Numerical Analysis and Calculations Related to the Paper "**Research on the Theory of Black-Body Thermal Radiation**" as an Exercise

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Objective

As an exercise to learn about physics research and, more importantly, to apply what has been learned in university classes or through studying various books and resources, I have approached this paper. With knowledge at the level of an undergraduate physics student halfway through the program, I aim to tackle it to challenge myself in analyzing a paper and understanding how to study it. We will proceed purely by analyzing the data, simulating, and applying the new method for the black-body radiation distribution presented in the paper using programming. I also utilized Jupyter notebooks collected from my GitHub repository to benefit from the experience of others who have worked on black-body analysis.

1 Overview of Black-Body Radiation and Planck's Law

First, we will design a simple plot of the traditional black-body radiation spectrum curve, as seen in Figure 1 of the paper, using Planck's distribution function:

$$e_b(\lambda, T) = C_1 \lambda^{-5} \left(e^{\frac{C_2}{\lambda T}} - 1 \right)^{-1} \quad \{C_1 = 2hc^2 \quad , \quad C_2 = \frac{hc}{k}\} \tag{1}$$

As mentioned in the paper, it can be shown that the maximum of Planck's energy density occurs at a wavelength given by $\lambda_{max} = \frac{b}{T}$, where b is $2897.8268\,\mu\text{m}\cdot\text{K}$.[1] [2]

The maximum of $e_b(\lambda, T)$ occurs when the first derivative of the function with respect to wavelength is zero:

$$\frac{\partial e_b(\lambda, T)}{\partial \lambda} = 0$$

Using `Sympy` in Python, we simplify the equation:

$$\frac{C_1 \left(C_2 e^{\frac{C_2}{T\lambda}} - 5T\lambda \left(e^{\frac{C_2}{T\lambda}} - 1 \right) \right)}{T\lambda^7 \left(e^{\frac{C_2}{T\lambda}} - 1 \right)^2} = 0 \tag{2}$$

Ultimately, this leads to

$$(x-5)e^x + 5 = 0 \quad \text{or} \quad 5(e^x - 1) = xe^x, \quad \text{where} \quad x = \frac{C_2}{\lambda T}$$

$$5(e^x - 1) = xe^x \tag{3}$$

[1] Example 1.1, Quantum Mechanics: Concepts and Applications, Nouredine Zettili
[2] Exercises 2.4 of Quantum Mechanics, Walter Greiner

1.1 Numerical Solution Methods

1.1.1 Binary Search Method

Considering Equation (3), we have:

$$\lambda_{max}T = \frac{C_2}{x} \approx 2897.8186\ \mu m \cdot K, \quad \text{where} \quad x \approx 4.9651 \tag{4}$$

1.1.2 Newton-Raphson Method

To solve the transcendental Equation (3) numerically using the Newton-Raphson method, we first write the equation in the standard form $f(x) = 0$:

$$f(x) = 5(e^x - 1) - xe^x$$

Its derivative with respect to x is:

$$f'(x) = 5e^x - \left(e^x + xe^x\right) = (4 - x)e^x$$

The Newton-Raphson method operates on the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In each step, the current approximate value x_n and the slope of the function at that point are used to calculate the new estimate x_{n+1}. This process continues until the difference between two successive estimates is less than a precision threshold ε.

In the `SciPy` library, this equation can be solved with the `optimize.newton()` function, where x_0 is the initial guess for the root. This built-in function applies the Newton-Raphson formula and returns the approximate value of the root after convergence. The final result is:

$$\lambda_{max}T = \frac{C_2}{x} \approx 2897.8186\ \mu m \cdot K, \quad \text{where} \quad x \approx 4.965114 \tag{5}$$

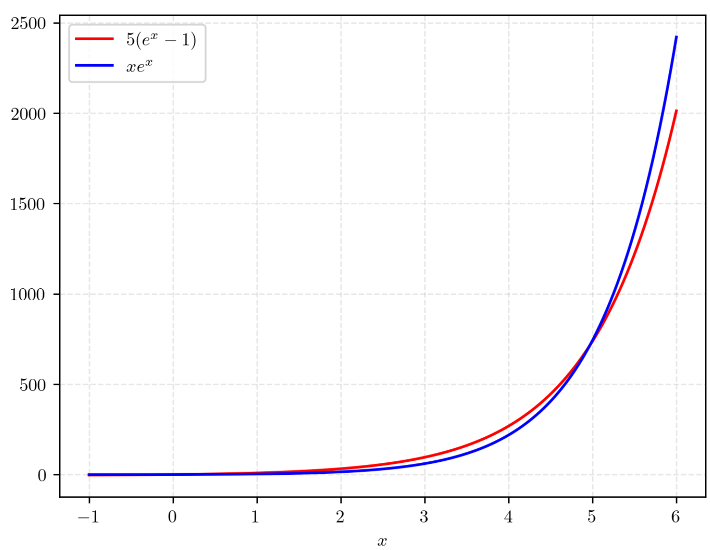

Figure 1: The two curves of the transcendental Equation (3).

