ij} \leq \bar{\Sigma}_{ij} + \kappa(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}}, \\ \forall i, j \end{array} \right\}$$

Reformulating the MVO Problem with Covariance Uncertainty

(optional)

Revised MVO Formulation:

$$\begin{aligned} \underset{\mathbf{w}}{\text{maximize}} \quad & \boldsymbol{\mu}^T \mathbf{w} - \lambda \max_{\Sigma \in U_\Sigma} \mathbf{w}^T \Sigma \mathbf{w} \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{1} = 1, \\ & w_i \geq 0, \ \forall i. \end{aligned}$$

Inner Maximization:

$$\begin{aligned} \max_{\Sigma \in U_\Sigma} \mathbf{w}^T \Sigma \mathbf{w} &= \max_{|\Delta_{ij}| \leq \kappa (\bar{\Sigma}_{ii} \bar{\Sigma}_{jj})^{\frac{1}{2}}} \mathbf{w}^T (\bar{\Sigma} + \Delta) \mathbf{w} \\ &= \mathbf{w}^T \bar{\Sigma} \mathbf{w} + \max_{|\Delta_{ij}| \leq \kappa (\bar{\Sigma}_{ii} \bar{\Sigma}_{jj})^{\frac{1}{2}}} \mathbf{w}^T \Delta \mathbf{w} \end{aligned}$$

Simplification of the Inner Maximization Problem

(optional)

Inner Maximization Simplified:

$$\max_{|\Delta_{ij}|\leq\kappa(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}}} \mathbf{w}^T\Delta\mathbf{w} = \max_{|\Delta_{ij}|\leq\kappa(\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}}} \sum_{ij} w_i w_j \Delta_{ij}$$
$$= \kappa \sum_{ij} |w_i w_j| (\bar{\Sigma}_{ii}\bar{\Sigma}_{jj})^{\frac{1}{2}}$$
$$= \kappa \left(\sum_i |w_i| (\bar{\Sigma}_{ii})^{1/2} \right)^2$$

Final MVO Formulation with Covariance Uncertainty:

$$\max_{\mathbf{w}} \left(\boldsymbol{\mu}^T\mathbf{w} - \lambda \left(\mathbf{w}^T\bar{\Sigma}\mathbf{w} + \kappa \left(\sum_i |w_i| (\bar{\Sigma}_{ii})^{1/2} \right)^2 \right) \right)$$

Interpretation: The robust MVO model accounts for maximum possible variance due to covariance estimation errors, enhancing portfolio resilience against estimation errors by considering the worst-case scenario of Σ within the defined uncertainty set U_Σ.

Elliptical Uncertainty Over Mean Return

(optional)

Motivation: The expected returns $\boldsymbol{\mu}$ are often estimated with error. To account for this uncertainty, we introduce an elliptical uncertainty set around the nominal estimates $\hat{\boldsymbol{\mu}}$.

Elliptical Uncertainty Set:

$$U_{\mu} = \{\boldsymbol{\mu} \mid (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^T \boldsymbol{S}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \delta^2\}$$

- **Interpretation**: $\boldsymbol{S}^{-1}$ shapes the ellipsoid; δ controls the radius, reflecting the degree of uncertainty in $\boldsymbol{\mu}$.

Geometric Representation:

$$U_{\mu} = \left\{\boldsymbol{\mu} = \hat{\boldsymbol{\mu}} + \delta \boldsymbol{S}^{1/2} \mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\right\}$$

- **Key Components**:
  - $\mathbf{u}$ is a unit vector defining the direction of deviation.
  - $\boldsymbol{S}^{1/2}$ scales this deviation according to the uncertainty in asset returns (e.g., $\boldsymbol{S} = \hat{\boldsymbol{\Sigma}}$, the covariance matrix).

Robust MVO Formulation with Elliptical Uncertainty

(optional)

Robust MVO Problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \min_{\boldsymbol{\mu} \in U_{\boldsymbol{\mu}}} \left(\mathbf{w}^T \boldsymbol{\mu} - \lambda \mathbf{w}^T \hat{\boldsymbol{\Sigma}} \mathbf{w} \right) \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0} \end{array}$$

Inner Minimization Simplification:

$$\min_{\boldsymbol{\mu} \in U_{\boldsymbol{\mu}}} \mathbf{w}^T \boldsymbol{\mu} = \mathbf{w}^T \hat{\boldsymbol{\mu}} - \delta \|(\boldsymbol{S}^{1/2})^T \mathbf{w}\|_2$$

- **Key Insight**: The worst-case expected return within the uncertainty set is the nominal return minus a risk adjustment term $-\delta \|(\boldsymbol{S}^{1/2})^T \mathbf{w}\|_2$.

