**Theorem 0.1.** *Function* $F_1 : (t, x, y, z) \mapsto F_1(t, x, y, z), F_2$ *and* $F_3$ *and its any partial derivative is continuous in some rectangle area* $\mathbb{R} : (t, x, y, z) \in [t_1, t_2] \times [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$*, containing* $(t_0, x_0, y_0, z_0)$*. Then the initial value problem*

$$x' = F_1(t, x, y, z)$$
$$y' = F_2(t, x, y, z)$$
$$z' = F_3(t, x, y, z)$$
$$x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0$$

*Proof.* Let's make a new sequence $\{x_n\}, \{y_n\}, \{z_n\}$ by following iteration.

$$x_0 = x_0, \quad x_{n+1} = x_0 + \int_{t_0}^{t} F_1(s, x_n(s), y_n(s), z_n(s))ds$$
$$y_0 = y_0, \quad y_{n+1} = y_0 + \int_{t_0}^{t} F_2(s, x_n(s), y_n(s), z_n(s))ds$$
$$z_0 = z_0, \quad z_{n+1} = z_0 + \int_{t_0}^{t} F_3(s, x_n(s), y_n(s), z_n(s))ds$$

By the continuity assumption, we can apply extreme value theorem and set $|F_1| \leq M, |F_2| \leq N$, and $|F_3| \leq P$. W.L.O.G, we will only show this theorem with respect to x.

$$|x_n - x_0| = \int_{t_0}^{t} F_1(s, x_n, y_n, z_n)ds \leq M \cdot |t - t_0|$$

Thus, choosing $|t - t_0| \leq \min(a, \frac{b}{M})$ guarantees the existence of the sequence inside the rectangle region, where $a$ and $b$ each represents $\min\{t_0 - t_1, t_2 - t_0\}$ and $\min\{x_0 - x_1, x_2 - x_0\}$.

Next we will show the convergence of this sequnce by using

$$x_n = x_0 + \sum_{1}^{n}(x_i - i_{i-1})$$

The difference between to adjacent terms $|x_{n+1} - x_n|$ is

$$|x_{n+1} - x_n| = \left| \int_{t_0}^{t} F_1(s, x_n, y_n, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_{n-1})ds \right|$$

By the extreme value theorem, we can get some maximum values of following derivates.

$$A = \max \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix}$$

$$\begin{aligned} F_1(s, x_n, y_n, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_{n-1}) &= F_1(s, x_n, y_n, z_n) - F_1(s, x_{n-1}, y_n, z_n) \\ &+ F_1(s, x_{n-1}, y_n, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_n) \\ &+ F_1(s, x_{n-1}, y_{n-1}, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_{n-1}) \end{aligned}$$

$$\frac{\partial F_1^*}{\partial x}|x_n - x_{n-1}| + \frac{\partial F_1^*}{\partial y}|y_n - y_{n-1}| + \frac{\partial F_1^*}{\partial z}|z_n - z_{n-1}| \le A_{11}|x_n - x_{n-1}| + A_{12}|y_n - y_{n-1}| + A_{13}|z_n - z_{n-1}|$$

We use these symbols from now on.

$$A_i \coloneqq \max(A_{i1}, A_{i2}, A_{i3}) \text{ for } i = 1, 2, 3, \qquad p_n \coloneqq |x_n - x_{n-1}| + |y_n - y_{n-1}| + |z_n - z_{n-1}| \text{ for } n \ge 1$$

By the process above,

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \int_{t_0}^{t} F_1(s, x_n, y_n, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_{n-1}) ds \right| \\ &= |t - t_0| \cdot (F_1(s, x_n, y_n, z_n) - F_1(s, x_{n-1}, y_{n-1}, z_{n-1}))^* \\ &\le |t - t_0| \cdot A_{11}|x_n - x_{n-1}| + A_{12}|y_n - y_{n-1}| + A_{13}|z_n - z_{n-1}| \\ &\le |t - t_0| \cdot A_1 \cdot p_n \end{aligned}$$

Using the same method and summing the left handed side, we can get

$$|x_{n+1} - x_n| + |y_{n+1} - y_n| + |z_{n+1} - z_n| = p_{n+1} \le |t - t_0| \cdot p_n \cdot (A_1 + A_2 + A_3)$$

Hence, choosing choosing $|t - t_0| \le \min(a, \frac{b}{M}, \frac{r}{A_1 + A_2 + A_3})$ for $0 < r < 1$ will prove that $p_{n+1} \le r \cdot p_n$, making the new sequence $\{p_n\}$ contracting.