1.1.3 Graphical Method (Plotting)

Using the graphical method, where one arrives at the value by observing the graph and narrowing down the interval, we ultimately have:

$$\lambda_{max}T = \frac{C_2}{x} \approx 2918.3649\ \mu m \cdot K, \quad \text{where} \quad x \approx 4.9301580 \tag{6}$$

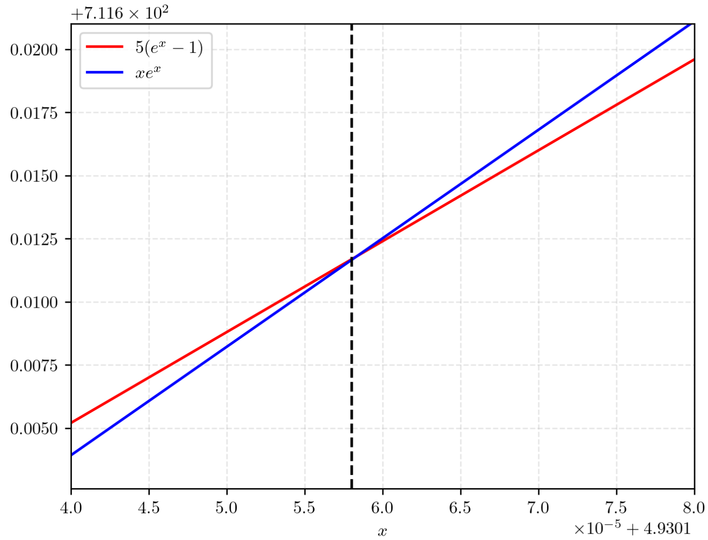

Figure 2: Intersection of the two curves.

1.1.4 Fixed-Point Iteration Method

To solve Equation (3), we must first rewrite it in a fixed-point form. By dividing both sides by e^x, we get:

$$5\left(1 - e^{-x}\right) = x$$

which takes the form

$$x = g(x) = 5\left(1 - e^{-x}\right)$$

Then, by choosing an initial guess x_0 and using the iterative relation $x_{n+1} = g(x_n)$, new values are calculated until the convergence condition $|x_{n+1} - x_n| < \varepsilon$ is met. With this method and choosing $x_0 = 6$, the final value of the root is:

$$\lambda_{max}T = \frac{C_2}{x} \approx 2897.8185\ \mu m \cdot K, \quad \text{where} \quad x \approx 4.9651142 \tag{7}$$

1.2 Plotting the Graphs

Following the provided Jupyter notebook, we plotted the graphs. First, Function (1) was plotted, with the following result:

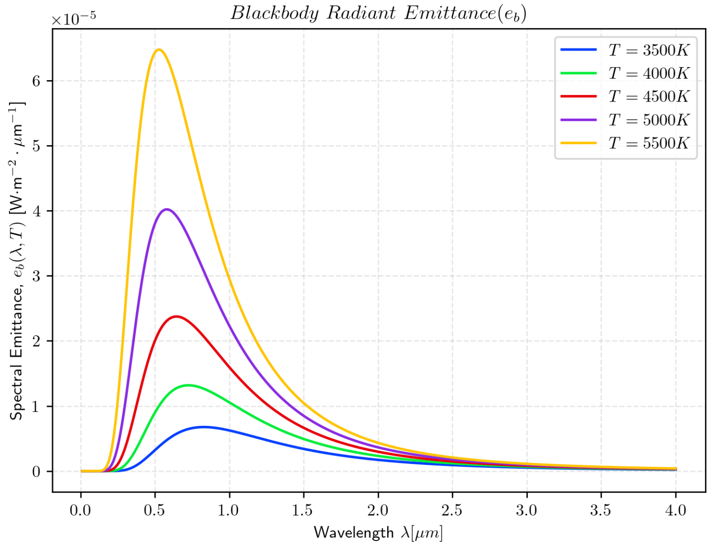

Figure 3: Graph of $e_b(\lambda, T)$.