Final Robust MVO Formulation:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^T \hat{\boldsymbol{\mu}} - \lambda \mathbf{w}^T \hat{\boldsymbol{\Sigma}} \mathbf{w} - \delta \|(\boldsymbol{S}^{1/2})^T \mathbf{w}\|_2 \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0} \end{array}$$

Interpretation: This formulation hedges against the risk of return underestimation by optimizing for the worst-case scenario within the uncertainty ellipsoid, enhancing portfolio robustness.

Frobenius Norm and Uncertainty in the Design Matrix

(optional)

Frobenius Norm: Measures the size of a matrix as the square root of the sum of the squares of its elements, encapsulating the matrix within a spherical boundary.

Uncertainty in Design Matrix $\boldsymbol{X}$:

$$\boldsymbol{X} = \hat{\boldsymbol{X}} + \Delta$$

- $\hat{\boldsymbol{X}}$: Nominal design matrix.
- Δ: Perturbation matrix representing uncertainty.

Frobenius Norm Constraint:

$$\|\Delta\|_F \leq \delta_{\boldsymbol{X}}$$

- Defines a spherical uncertainty set around $\hat{\boldsymbol{X}}$.

Uncertainty Set:

$$U_{\boldsymbol{X}} = \{\boldsymbol{X} \mid \|\boldsymbol{X} - \hat{\boldsymbol{X}}\|_F \leq \delta_{\boldsymbol{X}}\}$$

- $\delta_{\boldsymbol{X}}$: Radius controlling the level of uncertainty.

Impact on the Covariance Matrix and Risk Term

(optional)

Covariance Matrix $\boldsymbol{\Sigma}$ Relation:

$$\boldsymbol{\Sigma} = \frac{1}{T}\boldsymbol{X}^T\boldsymbol{X}$$

- Uncertainty in $\boldsymbol{X}$ directly affects $\boldsymbol{\Sigma}$.

Revised Risk Component:

$$\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = \frac{1}{T}\|\boldsymbol{X}\mathbf{w}\|_2^2 = \frac{1}{T}\|(\hat{\boldsymbol{X}} + \Delta)\mathbf{w}\|_2^2$$

Worst-Case Risk Maximization:

$$\max_{\boldsymbol{X}\in U_{\boldsymbol{X}}}\|\boldsymbol{X}\mathbf{w}\|_2^2 = \max_{\|\Delta\|_F\leq\delta_{\boldsymbol{X}}}\|(\hat{\boldsymbol{X}} + \Delta)\mathbf{w}\|_2^2$$

Reformulating the Robust MVO Problem

(optional)

Applying the Triangular Inequality:

$$\|(\hat{\boldsymbol{X}} + \Delta)\mathbf{w}\|_2 \leq \|\hat{\boldsymbol{X}}\mathbf{w}\|_2 + \|\Delta\mathbf{w}\|_2$$

Bounding the Norm:

$$\|\Delta\mathbf{w}\|_2 \leq \|\Delta\|_F\|\mathbf{w}\|_2 \leq \delta_{\boldsymbol{X}}\|\mathbf{w}\|_2$$

Simplified Worst-Case Risk Expression:

$$\max_{\|\Delta\|_F \leq \delta_{\boldsymbol{X}}} \|(\hat{\boldsymbol{X}} + \Delta)\mathbf{w}\|_2^2 = (\|\hat{\boldsymbol{X}}\mathbf{w}\|_2 + \delta_{\boldsymbol{X}}\|\mathbf{w}\|_2)^2$$

Final Robust MVO Formulation:

$$\begin{aligned} \underset{\mathbf{w}}{\text{maximize}} \quad & \boldsymbol{\mu}^T\mathbf{w} - \lambda(\|\hat{\boldsymbol{X}}\mathbf{w}\|_2 + \delta_{\boldsymbol{X}}\|\mathbf{w}\|_2)^2 \\ \text{subject to} \quad & \mathbf{w}^T\mathbf{1} = 1, \\ & w_i \geq 0, \quad \forall i. \end{aligned}$$

Conclusion: This SOCP (Second-Order Cone Problem) formulation robustly addresses portfolio optimization by considering worst-case deviations in the design matrix, enhancing the model's resilience to uncertainty.

Coding session