The next procedure is to evaluate the convergence of the sequence defined by following series

$$x_n = x_0 + \sum_{1}^{n} (x_i - x_{i-1})$$

by showing the absolute value of this series converges, and using the absolute convergence theorem guarantees the convergence of the original series.

$$x_n = x_0 + \sum_1^n (x_i - x_{i-1}) \leq x_0 + \sum_1^n |x_i - x_{i-1}| \leq x_0 + \sum_1^n p_i \leq x_0 + \sum_1^n r^i \cdot p_0$$

By applying both comparison test and absolute convergence theorem, the sequence $\{x_n\}$ converges into function $\xi(t)$. We can apply the same process to $\{y_n\}$ and $\{z_n\}$ to show they also converges into $\phi(t)$ and $\psi(t)$.

To apply fundamental theorem of calculus to the equation

$$x_0 = x_0, \quad x_{n+1} = x_0 + \int_{t_0}^t F_1(s, x_n(s), y_n(s), z_n(s))ds$$

we should show the continuity of the function near $t_0$ derived from the series above:

$$\xi(t) = x_0 + \sum_1^\infty (x_i - x_{i-1})$$

We should be able to find appropriate $\delta(\epsilon)$ to make this statement true: $\forall t \in (t_0 - \delta, t_0 + \delta)$, $|\xi(t) - \xi(t_0)| < \epsilon$ for any positive $\epsilon$.

$$|\xi(t) - \xi(t_0)| \leq |\xi(t) - x_n(t)| + |x_n(t) - x_n(t_0)| + |x_n(t_0) - \xi(t_0)| \qquad \text{by triangle inequality}$$

The first and third term could be evaluated by the following manipulation.

$$|\xi(t) - x_n(t)| = \sum_{n+1}^\infty (x_i - x_{i-1}) \leq \sum_{n+1}^\infty |x_i - x_{i-1}| \leq \sum_{n+1}^\infty p_i \leq \sum_{n+1}^\infty p_0 \cdot r^i = p_0 \cdot \frac{r^{n+1}}{1-r}$$

$$\exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |\xi - x_n| < \epsilon \text{ for any given } \epsilon > 0$$

In addition, the assumption that $\{x_n\}$ is continuous at $t_0$ makes the following statement true.

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |t - t_0| < \delta \longrightarrow |x_n(t) - x_n(t_0)| < \epsilon$$

Therefore, for any given $\epsilon_0 > 0$,

$$|\xi(t) - \xi(t_0)| \leq |\xi(t) - x_n(t)| + |x_n(t) - x_n(t_0)| + |x_n(t_0) - \xi(t_0)| < (\epsilon_0/3) \cdot 3 < \epsilon_0$$

making function $\xi(t)$ continuous near $t_0$. Applying fundamental theorem of calculus makes the following iteration reasonable after making the function $\phi(t), \psi(t)$.

$$\xi(t) = x_0 + \int_{t_0}^{t} F_1(s, \xi(s), \phi(s), \psi(s))ds$$

Now lets show the uniqueness of the set of function, $\{\xi(t), \phi(t), \psi(t)\}$. If we assume there are two different $\xi(t)$ and $\chi(t)$ for $x_n$. Then

$$\begin{aligned}
|\xi(t) - \chi(t)| &= \left| \int_{t_0}^{t} F_1(s, \xi(s), \phi(s), \psi(s) - F_1(s, \chi(s), \phi(s), \psi(s))ds \right| \\
&= |t - t_0| \cdot \frac{\partial F_1}{\partial x}(s^*)|\xi(s) - \chi(s)| \\
&\leq |t - t_0| \cdot A_{11} \cdot \ \max|\xi(s) - \chi(s)| \\
&\leq r \cdot \ \max|\xi(s) - \chi(s)|
\end{aligned}$$

which is contradictory since $0 < r < 1$, and the equation states $|\xi(t) - \chi(t)| < \max|\xi(t) - \chi(t)| \leq r \cdot \max|\xi(s) - \chi(s)|$. In the same way, we can prove that there is unique set of $\{\xi(t), \phi(t), \psi(t)\}$ satisfying the differential equation and initial value condition.

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