Then, according to the traditional black-body radiation spectrum curve in the paper, which had wavelength in nanometers, we plotted it based on the function below (similar to Eq. 8 in the paper):

$$u(\lambda) = \frac{\alpha}{\lambda^5 \left(e^{\beta} - 1\right)} \times 10^{-3} \quad (\alpha = 8\pi hc, \quad \beta = \frac{hc}{\lambda kT}) \tag{8}$$

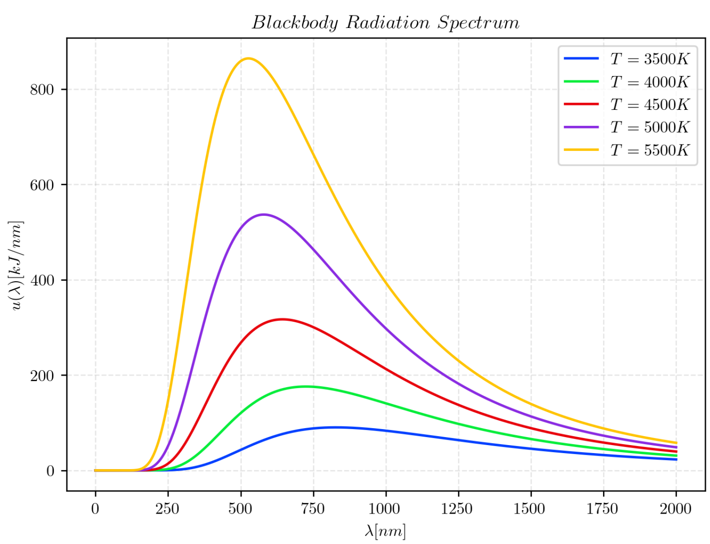

Figure 4: Graph of $u(\lambda)$.

1.3 The Importance of Calculating Inflection Points in the Paper

The paper calculates the inflection points of Planck's law curve to analyze the shape and symmetry of the black-body radiation curve. These points, where the second derivative of the radiation intensity becomes zero (with constants $\lambda_{li}T \approx 4082.66\,\mu\mathrm{m}\cdot\mathrm{K}$ and $\lambda_{ri}T \approx 1703.82\,\mu\mathrm{m}\cdot\mathrm{K}$), are used to define the relative width and symmetry factor of the curve. They mark the boundaries where the curve changes from a steep slope to a gentle one. Furthermore, the paper uses these points to develop a novel wavelength-based thermometry method, which is based on the ratio $\lambda_{li}/\lambda_{ri}$ or their position relative to λ_m (the peak wavelength), and can be more accurate than traditional methods like Wien's law.

These calculations are combined with normalization equations (like (6), (7), and (8) in the paper) to determine the effective wavelength range (e.g., from λ_{min} to λ_{max} for a small η), especially at temperatures from 200 to 6000 K. The paper shows that the inflection points can provide practical boundaries for radiation modeling, which forms the basis for practical innovations in thermal sensing and spectral analysis.

The inflection points were obtained using `nsolve`, and this can be used for different temperatures:

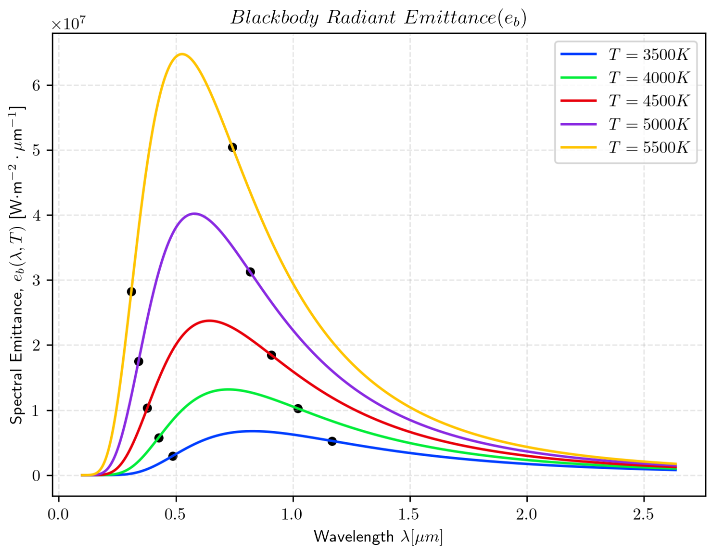

Figure 5: Inflection points at different temperatures.

The relationship between the left inflection point λ_{il} and the right inflection point λ_{ir} with temperature T has been extracted as follows:

$$\lambda_{il}T = 4082.6999\,\mu\mathrm{m}\cdot\mathrm{K} \tag{9}$$

$$\lambda_{ir}T = 1703.8229\,\mu\mathrm{m}\cdot\mathrm{K} \tag{10}$$